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ANALYSIS FOR DIFFUSION PROCESSES ON RIEMANNIAN MANIFOLDS

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Preface

As a cross research field of probability theory and Riemannian geometry, stochastic analysis on Riemannian manifolds devotes to providing probabilistic solutions of problems arising from differential geometry and developing a complete theory of diffusion processes on Riemannian manifolds. Since 1980s, many important contributions have been made in this field, which include, as two typical examples, probabilistic proofs of the Hörmander theorem and the Atiyah-Singer index theorem made by P. Malliavin and J. M. Bismut respectively. We would also like to mention three powerful tools developed in the literature: Malliavian calculus, Bakry-Emery's semigroup argument, and coupling method, which have led to numerous results for diffusion processes and applications to geometry analysis. For instance, as included in the present book, about twenty equivalent semigroup inequalities have been found for the curvature lower bound condition by using these tools, and these semigroup inequalities are crucial in the study of various different topics in the field.

Based on recent progresses made in the last decade, this book aims to present a self-contained theory concerning (reflecting) diffusion processes on Riemannian manifolds with or without boundary, and thus complements some earlier published books in the literature: [Bismut (1984)], [Emery (1989)], [Elworthy (1982)], [Hsu (2002a)], [Ikeda and Watanabe (1989)], [Malliavin (1997)], and [Stroock (2000)]. The author did not intend to include in the book all recent contributions in the field, materials of the book are selected systematically but mainly according to his own research interests.

The book consists of five chapters. The first chapter contains necessary preparations for the study, which include a collection of fundamental results from Riemannian manifold, coupling method and applications, and a brief theory of functional inequalities. The second chapter is devoted to the theory of diffusion processes on Riemannian manifolds without boundary, where various equivalent semigroup properties are presented for the curvature lower bound of the underlying diffusion operator. These equivalent properties have been applied to the study of functional inequalities, Harnack inequalities and applications, and transportation-cost inequalities. The third chapter aims to build up a corresponding theory for the reflecting diffusion processes on Riemannian manifold with boundary, for which equivalent semigroup properties are presented for both the curvature lower bound and the lower bound of the second fundamental form of the boundary. As applications, functional/Harnack/transportation-cost inequalities as well as the Robin semigroup are closely investigated. In Chapter 4 we investigate the stochastic analysis on the path space of the reflecting diffusion process on a Riemannian manifold with boundary. The main content includes the quasi-invariant flow induced by stochastic differential equations with reflection, integration by parts formula for the damped gradient operator, and the log-Sobolev/transpotation-cost inequalities. Finally, in Chapter 5, functional inequalities and regularity estimates for sub-elliptic diffusion processes are studied by using Malliavin calculus as well as arguments introduced in the previous chapters.

Most of the book is organized from the author's recent publications concerning diffusion processes on manifolds, including joint papers with colleagues who are gratefully acknowledged for their fruitful collaborations. I would like to thank Lijuan Cheng, Xiliang Fan, Huaiqian Lee, Jian Wang, Shaoqin Zhang and Ms. Lai Fun Kwong for reading earlier drafts of the book and corrections. A main part of Chapter 3 has been presented for a mini course in the Chinese Academy of Science. I would like to thank Xiang-Dong Li for the kind invitation and all audience who attended the mini course. I would also like to thank my colleagues from the probability groups of Beijing Normal University and Swansea University, in particular Mu-Fa Chen, Wenming Hong, Niels Jacob, Zenghu Li, Eugene Lytvynov, Yonghua Mao, Aubrey Truman, Jiang-Lun Wu, Chenggui Yuan and Yuhui Zhang. Their kind help and constant encouragement have provided an excellent working environment for me. Finally, financial support from the National Natural Science Foundation of China, Specialized Research Foundation for Doctorial Programs, the Fundamental Research Funds for the

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Feng-Yu Wang

Contents

Preface v

1. Preliminaries 1

1.1 Riemannian manifold 1
1.1.1 Differentiable manifold 1
1.1.2 Riemannian manifold 3
1.1.3 Some formulae and comparison results 9
1.2 Riemannian manifold with boundary 11
1.3 Coupling and applications 15
1.3.1 Transport problem and Wasserstein distance . . . 16
1.3.2 Optimal coupling and optimal map 18
1.3.3 Coupling for stochastic processes 19
1.3.4 Coupling by change of measure 22
1.4 Harnack inequalities and applications 24
1.4.1 Harnack inequality 24
1.4.2 Shift Harnack inequality 31
1.5 Harnack inequality and derivative estimate 33
1.5.1 Harnack inequality and entropy-gradient estimate 33
1.5.2 Harnack inequality and L^2-gradient estimate . . . 36
1.5.3 Harnack inequalities and gradient-gradient estimates 37
1.6 Functional inequalities and applications 39
1.6.1 Poincaré type inequality and essential spectrum . 39
1.6.2 Exponential decay in the tail norm 42
1.6.3 The F-Sobolev inequality 42
1.6.4 Weak Poincaré inequality 43
1.6.5 Equivalence of irreducibility and weak Poincaré inequality 45

2. Diffusion Processes on Riemannian Manifolds without Boundary 49

2.1 Brownian motion with drift 49
2.2 Formulae for ∇P_t and Ric_Z 54
2.3 Equivalent semigroup inequalities for curvature lower bound 60
2.4 Applications of equivalent semigroup inequalities 72
2.5 Transportation-cost inequality 77
2.5.1 From super Poincaré to weighted log-Sobolev inequalities 79
2.5.2 From log-Sobolev to transportation-cost inequalities 82
2.5.3 From super Poincaré to transportation-cost inequalities 87
2.5.4 Super Poincaré inequality by perturbations 92
2.6 Log-Sobolev inequality: Different roles of Ric and Hess . . 95
2.6.1 Exponential estimate and concentration of μ . . . 96
2.6.2 Harnack inequality and the log-Sobolev inequality 98
2.6.3 Hypercontractivity and ultracontractivity 102
2.7 Curvature-dimension condition and applications 109
2.7.1 Gradient and Harnack inequalities 109
2.7.2 HWI inequalities 120
2.8 Intrinsic ultracontractivity on non-compact manifolds . . 127
2.8.1 The intrinsic super Poincaré inequality 129
2.8.2 Curvature conditions for intrinsic ultracontractivity 131
2.8.3 Some examples 136

3. Reflecting Diffusion Processes on Manifolds with Boundary 141

3.1 Kolmogorov equations and the Neumann problem 142
3.2 Formulae for $\nabla P_t, \mathrm{Ric}_Z$ and $\mathbb{I}$ 146
3.2.1 Formula for ∇P_t 146
3.2.2 Formulae for Ric_Z and $\mathbb{I}$ 149
3.2.3 Gradient estimates 152
3.3 Equivalent semigroup inequalities for curvature condition and lower bound of $\mathbb{I}$ 159
3.3.1 Equivalent statements for lower bounds of Ric_Z and $\mathbb{I}$ 159
3.3.2 Equivalent inequalities for curvature-dimension condition and lower bound of $\mathbb{I}$ 165
3.4 Harnack inequalities for SDEs on $\mathbb{R}^d$ and extension to non-convex manifolds 167

3.4.1 Construction of the coupling 169
3.4.2 Harnack inequality on $\mathbb{R}^d$ 174
3.4.3 Extension to manifolds with convex boundary . . 176
3.4.4 Neumann semigroup on non-convex manifolds . . 180
3.5 Functional inequalities 181
3.5.1 Estimates for inequality constants on compact manifolds 181
3.5.2 A counterexample for Bakry-Emery criterion . . . 184
3.5.3 Log-Sobolev inequality on locally concave manifolds 186
3.5.4 Log-Sobolev inequality on non-convex manifolds . 190
3.6 Modified curvature tensors and applications 195
3.6.1 Equivalent semigroup inequalities for the modified curvature lower bound 196
3.6.2 Applications of Theorem 3.6.1 200
3.7 Generalized maximum principle and Li-Yau's Harnack inequality 204
3.7.1 A generalized maximum principle 206
3.7.2 Li-Yau type gradient estimate and Harnack inequality 210
3.8 Robin semigroup and applications 214
3.8.1 Characterization of $P_t^{Q,W}$ and $\mathcal{D}(\mathcal{E}_Q)$ 215
3.8.2 Some criteria on λ_Q for $\mu(M) = 1$ 218
3.8.3 Application to HWI inequality 222

4. Stochastic Analysis on Path Space over Manifolds with Boundary 227

4.1 Multiplicative functional 228
4.2 Damped gradient, quasi-invariant flows and integration by parts 234
4.2.1 Damped gradient operator and quasi-invariant flows 234
4.2.2 Integration by parts formula 236
4.3 The log-Sobolev inequality 239
4.3.1 Log-Sobolev inequality on W_x^T 239
4.3.2 Log-Sobolev inequality on the free path space . . 242
4.4 Transportation-cost inequalities on path spaces over convex manifolds 245
4.5 Transportation-cost inequality on the path space over nonconvex manifolds 251

4.5.1 The case with a diffusion coefficient 252
4.5.2 Non-convex manifolds 255

5. Subelliptic Diffusion Processes 257

5.1 Functional inequalities 259
5.1.1 Super and weak Poincaré inequalities 259
5.1.2 Nash and log-Sobolev inequalities 265
5.1.3 Gruschin type operator 275
5.1.4 Kohn-Laplacian type operator 277
5.2 Generalized curvature and applications 283
5.2.1 Derivative inequalities 285
5.2.2 Applications of Theorem 5.2.1 289
5.2.3 Examples 293
5.2.4 An extension of Theorem 5.2.1 300
5.3 Stochastic Hamiltonian system: Coupling method 303
5.3.1 Derivative formulae 304
5.3.2 Gradient estimates 309
5.3.3 Harnack inequality and applications 317
5.3.4 Integration by parts formula and shift Harnack inequality 322
5.4 Stochastic Hamiltonian system: Malliavin calculus 328
5.4.1 A general result 328
5.4.2 Explicit formula 332
5.4.3 Two specific cases 338
5.5 Gruschin type semigroups 343
5.5.1 Derivative formula 343
5.5.2 Log-Harnack inequality 353

Bibliography 365

Index 377

Chapter 1

Preliminaries

In this chapter we collect necessary preliminaries used in the book. §1.1 and §1.2 consist of some fundamental contents from Riemannian geometry (see e.g. [Chavel (1984)] and [Cheeger and Ebin (1975)]); §1.3 is a brief account for coupling arguments and applications organized from [Cranston and Greven (1995)], [Lindvall and Rogers (1986)], [Lindvall (1992)], [Wang (2010b)], [Wang (2012a)] and [Wang (2012d)]; §1.4 and §1.5 are mainly selected from [Wang (2012d); Wang and Yuan (2011)] for Harnack inequalities, derivative formulae and their applications; and §1.6 introduces some general results on functional inequalities and applications (see [Wang (2005a)]).

1.1 Riemannian manifold

1.1.1 *Differentiable manifold*

Let M be a Hausdorff topological space with a countable basis of open sets. For each open set $\mathcal{O} \subset M$, if $\varphi : \mathcal{O} \to \mathbb{R}^d$ is one-to-one and $\varphi(\mathcal{O})$ is open, then $(\mathcal{O}, \varphi)$ is called a *coordinate neighborhood* on M. A *d-dimensional differential structure* on M is a family $\mathcal{U} := \{(\mathcal{O}_\alpha, \varphi_\alpha)\}$ of coordinate neighborhoods such that

(i) $\bigcup_\alpha \mathcal{O}_\alpha \supset M$,
(ii) For any α, β, $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{O}_\beta \bigcap \mathcal{O}_\alpha) \to \varphi_\alpha(\mathcal{O}_\alpha \bigcap \mathcal{O}_\beta)$ is C^∞-smooth, i.e. $(\mathcal{O}_\alpha, \varphi_\alpha)$ and $(\mathcal{O}_\beta, \varphi_\beta)$ are C^∞*-compatible*,
(iii) If a coordinate neighborhood $(\mathcal{O}, \varphi)$ is C^∞-compatible with each $(\mathcal{O}_\alpha, \varphi_\alpha)$ in $\mathcal{U}$, then $(\mathcal{O}, \varphi) \in \mathcal{U}$.

If M is equipped with a differential structure, then it is called a d-dimensional *differentiable manifold*, and each $(\mathcal{O}, \varphi) \in \mathcal{U}$ is called a *local (coordinate) chart.*

For any $p \in \mathbb{Z}_+$, the set of all non-negative integers, and an open set $D \subset \mathbb{R}^d$, a function $h : D \to \mathbb{R}$ is called C^p-smooth and denoted by $h \in C^p(D)$, if it is continuous when $p = 0$ and has continuous derivatives up to order p when $p \geq 1$. A function $f : M \to \mathbb{R}$ is called C^p-smooth and denoted by $f \in C^p(M)$, if for any $(\mathcal{O}, \varphi) \in \mathcal{U}$ there holds $f \circ \varphi^{-1} \in C^p(\varphi(\mathcal{O}))$. Let $C^p(M)$ denote the set of all C^p-smooth functions on M, and $C_0^p(M)$ the set of such functions with compact supports. When $p = 0$ we denote $C(M) = C^p(M)$ and $C_0^p(M) = C_0(M)$. Moreover, let $C^\infty(M) = \bigcap_{p \geq 1} C^p(M)$ and $C_0^\infty(M) = \bigcap_{p \geq 1} C_0^p(M)$. Finally, given $x \in M$, let $C^p(x)$ be the set of C^p-smooth functions defined in a neighborhood of x. When $p = 0$ we denote $C(x) = C^p(x)$ and $C_0^p(x) = C_0(x)$. Moreover, let $C^\infty(x) = \bigcap_{p \in \mathbb{N}} C^p(x)$ and $C_0^\infty(x) = \bigcap_{p \in \mathbb{N}} C_0^p(x)$.

Definition 1.1.1. Let M be a differentiable manifold. The *tangent space* T_xM at a point $x \in M$ is the set of all mappings $X : C^\infty(x) \to \mathbb{R}$ satisfying:

(i) $X(c_1 f + c_2 g) = c_1 Xf + c_2 Xg, \quad f, g \in C^\infty(x), \ c_1, c_2 \in \mathbb{R}^d,$
(ii) $X(fg) = (Xf)g(x) + f(x)Xg, \quad f, g \in C^\infty(x).$

Obviously, T_xM is a vector space by the convention

$$(X + Y)f := Xf + Yf, \quad (cX)f := c(Xf), \quad c \in \mathbb{R}, \ f \in C^\infty(x).$$

Let $x \in \mathcal{O}$ with $(\mathcal{O}, \varphi) \in \mathcal{U}$. then for any vector Z at $\varphi(x)$ on $\mathbb{R}^d$. one may define $\varphi^* Z \in T_xM$ by

$$(\varphi^* Z)f := Z(f \circ \varphi^{-1}), \quad f \in C^\infty(x).$$

Let $(u_1, \ldots, u_d)$ be the Euclidean coordinate on $\varphi(\mathcal{O})$, and let $\frac{\partial}{\partial x_i} := \varphi^* \frac{\partial}{\partial u_i}$, $1 \leq i \leq d$. For any $X \in T_xM$ one has $X = \varphi^* \varphi_* X$, where $\varphi_* X$ is a vector at $\varphi(x)$ satisfying

$$(\varphi_* X)g := X(g \circ \varphi), \quad g \in C^\infty(\varphi(x)).$$

Then

$$X = \varphi^* \left(\sum_{i=1}^d \langle \varphi_* X, \frac{\partial}{\partial u_i} \rangle_{\mathbb{R}^d} \frac{\partial}{\partial u_i} \right) = \sum_{i=1}^d \langle \varphi_* X, \frac{\partial}{\partial u_i} \rangle_{\mathbb{R}^d} \frac{\partial}{\partial x_i}.$$

Therefore, $\frac{\partial}{\partial x_i}$ is a basis of T_xM.

Now, let $TM := \bigcup_{x\in M} T_xM$, which is called the *vector bundle* on M. A *vector field* on M is a mapping

$$X : M \to TM; \quad X_x \in T_xM, \quad x \in M.$$

Let $\Gamma(TM)$ be the set of all vector fields on M. A vector field X is called C^p-smooth if in any local chart there exist C^p-smooth functions $f_1, \dots, f_d$ such that

$$X = \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i}.$$

Let $\Gamma^p(TM)$ denote the set of all C^p-smooth vector fields.

Definition 1.1.2. Let M be a differentiable manifold. A mapping $\nabla : TM \times \Gamma^1(TM) \to TM$ is called a *connection* on M, if it is bilinear and $\nabla_X Y := \nabla(X, Y)$ has the following properties: for any $x \in M$,

(i) If $X \in T_xM$ and $Y \in \Gamma^1(TM)$, then $\nabla_X Y \in T_xM$;
(ii) For any $f \in C^1(M)$, $\nabla_X(fY) = (Xf)Y_x + f(x)\nabla_X Y$, $X \in T_xM$, $Y \in \Gamma^1(TM)$.

1.1.2 *Riemannian manifold*

Definition 1.1.3. Let M be a differentiable manifold. For each $x \in M$, let g_x be an *inner product* on the vector space T_xM. If for any local chart $(\mathcal{O}, \varphi)$ and any $X, Y \in \Gamma^\infty(TM)$, $g_x(X_x, Y_x)$ is C^∞-smooth in x, then g is called a *Riemannian metric* on M. A differentiable manifold equipped with a Riemannian metric is called a *Riemannian manifold.*

In the sequel, we also denote $g = \langle \cdot, \cdot \rangle$. It is clear that under a local chart $(\mathcal{O}, \varphi)$ a Riemannian metric has the representation

$$\Big\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \Big\rangle := g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = g_{ij}(x), \quad 1 \le i,\ j \le d,$$

so that $g_{ij} \in C^\infty(M)$ and $(g_{ij}(x))$ is strictly positive definite at each $x \in \mathcal{O}$. Moreover, the Riemannian metric determines a unique measure such that for any local chart $(\mathcal{O}, \varphi)$,

$$\mathrm{vol}(A) = \int_{\varphi(A)} \sqrt{\det g \circ \varphi^{-1}(u)}\, \mathrm{d}u, \quad A \subset U.$$

We call this measure the *volume measure* of M and simply denote it by $\mathrm{d}x$.

Theorem 1.1.1 (Levi-Civita). *If M is a Riemannian manifold, then there exists a unique connection ∇ (called Levi-Civita connection) satisfying*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad \nabla_X Y = \nabla_Y X + [X, Y]$$

for all $X, Y, Z \in \Gamma^1(TM)$, where $\langle , \rangle$ denotes the inner product under the Riemannian metric and $[X, Y] := XY - YX$.

Throughout the book, we only use the Levi-Civita connection. It is useful to note that $[X, Y]$ is a vector field for any $X, Y \in \Gamma^1(TM)$. A mapping $\gamma : [\alpha, \beta] \to M$ is called a C^p-curve on M if it is continuous and for any local chart $(\mathcal{O}, \varphi)$, $\varphi \circ \gamma : [\alpha, \beta] \bigcap \gamma^{-1}(\mathcal{O}) \to \mathbb{R}^d$ is C^p-smooth. For a C^1-curve γ, we may define the *tangent vector* along γ by

$$\dot{\gamma}_t f = \langle \dot{\gamma}_t, \nabla f(\gamma_t)\rangle := \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t), \quad f \in C^\infty(\gamma_t).$$

Definition 1.1.4. (1) Let $\gamma : [\alpha, \beta] \to M$ be a C^1-curve on M. A vector field X is said to be constant (or parallel) along γ if $\nabla_{\dot{\gamma}_t} X = 0$ for $t \in [\alpha, \beta]$. Given $V \in T_{\gamma_\alpha} M$, there exists a unique constant vector field X along γ satisfying $X_{\gamma_\alpha} = V$. We call this vector field the *parallel transportation* of V along γ. A C^2-curve γ is called *geodesic* if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

(2) For any $x \in M$ and any $X \in T_x M$, $X \neq 0$, there exists a unique geodesic $\gamma : [0, \infty) \to M$ such that $\gamma_0 = x$ and $\dot{\gamma}_0 = X$. We denote $\gamma_t := \exp_x(tX)$ and call $\exp_x : T_x M \to M$ the *exponential map* at x. By convention we set $\exp_x(0) = x$.

For any $x \neq y$, one may define the *Riemannian distance* between x and y by

$$\rho(x, y) := \inf\left\{ \int_0^1 |\dot{\gamma}_s| \mathrm{d}s : \gamma : [0, 1] \to M \right.$$
$$\left. \text{is a } C^1\text{-curve such that } \gamma_0 = x \text{ and } \gamma_1 = y \right\},$$

where $|X| = \langle X, X\rangle^{1/2} := g(X, X)^{1/2}$. Throughout the book we only consider connected M, i.e. $\rho(x, y) < \infty$ for any $x, y \in M$. In this case $\rho(x, y)$ can be reached by a geodesic. On the other hand, however, geodesics linking two points may not be unique. Thus, the one with length $\rho(x, y)$ is called the *minimal geodesic*. For any point $o \in M$, let $\rho_o = \rho(o, \cdot)$.

In many cases, the minimal geodesic is still not unique. For instance, for the unit sphere $\mathbb{S}^d$, each half circle linking the highest and the lowest points is a minimal geodesic. This fact leads to the following notion of cut-locus.

Definition 1.1.5. Let $x \in M$. For any $X \in \mathbb{S}_x := \{X \in T_xM : |X| = 1\}$, let

$$r(X) := \sup\{t > 0 : \rho(x, \exp_x(tX)) = t\}.$$

If $r(X) < \infty$ then we call $\exp_x(r(X)X)$ a cut-point of x. The set

$$\mathrm{cut}(x) := \{\exp_x(r(X)X) : X \in \mathbb{S}_x,\ r(X) < \infty\}$$

is called the *cut-locus* of the point x. Moreover, the quantity

$$\mathrm{i}_x := \inf\{r(X) : X \in \mathbb{S}_x\}$$

is called the *injectivity radius* of x. Finally, we call $\mathrm{i}_M := \inf_{x\in M} \mathrm{i}_x$ the injectivity radius of M.

The following result summarizes some properties of the cut-locus.

Theorem 1.1.2. (1) $\mathrm{cut}(x)$ *is closed and has volume zero.*
(2) $\rho(x, \cdot)$ *is* C^∞*-smooth on* $M \setminus (\{x\} \cup \mathrm{cut}(x))$.
(3) $\mathrm{i}_x > 0$ *for any* $x \in M$ *and the function* $\mathrm{i} : M \to (0, \infty]$ *is continuous.*
(4) *The set* $D_x := \exp_x^{-1}(M \setminus \mathrm{cut}(x))$ *is starlike in* T_xM *and*

$$\exp_x : D_x \to \exp_x(D_x)$$

is a diffeomorphism. Consequently, if $y \notin \mathrm{cut}(x)$ *then the minimal geodesic linking* x *and* y *is unique.*

We now introduce the curvature on M. For any $X, Y, Z \in \Gamma^2(TM)$, let

$$\mathcal{R}(X,Y)Z := \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,$$

where $[X, Y] := XY - YX$ is the Lie bracket of X and Y. For any $X_x, Y_x, Z_x \in T_xM$, let X, Y, Z be their smooth extensions respectively. Then the value of $\mathcal{R}(X,Y)Z$ at point x is independent of the choices of extensions and hence, $\mathcal{R}$ is a well-defined tensor which is called the *curvature* tensor of the connection ∇.

The curvature tensor satisfies the following identities:

$$\begin{aligned}
&\mathcal{R}(X,Y)Z + \mathcal{R}(Y,X)Z = 0,\\
&\mathcal{R}(X,Y)Z + \mathcal{R}(Z,X)Y + \mathcal{R}(Y,Z)X = 0,\\
&\langle \mathcal{R}(X,Y)Z, V\rangle = \langle \mathcal{R}(Z,V)X, Y\rangle = -\langle \mathcal{R}(X,Y)V, Z\rangle.
\end{aligned}$$

Definition 1.1.6. (1) For $X, Y \in T_xM$, the quantity

$$\mathrm{Sect}(X,Y) := \frac{\langle \mathcal{R}(X,Y)X, Y\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

is called the *sectional curvature* of the plane spanned by X and Y. If X is parallel to Y then we set $\mathrm{Sect}(X,Y)=0$.

(2) Let $\{W_i\}_{i=1}^d$ be an orthonormal basis on T_xM. The quantity

$$\mathrm{Ric}(X,Y):=\sum_{i=1}^d\langle\mathcal{R}(X,W_i)Y,W_i\rangle$$

is independent of the choice of $\{W_i\}$ and Ric is called the *Ricci curvature* tensor.

(3) Let γ be a geodesic. A smooth vector field J is called a *Jacobi field* along γ if

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J=-\mathcal{R}(\dot\gamma,J)\dot\gamma.$$

This equation is called the *Jacobi equation.*

Since the Jacobi equation is a second order ordinary equation, given $X,Y\in T_{\gamma_0}M$, there exists a Jacobi field along γ such that $J_0=X$, $\nabla_{\dot\gamma_t}J_t|_{t=0}=Y$. Moreover, let $\gamma:[0,t]\to M$ be a geodesic, for any $X\in T_{\gamma_0}M$ and $Y\in T_{\gamma_t}M$, there exists a Jacobi field J along γ satisfying $J_0=X$ and $J_t=Y$. Concerning the uniqueness of Jacobi fields, we introduce the notion of conjugate point.

Definition 1.1.7. Let $x\in M$. $y\in M$ is called a conjugate point of x, if there exists a nontrivial Jacobi field J along a minimal geodesic linking x and y such that J vanishes at x and y.

Proposition 1.1.3. $\mathrm{cut}(x)$ *consists of conjugate points of x and points having more than one minimal geodesics to x.*

To make analysis on Riemannian manifolds, let us introduce some fundamental operators including the divergence, the gradient and the Laplace operators.

Definition 1.1.8. Let $X\in\Gamma^1(TM)$, we define its *divergence* by

$$(\mathrm{div}X)(x)=(\mathrm{tr}\nabla X)(x)=\sum_{i=1}^d\langle\nabla_{W_i}X,W_i\rangle,$$

where $\{W_i\}$ is an orthonormal basis of T_xM. It is easy to check that $\mathrm{div}X$ is independent of the choice of $\{W_i\}$. For $f\in C^1(M)$, define its *gradient* $\nabla f\in\Gamma(TM)$ by

$$\langle\nabla f,X\rangle=Xf,\quad X\in\Gamma(TM).$$

Finally, the *Laplace operator* is defined by $\Delta=\mathrm{div}\nabla$ which acts well on C^2-functions.

Next, we introduce calculus for *differential forms.* We call a function on M a 0-form, and for $p \in \mathbb{N}$, an alternating linear functional on $\otimes^p TM$ is called a p-form, where a map $\theta : \otimes^p TM \to \mathbb{R}$ is called alternating, if for any $X_1, \dots, X_p \in TM$ and any permutation σ of $\{1, \dots, p\}$, there holds

$$\theta(X_{\sigma(1)}, \dots, X_{\sigma(p)}) = \mathrm{sgn}(\sigma)\theta(X_1, \dots, X_p).$$

We write $\mathrm{sgn}(\sigma) = 1$ if it is representable as even number of transposition and $\mathrm{sgn}(\sigma) = -1$ otherwise. It is clear that for $p > d$, there is only zero p-form; and for any 1-form θ there exists a dual vector field θ^* such that

$$\langle \theta^*, X \rangle = \theta(X), \quad X \in TM.$$

A linear functional Φ on $\otimes^p TM$ is called a *p-tensor.* For a p-tensor Φ and a q-tensor Ψ, we define their product $(p+q)$-tensor $\Phi \otimes \Psi$ by letting

$$(\Phi \otimes \Psi)(X_1, \dots, X_p, Y_1, \dots, Y_q) = \Phi(X_1, \dots, X_p)\Psi(Y_1, \dots, Y_q)$$

for $X_1, \dots, X_p, Y_1, \dots, Y_q \in TM$. To make a p-form from a p-tensor Φ, we introduce the alternating map $\mathcal{A}$ with

$$\mathcal{A}\Phi(X_1, \dots, X_p) = \frac{1}{p!} \sum_{\sigma} \mathrm{sgn}(\sigma)\Phi(X_{\sigma(1)}, \dots, X_{\sigma(p)}),$$

where σ runs over all permutations of $\{1, \dots, p\}$. Now, for a p-form θ and a q-form $\tilde{\theta}$, their *alternating product*

$$\theta \wedge \tilde{\theta} := \mathcal{A}\theta \otimes \tilde{\theta}$$

is a $(p+q)$-form.

A p-form θ is called smooth, if for any smooth vector fields $X_1, \dots, X_p$, $\theta(X_1, \dots, X_p)$ is a smooth function. Let $\Omega^p(M)$ denote the class of all smooth p-forms on M. The manifold is called *orientable* if there is a smooth d-form which is non-zero at any point. For a 0-form f, its exterior differential $\mathbf{d}f$ is defined by $\mathbf{d}f(X) := Xf = \langle \nabla f, X \rangle$ for $X \in TM$. When $p \in \mathbb{N}$, the *exterior differential* $\mathbf{d}$ is a linear map from $\Omega^p(M)$ to $\Omega^{p+1}(M)$ for $p \in \mathbb{N}$ such that for any smooth functions $f_0, f_1, \dots, f_p$,

$$\mathbf{d}\{f_0 \mathbf{d}f_1 \wedge \dots \wedge \mathbf{d}f_p\} = \mathbf{d}f_0 \wedge \mathbf{d}f_1 \wedge \dots \wedge \mathbf{d}f_p.$$

Since $\mathbf{d}$ is a local operator and on local charts a smooth p-form can be represented as combinations of p-forms like $f_0 \mathbf{d}x_{i_1} \wedge \dots \wedge \mathbf{d}x_{i_p}$ for $f_0 \in C^\infty(M)$ and $1 \le i_1 < i_2 < \dots < i_p \le d$, $\mathbf{d}$ is well-defined on $\Omega^p(M)$. Obviously, $\mathbf{d}^2 := \mathbf{dd} = 0$.

To introduce the *codifferential*, which is the formal adjoint of $\mathbf{d}$, we first define the covariant derivative. Let $\theta \in \Omega^p(M)$ and $X \in T_xM$. Then $\nabla_X\theta$ defined by

$$(\nabla_X\theta)(X_1,\ldots,X_p) = \frac{\mathrm{d}}{\mathrm{d}s}\theta(X_1(s),\ldots,X_p(s))|_{s=0}$$

is called the *covariant derivative* of θ along X, where $X_i(s)$ is the parallel transportation of X along the geodesic $s \mapsto \exp_x[sX]$. Now, the codifferential is defined by

$$(\mathbf{d}^*\theta)(X_1,\ldots,X_p) = -\sum_{i=1}^d (\nabla_{u^i}\theta)(u^i, X_1,\ldots,X_p),$$

for $\theta \in \Omega^p(M), x \in M, X_1,\ldots,X_p \in T_xM$ and $u = \{u^1,\ldots,u^d\}$ an orthonormal basis of T_xM. The operator

$$\Delta_p := -(\mathbf{dd}^* + \mathbf{d}^*\mathbf{d}) : \Omega^p(M) \to \Omega^p(M)$$

is called the (negative) *Hodge-de Rham Laplacian* on $\Omega^p(M)$.

Finally, we introduce the orthonormal frame bundle and the horizontal lift. For M being a connected complete Riemannian manifold of dimension d, $O(M) := \bigcup_{x\in M} O_x(M)$ is called the *orthonormal frame bundle over* M, where $O_x(M)$ is the space of all orthonormal bases of T_xM. Obviously, $O_x(M)$ is isometric to $O(d)$, the d-dimensional orthogonal group – the group of orthogonal $(d \times d)$-matrices. Thus, for each $x \in M$, $O_x(M)$ is a $\frac{1}{2}d(d-1)$-dimensional Riemannian manifold.

To see that $O(M)$ has a natural Riemannian structure, let $\mathbf{p} : O(M) \to M$ with $\mathbf{p}u := x$ if $u \in O_x(M)$, which is called the *canonical projection* from $O(M)$ onto M. Let $\{(\mathcal{O}_\alpha, \psi_\alpha)\}$ be the differential structure of M, one may define the local charts on $O(M)$ by letting $\tilde{\mathcal{O}}_\alpha := \bigcup_{x\in\mathcal{O}_\alpha} O_x(M)$ and $\tilde{\psi}_\alpha(u) := (\psi_\alpha(\mathbf{p}u), (\psi_\alpha)_*u)$, where $(\psi_\alpha)_*u := ((\psi_\alpha)_*X_1,\ldots,(\psi_\alpha)_*X_d) \in O(d)$ for $u = (X_1,\ldots,X_d) \in \tilde{\mathcal{O}}_\alpha$. Note that $O(d)$ is equipped with the Riemannian structure induced by the Euclidean metric on $\mathbb{R}^{d^2}$. Then the family $\{(\tilde{\mathcal{O}}_\alpha, \tilde{\psi}_\alpha)\}$ together with the Riemannian structure of $O(d)$ and the metric on M determines a unique Riemannian structure on $O(M)$. Therefore, $O(M)$ is a $\frac{1}{2}d(d+1)$-dimensional Riemannian manifold.

Now, given $e \in \mathbb{R}^d$, let us define the corresponding horizontal vector field H_e on $O(M)$. For any $u \in O(M)$ we have $ue \in T_{\mathbf{p}u}M$. Let u_s be the parallel transportation of u along the geodesic $\exp_{\mathbf{p}u}(sue), s \geq 0$. We obtain a vector $H_e(u) := \frac{\mathrm{d}}{\mathrm{d}s}u_s|_{s=0} \in T_uO(M)$. Thus, we have defined a vector field H_e on $O(M)$ which is indeed C^∞-smooth. In particular, let $\{e_i\}_{i=1}^d$ be an orthonormal basis on $\mathbb{R}^d$, define

$$\Delta_{O(M)} := \sum_{i=1}^d H_{e_i}^2.$$

It is easy to see that this operator is independent of the choice of the basis $\{e_i\}$. We call $\Delta_{O(M)}$ *the horizontal Laplace operator*. Moreover, for any vector field Z on M, we define its *horizontal lift* by $\mathbf{H}_Z(u) := H_{u^{-1}Z}(u), u \in O(M)$, where $u^{-1}Z$ is the unique vector $e \in \mathbb{R}^d$ such that $Z_{\mathbf{p}u} = ue$.

1.1.3 *Some formulae and comparison results*

We first introduce the *Bochner-Weitzenböck formula*, which formulates the Ricci curvature as the difference between the Hodge-de Rham Laplacian and the horizontal Laplacian on Ω^1. For any $p \geq 0$, the horizontal Laplacian $\Box_p := \mathrm{tr}\nabla^2$ is defined on Ω^p; that is, for any $x \in M$ and $\theta \in \Omega^p$,

$$(\Box_p\theta)(x) := \sum_{i=1}^d (\nabla_{u^i}\nabla_{u^i}\theta)(x)$$

for an orthonormal basis $u = \{u^1, \dots, u^d\}$ around x with $\nabla u^i(x) = 0, 1 \leq i \leq d$; we call such u a *normal frame* at x. In particular, $\Delta_0 = \Box_0 = \Delta$. But when $p \in \mathbb{N}$, Δ_p and $\Box_p$ might be no longer equal, and their difference gives rise to a curvature term.

Theorem 1.1.4 (Bochner-Weitzenböck formula). $\Delta_1 = \Box_1 - \mathrm{Ric}$, *where for any 1-form* θ, $\mathrm{Ric}(\theta)$ *is the 1-form defined by*

$$\mathrm{Ric}(\theta)(X) := \mathrm{Ric}(X, \theta^*), \quad X \in TM.$$

Consequently, for any smooth function f,

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle\nabla\Delta f, \nabla f\rangle = \|\mathrm{Hess}_f\|_{HS}^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

where $\mathrm{Hess}_f(X,Y) := \langle\nabla_X\nabla f, Y\rangle$ *for* $X, Y \in TM$ *and* $\|\cdot\|_{HS}$ *is the Hilbert-Schmidt norm.*

Now, we introduce some useful integral formulae for the above operators. For given $f \in C_b^\infty(M)$, the set of functions in $C^p(M)$ with bounded derivatives up to order p, by Sard's theorem the set of critical values in $f(M)$ has Lebesgue measure zero. In other words, $\{f = t\}$ is a $(d-1)$-dimensional submanifold of M for a.e. $t \in f(M)$. Let $\mathbf{A}$ denote the volume measure on a $(d-1)$-dimensional submanifold of M with the induced metric.

Theorem 1.1.5 (Coarea formula). *For any $f \in C_b^\infty(M)$ and any $h \in L^1(\mathrm{d}x)$,*

$$\int_M h|\nabla f|\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{d}t \int_{\{f=t\}} h\mathrm{d}\mathbf{A}.$$

In particular, if $\mathrm{d}\mu := h\mathrm{d}x$ is a finite measure and let $\mathrm{d}\mu_\partial := h\mathrm{d}\mathbf{A}$ then

$$\mu(|\nabla f|) = \int_{-\infty}^{\infty} \mu_\partial(\{f=t\})\mathrm{d}t.$$

Theorem 1.1.6 (Green formula or integration by parts formula). (1) *If $X \in \Gamma^1(TM)$ with compact support, then*

$$\int_M \mathrm{div}X(x)\mathrm{d}x = 0.$$

(2) *If $f, g \in C_0^2(M)$, then*

$$\int_M (f\Delta g)(x)\mathrm{d}x = \int_M (g\Delta f)(x)\mathrm{d}x = -\int_M \langle \nabla f, \nabla g\rangle(x)\mathrm{d}x.$$

(3) *Let $X \in \Gamma^1(TM)$ and $D \subset M$ a smooth open domain, i.e. an open domain with boundary a $(d-1)$-dimensional differential manifold. Then*

$$\int_D (\mathrm{div}X)(x)\mathrm{d}x = -\int_{\partial D} \langle X, N\rangle \mathrm{d}\mathbf{A},$$

where N is the inward pointing unit normal vector field on ∂D.

(4) *For a smooth open domain D,*

$$\int_D (f\Delta g + \langle \nabla f, \nabla g\rangle)(x)\mathrm{d}x = -\int_{\partial D} f(Ng)\mathrm{d}\mathbf{A}, \quad f \in C_0^1(\bar{D}), g \in C_0^2(\bar{D}).$$

Finally, we introduce two variational formulae for the Riemannian distance. Let $X \in T_xM$ and $Y \in T_yM$. We assume that $y \notin \mathrm{cut}(x)$ and let $\gamma : [0, \rho(x,y)] \to M$ be the unique minimal geodesic from x to y.

Theorem 1.1.7 (First variational formula). *We have*

$$(X+Y)\rho(x,y) := X\rho(\cdot,y)(x) + Y\rho(x,\cdot)(y) = \int_0^{\rho(x,y)} \langle \nabla_{\dot\gamma_s} V, \dot\gamma_s\rangle \mathrm{d}s$$

for any smooth vector field V along γ with $V_0 = X$ and $V_{\rho(x,y)} = Y$.

Theorem 1.1.8 (Second variational formula). *Let X and Y be two smooth vector fields with $\nabla X(x) = 0$ and $\nabla Y(y) = 0$. Let X and Y act on x and y respectively. Then*

$$(X+Y)^2\rho(x,y) = \int_0^{\rho(x,y)} (|\nabla_{\dot\gamma} J|^2 - \langle \mathcal{R}(\dot\gamma, J)\dot\gamma, J\rangle)_s \mathrm{d}s,$$

where J is the unique Jacobi field along γ with $J_0 = X$ and $J_{\rho(x,y)} = Y$.

In particular, the following Hessian comparison theorem and Laplacian comparison theorem are consequences of the second variational formula.

Theorem 1.1.9 (Hessian comparison theorem). *Assume that for any unit vector field Y along γ with $\langle Y, \dot\gamma\rangle = 0$ one has* $\mathrm{Sect}(\dot\gamma, Y) \le k$, *where $k \in \mathbb{R}$ is a constant. If $\rho(x,y) < \pi/\sqrt{k^+}$, then*

$$\mathrm{Hess}_{\rho(x,\cdot)}(Y,Y)(y) \ge \frac{f'}{f}(\rho(x,y))\Big\{1 - \big(Y\rho(x,\cdot)(y)\big)^2\Big\},$$

where

$$f(r) := \begin{cases} r, & \text{if } k = 0,\\ \sin(\sqrt{k}r)/\sqrt{k}, & \text{if } k > 0,\\ \sinh(\sqrt{-k}r)/\sqrt{-k}, & \text{if } k < 0. \end{cases} \tag{1.1.1}$$

Theorem 1.1.10 (Laplacian comparison theorem). *Let* $\mathrm{Ric}(\dot\gamma,\dot\gamma) \ge k(d-1)$ *hold for some $k \in \mathbb{R}$. Then*

$$\Delta\rho(x,\cdot)(y) \le \frac{(d-1)f'}{f}(\rho(x,y)).$$

If $\mathrm{Sect}(Y,\dot\gamma) \le k$ *for any unit vector field Y along γ with $\langle Y,\dot\gamma\rangle = 0$, then*

$$\Delta\rho(x,\cdot)(y) \ge \frac{(d-1)f'}{f}(\rho(x,y)).$$

The Hessian and Laplacian comparison theorems can be proved by using the second variational formula and the following index lemma. Let $\gamma : [0,t]\to M$ be a minimal geodesic, for any vector field X along γ, let

$$I(X,X) := \int_0^t \big(|\nabla_{\dot\gamma}X|^2 - \langle \mathcal{R}(\dot\gamma, X)\dot\gamma, X\rangle\big)_s \mathrm{d}s.$$

We call $I(X,X)$ the *index form* of X along γ.

Theorem 1.1.11 (Index lemma). *Let J be a Jacobi field along a minimal geodesic γ with $\gamma_0 \notin$ $\mathrm{cut}(\gamma_t)$. For any vector field X along γ with $X_0 = J_0$ and either $X_t = J_t$ or $\nabla_{\dot\gamma_0}X = \nabla_{\dot\gamma_0}J$, one has $I(X,X) \ge I(J,J)$.*

1.2 Riemannian manifold with boundary

Definition 1.2.1. Let M be a Hausdorff topological space with a countable basis of open sets having disjoint decomposition $M = (\partial M)\bigcup M^\circ$, where M° is a d-dimensional differential manifold, and for any point

$o \in \partial M$ there exists a neighborhood $\mathcal{O}$ of o and a homeomorphism $\varphi : \mathcal{O} \to H^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d \geq 0\}$ such that $\varphi(o) = 0$ and $\varphi(\mathcal{O} \bigcap \partial M) = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d = 0\}$. If these local charts, together with those on M°, are compatible with each other, then M is called a d-dimensional differential manifold with *boundary* ∂M. If moreover M is equipped with a smooth metric, then it is called a Reimennian manifold with boundary.

Obviously, if M is a d-dimensional differential manifold with boundary, then the boundary ∂M is a $(d-1)$-dimensional differential manifold. Simple examples for differential manifolds with boundary are smooth domains in a differential manifold. As before, $T_x M$ denotes the tangent space at point $x \in M$, and $TM := \bigcup_{x \in M} T_x M$. If $x \in \partial M$, then $T_x \partial M$, the tangent space of ∂M at point x, is a subspace of $T_x M$. Let $T\partial M = \bigcup_{x \in \partial M} T_x \partial M$. Obviously, when $x \in \partial M$ we have $T_x M = T_x \partial M \oplus \text{span}\{N_x\}$, where $N_x \in T_x M$ is a unitary vector orthogonal to $T_x \partial M$. Throughout the book, we will take N to be the inward pointing unit normal vector field of M, i.e. for any $x \in \partial M$, $N_x \in T_x M$ is unitary and orthogonal to $T_x \partial M$ such that $\exp_x[\varepsilon N] \in M^\circ$ holds for small $\varepsilon > 0$.

Definition 1.2.2. Let M be a Riemannian manifold with boundary, and let N be the inward pointing unit normal vector field of ∂M. Then the 2-tensor

$$\mathbb{I}(X, Y) := -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_x \partial M, x \in \partial M$$

is called the *second fundamental form* of the boundary. If $\mathbb{I} \geq 0$, i.e. $\mathbb{I}(X, X) \geq 0$ for any $X \in T\partial M$, then the manifold (or the boundary) is called *convex*.

For a connected Riemannian manifold with boundary, let ρ be the Riemannian distance defined as in the case without boundary; i.e. for any $x, y \in M$, $\rho(x, y)$ is the inf over the lengths of smooth curves in M linking these two points. In general, $\rho(x, y)$ might not be reached by a geodesic, but it is the case if ∂M is convex. To see this, we first extend M to a complete Riemannian manifold without boundary by using the polar coordinate around ∂M. Let $\partial_{\mathbf{r}}^+ = \{(\theta, r) : \theta \in \partial M, r \in [0, \mathbf{r}(\theta))\}$, $\mathbf{r} \in C^\infty(\partial M; (0, \infty))$ such that the exponential map

$$\partial_{\mathbf{r}}^+ \ni (\theta, r) \mapsto \exp(\theta, r) := \exp_\theta[rN] \in \exp(\partial_{\mathbf{r}}^+)$$

is diffeomorphic. Then under the polar coordinate we extend $\partial_{\mathbf{r}}^+$ to $\partial_{\mathbf{r}} := \{(\theta, r) : \theta \in \partial M, r \in (-\mathbf{r}(\theta), \mathbf{r}(\theta))\}$, so that

$$\tilde{M} := M \cup \partial_{\mathbf{r}} = M \cup \{(\theta, r) : \theta \in \partial M, r \in (-\mathbf{r}(\theta), 0)\}$$

is a differential manifold without boundary. Moreover, $\langle \cdot, \cdot \rangle$ extends naturally to a Riemannian metric on $\tilde{M}$ by using the metric on ∂M: under the polar coordinate such that $\{\partial\theta_i, \partial r : 1 \le i \le d-1\}$ on ∂M is orthonormal under the original metric, if at point (θ, r) one has $X = \sum_{i=1}^{d-1} f_i \partial\theta_i + f_0 \partial r$ and $Y = \sum_{i=1}^{d-1} g_i \partial\theta_i + g_0 \partial r$, then let $\langle X, Y \rangle = \sum_{i=0}^{d-1} f_i g_i$. To make $\tilde{M}$ complete, let $h \in C^\infty(\mathbb{R})$ be such that $h|_{[0,\infty)} = 1$, $h(r) > 0$ for $r > -1$ and $h(r) = 0$ for $r \le -1$. Then $\tilde{M}$ is complete under the metric $\tilde{h}^{-2}\langle \cdot, \cdot \rangle$, where $\tilde{h} \in C^\infty(\tilde{M})$ is such that $\tilde{h}|_M = 1$ and $\tilde{h}(\theta, r) = h(r/\mathbf{r}(\theta))$ for $(\theta, r) \in \partial_{\mathbf{r}}$. Therefore, according to Proposition 2.1.5 in [Wang (2005a)], we have the following result.

Theorem 1.2.1. *If ∂M is convex, then there exists a complete Riemannian manifold $(M_0, \langle \cdot, \cdot \rangle_0)$ without boundary, which extends $(M, \langle \cdot, \cdot \rangle)$ such that for any $x, y \in M$, the minimal geodesic linking x and y lies in M.*

According to Theorem 1.2.1, we can define the cut-locus and state the comparison theorems for ρ_o as in the case without boundary.

Let ρ_∂ be the Riemannian distance to the boundary. It is clear using local charts that ρ_∂ is smooth in a neighborhood of ∂M. We call

$$\mathrm{i}_\partial := \sup\big\{r > 0 : \ \rho_\partial \text{ is smooth on } \{\rho_\partial < r\}\big\}$$

the *injectivity radius* of ∂M. Obviously, $\mathrm{i}_\partial > 0$ if M is compact, but it could be zero in the non-compact case ($\sup \emptyset = 0$ by convention).

Let M be a Riemannian manifold with boundary. The frame bundle $O(M)$ is again a Riemannian manifold with boundary $\partial O(M) = \bigcup_{x \in \partial M} O_x(M)$. The following two results are essentially due to Kasue [Kasue (1982, 1984)] (see also Theorem A.1 in [Wang (2005b)]).

Theorem 1.2.2 (Hessian Comparison). (1) *Let $\theta, k \in \mathbb{R}$ be constants such that $\mathbb{I} \le \theta$ and* Sect $\le k$. *Let*

$$h(t) = \begin{cases} \cos\sqrt{k}\, t - \frac{\theta}{\sqrt{k}} \sin\sqrt{k}\, t & \text{if } k \ge 0, \\ \cosh\sqrt{-k}\, t - \frac{\theta}{\sqrt{-k}} \sinh\sqrt{-k}\, t & \text{if } k < 0, \end{cases} \qquad t \ge 0.$$

Let $h^{-1}(0)$ be the first zero point of h ($h^{-1}(0) := \infty$ if the zero point of h does not exist). Then for any $x \in M^\circ$ such that $\rho_\partial(x) < \mathrm{i}_\partial \wedge h^{-1}(0)$ and any unit $X \in T_xM$ orthogonal to $\nabla\rho_\partial(x)$,

$$\mathrm{Hess}_{\rho_\partial}(X, X) \ge \frac{h'}{h}(\rho_\partial(x)).$$

(2) *If* $\mathbb{I} \geq \theta$ *and* Sect $\geq k$, *then*

$$\mathrm{Hess}_{\rho_\partial}(X, X) \leq \frac{h'}{h}(\rho_\partial(x))$$

holds for any $x \in M$ *with* $\rho_\partial(x) < \mathrm{i}_\partial \wedge h^{-1}(0)$ *and unit* $X \in T_xM$ *orthogonal to* $\nabla \rho_\partial(x)$.

Proof. The proof of (1) can be found in the Appendix of [Wang (2005b)]. Below we include a brief proof of (2). Let p be the orthogonal projection of x on ∂M, and let $\gamma(s) = \exp_p[sN], s \in [0, \rho_\partial(x)]$ be the geodesic from p to x. Let $\{J(s)\}_{s\in[0,\rho_\partial(x)]}$ be the Jacobi field along γ such that $J(\rho_\partial(x)) = X$ and

$$\langle \dot{J}(0), V\rangle = -\mathbb{I}(J(0), V), \quad V \in T_p\partial M.$$

By the second variational formula we have (see e.g. page 321 in [Chavel (1995)])

$$\begin{aligned}&\mathrm{Hess}_{\rho_\partial}(X, X)\\ &= -\mathbb{I}(J(0), J(0)) + \int_0^{\rho_\partial(x)} \Big(|\dot{J}(s)|^2 - \langle \mathcal{R}(\dot\gamma(s), J(s))\dot\gamma(s), J(s)\rangle\Big)\mathrm{d}s.\end{aligned}$$

Let $\{X(s)\}_{s\in[0,\rho_\partial(x)]}$ be the parallel displacement of X along γ such that $X(\rho_\partial(x)) = X$. Define

$$\tilde{J}(s) = \frac{h(s)}{h(\rho_\partial(x))} X(s), \quad s \in [0, \rho_\partial(x)].$$

Then $\tilde{J}$ is orthogonal to $\nabla\rho_\partial$ along γ and $\tilde{J}(\rho_\partial(x)) = J(\rho_\partial(x)) = X$. By the index lemma (see the first display on page 322 in [Chavel (1995)]), we obtain

$$\begin{aligned}\mathrm{Hess}_{\rho_\partial}(X, X) &= -\mathbb{I}(\tilde{J}(0), \tilde{J}(0)) + \frac{1}{h(\rho_\partial(x))^2}\int_0^{\rho_\partial(x)} \big\{h'(s)^2 - kh(s)^2\big\}\mathrm{d}s\\ &= \frac{h'}{h}(\rho_\partial(x)).\end{aligned}$$

□

The following Laplacian comparison theorem is a direct consequence of Theorem 1.2.2.

Theorem 1.2.3 (Laplacian comparison). (1) *In the situation of Theorem 1.2.2(1),*

$$\Delta\rho_\partial(x) \geq \frac{(d-1)h'}{h}(\rho_\partial(x)), \quad \text{if } \rho_\partial(x) < \mathrm{i}_\partial \wedge h^{-1}(0).$$

(2) *In the situation of Theorem 1.2.2(2),*

$$\Delta\rho_\partial(x) \leq \frac{(d-1)h'}{h}(\rho_\partial(x)), \quad \text{if } \rho_\partial(x) < \mathrm{i}_\partial \wedge h^{-1}(0).$$

To do the stochastic analysis on non-convex manifolds, we will make use of a conformal change of metric such that the boundary becomes convex under the new metric. In general, for any strictly positive smooth function ϕ on M, the metric $\langle \cdot, \cdot \rangle' := \phi^{-2}\langle \cdot, \cdot \rangle$ is called a conformal change of the metric $g := \langle \cdot, \cdot \rangle$. The following results can be easily verified (see Theorem 1.159 in [Bess (1987)] and (3.2) in [Fang *et al* (2008)]).

Theorem 1.2.4. *Let ∇' and Ric' be the Levi-Civita connection and the Ricci curvature for the metric $\langle \cdot, \cdot \rangle'$. Then:*

(1) *For any two vector fields X, Y on M, $\nabla'_X Y = \nabla_X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi$.*
(2) $\mathrm{Ric}' = \mathrm{Ric} + (d-2)\phi^{-1}\mathrm{Hess}_\phi + (\phi^{-1}\Delta\phi - (d-3)|\nabla \log \phi|^2)\langle \cdot, \cdot \rangle$.

Theorem 1.2.5. *Let $\phi \in C^2(M)$ be strictly positive. If $\mathbb{I} \geq -N \log \phi$ then ∂M is convex under the metric $\langle \cdot, \cdot \rangle' := \phi^{-2}\langle \cdot, \cdot \rangle$.*

Proof. Since $\langle X, N \rangle = 0$ for $X \in T\partial M$ and noting that the inward unit normal vector field of ∂M under the metric $\langle \cdot, \cdot \rangle'$ is $N' := \phi N$, by Theorem 1.2.4(1),

$$\begin{aligned} -\langle \nabla'_X N', X \rangle' &= \phi^{-2}\langle N', \nabla \log \phi \rangle |X|^2 - \phi^{-2}\langle \nabla_X N', X \rangle \\ &= \phi^{-1}\big(\mathbb{I}(X, X) + (N \log \phi)|X|^2\big) \geq 0, \quad X \in T\partial M. \end{aligned}$$

□

1.3 Coupling and applications

A *coupling* for two distributions (i.e. probability measures) is nothing but a joint distribution of them. More precisely:

Definition 1.3.1. Let $(E, \mathcal{B})$ be a measurable space, and let $\mu, \nu \in \mathcal{P}(E)$, the set of all probability measures on $(E, \mathcal{B})$. A probability measure Π on the product space $(E \times E, \mathcal{B} \times \mathcal{B})$ is called a coupling of μ and ν, if

$$\Pi(A \times E) = \mu(A), \quad \Pi(E \times A) = \nu(A), \quad A \in \mathcal{B}.$$

We shall let $\mathcal{C}(\mu.\nu)$ stand for the set of all couplings of μ and ν. Obviously, the product measure $\mu \times \nu$ is a coupling of μ and ν, which is called the independent coupling. This coupling is too simple to have broad applications, but it at least indicates the existence of coupling. Before moving to more general applications of coupling, let us present a simple example to show that even this trivial coupling could have non-trivial applications.

Throughout the paper, we shall let $\mu(f)$ denote the integral of function f w.r.t. measure μ.

For a measurable space $(E, \mathcal{B})$, let $\mathcal{B}(E)$ (resp. $\mathcal{B}_b(E), \mathcal{B}_b^+(E)$) denote the set of all measurable (resp. bounded measurable, bounded non-negative measurable) functions on E. If moreover E is a topology space with $\mathcal{B}$ the Borel σ-field, let $C(E)$ (resp. $C_b(E), C_b^+(E)$) stand for the set of a continuous (resp. bounded continuous, bounded non-negative continuous) functions on E.

Example 1.3.1 (FKG inequality) Let μ and ν be probability measures on $\mathbb{R}$, then for any two bounded increasing functions f and g, one has

$$\mu(fg) + \nu(fg) \geq \mu(f)\nu(g) + \nu(f)\mu(g).$$

Proof. Since f and g are increasing, one has

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \quad x, y \in \mathbb{R}.$$

So, the desired inequality follows by taking integral w.r.t. the independent coupling $\mu \times \nu$. □

In the remainder of this section, we first link coupling to transport problem, which leads to the notions of optimal coupling and probability distances, then introduce coupling for stochastic processes and a coupling method to establish Harnack type inequalities.

1.3.1 *Transport problem and Wasserstein distance*

Let $x_1, x_2, \ldots, x_n$ be n places, and consider the distribution $\mu := \{\mu_i : i = 1, \ldots, n\}$ of some product among these places, i.e. μ_i refers to the ratio of the product at place x_i. We have $\mu_i \geq 0$ and $\sum_{i=1}^n \mu_i = 1$; that is, μ is a probability measure on $E := \{1, \ldots, n\}$. Now, due to market demand one wishes to transport the product among these places to the target distribution $\nu := \{\nu_i : 1 \leq i \leq n\}$, which is another probability measure on E. Let $\Pi := \{\Pi_{ij} : 1 \leq i, j \leq n\}$ be a transport scheme, where Π_{ij} refers to the amount to be transported from place x_i to place x_j. Obviously, the scheme is exact to transport the product from distribution μ into distribution ν if and only if Π satisfies

$$\mu_i = \sum_{j=1}^n \Pi_{ij}, \quad \nu_j = \sum_{i=1}^n \Pi_{ij}, \quad 1 \leq i, j \leq n.$$

Thus, a scheme transporting from μ to ν is nothing but a coupling of μ and ν, and vice versa.

Now, suppose ρ_{ij} is the cost to transport a unit product from place x_i to place x_j. Then it is reasonable that ρ gives rise to a distance on E. With the cost function ρ, the *transportation cost* for a scheme Π is

$$\sum_{i,j=1}^{n} \rho_{ij}\Pi_{ij} = \int_{E\times E} \rho \, \mathrm{d}\Pi.$$

Therefore, the minimal transportation cost between these two distributions is

$$W_1^\rho(\mu,\nu) := \inf_{\Pi\in\mathcal{C}(\mu,\nu)} \int_{E\times E} \rho \, \mathrm{d}\Pi,$$

which is called the L^1-*Wasserstein distance* between μ and ν induced by the cost function ρ.

In general, we have the following notion for L^p-transportation cost.

Definition 1.3.2. Let $(E,\mathcal{B})$ be a measurable space and ρ a non-negative measurable function on $E\times E$. For any $p\in[1,\infty]$,

$$W_p^\rho(\mu,\nu) := \left\{ \inf_{\Pi\in\mathcal{C}(\mu,\nu)} \int_{E\times E} \rho^p \mathrm{d}\Pi \right\}^{1/p} \tag{1.3.1}$$

is also called the L^p-transportation cost between probability measures μ and ν induced by the cost function ρ.

When ρ is a distance on E, it is also called the L^p-Wasserstein distance induced by ρ, since in this case W_p^ρ is a distance on $\mathcal{P}_p(E) := \{\mu\in\mathcal{P}(E) : \rho\in L^p(\mu\times\mu)\}$ (see e.g. [Chen (1992)]).

It is easy to see from (1.3.1) that any coupling provides an upper bound of the transportation cost, while the following Kontorovich dual formula enables one to find lower bound estimates.

Proposition 1.3.1 (Monge-Kontorovich dual formula). *For $p\geq 1$, let*

$$\mathcal{C}_p = \{(f,g) : f,g\in\mathcal{B}_b(E), f(x)\leq g(y)+\rho(x,y)^p,\ x,y\in E\}.$$

Then

$$W_p^\rho(\mu,\nu)^p = \sup_{(f,g)\in\mathcal{C}_p} \{\mu(f)-\nu(g)\}.$$

When (E,ρ) is a metric space, $\mathcal{B}_b(E)$ in the definition of $\mathcal{C}_p$ can be replaced by a sub-class of bounded measurable functions determining probability measures (e.g. bounded Lipschitzian functions), see e.g. [Rachev (1991)].

1.3.2 *Optimal coupling and optimal map*

Definition 1.3.3. Let $\mu, \nu \in \mathcal{P}(E)$ and $\rho \geq 0$ on $E \times E$ be fixed. If $\Pi \in \mathcal{C}(\mu, \nu)$ reaches the infimum in (1.3.1), then it is called an *L^p-optimal coupling* for μ and ν w.r.t. the cost function ρ. If a measurable map $\mathsf{T} : E \to E$ maps μ into ν (i.e. $\nu = \mu \circ \mathsf{T}^{-1}$), such that $\Pi(\mathrm{d}x, \mathrm{d}y) := \mu(\mathrm{d}x)\delta_x(\mathrm{d}y)$ is an optimal coupling, where δ_x is the *Dirac measure* at x, then T is called an *optimal transportation map* for the L^p-transportation cost.

To fix (or estimate) the transportation cost, it is crucial to construct the optimal coupling or optimal map. Below we introduce some results on existence and construction of the optimal coupling/map.

Proposition 1.3.2. *Let (E, ρ) be a Polish space. Then for any $\mu, \nu \in \mathcal{P}(E)$ and any $p \in [1, \infty)$, there exists an optimal coupling.*

The proof is fundamental. Since it is easy to see that the class $\mathcal{C}(\mu, \nu)$ is tight, for a sequence of couplings $\{\Pi_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \Pi_n(\rho^p) = W_p^\rho(\mu, \nu)^p,$$

there is a weakly convergent subsequence, whose weak limit gives an optimal coupling.

As for the optimal map, let us simply mention a result of McCann [McCann (1995)] for $E = \mathbb{R}^d$, see [Villani (2009a)] and references within for extensions and historical remarks.

Theorem 1.3.3. *Let $E = \mathbb{R}^d, \rho(x, y) = |x - y|$, and $p = 2$. Then for any two absolutely continuous probability measures $\mu(\mathrm{d}x) := f(x)\mathrm{d}x$ and $\nu(\mathrm{d}x) := g(x)\mathrm{d}x$ such that $f > 0$, there exists a unique optimal map, which is given by $T = \nabla V$ for a convex function V solving the equation*

$$f = g(\nabla V)\det\nabla_{ac}\nabla V$$

in the distribution sense, where ∇_{ac} is the gradient for the absolutely continuous part of a distribution.

Finally, we introduce the Wasserstein coupling which is optimal when ρ is the discrete distance on E; that is, this coupling is optimal for the total variation distance. We leave the proof as an exercise.

Proposition 1.3.4 (Wasserstein coupling). *Let $\rho(x, y) = 1_{\{x\neq y\}}$. We have*

$$W_p^\rho(\mu, \nu)^p = \frac{1}{2}\|\mu - \nu\|_{var} := \sup_{A\in\mathcal{B}} |\mu(A) - \nu(A)|,$$

and the Wasserstein coupling

$$\Pi(\mathrm{d}x, \mathrm{d}y) := (\mu \wedge \nu)(\mathrm{d}x)\delta_x(\mathrm{d}y) + \frac{(\mu-\nu)^+(\mathrm{d}x)(\mu-\nu)^-(\mathrm{d}y)}{(\mu-\nu)^-(E)}$$

is optimal, where $(\mu - \nu)^+$ *and* $(\mu - \nu)^-$ *are the positive and negative parts respectively in the Hahn-Jordan decomposition of* $\mu - \nu$*, and* $\mu \wedge \nu = \mu - (\mu - \nu)^+$.

1.3.3 *Coupling for stochastic processes*

Definition 1.3.4. Let $X := \{X_t\}_{t\geq 0}$ and $Y := \{Y_t\}_{t\geq 0}$ be two stochastic processes on E. A stochastic process $(\bar{X}, \bar{Y})$ on $E \times E$ is called a coupling of them if the distributions of $\bar{X}$ and $\bar{Y}$ coincide with those of X and Y respectively.

Let us observe that a coupling of two stochastic processes corresponds to a coupling of their distributions, so that the notion goes back to coupling of probability measures introduced above.

Let μ and ν be the distributions of X and Y respectively, which are probability measures on the path space $W := E^{[0,\infty)}$, equipped with the product σ-algebra

$$\mathcal{F}(W) := \sigma\big(w \mapsto w_t : \ t \in [0, \infty)\big).$$

For any $\Pi \in \mathcal{C}(\mu, \nu)$, $(W \times W, \mathcal{F}(W) \times \mathcal{F}(W), \Pi)$ is a probability space under which

$$(\bar{X}, \bar{Y})(w) := (w^1, w^2), \quad w = (w^1, w^2) \in W \times W$$

is a coupling for X and Y. Conversely, the distribution of a coupling for X and Y also provides a coupling for μ and ν.

Now, let P_t and $P_t(x, \mathrm{d}y)$ be the semigroup and transition probability kernel for a strong Markov process on a Polish space E. If $X := (X_t)_{t\geq 0}$ and $Y := (Y_t)_{t\geq 0}$ are two processes with the same transition probability kernel $P_t(x, \mathrm{d}y)$, then $(X, Y) = (X_t, Y_t)_{t\geq 0}$ is called a coupling of the strong Markov process with coupling time

$$T_{x,y} := \inf\{t \geq 0 : X_t = Y_t\}.$$

The coupling is called successful if $T_{x,y} < \infty$ a.s. For any $\mu \in \mathcal{P}(E)$, let $\mathbb{P}^\mu$ be the distribution of the Markov process with initial distribution μ, and let μP_t be the marginal distribution of $\mathbb{P}^\mu$ at time t.

Definition 1.3.5. If for any $x, y \in E$, there exists a successful coupling starting from (x, y), then the strong Markov process is said to have *successful coupling* (or to have the coupling property).

Let

$$\mathcal{T} = \bigcap_{t>0} \sigma(\omega \mapsto \omega_s : s \geq t)$$

be the tail σ-field. The following result includes some equivalent assertions for the coupling property (see [Cranston and Greven (1995); Lindvall (1992); Thorisson (1994)]).

Theorem 1.3.5. *Each of the following is equivalent to the coupling property:*

(1) *For any* $\mu, \nu \in \mathcal{P}(E)$, $\lim_{t\to\infty} \|\mu P_t - \nu P_t\|_{var} = 0$.
(2) *All bounded time-space harmonic functions are constant, i.e. a bounded measurable function* u *on* $[0, \infty) \times E$ *has to be constant if*
$$u(t, \cdot) = P_s u(t + s, \cdot), \quad s, t \geq 0.$$
(3) *The tail* σ*-algebra* $\mathcal{T}$ *is trivial, i.e.* $\mathbb{P}^\mu(X \in A) = 0$ *or* 1 *holds for* $\mu \in \mathcal{P}(E)$ *and* $A \in \mathcal{T}$.
(4) *For any* $\mu, \nu \in \mathcal{P}(E)$, $\mathbb{P}^\mu = \mathbb{P}^\nu$ *holds on* $\mathcal{T}$.

A weaker notion than the coupling property is the shift-coupling property.

Definition 1.3.6. The strong Markov process is said to have the *shift coupling* property, if for any $x, y \in E$ there is a coupling (X, Y) starting at (x, y) such that $X_{T_1} = Y_{T_2}$ holds for some finite stopping times T_1 and T_2.

Let

$$\mathcal{I} := \{A \in \mathcal{F}(W) : w \in A \text{ implies } w(t + \cdot) \in A, t \geq 0\}$$

be the shift-invariant σ-field. Below are some equivalent statements for the shift-coupling property (see [Aldous and Thorisson (1993); Cranston and Greven (1995); Thorisson (1994)]).

Theorem 1.3.6. *Each of the following is equivalent to the shift coupling property:*

(5) *For any* $\mu, \nu \in \mathcal{P}(E)$, $\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\mu P_s - \nu P_s\|_{var} \mathrm{d}s = 0$.
(6) *All bounded harmonic functions are constant, i.e. a bounded measurable function* f *on* E *has to be constant if* $P_t f = f$ *holds for all* $t \geq 0$.
(7) *The invariant* σ*-algebra of the process is trivial, i.e.* $\mathbb{P}^\mu(X \in A) = 0$ *or* 1 *holds for* $\mu \in \mathcal{P}(E)$ *and* $A \in \mathcal{I}$.
(8) *For any* $\mu, \nu \in \mathcal{P}(E)$, $\mathbb{P}^\mu = \mathbb{P}^\nu$ *holds on* $\mathcal{I}$.

According to Theorem 5 in [Cranston and Wang (2000)], the coupling property and the shift-coupling property are equivalent, and thus all above statements (1)-(8) are equivalent, provided there exist $s, t > 0$ and increasing function $\Phi \in C([0,1])$ with $\Phi(0) < 1$ such that

$$P_t f \leq \Phi(P_{t+s} f), \quad 0 \leq f \leq 1$$

holds.

By the strong Markov property, for a coupling (X, Y) with coupling time T, we may let $X_t = Y_t$ for $t \geq T$ without changing the transition probability kernel; that is, letting

$$\tilde{Y}_t = \begin{cases} Y_t, & \text{if } t \leq T, \\ X_t, & \text{if } t > T, \end{cases}$$

the process $(X, \tilde{Y})$ is again a coupling. Therefore, for any $x, y \in E$ and any coupling (X, Y) starting at (x, y) with coupling time $T_{x,y}$, we have

$$|P_t f(x) - P_t f(y)| = |\mathbb{E}(f(X_t) - f(\tilde{Y}_t))| \leq \operatorname{osc}(f)\mathbb{P}(T_{x,y} > t), \quad f \in \mathcal{B}_b(E),$$

where $\operatorname{osc}(f) := \sup f - \inf f$. This implies the following assertions, which are fundamentally crucial for applications of coupling in the study of Markov processes.

(i) If $\lim_{y \to x} \mathbb{P}(T_{x,y} > t) = 0, x \in E$, then P_t is strong Feller, i.e. $P_t \mathcal{B}_b(E) \subset C_b(E)$.
(ii) Let μ be an invariant probability measure. If the coupling time $T_{x,y}$ is measurable in (x, y), then

$$\|\nu P_t - \mu\|_{var} \leq 2 \int_{E \times E} \mathbb{P}(T_{x,y} > t)\Pi(\mathrm{d}x, \mathrm{d}y), \quad \Pi \in \mathcal{C}(\mu, \nu)$$

holds for $\nu \in \mathcal{P}(E)$.
(iii) The gradient estimate

$$\begin{aligned} |\nabla P_t f(x)| &:= \limsup_{y \to x} \frac{|P_t f(y) - P_t f(x)|}{\rho(x,y)} \\ &\leq \operatorname{osc}(f) \limsup_{y \to x} \frac{\mathbb{P}(T_{x,y} > t)}{\rho(x,y)}, \quad x \in E \end{aligned}$$

holds.

By constructing coupling such that $\mathbb{P}(T_{x,y} > t) \leq C\mathrm{e}^{-\lambda t}$ holds for some $C, \lambda > 0$, we derive lower bound estimates of the spectral gap in the symmetric case (see [Chen and Wang (1997a,b)]).

1.3.4 *Coupling by change of measure*

Finally, we introduce the notion of coupling by change of measure and applications to the dimension-free Harnack inequality [Wang (1997b)], Busmut formula [Bismut (1984)] and Driver's integration by parts formula [Driver (1997)]. The Harnack inequality has been investigated and applied by many authors for finite- and infinite-dimensional diffusions, see [Arnaudon *et al* (2006, 2009); Da Prato *et al* (2009); Aida and Zhang (2002); Kawabi (2005); Es-Sarhir *et al* (2009); Liu (2009); Liu and Wang (2008); Röckner and Wang (2010); Wang (2006, 2007a, 2011e, 2010d); Wang *et al* (2012); Wang and Xu (2013); Wang and Yuan (2011)], while the Bismut formula and the integration by parts formula are important tools in the study of regularity estimates of diffusion semigroups.

Definition 1.3.7. Let μ and ν be two probability measures on a measurable space $(E, \mathcal{B})$, and let X, Y be two E-valued random variables w.r.t. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) If the distribution of X is μ, while under another probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ the distribution of Y is ν, we call (X, Y) a *coupling by change of measure* for μ and ν with changed probability $\mathbb{Q}$.
(ii) If μ and ν are distributions of two stochastic processes with path space E, a coupling by change of measure for μ and ν is also called a coupling by change of measure for these processes. In this case X and Y are called the marginal processes of the coupling.

Theorem 1.3.7 (Harnack inequality). *Let P_t be a Markov semigroup and let $x, y \in E, T > 0$ be fixed. Let $\mathbb{P}^x$ and $\mathbb{P}^y$ be the distributions of the process starting at x and y respectively. If there is a coupling by change of measure (X, Y) of the Markov process with changed probability $\mathrm{d}\mathbb{Q} := R\mathrm{d}\mathbb{P}$, such that $X_0 = x, Y_0 = y$ and $X_T = Y_T$, then for any $f \in \mathcal{B}_b^+(E)$,*

$$(P_T f)^p(y) \le \big(P_T f^p(x)\big)\big(\mathbb{E}R^{p/(p-1)}\big)^{p-1}, \quad p > 1;$$
$$P_T f(y) \le \log P_T \mathrm{e}^f(x) + \mathbb{E}(R \log R).$$

Proof. Since $P_T f(x) = \mathbb{E}f(X_T), \mathbb{E}(Rf(Y_T)) = P_T f(y)$ and $X_T = Y_T$, by the Hölder inequality, we have

$$\begin{aligned}(P_T f)^p(y) &= \big(\mathbb{E}(Rf(Y_T))\big)^p = \big(\mathbb{E}(Rf(X_T))\big)^p \\ &\le \big(\mathbb{E}f^p(X_T)\big)(\mathbb{E}R^{p/(p-1)})^{p-1} = \big(P_T f^p(x)\big)\big(\mathbb{E}R^{p/(p-1)}\big)^{p-1}.\end{aligned}$$

Thus, the first inequality holds. By using the Young inequality

$$\mathbb{E}(Rf(X_T)) \le \log \mathbb{E}\mathrm{e}^{f(X_T)} + \mathbb{E}(R \log R)$$

instead of the Hölder inequality, we prove the second inequality. □

Moreover, the argument of coupling by change of measure can also be used to establish the Bismut type derivative formula.

Theorem 1.3.8 (Bismut formula). *Let P_t be a Markov semigroup and let $T > 0$ be fixed. Let $\gamma : [0, r_0] \to E$ with $r_0 > 0$ be a curve on E such that for any $\varepsilon \in (0, r_0)$ there exists a coupling by change of measure (X, X^ε) with changed probability $\mathrm{d}\mathbb{Q}_\varepsilon := R_\varepsilon \mathrm{d}\mathbb{P}$ of the Markov process with $X_0 = \gamma(0), X_0^\varepsilon = \gamma(\varepsilon)$ and $X_T = X_T^\varepsilon$. If*

$$M(T) := \lim_{\varepsilon\to 0} \frac{R_\varepsilon - 1}{\varepsilon}$$

exists in $L^1(\mathbb{P})$, then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} P_T f(\gamma(\varepsilon))\Big|_{\varepsilon=0} = \mathbb{E}\big[M(T) f(X_T)\big], \quad f \in \mathcal{B}_b(E).$$

Proof. Simply note that under the given conditions

$$\lim_{\varepsilon\to 0} \frac{P_T f(\gamma(\varepsilon)) - P_T f(x)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}[R_\varepsilon f(X_T^\varepsilon)] - \mathbb{E} f(X_T)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \mathbb{E}\big[f(X_T)(R_\varepsilon - 1)\big] = \mathbb{E}[M(T) f(X_T)].$$
□

Finally, we consider the integration by parts formula and shift Harnack inequalities.

Theorem 1.3.9. *Let E be a Banach space and $x, e \in E$ and $T > 0$ be fixed.*

(1) *For any coupling by change of measure (X, Y) with changed probability $\mathbb{Q} = R\mathbb{P}$ for the Markov process such that $X_0 = Y_0 = x$ and $Y_T = X_T + e$, there holds the shift Harnack inequality*

$$|P_T f(x)|^p \le P_T\{|f|^p(e + \cdot)\}(x)\big(\mathbb{E} R^{\frac{p}{p-1}}\big)^{p-1}, \quad p > 1, f \in \mathcal{B}_b(E),$$

and the shift log-Harnack inequality

$$P_T \log f(x) \le \log P_T\{f(e + \cdot)\}(x) + \mathbb{E}(R \log R), \quad f \in \mathcal{B}_b(E), f > 0.$$

(2) *Let $(X, X^\varepsilon), \varepsilon \in [0, 1]$, be a family of couplings by change of measure for $\mathbb{P}^x$ and $\mathbb{P}^x$ with changed probability $\mathbb{Q}_\varepsilon = R_\varepsilon \mathbb{P}$ such that*

$$X^\varepsilon(T) = X_T + \varepsilon e, \quad \varepsilon \in (0, 1].$$

If $R_0 = 1$ and $N_T := -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_\varepsilon|_{\varepsilon=0}$ exists in $L^1(\mathbb{P})$, then

$$P_T(\nabla_e f)(x) = \mathbb{E}\big\{f(X_T) N_T\big\}, \quad f, \nabla_e f \in \mathcal{B}_b(E).$$

Proof. The proof is similar to that we introduced above for the Harnack inequality and Bismut formula.

(1) Note that $P_T f(x) = \mathbb{E}\{Rf(Y_T)\} = \mathbb{E}\{Rf(X_T + e)\}$. We have

$$|P_T f(x)|^p \le \big(\mathbb{E}|f|^p(X_T+e)\big)\big(\mathbb{E}R^{\frac{p}{p-1}}\big)^{p-1} = P_T\{|f|^p(e+\cdot)\}(x)\big(\mathbb{E}R^{\frac{p}{p-1}}\big)^{p-1}.$$

Similarly, for positive f,

$$\begin{aligned} &P_T \log f(x) = \mathbb{E}\{R\log f(X_T + e)\} \\ &\le \log \mathbb{E}f(X_T + e) + \mathbb{E}(R\log R) = \log P_T\{f(e+\cdot)\}(x) + \mathbb{E}(R\log R). \end{aligned}$$

(2) Noting that $P_T f(x) = \mathbb{E}\big\{R_\varepsilon f(X^\varepsilon(T))\big\} = \mathbb{E}\big\{R_\varepsilon f(X_T + \varepsilon e)\big\}$, we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\big\{R_\varepsilon f(X_T + \varepsilon e)\big\}\Big|_{\varepsilon=0} = P_T(\nabla_e f)(x) - \mathbb{E}\big\{f(X_T)N_T\big\},$$

provided $R_0 = 1$ and $N_T := -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}R_\varepsilon|_{\varepsilon=0}$ exists in $L^1(\mathbb{P})$. □

1.4 Harnack inequalities and applications

In this section we consider the Harnack and shift Harnack inequalities for a bounded linear operator and applications. As results presented below are not yet well known, we include complete proofs (see also [Wang and Yuan (2011); Wang (2012d)]).

1.4.1 *Harnack inequality*

Definition 1.4.1. Let μ be a probability measure on $(E, \mathcal{B})$, and let P be a bounded linear operator on $\mathcal{B}_b(E)$.

(*i*) μ is called *quasi-invariant* of P, if μP is absolutely continuous w.r.t. μ, where $(\mu P)(A) := \mu(P1_A)$, $A \in \mathcal{B}$. If $\mu P = \mu$ then μ is called an *invariant probability measure* of P.

(*ii*) A measurable function p on E^2 is called the *kernel* of P w.r.t. μ, if

$$Pf = \int_E p(\cdot, y)f(y)\mu(\mathrm{d}y), \quad f \in \mathcal{B}_b(E).$$

(*iii*) Let E be a topology space. P is called a *Feller* operator, if $PC_b(E) \subset C_b(E)$, while it is called a *strong Feller* operator if $P\mathcal{B}_b(E) \subset C_b(E)$.

From now on, in this section we assume that E is a topology space and $\mathcal{B}$ is its Borel σ-field and P is a Markov operator (i.e. positivity-preserving, contraction linear operator with $P1 = 1$) given by

$$Pf(x) = \int_E f(y)P(x, \mathrm{d}y), \quad f \in \mathcal{B}_b(E), x \in E$$

for a transition probability measure $P(x, \mathrm{d}y)$. We will consider the following *Harnack* type inequality for P:

$$\Phi(Pf(x)) \leq \{P\Phi(f)(y)\}\mathrm{e}^{\Psi(x,y)}, \quad x, y \in E, f \in \mathcal{B}_b^+(E), \tag{1.4.1}$$

where $\Phi \in C([0, \infty))$ is non-negative and strictly increasing, and Ψ is a measurable non-negative function on E^2.

Theorem 1.4.1. *Let μ be a quasi-invariant probability measure of P. Let $\Phi \in C^1([0, \infty))$ be an increasing function with $\Phi'(1) > 0$ and $\Phi(\infty) := \lim_{r \to \infty} \Phi(r) = \infty$, such that (1.4.1) holds.*

(1) *If $\lim_{y \to x}\{\Psi(x, y) + \Psi(y, x)\} = 0$ holds for all $x \in E$, then P is strong Feller.*
(2) *P has a kernel p w.r.t. μ, so that any invariant probability measure of P is absolutely continuous w.r.t. μ.*
(3) *P has at most one invariant probability measure and if it has, the kernel of P w.r.t. the invariant probability measure is strictly positive.*
(4) *The kernel p of P w.r.t. μ satisfies*

$$\int_E p(x, \cdot)\Phi^{-1}\Big(\frac{p(x, \cdot)}{p(y, \cdot)}\Big)\mathrm{d}\mu \leq \Phi^{-1}(\mathrm{e}^{\Psi(x,y)}), \quad x, y \in E,$$

where $\Phi^{-1}(\infty) := \infty$ by convention.
(5) *If $r\Phi^{-1}(r)$ is convex for $r \geq 0$, then the kernel p of P w.r.t. μ satisfies*

$$\int_E p(x, \cdot)p(y, \cdot)\mathrm{d}\mu \geq \mathrm{e}^{-\Psi(x,y)}, \quad x, y \in E.$$

(6) *If μ is an invariant probability measure of P, then*

$$\sup_{f \in \mathcal{B}_b^+(E), \mu(\Phi(f)) \leq 1} Pf(x) \leq \frac{1}{\int_E \mathrm{e}^{-\Psi(x,y)}\mu(\mathrm{d}y)}, \quad x \in E.$$

Proof. Since (6) is obvious, below we prove (1)-(5) respectively.

(1) Let $f \in \mathcal{B}_b(E)$ be positive. Applying (1.4.1) to $1 + \varepsilon f$ in place of f for $\varepsilon > 0$, we have

$$\Phi(1 + \varepsilon Pf(x)) \leq \{P\Phi(1 + \varepsilon f)(y)\}\mathrm{e}^{\Psi(x,y)}, \quad x, y \in E, \varepsilon > 0.$$

By the Taylor expansion this implies

$$\Phi(1)+\varepsilon\Phi'(1)Pf(x)+\mathrm{o}(\varepsilon)\le\{\Phi(1)+\varepsilon\Phi'(1)Pf(y)+\mathrm{o}(\varepsilon)\}\mathrm{e}^{\Psi(x,y)} \tag{1.4.2}$$

for small $\varepsilon>0$. Letting $y\to x$ we obtain

$$\varepsilon Pf(x)\le\varepsilon\liminf_{y\to x}Pf(y)+\mathrm{o}(\varepsilon).$$

Thus, $Pf(x)\le\liminf_{y\to x}Pf(y)$ holds for all $x\in E$. Similarly, changing the roles of x and y we obtain $Pf(y)\ge\limsup_{x\to y}Pf(x)$ for any $y\in E$. Therefore, Pf is continuous.

(2) To prove the existence of a kernel, it suffices to prove that for any $A\in\mathcal{B}$ with $\mu(A)=0$ we have $P1_A\equiv0$. Applying (1.4.1) to $f=1+n1_A$, we obtain

$$\begin{aligned}&\Phi(1+nP1_A(x))\int_E\mathrm{e}^{-\Psi(x,y)}\mu(\mathrm{d}y)\\&\le\int_E\Phi(1+n1_A)(y)(\mu P)(\mathrm{d}y),\quad n\ge1.\end{aligned} \tag{1.4.3}$$

Since $\mu(A)=0$ and μ is quasi-invariant for P, we have $1_A=0,\mu P$-a.s. So, it follows from (1.4.3) that

$$\Phi(1+nP1_A(x))\le\frac{\Phi(1)}{\int_E\mathrm{e}^{-\Psi(x,y)}\mu(\mathrm{d}y)}<\infty,\quad x\in E,n\ge1.$$

Since $\Phi(1+n)\to\infty$ as $n\to\infty$, this implies that $P1_A(x)=0$ for all $x\in E$.

Now, for any invariant probability measure μ_0 of P, if $\mu(A)=0$ then $P1_A\equiv0$ implies that $\mu_0(A)=\mu_0(P1_A)=0$. Therefore, μ_0 is absolutely continuous w.r.t. μ.

(3) We first prove that the kernel of P w.r.t. an invariant probability measure μ_0 is strictly positive. To this end, it suffices to show that for any $x\in E$ and $A\in\mathcal{B}$, $P1_A(x)=0$ implies that $\mu_0(A)=0$. Since $P1_A(x)=0$, applying (1.4.1) to $f=1+nP1_A$ we obtain

$$\Phi(1+nP1_A(y))\le\{P\Phi(1+n1_A)(x)\}\mathrm{e}^{\Psi(y,x)}=\Phi(1)\mathrm{e}^{\Psi(y,x)},\quad y\in E,n\ge1.$$

Letting $n\to\infty$ we conclude that $P1_A\equiv0$ and hence, $\mu_0(A)=\mu_0(P1_A)=0$.

Next, let μ_1 be another invariant probability measure of P, by (2) we have $\mathrm{d}\mu_1=f\mathrm{d}\mu_0$ for some probability density function f. We aim to prove that $f=1,\mu_0$-a.e. Let $p(x,y)>0$ be the kernel of P w.r.t. μ_0, and let $P^*(x,\mathrm{d}y)=p(y,x)\mu_0(\mathrm{d}y)$. Then

$$P^*g=\int_E g(y)P^*(\cdot,\mathrm{d}y),\quad g\in\mathcal{B}_b(E)$$

is the adjoint operator of P w.r.t. μ_0. Since μ_0 is P-invariant, we have

$$\int_E gP^*1\,\mathrm{d}\mu_0 = \int_E Pg\,\mathrm{d}\mu_0 = \int_E g\,\mathrm{d}\mu_0, \quad g \in \mathcal{B}_b(E).$$

This implies that $P^*1 = 1$. μ_0-a.e. Thus, for μ_0-a.e. $x \in E$ the measure $P^*(x,\cdot)$ is a probability measure. On the other hand, since μ_1 is P-invariant, we have

$$\begin{aligned}\int_E (P^*f)g\,\mathrm{d}\mu_0 &= \int_E fPg\,\mathrm{d}\mu_0 = \int_E Pg\,\mathrm{d}\mu_1\\ &= \int_E g\,\mathrm{d}\mu_1 = \int_E fg\,\mathrm{d}\mu_0, \quad g \in \mathcal{B}_b(E).\end{aligned}$$

This implies that $P^*f = f$, μ_0-a.e. Therefore,

$$\int_E P^*\frac{1}{f+1}\,\mathrm{d}\mu_0 = \int_E \frac{1}{f+1}\,\mathrm{d}\mu_0 = \int_E \frac{1}{P^*f+1}\,\mathrm{d}\mu_0.$$

When $P^*(x,\cdot)$ is a probability measure, by the Jensen inequality one has $P^*\frac{1}{1+f}(x) \geq \frac{1}{P^*f+1}(x)$ and the equation holds if and only if f is constant $P^*(x,\cdot)$-a.s. Hence, f is constant $P^*(x,\cdot)$-a.s. for μ_0-a.e. x. Since $p(x,y) > 0$ for any $y \in E$ such that μ_0 is absolutely continuous w.r.t. $P^*(x,\cdot)$ for any $x \in E$, we conclude that f is constant μ_0-a.s. Therefore, $f = 1$ μ_0-a.s. since f is a probability density function.

(4) Applying (1.4.1) to

$$f = n \wedge \Phi^{-1}\Big(\frac{p(x,\cdot)}{p(y,\cdot)}\Big)$$

and letting $n \to \infty$, we obtain the desired inequality.

(5) Let $r\Phi^{-1}(r)$ be convex for $r \geq 0$. By the Jensen inequality we have

$$\int_E p(x,\cdot)\Phi^{-1}(p(x,\cdot))\mathrm{d}\mu \geq \Phi^{-1}(1).$$

So, applying (1.4.1) to

$$f = n \wedge \Phi^{-1}(p(x,\cdot))$$

and letting $n \to \infty$, we obtain

$$\int_E p(x,\cdot)p(y,\cdot)\mathrm{d}\mu \geq \mathrm{e}^{-\Psi(x,y)}\Phi\bigg(\int_E p(x,\cdot)\Phi^{-1}(p(x,\cdot))\mathrm{d}\mu\bigg) \geq \mathrm{e}^{-\Psi(x,y)}. \qquad \square$$

Let (E,ρ) be a metric space. We shall often consider the following Harnack inequality with a power $\alpha > 1$ (i.e. (1.4.1) with $\Phi(r) = r^\alpha$):

$$(Pf(x))^\alpha \leq (Pf^\alpha(y))\exp\Big[\frac{\alpha c\rho(x,y)^2}{\alpha-1}\Big], \quad f \in \mathcal{B}_b^+(E), x,y \in E, \quad (1.4.4)$$

where $c > 0$ is a constant. To state our next result, we shall assume that E is a length space, i.e. for any $x \neq y$ and any $s \in (0,1)$, there exists a sequence $\{z_n\} \subset E$ such that $\rho(x, z_n) \to s\rho(x,y)$ and $\rho(z_n, y) \to (1-s)\rho(x,y)$ as $n \to \infty$.

Theorem 1.4.2. *Assume that (E, ρ) is a length space and let $\alpha_1, \alpha_2 > 1$ be two constants. If (1.4.4) holds for $\alpha = \alpha_1, \alpha_2$, it holds also for $\alpha = \alpha_1\alpha_2$.*

Proof. Let

$$s = \frac{\alpha_1 - 1}{\alpha_1\alpha_2 - 1}, \text{ or equivalently, } 1 - s = \frac{\alpha_1(\alpha_2 - 1)}{\alpha_1\alpha_2 - 1},$$

and let $\{z_n\} \subset E$ such that $\rho(x, z_n) \to s\rho(x,y)$ and $\rho(z_n, y) \to (1-s)\rho(x,y)$ as $n \to \infty$. Since (1.4.4) holds for $\alpha = \alpha_1$ and $\alpha = \alpha_2$, for any $f \in \mathcal{B}_b^+(E)$ we have

$$\begin{aligned}(Pf(x))^{\alpha_1\alpha_2} &\leq (Pf^{\alpha_1}(z_n))^{\alpha_2} \exp\left[\frac{\alpha_1\alpha_2 c\rho(x, z_n)^2}{\alpha_1 - 1}\right] \\ &\leq (Pf^{\alpha_1\alpha_2}(y)) \exp\left[\frac{\alpha_1\alpha_2 c\rho(x, z_n)^2}{\alpha_1 - 1} + \frac{\alpha_2 c\rho(z_n, y)^2}{\alpha_2 - 1}\right].\end{aligned}$$

Letting $n \to \infty$ we arrive at

$$\begin{aligned}(Pf(x))^{\alpha_1\alpha_2} &\leq (Pf^{\alpha_1\alpha_2}(y)) \exp\left[\frac{\alpha_1\alpha_2 cs^2\rho(x,y)^2}{\alpha_1 - 1} + \frac{\alpha_2 c(1-s)^2\rho(x,y)^2}{\alpha_2 - 1}\right] \\ &= (Pf^{\alpha_1\alpha_2}(y)) \exp\left[\frac{\alpha_1\alpha_2 c\rho(x,y)^2}{\alpha_1\alpha_2 - 1}\right].\end{aligned}$$

□

As a consequence of Theorem 1.4.2, (1.4.4) implies the following log-Harnack inequality (1.4.5).

Corollary 1.4.3. *Let (E, ρ) be a length space. If (1.4.4) holds for some $\alpha > 1$, then*

$$P(\log f)(x) \leq \log Pf(y) + c\rho(x,y)^2, \quad x, y \in E, f \geq 1, f \in \mathcal{B}_b(E). \tag{1.4.5}$$

Proof. By Theorem 1.4.2, (1.4.4) holds for $\alpha^n (n \in \mathbb{N})$ in place of α. So,

$$Pf^{\alpha^{-n}}(x) \leq (Pf(y))^{\alpha^{-n}} \exp\left[\frac{c\rho(x,y)^2}{\alpha^n - 1}\right].$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned}P(\log f)(x) &= \lim_{n\to\infty} P\Big(\frac{f^{\alpha^{-n}} - 1}{\alpha^{-n}}\Big)(x) \\ &\leq \lim_{n\to\infty} \left\{\frac{(Pf(y))^{\alpha^{-n}} - 1}{\alpha^{-n}} + (Pf(y))^{\alpha^{-n}} \frac{\exp\left[\frac{c\rho(x,y)^2}{\alpha^n - 1}\right] - 1}{\alpha^{-n}}\right\} \\ &= \log Pf(y) + c\rho(x,y)^2.\end{aligned}$$

□

Obviously, each of (1.4.4) and (1.4.5) implies that $P(x,\cdot)$ and $P(y,\cdot)$ are equivalent to each other. Indeed, if $P(y,A)=0$ then applying (1.4.4) to $f=1_A$ or applying (1.4.5) to $f=1+n1_A$ and letting $n\to\infty$, we conclude that $P(x,A)=0$. By the same reason, $P(x,\cdot)$ and $P(y,\cdot)$ are equivalent for any $x,y\in E$ if

$$(Pf(x))^\alpha \le (Pf^\alpha(y))\Psi(x,y),\quad x,y\in E, f\in\mathcal{B}_b^+(E) \tag{1.4.6}$$

or

$$P(\log f)(x)\le \log Pf(y)+\Psi(x,y),\quad x,y\in E, f\ge 1, f\in\mathcal{B}_b(E) \tag{1.4.7}$$

holds for some positive function Ψ on $E\times E$. In these cases let

$$p_{x,y}(z)=\frac{P(x,\mathrm{d}z)}{P(y,\mathrm{d}z)}$$

be the Radon-Nikodym derivative of $P(x,\cdot)$ with respect to $P(y,\cdot)$.

Proposition 1.4.4. *Let Ψ be a positive function on $E\times E$.*

(1) (1.4.6) *holds if and only if $P(x,\cdot)$ and $P(y,\cdot)$ are equivalent and $p_{x,y}$ satisfies*

$$P\{p_{x,y}^{1/(\alpha-1)}\}(x)\le\Psi(x,y)^{1/(\alpha-1)},\quad x,y\in E. \tag{1.4.8}$$

(2) (1.4.7) *holds if and only if $P(x,\cdot)$ and $P(y,\cdot)$ are equivalent and $p_{x,y}$ satisfies*

$$P\{\log p_{x,y}\}(x)\le\Psi(x,y),\quad x,y\in E. \tag{1.4.9}$$

(3) *If* (1.4.7) *holds then for a P-invariant probability measure μ, the entropy-cost inequality*

$$\mu\big((P^*f)\log P^*f\big)\le W_1^\Psi(f\mu,\mu),\quad f\ge 0, \mu(f)=1$$

holds for P^* the adjoint operator of P in $L^2(\mu)$.

Proof. (1) Applying (1.4.6) to $f_n(z):=\{n\wedge p_{x,y}(z)\}^{1/(\alpha-1)}$, $n\ge 1$, we obtain

$$\begin{aligned}(Pf_n(x))^\alpha &\le \Psi(x,y)Pf_n^\alpha(y)=\Psi(x,y)\int_E\{n\wedge p_{x,y}(z)\}^{\alpha/(\alpha-1)}P(y,\mathrm{d}z)\\ &\le \Psi(x,y)\int_E\{n\wedge p_{x,y}(z)\}^{1/(\alpha-1)}P(x,\mathrm{d}z)=\Psi(x,y)Pf_n(x).\end{aligned}$$

Thus,

$$P\{p_{x,y}^{1/(\alpha-1)}\}(x)=\lim_{n\to\infty}Pf_n(x)\le\Psi(x,y)^{1/(\alpha-1)}.$$

So, (1.4.6) implies (1.4.8).

On the other hand, if (1.4.8) holds then for any $f \in \mathcal{B}_b^+(E)$, by the Hölder inequality

$$\begin{aligned} Pf(x) &= \int_E \{p_{x,y}\}(z) f(z) P(y, \mathrm{d}z) \\ &\leq (Pf^\alpha(y))^{1/\alpha} \bigg(\int_E p_{x,y}(z)^{\alpha/(\alpha-1)} P(y, \mathrm{d}z) \bigg)^{(\alpha-1)/\alpha} \\ &= (Pf^\alpha(y))^{1/\alpha} (P p_{x,y}^{1/(\alpha-1)}(x))^{(\alpha-1)/\alpha} \\ &\leq (Pf^\alpha(y))^{1/\alpha} \Psi(x,y)^{1/\alpha}. \end{aligned}$$

Therefore, (1.4.6) holds.

(2) We shall use the following *Young inequality*: for any probability measure ν on M, if $g_1, g_2 \geq 0$ with $\nu(g_1) = 1$, then

$$\nu(g_1 g_2) \leq \nu(g_1 \log g_1) + \log \nu(\mathrm{e}^{g_2}).$$

For $f \geq 1$, applying the above inequality for $g_1 = p_{x,y}, g_2 = \log f$ and $\nu = P(y, \cdot)$, we obtain

$$\begin{aligned} P(\log f)(x) &= \int_E \{p_{x,y}(z) \log f(z)\} P(y, \mathrm{d}z) \\ &\leq P(\log p_{x,y})(x) + \log Pf(y). \end{aligned}$$

So, (1.4.9) implies (1.4.7). On the other hand, applying (1.4.7) to $f_n = 1 + n p_{x,y}$, we arrive at

$$\begin{aligned} &P\{\log p_{x,y}\}(x) \leq P(\log f_n)(x) - \log n \\ &\leq \log Pf_n(y) - \log n + \Psi(x,y) = \log \frac{n+1}{n} + \Psi(x,y). \end{aligned}$$

Therefore, by letting $n \to \infty$ we obtain (1.4.9).

(3) Let $\Pi \in \mathcal{C}(f\mu, \mu)$. Applying (1.4.7) to $P^* f$ in place of f and integrating w.r.t. Π, we obtain

$$\begin{aligned} \mu\big((P^* f) \log P^* f\big) &= \int_{E \times E} P \log P^* f(x) \Pi(\mathrm{d}x, \mathrm{d}y) \\ &\leq \int_{E \times E} \log PP^* f(y) \Pi(\mathrm{d}x, \mathrm{d}y) + \Pi(\Psi) \\ &= \mu(\log PP^* f) + \Pi(\Psi) \\ &\leq \log \mu(PP^* f) + \Pi(\Psi) = \Pi(\Psi), \end{aligned}$$

where in the last two steps we have used the Jensen inequality and that μ is PP^*-invariant. This completes the proof. □

1.4.2 *Shift Harnack inequality*

Let $P(x, \mathrm{d}y)$ be a transition probability on a Banach space E. Let

$$Pf(x) = \int_{\mathbb{R}^d} f(y)P(x, \mathrm{d}y), \quad f \in \mathcal{B}_b(\mathbb{R}^d)$$

be the associated Markov operator. Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing and convex continuous function. Consider the shift Harnack inequality

$$\Phi(Pf(x)) \le P\{\Phi \circ f(e + \cdot)\}(x)\mathrm{e}^{C_\Phi(x,e)}, \quad f \in \mathcal{B}_b^+(E) \tag{1.4.10}$$

for some $x, e \in E$ and constant $C_\Phi(x, e) \ge 0$. Obviously, if $\Phi(r) = r^p$ for some $p > 1$ then this inequality reduces to the shift Harnack inequality with power p, while when $\Phi(r) = \mathrm{e}^r$ it becomes the shift log-Harnack inequality.

Theorem 1.4.5. *Let P be given above and satisfy* (1.4.10) *for all $x, e \in E := \mathbb{R}^d$ and some non-negative measurable function C_Φ on $\mathbb{R}^d \times \mathbb{R}^d$. Then*

$$\sup_{f \in \mathcal{B}_b^+(\mathbb{R}^d), \int_{\mathbb{R}^d} \Phi \circ f(x)\mathrm{d}x \le 1} \Phi(Pf)(x) \le \frac{1}{\int_{\mathbb{R}^d} \mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e}, \quad x \in \mathbb{R}^d. \tag{1.4.11}$$

Consequently:

(1) *If $\Phi(0) = 0$, then P has a transition density $\mathbf{p}(x, y)$ w.r.t. the Lebesgue measure such that*

$$\int_{\mathbb{R}^d} \mathbf{p}(x, y)\Phi^{-1}(\mathbf{p}(x, y))\mathrm{d}y \le \Phi^{-1}\left(\frac{1}{\int_{\mathbb{R}^d} \mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e}\right). \tag{1.4.12}$$

(2) *If $\Phi(r) = r^p$ for some $p > 1$, then*

$$\int_{\mathbb{R}^d} \mathbf{p}(x, y)^{\frac{p}{p-1}}\mathrm{d}y \le \frac{1}{\left(\int_{\mathbb{R}^d} \mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e\right)^{\frac{1}{p-1}}}. \tag{1.4.13}$$

Proof. Let $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \Phi(f)(x)\mathrm{d}x \le 1$. By (1.4.10) we have

$$\Phi(Pf)(x)\mathrm{e}^{-C_\Phi(x,e)} \le P(\Phi \circ f(e + \cdot))(x) = \int_{\mathbb{R}^d} \Phi \circ f(y + e)P(x, \mathrm{d}y).$$

Integrating both sides w.r.t. $\mathrm{d}e$ and noting that $\int_{\mathbb{R}^d} \Phi \circ f(y + e)\mathrm{d}e = \int_{\mathbb{R}^d} \Phi \circ f(e)\mathrm{d}e \le 1$, we obtain

$$\Phi(Pf)(x)\int_{\mathbb{R}^d} \mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e \le 1.$$

This implies (1.4.11).

When $\Phi(0)=0$, (1.4.11) implies that

$$\sup_{f\in\mathcal{B}_b^+(\mathbb{R}^d),\int_{\mathbb{R}^d}\Phi\circ f(x)\mathrm{d}x\le 1} Pf(x)\le \Phi^{-1}\bigg(\frac{1}{\int_{\mathbb{R}^d}\mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e}\bigg)<\infty \tag{1.4.14}$$

since by the strictly increasing and convex properties we have $\Phi(r)\uparrow\infty$ and $r\uparrow\infty$. Now, for any Lebesgue-null set A, taking $f_n=n1_A$ we obtain from $\Phi(0)=0$ that

$$\int_{\mathbb{R}^d}\Phi\circ f_n(x)\mathrm{d}x=0\le 1.$$

Therefore, applying (1.4.14) to $f=f_n$ we obtain

$$P(x,A)=P1_A(x)\le\frac{1}{n}\Phi^{-1}\bigg(\frac{1}{\int_{\mathbb{R}^d}\mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e}\bigg),$$

which goes to zero as $n\to\infty$. Thus, $P(x,\cdot)$ is absolutely continuous w.r.t. the Lebesgue measure, so that the density function $\mathbf{p}(x,y)$ exists, and (1.4.12) follows from (1.4.11) by taking $f(y)=\Phi^{-1}(\mathbf{p}(x,y))$.

Finally, let $\Phi(r)=r^p$ for some $p>1$. For fixed x, let

$$f_n(y)=\frac{\{n\wedge\mathbf{p}(x,y)\}^{\frac{1}{p-1}}}{\big(\int_{\mathbb{R}^d}\{n\wedge\mathbf{p}(x,z)\}^{\frac{p}{p-1}}\mathrm{d}z\big)^{\frac{1}{p}}},\quad n\ge 1.$$

It is easy to see that $\int_{\mathbb{R}^d}f_n^p(y)\mathrm{d}y=1$. Then it follows from (1.4.11) with $\Phi(r)=r^p$ that

$$\int_{\mathbb{R}^d}\{n\wedge\mathbf{p}(x,y)\}^{\frac{p}{p-1}}\mathrm{d}y\le\big(Pf_n(x)\big)^{\frac{p}{p-1}}\le\frac{1}{\big(\int_{\mathbb{R}^d}\mathrm{e}^{-C_\Phi(x,e)}\mathrm{d}e\big)^{\frac{1}{p-1}}}.$$

Then (1.4.13) follows by letting $n\to\infty$. □

Finally, we consider applications of the shift Harnack inequality to distribution properties of the underlying transition probability.

Theorem 1.4.6. *Let P be given as above for some Banach space E, and let* (1.4.10) *hold for some $x,e\in E$, finite $C_\Phi(x,e)$ and some strictly increasing and convex continuous function Φ with $\Phi(0)=0$.*

(1) *$P(x,\cdot)$ is absolutely continuous w.r.t. $P(x,\cdot-e)$.*
(2) *If $\Phi(r)=r\Psi(r)$ for some strictly increasing positive continuous function Ψ on $(0,\infty)$, then the density $\mathbf{p}(x,e;y):=\frac{P(x,\mathrm{d}y)}{P(x,\mathrm{d}y-e)}$ satisfies*

$$\int_E\Phi(\mathbf{p}(x,e;y))P(x,\mathrm{d}y-e)\le\Psi^{-1}\big(\mathrm{e}^{C_\Phi(x,e)}\big).$$

Proof. For $P(x, \cdot - e)$-null set A, let $f = 1_A$. Then (1.4.10) implies that $\Phi(P(x,A)) \le 0$; hence $P(x,A) = 0$ since $\Phi(r) > 0$ for $r > 0$. Therefore, $P(x,\cdot)$ is absolutely continuous w.r.t. $P(x, \cdot - e)$. Next, let $\Phi(r) = r\Psi(r)$. Applying (1.4.10) for $f(y) = \Psi(n \wedge \mathbf{p}(x,e;y))$ and noting that

$$\begin{aligned} Pf(x) &= \int_E \{\Psi(n \wedge \mathbf{p}(x,e;y))\} P(x,\mathrm{d}y) \\ &\ge \int_E \Phi(n \wedge \mathbf{p}(x,e;y)) P(x, \mathrm{d}y - e), \end{aligned}$$

we obtain

$$\int_E \Phi(n \wedge \mathbf{p}(x,e;y)) P(x, \mathrm{d}y - e) \le \Psi^{-1}\big(\mathrm{e}^{C_\Phi(x,e)}\big).$$

Then the proof is completed by letting $n \to \infty$. □

1.5 Harnack inequality and derivative estimate

In this section, we consider the relationship between Harnack inequalities and derivative estimates of Markov operators on a geodesic space. The main results are reorganized from [Arnaudon *et al* (2009); Röckner and Wang (2010); Wang (2012b,d)] where different type of Harnack inequalities are considered.

Recall that a metric space (E,ρ) is called a *geodesic space*, if for any $x, y \in E$, there exists a map $\gamma : [0,1] \to E$ such that $\gamma(0) = x, \gamma(1) = y$ and $\rho(\gamma(s),\gamma(t)) = |t-s|\rho(x,y)$ for $s,t \in [0,1]$. A map $\gamma : [0, r_0] \to E$ with $\gamma(0) = x$ for some $r_0 > 0$ and $x \in E$ is called a minimal geodesic from x with speed $c \ge 0$, if $\rho(\gamma(s),\gamma(t)) = c|t-s|$ holds for $s,t \in [0,r_0]$. For a function f on E, we define $|\nabla f|(x)$ as the local Lipschitz constant of f at point x, i.e.

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{\rho(x,y)}.$$

Obviously, $|\nabla f| \ge 0$ and $|f(x) - f(y)| \le \rho(x,y)\|\nabla f\|_\infty$. Let P be a Markov operator on $\mathcal{B}_b(E)$.

1.5.1 *Harnack inequality and entropy-gradient estimate*

Proposition 1.5.1. *Let $\delta_0 \ge 0$ and $\beta \in C((\delta_0,\infty) \times E; [0,\infty))$. The following two statements are equivalent.*

(1) *For any strictly positive* $f \in \mathcal{B}_b(E)$,

$$|\nabla Pf| \le \delta\big\{P(f\log f) - (Pf)\log Pf\big\} + \beta(\delta,\cdot)Pf, \quad \delta > \delta_0.$$

(2) *For any* $p > 1$ *and* $x, y \in E$ *such that* $\rho(x,y) \le \frac{p-1}{p\delta_0}$, *and for any positive* $f \in \mathcal{B}_b(E)$,

$$(Pf)^p(x) \le \big\{Pf^p(y)\big\}$$
$$\exp\left[\int_0^1 \frac{p\rho(x,y)}{1+(p-1)s}\beta\Big(\frac{p-1}{\rho(x,y)\{1+(p-1)s\}}, \gamma(s)\Big)\mathrm{d}s\right],$$

where $\gamma : [0,1] \to E$ *is a minimal geodesic from* x *to* y *with speed* $\rho(x,y)$.

Proof. For $p > 1$, let $\alpha(s) = 1 + (p-1)s$. We have $\delta(s) := \frac{p-1}{\alpha(s)\rho(x,y)} > \delta_0$ for $s \in [0,1)$. Then (1) implies that

$$[0,1) \ni s \mapsto \log(Pf^{\alpha(s)})^{p/\alpha(s)}(\gamma(s))$$

is Lipschitz continuous and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\log(Pf^{\alpha(s)})^{p/\alpha(s)}(\gamma(s))\\
&\ge \frac{p(p-1)\{P(f^{\alpha(s)}\log f^{\alpha(s)}) - (Pf^{\alpha(s)})\log Pf^{\alpha(s)}\}}{\alpha(s)^2 Pf^{\alpha(s)}}(\gamma(s))\\
&\quad - \frac{p\rho(x,y)|\nabla Pf^{\alpha(s)}|}{\alpha(s)Pf^{\alpha(s)}}(\gamma(s))\\
&\ge -\frac{p\rho(x,y)}{\alpha(s)}\beta\Big(\frac{p-1}{\alpha(s)\rho(x,y)},\ \gamma(s)\Big), \quad s \in [0,1).
\end{aligned}$$

Integrating over $[0,1)$ we obtain (2).

On the other hand, for any $z \in E$, let γ be a minimal geodesic from z with $\rho(\gamma(r), z) = r$ for small $r > 0$, and

$$|\nabla f|(z) = \limsup_{r\to 0}\frac{|f(\gamma(r)) - f(z)|}{r}.$$

We have either (i) or (ii):

(i) $|\nabla f|(z) = \limsup_{r\to 0}\frac{f(\gamma(r))-f(z)}{r}$,
(ii) $|\nabla f|(z) = \limsup_{r\to 0}\frac{f(z)-f(\gamma(r))}{r}$.

For any $\delta > \delta_0$, let $p = 1 + \delta r$. We have $\delta \ge \delta_0(1+\delta r)$ and thus, $\rho(\gamma(r), z) = r \le \frac{p-1}{p\delta_0}$ for small $r > 0$. Applying (2) to $x = \gamma(r)$ and $y = z$, we obtain from (i) that

$$\delta\big\{(Pf)\log Pf\big\}(z) + |\nabla Pf|(z) = \limsup_{r\to 0} \frac{(Pf)^{1+\delta r}(\gamma(r)) - Pf(z)}{r}$$
$$\le \limsup_{r\to 0} \frac{(Pf^{1+\delta r})(z)\exp\Big[\int_0^1 \frac{(1+\delta r)r}{1+\delta rs}\beta\Big(\frac{\delta}{1+\delta rs}, \gamma(sr)\Big)\mathrm{d}s\Big] - Pf(z)}{r}$$
$$= \delta P(f\log f)(z) + \beta(\delta, z)Pf(z).$$

Similarly, if (ii) holds then

$$|\nabla Pf|(z) - \delta P(f\log f)(z) = \limsup_{r\to 0} \frac{(Pf)(z) - (Pf^{1+\delta r})(\gamma(r))}{r}$$
$$\le \limsup_{r\to 0} \frac{(Pf)(z) - (Pf)^{1+\delta r}(z)\exp\Big[\int_0^1 \frac{(1+\delta r)r}{1+\delta rs}\beta\Big(\frac{\delta}{1+\delta rs}, \gamma(sr)\Big)\mathrm{d}s\Big]}{r}$$
$$= \beta(\delta, z)Pf(z) - \delta\big\{(Pf)\log Pf\big\}(z).$$

Therefore, (1) holds. □

Similarly, we have the following result on the shift Harnack inequality (see also Proposition 5.3.8 in Chapter 5).

Proposition 1.5.2. *Let E be a Banach space. Let $e \in E, \delta_e \in (0,1)$ and $\beta_e \in C((\delta_e, \infty) \times E; [0,\infty))$. Then the following assertions are equivalent.*

(1) *For any positive $f \in C_b^1(E)$,*

$$|P(\nabla_e f)| \le \delta\big\{P(f\log f) - (Pf)\log Pf\big\} + \beta_e(\delta, \cdot)Pf, \quad \delta \ge \delta_e.$$

(2) *For any positive $f \in \mathcal{B}_b(E), r \in (0, \frac{1}{\delta_e})$ and $p \ge \frac{1}{1-r\delta_e}$,*

$$(Pf)^p \le \big(P\{f^p(re + \cdot)\}\big)$$
$$\exp\bigg[\int_0^1 \frac{pr}{1+(p-1)s}\beta_e\Big(\frac{p-1}{r+r(p-1)s}, \cdot + sre\Big)\mathrm{d}s\bigg].$$

Proof. The proof from (1) to (2) is completely similar to the first part of the proof in Proposition 1.5.1. To prove (1) from (2), we let $z, e \in E$ be fixed and assume that $P(\nabla_e f)(z) \ge 0$ (otherwise, simply use $-e$ to replace

e). Then (2) with $p = 1 + \delta r$ for $\delta \geq \delta_e$ implies that

$$\begin{aligned}
&\delta\big\{(Pf)\log Pf\big\}(z) + |P(\nabla_e f)|(z)\\
&= \limsup_{r\to 0} \frac{(P\{f(re+\cdot)\})^{1+\delta r}(z) - Pf(z)}{r}\\
&\leq \limsup_{r\to 0} \frac{\Big\{(Pf^{1+\delta r})(z)\exp\Big[\int_0^1 \frac{(1+\delta r)r}{1+\delta rs}\beta_e\Big(\frac{\delta}{1+\delta rs}, \gamma(r)\Big)\mathrm{d}r\Big] - Pf(z)\Big\}}{r}\\
&= \delta P(f\log f)(z) + \beta_e(\delta, z)Pf(z).
\end{aligned}$$

Therefore, (1) holds. □

1.5.2 Harnack inequality and L^2-gradient estimate

Proposition 1.5.3. *For any constant $C > 0$, the L^2-gradient estimate*

$$|\nabla Pf|^2 \leq C^2 Pf^2, \quad f \in \mathcal{B}_b(E) \tag{1.5.1}$$

is equivalent to the Harnack type inequality

$$Pf(z') \leq Pf(z) + C\rho(z,z')\sqrt{Pf^2(z')}, \tag{1.5.2}$$

holds for all $z, z' \in E$, $f \geq 0$, $f \in \mathcal{B}_b(E)$.

Proof. **(1.5.1) ⇒ (1.5.2)**. Let $\gamma : [0,1] \to E$ be a minimal geodesic such that $\gamma(0) = z, \gamma(1) = z'$. By (1.5.1), for any positive $f \in \mathcal{B}_b(E)$ and constant $r > 0$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}P\Big(\frac{f}{1+rsf}\Big)(\gamma(s))\\
&\leq -rP\Big(\frac{f^2}{(1+rsf)^2}\Big)(\gamma_s) + C\rho(z,z')\sqrt{P\Big(\frac{f}{1+rsf}\Big)^2(\gamma_s)}\\
&\leq \frac{C^2\rho(z,z')^2}{4r}.
\end{aligned}$$

So,

$$P\Big(\frac{f}{1+rf}\Big)(z') \leq Pf(z) + \frac{C^2\rho(z,z')^2}{4r}.$$

Combining this with the fact that

$$\frac{f}{1+rf} = f - \frac{rf^2}{1+rf} \geq f - rf^2,$$

we obtain

$$Pf(z') \leq Pf(z) + \frac{C^2\rho(z,z')^2}{4r} + rPf^2(z').$$

Minimizing the right-hand side in $r > 0$ we prove (1.5.2).

(1.5.2) $\Rightarrow$ (1.5.1). By (1.5.2), we have

$$|Pf(z) - Pf(z')| \leq C\rho(z,z')\|f\|_\infty, \quad f \in C_b(M).$$

So, Pf is Lipschitz continuous for any $f \in \mathcal{B}_b(E)$. Let $z \in E$ and $\gamma : [0,1] \to M$ be a minimal geodesic such that $\gamma(0) = z, \rho(\gamma_0, \gamma_s) = s$ and

$$\limsup_{s\to 0} \frac{Pf(\gamma(s)) - Pf(z)}{s} = |\nabla Pf|(z).$$

Then it follows from (1.5.2) that

$$\begin{aligned} |\nabla Pf|(z) &= \limsup_{s\to 0} \frac{Pf(\gamma(s)) - Pf(\gamma(0))}{s} \\ &\leq C \lim_{s\to 0} \sqrt{Pf^2(\gamma(s))} = C\sqrt{Pf^2(z)}. \end{aligned}$$

Therefore, (1.5.1) holds. □

Correspondingly, we have the following result concerning the shift Harnack inequality.

Proposition 1.5.4. *Let E be a Banach space and $C \geq 0$ be a constant. Then*

$$|P(\nabla_e f)|^2 \leq CPf^2, \quad f \in C_b^1(E), f \geq 0$$

is equivalent to

$$Pf \leq P\{f(re + \cdot)\} + |r|\sqrt{CPf^2}, \quad r \in \mathbb{R}, f \in \mathcal{B}_b^+(E).$$

1.5.3 *Harnack inequalities and gradient-gradient estimates*

In this subsection we consider diffusion semigroup P_t with generator $(L, \mathcal{D}(L))$ on a geodesic space (E,ρ) in the following sense: there exists a subclass $\mathcal{A}_0 \subset \mathcal{D}(L)$ of $\mathcal{B}_b(E)$, such that for any $f \in \mathcal{A}_0$ and $\varphi \in C^\infty([\inf f, \sup f])$ one has $P_t f, \varphi\circ f \in \mathcal{A}_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t Lf = P_t Lf = LP_t f, \quad L\varphi\circ f = \varphi'\circ f Lf + \varphi''\circ f|\nabla f|^2, \quad t \geq 0. \quad (1.5.3)$$

A typical example is a non-explosive elliptic diffusion process on a differential manifold E. In this case we take ρ to be the intrinsic metric induced by the square field of the diffusion, and let

$$\mathcal{A}_0 = \big\{P_t f : \ t \geq 0, f \in C^\infty, \mathrm{d}f \text{ has compact support}\big\}.$$

Proposition 1.5.5. *Assume that* (1.5.3) *holds. Let* ξ *be a positive measurable function on* $[0,\infty)$, *and let* $g \in C^1([0,t])$ *be increasing with* $g(0)=0$ *and* $g(t)=1$.

(1) *If*

$$|\nabla P_t f|^2 \le \xi(t)^2 P_t|\nabla f|^2, \quad f\in\mathcal{A}_0,\ t\ge 0. \tag{1.5.4}$$

Then

$$P_t f(y) \le \log P_t \mathrm{e}^f(x) + \frac{\rho(x,y)^2}{4}\int_0^t |g'(s)\xi(s)|^2\mathrm{d}s, \quad t>0, f\in\mathcal{A}_0. \tag{1.5.5}$$

(2) *If*

$$|\nabla P_t f| \le \xi(t) P_t|\nabla f|, \quad f\in\mathcal{A}_0,\ t\ge 0. \tag{1.5.6}$$

Then

$$(P_t f)^p(x) \le \big(P_t f^p(y)\big)\exp\left[\frac{p\rho(x,y)^2}{4(p-1)}\int_0^t \big|\xi(s)g'(s)\big|^2\mathrm{d}s\right] \tag{1.5.7}$$

holds for $t>0$ *and nonnegative* $f\in\mathcal{A}_0$.

Proof. Let $\gamma:[0,1]\to E$ be a minimal geodesic from x to y with constant speed $\rho(x,y)$.

(1) By (1.5.3) and (1.5.4) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}P_s\log P_{t-s}\mathrm{e}^f(\gamma\circ g(s))\\
&\le \Big\{\rho(x,y)|g'(s)|\cdot|\nabla P_s\log P_{t-s}\mathrm{e}^f| - P_s|\nabla\log P_{t-s}\mathrm{e}^f|^2\Big\}(\gamma\circ g(s))\\
&\le \Big\{\rho(x,y)|g'(s)|\xi(s)\sqrt{P_s|\nabla\log P_{t-s}\mathrm{e}^f|^2} - P_s|\nabla\log P_{t-s}\mathrm{e}^f|^2\Big\}(\gamma\circ g(s))\\
&\le \frac{\rho(x,y)^2\xi(s)^2|g'(s)|^2}{4}.
\end{aligned}$$

Integrating over $[0,t]$ we obtain (1.5.5).

(2) Similarly, by (1.5.3) and (1.5.6) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}f)^p(\gamma\circ g(s))\\
&\ge P_s\big\{p(p-1)(P_{t-s}f)^{p-2}|\nabla P_{t-s}f|^2\big\}(\gamma\circ g(s))\\
&\qquad -\rho(x,y)|g'(s)|\cdot\big|\nabla P_s(P_{t-s}f)^p(\gamma\circ g(s))\big|\\
&\ge pP_s\left\{(P_{t-s}f)^p\Big(\frac{(p-1)|\nabla P_{t-s}f|^2}{(P_{t-s}f)^2} - \frac{|g'(s)|\rho(x,y)|\nabla P_{t-s}f|}{P_{t-s}f}\Big)\right\}(\gamma\circ g(s))\\
&\ge -\frac{p\rho(x,y)^2\xi(s)^2|g'(s)|^2}{4(p-1)}P_s(P_{t-s}f)^p(\gamma\circ g(s)), \quad s\in[0,t].
\end{aligned}$$

This implies (1.5.7). □

1.6 Functional inequalities and applications

Let $\mathbb{H}$ be a separable Hilbert space and $(L, \mathcal{D}(L))$ a negatively definite self-adjoint operator on $\mathbb{H}$ generating a contraction C_0-semigroup P_t. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the associated quadric form. We have $\mathcal{E}(f, g) = -\langle f, Lg\rangle$ for $f, g \in \mathcal{D}(L)$. It is well known that $\|P_t\| \le e^{-t/C}$ if and only if the *Poincaré inequality*

$$\|f\|^2 \le C\mathcal{E}(f, f), \quad f \in \mathcal{D}(\mathcal{E})$$

holds. This inequality is also equivalent to $\inf \sigma(-L) \ge 1/C$, where $\sigma(\cdot)$ stands for the spectrum of a linear operator.

We first introduce a Poincaré type inequality to describe the essential spectrum of L and the exponential decay of P_t in the tail norm, then introduce the weak Poincaré inequality to describe general convergence rates of P_t.

1.6.1 *Poincaré type inequality and essential spectrum*

Let $(L, \mathcal{D}(L))$ be a negative definite self-adjoint operator on a separable Hilbert space $\mathbb{H}$, and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the associate quadratic form. For $B \subset \mathbb{H}$, let

$$\|f\|_{B^*} = \sup\{|\langle f, g\rangle| : g \in B\}, \quad f \in \mathbb{H}.$$

We shall use the following Poincaré type inequality to study the essential spectrum of L:

$$\|f\|^2 \le r\mathcal{E}(f, f) + \beta(r)\|f\|_{B^*}^2, \quad r > r_0, f \in \mathcal{D}(\mathcal{E}), \tag{1.6.1}$$

where $r_0 \ge 0$ is a constant and $\beta : (r_0, \infty) \to (0, \infty)$ is a (decreasing) function.

Let $\sigma_{ess}(L)$ be the *essential spectrum* of L, which consists of limit points in the *spectrum* $\sigma(L)$ and isolated eigenvalues of L with infinite multiplicity.

The following result is due to [Wang (2004b)], which provides a correspondence between upper bound of the essential spectrum for $-L$ and the *Poincaré type inequality* (1.6.1).

Theorem 1.6.1. *Let $r_0 \ge 0$. Then the following statements are equivalent:*

(1) $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$.
(2) *There exist a compact set $B \subset \mathbb{H}$ and a function $\beta : (r_0, \infty) \to (0, \infty)$ such that* (1.6.1) *holds.*

(3) *There exist* $t > 0$ *and* $B \subset \mathbb{H}$ *such that* $P_t B$ *is relatively compact and* (1.6.1) *holds for some* $\beta : (r_0, \infty) \to (0, \infty)$.

Recall that a linear operator on a Banach space is called *compact*, if it sends bounded sets into relatively compact sets. Let $P_t = e^{tL}$. It is well known that P_t is compact for some/all $t > 0$ if and only if $\sigma_{ess}(L) = \emptyset$.

Theorem 1.6.2. *The following statements are equivalent to each other:*

(1) $\sigma_{ess}(L) = \emptyset$.
(2) (1.6.1) *holds for* $r_0 = 0$, *some compact set* B *and some* $\beta : (0, \infty) \to (0, \infty)$.
(3) *There exists* $t > 0$ *and* $B \subset \mathbb{H}$ *such that* $P_t B$ *is relatively compact and* (1.6.1) *holds for* $r_0 = 0$ *and some* $\beta : (0, \infty) \to (0, \infty)$.
(4) P_t *is compact for some* $t > 0$.
(5) P_t *is compact for all* $t > 0$.

Now, we apply the above results to Dirichlet forms on $\mathbb{H} := L^2(\mu)$ for a σ-finite complete measure space $(E, \mathcal{B}, \mu)$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form in $L^2(\mu)$. We shall study (1.6.1) for

$$B = B_\phi := \{g : \ |g| \le \phi\},$$

where $\phi > 0$ is a fixed function in $L^2(\mu)$. In this case

$$\|f\|_{B^*} = \sup_{|g| \le \phi} |\mu(gf)| = \mu(\phi|f|).$$

In particular, if μ is finite we may take $\phi = 1$ such that $\mu(\phi|f|) = \|f\|_1$. The following result is taken from [Wang (2002a)].

Theorem 1.6.3. *Let* $r_0 \ge 0$. *If* $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$, *then for any* $\phi \in L^2(\mu)$ *with* $\phi > 0$ μ*-a.e. there exists* $\beta : (r_0, \infty) \to (0, \infty)$ *such that*

$$\mu(f^2) \le r\mathcal{E}(f, f) + \beta(r)\mu(\phi|f|)^2, \quad r > r_0, f \in \mathcal{D}(\mathcal{E}). \tag{1.6.2}$$

When $r_0 = 0$, the inequality (1.6.2) is called the *super Poincaré inequality* for $\phi = 1$, and the *intrinsic super Poincaré* inequality if ϕ is the ground state of L, i.e. the positive unit eigenfunction of $\inf \sigma(-L)$. Of course, the ground state might not exist.

According to Theorem 1.6.1, to prove that (1.6.2) implies $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$, we need to verify that $P_t B_\phi$ is relatively compact for some $t > 0$. To this end, we assume that P_t has a density $p_t(x, y)$ with respect to μ. The following theorem is due to [Wang (2000b)] and [Gong and Wang (2002)].

Theorem 1.6.4. *Assume that for some $t > 0$ the operator P_t has a density $p_t(x, y)$ with respect to μ, i.e.*

$$P_t f = \int_E p_t(\cdot, y) f(y) \mu(\mathrm{d}y)$$

holds in $L^2(\mu)$. Then for any positive $\phi \in L^2(\mu)$ and $\beta : (r_0, \infty) \to (0, \infty)$, (1.6.2) implies $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$.

Remark 1.6.1. The assumption on the existence of density in Theorem 1.6.4 can be replaced by the existence of asymptotic density: *a linear operator is said to have asymptotic density w.r.t. μ if there exists a sequence of linear operators $\{P_n\}$ having densities w.r.t. μ such that $\|P_n - P\|_2 \to 0$ as $n \to \infty$.* A fact is that *any compact operator has asymptotic density.* See §3.1.3 in [Wang (2005a)] for details.

Due to this Remark, we have the following consequences.

Corollary 1.6.5. *Assume that P_t has asymptotic density for some $t > 0$. Then the following are equivalent:*

(1) *There exist $\phi > 0$ in $L^2(\mu)$ and some $\beta : (r_0, \infty) \to (0, \infty)$ such that (1.6.2) holds.*
(2) *For any $\phi > 0$ in $L^2(\mu)$, there exists $\beta : (r_0, \infty) \to (0, \infty)$ such that (1.6.2) holds.*
(3) $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$.

Corollary 1.6.6. *The following statements are equivalent to each other:*

(1) *P_t is compact for some/any $t > 0$.*
(2) *P_t has asymptotic density for some $t > 0$ and there exist $\phi > 0$ in $L^2(\mu)$ and some $\beta : (0, \infty) \to (0, \infty)$ such that (1.6.2) holds for $r_0 = 0$.*
(3) *P_t has asymptotic density for any $t > 0$, and for any $\phi > 0$ in $L^2(\mu)$ there exists $\beta : (0, \infty) \to (0, \infty)$ such that (1.6.2) holds for $r_0 = 0$.*
(4) $\sigma_{ess}(-L) = \emptyset$.

As a conclusion of this section, we present the following result on (1.6.2) which can be easily verified by splitting arguments.

Proposition 1.6.7. *Let $r_0 \geq 0$. If (1.6.2) holds for some positive $\phi \in L^2(\mu)$ and some $\beta : (r_0, \infty) \to (0, \infty)$, then for any $\tilde{\phi} > 0$ such that $\tilde{\phi} \in L^2(\mu)$ there exists $\tilde{\beta} : (r_0, \infty) \to (0, \infty)$ such that*

$$\mu(f^2) \leq r\mathcal{E}(f, f) + \tilde{\beta}(r)\mu(\tilde{\phi}|f|)^2, \quad f \in \mathcal{D}(\mathcal{E}), r > r_0.$$

1.6.2 Exponential decay in the tail norm

Let P be a bounded linear operator on $L^2(\mu)$. For any $\phi, \psi > 0$ with $\phi, \psi \in L^2(\mu)$ we have

$$\lim_{R\to\infty} \sup_{\|f\|_2\le 1} \|(Pf)1_{\{|Pf|>R\psi\}}\|_2 = \lim_{R\to\infty} \sup_{\|f\|_2\le 1} \mu\big((|Pf| - R\phi)^{+2}\big)^{1/2}.$$

So, the above limits are independent of the choices of ϕ and ψ. We call the limit *tail norm* of P, and denote it by $\|P\|_T$.

Theorem 1.6.8. *Let $r_0 \ge 0$ be fixed. Then*

(1) (1.6.2) *implies $\|P_t\|_T \le e^{-t/r_0}$ for all $t \ge 0$.*
(2) *If $\|P_t\|_T \le e^{-t/r_0}$ holds for some $t > 0$, then for any strictly positive $\phi \in L^2(\mu)$ there exists $\beta : (r_0, \infty) \to (0, \infty)$ such that* (1.6.2) *holds.*

Corollary 1.6.9. *The following statements are equivalent to each other:*

(1) (1.6.2) *holds for $r_0 = 0$, some positive $\phi \in L^2(\mu)$ and some $\beta : (0, \infty) \to (0, \infty)$.*
(2) *$\|P_t\|_T = 0$ for all $t > 0$.*
(3) *$\|P_t\|_T = 0$ for some $t > 0$.*

1.6.3 The F-Sobolev inequality

Let $F \in C(0, \infty)$ be an increasing function such that $\sup_{r\in(0,1]} |rF(r)| < \infty$ and $F(\infty) := \lim_{r\to\infty} F(r) = \infty$. We say that the *$F$-Sobolev inequality* holds if there exist two constants $C_1 > 0, C_2 \ge 0$ such that

$$\mu(f^2 F(f^2)) \le C_1 \mathcal{E}(f, f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}),\ \mu(f^2) = 1. \tag{1.6.3}$$

In particular, if $F = \log$, we call (1.6.3) the (defective when $C_2 \ne 0$) *log-Sobolev inequality.* We will provide a correspondence between (1.6.3) and

$$\mu(f^2) \le r\mathcal{E}(f, f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}). \tag{1.6.4}$$

Theorem 1.6.10. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mu)$.*

(1) *If the F-Sobolev inequality* (1.6.3) *holds with $F \ge 0$, then* (1.6.4) *holds with $\beta(r) = c_1 F^{-1}(c_2(1 + r^{-1}))$ for some $c_1, c_2 > 0$, where $F^{-1}(r) = \inf\{s \ge 0 : F(s) \ge r\}$ and $\inf \emptyset := \infty$.*
(2) *If* (1.6.4) *holds, then* (1.6.3) *holds with*

$$F(r) = \frac{c_1(\varepsilon)}{r} \int_0^r \xi(\varepsilon t)\mathrm{d}t - c_2(\varepsilon)$$

for any $\varepsilon \in (0,1)$ *and some* $c_1(\varepsilon), c_2(\varepsilon) > 0$, *where*

$$\xi(t) = \sup_{r>0}\Big(\frac{1}{r} - \frac{\beta(r)}{rt}\Big) \geq 0, \quad t > 0.$$

The following is a direct consequence of Theorems 1.6.10.

Corollary 1.6.11. *Assume that* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is a Dirichlet form.*

(1) *Let* $\delta > 0$. *Then* (1.6.3) *holds with* $F(r) = [\log(1+r)]^\delta$ *if and only if* (1.6.4) *holds with* $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ *for some* $c > 0$.
(2) *Let* $p > 0$. *Then* (1.6.3) *holds with* $F(r) = r^{2/p}$ *if and only if* (1.6.4) *holds with* $\beta(r) = c(1 + r^{-p/2})$ *for some* $c > 0$. *They are all equivalent to the Nash inequality*

$$\mu(f^2) \leq c_1 + c_2\mathcal{E}(f,f)^{p/(2+p)}, \quad f \in \mathcal{D}(\mathcal{E}), \mu(|f|) = 1$$

for some $c_1, c_2 > 0$, *and hence also to the classical Sobolev inequality if* $p > 2$:

$$\|f\|^2_{2p/(p-2)} \leq c_1\mu(f^2) + c_2\mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E})$$

for some $c_1, c_2 > 0$.

1.6.4 *Weak Poincaré inequality*

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $(L, \mathcal{D}(L))$ a linear operator generating a C_0-contraction semigroup P_t. Let $\mathcal{E}(f,g) := \langle g, Lf\rangle$ for $f, g \in \mathcal{D}(L)$. The following inequality is called the *weak Poincaré inequality*:

$$\|f\|^2 \leq \alpha(r)\mathcal{E}(f,f) + r\Phi(f), \quad f \in \mathcal{D}(L), r > 0, \tag{1.6.5}$$

where α is a nonnegative and decreasing function on $(0,\infty)$, and $\Phi : \mathbb{H} \to [0,\infty]$ satisfies $\Phi(cf) = c^2\Phi(f)$ for all $c \in \mathbb{R}$ and $f \in \mathbb{H}$.

Corresponding to the equivalence of the Poincaré inequality and the existence of spectral gap, the weak Poincaré inequality describes a "weak spectral gap" property. More precisely, for a conservative Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mu)$ and $\Phi(f) := \|f\|^2_\infty$, (1.6.5) with $\mathbb{H} := \{f \in L^2(\mu) : \mu(f) = 0\}$ is equivalent to Kusuoka-Aida's "weak spectral gap property" (**WSGP** for short, see [Aida (1998)]): *for any sequence* $\{f_n\} \subset \mathcal{D}(\mathcal{E})$ *such that* $\mu(f_n^2) \leq 1, \mu(f_n) = 0$, *and* $\mathcal{E}(f_n, f_n) \to 0$ *as* $n \to \infty$, *we have* $f_n \to 0$ *in probability.*

Proposition 1.6.12. *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a conservative Dirichlet form on* $L^2(\mu)$ *w.r.t. the probability space* $(E, \mathcal{B}, \mu)$. *Let* $\mathbb{H} := \{f \in L^2(\mu) : \mu(f) = 0\}$. *Then* **WSGP** *is equivalent to* (1.6.5) *for some* α *and* $\Phi(f) := \|f\|^2_\infty$.

The next result indicates that the weak Poincaré inequality together with the defective Poincaré inequality imply the Poincaré inequality.

Proposition 1.6.13. *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(\mu)$. Let $\mathbb{H}$ be either $L^2(\mu)$, or the orthogonal complement of constants when μ is a probability measure and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is conservative. Assume that there exist four constants $C_1, C_2, C_1', C_2' > 0$ such that*

$$\mu(f^2) \le C_1 \mathcal{E}(f,f) + C_2 \|f\|_1^2, \quad f \in \mathcal{D}(\mathcal{E})$$

and

$$\mu(f^2) \le C_1' \mathcal{E}(f,f) + C_2' \|f\|_\infty^2, \quad f \in \mathcal{D}(\mathcal{E}) \bigcap \mathbb{H}$$

hold.

(1) *If $\mathbb{H} = L^2(\mu)$ then $C_2 C_2' < 1$ implies*

$$\mu(f^2) \le \frac{2(C_1 + C_1')}{1 - \sqrt{C_2 C_2'}} \mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E}).$$

(2) *Let μ be a probability measure, $\mathcal{E}$ be conservative and $\mathbb{H} := \{f \in L^2(\mu), \mu(f) = 0\}$. If $c := \frac{1}{2}\big(1 + C_2' + \sqrt{(C_2 + 1 + C_2')C_2'}\big) < 1$ then the Poincaré inequality*

$$\mu(f^2) \le C \mathcal{E}(f,f) + \mu(f)^2, \quad f \in \mathcal{D}(\mathcal{E}) \tag{1.6.6}$$

holds for $C = (C_1 + C_1')/(1-c)$.

Now, let us describe the convergence rate of P_t by using (1.6.5).

Theorem 1.6.14. *Assume that* (1.6.5) *holds. Then*

$$\|P_t f\|^2 \le \inf_{r>0} \Big\{ r \sup_{s \in [0,t]} \Phi(P_s f) + \exp[-2t/\alpha(r)] \|f\|^2 \Big\} \tag{1.6.7}$$

holds for $t > 0, f \in \mathcal{D}(L)$. Consequently, if $\Phi(P_t f) \le \Phi(f)$ for any $t \ge 0$ and $f \in \mathbb{H}$, then

$$\|P_t f\|^2 \le \xi(t)[\Phi(f) + \|f\|^2], \quad t > 0, f \in \mathcal{D}(L), \tag{1.6.8}$$

where $\xi(t) := \inf\{r > 0 : -\frac{1}{2}\alpha(r) \log r \le t\}$ for $t > 0$. In particular, $\xi(t) \downarrow 0$ as $t \uparrow \infty$.

The following is a converse result of Theorem 1.6.14, which says that at least when L is normal a convergence rate of P_t also implies the weak Poincaré inequality.

Theorem 1.6.15. *Assume that L is normal, i.e. $LL^* = L^*L$. If there exist $\Psi : \mathbb{H} \to [0,\infty]$ and decreasing $\xi : [0,\infty) \to (0,\infty)$ such that $\Psi(cf) = c^2\Psi(f)$ for $c \in \mathbb{R}$ and $f \in \mathbb{H}, \xi(t) \downarrow 0$ as $t \uparrow \infty$, and*

$$\|P_t f\|^2 \le \xi(t)\Psi(f), \qquad t > 0,\ f \in \mathcal{D}(L), \tag{1.6.9}$$

then (1.6.5) *holds with $\Phi = \Psi$ and*

$$\alpha(r) = 2r \inf_{s>0} \frac{1}{s}\xi^{-1}(s\exp[1 - s/r]), \tag{1.6.10}$$

where $\xi^{-1}(t) := \inf\{r > 0 : \xi(r) \le t\}$. If in particular (1.6.9) *holds for $\xi(t) = \exp[-\delta t]$ for some $\delta > 0$, then the Poincaré inequality* (1.6.6) *holds for $C = 2/\delta$ and all $f \in \mathcal{D}(L)$ with $\Psi(f) < \infty$.*

Finally, we present an analogue of Theorem 1.6.15 for a class of operators L, which are not necessarily normal, but are such that

$$\mathcal{E}(P_t f, P_t f) \le h(t)\mathcal{E}(f,f), \quad t \ge 0, f \in \mathcal{D}(L) \tag{1.6.11}$$

for some positive $h \in C[0,\infty)$. It is well-known that (1.6.11) holds for $h = 1$ provided L is self-adjoint. Moreover, it also holds for diffusion processes under certain curvature condition (see Theorem 2.3.1(2) below).

Theorem 1.6.16. *Assume that* (1.6.11) *holds. Then* (1.6.9) *implies* (1.6.5) *with $\Phi = \Psi$ and*

$$\alpha(r) = 2\int_0^{\xi^{-1}(r)} h(s)ds, \quad r > 0.$$

1.6.5 *Equivalence of irreducibility and weak Poincaré inequality*

Let $(\mathcal{E}, \mathcal{F}, \mu)$ be a σ-finite measure space. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mu)$ is called non-conservative if either $1 \notin \mathcal{D}(\mathcal{E})$ or $\mathcal{E}(1,1) > 0$, while it is called irreducible if $f \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}(f,f) = 0$ implies $f = 0$. We shall prove that the irreducibility is equivalent to the validity of the weak Poincaré inequality of type

$$\mu(f^2) \le \alpha(r)\mathcal{E}(f,f) + r(\|f\|_\infty \vee \|f\|_1)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}). \tag{1.6.12}$$

Here, the L^1-norm appears in the right-hand side since in this case μ is allowed to be infinite and thus the L^∞-norm is no longer larger than the

L^2-norm. Of course, the L^1-norm can be dropped from (1.6.12) when μ is finite.

Theorem 1.6.17. *A non-conservative Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible if and only if there exists $\alpha : (0, \infty) \to (0, \infty)$ such that (1.6.12) holds. Consequently, for any symmetric (sub-) Markov semigroup P_t on $L^2(\mu)$, $\|P_t f\|_2 \to 0$ for any $f \in L^2(\mu)$ as $t \to \infty$ if and only if*

$$\lim_{t\to\infty} \sup_{\|f\|_1 \vee \|f\|_\infty \le 1} \|P_t f\|_2 = 0.$$

Proof. (a) Let P_t be the associated semigroup of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible if and only if $\mu((P_t f)^2) \to 0$ as $t \to \infty$ for any $f \in L^2(\mu)$. On the other hand, by Theorem 2.1 in [Röckner and Wang (2001)] with $\Phi(f) = \|f\|_1^2 \vee \|f\|_\infty^2$, (1.6.12) holds for some α if and only if

$$\lim_{t\to\infty} \sup_{\|f\|_1 \vee \|f\|_\infty \le 1} \mu((P_t f)^2) = 0.$$

So, the second assertion follows from the first one.

(b) Let $f \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}(f, f) = 0$. For any $\varepsilon > 0$ let $f_\varepsilon = (|f| - \varepsilon)^+ \wedge 1$. We have $\mathcal{E}(f_\varepsilon, f_\varepsilon) = 0$ and by the Schwarz inequality

$$\|f_\varepsilon\|_1^2 \le \mu(f^2)\mu(|f| > \varepsilon) \le \frac{\mu(f^2)^2}{\varepsilon^2}.$$

So, applying (1.6.12) to f_ε we obtain $\mu(f_\varepsilon^2) \le r(1 + \varepsilon^{-2}\mu(|f|)^2)$ for all $r > 0$. This implies $f_\varepsilon = 0$ for all $\varepsilon > 0$ and thus, $f = 0$.

(c) Now, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be irreducible, we claim that (1.6.12) holds for some function $\alpha : (0, \infty) \to (0, \infty)$. Otherwise, there exist some $r > 0$ and a sequence $\{f_n\} \subset \mathcal{D}(\mathcal{E})$ such that

$$1 = \mu(f_n^2) > n\mathcal{E}(f_n, f_n) + r(\|f_n\|_1 \vee \|f_n\|_\infty)^2, \quad n \ge 1. \tag{1.6.13}$$

Since $\mathcal{E}(|f_n|, |f_n|) \le \mathcal{E}(f_n, f_n)$, we may and do assume that $f_n \ge 0$ for all $n \ge 1$. Since $\{f_n\}$ is bounded both in $L^2(\mu)$ and $L^1(\mu)$, there exist two functions $f \in L^2(\mu), \tilde{f} \in L^1(\mu)$ and a subsequence $\{f_{n_k}\}$ such that f_{n_k} converges weakly to f in $L^2(\mu)$ and $\tilde{f}$ in $L^1(\mu)$ respectively. Obviously, $\mu(fg) = \mu(\tilde{f}g)$ for all $g \in L^2(\mu) \cap L^\infty(\mu)$, so that $f = \tilde{f}$.

Let P_t be the (sub-) Markov semigroup and $(L, \mathcal{D}(L))$ the generator associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then $P_t f \in \mathcal{D}(L)$ for any $t > 0$. By the symmetry of P_t and the weak convergence of $\{f_{n_k}\}$ to f in $L^2(\mu)$, we have

$$\lim_{k\to\infty} \mu((P_t f_{n_k})g) = \lim_{k\to\infty} \mu(f_{n_k} P_t g) = \mu(f P_t g) = \mu((P_t f)g), \quad g \in L^2(\mu).$$

This implies

$$\begin{aligned}\lim_{k\to\infty}\mathcal{E}(P_tf_{n_k},g) &= -\lim_{k\to\infty}\mu((P_tf_{n_k})Lg)\\ &= -\mu((P_tf)Lg) = \mathcal{E}(P_tf,g), \quad g\in\mathcal{D}(L).\end{aligned} \tag{1.6.14}$$

Moreover, due to (1.6.13) and the symmetry of $\mathcal{E}$,

$$\begin{aligned}\lim_{k\to\infty}\mathcal{E}(P_tf_{n_k},g)^2 &\le \lim_{k\to\infty}\mathcal{E}(P_tf_{n_k},P_tf_{n_k})\mathcal{E}(g,g)\\ &\le \lim_{k\to\infty}\mathcal{E}(f_{n_k},f_{n_k})\mathcal{E}(g,g)=0.\end{aligned}$$

Combining this with (1.6.14) we conclude that $\mathcal{E}(P_tf,P_tf)=0$ for all $t>0$. Thus, by the irreducibility, $P_tf=0$ holds for all $t>0$. This implies $f=0$ by the strong continuity of P_t in $L^2(\mu)$. Since (1.6.13) implies $f_n\le r^{-1/2}$, by the weak convergence of $\{f_{n_k}\}$ to $f=0$ in $L^1(\mu)$ we obtain

$$\lim_{k\to\infty}\mu(f_{n_k}^2)\le r^{-1/2}\lim_{k\to\infty}\mu(f_{n_k})=0.$$

This contradicts to the assumption that $\mu(f_n^2)=1$ for all $n\ge 1$. Therefore, (1.6.12) holds for some function $\alpha:(0,\infty)\to(0,\infty)$. □

Chapter 2

Diffusion Processes on Riemannian Manifolds without Boundary

In this chapter we aim to study the diffusion semigroup on Riemannian manifolds by using Bakry-Emery's curvature condition. By establishing the asymptotic formulae for the curvature operator, various equivalent semigroup inequalities and applications are presented for the curvature lower bound condition. Transportation-cost inequalities, functional inequalities for curvature unbounded below, and intrinsic Harnack ultracontractivity on non-compact manifolds are also investigated. The main tools of the study are the Itô formula for SDEs on Riemannian manifolds and the coupling method.

2.1 Brownian motion with drift

Let M be a complete connected Riemannian manifold of dimension d, and let Z be a C^1-smooth vector field on M. We will study the diffusion process generated by $L := \Delta + Z$. To this end, we first construct the corresponding *horizontal diffusion process* generated by $\Delta_{O(M)} + \mathbf{H}_Z$ on $O(M)$ by solving the Stratonovich stochastic differential equation (SDE)

$$\mathrm{d}u_t = \sqrt{2}\sum_{i=1}^d H_{e_i}(u_t) \circ \mathrm{d}B_t^i + \mathbf{H}_Z(u_t)\mathrm{d}t, \quad u_0 = u \in O(M),$$

where $B_t := (B_t^1, \dots, B_t^d)$ is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Since $\mathbf{H}_Z$ is C^1, it is well known that (see e.g. [Ikeda and Watanabe (1989); Elworthy (1982)]) the equation has a unique solution up to the life time $\zeta := \lim_{n\to\infty} \zeta_n$, where

$$\zeta_n := \inf\{t \geq 0 : \rho(\mathbf{p}u, \mathbf{p}u_t) \geq n\}, \quad n \geq 1.$$

Let $X_t = \mathbf{p}u_t$. Then X_t solves the equation

$$\mathrm{d}X_t = \sqrt{2}\, u_t \circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t, \quad X_0 = x := \mathbf{p}u_0 \tag{2.1.1}$$

up to the life time ζ. By the Itô formula, for any $f \in C_0^2(M)$,

$$f(X_t) - f(x) - \int_0^t Lf(X_s)\mathrm{d}s = \sqrt{2} \int_0^t \langle u_s^{-1}\nabla f(X_s), \mathrm{d}B_s\rangle$$

is a martingale up to the life time ζ; that is, X_t is the *diffusion process* generated by L, and we call it the L-diffusion process. When $Z = 0$, then $\bar{X}_t := X_{t/2}$ is generated by $\frac{1}{2}\Delta$ and is called the Brownian motion on M.

Throughout the book except §2.8 where the intrinsic ultracontractivity is considered, we only consider non-explosive (i.e. $\zeta = \infty$) diffusion processes. In this case

$$P_t f(x) := \mathbb{E}^x f(X_t), \quad x \in M, t \geq 0, f \in \mathcal{B}_b(M)$$

gives rise to a Markov semigroup $\{P_t\}_{t\geq 0}$ on $\mathcal{B}_b(M)$, which is called the diffusion semigroup generated by L. Here and in what follows, $\mathbb{E}^x$ (resp. $\mathbb{P}^x$) stands for the expectation (resp. probability) taken for the underlying process starting from point x. Below we present a criterion for the non-explosion.

Theorem 2.1.1. *Let $\psi \in C(0,\infty)$ be non-negative such that $L\rho_o \leq \psi \circ \rho_o$ holds outside* $\mathrm{cut}(o)$. *If*

$$\int_1^\infty \mathrm{d}t \int_1^t \exp\left[-\int_r^t \psi(s)\mathrm{d}s\right]\mathrm{d}r = \infty, \tag{2.1.2}$$

then the diffusion process generated by L is non-explosive.

Proof. Let

$$f = \int_1^{\rho_o} \mathrm{d}t \int_1^t \exp\left[-\int_r^t \psi(s)\mathrm{d}s\right]\mathrm{d}r =: g \circ \rho_o.$$

It is easy to see that $Lf \leq 1$ holds outside $\mathrm{cut}(o)$. Then, by (2.1.1) and Kendall's Itô formula for the radial part (see [Kendall (1987)]),

$$\begin{aligned} \mathrm{d}f(X_t) &\leq \sqrt{2}\langle u_t^{-1}\nabla f(X_t), \mathrm{d}B_t\rangle + Lf(X_t)\mathrm{d}t \\ &\leq \sqrt{2}\,\langle u_t^{-1}\nabla f(X_t), \mathrm{d}B_t\rangle + \mathrm{d}t \end{aligned}$$

holds up to the life time ζ. In particular, if $X_0 = x \in M$, then

$$g(n)\mathbb{P}(\zeta_n \leq t) \leq \mathbb{E}f(X_{t\wedge\zeta_n}) \leq f(x) + t.$$

Since $g(n) \to \infty$ as $n \to \infty$, this implies that

$$\mathbb{P}(\zeta \leq t) \leq \lim_{n\to\infty} \mathbb{P}(\zeta_n \leq t) \leq \lim_{n\to\infty} \frac{f(x)+t}{g(n)} = 0, \ t \geq 0.$$

Therefore, $\mathbb{P}(\zeta = \infty) = 1$. □

As a consequence of Theorem 2.1.1, the following result includes two explicit curvature conditions for the non-explosion, see [Hsu (2002a, 2003); March (1986)] for the study of the non-explosion of the Brownian motion and relations to the Dirichlet problem at infinity. Let

$$\mathrm{Ric}_Z(X,Y) = \mathrm{Ric}(X,Y) - \langle \nabla_X Z, Y\rangle, \quad X, Y \in T_xM, x \in M.$$

For any two-tensor $\mathbf{T}$ and any function f, we write $\mathbf{T} \geq f$ if $\mathbf{T}(X,X) \geq f|X|^2$ holds for $X \in TM$.

Corollary 2.1.2. *The diffusion process is non-explosive in each of the following situations:*

(a) *There exist non-negative functions* $\varphi, \psi \in C(0,\infty)$ *such that* (2.1.2) *holds,* $\mathrm{Ric} \geq -\varphi(\rho_o)$, *and*

$$\langle Z, \nabla\rho_o\rangle + \sqrt{(d-1)\varphi(\rho_o)}\coth\Big(\sqrt{\varphi(\rho_o)/(d-1)}\,\rho_o\Big) \leq \psi\circ\rho_o$$

holds outside $\mathrm{cut}(o)$. *In particular, it is the case if* $\mathrm{Ric} \geq -c(1+\rho_0^2)\log^2(\mathrm{e}+\rho_o)$ *and* $\langle Z, \nabla\rho_o\rangle \leq c(1+\rho_o)\log(\mathrm{e}+\rho_o)$ *outside* $\mathrm{cut}(o)$ *hold.*

(b) *There exists a non-negative* $h \in C([0,\infty))$ *such that* $\mathrm{Ric}_Z \geq -h\circ\rho_o$ *and* (2.1.2) *holds for* $\psi(s) := \int_0^s h(r)\mathrm{d}r$. *In particular, it is the case if* $\mathrm{Ric}_Z \geq -c\log(\mathrm{e}+\rho_o)$ *holds for some constant* $c > 0$.

Proof. The first assertion follows from the Laplacian comparison theorem, Theorem 1.1.10, and Theorem 2.1.1. To prove the second assertion, let $x \notin \mathrm{cut}(o)$ such that $\rho_o(x) > 0$, and let $\gamma : [0, \rho_o(x)] \to M$ be the unique minimal geodesic from o to x. For simplicity, we will write $\rho_o = \rho_o(x)$. Let $u := (u^1, \ldots, u^d) \in O_x(M)$ such that $u^d = \dot\gamma(\rho_o)$, and let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along γ such that $J_i(0) = 0$ and $J_i(\rho_o) = u^i, 1 \leq i \leq d-1$. By the second variational formula Theorem 1.1.8 we have

$$\Delta\rho_o = \sum_{i=1}^{d-1}\int_0^{\rho_o} \big(|\nabla_{\dot\gamma}J_i|^2 - \langle \mathcal{R}(\dot\gamma, J_i)\dot\gamma, J_i\rangle\big)(s)\mathrm{d}s.$$

Let U_i be the constant vector field along γ such that $U_i(\rho_o) = u_i$, and let $f(s) = 1 \wedge \frac{s}{\rho_o\wedge 1}$. By the index lemma (Lemma 1.1.11) for $X_i = fU_i$, we obtain

$$\Delta\rho_o \leq \int_0^{\rho_o}\Big((d-1)(f')^2 - f^2\mathrm{Ric}(\dot\gamma,\dot\gamma)\Big)(s)\mathrm{d}s.$$

On the other hand,

$$\begin{aligned}\rho_o &= \int_0^{\rho_o} \frac{\mathrm{d}}{\mathrm{d}s}\{f^2\langle Z\circ\gamma, \dot\gamma\rangle\}(s)\mathrm{d}s\\ &= \int_0^{\rho_o} \{2ff'\langle Z\circ\gamma,\dot\gamma\rangle + f^2\langle\nabla_{\dot\gamma} Z\circ\gamma,\dot\gamma\rangle\}(s)\mathrm{d}s.\end{aligned}$$

Therefore, there exists a constant $C>0$ such that

$$L\rho_o \le C + \frac{d-1}{\rho_o} + \int_0^{\rho_o} h(s)\mathrm{d}s =: \bar\psi(\rho_o). \tag{2.1.3}$$

It is easy to see that (2.1.2) holds for $\psi(s) := \int_0^s h(r)\mathrm{d}r$ if and only if it holds for $\bar\psi$ in place of ψ. Then the desired assertion follows from Theorem 2.1.1. □

Since L is elliptic, according to the Malliavin calculus, for any $f\in\mathcal{B}_b(M)$, $P_\cdot f$ is smooth on $(0,\infty)\times M$, see [Malliavin (1997); Nualart (1995)].

Theorem 2.1.3. *For any $f\in\mathcal{B}_b(M)$, the backward Kolmogorov equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f = LP_t f,\quad t>0 \tag{2.1.4}$$

holds. If moreover $f\in C^2(M)$ such that Lf is bounded, there also holds the forward Kolmogorov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f = P_t Lf,\quad t\ge 0. \tag{2.1.5}$$

To prove this theorem, we will make use of the following simple lemma concerning the exit time. For $r>0$, let

$$\sigma_r = \inf\{t\ge 0: X_t\notin B(X_0,r)\},$$

where for $x\in M$ and $r>0$, $B(x,r) := \{y\in M:\rho(x,y)<r\}$ is the geodesic ball at x with radius r.

Lemma 2.1.4. *For any $x\in M$ and $r>0$, there exists a constant $c>0$ such that $\mathbb{P}^x(\sigma_r\le t)\le \mathrm{e}^{-cr^2/t}$ holds for $t\in(0,1]$.*

Proof. There exists a constant $c_1>0$ such that $L\rho_x^2\le c_1$ holds on $B(x,r)$ outside the cut-locus of x. Let $\gamma_t := \rho_x(X_t)$, $t\ge 0$. By Kendall's Itô formula [Kendall (1987)], there exists a one-dimensional Brownian motion b_t such that

$$\mathrm{d}\gamma_t^2 \le 2\sqrt{2}\,\gamma_t\,\mathrm{d}b_t + c_1\,\mathrm{d}t,\quad t\le\sigma_r.$$

Thus, for fixed $t > 0$ and $\delta > 0$,

$$Z_s := \exp\left(\frac{\delta}{t}\gamma_s^2 - \frac{\delta}{t}c_1 s - 4\frac{\delta^2}{t^2}\int_0^s \gamma_u^2 \mathrm{d}u\right), \quad s \le \sigma_r$$

is a supermartingale. Therefore,

$$\begin{aligned}\mathbb{P}(\sigma_r \le t) &= \mathbb{P}\left\{\max_{s\in[0,t]} \gamma_{s\wedge\sigma_r} \ge r\right\}\\ &\le \mathbb{P}\left\{\max_{s\in[0,t]} Z_{s\wedge\sigma_r} \ge \mathrm{e}^{\delta r^2/t - \delta c_1 - 4\delta^2 r^2/t}\right\}\\ &\le \exp\left(c_1\delta - \frac{1}{t}(\delta r^2 - 4\delta^2 r^2)\right).\end{aligned}$$

The proof is completed by taking $\delta := 1/8$. □

Proof. [Proof of Theorem 2.1.3] For fixed $x \in M$, let $h \in C_0^\infty(M)$ such that $h|_{B(x,1)} = 1$. By the Itô formula,

$$\mathrm{d}(hP_{t-s}f)(X_s) = \mathrm{d}M_s + \left\{L(hP_{t-s}f) + h\frac{\mathrm{d}}{\mathrm{d}s}P_{t-s}f\right\}(X_s)\mathrm{d}s, \quad s \in [0,t]$$

holds for martingale $\mathrm{d}M_s := \sqrt{2}\,\langle\nabla(hP_{t-s}f)(X_s), u_s\mathrm{d}B_s\rangle$. Thus,

$$\begin{aligned}&\lim_{s\downarrow 0}\frac{\mathbb{E}^x(hP_{t-s}f)(X_s) - P_t f(x)}{s}\\ &= \lim_{s\downarrow 0}\mathbb{E}^x\frac{1}{s}\int_0^s\left\{L(hP_{t-r}f) + h\frac{\mathrm{d}}{\mathrm{d}r}P_{t-r}f\right\}(X_r)\mathrm{d}r \qquad (2.1.6)\\ &= \left\{LP_t f - \frac{\mathrm{d}}{\mathrm{d}t}P_t f\right\}(x).\end{aligned}$$

On the other hand, by Lemma 2.1.4,

$$\begin{aligned}|\mathbb{E}^x(hP_{t-s}f)(X_s) - P_t f(x)| &\le \|f(h-1)\|_\infty \mathbb{P}^x(\sigma_1 < s)\\ &\le \|f(h-1)\|_\infty \mathrm{e}^{-c/s}\end{aligned}$$

holds for some constant $c > 0$ and $s \in (0,1]$. Therefore, (2.1.4) follows from (2.1.6).

Next, if $f \in C^2(M)$ such that Lf is bounded, then by the Itô formula, $f(X_t) - f(X_0) - \int_0^t Lf(X_s)\mathrm{d}s$ is a martingale. So,

$$P_t f = f + \int_0^t P_s Lf \mathrm{d}s, \quad s \ge 0.$$

This implies (2.1.5). □

2.2 Formulae for ∇P_t and Ric_Z

The *derivative formula* of P_t is known as Bismut-Elworthy-Li formula [Bismut (1984); Elworthy and Li (1994)]. The formula we are introducing is a more general version due to Thalmaier [Thalmaier (1997)]. Let us introduce the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process $\{Q_t\}_{t\geq 0}$, which solves the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t = -\mathrm{Ric}_Z^\#(u_t)Q_t, \quad Q_0 = I, \tag{2.2.1}$$

where u_t is the horizontal L-diffusion process with $\mathbf{p}u_0 = x$, and $\mathrm{Ric}_Z^\#(u_t)$ is a random variable on $\mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\langle \mathrm{Ric}_Z^\#(u_t)a, b\rangle_{\mathbb{R}^d} = \mathrm{Ric}_Z(u_t b, u_t a), \quad a, b \in \mathbb{R}^d. \tag{2.2.2}$$

Since $\mathrm{Ric}_Z^\#$ is continuous and the process is non-explosive, this equation has a unique solution. In particular, let $K \in C(M)$ be such that $\mathrm{Ric}_Z \geq K$, then

$$\|Q_t\| \leq \exp\left[-\int_0^t K(X_s)\mathrm{d}s\right], \quad t \geq 0, \tag{2.2.3}$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$.

Theorem 2.2.1. *Let $t > 0, x \in M$ and D be a compact domain such that $x \in D^\circ$, the interior of D. Let τ_D be the first hitting time of X_t to ∂D, where $X_0 = x$. Let $F \in C^2([0,t] \times D)$ satisfy the heat equation*

$$\partial_s F(\cdot, x)(s) = LF(s, \cdot)(x), \quad s \in [0,t], x \in D. \tag{2.2.4}$$

Then for any adapted absolutely continuous $\mathbb{R}_+$-valued process h such that $h(0) = 0, h(s) = 1$ for $s \geq t \wedge \tau_D$ and $\mathbb{E}(\int_0^t h'(s)^2\mathrm{d}s)^\alpha < \infty$ for some $\alpha > \frac{1}{2}$, there holds

$$u_0^{-1}\nabla F(t, \cdot)(x) = \frac{1}{\sqrt{2}}\mathbb{E}\left\{F(0, X_{t\wedge\tau_D})\int_0^t h'(s)Q_s^*\mathrm{d}B_s\right\}.$$

Proof. Let $F_s = F(s, \cdot)$. By Theorem 2.1.3 and Theorem 1.1.4, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{d}F_s) &= \mathbf{d}LF_s = \mathbf{d}\{-\mathbf{d}^*\mathbf{d}F_s + (\mathbf{d}F_s)(Z)\} \\ &= \Delta_1(\mathbf{d}F_s) + \nabla_Z(\mathbf{d}F_s) + \langle \nabla . Z, \nabla F_s\rangle \\ &= (\square_1 + \nabla_Z)(\mathbf{d}F_s) - \mathrm{Ric}_Z(\cdot, \nabla F_s).\end{aligned} \tag{2.2.5}$$

On the other hand, by the Itô formula,

$$\mathrm{d}(\mathbf{d}f)(X_s) = (\square_1 + \nabla_Z)(\mathbf{d}f)(X_s)\mathrm{d}s + \sqrt{2}\nabla_{u_s\mathrm{d}B_s}(\mathbf{d}f)(X_s), \quad f \in C^2(M).$$

Combining this with (2.2.5), we obtain

$$\mathrm{d}(\mathbf{d}F_{t-s})(X_s) = \sqrt{2}\nabla_{u_s\mathrm{d}B_s}(\mathbf{d}F_{t-s})(X_s) + \mathrm{Ric}_Z(\cdot, \nabla F_{t-s}(X_s))\mathrm{d}s. \quad (2.2.6)$$

Now, for any $a \in \mathbb{R}^d$, letting $v_s = u_s Q_s a \in T_{X_s}M, s \in [0,t]$, it follows from (2.2.6) and (2.2.1) that

$$\mathrm{d}\langle \nabla F_{t-s}(X_s), u_s Q_s a\rangle = \sqrt{2}\mathrm{Hess}_{F_{t-s}}(u_s\mathrm{d}B_s, u_s Q_s a)(X_s) \qquad (2.2.7)$$

is a local martingale. Moreover, by the Itô formula we have

$$\mathrm{d}F_{t-s}(X_s) = \sqrt{2}\,\langle \nabla P_{t-s}F_0(X_s), u_s\mathrm{d}B_s\rangle,$$

so that

$$F_0(X_{t\wedge\tau_D}) = F_t(x) + \sqrt{2}\int_0^t \langle \nabla F_{t-s\wedge\tau_D}(X_{s\wedge\tau_D}), u_s\mathrm{d}B_s\rangle.$$

Therefore, noting that $h'(s) = 0$ for $s \geq t\wedge\tau_D$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbb{E}\bigg\{F_0(X_{t\wedge\tau_D})\int_0^t \langle h'(s)Q_s a, \mathrm{d}B_s\rangle\bigg\}\\
&= \mathbb{E}\bigg\{\int_0^t \langle \nabla P_{t-s}F_0(X_s), v_s\rangle (h-1)'(s)\mathrm{d}s\bigg\}\\
&= \mathbb{E}\big\{\langle \nabla F_{t-s}(X_s), v_s\rangle (h-1)(s)\big\}\big|_0^t \qquad (2.2.8)\\
&\qquad - \mathbb{E}\int_0^t (h-1)(s)\mathrm{d}\langle \nabla F_{t-s}(X_s), v_s\rangle\\
&= \langle \nabla F_t, u_0 a\rangle,
\end{aligned}$$

where the last step follows from the fact that $(h-1)(s)\mathrm{d}\langle \nabla F_{t-s}(X_s), v_s\rangle$ is a martingale for $s \in [0,t]$ according to (2.2.7) and our assumption on h. □

In [Thalmaier and Wang (1998)] some explicit processes h required in Theorem 2.2.1 have been constructed according to the geometry on D, from which one obtains explicit gradient estimates of $P_t f$ only using local geometry of the manifold. For instance, we have the following result.

Corollary 2.2.2. *Let* $\mathrm{Ric}_Z \geq K$ *for some* $K \in C(M)$. *For any* $x \in M$ *let* $\kappa(x) = \sup_{B(x,1)}(K^- + |Z|)$. *Then there exists a constant* $c > 0$ *such that*

$$|\nabla P_t f| \leq \frac{\|f\|_\infty \exp[c(1+\kappa)]}{\sqrt{t\wedge 1}}, \quad t > 0, f \in \mathcal{B}_b(M).$$

Proof. By the semigroup property and the contraction of P_t, it suffices to prove for $t \le 1$. We will apply Theorem 2.2.1 to $D = B(x,1)$. Let $f = \cos(\pi\rho_x/2)$. Let $X_0 = x$ and

$$T(s) = \int_0^s f^{-2}(X_r)\mathrm{d}r, \quad s \le \tau_D$$

and set $T(s) = \infty$ for $s > \tau_D$. Let

$$\tau(s) = \inf\{r \ge 0 : T(r) \ge s\}, \quad s \ge 0.$$

Then $\tau \circ T(s) = T \circ \tau(s) = s$ for $s \le \tau_D$. Since $f \le 1$, we have $T(s) \ge s$ and $\tau(s) \le s$. Moreover,

$$\tau'(s) = \frac{1}{T' \circ \tau(s)} = f^2(X_{\tau(s)}), \quad s \le \tau_D. \tag{2.2.9}$$

Define

$$h(s) = 1 - \frac{1}{t}\int_0^{s\wedge\tau(t)} f^{-2}(X_r)\mathrm{d}r.$$

Then h meets the requirement of Theorem 2.2.1, and

$$\begin{aligned}\int_0^{\tau(t)} h'(s)^2\mathrm{d}s &= \frac{1}{t^2}\int_0^{\tau(t)} f^{-4}(X_s)\mathrm{d}s \\ &= \frac{1}{t^2}\int_0^{\tau(t)} f^{-2}(X_s)\mathrm{d}T(s) = \frac{1}{t^2}\int_0^t f^{-2}(X_{\tau(s)})\mathrm{d}s.\end{aligned} \tag{2.2.10}$$

It is easy to see from the Itô formula that $s \mapsto X_{\tau(s)}$ is generated by f^2L, which is non-explosive on $B(x,1)$. So, it follows from Kendall's Itô formula that

$$\mathrm{d}f^{-2}(X_{\tau(s)}) \le \mathrm{d}M_s + (f^2Lf^{-2})(X_{\tau(s)})\mathrm{d}s \tag{2.2.11}$$

holds for some local martingale M_s. By Theorem 1.1.10 and the definition of κ, there exists a constant $c_1 > 0$ such that

$$\big(\sin[\pi\rho_x/2]\big)L\rho_x \le c_1(1+\kappa(x))$$

holds on $B(x,1)$. Thus, there exists a constant $c_2 > 0$ such that

$$f^2Lf^{-2} = -2f^{-1}Lf + 6f^{-2}|\nabla f|^2 \le c_2(1+\kappa(x))f^{-2}$$

holds on $B(x,1)$. Therefore, (2.2.10) and (2.2.11) yield

$$\begin{aligned}\mathbb{E}^x\int_0^{\tau(t)} h'(s)^2\mathrm{d}s &\le \frac{1}{t^2}\int_0^t \mathbb{E}^x f^{-2}(X_{\tau(s)})\mathrm{d}s \\ &\le \frac{1}{t^2}\int_0^t \mathrm{e}^{c_2(1+\kappa(x))s}\mathrm{d}s \le \frac{c_3}{t}\mathrm{e}^{c_3(1+\kappa(x))}, \quad t \in (0,1]\end{aligned}$$

for some constant $c_3 > 0$. Let $v \in T_xM$ such that $|v| = 1$. Since by (2.2.3) and $\mathrm{Ric}_Z \ge -\kappa(x)$ on $B(x,1)$

$$|v_s| \le |v|\mathrm{e}^{c\kappa(x)}, \quad s \le \tau(t), t \le 1$$

holds for $v_s := u_s Q_s u_0^{-1} v$, it follows from Theorem 2.2.1 that

$$|\langle \nabla P_t f(x), v\rangle| \le \|f\|_\infty \mathrm{e}^{c\kappa(x)} \left(\mathbb{E}\int_0^{\tau(t)} h'(s)^2 \mathrm{d}s\right)^{1/2} \le \frac{\|f\|_\infty c_4 \mathrm{e}^{c_4(1+\kappa(x))}}{\sqrt{t}}$$

holds for some constant $c_4 > 0$ and all $t \in (0,1]$. This completes the proof.

□

Next, we present derivative formulae of P_t without using hitting times, which are essentially due to Bismut [Bismut (1984)] and Elworthy-Li [Elworthy and Li (1994)].

Theorem 2.2.3. *Assume that $L\rho_o^2 \le c(1+\rho_o^2)$ holds outside* cut(o) *for some constant $c > 0$. If*

$$\mathrm{Ric}_Z \ge c' - 16\mathrm{e}^{-(c+16)t}\rho_o^2 \tag{2.2.12}$$

holds for some constant $c' \in \mathbb{R}$, then for any $h \in C^1([0,t])$ such that $h(0) = 0, h(t) = 1$,

$$\begin{aligned} u_0^{-1}\nabla P_t f(x) &= \mathbb{E}^x\{u_t^{-1}\nabla f(X_t)\} \\ &= \frac{1}{\sqrt{2}}\mathbb{E}^x\left\{f(X_t)\int_0^t h'(s)Q_s^* \mathrm{d}B_s\right\} \end{aligned} \tag{2.2.13}$$

holds for $f \in C_b^1(M), x \in M$ and $v \in T_xM$. In particular, taking $h(s) = 1 \wedge \frac{s}{t}$,

$$u_0^{-1}\nabla P_t f(x) = \frac{1}{t\sqrt{2}}\mathbb{E}^x\left\{f(X_t)\int_0^t Q_s^* \mathrm{d}B_s\right\}.$$

Proof. By Kendall's Itô formula [Kendall (1987)],

$$\mathrm{d}\rho_o(X_t)^2 \le 2\sqrt{2}\rho_o(X_t)\mathrm{d}b_t + c(1+\rho_o(X_t)^2)\mathrm{d}t$$

holds for some one-dimensional Brownian motion b_t. Letting $\lambda = 16 + c$, we obtain

$$\mathrm{d}\{\mathrm{e}^{-\lambda t}\rho_o(X_t)^2\} \le 2\sqrt{2}\rho_o(X_t)\mathrm{e}^{-\lambda t}\mathrm{d}b_t + c\mathrm{e}^{-\lambda t}\mathrm{d}t - 16\mathrm{e}^{-\lambda t}\rho_o(X_t)^2\mathrm{d}t.$$

Therefore, letting $C(t,x) = \mathrm{e}^{\rho_o(x)^2 + ct}$,

$$\begin{aligned} \mathbb{E}^x e^{16\int_0^{t\wedge\zeta_n}\exp[-\lambda s]\rho_o(X_s)^2\mathrm{d}s} &\le C(t,x)\mathbb{E}^x\mathrm{e}^{2\sqrt{2}\int_0^{t\wedge\zeta_n}\rho_o(X_s)\exp[-\lambda s]\mathrm{d}b_s} \\ &\le C(t,x)\big(\mathbb{E}^x\mathrm{e}^{16\int_0^{t\wedge\zeta_n}\exp[-2\lambda s]\rho_o(X_s)^2\mathrm{d}s}\big)^{1/2}. \end{aligned}$$

This implies that

$$\mathbb{E}^x \mathrm{e}^{16\int_0^{t\wedge\zeta_n} \exp[-\lambda s]\rho_o(X_s)^2 \mathrm{d}s} \leq C(t,x)^2.$$

Letting $n \to \infty$, we arrive at

$$\mathbb{E}^x \mathrm{e}^{16\exp[-\lambda t]\int_0^t \rho_o(X_s)^2 \mathrm{d}s} \leq C(t,x)^2.$$

Combining this with (2.2.12) we conclude that for $K := c' - 16\mathrm{e}^{-(c+16)t}\rho_o^2$, one has $\mathrm{Ric}_Z \geq K$ and

$$\sup_{x\in\mathbf{K}} \mathbb{E}^x \mathrm{e}^{\int_0^t K^-(X_s)\mathrm{d}s} < \infty, \quad \mathbf{K} \subset M \text{ is compact.} \tag{2.2.14}$$

Therefore, due to (2.2.1) and Theorem A.6(i) in [Elworthy (1982)] (see also Theorems 3.1, 9.1 in [Li, X.-M. (1994)]), we have $\sup_{s\in[0,t]} \|\nabla P_s f\|_\infty < \infty$. Then the first equality in (2.2.13) follows from (2.2.7) by taking $F(x,s) = \langle \nabla P_{t-s} f(x), v_s\rangle$ for $v_s = u_s Q_s u_0^{-1} v, v \in T_x M$. Next, by the first equality in (2.2.13) and (2.2.14), we obtain

$$\mathbb{E}^x \sup_{s\in[0,t]} |\langle \nabla P_{t-s} f(X_s), v_s\rangle| \leq \|\nabla f\|_\infty \mathbb{E}^x \mathrm{e}^{\int_0^t K^-(X_s)\mathrm{d}s} < \infty.$$

So, for the above v_s it follows from (2.2.7) that $\langle \nabla P_{t-s} f(X_s), v_s\rangle, s \in [0,t]$ is a uniformly integrable martingale, and thus, (2.2.8) holds for t in place of $t \wedge \tau_D$ and any $h \in C^1([0,t])$ with $h(0) = 0, h(t) = 1$. Therefore, the second equality in (2.2.13) holds. □

According to (2.1.3) and the Laplacian comparison theorem, Theorem 1.1.10, the assumption $L\rho_o^2 \leq c(1+\rho_o^2)$ in Theorem 2.2.3 is ensured by the assumption

(A2.2.1) $\exists C > 0$ such that either $\mathrm{Ric}_Z \geq -C$, or $\mathrm{Ric} \geq -C(1+\rho_o^2)$ and $\langle Z, \nabla\rho_o\rangle \leq C(1+\rho_o)$.

The above two theorems describe the gradient of P_t by using the curvature Ric_Z. Below we present characterizations of Ric_Z using the gradient of P_t.

Theorem 2.2.4. *Let $x \in M$ and $X \in T_x M$ with $|X| = 1$. Let $f \in C_0^\infty(M)$ such that $\nabla f(x) = X$ and $\mathrm{Hess}_f(x) = 0$, and let $f_n = n + f$ for $n \geq 1$. Then:*

(1) *For any $p > 0$,* $\mathrm{Ric}_Z(X,X) = \lim_{t\to 0} \dfrac{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt}$;

(2) *For any* $p > 1$,

$$\begin{aligned}\mathrm{Ric}_Z(X,X) &= \lim_{n\to\infty}\lim_{t\to 0}\frac{1}{t}\Big(\frac{p\{P_t f_n^2 - (P_t f_n^{2/p})^p\}}{4(p-1)t} - |\nabla P_t f|^2\Big)(x)\\ &= \lim_{n\to\infty}\lim_{t\to 0}\frac{1}{t}\Big(P_t|\nabla f|^2 - \frac{p\{P_t f_n^2 - (P_t f_n^{2/p})^p\}}{4(p-1)t}\Big)(x).\end{aligned}$$

(3) $\mathrm{Ric}_Z(X,X)$ *is equal to each of the following limits:*

$$\lim_{n\to\infty}\lim_{t\to 0}\frac{1}{t^2}\big[(P_t f_n)\{P_t(f_n\log f_n) - (P_t f_n)\log P_t f_n\} - t|\nabla P_t f|^2\big](x),$$

$$\lim_{n\to\infty}\lim_{t\to 0}\frac{1}{4t^2}\big[4tP_t|\nabla f|^2 + (P_t f_n^2)\log P_t f_n^2 - P_t\{f_n^2\log f_n^2\}\big](x).$$

Proof. Since $\nabla f(x) = X$ and $\mathrm{Hess}_f(x) = 0$, by the Bochner-Weitzenböck formula Theorem 1.1.4,

$$\Gamma_2(f,f)(x) := \frac{1}{2}L|\nabla f|^2(x) - \langle\nabla f, \nabla Lf\rangle(x) = \mathrm{Ric}_Z(X,X). \tag{2.2.15}$$

Therefore, the first assertion follows from the Kolmogorov equation Theorem 2.1.3 and the Taylor expansions at point x (recall that $\mathrm{Hess}_f(x) = 0$):

$$\begin{aligned}P_t|\nabla f|^p &= |\nabla f|^p + tL|\nabla f|^p + \mathrm{o}(t) = |\nabla f|^p + \frac{pt}{2}|\nabla f|^{p-2}L|\nabla f|^2 + \mathrm{o}(t),\\ |\nabla P_t f|^p &= |\nabla f|^p + pt|\nabla f|^{p-2}\langle\nabla Lf, \nabla f\rangle + \mathrm{o}(t)\end{aligned}$$

for small $t > 0$.

Next, let $f_n = n + f$, which is positive for large n. We have, for small $t > 0$ and large n,

$$\begin{aligned}P_t f_n^2 - (P_t f_n^{2/p})^p &= t\big(Lf_n^2 - pf_n^{2(p-1)/p}Lf_n^{2/p}\big)\\ &\quad + \frac{t^2}{2}\big(L^2 f_n^2 - p(p-1)f_n^{2(p-2)/p}(Lf_n^{2/p})^2 - pf_n^{2(p-1)/p}L^2 f_n^{2/p}\big) + \mathrm{o}(t^2)\\ &= \frac{4(p-1)t}{p}|\nabla f|^2 + \frac{4(p-1)t^2}{p}\Gamma_2(f,f)\\ &\quad + \frac{8(p-1)t^2}{p}\langle\nabla f, \nabla Lf\rangle + t^2\mathrm{O}(n^{-1}) + \mathrm{o}(t^2),\end{aligned}$$

where $\mathrm{o}(t^2)$ depends on n but $\mathrm{O}(n^{-1})$ is independent of t such that $n\mathrm{O}(n^{-1})$ is bounded for $n \geq 1$. Combining this with (2.2.15) and

$$|\nabla P_t f|^2 = |\nabla f|^2 + 2t\langle\nabla f, \nabla Lf\rangle + \mathrm{o}(t), \tag{2.2.16}$$

we prove the first equality in (2). Similarly, the second equality follows since

$$P_t|\nabla f|^2 = |\nabla f|^2 + tL|\nabla f|^2 + \mathrm{o}(t). \tag{2.2.17}$$

Finally, (3) can be proved by combining (2.2.16) and (2.2.17) with the following two asymptotic formulae respectively:

$$\begin{aligned}
&(P_t f_n)\{P_t(f_n \log f_n) - (P_t f_n)\log P_t f_n\}\\
&= \big(f_n + t\mathrm{O}(1) + \mathrm{o}(t)\big)\Big\{t[L(f_n\log f_n) - (1+\log f_n)Lf_n]\\
&\qquad + \frac{t^2}{2}\big[L^2(f_n\log f_n) - (1+\log f_n)L^2 f_n - f_n^{-1}(Lf_n)^2\big] + \mathrm{o}(t^2)\Big\}\\
&= t|\nabla f|^2 + t^2\Gamma_2(f,f) + 2t^2\langle\nabla f, \nabla Lf\rangle + t^2\mathrm{O}(n^{-2}) + \mathrm{o}(t^2);
\end{aligned}$$

and

$$\begin{aligned}
&(P_t f_n^2)\log P_t f_n^2 - P_t(f_n^2\log f_n^2)\\
&= t\big[(1+\log f_n^2)Lf_n^2 - L(f_n^2\log f_n^2)\big]\\
&\quad + \frac{t^2}{2}\big[f_n^{-2}(Lf_n^2)^2 + (1+\log f_n^2)L^2 f_n^2 - L^2(f_n^2\log f_n^2)\big]\\
&= -4t|\nabla f|^2 - 4t^2\langle\nabla Lf, \nabla f\rangle - 2t^2 L|\nabla f|^2 + \mathrm{o}(t^2) + t^2\mathrm{O}(n^{-1}). \qquad \square
\end{aligned}$$

2.3 Equivalent semigroup inequalities for curvature lower bound

In this section we aim to provide various equivalent semigroup properties for the curvature lower bound. Basing on Theorem 2.2.4, we first introduce equivalent gradient inequalities.

Theorem 2.3.1. *Assume* **(A2.2.1)** *and let* $p \in [1,\infty)$ *and* $\tilde p = p \wedge 2$. *Then for any* $K \in C(M)$ *such that* $K^-/\rho_o^2 \to 0$ *as* $\rho_o \to \infty$, *the following statements are equivalent to each other:*

(1) $\mathrm{Ric}_Z \ge K$.

(2) $|\nabla P_t f(x)|^p \le \mathbb{E}^x\Big\{|\nabla f|^p(X_t)\exp\big[-p\int_0^t K(X_s)\mathrm{d}s\big]\Big\}$ *holds for* $t \ge 0, x \in M, f \in C_b^1(M)$.

(3) *For any* $t \ge 0, x \in M$ *and positive* $f \in C_b^1(M)$,

$$\frac{\tilde p[P_t f^2 - (P_t f^{2/\tilde p})^{\tilde p}](x)}{4(\tilde p - 1)} \le \mathbb{E}^x\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{-2\int_s^t K(X_r)\mathrm{d}r}\mathrm{d}s\Big\},$$

where when $p = 1$ *the inequality is understood as its limit as* $p \downarrow 1$:

$$\begin{aligned}
&P_t(f^2\log f^2)(x) - (P_t f^2(x))\log P_t f^2(x)\\
&\le 4\mathbb{E}^x\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{-2\int_s^t K(X_r)\mathrm{d}r}\mathrm{d}s\Big\}.
\end{aligned}$$

(4) *For any* $t \geq 0, x \in M$ *and positive* $f \in C_b^1(M)$,

$$|\nabla P_t f|^2(x) \leq \frac{P_t f^{\tilde p} - (P_t f)^{\tilde p}}{\tilde p(\tilde p - 1)\int_0^t (\mathbb{E}^x\{(P_{t-s}f)^{2-\tilde p}(X_s)\exp[-2\int_0^s K(X_r)\mathrm{d}r]\})^{-1}\mathrm{d}s},$$

where when $p = 1$ *(hence,* $\tilde p = 1$*), the inequality is understood as its limit by taking* $p \downarrow 1$*:*

$$|\nabla P_t f|^2(x) \leq \frac{[P_t(f\log f) - (P_t f)\log P_t f](x)}{\int_0^t (\mathbb{E}^x\{P_{t-s}f(X_s)\exp[-2\int_0^s K(X_r)\mathrm{d}r]\})^{-1}\mathrm{d}s}.$$

Proof. According to the proof of Theorem 2.2.3, $\mathbb{E}^x \mathrm{e}^{p\int_0^t K^-(X_s)\mathrm{d}s} < \infty$ holds for any $p, t > 0$ and $x \in M$. So, according to Theorem 2.2.4, we obtain (1) by applying (2) to $f \in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x) = 0$, or applying (3) to $n+f$ in place of f, or applying (4) to $(n+f)^{2/p}$ when $p > 1$ (resp. $n + f$ when $p = 1$) in place of f. So, it suffices to show that (1) implies (2)-(4).

Firstly, (2) follows from (1) according to the first equality in (2.2.13) and (2.2.1). To prove (3) and (4), let $p \in (1, 2]$. By an approximation argument we assume that $f \in C^\infty(M)$ and is constant outside a compact set such that Lf^p is bounded for any $p > 0$. In this case, by Theorem 2.1.3, (2) for $p = 1$, and the Hölder inequality, we obtain at point x that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} P_s(P_{t-s}f^{2/p})^p = p(p-1)P_s\{|\nabla P_{t-s}f^{2/p}|^2(P_{t-s}f^{2/p})^{p-2}\} \\
&\leq p(p-1)\mathbb{E}^x\big\{\big(\mathbb{E}^{X_s}|\nabla f^{2/p}|(X_{t-s})\mathrm{e}^{-\int_0^{t-s}K(X_r)\mathrm{d}r}\big)^2(P_{t-s}f^{2/p})^{p-2}(X_s)\big\} \\
&\leq \frac{4(p-1)}{p}\mathbb{E}^x\Big\{\big(\mathbb{E}^{X_s}(|\nabla f|^2(X_{t-s})\mathrm{e}^{-2\int_0^{t-s}K(X_r)\mathrm{d}r})\big) \\
&\qquad\qquad \times \big((P_{t-s}f^{\frac{2(2-p)}{p}})(P_{t-s}f^{2/p})^{p-2}\big)(X_s)\Big\}.
\end{aligned}$$

Since $2 - p \in [0, 1]$, by the Jensen inequality

$$P_{t-s}f^{\frac{2(2-p)}{p}} \leq (P_{t-s}f^{2/p})^{2-p},$$

so that by the Markov property,

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}f^{2/p})^p \leq \frac{4(p-1)}{p}\mathbb{E}^x\{|\nabla f|^2(X_t)\mathrm{e}^{-2\int_s^t K(X_r)\mathrm{d}r}\}$$

holds for $s \in [0, t]$. This implies (3) by taking integral over $[0, t]$. Similarly,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}f)^p &= p(p-1)P_s\big\{(P_{t-s}f)^{p-2}|\nabla P_{t-s}f|^2\big\} \\
&\geq \frac{p(p-1)\big(\mathbb{E}^x|\nabla P_{t-s}f|(X_s)\mathrm{e}^{-\int_0^s K(X_r)\mathrm{d}r}\big)^2}{\mathbb{E}^x(P_{t-s}f)^{2-p}(X_s)\mathrm{e}^{-2\int_0^s K(X_r)\mathrm{d}r}} \\
&\geq \frac{p(p-1)|\nabla P_t f|^2}{\mathbb{E}^x(P_{t-s}f)^{2-p}(X_s)\mathrm{e}^{-2\int_0^s K(X_r)\mathrm{d}r}}, \quad s \in [0, t].
\end{aligned}$$

Integrating over $[0,t]$ we prove (4). $\square$

When K is constant and $p \in [1,2]$, the above equivalences are well known according to Bakry-Emery and Ledoux (see e.g. [Bakry (1994, 1997); Bakry and Emery (1984); Bakry and Ledoux (1996a); Ledoux (2000)]), while when $p > 2$ and K is constant the equivalence of (1) and (2) is first observed by von Renesse and Sturm in [von Renesse and Strum (2005)]. The present general version of these equivalences appear here for the first time.

Next, we aim to present equivalent Harnack and cost inequalities for the curvature lower bound. To this end, let us first introduce two useful couplings for the diffusion process generated by L, namely, the coupling by parallel displacement and the coupling by reflection. These couplings were first introduced by Kendall (see [Kendall (1986)]) by taking independent coupling in a neighborhood of the cut-locus, and then refined by Cranston [Cranston (1991)] by taking limit as the neighborhood converges to the cut-locus. Here, we adopt the formulation of [Wang (2005a)] where these couplings were constructed by solving SDEs on $M \times M$ which are singular on the cut-locus, see proofs of Theorem 2.1.1 and Proposition 2.5.1 in [Wang (2005a)] which work also for the slightly more general framework in Theorem 2.3.2 below (cf. Section 3 in [Arnaudon *et al* (2006)]).

For $x, y \in M$ such that $(x,y) \notin \mathrm{cut} := \{(x',y') \in M \times M : x' \in \mathrm{cut}(y')\}$, let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic γ from x to y such that at x and y $\{J_i, \dot\gamma : 1 \le i \le d-1\}$ is an orthonormal basis. Let

$$I_Z(x,y) = \sum_{i=1}^{d-1} I(J_i, J_i) + Z\rho(\cdot, y)(x) + Z\rho(x, \cdot)(y).$$

Moreover, let $P_{x,y} : T_xM \to T_yM$ be the parallel transform along the geodesic γ, and let

$$M_{x,y} : T_xM \to T_yM;\ v \mapsto P_{x,y}v - 2\langle v, \dot\gamma\rangle(x)\dot\gamma(y)$$

be the mirror reflection. Then $P_{x,y}$ and $M_{x,y}$ are smooth outside cut and $\mathbf{D} := \{(x,x) : x \in M\}$. For convenience, we let $P_{x,x}$ and $M_{x,x}$ be the identity for any $x \in M$.

Theorem 2.3.2. *Let $x \neq y$ and $T > 0$ be fixed. Let $U : [0,T) \times M^2 \to TM^2$ be C^1-smooth in $[0,T) \times (\mathrm{cut} \cup \mathbf{D})^c$.*

(1) *There exist two Brownian motions B_t and $\bar B_t$ on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ such that*

$$1_{\{(X_t,\bar X_t)\notin \mathrm{cut}\}} \mathrm{d}\bar B_t = 1_{\{(X_t,\bar X_t)\notin \mathrm{cut}\}} \tilde u_t^{-1} P_{X_t,\bar X_t} u_t \mathrm{d}B_t$$

holds, where X_t *with lift* u_t *and* $\bar{X}_t$ *with lift* $\bar{u}_t$ *solve the equation*

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\,u_t \circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t, \ X_0 = x \\ \mathrm{d}\bar{X}_t = \sqrt{2}\,\bar{u}_t \circ \mathrm{d}\bar{B}_t + \{Z(\bar{X}_t) + U(t, X_t, \bar{X}_t)1_{\{X_t \neq \bar{X}_t\}}\}\mathrm{d}t, \ \bar{X}_0 = y. \end{cases}$$

Moreover,

$$\mathrm{d}\rho(X_t, \bar{X}_t) \le \{I_Z(X_t, \bar{X}_t) + \langle U(t, X_t, \bar{X}_t), \nabla\rho(X_t, \cdot)(\bar{X}_t)\rangle 1_{\{X_t \neq \bar{X}_t\}}\}\mathrm{d}t.$$

(2) *The first assertion in* (1) *holds by using* $M_{X_t, \bar{X}_t}$ *to replace* $P_{X_t, \bar{X}_t}$. *In this case*

$$\mathrm{d}\rho(X_t, \bar{X}_t) \\ \le 2\sqrt{2}\,\mathrm{d}b_t + \{I_Z(X_t, \bar{X}_t) + \langle U(t, X_t, \bar{X}_t), \nabla\rho(X_t, \cdot)(\bar{X}_t)\rangle 1_{\{X_t \neq \bar{X}_t\}}\}\mathrm{d}t$$

holds for some one-dimensional Brownian motion b_t.

Definition 2.3.1. The couplings in Theorem 2.3.2 (1) and (2) are called the *coupling by parallel displacement* and the *coupling by reflection* respectively.

The coupling by reflection was first introduced by Lindvall and Rogers [Lindvall and Rogers (1986)], see [Chen and Li (1989)] for more couplings of diffusions on $\mathbb{R}^d$. The next result provides some additional equivalent statements for $\mathrm{Ric}_Z \ge K$ for some constant K, where the equivalence of (1) and (2) is due to [von Renesse and Strum (2005)]. It is easy to see that (2) is also equivalent to

$$W_p^\rho(\mu P_t, \nu P_t) \le W_p^\rho(\mu, \nu)\mathrm{e}^{-Kt}, \quad \mu, \nu \in \mathcal{P}(M),$$

where $\mu P_t \in \mathcal{P}(M)$ is defined by $(\mu P_t)(A) = \mu(P_t 1_A)$ for measurable set A. (3) was initiated in [Wang (1997b)] while the equivalences of (1)-(4) are essentially due to [Wang (2004a, 2010b)], and (7)-(8) are found in [Bakry *et al* (2011)]. See Theorem 4.4.2 in Chapter 4 for 7 more equivalent transportation-cost inequalities. Moreover, (12) and (13) are taken from [Bakry and Ledoux (1996a)], which provide gradient inequalities using the Gaussian isoperimetric function

$$I_G := \Phi_G' \circ \Phi_G, \text{ where } \Phi_G(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s \mathrm{e}^{-\frac{u^2}{2}}\mathrm{d}u, \ s \in \mathbb{R}.$$

Finally, (14) and (15), first presented in [Bakry *et al* (2012)], are the Harnack type inequalities corresponding to (12) and (13) respectively.

Theorem 2.3.3. *Let* $p \in [1, \infty)$ *and* $K \in \mathbb{R}$ *be constants, and let* $p_t(x, y)$ *be the heat kernel of* P_t *w.r.t. a measure* μ *equivalent to the volume measure. Then the following assertions are equivalent to each other:*

(1) $\mathrm{Ric}_Z \geq K$.
(2) *For any* $x, y \in M$ *and* $t \geq 0$, $W_p^\rho(\delta_x P_t, \delta_y P_t) \leq \rho(x,y)\mathrm{e}^{-Kt}$ *holds.*
(2′) *For any* $\nu_1, \nu_2 \in M$ *and* $t \geq 0$, $W_p^\rho(\nu_1 P_t, \nu_2 P_t) \leq W_p^\rho(\nu_1, \nu_2)\mathrm{e}^{-Kt}$ *holds.*
(3) *When* $p > 1$, *for any* $f \in \mathcal{B}_b^+(M)$,

$$(P_t f)^p(x) \leq P_t f^p(y) \exp\left[\frac{Kp\rho(x,y)^2}{2(p-1)(\mathrm{e}^{2Kt}-1)}\right], \quad x, y \in M, t > 0.$$

(4) *For any* $f \in \mathcal{B}_b(M)$ *with* $f \geq 1$,

$$P_t \log f(x) \leq \log P_t f(y) + \frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)}, \quad x, y \in M, t > 0.$$

(5) *When* $p > 1$, *for any* $t > 0$ *and* $x, y \in M$,

$$\int_M p_t(x,z)\Big(\frac{p_t(x,z)}{p_t(y,z)}\Big)^{\frac{1}{p-1}} \mu(\mathrm{d}z) \leq \exp\left[\frac{Kp\rho(x,y)^2}{2(p-1)^2(\mathrm{e}^{2Kt}-1)}\right].$$

(6) *For any* $t > 0$ *and* $x, y \in M$,

$$\int_M p_t(x,z) \log\frac{p_t(x,z)}{p_t(y,z)} \mu(\mathrm{d}z) \leq \frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)}.$$

(7) *For any* $0 < s \leq t$ *and* $1 < q_1 \leq q_2$ *such that*

$$\frac{q_2 - 1}{q_1 - 1} = \frac{\mathrm{e}^{2Kt}-1}{\mathrm{e}^{2Ks}-1}, \tag{2.3.1}$$

there holds

$$\big\{P_s(P_{t-s}f)^{q_2}\big\}^{\frac{1}{q_2}} \leq (P_t f^{q_1})^{\frac{1}{q_1}}, \quad f \geq 0, f \in \mathcal{B}_b(M).$$

(8) *For any* $0 < s \leq t$ *and* $0 < q_2 \leq q_1$ *or* $q_2 \leq q_1 < 0$ *such that* (2.3.1) *holds,*

$$(P_t f^{q_1})^{\frac{1}{q_1}} \leq \big\{P_s(P_{t-s}f)^{q_2}\big\}^{\frac{1}{q_2}}, \quad f > 0, f \in \mathcal{B}_b(M).$$

(9) $|\nabla P_t f|^p \leq \mathrm{e}^{-pKt} P_t |\nabla f|^p, \quad f \in C_b^1(M), t \geq 0.$
(10) *For any* $t \geq 0$ *and positive* $f \in C_b^1(M)$,

$$\frac{(p\wedge 2)\{P_t f^2 - (P_t|f|^{\frac{2}{p\wedge 2}})^{p\wedge 2}\}}{4(p\wedge 2 - 1)} \leq \frac{1-\mathrm{e}^{-2Kt}}{2K} P_t|\nabla f|^2.$$

When $p = 1$ *the inequality reduces to the log-Sobolev inequality*

$$P_t(f^2 \log f^2) - (P_t f^2)\log P_t f^2 \leq \frac{2(1-\mathrm{e}^{-2Kt})}{K} P_t|\nabla f|^2.$$

(11) *For any $t > 0$ and positive $f \in C_b^1(M)$,*

$$|\nabla P_t f|^2 \le \frac{2K\{P_t f^{(p\wedge 2)} - (P_t f)^{p\wedge 2}\}(P_t f)^{(2-p)^+}}{(p\wedge 2)(p\wedge 2-1)(\mathrm{e}^{2Kt}-1)}.$$

When $p = 1$ the inequality reduces to

$$|\nabla P_t f|^2 \le \frac{2K\{P_t(f\log f) - (P_t f)\log P_t f\}P_t f}{\mathrm{e}^{2Kt}-1}.$$

(12) *For any $f \in \mathcal{B}_b(M)$ and $t > 0$,*

$$|\nabla P_t f|^2 \le \frac{K}{\mathrm{e}^{2Kt}-1}\{(I_G(P_t f))^2 - (P_t I_G(f))^2\}.$$

(13) *For any $f \in C_b^1(M)$ and $t \ge 0$,*

$$I_G(P_t f) \le P_t\sqrt{I_G(f)^2 + \frac{1-\mathrm{e}^{-2Kt}}{K}|\nabla f|^2}.$$

(14) *For any $f \in \mathcal{B}_b(M)$ and $t > 0$,*

$$\Phi_G^{-1}(P_t f)(x) \le \Phi_G^{-1}(P_t f)(y) + \rho(x,y)\sqrt{\frac{K}{\mathrm{e}^{2Kt}-1}}, \quad x, y \in M.$$

(15) *For any smooth domain $A \subset M$ and $A(r) := \{z \in M : \rho(z, A) \le r\}$ for $r \ge 0$,*

$$P_t 1_A(x) \le P_t 1_{A(\mathrm{e}^{-Kt}\rho(x,y))}(y), \quad t \ge 0, x, y \in M.$$

Proof. The equivalence of (1), (9) and (10) follows directly from Theorem 2.3.1 with constant K. The proof of (11) implying (1) is the same as that of Theorem 2.3.1(4) implying (1), while (11) follows from Theorem 2.3.1(4) since by the Jensen inequality we have

$$\mathbb{E}\big\{(P_{t-s}f)^{(2-p)^+}(X_s)\big\} \le \big(\mathbb{E}P_{t-s}f(X_s)\big)^{(2-p)^+} = (P_t f)^{(2-p)^+}.$$

By Theorem 1.4.1, (3) and (4) are equivalent to (5) and (6) respectively. Moreover, according to Corollary 1.4.3, we see that (3) implies (4). Therefore, it remains to prove that (1) is equivalent to (2)/(2′), (1) implies (3), (4) implies (1), and (10) with $p = 1$ is equivalent to each of (7) and (8), (1) is equivalent to (12), (12) is equivalent to (14), (1) implies (13), (13) and (14) imply (15), and (15) implies (9) with $p = 1$.

(a) **(1) is equivalent to (2), (2′).** By (1) and the index lemma Theorem 1.1.11, we have $I_Z(x,y) \le -K\rho(x,y)$. So, using the coupling by parallel displacement and Theorem 2.3.2 with $U = 0$, we obtain from (1) that

$$W_p^\rho(\delta_x P_t, \delta_y P_t) \le (\mathbb{E}\rho(X_t, Y_t)^p)^{1/p} \le \rho(x,y)\mathrm{e}^{-Kt}.$$

That is, (1) implies (2). Obviously, (2′) implies (2). It is also easy to see that (2) implies (2′), so that they are equivalent. Indeed, let $\pi \in \mathcal{C}(\nu_1, \nu_2)$ such that $W_p^\rho(\nu_1 P_t, \nu_2 P_t) = \pi(\rho^p)^{1/p}$. Then from Proposition 1.3.1 and (2) we obtain

$$\begin{aligned} W_p^\rho(\nu_1 P_t, \nu_2 P_t)^p &\le \int_{M\times M} W_p^\rho(\delta_x P_t, \delta_y P_t)^p \pi(\mathrm{d}x, \mathrm{d}y) \\ &\le \mathrm{e}^{-pKt} W_p^\rho(\nu_1, \nu_2)^p. \end{aligned}$$

On the other hand, if (2) holds then letting $\Pi_{x,y}$ be the optimal coupling for $\delta_x P_t$ and $\delta_y P_t$ for the L^p-transportation cost, for $f \in C_b^1(M)$ we have

$$\begin{aligned} |\nabla P_t f(x)| &\le \lim_{y\to x} \frac{\int_{M\times M} |f(x') - f(y')| \Pi_{x,y}(\mathrm{d}x', \mathrm{d}y')}{\rho(x,y)} \\ &\le \lim_{y\to x} \bigg(\int_{M\times M} \Big(\frac{|f(x') - f(y')|}{\rho(x',y')} \Big)^{p/(p-1)} \Pi_{x,y}(\mathrm{d}x', \mathrm{d}y') \bigg)^{(p-1)/p} \\ &\quad \times \frac{W_p^\rho(\delta_x P_t, \delta_y P_t)}{\rho(x,y)} \\ &\le \mathrm{e}^{-Kt} (P_t |\nabla f|^{p/(p-1)})^{(p-1)/p}(x). \end{aligned}$$

Thus, (9) holds, which is equivalent to (1) as mentioned above according to Theorem 2.3.1.

(b) **(9) implies (3).** By approximations and the monotone class theorem, we may assume that $f \in C_b^2(M), \inf f > 0$ and f is constant outside a compact set. Given $x \ne y$ and $t > 0$, let $\gamma : [0,t] \to M$ be the geodesic from x to y with length $\rho(x,y)$. Letting $v_s = \mathrm{d}\gamma_s/\mathrm{d}s$, we have $|v_s| = \rho(x,y)/t$. Let

$$h(s) = \frac{t(\exp[2Ks] - 1)}{\exp[2Kt] - 1}, \quad s \in [0,t].$$

Then $h(0) = 0, h(t) = t$. Let $y_s = \gamma_{h(s)}$. Define

$$\varphi(s) = \log P_s (P_{t-s} f)^p (y_s), \quad s \in [0,t].$$

By (9) with $p = 1$ we have $|\nabla P_t f| \le \mathrm{e}^{-Kt} P_t |\nabla f|$. Combining this with the Kolmogorov equations, we obtain (simply denote $\rho(x, y)$ by ρ)

$$\begin{aligned}
\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} &= \frac{1}{P_s(P_{t-s}f)^p}\Big\{p(p-1)P_s(P_{t-s}f)^{p-2}|\nabla P_{t-s}f|^2 \\
&\quad + h'(s)\langle \nabla P_s(P_{t-s}f)^p, v_{h(s)}\rangle\Big\} \\
&\ge \frac{p}{P_s(P_{t-s}f)^p} P_s\Big\{(p-1)(P_{t-s}f)^{p-2}|\nabla P_{t-s}f|^2 \\
&\quad - \frac{\rho}{t}\mathrm{e}^{-Ks}h'(s)(P_{t-s}f)^{p-1}|\nabla P_{t-s}f|\Big\} \\
&= \frac{p}{P_s(P_{t-s}f)^p} P_s\Big\{(P_{t-s}f)^p\big((p-1)|\nabla \log P_{t-s}f|^2 \\
&\quad - \frac{\rho}{t}h'(s)\mathrm{e}^{-Ks}|\nabla \log P_{t-s}f|\big)\Big\} \\
&\ge -\frac{pK^2\rho^2\exp[2Ks]}{(p-1)(\exp[2Kt]-1)^2}, \quad s \in [0, t].
\end{aligned}$$

By integrating over s from 0 to t, we complete the proof.

(c) **(4) implies (1).** Let $x \in M$ and $X \in T_x M$ be fixed. For any $n \ge 1$ we may take $f \in C_b^\infty(M)$ such that f is constant outside a compact set, and

$$\nabla f(x) = X, \quad \mathrm{Hess}_f(x) = 0, \quad f \ge n. \tag{2.3.2}$$

Taking $\gamma_t = \exp_x[-2t\nabla \log f(x)]$, we have $\rho(x, \gamma_t) = 2t|\nabla \log f|(x)$ for $t \in [0, t_0]$, where $t_0 > 0$ is such that $2t_0|X| < r_0 f(x)$. By (4) with $y = \gamma_t$, we obtain

$$P_t(\log f)(x) \le \log P_t f(\gamma_t) + \frac{2Kt^2|\nabla \log f|^2(x)}{\mathrm{e}^{2Kt}-1}, \quad t \in (0, t_0]. \tag{2.3.3}$$

Since $Lf \in C_0^2(M)$ and $\mathrm{Hess}_f(x) = 0$ implies $\nabla|\nabla f|^2(x) = 0$, at point x we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}P_t \log f|_{t=0} = L\log f = \frac{Lf}{f} - |\nabla \log f|^2, \\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}P_t \log f|_{t=0} = L^2 \log f \\
&= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} + \frac{2|\nabla f|^2 Lf}{f^3} + 2\langle \nabla Lf, \nabla f^{-1}\rangle - \frac{L|\nabla f|^2}{f^2} \\
&\quad + \frac{2|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} - 2\langle \nabla|\nabla f|^2, \nabla f^{-2}\rangle \\
&= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - \frac{2}{f^2}\langle \nabla Lf, \nabla f\rangle - \frac{L|\nabla f|^2}{f^2} + \frac{4|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} =: A.
\end{aligned}$$

Thus, by Taylor's expansions,

$$P_t(\log f)(x) = \log f(x) + t\big(f^{-1}Lf - |\nabla \log f|^2\big)(x) + \frac{t^2}{2}A + \mathrm{o}(t^2) \quad (2.3.4)$$

holds for small $t > 0$. On the other hand, let $N_t = P_{x,\gamma_t}\nabla \log f(x)$, where P_{x,γ_t} is the parallel displacement along the geodesic $t \mapsto \gamma_t$. We have $\dot\gamma_t = -2N_t$ and $\nabla_{\dot\gamma_t} N_t = 0$. So,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log P_t f(\gamma_t)|_{t=0} &= \Big(\frac{LP_tf}{P_tf}(\gamma_t) - \frac{2\langle \nabla P_t f, N_t\rangle}{P_t f}(\gamma_t)\Big)\Big|_{t=0}\\
&= \frac{Lf}{f} - 2|\nabla \log f|^2,\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log P_t f(\gamma_t)|_{t=0} &= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - 2\langle \nabla(f^{-1}Lf), \nabla \log f\rangle\\
&\quad - \frac{2}{f}\langle \nabla Lf, \nabla \log f\rangle + \frac{2}{f^2}\langle \nabla f, \nabla \log f\rangle Lf\\
&\quad + 4\mathrm{Hess}_{\log f}(\nabla \log f, \nabla \log f)\\
&= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - 4\frac{\langle \nabla Lf, \nabla f\rangle}{f^2}\\
&\quad + 4\frac{|\nabla f|^2 Lf}{f^3} - 4\frac{|\nabla f|^4}{f^4}\\
&=: B,
\end{aligned}$$

where, as in above, the functions take value at point x and we have used $\mathrm{Hess}_f(x) = 0$ in the last step. Thus, we have

$$\log P_t f(\gamma_t) = \log f(x) + t\big(f^{-1}Lf - 2|\nabla \log f|^2\big)(x) + \frac{t^2}{2}B + \mathrm{o}(t^2).$$

Combining this with (2.3.3) and (2.3.4), we arrive at

$$\begin{aligned}
&\frac{1}{t}\Big(1 - \frac{2Kt}{\mathrm{e}^{2Kt}-1}\Big)|\nabla \log f|^2(x)\\
&\le \frac{1}{2}\Big(\frac{L|\nabla f|^2 - 2\langle \nabla Lf, \nabla f\rangle}{f^2} + \frac{2|\nabla f|^4}{f^4}\Big)(x) + \mathrm{o}(1).
\end{aligned}$$

Letting $t \to 0$ we obtain

$$\Gamma_2(f,f)(x) := \frac{1}{2}L|\nabla f|^2(x) - \langle \nabla Lf, \nabla f\rangle(x) \ge K|\nabla f|^2(x) - \frac{|\nabla f|^4}{f^2}(x).$$

Denote $\Gamma_2(f,f)$ by $\Gamma_2(f)$ for simplicity. Since by the Bochner-Weitzenböck formula and (2.3.2) we have $\nabla f(x) = X, f(x) \ge n$ and

$$\Gamma_2(f,f)(x) = \mathrm{Ric}(X,X) - \langle \nabla_X Z, X\rangle,$$

it follows that

$$\mathrm{Ric}(X,X) - \langle \nabla_X Z, X\rangle \geq K|X|^2 - \frac{|X|^4}{n}, \quad n \geq 1.$$

This implies (1) by letting $n \to \infty$.

(d) **(10) with $p = 1$ implies (7) and (8).** It suffices to prove for $f \in C_b^\infty(M)$ such that $\inf f > 0$ and Lf is bounded. In this case, for any $t > 0$, let

$$q(s) = 1 + \frac{(q_1 - 1)(\mathrm{e}^{2Kt} - 1)}{\mathrm{e}^{2Ks} - 1}, \; \psi(s) = \big\{P_s(P_{t-s}f)^{q(s)}\big\}^{\frac{1}{q(s)}}, \quad s \in (0,t].$$

Then

$$\frac{1 - \mathrm{e}^{-2Ks}}{2K} + \frac{q(s) - 1}{q'(s)} = 0,$$

so that (10) with $p = 1$ implies

$$\begin{aligned}
&\Big(\frac{\psi'\psi^{q-1}q^2}{q'}\Big)(s)\\
&= P_s(P_{t-s}f)^{q(s)}\log(P_{t-s}f)^{q(s)} - \{P_s(P_{t-s}f)^{q(s)}\}\log P_s(P_{t-s}f)^{q(s)}\\
&\quad + \frac{q(s)^2(q(s)-1)}{q'(s)}P_s\big\{(P_{t-s}f)^{q(s)-2}|\nabla P_{t-s}f|^2\big\}\\
&\leq q(s)^2\Big(\frac{1 - \mathrm{e}^{-2Ks}}{2K} + \frac{q(s)-1}{q'(s)}\Big)P_s\big\{(P_{t-s}f)^{q(s)-2}|\nabla P_{t-s}f|^2\big\} = 0.
\end{aligned}$$

Therefore, in case (7) one has $q'(s) < 0$ so that $\psi'(s) \geq 0$, while in case (8) one has $q'(s) > 0$ so that $\psi'(s) \leq 0$. Hence, the inequalities in (7) and (8) hold.

(e) **(7) or (8) implies (10) with $p = 1$.** We only prove that (7) implies (10), since (8) implying (10) can be proved in a similar way. Let $q_1 = 2$ and $q_2 = 2(1+\varepsilon)$ for small $\varepsilon > 0$. According to (2.3.1) we take

$$\begin{aligned}
s(\varepsilon) &= \frac{1}{2K}\log\Big(1 + \frac{(q_1 - 1)(\mathrm{e}^{2Kt} - 1)}{q_2 - 1}\Big) = \frac{1}{2K}\log\Big(1 + \frac{\mathrm{e}^{2Kt} - 1}{1 + 2\varepsilon}\Big)\\
&= t + \frac{1}{2K}\log\Big(1 - \frac{2\varepsilon(1 - \mathrm{e}^{-2Kt})}{1 + 2\varepsilon}\Big) = t - \frac{\varepsilon(1 - \mathrm{e}^{-2Kt})}{K} + \mathrm{o}(\varepsilon).
\end{aligned}$$

So, we obtain from (7) that

$$\begin{aligned}
0 &\geq \lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\Big\{\big(P_{s(\varepsilon)}(P_{t-s(\varepsilon)}f)^{2(1+\varepsilon)}\big)^{\frac{1}{1+\varepsilon}} - P_t f^2\Big\}\\
&= P_t f^2 \log f^2 - (P_t f^2)\log P_t f^2 - \frac{2(1 - \mathrm{e}^{-2Kt})}{K}P_t|\nabla f|^2.
\end{aligned}$$

Therefore, (10) with $p = 1$ holds.

(f) **(1) is equivalent to (12).** As (1) is equivalent to (9) for $p = 1$, it implies that $P_s|\nabla P_{t-s}f| \geq e^{Ks}|\nabla P_t f|$. Combining this with the Kolmogorov equation, the fact that $I_G I_G'' = -1$ and using the Schwarz inequality, we obtain

$$-\frac{d}{ds}\{P_s I_G(P_{t-s}f)\}^2 = -2\{P_s I_G(P_{t-s}f)\}P_s\{I_G''(P_{t-s}f)|\nabla P_{t-s}f|^2\}$$
$$= 2\{P_s I_G(P_{t-s}f)\}P_s\frac{|\nabla P_{t-s}f|^2}{I_G(P_{t-s}f)} \geq \{P_s|\nabla P_{t-s}f|\}^2 \geq e^{2Ks}|\nabla P_t f|^2.$$

Integrating w.r.t. ds over the interval $[0,t]$, we prove (12).

On the other hand, as observed in [Hino (2002)], using εf in place of f and letting $\varepsilon \to 0$, it is easy to derive (11) for $p = 2$ from (12), and (11) is equivalent to (1) as observed above.

(g) **(12) is equivalent to (14).** Since $|\nabla \Phi_G^{-1} \circ P_t f|^2 = \frac{|\nabla P_t f|^2}{(I_G \circ P_t f)^2}$, it is easy to see that (12) is equivalent to

$$|\nabla \Phi_G^{-1} \circ P_t f|^2 \leq \frac{K}{e^{2Kt}-1}, \quad t > 0,$$

which is obviously equivalent to (14).

(h) **(1) is equivalent to (13).** Let $h(s) = \frac{1-e^{-2Ks}}{K}$. Noting that $I_G I_G'' = -1$ and (1) implies

$$\Gamma_2(P_{t-s}f, P_{t-s}f) \geq K|\nabla P_{t-s}f|^2 + \frac{|\nabla|\nabla P_{t-s}f|^2|^2}{4|\nabla P_{t-s}f|^2}, \quad s \in [0,t],$$

we have

$$\frac{d}{ds}P_s\sqrt{I_G(P_{t-s}f)^2 + h(s)|\nabla P_{t-s}f|^2}$$
$$= P_s\frac{(I_G I_G'' + I_G'^2)(P_{t-s}f)|\nabla P_{t-s}f|^2 + h(s)\Gamma_2(P_{t-s}f) + e^{-2Ks}|\nabla P_{t-s}f|^2}{\sqrt{I_G(P_{t-s}f)^2 + h(s)|\nabla P_{t-s}f|^2}}$$
$$- P_s\frac{|(I_G' I_G)(P_{t-s}f)\nabla P_{t-s}f + h(s)\nabla|\nabla P_{t-s}f|^2|^2}{\{I_G(P_{t-s}f)^2 + h(s)|\nabla P_{t-s}f|^2\}^{3/2}}$$
$$\geq P_s\frac{(I_G'(P_{t-s}f)|\nabla P_{t-s}f|)^2 + \frac{h(s)|\nabla|\nabla P_{t-s}f|^2|^2}{4|\nabla P_{t-s}f|^2}}{\sqrt{I_G(P_{t-s}f)^2 + h(s)|\nabla P_{t-s}f|^2}}$$
$$- P_s\frac{|(I_G' I_G)(P_{t-s}f)\nabla P_{t-s}f + h(s)\nabla|\nabla P_{t-s}f|^2|^2}{\{I_G(P_{t-s}f)^2 + h(s)|\nabla P_{t-s}f|^2\}^{3/2}}$$
$$\geq 0.$$

Thus, (13) holds.

On the other hand, as observed in [Bakry and Ledoux (1996a)], using εf in place of f and letting $\varepsilon \to 0$, we derive (10) from (13), and (10) is equivalent to (1) as already proved above.

(i) **(13) and (14) imply (15)**. By (13) we have

$$I_G(P_t f) \le P_t I_G(f) + \sqrt{h(t)} P_t |\nabla f|.$$

Taking $f = 1_A$ for a smooth domain A and noting that $I_G(0) = I_G(1) = 0$, we obtain

$$I_G(P_t 1_A) \le \sqrt{h(t)} P_t 1_{\partial A}.$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big\{\Phi_G^{-1}(P_t 1_{A(r)}) - \frac{r}{\sqrt{h(t)}}\Big\} = \frac{P_t 1_{\partial A(r)}}{I(P_t 1_{A(r)})} - \frac{1}{\sqrt{h(t)}} \ge 0.$$

This implies

$$P_t 1_{A(r)} \ge \Phi_G\Big(\Phi_G^{-1}(P_t 1_A) + \frac{r}{\sqrt{h(t)}}\Big), \quad r \ge 0.$$

Combining this with (14) we obtain

$$P_t 1_A(x) \le \Phi_G\Big(\Phi_G^{-1}(P_t 1_A(y)) + \frac{\rho(x,y)\mathrm{e}^{-Kt}}{\sqrt{h(t)}}\Big) \le P_t 1_{A(\rho(x,y)\mathrm{e}^{-Kt})}.$$

(j) **(15) implies (9) for $p = 1$**. For any unit $v \in T_x M$, (15) implies

$$P_t 1_A(\exp_x[\varepsilon v]) - P_t 1_A(x) \le P_t 1_{\partial_{\varepsilon \mathrm{e}^{-Kt}} A}(x), \quad \varepsilon > 0,$$

where $\partial_r A := \{z : d(z, A) \in (0, r)\}$. Multiplying both sides by $\frac{1}{\varepsilon}$ and letting $\varepsilon \to 0$, we obtain

$$|\nabla P_t 1_A|(x) \le \mathrm{e}^{-Kt}(P_t(x,\cdot))\mu_\partial(\partial A),$$

where $\mu_\partial(\partial A)$ is the area of ∂A induced by a measure μ. Therefore, for any smooth $f > 0$,

$$\begin{aligned}|\nabla P_t f|(x) &\le \int_0^\infty |\nabla P_t 1_{\{f>s\}}|(x)\mathrm{d}s \\ &\le \mathrm{e}^{-Kt}\int_0^\infty (P_t(x,\cdot))\mu_\partial(\{f=s\})\mathrm{d}s = \mathrm{e}^{-Kt} P_t|\nabla f|(x).\end{aligned}$$

□

2.4 Applications of equivalent semigroup inequalities

Throughout this section, we assume that P_t has a non-trivial invariant measure μ. Then there exists $V \in C(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$, where and throughout, $\mathrm{d}x$ is the Riemannian volume measure, see [Bogachev, Krylov and Röckner (2001); Bogachev, Röckner and Wang (2001)] and references within. In particular, if $Z = \nabla V$ for some $V \in C^2(M)$, then P_t is symmetric in $L^2(\mu)$ for $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$.

Theorem 2.4.1. *Let* $\mathrm{Ric}_Z \geq K$ *for some* $K \in \mathbb{R}$ *and assume that* μ *is a probability measure.*

(1) *If* $K > 0$ *then for any* $p \in [1, \infty)$,

$$\frac{p(\mu(f^2) - \mu(f^{2/p})^p)}{p-1} \leq \frac{2}{K}\mu(|\nabla f|^2), \quad f \geq 0, f \in C_b^1(M).$$

In particular, the following log-Sobolev *inequality holds:*

$$\mu(f^2 \log f^2) \leq \frac{2}{K}\mu(|\nabla f|^2), \quad f \in C_b^1(M), \mu(f^2) = 1.$$

(2) *Let* P_t^* *be the adjoint operator of* P_t *in* $L^2(\mu)$*. Then*

$$\mu((P_t^* f^2) \log P_t^* f^2) \leq \frac{K W_2^\rho (f^2\mu, \mu)^2}{2(\mathrm{e}^{2Kt} - 1)}, \quad \mu(f^2) = 1.$$

(3) *If* P_t *is symmetric in* $L^2(\mu)$*, then the following HWI inequality holds:*

$$\mu(f^2 \log f^2) \leq 2\sqrt{\mu(|\nabla f|^2)} W_2^\rho(f^2\mu, \mu) - \frac{K}{2} W_2^\rho(f^2\mu, \mu)^2,$$

for all $f \in C_b^1(M)$, $\mu(f^2) = 1$.

Proof. Since the Poincaré inequality holds on any compact connected smooth domains, by Theorem 3.1 in [Röckner and Wang (2001)] there holds a weak Poincaré inequality, namely

$$\mu(f^2) \leq \alpha(r)\mu(|\nabla f|^2) + r\|f\|_\infty^2, \quad f \in C_b^1(M), \mu(f) = 0, r > 0$$

holds for some positive function α on $(0, \infty)$. Then, according to Theorem 1.6.14 for $\Phi(f) = \|f\|_\infty^2$, $P_t f \to \mu(f)(t \to \infty)$ in $L^2(\mu)$ for any $f \in \mathcal{B}_b(M)$. Hence, (1) follows from Theorem 2.3.1(3) with $p = 1$ by letting $t \to \infty$.

Next, applying Theorem 2.3.3(4) to $P_t^* f^2$ in place of f, we obtain

$$P_t(\log P_t^* f^2)(x) \leq \log(P_t P_t^* f^2)(y) + \frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt} - 1)}, \quad x, y \in M.$$

Integrating with respect to the optimal coupling $\Pi(dx.dy)$ of $f^2\mu$ and μ for the L^2-transportation cost, and using the Jensen inequality, we obtain

$$\begin{aligned}\mu((P_t^* f^2)\log P_t^* f^2) &\le \frac{KW_2^\rho(f^2\mu,\mu)^2}{2(e^{2Kt}-1)} + \mu(\log(P_t P_t^* f^2))\\ &\le \frac{KW_2^\rho(f^2\mu,\mu)^2}{2(e^{2Kt}-1)}.\end{aligned}$$

Therefore, (2) holds.

Finally, By Theorem 2.3.1(3), we have

$$P_t(f^2\log f^2) \le (P_t f^2)\log P_t f^2 + \frac{2(1-e^{-2Kt})}{K} P_t|\nabla f|^2.$$

Integrating w.r.t. μ leads to

$$\mu(f^2\log f^2) \le \frac{2(1-e^{-2Kt})}{K}\mu(|\nabla f|^2) + \mu((P_t f^2)\log P_t f^2).$$

If $P_t = P_t^*$, combining this with (2) we arrive at

$$\begin{aligned}\mu(f^2\log f^2) &\le \frac{2(1-e^{-2Kt})}{K}\mu(|\nabla f|^2) + \frac{KW_2^\rho(f^2\mu,\mu)^2}{2(e^{2Kt}-1)}\\ &= r_t\mu(|\nabla f|^2) + \frac{W_2^\rho(f^2\mu,\mu)^2}{r_t} - \frac{K}{2}W_2^\rho(f^2\mu,\mu)^2, \quad t>0,\end{aligned}$$

where

$$r_t = \frac{2(1-e^{-2Kt})}{K}.$$

Let f be non-constant. Taking $t \in (0,\infty]$ such that

$$r_t = \frac{W_2^\rho(f^2\mu,\mu)}{\sqrt{\mu(|\nabla f|^2)}},$$

we complete the proof. Note that if $K \le 0$ then $\{r_t : t \in (0,\infty]\} = (0,\infty]$ so that such the required t exists. If $K > 0$ then the range of r_t is $(0, \frac{2}{K}]$, and in this case the log-Sobolev inequality in (1) implies the Talagrand inequality (see [Otto and Villani (2000); Bobkov *et al* (2001)])

$$W_2^\rho(f^2\mu,\mu)^2 \le \frac{2}{K}\mu(f^2\log f^2).$$

This, together with the log-Sobolev inequality, implies that

$$\frac{W_2^\rho(f^2\mu,\mu)}{\sqrt{\mu(|\nabla f|^2)}} \le \frac{2}{K}.$$

Therefore, the required t exists for $K > 0$ as well. $\square$

Theorem 2.4.1(1) is well known as the Bakry-Emery criterion [Bakry and Emery (1984)] and was extended in [Chen and Wang (1997b)] to the situation that Ric$-$Hess$_V$ is uniformly positive outside a compact set, while the third assertion (known as the HWI inequality) is first proved in [Otto and Villani (2000)]. Our proof of Theorem 2.4.1 is taken from [Bobkov *et al* (2001)]. Next, we apply the dimension free Harnack inequality in Theorem 2.3.3 (see [Röckner and Wang (2003a)]).

Theorem 2.4.2. *Let* $\mathrm{Ric}_Z \geq K$ *for some* $K \in \mathbb{R}$ *and assume that* μ *is a probability measure. Then for any* $o \in M$*:*

(1) P_t *is* ultracontractive, *i.e.* $\|P_t\|_{2\to\infty} < \infty$ *for all* $t > 0$, *if and only if* $\|P_t \exp[\lambda\rho_o^2]\|_\infty < \infty$ *for any* $\lambda, t > 0$.
(2) P_t *is* supercontractive, *i.e.* $\|P_t\|_{2\to 4} < \infty$ *for all* $t > 0$, *if and only if* $\mu(\exp[\lambda\rho_o^2]) < \infty$ *for any* $\lambda > 0$.
(3) *If there exists* $\lambda > -K/2$ *such that* $\mu(\exp[\lambda\rho_o^2]) < \infty$, *then* P_t *is* hypercontractive, *i.e.* $\|P_t\|_{2\to 4} \leq 1$ *holds for some* $t > 0$.

Proof. By Theorem 2.3.3(3),

$$|P_s f(x)|^2 \leq P_s f^2(y) \exp\Big[\frac{K\rho(x,y)^2}{\exp[2Ks]-1}\Big]. \tag{2.4.1}$$

This implies that

$$\begin{aligned} 1 &\geq |P_s f(x)|^2 \int_M \exp\Big[-\frac{K\rho(x,y)^2}{\exp[2Ks]-1}\Big]\mu(\mathrm{d}y) \\ &\geq |P_s f(x)|^2 \mu(B(o,1)) \exp\Big[-\frac{K(\rho_o(x)+1)^2}{\exp[2Ks]-1}\Big]. \end{aligned}$$

Then there exist $c_1, c_2 > 0$ such that

$$|P_s f| \leq \exp\big[(c_1 + c_2\rho_o^2)/s\big], \quad s \in (0,1]. \tag{2.4.2}$$

By (2.4.2) we have

$$\|P_t\|_{2\to\infty} \leq \|P_{t/2}\exp[2(c_1+c_2\rho_o^2)/t]\|_\infty < \infty, \quad t \in (0,1], \tag{2.4.3}$$

provided $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ for any $t, \lambda > 0$. On the other hand, if P_t is ultracontractive, then it is supercontractive and thus (cf. [Aida *et al* (1994)] or [Aida and Stroock (1994)]) $\exp[\lambda\rho_o^2] \in L^2(\mu)$ for any $\lambda > 0$. Therefore, $\|P_t \exp[\lambda\rho^2]\|_\infty \leq \|P_t\|_{2\to\infty}\|\exp[\lambda\rho^2]\|_2 < \infty$ for any $t, \lambda > 0$. Similarly, (2) also follows from (2.4.2).

Now, if there exists $\lambda > -K/2$ such that $\mu(\exp[\lambda\rho^2]) < \infty$, then there exists $t > 0$ and $q > p \geq 2$ such that $\|P_t\|_{p\to q} < \infty$. This can be proved by the argument leading to (2.4.2), just consider f with $\|f\|_p = 1$ instead of $\|f\|_2 = 1$, and apply the dimension-free Harnack inequality Theorem 2.3.3(3). Then, by Riesz-Thorin's interpolation theorem, we have $\|P_t\|_{2\to q} < \infty$ for some $t > 0$ and some $q > 2$. This implies the defective log-Sobolev inequality (see e.g. [Gross (1976, 1993); Davies (1989); Davies and Simon (1984)])

$$\mu(f^2 \log f^2) \le C_1\mu(|\nabla f|^2) + C_2, \quad f \in C_b^1(M), \mu(f^2) = 1$$

for some constants $C_1, C_2 > 0$. Since M is connected so that there holds a weak Poincaré inequality (see Theorem 3.1 in [Röckner and Wang (2001)]), this and Proposition 1.6.13 imply the log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(|\nabla f|^2), \quad f \in C_b^1(M), \mu(f^2) = 1$$

for some constant $C > 0$. Therefore, due to Gross [Gross (1976)], P_t is hypercontractive. □

Although Theorem 2.4.2 provide exact criteria (i.e. sufficient and necessary conditions) for the ultracontracitity and the supercontractivity, it merely provides a sufficient condition for the hypercontractivity. It was shown in [Chen and Wang (2007)] that this sufficient condition is already sharp in the sense that if for any $K < 0$ and $\varepsilon > 0$, there exists an example of M and V such that $\mathrm{Ric}_Z \ge K$ for $Z = \nabla V, \mu(\mathrm{e}^{(K^- -\varepsilon)\rho_o^2}) < \infty$ but P_t is not hypercontractive. It was shown in [Röckner and Wang (2003a)] that when Ric_Z is bounded below, the hypercontractivity of P_t is equivalent to the validity of the log-Sobolev inequality. So, according to the concentration property of the log-Sobolev inequality (see [Aida *et al* (1994)]), Theorem 2.4.2 also implies that when $\mathrm{Ric}_Z \ge 0$, P_t is hypercontractive if and only if $\mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty$ holds for some $\lambda > 0$.

Finally, we apply the dimension-free Harnack inequality to heat kernel estimates. To this end, we need the following lemma (see [Grigor'yan (1997)]).

Lemma 2.4.3. *Assume that $Z = \nabla V$ for some $V \in C^2(M)$ and let $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$. For $x \in M, T > 0, p > 1, q = p/(2(p-1))$, let*

$$\eta(s,y) = -\frac{\rho(x,y)^2}{2(T-qs)}, \quad y \in M, s < \frac{T}{q}.$$

Then for any $f \in \mathcal{B}_b^+(M)$,

$$\int_M (P_t f)^p(y)\mathrm{e}^{\eta(t,y)}\mu(\mathrm{d}y) \le \int_M f^p(y)\mathrm{e}^{-\rho(x,y)^2/(2T)}\mu(\mathrm{d}y), \quad t < \frac{T}{q}.$$

Proof. By an approximation argument, it suffices to prove for finite μ. Indeed, we may take $\{V_n\} \subset C^2(M)$ such that $V_n \uparrow V$ and for each n, $\mu_n(\mathrm{d}x) := \exp[V_n(x)]\mathrm{d}x$ is finite and $V_n = V$ on $B(x,n)$. If the desired inequality holds for μ_n and P_t^n generated by $\Delta + \nabla V_n$, then it holds for P_t and μ as well by letting $n \to \infty$. Since μ is finite, we may assume that $f \geq c > 0$ for some constant c. Let

$$I(s) = \int_M (P_s f(y))^p \exp[\eta(s,y)]\mu(\mathrm{d}y), \quad s \in [0, T/q).$$

It is easy to see that

$$\begin{aligned} I'(s) &\leq -\int p(p-1)(P_s f)^p \mathrm{e}^{\eta(s,\cdot)}\Big(\frac{|\nabla P_s f|}{P_s f} - \frac{\rho(x,\cdot)}{2(p-1)(T-qs)}\Big)^2 \mathrm{d}\mu \\ &\leq 0, \qquad s < \frac{T}{q}. \end{aligned}$$

This completes the proof by taking integral over $[0,t]$. □

Theorem 2.4.4. *Let P_t be symmetric in $L^2(\mu)$, and let p_t be the heat kernel of P_t w.r.t. μ. Let $\mathrm{Ric}_Z \geq K$ for some $K \in \mathbb{R}$.*

(1) *For any $\delta > 2$ there exists $c(\delta) > 0$ such that*

$$p_t(x,y) \leq \frac{1}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}} \exp\left[c(\delta)(1+t) - \frac{\rho(x,y)^2}{2\delta t}\right],$$

for all $x, y \in M, t > 0$.

(2) *If μ is a probability measure, then*

$$p_t(x,y) \geq \exp\left[-\frac{K\rho(x,y)^2}{2(\mathrm{e}^{Kt}-1)}\right], \quad x, y \in M, t > 0.$$

Proof. (1) For $\delta > 2$, let $p \in (1,2)$ such that $q := p/[2(p-1)] < \delta/2$. By Lemma 2.4.3 for $T = \delta t/2$ and applying Theorem 2.3.3(3),

$$\begin{aligned} &(P_t f)^2(x)\mu(B(x,\sqrt{2t}))\exp\left[-\frac{2p^2Kt}{(2-p)(\mathrm{e}^{2Kt}-1)} - \frac{1}{\delta/2-q}\right] \\ &\leq \int_M (P_t f)^2(x)\exp\left[-\frac{p^2K\rho(x,y)^2}{(2-p)(\mathrm{e}^{2Kt}-1)} - \frac{\rho(x,y)^2}{2(T-qt)}\right]\mu(\mathrm{d}y) \\ &\leq \int_M (P_t f^{2/p})^p(y)\exp\left[-\frac{\rho(x,y)^2}{2(T-qt)}\right]\mu(\mathrm{d}y) \\ &\leq \int_M f^2(y)\mathrm{e}^{-\rho(x,y)^2/(2T)}\mu(\mathrm{d}y). \end{aligned}$$

Taking

$$f(y) = (n \wedge p_t(x,y))\mathrm{e}^{(n\wedge\rho(x,y)^2)/(2T)}, \quad y \in M,$$

we obtain

$$\int_M (n \wedge p_t(x,y))^2 \mathrm{e}^{(n\wedge\rho(x,y)^2)/(\delta t)}\mu(\mathrm{d}y) \le \frac{\mathrm{e}^{c(\delta)(1+t)}}{\mu(B(x,\sqrt{2t}))}$$

for some constant $c(\delta) > 0$. Letting $n \to \infty$, we arrive at

$$\int_M p_t(x,y)^2 \exp\left[\frac{\rho(x,y)^2}{\delta t}\right]\mu(\mathrm{d}y) \le \frac{\mathrm{e}^{c(\delta)(1+t)}}{\mu(B(x,\sqrt{2t}))}.$$

Applying this inequality for $t/2$ in place of t, we arrive at

$$\begin{aligned}
&\mathrm{e}^{\rho(x,y)^2/(2\delta t)}p_t(x,y) \\
&= \mathrm{e}^{\rho(x,y)^2/(2\delta t)}\int_M p_{t/2}(x,z)p_{t/2}(y,z)\mu(\mathrm{d}z) \\
&\le \int_M \left(p_{t/2}(x,z)\mathrm{e}^{\rho(x,z)^2/(\delta t)}\right)\left(p_{t/2}(y,z)\mathrm{e}^{\rho(y,z)^2/(\delta t)}\right)\mu(\mathrm{d}z) \\
&\le \left(\int_M p_{t/2}(x,z)^2\mathrm{e}^{2\rho(x,z)^2/(\delta t)}\mu(\mathrm{d}z)\right)^{1/2} \\
&\quad \times \left(\int_M p_{t/2}(y,z)^2\mathrm{e}^{2\rho(y,z)^2/(\delta t)}\mu(\mathrm{d}z)\right)^{1/2} \\
&\le \frac{\mathrm{e}^{\rho(x,y)^2/(2\delta t)}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}\exp\left[c(\delta)(1+t) - \frac{\rho(x,y)^2}{2\delta t}\right]
\end{aligned}$$

for some constant $c(\delta) > 0$.

(2) Let μ be a probability measure. Applying Theorem 2.3.3(4) to $f(z) = p_t(x,z) \wedge n$ and letting $n \to \infty$, we obtain

$$\begin{aligned}
\log p_{2t}(x,y) &\ge -\frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)} + \int_M p_t(x,z)\log p_t(x,z)\mu(\mathrm{d}z) \\
&\ge -\frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)}.
\end{aligned}$$

□

2.5 Transportation-cost inequality

This section is essentially reorganized from [Wang (2004c, 2008a)]. Let (E, ρ) be a Polish space and μ a probability measure on E. Recall that for any $p \in [1, \infty)$, the L^p-Wasserstein distance (or the L^p-transportation cost) between two probability measures μ_1 and μ_2 is

$$W_p^\rho(\mu_1, \mu_2) := \left\{ \inf_{\Pi \in \mathcal{C}(\mu_1,\mu_2)} \int_{E\times E} \rho(x,y)^p \Pi(\mathrm{d}x, \mathrm{d}y) \right\}^{1/p}.$$

According to Corollary 4 in [Bolley and Villani (2005)], the transportation cost inequality

$$W_p^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1$$

holds for some $C > 0$ provided $\mu(\mathrm{e}^{\lambda \rho_o^{2p}}) < \infty$ for some $\lambda > 0$, where $o \in E$ is a fixed point. See also [Djellout *et al* (2004)] for $p = 1$. Furthermore, applying Theorem 1.15 in [Gozlan (2006)] with $c(x,y) = \rho(x,y)^q$ and $\alpha(r) = r^{2p}$, we conclude that for any $q \in [1, 2p)$,

$$W_q^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1 \tag{2.5.1}$$

holds for some $C > 0$ if and only if $\mu(\mathrm{e}^{\lambda \rho(o,\cdot)^{2p}}) < \infty$ for some $\lambda > 0$.

In general, however, this concentration of μ does not imply (2.5.1) for $q = 2p$. For instance, due to [Bakry *et al* (2007)], there exist plentiful examples with $\mu(\mathrm{e}^{\lambda \rho(o,\cdot)^2}) < \infty$ for some $\lambda > 0$ but the Poincaré inequality does not hold, which is weaker than the Talagrand inequality (see Section 7 in [Otto and Villani (2000)] or Section 4.1 in [Bobkov *et al* (2001)])

$$W_2^\rho(f\mu, \mu)^2 \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1. \tag{2.5.2}$$

Therefore, to derive (2.5.1) with $q = 2p$, one needs something stronger than the corresponding concentration of μ.

In this section, we aim to derive (2.5.1) with $q = 2p$, i.e.

$$W_{2p}^\rho(f\mu, \mu)^{2p} \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1, \tag{2.5.3}$$

on a connected complete Riemnnian manifold M for the Riemannian distance ρ, by using the super Poincaré inequalities

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in C_b^1(M), \tag{2.5.4}$$

where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function. The advantage of (2.5.3) is its *tensorization property*. More precisely, due to the induction argument in Section 3 in [Talagrand (1996)], if (2.5.3) holds for couples $(\mu_i, \rho_i), i = 1, \ldots, n$, then it also holds for the product measure $\mu_1 \times \ldots \times \mu_n$ and

$$\rho_n(x_1, \ldots, x_n; y_1, \ldots, y_n) := \left\{ \sum_{i=1}^n \rho_i(x_i, y_i)^{2p} \right\}^{1/2p}.$$

To derive (2.5.3) from (2.5.4), we first prove the *weighted log-Sobolev inequality*

$$\mu(f^2 \log f^2) \le C\mu(\alpha \circ \rho(o, \cdot)|\nabla f|^2), \quad \mu(f^2) = 1, \tag{2.5.5}$$

where α is a positive function determined by β in (2.5.4), then establish the transportation-cost inequality using log-Sobolev type inequalities, and finally, make links between the super Poincaré inequality and the transportation-cost inequality.

2.5.1 *From super Poincaré to weighted log-Sobolev inequalities*

We shall work with a diffusion framework as in [Bakry *et al* (2007)]. Let $(E, \mathcal{F}, \mu)$ be a separable complete probability space, and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ with domain $\mathcal{D}(\mathcal{E})$ in the following sense. Let $\mathcal{A}$ be a dense subspace of $\mathcal{D}(\mathcal{E})$ under the $\mathcal{E}_1^{1/2}$-norm ($\mathcal{E}_1(f,f) = \|f\|_2^2 + \mathcal{E}(f,f)$) which is composed of bounded functions, stable under products and composition with Lipschitz functions on $\mathbb{R}$. Let $\Gamma : \mathcal{A} \times \mathcal{A} \to \mathcal{B}_b(E)$ be a bilinear mapping, such that

(1) $\Gamma(f,f) \ge 0$ and $\mathcal{E}(f,g) = \mu(\Gamma(f,g))$ for $f,g \in \mathcal{A}$;
(2) $\Gamma(\phi \circ f, g) = \phi'(f)\Gamma(f,g)$ for $f,g \in \mathcal{A}$ and $\phi \in C_b^\infty(\mathbb{R})$;
(3) $\Gamma(fg,h) = g\Gamma(f,h) + f\Gamma(g,h)$ for $f,g,h \in \mathcal{A}$ with $fg \in \mathcal{A}$.

It is easy to see that the positivity and the bilinear property imply $\Gamma(f,g)^2 \le \Gamma(f,f)\Gamma(g,g)$ for all $f,g \in \mathcal{A}$. We shall denote by $\mathcal{A}_{\rm loc}$ the set of functions f such that for any integer n, the truncated function $f_n = \min(n, \max(f, -n))$ is in $\mathcal{A}$. For such functions, the bilinear map Γ automatically extends and shares the same properties than for functions in $\mathcal{A}$.

Next, let $\varrho \in \mathcal{A}_{\rm loc}$ be positive such that $\Gamma(\varrho,\varrho) \le 1$. We shall start from the super Poincaré inequality

$$\mu(f^2) \le r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0. \tag{2.5.6}$$

To derive the desired weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f,f)\alpha \circ \varrho), \quad \mu(f^2) = 1, \tag{2.5.7}$$

we shall also need the following Poincaré inequality

$$\mu(f^2) \le C_0\mathcal{E}(f,f) + \mu(f)^2 \tag{2.5.8}$$

for some $C_0 > 0$. Here and in what follows, the reference function f is taken from $\mathcal{A}$.

Theorem 2.5.1. *Assume* (2.5.8) *holds for some* $C_0 > 0$. *Then* (2.5.6) *implies* (2.5.7) *for some constant* $C > 0$ *and*

$$\alpha(s) := \sup\Big\{\eta(t):\ t \geq \frac{1}{\mu(\rho(o,\cdot) \geq s-2)}\Big\}, \quad s \geq 0,$$

where

$$\eta(s) = \big(\log(2s)\big)\big(1 \wedge \beta^{-1}(s/2)\big), \quad s \geq 1$$

for $\beta^{-1}(s) := \inf\{t \geq 0:\ \beta(t) \leq s\}$.

Proof. (a) Let $\Phi(s) = \mu(\varrho \geq s)$ which decreases to zero as $s \to \infty$. We may take $r_0 > 0$ such that

$$r_0(1 + \sup_{s\geq 1}\eta(s)) \leq \frac{1}{32} \tag{2.5.9}$$

and

$$\beta^{-1}(e^{r_0^{-1}}/4) \leq 1. \tag{2.5.10}$$

For a fixed number $r \in (0, r_0]$ we define $u_r = \Phi^{-1}(2e^{-r^{-1}})$ and let

$$\begin{aligned} h_n &= \big((\varrho - u_r - n)_+ \wedge 1\big)\big((n+2+u_r-\varrho)_+ \wedge 1\big), \\ \delta_n &= \Big(\log\frac{2}{\Phi(n+u_r)}\Big)\beta^{-1}\Big(\frac{1}{2\Phi(n+u_r)}\Big), \\ B_n &= \{n \leq \varrho - u_r \leq n+2\}, \quad n \geq 0. \end{aligned}$$

Then

$$\sum_{n=0}^{\infty} h_n^2 \geq \frac{1}{2}1_{\{\varrho \geq 1+u_r\}}. \tag{2.5.11}$$

By (2.5.6) and noting that

$$\mu(|f|h_n)^2 \leq \mu(f^2h_n^2)\mu(\varrho > n+u_r) \leq \mu(f^2h_n^2)\Phi(n+u_r),$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty}\mu(f^2h_n^2) &\leq \sum_{n=0}^{\infty}\Big\{r_n\mu(\Gamma(fh_n,fh_n)) + \beta(r_n)\mu(|f|h_n)^2\Big\} \\ &\leq \sum_{n=0}^{\infty}\Big\{\frac{2r_n}{\delta_n}\mu(\Gamma(f,f)\delta_n 1_{B_n}) + 2r_n\mu(f^2 1_{B_n}) + \beta(r_n)\Phi(n+u_r)\mu(f^2h_n^2)\Big\} \end{aligned}$$

for $r_n > 0$. Since by (2.5.10) and the definition of α

$$\alpha(s) \geq \delta_n \text{ for } s \leq n + 2 + u_r,$$

letting $r_n = \delta_n r$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \leq \sum_{n=0}^{\infty} \Big\{ & 2r\mu(\Gamma(f,f)\alpha \circ \varrho 1_{B_n}) + 2r\delta_n \mu(f^2 1_{B_n}) \\ & + \beta(r\delta_n)\Phi(n+u_r)\mu(f^2 h_n^2)\Big\}.\end{aligned} \tag{2.5.12}$$

Noting that

$$A := r \log \frac{2}{\Phi(n+u_r)} \geq r \log \frac{2}{\Phi(\Phi^{-1}(2e^{-r^{-1}}))} = 1,$$

we have

$$\beta(\delta_n r) = \beta\Big(A\beta^{-1}\Big(\frac{1}{2\Phi(n+u_r)}\Big)\Big) \leq \frac{1}{2\Phi(n+u_r)}.$$

Thus, by (2.5.12) and (2.5.9) and the fact that $\delta_n \leq \sup \eta$, we arrive at

$$\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \leq \sum_{n=0}^{\infty} 2r\mu(\Gamma(f,f)\alpha \circ \varrho 1_{B_n}) + \frac{1}{8}\mu(f^2) + \frac{1}{2}\sum_{n=0}^{\infty} \mu(f^2 h_n^2).$$

It follows from this and (2.5.11) that

$$\mu(f^2 1_{\{\varrho \geq 1+u_r\}}) \leq 16r\mu(\Gamma(f,f)\alpha \circ \varrho) + \frac{1}{2}\mu(f^2). \tag{2.5.13}$$

(b) On the other hand, since α is decreasing

$$\begin{aligned}&\mu(f^2 1_{\{\varrho \leq 1+u_r\}}) \leq \mu(f^2\{(2+u_r-\varrho)_+^2 \wedge 1\}) \\ &\leq 2s\mu(\Gamma(f,f)1_{\{\varrho \leq 2+u_r\}}) + 2s\mu(f^2) + \beta(s)\mu(|f|)^2 \\ &\leq \frac{2s}{\alpha(2+u_r)}\mu(\Gamma(f,f)\alpha \circ \varrho) + 2s\mu(f^2) + \beta(s)\mu(|f|)^2, \quad s > 0.\end{aligned}$$

Taking

$$s = r\alpha(2+u_r) \leq \frac{1}{32}$$

due to (2.5.9), we obtain

$$\mu(f^2 1_{\{\varrho \leq 1+u_r\}}) \leq 2r\mu(\Gamma(f,f)\alpha \circ \varrho) + \frac{1}{16}\mu(f^2) + \beta\big(r\alpha(2+u_r)\big)\mu(|f|)^2.$$

Since by (2.5.10) and the definitions of α and u_r

$$\begin{aligned}r\alpha(2+u_r) &\geq \Big(r \log \frac{2}{\Phi(\Phi^{-1}(2e^{-r^{-1}}))}\Big)\beta^{-1}\Big(\frac{1}{2\Phi(\Phi^{-1}(2e^{-r^{-1}}))}\Big) \\ &= \beta^{-1}\Big(\frac{e^{r^{-1}}}{4}\Big),\end{aligned}$$

we obtain

$$\mu(f^2 1_{\{\varrho\le 1+u_r\}}) \le 2r\mu(\Gamma(f,f)\alpha\circ\varrho) + \frac{1}{16}\mu(f^2) + \frac{e^{r^{-1}}}{4}\mu(|f|)^2.$$

Combining this with (2.5.13) we conclude that

$$\mu(f^2) \le 40r\mu(\Gamma(f,f)\alpha\circ\varrho) + e^{r^{-1}}\mu(|f|)^2, \quad r\in(0,r_0].$$

Therefore, there exists a constant $c>0$ such that

$$\mu(f^2) \le r\mu(\Gamma(f,f)\alpha\circ\varrho) + e^{c(1+r^{-1})}\mu(|f|)^2, \quad r>0. \tag{2.5.14}$$

According to e.g. Corollary 1.3 in [Wang (2000b)], this is equivalent to the defective weighted log-Sobolev inequality

$$\mu(f^2\log f^2) \le C_1\mu(\Gamma(f,f)\alpha\circ\varrho) + C_2, \quad \mu(f^2)=1. \tag{2.5.15}$$

(c) Finally, for any f with $\mu(f)=0$, it follows from (2.5.8) that

$$\begin{aligned}\mu(f^2) &\le \mu(f^2\{(1+R-\varrho)_+^2\wedge 1\}) + \|f\|_\infty^2\mu(\varrho\ge R)\\ &\le 2C_0\mu(\Gamma(f,f)1_{\{\varrho\le 1+R\}}) + (2C_0+1)\|f\|_\infty^2\mu(\varrho\ge R)\\ &\quad + \mu(f\{(\varrho-R)_+\wedge 1\})^2\\ &\le \frac{2C_0}{\alpha(1+R)}\mu(\Gamma(f,f)\alpha\circ\varrho) + 2(C_0+1)\|f\|_\infty^2\mu(\varrho\ge R), \quad R>0.\end{aligned}$$

Since $\mu(\varrho\ge R)\to 0$ as $R\to\infty$, the weighted weak Poincaré inequality

$$\mu(f^2) \le \tilde\beta(r)\mu(\Gamma(f,f)\alpha\circ\varrho) + r\|f\|_\infty^2, \quad r>0, \mu(f)=0$$

holds for some positive function $\tilde\beta$ on $(0,\infty)$. By Proposition 1.3 in [Röckner and Wang (2001)], this and (2.5.14) implies the weighted Poincaré inequality

$$\mu(f^2) \le C'\mu(\Gamma(f,f)\alpha\circ\varrho) + \mu(f)^2$$

for some constant $C'>0$. Combining this with (2.5.15) we obtain the desired weighted log-Sobolev inequality (2.5.7). □

2.5.2 *From log-Sobolev to transportation-cost inequalities*

Let $V\in C(M)$ be such that $\mu := e^{V(x)}dx$ is a probability measure, and let $A: TM\to TM$ be a continuous mapping such that $A(x)$ is a strictly positive definite, symmetric linear operator on T_xM for each $x\in M$. Define

$$\mathcal{E}(f,g) = \mu(\Gamma(f,g)), \quad f,g\in C_0^\infty(M),$$

where $\Gamma(f,g) := \langle A\nabla f, \nabla g\rangle$ for $f,g\in C^1(M)$. Then $(\mathcal{E}, C_0^\infty(M))$ is closable in $L^2(\mu)$. Indeed, we may assume that V and A are C^∞-smooth since the closability does not change if we replace V and A by smooth $\tilde{V}$ and $\tilde{A}$ such that $\|V - \tilde{V}\|_\infty < \infty$ and $c_1 A \le \tilde{A} \le c_2 A$ for some constant $c_1, c_2 > 0$. In the smooth case the Dirichlet form of the diffusion process generated by $L := \mathrm{div}(A\nabla) + A\nabla V$, which is symmetric in $L^2(\mu)$, is a closed extension to $(\mathcal{E}, C_0^\infty(M))$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the closure which is a Dirichlet form on $L^2(\mu)$.

Next, let ρ_A be the distance induced by A, i.e. for all $x, y \in M$,

$$\begin{aligned}\rho_A(x,y) &= \sup\{f(x) - f(y) : f \in C^1(M),\ \Gamma(f,f) \le 1\}\\ &= \inf\left\{\int_0^1 \sqrt{\langle A^{-1}\dot\gamma_s, \dot\gamma_s\rangle}\,\mathrm{d}s : \gamma \in C^1([0,1];M), \gamma_0 = x, \gamma_1 = y\right\}.\end{aligned}$$

We first consider the case where A and V are smooth. In this case, M with metric $g_A(X,Y) := \langle A^{-1}X, Y\rangle$ is a complete Riemannian manifold, and the associated Markov semigroup P_t of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is generated by $L = \mathrm{div}(A\nabla) + A\nabla V$. Let $f \in C_c^\infty(M) := \{f + C : f \in C_0^\infty(M), C \in \mathbb{R}\}$ such that $\mu(f) = 1$ and $\varepsilon^{-1} \ge f \ge \varepsilon$ for some $\varepsilon \in (0,1)$. Let $\mu_t := (P_t f)\mu$ which is a probability measure for each $t \ge 0$. Let us fix $t > 0$. To estimate the Wasserstein distance between μ_t and μ_{t+s} for $s > 0$, Otto and Villani constructed a coupling (for $A = \mathbf{I}$) in the following way. Let $\xi_{t+s}(x) := \nabla \log P_{t+s} f(x)$. Then the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_s = -(A\xi_{t+s}) \circ \phi_s, \qquad \phi_0 = \mathrm{I},\ s \ge 0 \tag{2.5.16}$$

has a unique solution. We will prove that

$$\pi_s(\mathrm{d}x, \mathrm{d}y) := \mu_t(\mathrm{d}x)\delta_{\phi_s(x)}(\mathrm{d}y) \tag{2.5.17}$$

provides a coupling of μ_t and μ_{t+s} which is called *Otto-Villani's coupling,* where $\delta_{\phi_s(x)}$ denotes the Dirac measure at point $\phi_s(x)$. This was done by Otto and Villani [Otto and Villani (2000)] under the assumption that $\mathrm{Ric}_{\nabla V}$ is bounded below. To avoid this additional assumption we follow the line of [Wang (2004c)] (see also [Wang (2005a)]).

Lemma 2.5.2. *Let V and A be smooth such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure and (M, ρ_A) is complete. For $f \in C_c^\infty(M)$ with $\varepsilon^{-1} \ge f \ge \varepsilon$ for some constant $\varepsilon \in (0,1)$, the unique solution to* (2.5.16) *is nonexplosive with $\rho(x, \phi_s(x)) \le c\sqrt{s(s+1)}$ for some $c > 0$, all $x \in M$ and all $s \ge 0$. Moreover, for each $s \ge 0, \phi_s : M \to M$ is a diffeomorphism whose inverse solves the equation*

$$\frac{\mathrm{d}}{\mathrm{d}u}\tilde\phi_u = \xi_{t+s-u} \circ \tilde\phi_u, \quad \tilde\phi_0 = I. \tag{2.5.18}$$

Proof. It suffices to prove for noncompact M. Let $x \in M$ be fixed, and let

$$\tau_x^n := \inf\{s \geq 0 : \rho_A(x, \phi_s(x)) \geq n\}, \quad n \geq 1.$$

If $\tau_x := \lim_{n\to\infty} \tau_x^n < \infty$, then there is a sequence $\{s_n\} \subset (0, \tau_x)$ such that $\rho_A(x, \phi_{s_n}(x)) \geq n$. But for $s < \tau_x$, it follows from the Kolmogorov equation (2.1.5) that

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}s}(\log(P_{t+s}f))(\phi_s(x)) &= -\Big\langle \xi_{t+s} \circ \phi_s, \frac{\mathrm{d}}{\mathrm{d}s}\phi_s(x)\Big\rangle - \Big(\frac{P_{t+s}Lf}{P_{t+s}f}\Big) \circ \phi_s(x) \\
&\geq \langle (A\xi_{t+s}) \circ \phi_s(x), \xi_{t+s} \circ \phi_s(x)\rangle - \frac{\|Lf\|_\infty}{\varepsilon}.
\end{aligned}$$

Then

$$\int_0^{s_n} \langle (A\xi_{t+s}) \circ \phi_s(x), \xi_{t+s} \circ \phi_s(x)\rangle \mathrm{d}s \leq \frac{\|Lf\|_\infty s_n}{\varepsilon} + \log \varepsilon^{-2}.$$

Therefore, letting $|X|_A := \sqrt{\langle A^{-1}X, X\rangle}$ for $X \in TM$, we obtain

$$\begin{aligned}
n^2 \leq \rho_A(x, \phi_{s_n}(x))^2 &= \Big(\int_0^{s_n} \Big[\frac{\mathrm{d}}{\mathrm{d}s}\rho_A(x, \phi_s(x))\Big]\mathrm{d}s\Big)^2 \\
&\leq \Big(\int_0^{s_n} \Big|\frac{\mathrm{d}}{\mathrm{d}s}\phi_s\Big|_A \mathrm{d}s\Big)^2 \leq s_n \int_0^{s_n} \langle (A\xi_{s+t}) \circ \phi_s(x), \xi_{s+t} \circ \phi_s(x)\rangle \,\mathrm{d}s \\
&\leq \frac{\|Lf\|_\infty s_n^2}{\varepsilon} + s_n \log \varepsilon^{-2}.
\end{aligned}$$

Letting $n \to \infty$ we prove that $\tau_x = \infty$. Moreover, replacing s_n by s we obtain that $\rho_A(x, \phi_s(x)) \leq c\sqrt{s(s+1)}$ for some $c > 0$ and all $s \geq 0, x \in M$.

Finally, for fixed $s > 0$, let $\{\tilde{\phi}_u : u \in [0, s]\}$ solve (2.5.18). It is easy to check that $\tilde{\phi}_s = \phi_s^{-1}$, the inverse map of ϕ_s. Indeed, one has $\phi_{s-u} = \tilde{\phi}_u \circ \phi_s$ (resp. $\tilde{\phi}_{s-u} = \phi_u \circ \tilde{\phi}_s$) for all $u \in [0, s]$, since both of them solve (2.5.18) (resp. (2.5.16)) with initial value ϕ_s (resp. $\tilde{\phi}_s$). Hence ϕ_s is a homeomorphism on M. □

Proposition 2.5.3. *In the situation of Lemma* 2.5.2 *let* ϕ_s *solve* (2.5.16), *then* (2.5.17) *determines a coupling* π_s *for* μ_t *and* μ_{t+s}, *i.e.* $\pi_s \in \mathcal{C}(\mu_t, \mu_{t+s})$.

Proof. It suffices to prove that for any $h \in C_0^1(M)$ one has

$$\int_M h \circ \phi_s^{-1} \mathrm{d}\mu_{t+s} = \int_M h \mathrm{d}\mu_t. \tag{2.5.19}$$

Letting $h_s := h \circ \phi_s^{-1}$, we have $h_s \circ \phi_s = h$ and hence,

$$\frac{\mathrm{d}}{\mathrm{d}s}h_s - \langle A\xi_{t+s}, \nabla h_s\rangle = 0.$$

Since h has compact support, by Lemma 2.5.2 and the completeness of (M,ρ_A), there is a compact set B such that $h_r|B^c = 0$ for all $r \in [0,s]$. Thus, by the symmetry of L,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\int_M h_r \mathrm{d}\mu_{t+r} &= \frac{\mathrm{d}}{\mathrm{d}r}\int_B h_r P_{t+r} f \mathrm{d}\mu \\ &= \int_B \Big[(P_{t+r}f)\frac{\mathrm{d}}{\mathrm{d}r}h_r + h_r(LP_{t+r}f)\Big]\mathrm{d}\mu \\ &= \int_B \Big[\frac{\mathrm{d}}{\mathrm{d}r}h_r - \langle A\xi_{t+r}, \nabla h_r\rangle\Big]\mathrm{d}\mu_{t+r} = 0.\end{aligned}$$

Therefore, (2.5.19) holds. □

We now adopt the above constructed coupling to establish transportation-cost inequality from the log-Sobolev inequality

$$\Psi(\mu(f^2\log f^2)) \le \mathcal{E}(f,f), \quad f \in C_b^1(M), \mu(f^2) = 1, \tag{2.5.20}$$

where $\Psi : [0,\infty) \to [0,\infty)$.

Theorem 2.5.4. *Let $\Psi \in C([0,\infty);[0,\infty))$ be increasing such that $\int_0^1 \Psi(s)^{-1/2}\mathrm{d}s < \infty$. Then* (2.5.20) *implies*

$$W_2^{\rho_A}(f^2\mu,\mu) \le \Phi(\mu(f^2\log f^2)), \quad \mu(f^2) = 1 \tag{2.5.21}$$

for

$$\Phi(r) := \frac{1}{2}\int_0^r \frac{\mathrm{d}s}{\sqrt{\Psi(s)}}, \quad r \ge 0.$$

In particular, if $\int_0^\infty \Psi(s)^{-1/2}\mathrm{d}s < \infty$ then (M,ρ_A) is compact with diameter $D \le \int_0^\infty \Psi(s)^{-1/2}\mathrm{d}s$.

Proof. By an approximation procedure mentioned above we assume that both V and A are smooth and $f \in C_c^\infty(M)$ such that $\varepsilon^{-1} \ge f^2 \ge \varepsilon$ for some $\varepsilon \in (0,1)$ and $\mu(f^2) = 1$. Let $o \in M$ be a fixed point and denote $\rho_{A,o} = \rho_A(o,\cdot)$. We first note that (2.5.20) with $\Psi(\infty) := \lim_{r\to\infty}\Psi(r) > 0$ implies that $\mu(\rho_{A,o}^2) < \infty$. Indeed, applying (2.5.20) for $f_{n,m} := \frac{(\rho_{A,o}-n)^+\wedge m}{c_{n,m}^{1/2}}, n,m > 0$, where $c_{n,m} := \mu([(\rho_{A,o}-n)^+\wedge m]^2)$ is the normalization, we obtain

$$\Psi(\mu(f_{n,m}^2\log f_{n,m}^2)) \le \frac{\mu(B_n^c)}{c_{n,m}}, \tag{2.5.22}$$

where $B_n := \{\rho_{A,o} \le n\}$. Since by Jensen's inequality

$$\mu(f_{n,m}^2\log f_{n,m}^2) = \mu(1_{B_n^c} f_{n,m}^2 \log f_{n,m}^2) \ge \log\mu(B_n^c)^{-1} \to \infty$$

as $n \to \infty$, there exists $n, c > 0$ such that

$$\Psi(\mu(f_{n,m}^2 \log f_{n,m}^2)) \geq c, \quad m > 0.$$

Thus, it follows from (2.5.22) that

$$c \leq \frac{\mu(B_n^c)}{c_{n,m}}, \quad m > 0.$$

Letting $m \to \infty$ we obtain $\mu((\rho_{A,o} - n)^{+2}) < \infty$ and hence, $\mu(\rho_{A,o}^2) < \infty$.

By Proposition 2.5.3 and (2.5.16) we have

$$\begin{aligned}
&\frac{1}{s^2} W_2^{\rho_A}(\mu_t, \mu_{t+s})^2 \leq \frac{1}{s^2} \int_M \rho_A(x, \phi_s(x))^2 \mu_t(\mathrm{d}x) \\
&\leq \int_M \left(\frac{1}{s} \int_0^s \left| \frac{\mathrm{d}}{\mathrm{d}r} \rho_A(x, \phi_r(x)) \right| \mathrm{d}r \right)^2 \mu_t(\mathrm{d}x) \\
&\leq \frac{1}{s} \int_0^s \mathrm{d}r \int_M \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{(P_{t+r}f)^2} (\phi_r(x)) \mu_t(\mathrm{d}x) \\
&= \frac{1}{s} \int_M \mathrm{d}\mu_t \int_0^s \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{(P_{t+r}f)^2} \circ \phi_r \mathrm{d}r.
\end{aligned} \tag{2.5.23}$$

Since due to (2.5.19) and $f \geq \varepsilon$ one has

$$\begin{aligned}
&\int_M \left\{ \frac{1}{s} \int_0^s \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{(P_{t+r}f)^2} \circ \phi_r \mathrm{d}r \right\} \mathrm{d}\mu_t \\
&= \frac{1}{s} \int_0^s \mathrm{d}r \int_M \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{P_{t+r}f} \mathrm{d}\mu \\
&\leq \frac{1}{\varepsilon s} \int_0^s \mathcal{E}(P_{t+r}f, P_{t+r}f) \mathrm{d}r \leq \frac{1}{\varepsilon} \mathcal{E}(f, f) < \infty,
\end{aligned}$$

we conclude that

$$\left\{ \frac{1}{s} \int_0^s \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{(P_{t+r}f)^2} \circ \phi_r \mathrm{d}r : \quad s \in [0, 1] \right\}$$

is uniformly integrable w.r.t. μ_t. Then it follows from (2.5.23) and the dominated convergence theorem that

$$\begin{aligned}
\limsup_{s \to 0+} \frac{W_2^{\rho_A}(\mu_t, \mu_{t+s})^2}{s^2} &\leq \int_M \mathrm{d}\mu_t \limsup_{s \to 0+} \frac{1}{s} \int_0^s \frac{\langle A\nabla P_{t+r}f, \nabla P_{t+r}f \rangle}{(P_{t+r}f)^p} \circ \phi_r \mathrm{d}r \\
&= \int_M \frac{\langle A\nabla P_t f, \nabla P_t f \rangle}{P_t f} \mathrm{d}\mu.
\end{aligned}$$

Combining this with (2.5.20) we arrive at

$$\limsup_{s \downarrow 0} \frac{1}{s} W_2^{\rho_A}(\mu_t, \mu_{t+s}) \leq \frac{2\mathcal{E}(\sqrt{P_t f^2}, \sqrt{P_t f^2})}{\sqrt{\Psi(\mu((P_t f^2) \log P_t f^2))}}. \tag{2.5.24}$$

Since $\mu(\rho_{A,o}^2) < \infty$ and $f^2 \in [\varepsilon, \varepsilon^{-1}]$, we have $W_2^{\rho_A}(\mu_t, \mu) < \infty$ for all $t \geq 0$ and

$$W_2^{\rho_A}(\mu, \mu_t) - W_2^{\rho_A}(\mu, \mu_{t+s}) \leq W_2^{\rho_A}(\mu_t, \mu_{t+s}),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mu((P_t f^2)\log P_t f^2) = -4\mathcal{E}\big(\sqrt{P_t f^2}, \sqrt{P_t f^2}\big), \quad t \geq 0.$$

Thus, letting $h_t := \mu((P_t f^2)\log P_t f^2)$ we obtain from (2.5.24) that

$$\begin{aligned}\frac{d^+}{\mathrm{d}t}\{-W_2^{\rho_A}(\mu, \mu_t)\} &:= \lim_{s\downarrow 0}\frac{1}{s}(W_2^{\rho_A}(\mu, \mu_t) - W_2(\mu, \mu_{t+s}))\\ &\leq -\frac{h_t'}{2\Psi(h_t)^{1/2}}, \quad t \geq 0.\end{aligned} \tag{2.5.25}$$

Since $P_t f^2 \to \mu(f^2) = 1$ in $L^2(\mu)$ as $t \to \infty$ as explained in the proof of Theorem 2.4.2(3), and since f^2 is bounded and $\mu(\rho_{A,o}^2) < \infty$, we have $h_t \to 0$ and $W_2(\mu, \mu_t) \to 0$ as $t \to \infty$. Therefore, (2.5.25) implies

$$W_2^{\rho_A}(\mu, f^2\mu) \leq -\frac{1}{2}\int_0^\infty \frac{h_t'\mathrm{d}t}{\Psi(h_t)^{1/2}} = \frac{1}{2}\int_0^{\mu(f^2\log f^2)} \frac{\mathrm{d}r}{\Psi(r)^{1/2}}. \tag{2.5.26}$$

This implies the first assertion.

Finally, for any $x \in M$, by taking f such that $f^2\mu \to \delta_x$ weakly we obtain from (2.5.26) that

$$W_2^{\rho_A}(\mu, \delta_x) \leq \frac{1}{2}\int_0^\infty \frac{\mathrm{d}r}{\Psi(r)^{1/2}}.$$

Hence the proof is completed since this implies

$$\begin{aligned}\rho_A(x,y) = W_2^{\rho_A}(\delta_x, \delta_y) &\leq W_2^{\rho_A}(\mu, \delta_x) + W_2^{\rho_A}(\mu, \delta_y)\\ &\leq \int_0^\infty \frac{\mathrm{d}r}{\Psi(r)^{1/2}}, \quad x, y \in M.\end{aligned}$$

□

2.5.3 *From super Poincaré to transportation-cost inequalities*

Theorem 2.5.5. *Let $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ for some $V \in C(M)$ be a probability measure on M. Assume that* (2.5.4) *holds for some positive decreasing $\beta \in C((0,\infty))$ such that*

$$\eta(s) := \big(\log(2s)\big)\big(1 \wedge \beta^{-1}(s/2)\big), \quad s \geq 1$$

is bounded, where $\beta^{-1}(s) := \inf\{t \geq 0 : \ \beta(t) \leq s\}$. Then (2.5.5) *holds for some $C > 0$ and*

$$\alpha(s) := \sup\Big\{\eta(t) : \ t \geq \frac{1}{\mu(\rho_o \geq s-2)}\Big\}, \quad s \geq 0.$$

Consequently,

$$W_2^{\rho_\alpha}(f\mu,\mu)^2 \le C\mu(f\log f), \quad f \ge 0, \mu(f^2) = 1, \tag{2.5.27}$$

where ρ_α is the Riemannian distance induced by the metric

$$\langle X,Y\rangle' := \frac{1}{\alpha \circ \rho_o(x)}\langle X,Y\rangle, \quad X,Y \in T_xM, \ x \in M. \tag{2.5.28}$$

Proof. Since α is bounded, the completeness of the original metric implies that of the weighted one given by (2.5.28). So, (2.5.27) follows from (2.5.5) according to Theorem 2.5.4 for $\Psi(r) = r/C$. Thus, by Theorem 2.5.1 with $E = M$ and $\Gamma(f,f) = |\nabla f|^2$, it suffices to prove that (2.5.4) implies the Poincaré inequality (2.5.8) for some $C_0 > 0$. Due to Theorem 1.6.2, the super Poincaré inequality (2.5.4) implies that the spectrum of L is discrete. Moreover, since M is connected, the corresponding Dirichlet form is irreducible so that 0 is a simple eigenvalue. Therefore, L possesses a spectral gap, which is equivalent to the desired Poincaré inequality. □

Since $\mu(\rho_o \ge s-2)$ can be estimated by using known concentration of μ induced by the super Poincaré inequality, one may determine the function α in Theorem 2.5.5 by using β only. To present specific consequences of this result, we need the following lemma in the spirit of [Marton (1986); Bobkov and Götze (1999)].

Proposition 2.5.6. *Let $\bar\rho : M \times M \to [0,\infty)$ be measurable. For any $r > 0$ and measurable set $A \subset M$ with $\mu(A) > 0$, let*

$$A_r = \{x \in M : \bar\rho(x,y) \ge r \text{ for some } y \in A\}, \quad r > 0.$$

If

$$W_1^{\bar\rho}(f\mu,\mu) \le \Phi \circ \mu(f\log f), \quad f \ge 0, \mu(f) = 1 \tag{2.5.29}$$

holds for some positive increasing $\Phi \in C([0,\infty))$, then

$$\mu(A_r) \le \exp\left[-\Phi^{-1}(r - \Phi \circ \log\mu(A)^{-1})\right], \quad r > \Phi \circ \log\mu(A)^{-1},$$

where $\Phi^{-1}(r) := \inf\{s \ge 0 : \ \Phi(s) \ge r\}$, $r \ge 0$.

Proof. It suffices to prove for $\mu(A_r) > 0$. In this case, letting $\mu_A = \mu(\cdot \cap A)/\mu(A)$ and $\mu_{A_r} = \mu(\cdot \cap A_r)/\mu(A_r)$, we obtain from (2.5.29) that

$$\begin{aligned} r &\le W_1^{\bar\rho}(\mu_A,\mu_{A_r}) \le W_1^{\bar\rho}(\mu_A,\mu) + W_1^{\bar\rho}(\mu_{A_r},\mu) \\ &\le \Phi \circ \log\mu(A)^{-1} + \Phi \circ \log\mu(A_r)^{-1}. \end{aligned}$$

This completes the proof. □

Corollary 2.5.7. *Let $\delta \in (1,2)$.*

(a) (2.5.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ implies (2.5.5) with

$$\alpha(s) := (1+s)^{-2(\delta-1)/(2-\delta)}$$

and (2.5.27) with $\rho_\alpha(x,y)$ replaced by

$$\rho(x,y)(1+\rho_o(x)\vee\rho_o(y))^{(\delta-1)/(2-\delta)}.$$

Consequently, it implies

$$W^{\rho}_{2/(2-\delta)}(f\mu,\mu)^{2/(2-\delta)} \le C\mu(f\log f), \quad \mu(f)=1, f\ge 0 \tag{2.5.30}$$

for some constant $C>0$.

(b) If $V \in C^2(M)$ with $\mathrm{Ric}-\mathrm{Hess}_V$ bounded below, then the following are equivalent to each other:

(1) *(2.5.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some constant $c>0$;*
(2) *(2.5.5) with $\alpha(s) := (1+s)^{-2(\delta-1)/(2-\delta)}$ for some $C>0$;*
(3) *(2.5.27) for some $C>0$ and $\rho_\alpha(x,y)$ replaced by $\rho(x,y)(1+\rho_o(x)\vee\rho_o(y))^{(\delta-1)/(2-\delta)}$;*
(4) *(2.5.30) for some $C>0$;*
(5) *$\mu(\exp[\lambda\rho_o^{2/(2-\delta)}])<\infty$ for some $\lambda>0$.*

Proof. (a) Let $\beta(r) = \mathrm{e}^{c(1+r^{-1/\delta})}$ for some $c>0$ and $\delta>1$. It is easy to see that

$$1\wedge\beta^{-1}(s/2) \le c_1\log^{-\delta}(2s), \quad s\ge 1$$

holds for some constant $c_1>0$. Next, by Corollary 5.3 in [Wang (2000b)], (2.5.4) with this specific function β implies

$$\mu(\rho_o \ge s-2) \le c_2\exp[-c_3 s^{2/(2-\delta)}], \quad s\ge 0$$

for some constants $c_2, c_3>0$. Therefore,

$$\alpha(s) \le c_4(1+s)^{-2(\delta-1)/(2-\delta)}, \quad s\ge 0$$

holds for some constant $c_4>0$.

On the other hand, for any $x_1,x_2\in M$ let $i\in\{1,2\}$ such that $\rho_o(x_i) = \rho_o(x_1)\vee\rho_o(x_2)$. Define

$$f(x) = \Big(\rho(x,x_i)\wedge\frac{\rho_o(x_i)}{2}\Big)(1+\rho_o(x_i))^{(\delta-1)/(2-\delta)}, \quad x\in\mathbb{R}^d.$$

Then

$$\begin{aligned}&\alpha\circ\rho_o|\nabla f|^2 \le c_4(1+\rho_o)^{-2(\delta-1)/(2-\delta)}|\nabla f|^2\\ &\le c_4 1_{\{\rho(o,x_i)/2\le\rho_o\le 3\rho(o,x_i)/2\}}(1+\rho_o)^{-2(\delta-1)/(2-\delta)}(1+\rho_o(x_i))^{2(\delta-1)/(2-\delta)}\\ &\le c_5\end{aligned}$$

for some constant $c_5 > 0$. Since by the triangle inequality $\rho_o(x_i) \geq \frac{1}{2}\rho(x_1, x_2)$, this implies that the intrinsic distance ρ_α satisfies

$$\begin{aligned}\rho_\alpha(x_1, x_2)^2 &\geq \frac{|f(x_1) - f(x_2)|^2}{c_5} \\ &\geq c_6\rho(x_1, x_2)^2(1 + \rho_o(x_1) \vee \rho_o(x_2))^{2(\delta-1)/(2-\delta)} \\ &\geq c_7\rho(x_1, x_2)^{2/(2-\delta)}\end{aligned}$$

for some constants $c_6, c_7 > 0$. Hence the proof of (a) is completed by Theorem 2.5.5.

(b) Now, assume that

$$\mathrm{Ric} - \mathrm{Hess}_V \geq -K$$

for some $K \geq 0$. By (a) and Proposition 2.5.6, which ensures the implication from (4) to (5), it suffices to deduce (1) from (5). Let

$$h(r) = \mu(\mathrm{e}^{r\rho_o^2}), \quad r > 0.$$

By Theorem 5.7 in [Wang (2000b)], the super Poincaré inequality (2.5.4) holds with

$$\beta(r) := c_0 \inf_{0<r_1<r} r_1 \inf_{s>0} \frac{1}{s} h(2K + 12s^{-1})\mathrm{e}^{s/r_1 - 1}, \quad r > 0$$

for some constant $c_0 > 0$. Since for any $\lambda > 0$ there exists $c(\lambda) > 0$ such that

$$rt^2 \leq \lambda t^{2/(2-\delta)} + c(\lambda)r^{1/(\delta-1)}, \quad r > 0,$$

it follows from (5) that

$$h(r) \leq c_1 \exp[c_1 r^{1/(\delta-1)}], \quad r > 0$$

for some constant $c_1 > 0$. Therefore,

$$\beta(r) \leq c_2 \inf_{0<r_1<r} r_1 \inf_{s>0} \frac{1}{s} \exp[c_2 s^{-1/(\delta-1)} + s/r_1], \quad r > 0$$

for some $c_2 > 0$. Taking $s = r^{(\delta-1)/\delta}$ and $r_1 = r$, we conclude that

$$\beta(r) \leq \mathrm{e}^{c(1+r^{-1/\delta})}, \quad r > 0$$

for some $c > 0$. Thus, (1) holds. $\square$

We remark that (2.5.4) with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ for some $c > 0$ is equivalent to the following $\log^\delta$-Sobolev inequality (see [Wang (2000a,b); Gong and Wang (2002); Wang (2005a)] for more general results on (2.5.4) and the F-Sobolev inequality)

$$\mu(f^2 \log^\delta(1 + f^2)) \leq C_1\mu(|\nabla f|^2) + C_2, \quad \mu(f^2) = 1.$$

Since due to Corollary 5.3 in [Wang (2000b)] if (2.5.4) holds with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $\delta > 2$ then M has to be compact, as a complement to Corollary 2.5.7 we consider the critical case $\delta = 2$ in the next Corollary.

Corollary 2.5.8. (2.5.4) *with* $\beta(r) = \exp[c(1+r^{-1/2})]$ *for some* $c > 0$ *implies* (2.5.5) *with* $\alpha(s) := \mathrm{e}^{-c_1 s}$ *for some* $c_1 > 0$ *and* (2.5.27) *with* $\rho_\alpha(x, y)$ *replaced by*

$$\rho(x, y)\mathrm{e}^{c_2[\rho_o(x)\vee\rho_o(y)]}$$

for some c_2. *If* Ric − Hess$_V$ *is bounded below, they are all equivalent to the concentration* $\mu(\exp[\mathrm{e}^{\lambda\rho_o}]) < \infty$ *for some* $\lambda > 0$.

Proof. The proof is similar to that of Corollary 2.5.7 by noting that (2.5.4) with $\beta(r) = \exp[c(1+r^{-1/2})]$ implies $\mu(\rho_o \geq s) \leq \exp[-c\mathrm{e}^{c_1 s}]$ for some $c_1 > 0$, see Corollary 5.3 in [Wang (2000b)]. □

Finally, we present two examples to illustrate the above results.

Example 2.5.1. Let Ric be bounded below. Let $V \in C(M)$ be such that $V + a\rho_o^r$ is bounded for some $a > 0$ and $r \geq 2$. By Corollaries 2.5 and 3.3 in [Wang (2000a)], (2.5.4) holds for $\beta(r) = \exp[c(1+r^{-\sigma/[2(\sigma-1)]}]$. Then Corollary 2.5.7 implies

$$W_r^\rho(f\mu, \mu)^\sigma \leq C\mu(f\log f), \quad f \geq 0, \mu(f) = 1$$

for some constant $C > 0$. In this inequality r could not be replaced by any larger number, since $W_r^\rho \geq W_1^\rho$ and for any $p \in [1, \infty)$ the inequality

$$W_1^\rho(f\mu, \mu)^p \leq C\mu(f\log f), \quad f \geq 0, \mu(f) = 1$$

implies $\mu(\mathrm{e}^{\lambda\rho_o^p}) < \infty$ for some $\lambda > 0$, which fails when $p > r$ for μ specified above.

Example 2.5.2. In the situation of Example 2.5.1 but let $V + \exp[c\rho_o]$ be bounded for some $c > 0$. Then by Corollaries 2.5 and 3.3 in [Wang (2000a)], (2.5.4) holds with $\beta(r) = \exp[c'(1+r^{-1/2})]$ for some $c' > 0$. Hence, by Corollary 2.5.8, there are some $c_1,\ C > 0$, such that

$$\inf_{\pi\in C(\mu, f\mu)} \int_{M\times M} \rho(x, y)^2 \mathrm{e}^{c_1\rho(x,y)}\pi(\mathrm{d}x, \mathrm{d}y) \leq C\mu(f\log f), \tag{2.5.31}$$

holds for all $f \geq 0, \mu(f) = 1$. On the other hand, it is easy to see from Proposition 2.5.6 that (2.5.31) implies

$$\mu(\exp[\exp(\lambda\rho_o)]) < \infty, \quad \lambda > 0,$$

which is the exact concentration property of the given measure μ.

2.5.4 *Super Poincaré inequality by perturbations*

In this subsection we aim to present explicit estimates on β in the super Poincaré inequality by perturbation from a given Nash inequality, which is in particular available if the injectivity radius of the manifold is positive or Ricci curvature is bounded below. See [Wang (2000a,b, 2005a); Cattiaux *et al* (2009)] for more results in this direction.

Let $o \in M$ be fixed. For any $\lambda > 0$, let

$$\begin{aligned}\lambda(r) &= \inf \sigma(-L|_{B(o,r)^c})\\ &= \inf\{\mu(|\nabla f|^2) : f \in C_0^1(M), \mu(f^2) = 1, f|_{B(o,r)} = 0\}.\end{aligned} \tag{2.5.32}$$

According to [Wang (2000a)], we have $\lambda(\infty) := \lim_{r\to\infty} \lambda(r) = \infty$ if and only if the super Poincaré inequality

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in C_b^1(M) \tag{2.5.33}$$

holds for some decreasing function $\beta : (0,\infty) \to (0,\infty)$. In the following result, we estimate β by using λ and a prior Nash inequality.

Theorem 2.5.9. *Let $\lambda(r)$ be defined by* (2.5.32). *Assume that there exists a locally Lipschitz continuous function W on M such that the following Nash inequality holds for $\mathrm{d}\nu = \exp[W]\mathrm{d}x$ and some $p > 0$:*

$$\nu(f^2) \le c\{\nu(|\nabla f|^2) + \nu(f^2)\}^{p/(p+2)}, \quad \nu(|f|) = 1. \tag{2.5.34}$$

Put $\phi(r) = \sup_{B(o,r)} \exp[W - V]$ and let ψ be an increasing function such that

$$\psi(r) \ge \frac{1}{4}\{|\nabla W|^2 - |\nabla V|^2 - 2\Delta(V - W)\} \quad \text{on } B(o,r)$$

in the distribution sense. If ψ is finite, then there exists $c > 0$ such that (2.5.33) *holds with*

$$\beta(r) = c\big[1 + \psi(2 + \lambda^{-1}(8r^{-1})) + r^{-1}\big]^{p/2}\phi(\lambda^{-1}(8r^{-1}) + 2), \ r \le 1, \tag{2.5.35}$$

where $\lambda^{-1}(r) = \inf\{s \ge 0 : \lambda(s) \ge r\}$. Consequently, letting $\gamma(r) = -\sup_{B(o,r)^c} L\rho_o$, $r > 0$, the result remains true with λ replaced by $(\gamma^+)^2/4$.

Proof. By (2.5.34) and Corollary 1.6.11, there exists $c_1 > 0$ such that

$$\nu(f^2) \le r\nu(|\nabla f|^2) + c_1(1 + r^{-1})^{p/2}, \quad \nu(|f|) = 1. \tag{2.5.36}$$

For any $R > 0$, let $h = (\rho - R)^+ \wedge 1$, we then have, for any $f \in C_0^\infty(M)$,

$$\mu(f^2h^2) \le \frac{2}{\lambda(R)}\mu(|\nabla f|^2 + f^2). \tag{2.5.37}$$

Next, let $h_1 = (\rho - R - 2)^- \wedge 1$, and assume that $\mu(|f|) = 1$. It follows from (2.5.36) with test function $fh_1 \exp[(V-W)/2]$ that

$$\begin{aligned}\mu(f^2h_1^2) &\le r\mu\Big(\frac{1}{4}(fh_1)^2|\nabla(V-W)|^2 + \frac{1}{2}\langle\nabla(fh_1)^2, \nabla(V-W)\rangle \\ &\quad + |\nabla(fh_1)|^2\Big) + c_1(1+r^{-1})^{p/2}\mu\big(|fh_1|\exp[(W-V)/2]\big)^2 \\ &\le 2r\mu(|\nabla f|^2 + f^2) + \frac{r}{4}\mu\big(f^2h_1^2\{|\nabla W|^2 - |\nabla V|^2 - 2\Delta(V-W)\}\big) \\ &\quad + c_1(1+r^{-1})^{p/2}\phi(R+2) \\ &\le 2r\mu(|\nabla f|^2) + \big[2r + r\psi(R+2)\big]\mu(f^2) + c_1(1+r^{-1})^{p/2}\phi(R+2).\end{aligned}$$

By combining this with (2.5.37), we obtain

$$\mu(f^2) \le \frac{2(\lambda(R)^{-1} + r)\mu(|\nabla f|^2) + c_1(1+r^{-1})^{p/2}\phi(R+2)}{[1 - 2\lambda(R)^{-1} - 2r - r\psi(R+2)]^+}. \tag{2.5.38}$$

For any $\varepsilon \in (0,1]$, put

$$R = \lambda^{-1}\Big(\frac{4(1+\varepsilon)}{\varepsilon}\Big), \quad r = \frac{\varepsilon}{4(1+\varepsilon) + 2\varepsilon\psi(R+2)}.$$

Then (2.5.38) implies that

$$\mu(f^2) \le \varepsilon\mu(|\nabla f|^2) + \beta(\varepsilon)$$

with β determined by (2.5.35). Finally, if ∂M is either convex or empty, by Cheeger's inequality, $\lambda(r) \ge \frac{1}{4}(\gamma(r)^+)^2$, the second assertion then follows. $\square$

Remark. According to Croke's isoperimetric inequality [Croke (1980)], when $\partial M = \emptyset$, (2.5.34) holds for $W = 0$ and $p = d$ provided either $\mathrm{i}(M) = \infty$ or $\mathrm{i}(M) > 0$ and the Ricci curvature is bounded from below, where $\mathrm{i}(M)$ denotes the injectivity radius of M. Here, we present below a result based on a result due to [Wang (2001)]: (2.5.34) holds for $p = d$ and $W = c\rho$ whenever $\mathrm{Ric} \ge -K$ for some $K \ge 0$ and $c > \sqrt{(d-1)K}$.

Corollary 2.5.10. *Assume that* $\mathrm{Ric}(X,X) \ge -K|X|^2$ *for some* $K \ge 0$ *and all* $X \in TM$. *If* ∂M *is either convex or empty, then the results in Theorem* 2.5.9 *hold for* $p = d$ *and any smooth* W *with* $\|W - c\rho\|_\infty < \infty$ *for some* $c > \sqrt{(d-1)K}$. *Consequently, consider* $V = -\alpha\rho^\delta (\alpha > 0, \delta > 1)$, (2.5.33) *holds with*

$$\beta(r) = \exp[c'(1 + r^{-\lambda})]$$

for some $c' > 0$ *if and only if* $\lambda \ge \delta/[2(\delta - 1)]$. *Moreover, if* $V = -\exp[\alpha\rho]$ *for some* $\alpha > 0$, *then* (2.5.33) *holds with the above* β *for* $\lambda = \frac{1}{2}$.

Proof. The first assertion follows from Theorem 2.5.9 and the above Remark. Now, consider $V = -\alpha\rho^\delta (\alpha > 0, \delta > 1)$. Since the Ricci curvature is bounded from below, there exists $c_1 > 0$ such that $L\rho \le -c_1\rho^{\delta-1}$ for big ρ. Then $\lambda(r) \ge \frac{c_1^2}{4} r^{2(\delta-1)}$ for big r. Therefore, there exists $c_2 > 0$ such that

$$\lambda^{-1}(8r^{-1}) \le c_2 r^{-1/(2(\delta-1))}, \quad r \le 1.$$

Next, By Green and Wu's approximation theorem [Greene and Wu (1979)], there exists globally Lipschitz function $W \in C^\infty$ such that $\|W - c\rho\|_\infty \le 1$ and that ΔW is bounded from above. Then, there exists $c_3 > 0$ such that

$$\phi(r) \le \exp[c_3 r^\delta], \quad \psi(r) \le c_3, \quad r \ge 1.$$

By Theorem 2.5.9, (2.5.33) holds with $\beta(r) = \exp[c'(1 + r^{-\delta/(2(\delta-1))})]$ for some $c' > 0$.

On the other hand, if (2.5.33) holds with $\beta(r) = \exp[c'(1 + r^{-\lambda})]$ for some $c' > 0$ and $\lambda < \delta/(2(\delta - 1))$, by the concentration of μ (see Corollary 5.1 in [Wang (2000b)]), $\mu(\exp[\varepsilon\rho^{2\lambda/(2\lambda-1)}]) < \infty$ for some $\varepsilon > 0$. This is impossible since $\frac{2\lambda}{2\lambda-1} > \delta$ and hence

$$\mu\big(\exp\left[\varepsilon\rho^{2\lambda/(2\lambda-1)}\right]\big) = Z^{-1}\int \exp\left[\varepsilon\rho^{2\lambda/(2\lambda-1)} - \rho^\delta\right]\mathrm{d}x = \infty$$

by the volume comparison theorem due to [Cheeger *et al* (1982)]. The proof for the case that $V = \exp[-\alpha\rho]$ is similar. □

Finally, we look at the super Poincaré inequality by perturbations. Obviously, (2.5.33) does not change qualitatively if V is perturbed by bounded functions. Our next result says that this is also true if the perturbation is Lipshitz continuous. For optimal perturbation results using growth conditions, we are referred to [Bakry *et al* (2007)] and references within.

Proposition 2.5.11. *Assume that* (2.5.33) *holds. If U is a Lipschitz function, then* (2.5.33) *also holds for* $\mathrm{d}\bar\mu := \exp[U]\mathrm{d}\mu$ *with*

$$\bar\beta(r) = c_1\beta(c_2(1 + r))^2$$

for some $c_1, c_2 > 0$.

Proof. By e.g. Corollary 5.1 in [Wang (2000b)], (2.5.33) implies that $\mu(\mathrm{e}^{|U|}) < \infty$ so that $\bar\mu$ is a finite measure since $|U| \le c(1 + \rho)$ for some $c > 0$. For any f with $\bar\mu(|f|) = 1$, applying (2.5.33) to $f\exp[U/2]$ we obtain

$$\bar\mu(f^2) \le 2r\bar\mu(|\nabla f|^2) + \frac{r}{2}\bar\mu(f^2|\nabla U|^2) + \beta(r)\bar\mu(\exp[-U/2]|f|)^2. \quad (2.5.39)$$

On the other hand,

$$\begin{aligned}\bar\mu(\exp[-U/2]|f|)^2 &\le \Big\{e^{N/2}\bar\mu(|f|)+\bar\mu\big(|f|e^{-U/2}1_{\{|U|>N\}}\big)\Big\}^2\\ &\le 2e^N+2\bar\mu(f^2)\mu(|U|>N)\\ &\le 2e^N+2\mu(e^{|U|})e^{-N}\bar\mu(f^2),\quad N>0.\end{aligned}$$

Since $\mu(e^{|U|})<\infty$, this implies that

$$\bar\mu(\exp[-U/2]|f|)^2 \le \varepsilon\bar\mu(f^2)+\frac{c}{\varepsilon},\quad \varepsilon>0 \tag{2.5.40}$$

holds for some constant $c>0$. Taking $\varepsilon=[2\beta(r)]^{-1}$ in (2.5.40) and then substituting it into (2.5.39), we obtain

$$\bar\mu(f^2)=\frac{4r\bar\mu(|\nabla f|^2)}{1-\|\nabla U\|_\infty^2 r}+\frac{4c\beta(r)^2}{1-\|\nabla U\|_\infty^2 r},\quad r<\frac{1}{\|\nabla U\|_\infty^2}.$$

This proves the proposition. □

2.6 Log-Sobolev inequality: Different roles of Ric and Hess

Let $Z=\nabla V$ for some $V\in C^2(M)$ such that $\mu(dx):=e^{V(x)}dx$ is a probability measure. In previous sections we have described some properties of the diffusion process by using the Bakry-Emery curvature $\mathrm{Ric}_Z=\mathrm{Ric}-\mathrm{Hess}_V$, in which Ric and $-$Hess are taking the same role. In this section, we intend to show that at least for functional inequalities (e.g. the log-Sobolev inequality), these two tensors indeed play very different roles (see also [Wang (2009a)]).

According to Theorem 2.4.1(1), if $\mathrm{Ric}-\mathrm{Hess}_V\ge K$ holds for some constant $K>0$, then the log-Sobolev inequality

$$\mu(f^2\log f^2)\le C\mu(|\nabla f|^2),\quad \mu(f^2)=1, f\in C^1(M) \tag{2.6.1}$$

holds for $C=2/K$. We aim to prove the log-Sobolev inequality for unbounded below $\mathrm{Ric}-\mathrm{Hess}_V$ by using conditions on Ric and Hess_V separately.

Since the log-Sobolev inequality implies $\mu(e^{\lambda\rho_o^2})<\infty$ for some $\lambda>0$, to ensure the log-Sobolev inequality a reasonable condition of Hess_V is

$$-\mathrm{Hess}_V\ge\delta \text{ outside a compact set} \tag{2.6.2}$$

holds for some constant $\delta>0$. Under this condition we are going to search for the weakest lower bound condition of Ric for the log-Sobolev inequality to hold. It turns out that under (2.6.2) the optimal curvature lower bound condition will be of type

$$\mathrm{Ric} \geq -C - r^2\rho_o^2 \tag{2.6.3}$$

for some constants $C, r > 0$, where r will be explicitly given by δ in (2.6.2), see Theorem 2.6.5 below for details.

As already shown in the proof of Theorem 2.4.2(3), to ensure the log-Sobolev inequality (equivalently, the hypercontractivity of P_t) we need to establish the Harnack inequality and to verify the concentration of μ. We first investigate the exponential estimate of the diffusion process, which turns out to provide reasonable concentration property of μ; then establish the Harnack inequality by using the coupling method developed in [Arnaudon *et al* (2006)].

2.6.1 *Exponential estimate and concentration of* μ

We first study the concentration of μ by using (2.6.2) and (2.6.3), for which we need to estimate $L\rho_o$ from above.

Lemma 2.6.1. *If* (2.6.2) *and* (2.6.3) *hold then there exists a constant* $C_1 > 0$ *such that*

$$L\rho_o^2 \leq C_1(1+\rho_o) - 2\big(\delta - r\sqrt{d-1}\big)\rho_o^2 \tag{2.6.4}$$

holds outside $\mathrm{cut}(o)$, *the cut-locus of* o. *If moreover* $\delta > r\sqrt{d-1}$ *then* $\mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty$ *for all* $\lambda < \frac{1}{2}(\delta - r\sqrt{d-1})$.

Proof. According to (2.6.3) and the Laplacian comparison theorem (Theorem 1.1.10),

$$\Delta\rho_o \leq \sqrt{(c+r^2\rho_o^2)(d-1)}\coth\left[\sqrt{(c+r^2\rho_o^2)/(d-1)}\ \rho_o\right]$$

holds outside $\mathrm{cut}(o)$. Thus, outside $\mathrm{cut}(o)$ one has

$$\begin{aligned}\Delta\rho_o^2 &\leq 2\rho_o\sqrt{(c+r^2\rho_o^2)(d-1)}\coth\left[\sqrt{(c+r^2\rho_o^2)/(d-1)}\ \rho_o\right] + 2\\ &\leq 2d + 2\rho_o\sqrt{(c+r^2\rho_o^2)(d-1)},\end{aligned} \tag{2.6.5}$$

where the second inequality follows from the fact that

$$r\cosh r \leq (1+r)\sinh r, \quad r \geq 0.$$

On the other hand, for $x \notin \mathrm{cut}(o)$ and U the unit tangent vector along the unique minimal geodesic ℓ form o to x, by (2.6.2) there exists a constant $c_1 > 0$ independent of x such that

$$\langle\nabla V, \nabla\rho_o\rangle(x) = \langle\nabla V, U\rangle(o) + \int_0^{\rho_o(x)} \mathrm{Hess}_V(U,U)(\ell_s)\mathrm{d}s \leq c_1 - \delta\rho_o(x).$$

Combining this with (2.6.5) we prove (2.6.4).

Finally, let $\delta > r\sqrt{d-1}$ and $0 < \lambda < \frac{1}{2}(\delta - r\sqrt{d-1})$. By (2.6.4) we have

$$\begin{aligned} L\mathrm{e}^{\lambda\rho_o^2} &\le \lambda\mathrm{e}^{\lambda\rho_o^2}\big(C_1(1+\rho_o) - 2\big(\delta - r\sqrt{d-1}\big)\rho_o^2 + 4\lambda\rho_o^2\big) \\ &\le c_2 - c_3\rho_o^2\mathrm{e}^{\lambda\rho_o^2} \end{aligned}$$

for some constants $c_2, c_3 > 0$. This implies (see Proposition 3.2 in [Bogachev, Röckner and Wang (2001)])

$$\int_M \rho_o^2\mathrm{e}^{\lambda\rho_o^2}\mathrm{d}\mu \le \frac{c_2}{c_3} < \infty.$$

□

Lemma 2.6.2. *Let X_t be the L-diffusion process with $X_0 = x \in M$. If* (2.6.2) *and* (2.6.3) *hold with $\delta > r\sqrt{d-1}$, then for any $\delta_0 \in (r\sqrt{d-1}, \delta)$ there exists a constant $C_2 > 0$ such that*

$$\begin{aligned} &\mathbb{E}\exp\left[\frac{(\delta_0 - r\sqrt{d-1})^2}{4}\int_0^T \rho_o(X_t)^2\mathrm{d}t\right] \\ &\le \exp\left[C_2T + \frac{1}{4}(\delta_0 - r\sqrt{d-1})\rho_o(x)^2\right], \end{aligned}$$

for all $T > 0, x \in M$.

Proof. By Lemma 2.6.1, we have

$$L\rho_o^2 \le C - 2(\delta_0 - r\sqrt{d-1})\rho_o^2$$

outside cut(o) for some constant $C > 0$. Then by Kendall's Itô formula [Kendall (1987)] we have

$$\mathrm{d}\rho_o^2(X_t) \le 2\sqrt{2}\rho_o(X_t)\mathrm{d}b_t + \big[C - 2(\delta_0 - r\sqrt{d-1})\rho_o^2(X_t)\big]\mathrm{d}t \qquad (2.6.6)$$

for some Brownian motion b_t on $\mathbb{R}$. In particular, the L-diffusion process is non-explosive (see Theorem 2.1.1), i.e.

$$\zeta_n := \inf\{t \ge 0 : \rho_o(X_t) \ge n\} \uparrow \infty \quad \text{as } n \uparrow \infty.$$

For any $\lambda > 0$ and $n \ge 1$, it follows from (2.6.6) that

$$\begin{aligned} &\mathbb{E}\exp\left[2\lambda\big(\delta_0 - r\sqrt{d-1}\big)\int_0^{T\wedge\zeta_n} \rho_o^2(X_t)\mathrm{d}t\right] \\ &\le \mathrm{e}^{\lambda\rho_o^2(x)+C\lambda T}\mathbb{E}\exp\left[2\sqrt{2}\,\lambda\int_0^{T\wedge\zeta_n} \rho_o(X_t)\mathrm{d}b_t\right] \\ &\le \mathrm{e}^{\lambda\rho_o^2(x)+C\lambda T}\left(\mathbb{E}\exp\left[16\lambda^2\int_0^{T\wedge\zeta_n} \rho_o^2(X_t)\mathrm{d}t\right]\right)^{1/2}, \end{aligned}$$

where in the last step we have used the inequality

$$\mathbb{E}e^{M_t} \le (\mathbb{E}e^{2\langle M\rangle_t})^{1/2}$$

for $M_t = 2\sqrt{2}\lambda\int_0^{t\wedge\zeta_n}\rho_o(X_s)\mathrm{d}b_s$. This follows immediately from the Schwarz inequality and the fact that $\exp[2M_t - 2\langle M\rangle_t]$ is a martingale. Thus, taking

$$\lambda = \frac{1}{8}(\delta_0 - r\sqrt{d-1})$$

we obtain

$$\begin{aligned}\mathbb{E}\exp&\left[\frac{1}{4}(\delta_0 - r\sqrt{d-1})^2\int_0^{T\wedge\zeta_n}\rho_o^2(X_t)\mathrm{d}t\right]\\&\le \exp\left[\frac{1}{4}(\delta_0 - r\sqrt{d-1})\rho_o^2(x) + C_2T\right]\end{aligned}$$

for some $C_2 > 0$. Then the proof is completed by letting $n \to \infty$. □

2.6.2 *Harnack inequality and the log-Sobolev inequality*

According to Theorem 1.3.7, to establish the Harnack inequality we first construct a coupling by change of measure. As shown in [Arnaudon *et al* (2006)] the underlying changed probability measure will be given by a Girsanov transform.

Let $T > 0$ and $x \ne y \in M$ be fixed. Due to Theorem 2.3.2 we consider the following coupling by parallel displacement (for simplicity, we assume that cut $= \emptyset$)

$$\begin{aligned}&\mathrm{d}X_t = \sqrt{2}\,u_t \circ \mathrm{d}B_t + \nabla V(X_t)\mathrm{d}t, \quad X_0 = x,\\&\mathrm{d}Y_t = \sqrt{2}\,\bar{u}_t \circ (\bar{u}_t^{-1}P_{X_t,Y_t}u_t\mathrm{d}B_t) + \{\nabla V(Y_t) + \xi_t U(X_t,Y_t)\}\mathrm{d}t, \quad Y_0 = y,\end{aligned}$$

where P_{X_t,Y_t} is the parallel transformation along the unique minimal geodesic from X_t to Y_t, $U(X_t,Y_t) = -\nabla\rho(X_t,\cdot)(Y_t)1_{\{X_t\ne Y_t\}}$, $\xi_t \ge 0$ is a Lipschitzian function of X_t, and

$$\tau := \inf\{t \ge 0 : X_t = Y_t\}.$$

Lemma 2.6.3. *Assume that* (2.6.2) *and* (2.6.3) *hold with* $\delta \ge 2r\sqrt{d-1}$. *Then there exists a constant* $C_3 > 0$ *independent of* x, y *and* T *such that* $X_T = Y_T$ *holds for* $\xi_t := C_3 + 2r\sqrt{d-1}\rho_o(X_t) + \frac{\rho(x,y)}{T}$.

Proof. According to Theorem 2.3.2, we have

$$\begin{aligned}\mathrm{d}\rho(X_t,Y_t) = \big\{&I(X_t,Y_t) + \langle\nabla V, \nabla\rho(\cdot,Y_t)\rangle(X_t)\\&+ \langle\nabla V, \nabla\rho(X_t,\cdot)\rangle(Y_t) - \xi_t\big\}\mathrm{d}t, \quad t < \tau.\end{aligned}\tag{2.6.7}$$

By (2.6.3) and letting

$$K(X_t, Y_t) = \sup_{\ell([0,\rho(X_t,Y_t)])} \{c + r^2\rho_o^2\},$$

where ℓ is the minimal geodesic from X_t to Y_t, as in the proof of Corollary 2.1.2 by taking $\{J_i\}$ such that $\{J_i, \dot{\ell}\}$ is orthonomal basis at X_t and Y_t, we obtain from

$$\begin{aligned} I(X_t, Y_t) &\le 2\sqrt{K(X_t, Y_t)(d-1)} \\ &\qquad \times \tanh\left[\frac{\rho(X_t, Y_t)}{2}\sqrt{K(X_t, Y_t)/(d-1)}\right]. \end{aligned} \tag{2.6.8}$$

Moreover, by (2.6.2) there exist two constants $r_0, r_1 > 0$ such that $-\mathrm{Hess}_V \ge \delta$ outside $B(o, r_0)$ but $\le r_1$ on $B(o, r_0)$, where $B(o, r_0)$ is the closed geodesic ball at o with radius r_0. Since the length of ℓ contained in $B(o, r_0)$ is less than $2r_0$, we conclude that

$$\begin{aligned} &\langle \nabla V, \nabla\rho(\cdot, Y_t)\rangle(X_t) + \langle \nabla V, \nabla\rho(X_t, \cdot)\rangle(Y_t) \\ &= \int_0^{\rho(X_t,Y_t)} \mathrm{Hess}_V(\dot{\ell}_s, \dot{\ell}_s)\mathrm{d}s \le 2r_0r_1 - (\rho(X_t, Y_t) - 2r_0)^+\delta \\ &\le c_1 - \delta\rho(X_t, Y_t) \end{aligned}$$

for some constant $c_1 > 0$. Combining this with (2.6.7), (2.6.8) and

$$\xi_t = C_3 + 2r\sqrt{d-1}\,\rho_o(X_t) + \frac{\rho(x, y)}{T},$$

we arrive at

$$\begin{aligned} \mathrm{d}\rho(X_t, Y_t) \le \Big\{&2\sqrt{K(X_t, Y_t)(d-1)} + c_1 - \delta\rho(X_t, Y_t) \\ &- C_3 - 2r\sqrt{d-1}\rho_o(X_t) - \frac{\rho(x, y)}{T}\Big\}\mathrm{d}t \end{aligned}$$

for $t < \tau$. Noting that

$$\begin{aligned} \sqrt{K(X_t, Y_t)} &\le \left(c + r^2\big[\rho_o(X_t) + \rho(X_t, Y_t)\big]^2\right)^{1/2} \\ &\le \sqrt{c} + r\big[\rho_o(X_t) + \rho(X_t, Y_t)\big] \end{aligned}$$

and $\delta \ge 2r\sqrt{d-1}$, one has

$$2\sqrt{K(X_t, Y_t)(d-1)} - \delta\rho(X_t, Y_t) - 2r\sqrt{d-1}\rho_o(X_t) \le 2\sqrt{c(d-1)}.$$

Thus, when $C_3 \ge c_1 + 2\sqrt{c(d-1)}$ we have

$$\mathrm{d}\rho(X_t, Y_t) \le -\frac{\rho(x, y)}{T}\mathrm{d}t, \quad t < \tau$$

so that

$$0 = \rho(X_\tau, Y_\tau) \le \rho(x, y) - \int_0^\tau \frac{\rho(x, y)}{T}\mathrm{d}t = \frac{T - \tau}{T}\rho(x, y)$$

which implies that $\tau \le T$ and hence, $X_T = Y_T$. □

Proposition 2.6.4. *Assume that* (2.6.2) *and* (2.6.3) *hold with* $\delta > (1+\sqrt{2})r\sqrt{d-1}$. *Then there exist* $C>0$ *and* $p>1$ *such that*

$$(P_T f(y))^p \le (P_T f^p(x)) \exp\Big[\frac{C}{T}\rho(x,y)^2 + C(T+\rho_o(x)^2)\Big] \tag{2.6.9}$$

holds for all $x, y \in M, T>0$ *and nonnegative* $f \in C_b(M)$.

Proof. According to Lemma 2.6.3, we take

$$\xi_t = C_3 + 2r\sqrt{d-1}\rho_o(X_t) + \frac{\rho(x,y)}{T}$$

such that $\tau \le T$ and $X_T = Y_T$. Obviously, Y_t solves the equation

$$\mathrm{d}Y_t = \sqrt{2}\,\bar{u}_t \circ \mathrm{d}\tilde{B}_t + \nabla V(Y_t)\mathrm{d}t$$

for

$$\mathrm{d}\tilde{B}_t := \mathrm{d}B_t + \frac{1}{\sqrt{2}}\bar{u}_t^{-1}\xi_t U(X_t, Y_t)1_{\{t<\tau\}}\mathrm{d}t.$$

By the Girsanov theorem and the fact that $\tau \le T$, the process $\{\tilde{B}_t : t \in [0,T]\}$ is a d-dimensional Brownian motion under the probability measure $R\mathbb{P}$ for

$$R := \exp\Big[-\frac{1}{\sqrt{2}}\int_0^\tau \langle P_{X_t,Y_t}u_t\mathrm{d}B_t, \xi_t U(X_t,Y_t)\rangle - \frac{1}{4}\int_0^\tau \xi_t^2\mathrm{d}t\Big].$$

By Theorem 1.3.7, we have

$$(P_T f(y))^p \le (P_T f^p(x))(\mathbb{E}R^{p/(p-1)})^{p-1}. \tag{2.6.10}$$

Since for any continuous exponential integrable martingale M_t and any $\beta, p > 1$, the process $\exp[\beta p M_t - \frac{p^2\beta^2}{2}\langle M\rangle_t]$ is a martingale, by the Hölder inequality one has

$$\begin{aligned}\mathbb{E}e^{\beta M_t - \frac{\beta}{2}\langle M\rangle_t} &= \mathbb{E}\big[e^{\beta M_t - \frac{\beta^2 p}{2}\langle M\rangle_t}\cdot e^{\frac{\beta(\beta p-1)}{2}\langle M\rangle_t}\big]\\ &\le \mathbb{E}\big(e^{\frac{\beta p(\beta p-1)}{2(p-1)}\langle M\rangle_t}\big)^{(p-1)/p}.\end{aligned} \tag{2.6.11}$$

By taking $\beta = p/(p-1)$ we obtain

$$(\mathbb{E}R^{\frac{p}{(p-1)}})^{p-1} \le \Big\{\mathbb{E}\exp\Big[\frac{pq(pq-p+1)}{8(q-1)(p-1)^2}\int_0^T \xi_t^2\mathrm{d}t\Big]\Big\}^{\frac{(p-1)(q-1)}{q}} \tag{2.6.12}$$

holds for all $q>1$. Since $\delta > (1+\sqrt{2})r\sqrt{d-1}$, we may take $\delta_0 \in ((1+\sqrt{2})r\sqrt{d-1}, \delta)$, small $\varepsilon' > 0$ and large $C_4 > 0$, independent of T, x and y, such that

$$\begin{aligned}\xi_t^2 &= \Big(C_3 + 2r\sqrt{d-1}\rho_o(X_t) + \frac{\rho(x,y)}{T}\Big)^2\\ &\le (1-\varepsilon')\Big[C_4 + \frac{C_4\rho(x,y)^2}{T^2} + 2(\delta_0 - r\sqrt{d-1})^2\rho_o(X_t)^2\Big]\end{aligned}$$

holds. Moreover, since

$$\lim_{q\downarrow 1}\lim_{p\uparrow\infty}\frac{pq(pq-p+1)}{8(q-1)(p-1)^2}=\frac{1}{8}, \tag{2.6.13}$$

there exist $p, q > 1$ such that

$$\begin{aligned}&\frac{pq(pq-p+1)}{8(q-1)(p-1)^2}\int_0^T \xi_t^2\mathrm{d}t\\&\le C_4T+\frac{C_4\rho(x,y)^2}{T}+\frac{(\delta_0-r\sqrt{d-1})^2}{4}\int_0^T\rho_o(X_t)^2\mathrm{d}t.\end{aligned}$$

Combining this with (2.6.12) and Lemma 2.6.2, we obtain

$$(\mathbb{E}R^{p/(p-1)})^{p-1}\le\exp\Big[C_5T+\frac{C_5\rho(x,y)}{T}+C_5\rho_o(x)^2\Big],\quad T>0, x\in M$$

for some constant $C_5 > 0$. This completes the proof by (2.6.10). □

Theorem 2.6.5. *Assume that* (2.6.2) *and* (2.6.3) *hold for some constants* $c, \delta, r > 0$ *with* $\delta > (1+\sqrt{2})r\sqrt{d-1}$. *Then* (2.6.1) *holds for some* $C > 0$.

Proof. By Proposition 2.6.4, let $p > 1$ and $C > 0$ such that (2.6.9) holds. Since $\delta > r\sqrt{d-1}$, we may take $T > 0$ such that

$$\frac{C}{T}\le\varepsilon:=\frac{1}{8}\big(\delta-r\sqrt{d-1}\big).$$

Then for any nonnegative $f\in C_b(M)$ with $\mu(f^p)=1$, since μ is P_T-invariant, it follows from (2.6.9) that

$$\begin{aligned}1&=\int_M P_Tf^p(x)\mu(\mathrm{d}x)\\&\ge(P_Tf(y))^p\int_M \mathrm{e}^{-\varepsilon\rho(x,y)^2-C(1+\rho_o(x)^2)}\mu(\mathrm{d}x)\\&\ge(P_Tf(y))^p\int_{\{\rho_o\le1\}}\mathrm{e}^{-\varepsilon(1+\rho_o(y))^2-2C}\mu(\mathrm{d}x)\\&\ge\varepsilon'(P_Tf(y))^p\exp[-2\varepsilon\rho_o(y)^2],\qquad y\in M\end{aligned}$$

for some constant $\varepsilon' > 0$. Thus,

$$\int_M(P_Tf(y))^{2p}\mu(\mathrm{d}y)\le\frac{1}{\varepsilon'}\int_M\mathrm{e}^{4\varepsilon\rho_o(y)^2}\mu(\mathrm{d}y)<\infty$$

according to Lemma 2.6.1. This implies that

$$\|P_T\|_{L^p(\mu)\to L^{2p}(\mu)}<\infty.$$

Therefore, the log-Sobolev inequality (2.6.1) holds as explained in the proof of Theorem 2.4.2(3). □

To verify the sharpness of this theorem, let $\theta_0 > 0$ be the smallest positive constant such that for any connected complete non-compact Riemannian manifold M and $V \in C^2(M)$ such that $\int_M e^{V(x)}dx = 1$, the conditions (2.6.2) and (2.6.3) with $\delta > r\theta_0\sqrt{d-1}$ imply (2.6.1) for some $C > 0$. Due to Theorem 2.6.5 and the following example, we conclude that

$$\theta_0 \in [1, 1+\sqrt{2}].$$

The exact value of θ_0 is however unknown.

Example 2.6.1. Let $M = \mathbb{R}^2$ be equipped with the rotationally symmetric metric

$$ds^2 = dr^2 + \{re^{kr^2}\}^2 d\theta^2$$

under the polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{S}^1$ at 0, where $k > 0$ is a constant. Then (see e.g. [Gong and Wang (2002)])

$$\mathrm{Ric} = -\frac{\frac{d^2}{dr^2}(re^{kr^2})}{re^{kr^2}} = -4k - 4k^2r^2.$$

Thus, (2.6.3) holds for $r = 2k$. Next, take $V = -k\rho_o^2 - \lambda(\rho_o^2+1)^{1/2}$ for some $\lambda > 0$. By the Hessian comparison theorem and the negativity of the sectional curvature, we obtain (2.6.2) for $\delta = 2k$. Since $d = 2$ and

$$e^{V(x)}dx = re^{-\lambda(1+r^2)^{1/2}}drd\theta, \tag{2.6.14}$$

one has $Z < \infty$ and $\delta = 2k = r\sqrt{d-1}$. But the log-Sobolev inequality is not valid since by Herbst's inequality it implies $\mu(e^{r\rho_o^2}) < \infty$ for some $r > 0$, which is however not the case due to (2.6.14). Since in this example one has $\delta > r\theta\sqrt{d-1}$ for any $\theta < 1$, according to the definition of θ_0 we conclude that $\theta_0 \geq 1$.

2.6.3 *Hypercontractivity and ultracontractivity*

Recall that P_t is called supercontractive if $\|P_t\|_{2\to4} < \infty$ for all $t > 0$ while ultracontractive if $\|P_t\|_{2\to\infty} < \infty$ for all $t > 0$ (see [Davies and Simon (1984)]). In the present framework these two properties are stronger than the hypercontractivity: $\|P_t\|_{2\to4} \leq 1$ for some $t > 0$, which is equivalent to (2.6.1) due to Gross [Gross (1976, 1993)].

Proposition 2.6.6. *Under* (2.6.2) *and* (2.6.3). *P_t is supercontractive if and only if*

$$\mu(\exp[\lambda\rho_o^2]) < \infty, \quad \lambda > 0,$$

while it is ultracontractive if and only if $\|P_t \exp[\lambda\rho_o^2]\|_\infty < \infty$ for all $t, \lambda > 0$.

Proof. The proof is similar to that of Theorem 2.3 in [Röckner and Wang (2003a)]. Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$. By (2.6.9) for $p = 2$ and noting that μ is P_t-invariant, we obtain

$$\begin{aligned} 1 &\geq (P_T f(y))^2 \int_M \exp\Big[-\frac{C}{T}\rho(x,y)^2 - C(T+\rho_o(x)^2)\Big]\mu(\mathrm{d}x) \\ &\geq (P_T f(y))^2 \exp\Big[-\frac{2C}{T}(\rho_o(y)^2+1) - C(T+1)\Big]\mu(B(o,1)). \end{aligned}$$

Hence, for any $T > 0$ there exists a constant $\lambda_T > 0$ such that

$$|P_T f| \leq \exp[\lambda_T(1+\rho_o^2)], \quad T > 0, \mu(f^2) = 1. \tag{2.6.15}$$

(1) If $\mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty$ for any $\lambda > 0$, (2.6.15) yields that

$$\|P_T\|_{2\to 4}^4 \leq \mu(\mathrm{e}^{4\lambda_T(1+\rho_o^2)}) < \infty, \quad T > 0.$$

Conversely, if P_t is supercontractive then the super log-Sobolev inequality (cf. [Davies and Simon (1984)])

$$\mu(f^2 \log f^2) \leq r\mu(|\nabla f|^2) + \beta(r), \quad r > 0, \mu(f^2) = 1$$

holds for some $\beta : (0,\infty) \to (0,\infty)$. By [Aida *et al* (1994)] (see also [Liu (2009); Röckner and Wang (2003a)]), this inequality implies $\mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty$ for all $\lambda > 0$.

(2) By (2.6.15) and the semigroup property,

$$\|P_T\|_{2\to\infty} \leq \big\|P_{T/2}\mathrm{e}^{\lambda_{T/2}(1+\rho_o^2)}\big\|_\infty < \infty, \quad T > 0$$

provided $\|P_t \mathrm{e}^{\lambda\rho_o^2}\|_\infty < \infty$ for any $t, \lambda > 0$. Conversely, since the ultracontractivity is stronger than the supercontractivity, it implies that $\mathrm{e}^{\lambda\rho_o^2} \in L^2(\mu)$ for any $\lambda > 0$ as explained above. Therefore,

$$\|P_t \mathrm{e}^{\lambda\rho_o^2}\|_\infty \leq \|P_t\|_{2\to\infty}\|\mathrm{e}^{\lambda\rho_o^2}\|_2 < \infty, \quad \lambda > 0.$$

Then the proof is completed. □

To derive explicit conditions for the supercontractivity and ultracontractivity, we consider the following stronger version of (2.6.2):

$$-\mathrm{Hess}_V \geq \Phi \circ \rho_o \text{ holds outside a compact subset of } M, \tag{2.6.16}$$

for a positive increasing function Φ with $\Phi(r) \uparrow \infty$ as $r \uparrow \infty$. We then aim to search for reasonable conditions on positive increasing function Ψ such that

$$\mathrm{Ric} \geq -\Psi \circ \rho_o \tag{2.6.17}$$

implies the supercontractivity and/or ultracontractivity.

Theorem 2.6.7. *If* (2.6.16) *and* (2.6.17) *hold for some increasing positive functions* Φ *and* Ψ *such that*

$$\lim_{r\to\infty}\Phi(r)=\lim_{r\to\infty}\frac{(\int_0^r\Phi(s)\mathrm{d}s)^2}{\Phi(r)}=\infty, \tag{2.6.18}$$

$$\sqrt{\Psi(r+t)(d-1)}\le\theta\int_0^r\Phi(s)\mathrm{d}s+\frac{1}{2}\int_0^{t/2}\Phi(s)\mathrm{d}s+C,\quad r,t\ge 0 \tag{2.6.19}$$

for some constants $\theta\in(0,1/(1+\sqrt{2}))$ *and* $C>0$. *Then* P_t *is supercontractive. If furthermore*

$$\int_1^\infty\frac{\mathrm{d}s}{\sqrt{s}\int_0^{\sqrt{r}}\Phi(u)\mathrm{d}u}<\infty, \tag{2.6.20}$$

then P_t *is ultracontractive. More precisely, for*

$$\Gamma_1(r):=\frac{1}{\sqrt{r}}\int_0^{\sqrt{r}}\Phi(s)\mathrm{d}s,\quad \Gamma_2(r):=\int_r^\infty\frac{\mathrm{d}\ s}{\sqrt{s}\int_0^{\sqrt{s}}\Phi(u)\mathrm{d}u},\qquad r>0,$$

(2.6.20) *implies*

$$\|P_t\|_{2\to\infty}\le\exp\Big[c+\frac{c}{t}\big(1+\Gamma_1^{-1}(c/t)+\Gamma_2^{-1}(t/c)\big)\Big]<\infty,\quad t>0 \tag{2.6.21}$$

for some constant $c>0$ *and*

$$\Gamma_1^{-1}(s):=\inf\{t\ge 0:\ \Gamma_1(t)\ge s\},\quad s\ge 0.$$

Proof. (a) Replacing $c+\rho_o^2$ by $\Psi\circ\rho_o$ and noting that $\mathrm{Hess}_V\le-\Phi\circ\rho_o$ for large ρ_o, the proof of Lemma 2.6.1 implies

$$L\rho_o^2\le c_1(1+\rho_o)-2\rho_o\bigg(\int_0^{\rho_o}\Phi(s)\mathrm{d}s-\sqrt{\Psi\circ\rho_o(d-1)}\bigg) \tag{2.6.22}$$

for some constant $c_1>0$. Combining this with (2.6.19) and noting that $\frac{1}{\rho_o}\int_0^{\rho_o}\Phi(s)\mathrm{d}s\to\infty$ as $\rho_o\to\infty$, we conclude that for any $\lambda>0$

$$\begin{aligned}Le^{\lambda\rho_o^2}&\le C-\frac{2\lambda\rho_o\sqrt{2}}{1+\sqrt{2}}\mathrm{e}^{\lambda\rho_o^2}\int_0^{\rho_o}\Phi(s)\mathrm{d}s+4\lambda^2\rho_o^2\mathrm{e}^{\lambda\rho_o^2}\\&\le C+C(\lambda)-\lambda\rho_o\mathrm{e}^{\lambda\rho_o^2}\int_0^{\rho_o}\Phi(s)\mathrm{d}s,\end{aligned} \tag{2.6.23}$$

where $C>0$ is a universal constant and

$$\begin{aligned}C(\lambda)&:=\sup_{r>0}r\mathrm{e}^{\lambda r^2}\bigg\{4\lambda^2r-\frac{\lambda}{(1+\sqrt{2})^2}\int_0^r\Phi(s)\mathrm{d}s\bigg\}\\&=\sup_{r^2\le\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)}r\mathrm{e}^{\lambda r^2}\bigg\{4\lambda^2r-\frac{\lambda}{(1+\sqrt{2})^2}\int_0^r\Phi(s)\mathrm{d}s\bigg\}\\&\le 4\lambda^2\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)\exp\big[\lambda\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)\big]\\&\le\exp\big[4\lambda+2\lambda\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)\big]<\infty.\end{aligned} \tag{2.6.24}$$

Therefore,

$$\mu(e^{\lambda\rho_o^2}) < \infty, \qquad \lambda > 0. \tag{2.6.25}$$

(b) By (2.6.19), (2.6.22) and Kendall's Itô formula [Kawabi (2005)] as in the proof of Lemma 2.6.2, we have

$$d\rho_o^2(X_t) \le 2\sqrt{2}\rho_o(X_t)db_t + \left(C_1 - \frac{2\sqrt{2}\rho_o(X_t)(1+\varepsilon)}{1+\sqrt{2}}\int_0^{\rho_o(X_t)}\Phi(s)ds\right)dt$$

for some constants $\varepsilon, C_1 > 0$, where X_t and b_t are in the proof of Lemma 2.6.2. Let

$$\varphi(r) = \int_0^r \frac{ds}{\sqrt{s}}\int_0^{\sqrt{s}}\Phi(t)dt, \quad r \ge 0. \tag{2.6.26}$$

We arrive at

$$\begin{aligned} d\varphi\circ\rho_o^2(X_t) &\le 2\sqrt{2}\rho_o(X_t)\varphi'\circ\rho_o^2(X_t)db_t + 4\rho_o^2(X_t)\varphi''\circ\rho_o^2(X_t)dt \\ &\quad + \varphi'\circ\rho_o^2(X_t)\left(C_1 - \frac{2\sqrt{2}\rho_o(X_t)(1+\varepsilon)}{1+\sqrt{2}}\int_0^{\rho_o(X_t)}\Phi(s)ds\right)dt. \end{aligned}$$

From (2.6.18) we see that

$$\frac{\rho_o\varphi''\circ\rho_o^2}{\varphi'\circ\rho_o^2\int_0^{\rho_o}\Phi(s)ds} \le \frac{\Phi\circ\rho_o}{2(\int_0^{\rho_o}\Phi(s)ds)^2}$$

which goes to zero as $\rho_o \to \infty$. Then there exists a constant $C_2 > C_1$ such that

$$\begin{aligned} d\varphi\circ\rho_o^2(X_t) &\le 2\sqrt{2}\left(\int_0^{\rho_o(X_t)}\Phi(s)ds\right)db_t + C_2dt \\ &\quad - \frac{2\sqrt{2}}{1+\sqrt{2}}\left(\int_0^{\rho_o(X_t)}\Phi(s)ds\right)^2dt. \end{aligned}$$

This implies that for any $\lambda > 0$

$$\begin{aligned} &\mathbb{E}\exp\left[\frac{2\sqrt{2}\,\lambda}{1+\sqrt{2}}\int_0^T\left(\int_0^{\rho_o(X_t)}\Phi(s)ds\right)^2dt\right] \\ &\le e^{C_2\lambda T+\lambda\varphi\circ\rho_o^2(x)}\mathbb{E}\exp\left[2\sqrt{2}\,\lambda\int_0^T\left(\int_0^{\rho_o(X_t)}\Phi(s)ds\right)db_t\right] \\ &\le e^{C_2\lambda T+\lambda\varphi\circ\rho_o^2(x)}\left(\mathbb{E}\exp\left[16\lambda^2\int_0^T\left(\int_0^{\rho_o(X_t)}\Phi(s)ds\right)^2dt\right]\right)^{1/2}. \end{aligned}$$

Taking

$$\lambda = \frac{\sqrt{2}}{8(1+\sqrt{2})},$$

we arrive at

$$\begin{aligned}&\mathbb{E}\exp\left[\frac{1}{2(1+\sqrt{2})^2}\int_0^T\Big(\int_0^{\rho_o(X_t)}\Phi(s)\mathrm{d}s\Big)^2\mathrm{d}t\right]\\&\le \mathrm{e}^{2C_2T+\varphi\circ\rho_o^2(x)\sqrt{2}/8(1+\sqrt{2})}.\end{aligned}\tag{2.6.27}$$

(c) Let $\gamma:[0,\rho(X_t,Y_t)]\to M$ be the minimal geodesic from X_t to Y_t. By (2.6.16) we have

$$\begin{aligned}&\langle\nabla V,\nabla\rho(\cdot,Y_t)\rangle(X_t)+\langle\nabla V,\nabla\rho(X_t,\cdot)\rangle(Y_t)\\&=\int_0^{\rho(X_t,Y_t)}\mathrm{Hess}_V(\dot\gamma_s,\dot\gamma_s)\mathrm{d}s\le C_3-\int_0^{\rho(X_t,Y_t)}\Phi\circ\rho_o(\gamma_s)\mathrm{d}s\\&\le C_3-\int_0^{\rho(X_t,Y_t)/2}\Phi(s)\mathrm{d}s.\end{aligned}\tag{2.6.28}$$

To understand the last inequality, we assume, for instance, that $\rho_o(X_t)\ge\rho_o(Y_t)$ so that by the triangle inequality,

$$\rho_o(\gamma_s)\ge\rho_o(X_t)-s\ge\rho(X_t,Y_t)/2-s,\quad s\in[0,\rho(X_t,Y_t)/2].$$

For the coupling constructed in the above subsection, one concludes from (2.6.28) and the proof of Lemma 2.6.3 that

$$\begin{aligned}\mathrm{d}\rho(X_t,Y_t)\le\Big\{&2\sqrt{K(X_t,Y_t)(d-1)}+C_4\\&-\int_0^{\rho(X_t,Y_t)/2}\Phi(s)\mathrm{d}s-\xi_t\Big\}\mathrm{d}t,\quad t<\tau\end{aligned}\tag{2.6.29}$$

holds for some constant $C_4>0$, where

$$K(X_t,Y_t):=\sup_{\ell([0,\rho(X_t,Y_t)])}\Psi\circ\rho_o\le\Psi(\rho_o(X_t)+\rho(X_t,Y_t))$$

and ℓ is the minimal geodesic from X_t to Y_t. Combining this (2.6.19) and (2.6.29), we obtain

$$\mathrm{d}\rho(X_t,Y_t)\le\Big\{C_4+2\theta\int_0^{\rho_o(X_t)}\Phi(s)\mathrm{d}s-\xi_t\Big\}\mathrm{d}t,\quad t<\tau.$$

So, taking

$$\xi_t=C_4+2\theta\int_0^{\rho_o(X_t)}\Phi(s)\mathrm{d}s+\frac{\rho(x,y)}{T},$$

we arrive at

$$\mathrm{d}\rho(X_t,Y_t)\le-\frac{\rho(x,y)}{T}\mathrm{d}t,\quad t<\tau.$$

This implies $\tau\le T$ and hence $X_T=Y_T$ a.s.

Combining (2.6.19) with (2.6.12) and (2.6.13) we conclude that for the present choice of ξ_t there exist $p, q, C_5 > 1$ such that

$$\left(\mathbb{E}R^{p/(p-1)}\right)^{q/(q-1)} \le \mathbb{E}\exp\left[\frac{1}{2(1+\sqrt{2})^2}\int_0^T \left(\int_0^{\rho_o(X_t)} \Phi(s)\mathrm{d}s\right)^2 \mathrm{d}t \right.$$
$$\left. + C_5 T + \frac{C_5}{T}\rho(x,y)^2\right].$$

Combining this with (2.6.27) and (2.6.10) we obtain

$$(P_T f(y))^p \le (P_T f^p(x)) \exp\left[CT + \frac{C}{T}\rho(x,y)^2 + C\varphi \circ \rho_o^2(x)\right] \tag{2.6.30}$$

holds for some $p, C > 1$, any positive $f \in C_b(M)$ and all $x, y \in M, T > 0$.

(d) For any positive $f \in C_b(M)$ with $\mu(f^p) = 1$, (2.6.30) implies that

$$(P_T f(y))^p \int_{B(o,1)} \exp\left[-CT - \frac{C}{T}\rho(x,y)^2 - C\varphi \circ \rho_o^2(x)\right]\mu(\mathrm{d}x) \le 1.$$

Therefore, there exists a constant $C' > 0$ such that

$$(P_T f(y))^p \le \exp\left[C'(1+T) + \frac{C'}{T}\rho_o(y)^2\right], \quad y \in M, T > 0. \tag{2.6.31}$$

Combining this with (2.6.25) we obtain

$$\|P_T\|_{p\to qp} < \infty, \quad T > 0, q > 1.$$

This is equivalent to the supercontactivity by the Riesz-Thorin interpolation theorem and $\|P_t\|_{1\to 1} = 1$. Thus, the first assertion holds.

(e) To prove (2.6.21), it suffices to consider $t \in (0,1]$ since $\|P_t\|_{2\to\infty}$ is decreasing in $t > 0$. So, below we assume that $T \le 1$. By (2.6.31) and the fact that $(P_{2T}f)^p \le P_T(P_T f)^p$, we have

$$\|P_{2T}\|_{p\to\infty} \le \|P_T \mathrm{e}^{2C'\rho_o^2/T}\|_\infty \mathrm{e}^{C'(1+T)}, \quad T > 0. \tag{2.6.32}$$

So, by the Riesz-Thorin interpolation theorem and $\|P_t\|_{1\to 1} = 1$, for the ultracontractivity it suffices to show that

$$\|P_T \mathrm{e}^{\lambda\rho_o^2}\|_\infty < \infty, \quad \lambda, T > 0. \tag{2.6.33}$$

Since Φ is increasing, it is easy to check that

$$\eta(r) := \sqrt{r}\int_0^{\sqrt{r}} \Phi(s)\mathrm{d}s, \quad r \ge 0$$

is convex, and so is $s \mapsto s\eta(\frac{\log s}{\lambda})$ for $\lambda > 0$. Thus, it follows from (2.6.23) and the Jensen inequality that

$$h_{\lambda,x}(t) := \mathbb{E}\mathrm{e}^{\lambda\rho_o^2(X_t)} < \infty, \quad X_0 = x \in M, \lambda, t > 0$$

and

$$\frac{\mathrm{d}^+}{\mathrm{d}t} h_{\lambda,x}(t) \le C + C(\lambda) - \lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)), \quad t > 0.$$

This implies (2.6.33) provided (2.6.20) holds. This can be done by considering the following two situations.

(1) Since $h_{\lambda,x}(t)$ is decreasing provided $\lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)) > C + C(\lambda)$, if

$$\lambda h_{\lambda,x}(0)\eta(\lambda^{-1}\log h_{\lambda,x}(0)) \le 2C + 2C(\lambda)$$

then

$$h_{\lambda,x}(t) \le \sup\{r \ge 1 : \lambda r\eta(\lambda^{-1}\log r) \le 2C + 2C(\lambda)\} \le \frac{1}{\lambda}(2C + 2C(\lambda)) + C''$$

for some constant $C'' > 0$.

(2) If $\lambda h_{\lambda,x}(0)\eta(\lambda^{-1}\log h_{\lambda,x}(0)) > 2C + 2C(\lambda)$, then $h_{\lambda,x}(t)$ is decreasing in t up to

$$t_\lambda := \inf\{t \ge 0 : \lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)) \le 2C + 2C(\lambda)\}.$$

Indeed,

$$\frac{\mathrm{d}^+}{\mathrm{d}t} h_{\lambda,x}(t) \le -\frac{\lambda}{2} h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)), \quad t \le t_\lambda.$$

Thus,

$$\int_{h_{\lambda,x}(T\wedge t_\lambda)}^\infty \frac{\mathrm{d}r}{r\eta(\lambda^{-1}\log r)} \ge \frac{\lambda}{2}(T \wedge t_\lambda).$$

This is equivalent to

$$\Gamma_2(\lambda^{-1}\log h_{\lambda,x}(T \wedge t_\lambda)) \ge \frac{1}{2}(T \wedge t_\lambda).$$

Hence,

$$h_{\lambda,x}(T \wedge t_\lambda) \le \exp\left[\lambda\Gamma_2^{-1}\left(\frac{1}{2}(T \wedge t_\lambda)\right)\right].$$

Since it is reduced to case (1) if $T > t_\lambda$ by regarding t_λ as the initial time, in conclusion we have

$$\sup_{x\in M} h_{\lambda,x}(T) \le \max\left\{\exp\left[\lambda\Gamma_2^{-1}(T/2)\right],\ C'' + \frac{1}{\lambda}(2C + 2C(\lambda))\right\}.$$

Therefore, (2.6.21) follows from (2.6.32), (2.6.24) with $\lambda = 2C'/T$, and the Riesz interpolation theorem. □

Finally, we note that a simple example for conditions in Theorem 2.6.7 to hold is

$$\Phi(s) = s^{p-1}, \quad \Psi(s) = \varepsilon s^{2p}$$

for $p > 1$ and small enough $\varepsilon > 0$. In this case P_t is ultracontractive with

$$\|P_t\|_{2\to\infty} \le \exp[c(1 + t^{-(p+1)/(p-1)})], \quad t > 0$$

for some $c > 0$.

2.7 Curvature-dimension condition and applications

In this section we characterize semigroup properties by using the following curvature-dimension condition

$$\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geq K|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C^\infty(M), \tag{2.7.1}$$

where $K \in \mathbb{R}$ and $n \geq d$ provide a curvature lower bound and a dimension upper bound of L respectively. When $Z = 0$ this condition is equivalent to $\mathrm{Ric} \geq -K$. In this case (2.7.1) holds for $n = d$. When $Z \neq 0$, n is essentially larger than d. Indeed, (2.7.1) is equivalent to

$$\mathrm{Ric}(X, X) - \langle \nabla_X Z, X\rangle \geq K|X|^2 + \frac{\langle Z, X\rangle^2}{n-d}, \quad X \in TM. \tag{2.7.2}$$

In particular, when $n = \infty$, (2.7.1) reduces to the curvature condition $\mathrm{Ric}_Z \geq K$.

When $n < \infty$, the curvature-dimension condition (2.7.1) has been used in the study of the Sobolev inequality, the first eigenvalue and the diameter estimates, and Li-Yau type Harnack inequalities see e.g. [Bakry and Qian (1999, 2000); Li (2005); Saloff-Coste (1994)] and references within for applications of the curvature-dimension condition. The purpose of this section is to present inequalities of P_t for (2.7.1), and to establish the corresponding transportation-cost inequalities. Results presented in this section are mainly due to [Wang (2011c)].

2.7.1 *Gradient and Harnack inequalities*

Correspondingly to Theorems 2.3.1 and 2.3.3 where a number of equivalent inequalities for the curvature lower bound are presented, the following result includes some equivalent inequalities for the curvature-dimension condition.

Theorem 2.7.1. *Let $K \in \mathbb{R}$ and $n \geq d$ be two constants. Then each of the following statements is equivalent to* (2.7.1)*:*

(1) $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t|\nabla f|^2 - \frac{2}{n}\int_0^t \mathrm{e}^{-2Ks} P_s(P_{t-s}Lf)^2\mathrm{d}s$ *holds for all* $f \in C_0^2(M), t \geq 0$.
(2) $|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t|\nabla f|^2 - \frac{1-\mathrm{e}^{-2Kt}}{Kn}(P_t Lf)^2$ *holds for all* $f \in C_0^2(M), t \geq 0$.
(3) $P_t f^2 - (P_t f)^2 \leq \frac{1-\mathrm{e}^{-2Kt}}{K} P_t|\nabla f|^2 - \frac{\mathrm{e}^{-2Kt}-1+2Kt}{K^2 n}(P_t Lf)^2$ *holds for all* $f \in C_0^2(M), t \geq 0$.
(4) $P_t f^2 - (P_t f)^2 \geq \frac{\mathrm{e}^{2Kt}-1}{K}|\nabla P_t f|^2 + \frac{\mathrm{e}^{2Kt}-1-2Kt}{K^2 n}(P_t Lf)^2$ *holds for all* $f \in C_0^2(M), t \geq 0$.

(5) $e^{-Kt}P_t|\nabla f| \geq |\nabla P_t f| + \frac{1}{n-d}\int_0^t e^{-Ks}P_s\frac{\langle Z,\nabla P_{t-s}f\rangle^2}{|\nabla P_{t-s}f|}ds$ *holds for all* $f \in C_0^2(M), t \geq 0$.

(6) *For any* $t > 0$ *and increasing* $\varphi \in C^1([0,t])$ *with* $\varphi(0) = 0$ *and* $\varphi'(0) = 1$*, the log-Harnack inequality*

$$P_{\varphi(t)}\log f(y) \leq \log P_t f(x) + \frac{\rho(x,y)^2}{4\int_0^t e^{2K\varphi(s)}ds} + \frac{Kn}{4}\int_0^t \frac{(\varphi'(s)-1)^2}{e^{2K\varphi(s)}-1}ds$$

holds for any positive function f *with* $\inf f > 0$ *and all* $x, y \in M$.

Proof. By the Jensen inequality, (2) follows from (1) immediately. So, it suffices to show that (2.7.1) implies (1), (2) implies (3) and (4), each of (3) and (4) implies (2.7.1), (5) is equivalent to (2.7.2), (2) implies (6), and (6) implies (2.7.1). Below we prove these implications respectively.

(2.7.1) implies (1). By (2.7.1) and using the Kolmogorov equations (Theorem 2.1.3) we have

$$\begin{aligned}\frac{d}{ds}P_s|\nabla P_{t-s}f|^2 &= P_s\{L|\nabla P_{t-s}f|^2 - 2\langle\nabla P_{t-s}f, \nabla LP_{t-s}f\rangle\}\\ &\geq 2KP_s|\nabla P_{t-s}f|^2 + \frac{2}{n}P_s(P_{t-s}Lf)^2, \quad s \in [0,t].\end{aligned}$$

By the Gronwall lemma, this implies (1) immediately.

(2) implies (3) and (4). Obviously,

$$\begin{aligned}\frac{d}{ds}P_s(P_{t-s}f)^2 &= P_s\{L(P_{t-s}f)^2 - 2(P_{t-s}f)LP_{t-s}f\}\\ &= 2P_s|\nabla P_{t-s}f|^2.\end{aligned} \tag{2.7.3}$$

Next, according to (2) and noting that $P_s(P_{t-s}Lf)^2 \geq (P_tLf)^2$, we have

$$P_s|\nabla P_{t-s}f|^2 \leq e^{-2K(t-s)}P_t|\nabla f|^2 - \frac{1-e^{-2K(t-s)}}{Kn}P_s(P_{t-s}Lf)^2,$$

$$P_s|\nabla P_{t-s}f|^2 \geq e^{2Ks}|\nabla P_t f|^2 + \frac{e^{2Ks}-1}{Kn}(P_tLf)^2.$$

Combining these with (2.7.3) and integrating w.r.t. ds over $[0,t]$, we prove (3) and (4).

(3) or (4) implies (2.7.1). For small $t > 0$ we have

$$\begin{aligned}P_t f^2 &= f^2 + tLf^2 + \frac{t^2}{2}L^2f^2 + o(t^2),\\ (P_tf)^2 &= \Big(f + tLf + \frac{t^2}{2}L^2f + o(t^2)\Big)^2\\ &= f^2 + t^2(Lf)^2 + 2tfLf + t^2fL^2f + o(t^2).\end{aligned}$$

So,

$$P_t f^2 - (P_t f)^2 = 2t|\nabla f|^2 + t^2\{2\langle \nabla Lf, \nabla f\rangle + L|\nabla f|^2\} + \mathrm{o}(t^2). \tag{2.7.4}$$

On the other hand,

$$\begin{aligned}\frac{1-\mathrm{e}^{-2Kt}}{K} P_t|\nabla f|^2 &= \{2t - 2Kt^2 + \mathrm{o}(t^2)\}\cdot\{|\nabla f|^2 + tL|\nabla f|^2 + \mathrm{o}(t)\}\\ &= 2t|\nabla f|^2 + 2t^2\{L|\nabla f|^2 - K|\nabla f|^2\} + \mathrm{o}(t^2).\end{aligned}$$

Moreover, it is easy to see that

$$\frac{\mathrm{e}^{-2Kt} - 1 + 2Kt}{K^2 n}(P_t Lf)^2 = \frac{2}{n} t^2 (Lf)^2 + \mathrm{o}(t^2).$$

Combining these with (2.7.4), we see that (3) implies

$$2t^2\Big\{\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle - K|\nabla f|^2 - \frac{(Lf)^2}{n}\Big\} + \mathrm{o}(t^2) \geq 0.$$

Therefore, (2.7.1) holds.

Next, it is easy to see that

$$\begin{aligned}&\frac{\mathrm{e}^{2Kt}-1}{K}|\nabla P_t f|^2 + \frac{\mathrm{e}^{2Kt}-1-2Kt}{K^2 n}(P_t Lf)^2\\ &= \{2t + 2Kt^2 + \mathrm{o}(t^2)\}\cdot|\nabla f + t\nabla Lf + \mathrm{o}(t)|^2 + \frac{2t^2}{n}(Lf)^2 + \mathrm{o}(t^2)\\ &= 2t|\nabla f|^2 + 2t^2\Big\{2\langle \nabla f, \nabla Lf\rangle + \frac{(Lf)^2}{n} + K|\nabla f|^2\Big\} + \mathrm{o}(t^2).\end{aligned}$$

Combining this with (2.7.4) and (4) we prove (2.7.1).

(5) is equivalent to (2.7.2). Using $\sqrt{|\nabla P_{t-s} f|^2 + \varepsilon}$ to replace $|\nabla P_{t-s} f|$ and letting $\varepsilon \to 0$, in the following calculations we may assume that $|\nabla P_{t-s} f|$ is positive and smooth, so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} P_s|\nabla P_{t-s} f| &= P_s\Big\{L|\nabla P_{t-s} f| - \frac{\langle \nabla L P_{t-s} f, \nabla P_{t-s} f\rangle}{|\nabla P_{t-s} f|}\Big\}\\ &= P_s\Big\{\frac{\frac{1}{2}L|\nabla P_{t-s} f|^2 - \langle \nabla L P_{t-s} f, \nabla P_{t-s} f\rangle - \big|\nabla|\nabla P_{t-s} f|\big|^2}{|\nabla P_{t-s} f|}\Big\}.\end{aligned} \tag{2.7.5}$$

Since

$$\begin{aligned}&\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle\\ &= \mathrm{Ric}(\nabla f, \nabla f) - \langle \nabla_{\nabla f} Z, \nabla f\rangle + \|\mathrm{Hess}_f\|_{HS}^2,\end{aligned} \tag{2.7.6}$$

$$\big|\nabla|\nabla f|\big|^2 = \Big\|\mathrm{Hess}_f\Big(\frac{\nabla f}{|\nabla f|}, \cdot\Big)\Big\|^2 \leq \|\mathrm{Hess}_f\|_{HS}^2,$$

it follows from (2.7.2) and (2.7.5) that

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s|\nabla P_{t-s}f| \geq KP_s|\nabla P_{t-s}f| + \frac{1}{n-d}P_s\frac{\langle Z, \nabla P_{t-s}f\rangle^2}{|\nabla P_{t-s}f|}.$$

This implies (5).

On the other hand, since when $t = 0$ the equality in (5) holds, one may take derivatives at $t = 0$ for both sides of (5) to derive at points such that $|\nabla f| > 0$

$$-K|\nabla f| + L|\nabla f| \geq \frac{\langle \nabla Lf, \nabla f\rangle}{|\nabla f|} + \frac{\langle Z, \nabla f\rangle^2}{(n-d)|\nabla f|}.$$

Thus,

$$\frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geq K|\nabla f|^2 + \frac{\langle Z, \nabla f\rangle^2}{n-d}.$$

Combining this with (2.7.6) we obtain

$$\mathrm{Ric}(\nabla f, \nabla f) - \langle \nabla_{\nabla f} Z, \nabla f\rangle \geq K|\nabla f|^2 + \frac{\langle Z, \nabla f\rangle^2}{n-d}, \quad f \in C^\infty(M),$$

which is equivalent to (2.7.2).

(2) implies (6). By the monotone class theorem, we may assume that $f \in C^2(M)$ which is constant outside a compact set. Let $\gamma : [0,1] \to M$ be the minimal geodesic from x to y, and let

$$h(s) = \frac{\int_0^s \mathrm{e}^{2K\varphi(r)}\mathrm{d}r}{\int_0^t \mathrm{e}^{2K\varphi(r)}\mathrm{d}r}, \quad s \in [0,t].$$

By (2) and using the Kolmogorov equations we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}P_{\varphi(s)}\log P_{t-s}f(\gamma_{h(s)}) \\
&= P_{\varphi(s)}\Big\{\varphi'(s)L\log P_{t-s}f - \frac{LP_{t-s}f}{P_{t-s}f}\Big\}(\gamma_{h(s)}) \\
&\quad + h'(s)\langle \dot\gamma_{h(s)}, \nabla P_{\varphi(s)}\log P_{t-s}f(\gamma_{h(s)})\rangle \\
&\leq P_{\varphi(s)}\{(\varphi'(s)-1)L\log P_{t-s}f - |\nabla \log P_{t-s}f|^2\}(\gamma_{h(s)}) \\
&\quad + \{|h'(s)|\rho(x,y)\}|\nabla P_{\varphi(s)}\log P_{t-s}f|(\gamma_{h(s)}) \\
&\leq \Big\{|\varphi'(s)-1|\cdot\big|P_{\varphi(s)}L\log P_{t-s}f\big| - \frac{\mathrm{e}^{2K\varphi(s)}-1}{Kn}(P_{\varphi(s)}L\log P_{t-s}f)^2\Big\}(\gamma_{h(s)}) \\
&\quad + \Big\{\rho(x,y)h'(s)\big|\nabla P_{\varphi(s)}\log P_{t-s}f\big| - \mathrm{e}^{2K\varphi(s)}\big|\nabla P_{\varphi(s)}\log P_{t-s}f\big|^2\Big\}(\gamma_{h(s)}) \\
&\leq \frac{\mathrm{e}^{-2K\varphi(s)}\rho(x,y)^2h'(s)^2}{4} + \frac{Kn(\varphi'(s)-1)^2}{4(\mathrm{e}^{2K\varphi(s)}-1)}.
\end{aligned}$$

This completes the proof by integrating w.r.t. $\mathrm{d}s$ over $[0,t]$.

(6) implies (2.7.1). For fixed $x \in M$ and strictly positive $f \in C^\infty(M)$ which is constant outside a compact set. Let

$$\varphi(s) = s + \frac{2L(\log f)(x)}{n}s^2, \quad \gamma_s = \exp_x[-2s\nabla \log f(x)], \quad s \geq 0.$$

According to (6), for small $t > 0$ we have

$$\begin{aligned} P_{\varphi(t)}(\log f)(x) \leq \log P_t f(\gamma_t) + \frac{t^2|\nabla \log f|^2(x)}{\int_0^t e^{2K\varphi(s)}ds} \\ + \frac{Kn}{4}\int_0^t \frac{(\varphi'(s)-1)^2}{e^{2K\varphi(s)}-1}ds. \end{aligned} \tag{2.7.7}$$

Since $\varphi(t)^2 = t^2 + o(t^2)$, we have

$$\begin{aligned} &P_{\varphi(t)}(\log f)(x) = \Big\{\log f + \varphi(t)L\log f + \frac{\varphi(t)^2}{2}L^2 \log f\Big\}(x) + o(t^2) \\ &= \log f(x) + \varphi(t)L\log f(x) + o(t^2) \\ &\quad + \frac{t^2}{2}\Big\{\frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - \frac{2\langle \nabla Lf, \nabla f\rangle}{f^2} - \frac{L|\nabla f|^2}{f^2} \\ &\quad + \frac{4|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} + \frac{8\mathrm{Hess}_f(\nabla f, \nabla f)}{f^3}\Big\}(x). \end{aligned} \tag{2.7.8}$$

On the other hand,

$$\begin{aligned} \log P_t f(\gamma_t) &= \log f(x) + t\Big\{\frac{Lf}{f} - \frac{2|\nabla f|^2}{f^2}\Big\}(x) \\ &+ \frac{t^2}{2}\frac{d}{dt}\Big(\frac{LP_t f(\gamma_t) - 2\langle P_{x,\gamma_t}\nabla \log f(x), \nabla P_t f(\gamma_t)\rangle}{P_t f(\gamma_t)}\Big)\Big|_{t=0} + o(t^2) \\ &= \log f(x) + t\{L\log f - |\nabla \log f|^2\}(x) + o(t^2) \\ &\quad + \frac{t^2}{2}\Big\{\frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - \frac{4\langle \nabla Lf, \nabla f\rangle}{f^2} + \frac{4|\nabla f|^2 Lf}{f^3} \\ &\quad - \frac{8|\nabla f|^4}{f^4} + \frac{8\mathrm{Hess}_f(\nabla f, \nabla f)}{f^3}\Big\}(x). \end{aligned} \tag{2.7.9}$$

Finally, since it is easy to see that

$$\lim_{t\to 0}\frac{Kn}{4t^2}\int_0^t \frac{(\varphi'(s)-1)^2}{e^{2K\varphi(s)}-1}ds = \frac{1}{n}(L\log f)^2(x),$$

we have

$$\frac{Kn}{4}\int_0^t \frac{(\varphi'(s)-1)^2}{e^{2K\varphi(s)}-1}ds = \frac{1}{n}t^2(L\log f)^2(x) + o(t^2). \tag{2.7.10}$$

Substituting (2.7.8), (2.7.9) and (2.7.10) into (2.7.7), and noting that

$$(\varphi(t)-t)L(\log f)(x)=\frac{2t^2}{n}(L\log f)^2(x),$$

we arrive at

$$\begin{aligned}&\frac{1}{t}\Big(1-\frac{t}{\int_0^t e^{2K\varphi(s)}ds}\Big)|\nabla \log f|^2(x)+\frac{(L\log f)^2(x)}{n}\\ &\le \frac{1}{2}\Big(\frac{L|\nabla f|^2-2\langle\nabla Lf,\nabla f\rangle}{f^2}-\frac{2|\nabla f|^4}{f^4}\Big)(x)+o(1).\end{aligned}$$

Letting $t\to 0$ and multiplying both sides by f^2, we obtain

$$\begin{aligned}&K|\nabla f|^2(x)+\frac{(Lf-|\nabla f|^2/f)^2(x)}{n}\\ &\le \Big(\frac{1}{2}L|\nabla f|^2-\langle\nabla Lf,\nabla f\rangle-\frac{|\nabla f|^4}{f^2}\Big)(x).\end{aligned}$$

Replacing f by $f+m$ and letting $m\to\infty$, this implies that

$$K|\nabla f|^2(x)+\frac{(Lf)^2(x)}{n}\le\frac{1}{2}L|\nabla f|^2(x)-\langle\nabla Lf,\nabla f\rangle(x).$$

Therefore, (2.7.1) holds. □

In particular, when $K=0$ Corollary 2.7.3 below includes six more equivalent inequalities for (2.7.1), where (1)-(3) are due to [Bakry and Ledoux (2006)], (4)-(5) go back to [Bakry *et al* (2011)], and (6) is taken from [Qian, B. (2013)]. To formulate and prove this result, we first introduce the Hamilton-Jacob semigroup $\mathbf{Q}_t$ generated by $\frac{1}{2}|\nabla\cdot|^2$, i.e. for any $f\in C_b^1(M)$, $\mathbf{Q}_t f$ solves the equation

$$\frac{d\mathbf{Q}_t f}{dt}=-\frac{1}{2}|\nabla \mathbf{Q}_t f|^2,\quad \mathbf{Q}_0 f=f. \tag{2.7.11}$$

Using the Hopf-Lax formula we have

$$\mathbf{Q}_t f(x)=\inf_{y\in M}\Big\{f(y)+\frac{\rho(x,y)^2}{2t}\Big\},\quad t>0,x\in M,f\in\mathcal{B}_b(M). \tag{2.7.12}$$

Lemma 2.7.2. *For any $f\in C_b^2(M)$, there exists a constant $c>0$ such that*

$$\Big\|\mathbf{Q}_t(\varepsilon f)-\varepsilon f+\frac{\varepsilon^2 t}{2}|\nabla f|^2\Big\|_\infty\le ct^2\varepsilon^3,\quad \varepsilon,t\ge 0.$$

Proof. It suffices to prove for $\varepsilon, t > 0$. By (2.7.12), for any $z \in M$ there exists $y \in M$ such that

$$\mathbf{Q}_t(\varepsilon f)(z) = \varepsilon f(y) + \frac{\rho(z,y)^2}{2t}.$$

Then

$$\rho(z,y)^2 \le 2t(\varepsilon f(z) - \varepsilon f(y)) \le 4t\varepsilon \|f\|_\infty.$$

So, by the Taylor expansion,

$$\begin{aligned}
\mathbf{Q}_t(\varepsilon f)(z) &= \varepsilon f(y) + \frac{\rho(z,y)^2}{2t}\\
&\ge \varepsilon f(z) + \varepsilon \rho(z,y)\langle \nabla f(z), \nabla \rho(y,\cdot)(z)\rangle - \frac{\varepsilon}{2}\rho(z,y)^2 \|\mathrm{Hess}_f\|_\infty + \frac{\rho(z,y)^2}{2t}\\
&\ge \varepsilon f(z) - \frac{\varepsilon^2 t}{2}|\nabla f|^2(z) - ct^2\varepsilon^3
\end{aligned}$$

holds for some constant $c > 0$ and all $z \in M, \varepsilon, t > 0$.

On the other hand, taking $y = \exp[-\varepsilon t \nabla f(z)]$, we obtain from (2.7.12) that

$$\begin{aligned}
\mathbf{Q}_t(\varepsilon f)(z) &\le \varepsilon f\big(\exp[-\varepsilon t \nabla f(z)]\big) + \frac{\varepsilon^2 t}{2}|\nabla f|^2(z)\\
&\le \varepsilon f(z) - \frac{\varepsilon^2 t}{2}|\nabla f|^2(z) + ct^2\varepsilon^3, \quad z \in M, \varepsilon, t > 0
\end{aligned}$$

for some constant $c > 0$. Therefore, the proof is completed. $\square$

Corollary 2.7.3. *For any $n \ge d$, each of the following inequalities is equivalent to* (2.7.1) *with $K = 0$:*

(1) $(P_t f)L(\log P_t f) \ge P_t(f \log f)\Big(1 + \frac{2t}{n} L \log P_t f\Big)$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*
(2) $tLP_t f - \frac{n}{2}(P_t f)\log\Big(1 + \frac{2t}{n} L \log P_t f\Big) \le P_t(f\log f) - (P_t f)\log P_t f$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*
(3) $P_t(f\log f) - (P_t f)\log P_t f \le tLP_t f + \frac{n}{2}(P_t f)\log\Big(1 - \frac{2tP_t(fL\log f)}{nP_t f}\Big)$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*
(4) *For any $q_1 > q_2 > 0$ and $t_1, t_2 > 0$ such that $t := 2(t_1 q_1 - t_2 q_2) > 0$,*
$$(P_{t_1} e^{q_1 \mathbf{Q}_t f})^{\frac{1}{q_1}} \le (P_{t_2} e^{q_2 f})^{\frac{1}{q_2}} t_1^{\frac{n}{2q_2}} t_2^{-\frac{n}{2q_1}} \Big(\frac{2(q_1 - q_2)}{t}\Big)^{\frac{n(q_1-q_2)}{2q_1q_2}}, \quad f \in \mathcal{B}_b(M).$$
(5) *For any $o \in M, t_1, t_2 > 0$ and positive $f \in \mathcal{B}_b(M)$ with $P_{t_1} f(o) = 1$,*
$$W_2^\rho(fP_{t_1}(o,\cdot), P_{t_2}(o,\cdot))^2 \le 4t_1\Big\{P_{t_1}(f\log f)(o) + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big)\Big\}.$$

(6) *For any* $o \in M, t_1, t_2 > 0$ *and strictly positive* $f \in C_b^\infty(M)$ *with* $P_t f(o) = 1$,

$$\begin{aligned} &W_2^\rho(fP_{t_1}(o,\cdot), P_{t_2}(o,\cdot))^2 \\ &\le 4t_1\Big\{t_1 P_{t_1} Lf + \frac{n}{2}(P_{t_1}f)\log\Big(1 - \frac{2t_1 P_{t_1} fL\log f}{nP_{t_1}f}\Big) \\ &\qquad + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big)\Big\}. \end{aligned}$$

Proof. (a) **Each of (1), (2), (3) implies (2.7.1) for** $K = 0$. For any $f \in C_0^\infty(M)$, applying (1), (2) and (3) for $1 + \varepsilon f$ in place of f, and letting $\varepsilon \to 0$, we obtain, respectively,

$$\begin{aligned} &|\nabla P_t f|^2 \le P_t|\nabla f|^2 - \frac{2t}{n}(LP_t f)^2; \\ &(P_t f^2) - (P_t f)^2 \ge 2t|\nabla P_t f|^2 + \frac{2t^2}{n}(LP_t f)^2; \\ &P_t f^2 - (P_t f)^2 \le 2tP_t|\nabla f|^2 - \frac{2t^2}{n}(LP_t f)^2. \end{aligned}$$

According to Theorem 2.7.1, each of these inequalities implies (2.7.1) for $K = 0$.

(b) **(2.7.1) with** $K = 0$ **implies (1), (2), (3).** It suffice to prove for f being constant outside a compact set. Set

$$\phi(s) = P_s\{(P_{t-s}f)|\nabla \log P_{t-s}f|^2\}, \quad s \in [0,t].$$

By (2.7.1) for $K = 0$, and using the Schwarz inequality, we obtain

$$\begin{aligned} \phi'(s) &= 2P_s\Big\{(P_{t-s}f)\Big(\frac{1}{2}L|\nabla\log P_{t-s}f|^2 - \langle\nabla\log P_{t-s}f, \nabla L\log P_{t-s}f\rangle\Big)\Big\} \\ &\ge \frac{2}{n}P_s\{(P_{t-s}f)(L\log P_{t-s}f)^2\} \\ &= \frac{2}{n}P_s\Big\{\frac{1}{P_{t-s}f}\Big(LP_{t-s}f - \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\Big)^2\Big\} \\ &\ge \frac{2\big\{P_s\big(LP_{t-s}f - \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\big)\big\}^2}{nP_t f} \\ &= \frac{2}{nP_t f}(\phi(s) - LP_t f)^2, \quad s \in [0,t]. \end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{2s}{nP_t f} + \frac{1}{\phi(s) - LP_t f}\Big) \le 0, \quad s \in [0,t]. \tag{2.7.13}$$

In particular,

$$\frac{2t}{nP_tf} \le \frac{1}{\phi(0) - LP_tf} - \frac{1}{\phi(t) - LP_tf} = \frac{1}{P_t(fL\log f)} - \frac{1}{(P_tf)L\log P_tf}.$$

This implies the inequality in (1).

Moreover, it follows from (2.7.13) that

$$\frac{1}{\phi(0) - LP_tf} - \frac{2s}{nP_tf} \ge \frac{1}{\phi(s) - LP_tf} \ge \frac{2(t-s)}{nP_tf} + \frac{1}{\phi(t) - LP_tf}, \quad s \in [0,t].$$

Thus, letting $\alpha = \frac{2}{nP_tf}$, we have, for all $s \in [0,t]$,

$$\frac{1}{(\phi(0) - LP_tf)^{-1} - \alpha s} \le \phi(s) - LP_tf \le \frac{1}{\alpha(t-s) + (\phi(t) - LP_tf)^{-1}},$$

Integrating w.r.t. ds over $[0,t]$ and noting that

$$\int_0^t \phi(s)\mathrm{d}s = P_t(f\log f) - (P_tf)\log P_tf,$$

we prove the inequalities in (2) and (3).

(c) **(3) implies (4).** Let $\alpha = \frac{2(t_1 - t_2)}{t}$ and for $s \in [0,t]$,

$$q(s) = \frac{s + 2t_2q_2}{\alpha s + 2t_2}, \quad \theta(s) = \frac{(\alpha s + 2t_2)(s + 2t_2q_2)}{2(s + 2t_2q_2)}, \quad \lambda(s) = \frac{1}{1 - \alpha q(s)}.$$

It is easy to see that

$$(\theta q)'(s) = (\theta\lambda q')(s) = \frac{1}{2}, \quad q'(s) > 0, \quad \lambda(s) > 0, \quad s \in [0,t]. \tag{2.7.14}$$

Let

$$\Psi(s) = \left(P_{\theta(s)}\mathrm{e}^{q(s)\mathbf{Q}_sf}\right)^{\frac{1}{q(s)}}, \quad s \in [0,t].$$

By (2.7.11) we obtain

$$\begin{aligned}\Psi'(s) = \frac{q'(s)\Psi(s)^{1-q(s)}}{q(s)^2}\Big\{ &I(s) + \Big(\frac{\theta' q^2}{q'}\Big)(s)P_{\theta(s)}\big(\mathrm{e}^{q(s)\mathbf{Q}_sf}L\mathbf{Q}_sf\big) \\ &+ q(s)^2\Big(\frac{\theta' q}{q'} - \frac{1}{2q'}\Big)(s)P_{\theta(s)}\big(\mathrm{e}^{q(s)\mathbf{Q}_sf}|\nabla\mathbf{Q}_sf|^2\big)\Big\},\end{aligned} \tag{2.7.15}$$

where, according to (3) and $\lambda(s) > 0$,

$$\begin{aligned} I(s) := &\, P_{\theta(s)}\big(\mathrm{e}^{q(s)\mathbf{Q}_sf}\log \mathrm{e}^{q(s)\mathbf{Q}_sf}\big) \\ &- \big(P_{\theta(s)}\mathrm{e}^{q(s)\mathbf{Q}_sf}\big)\log P_{\theta(s)}\mathrm{e}^{q(s)\mathbf{Q}_sf} \\ \le &\, \theta(s)q(s)(1-\lambda(s))P_{\theta(s)}\big(\mathrm{e}^{q(s)\mathbf{Q}_sf}L\mathbf{Q}_sf\big) \\ &+ \theta(s)q(s)^2P_{\theta(s)}\big(\mathrm{e}^{q(s)\mathbf{Q}_sf}|\nabla\mathbf{Q}_sf|^2\big) \\ &+ \frac{n}{2}\big(\lambda - 1 - \log\lambda\big)(s)P_{\theta(s)}\mathrm{e}^{q(s)\mathbf{Q}_sf}. \end{aligned} \tag{2.7.16}$$

Substituting (2.7.16) into (2.7.15) and using (2.7.14), we arrive at

$$\Psi'(s) \le \frac{nq'(s)}{2q(s)^2}\big(\lambda - 1 - \log\lambda\big)(s)\Psi(s), \quad s \in [0,t].$$

Therefore,

$$\Psi(t) \le \Psi(0)\exp\left[\frac{n}{2}\int_0^t \frac{q'(s)}{q(s)^2}\big(\lambda(s) - 1 - \log\lambda(s)\big)\mathrm{d}s\right].$$

This is equivalent to the inequality in (4) since according to the definition of functions Ψ, θ, q and λ, we have

$$\begin{aligned}
&\int_0^t \frac{q'(s)}{q(s)^2}\big(\lambda(s) - 1 - \log\lambda(s)\big)\mathrm{d}s = \int_0^t \Big(-\frac{1}{q}\Big)'(s)\big(\lambda(s) - 1 - \log\lambda(s)\big)\mathrm{d}s \\
&= \frac{\lambda(0) - 1 - \log\lambda(0)}{q(0)} - \frac{\lambda(t) - 1 - \log\lambda(t)}{q(t)} + \int_0^t \frac{\alpha^2 q'(s)}{(1-\alpha q(s))^2}\mathrm{d}s \\
&= \frac{\alpha}{1-\alpha q_2} - \frac{\alpha}{1-\alpha q_1} - \log\frac{(1-\alpha q_2)^{\frac{1}{q_2}}}{(1-\alpha q_1)^{\frac{1}{q_1}}} + \frac{\alpha^2 t}{2t_2(1-\alpha q_2)} \\
&= \log\frac{(1-\alpha q_2)^{\frac{1}{q_2}}}{(1-\alpha q_1)^{\frac{1}{q_1}}} = \log\left\{\frac{t_1^{\frac{1}{q_2}}}{t_2^{\frac{1}{q_1}}}\Big(\frac{2(q_1-q_2)}{t}\Big)^{\frac{q_1-q_2}{q_1q_2}}\right\}.
\end{aligned}$$

(d) **(3) and (4) imply (5) and (6).** Since (6) follows immediately from (3) and (5), it suffices to prove (5). Applying (4) to $q_1 = \frac{2t_2q_2+1}{2t_1}$ and letting $q_2 \to 0$, we obtain

$$P_{t_1}\mathrm{e}^{\frac{1}{2t_1}\mathbf{Q}_1 f} \le \exp\left[\frac{1}{2t_1}P_{t_2}f + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big)\right].$$

Combining this with the Young inequality (see Lemma 2.4 in [Arnaudon *et al* (2009)] or [Stroock (2000)]), for any positive function g with $P_{t_1}g(o) = 1$ we have

$$\begin{aligned}
&\frac{1}{2t_1}\big(P_{t_1}(g\mathbf{Q}_1 f)(o) - P_{t_2}f(o)\big) \\
&\le P_{t_1}(g\log g)(o) + \log P_{t_1}\mathrm{e}^{\frac{1}{2t_1}\mathbf{Q}_1 f}(o) - \frac{1}{2t_1}P_{t_2}f(o) \\
&\le P_{t_1}(g\log g)(o) + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big).
\end{aligned}$$

This implies (5) by Proposition 1.3.1 and (2.7.12).

(e) **Each of (5) and (6) implies (2.7.1) for $K = 0$.** According to Theorem 2.7.1 for $K = 0$, it suffices to prove that for any $t > 0, o \in M$ and $f \in C_b(M)$ with $P_t f(o) = 0$,

$$P_t f^2(o) \le 2tP_t|\nabla f|^2(o) - \frac{2t^2(P_tLf)^2(o)}{n}. \tag{2.7.17}$$

Let $f_\varepsilon = 1 + \varepsilon f, t_1 = t, t_2 = (1 + a\varepsilon)t$, where $a \in \mathbb{R}$ is to be determined. Then, for small $\varepsilon > 0$ we have $f_\varepsilon > 0, P_t f_\varepsilon(o) = 1$, and by Proposition 1.3.1 and (2.7.12)

$$\frac{1}{2}W_2^\rho(f_\varepsilon P_t(o,\cdot), P_{(1+a\varepsilon)t}(o,\cdot))^2 \geq P_t\{f_\varepsilon \mathbf{Q}_1(4\varepsilon t f)\}(o) - P_{(1+a\varepsilon)t}(4\varepsilon t f)(o).$$

Combining this with Lemma 2.7.2, we obtain

$$\begin{aligned} &W_2^\rho(f_\varepsilon P_t(o,\cdot), P_{(1+a\varepsilon)t}(o,\cdot))^2 \\ &\geq 2P_t\big\{(1+\varepsilon f)(4\varepsilon t f - 4\varepsilon^2 t^2|\nabla f|^2)\big\}(o) \\ &\qquad - 8\varepsilon^2 t^2 a P_t L f(o) + \mathrm{o}(\varepsilon^2) \\ &= 8\varepsilon^2 t P_t f^2(o) - 8\varepsilon^2 t^2 P_t|\nabla f|^2(o) \\ &\qquad + 8\varepsilon^2 t^2 a P_t L f(o) + \mathrm{o}(\varepsilon^2). \end{aligned} \tag{2.7.18}$$

Combining this with (5) for f_ε in place of f, we arrive at

$$\begin{aligned} P_t f^2(o) &\leq t P_t|\nabla f|^2(o) + atP_t L f(o) \\ &\qquad + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big\{P_t(f_\varepsilon \log f_\varepsilon)(o) + \frac{n}{2}(\varepsilon a - \log(1+\varepsilon a))\Big\} \\ &= tP_t|\nabla f|^2(o) + atP_t L f(o) + \frac{1}{2}P_t f^2(o) + \frac{na^2}{4}. \end{aligned}$$

Taking $a = -\frac{2tP_tLf(o)}{n}$, we prove (2.7.17). Similarly, (2.7.17) follows by combining (2.7.18) with (6) for f_ε in place of f, and noting that $P_t f_\varepsilon(o) = 1$. □

Next, we consider applications of the above equivalent inequalities. We first present some consequences of Theorem 2.7.1(6) for heat kernel bounds. According to Li-Yau's Harnack inequality [Li and Yau (1986); Bakry and Qian (1999)], if (2.7.1) holds then P_t can be dominated by P_{t+s} for $s, t > 0$. A nice point of (6) is that we are also able to dominate P_{t+s} by P_t with help of the logarithmic function. With concrete choices of φ we have the following explicit log-Harnack inequalities.

Corollary 2.7.4. *If* (2.7.1) *holds, then for any* $s \geq 0, t > 0$,

$$P_{t+s}\log f(y) \leq \log P_t f(x) + \frac{K(t+2s)\rho(x,y)^2}{2t(e^{2K(t+s)}-1)} + \frac{nKs^2}{2t(e^{Kt}-1)}, \tag{2.7.19}$$

and

$$P_t \log f(y) \leq \log P_{t+s}f(x) + \frac{K\rho(x,y)^2}{2(e^{2Kt}-1)+4Kse^{2Kt}} + \frac{Kns}{4(e^{2Kt}-1)} \tag{2.7.20}$$

hold for $x, y \in M$ *and bounded measurable function* f *with* $\inf f > 0$.

Proof. Let $t_0 \in (0,t)$. Taking

$$\varphi(r) = r \wedge \frac{t}{2} + \frac{t+2s}{t}\Big(r - \frac{t}{2}\Big)^+, \quad r \in [0,t],$$

we have

$$\begin{aligned}\int_0^t e^{2K\varphi(r)}dr &= \frac{e^{Kt}-1}{2K} + \frac{t(e^{2K(t+s)}-e^{Kt})}{2K(t+2s)}\\ &\geq \frac{t(e^{2K(t+s)}-1)}{2K(t+2s)},\end{aligned}$$

and

$$\begin{aligned}K\int_0^t \frac{(\varphi'(r)-1)^2}{e^{2K\varphi(r)}-1}dr &= \frac{4Ks^2}{t^2}\int_{t/2}^t \frac{dr}{\exp[\frac{2K(t+2s)}{t}(r-\frac{t}{2})+Kt]-1}\\ &\leq \frac{2Ks^2}{t(e^{Kt}-1)}.\end{aligned}$$

Thus, (2.7.19) follows from Theorem 2.7.1(6).

Next, applying Theorem 2.7.1(6) to $t+s$ in place of t and taking $\varphi(r) = r \wedge t$, we prove (2.7.20). □

According to Proposition 1.4.4, for any $t > 0, s \geq 0$ and $x, y \in M$, (2.7.19) and (2.7.20) are equivalent to the following heat kernel inequalities (2.7.21) and (2.7.22) respectively, where ν is a measure equivalent to dx and p_t^ν is the heat kernel of P_t w.r.t. ν:

$$\begin{aligned}\int_M p_{t+s}^\nu(y,z)\log\frac{p_{t+s}^\nu(y,z)}{p_t^\nu(x,z)}\,\nu(dz)&\\ \leq \frac{K(t+2s)\rho(x,y)^2}{2t(e^{2K(t+s)}-1)} + \frac{nKs^2}{2t(e^{Kt}-1)},&\end{aligned}\tag{2.7.21}$$

$$\begin{aligned}\int_M p_t^\nu(y,z)\log\frac{p_t^\nu(y,z)}{p_{t+s}^\nu(x,z)}\,\nu(dz)&\\ \leq \frac{K\rho(x,y)^2}{2(e^{2Kt}-1)+4Kse^{2Kt}} + \frac{Kns}{4(e^{2Kt}-1)}.&\end{aligned}\tag{2.7.22}$$

2.7.2 *HWI inequalities*

In this subsection we aim to establish the HWI inequality using the curvature-dimension condition, which corresponds to Theorem 2.4.1(3) using the curvature condition. Again let $Z = \nabla V$ such that $\mu(dx) := e^{V(x)}dx$ is a probability measure. Let $\mathcal{P}(M)$ be the set of all probability measures on M and $\mathcal{C}(\mu_1,\mu_2)$ is the set of all couplings for μ_1 and μ_2.

Corollary 2.7.5. *Let $Z = \nabla V$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure. If (2.7.1) holds, then for any $f \in C^1(M)$ with $\mu(f^2) = 1$,*

$$\begin{aligned}\mu(f^2 \log f^2) \le r\mu(|\nabla f|^2) + \frac{\sqrt{n}(1-Kr)}{4\sqrt{r}}\Big(W_2^\rho(f^2\mu,\mu) - \frac{\sqrt{rn}}{2}\Big)^+ \\ + \frac{(1-Kr)W_2^\rho(f^2\mu,\mu)}{2r}\Big(W_2^\rho(f^2\mu,\mu) \wedge \frac{\sqrt{rn}}{2}\Big),\end{aligned} \tag{2.7.23}$$

for $r \in (0,\infty) \cap \Big(0, \frac{1}{K^+}\Big]$, *where $K^+ := \max\{0, K\}$. Consequently,*

$$\begin{aligned}\mu(f^2 \log f^2) \le \frac{3}{2}W_2^\rho(f^2\mu,\mu)\sqrt{\mu(|\nabla f|^2)} - \frac{K}{2}W_2^\rho(f^2\mu,\mu)^2 \\ - \frac{2\sqrt{\mu(|\nabla f|^2)} - KW_2^\rho(f^2\mu,\mu)}{2} \\ \times \Big(\sqrt{W_2^\rho(f^2\mu,\mu)} - \frac{\sqrt{n}}{2\sqrt{2}\mu(|\nabla f|^2)^{1/4}}\Big)^{+2}.\end{aligned} \tag{2.7.24}$$

Proof. Applying (2.7.20) for $P_t f^2 + \varepsilon$ in place of f and letting $\varepsilon \to 0$, we obtain, for all $s \ge 0$,

$$(P_t \log P_t f^2)(x) \le \log P_{2t+s} f^2(y) + \frac{\rho(x,y)^2 K}{2(\mathrm{e}^{2Kt}-1) + 4sK\mathrm{e}^{2Kt}} + \frac{Kns}{4(\mathrm{e}^{2Kt}-1)}.$$

Let $\Pi \in C(f^2\mu,\mu)$ be the optimal coupling for $W_2^\rho(f^2\mu,\mu)$, integrating both sides w.r.t. Π and noting that due to the Jensen inequality and $\mu(f^2) = 1$ it follows that $\mu(\log P_{2t+s}f^2) \le 0$, we arrive at

$$\mu((P_t f^2)\log P_t f^2) \le \frac{W_2^2 K}{2(\mathrm{e}^{2Kt}-1) + 4s\mathrm{e}^{2Kt}K} + \frac{Kns}{4(\mathrm{e}^{2Kt}-1)}, \tag{2.7.25}$$

where and in the remainder of the proof, W_2 stands for $W_2^\rho(f^2\mu,\mu)$ for simplicity. On the other hand, according to Theorem 2.3.1(3), (2.7.1) implies

$$P_t(f^2 \log f^2) \le (P_t f^2)\log P_t f^2 + \frac{1-\mathrm{e}^{-2Kt}}{K}P_t|\nabla f|^2.$$

Integrating both sides w.r.t. μ and using (2.7.25) we obtain

$$\mu(f^2 \log f^2) \le \frac{1-\mathrm{e}^{-2Kt}}{K}\mu(|\nabla f|^2) + \frac{W_2^2 K}{2(\mathrm{e}^{2Kt}-1) + 4s\mathrm{e}^{2Kt}K} + \frac{Kns}{4(\mathrm{e}^{2Kt}-1)}.$$

Letting $r = (1-\mathrm{e}^{-2Kt})/K$ which runs over all $(0, \frac{1}{K^+})$ as t varies in $(0,\infty)$, and using rs to replace s, we get

$$\mu(f^2 \log f^2) \le r\mu(|\nabla f|^2) + (1-Kr)\Big\{\frac{W_2^2}{2(1+2s)r} + \frac{ns}{4}\Big\},\ \ 0 < r \le \frac{1}{K^+}, s > 0.$$

Taking

$$s = \frac{1}{2}\Big(\frac{2W_2}{\sqrt{rn}} - 1\Big)^+,$$

we prove (2.7.23). To prove (2.7.24), let

$$\delta = \mu(|\nabla f|^2), \quad r = \frac{W_2}{2\sqrt{\delta}}.$$

Since according to Theorems 2.4.1(1) and Theorem 2.5.4 for constant Ψ,

$$\frac{K^+}{2} W_2^\rho(f^2\mu,\mu)^2 \le \mu(f^2 \log f^2) \le \frac{2}{K^+}\mu(|\nabla f|^2),$$

so that $r \le \frac{1}{K^+}$. Thus, (2.7.23) applies to this specific r. Therefore, (2.7.24) follows by noting that

$$\begin{aligned}
r\delta &+ \frac{(1-Kr)W_2}{2r}\Big(W_2 \wedge \frac{\sqrt{rn}}{2}\Big) + \frac{\sqrt{n}(1-Kr)}{4\sqrt{r}}\Big(W_2 - \frac{\sqrt{rn}}{2}\Big)^+ \\
&= \delta r + \frac{(1-Kr)W_2^2}{2r} - \frac{(1-Kr)W_2}{2r}\Big(W_2 - \frac{\sqrt{rn}}{2}\Big)^+ \\
&\quad + \frac{\sqrt{n}(1-Kr)}{4\sqrt{r}}\Big(W_2 - \frac{\sqrt{rn}}{2}\Big)^+ \\
&= \delta r + \Big(\frac{1}{2r} - \frac{K}{2}\Big)W_2^2 - \frac{1-Kr}{2r}\Big(W_2 - \frac{\sqrt{rn}}{\sqrt{2}}\Big)^{+2} \\
&= \frac{3}{2}W_2\sqrt{\delta} - \frac{K}{2}W_2^2 - \frac{2\sqrt{\delta} - KW_2}{2}\Big(\sqrt{W_2} - \frac{\sqrt{n}}{2\sqrt{2}\delta^{1/4}}\Big)^{+2}.
\end{aligned}$$

□

By Theorem 2.4.1(3), $\mathrm{Ric} - \mathrm{Hess}_V \ge K$ (i.e. (2.7.1) for $n = \infty$) implies

$$\mu(f^2 \log f^2) \le 2W_2^\rho(f^2\mu,\mu)\sqrt{\mu(|\nabla f|^2)} - \frac{K}{2}W_2^\rho(f^2\mu,\mu)^2$$

for all $f \in C^1(M)$ with $\mu(f^2) = 1$. According to (2.7.24), the dimension n contributes to a negative term in the right-hand side since $2\sqrt{\mu(|\nabla f|^2)} \ge KW_2^\rho(f^2\mu,\mu)$ as explained in the proof of (2.7.24).

Next, we introduce a Sobolev-type HWI inequality derived in [Wang (2008b)] using (2.7.1).

Theorem 2.7.6. *If* (2.7.1) *holds for some* $K \le 0$ *and* $n > 0$, *then for any* $\delta > 0$, $\mu(f^2) = 1$,

$$\mu(f^2 \log f^2) \le \frac{n}{2}\log(1 + \delta\mu(|\nabla f|^2)) + C(\delta, n, K)W_2(f^2\mu,\mu), \qquad (2.7.26)$$

where

$$
\begin{aligned}
C(\delta, n, K) := \inf_{R,r>0} \Big\{ & \frac{2}{\delta} R e^{\sqrt{-K}\,R/r} + \frac{n}{2} r\sqrt{-K} \\
& + \frac{1}{4R}\big(1+\sqrt{-nK}\,R\big)^2 - \frac{R}{4} nK \Big\} \\
\le \ & \frac{\sqrt{2}}{\sqrt{\delta}} + \frac{3}{2}\sqrt{-nK}.
\end{aligned}
$$

Proof. Let f be a fixed smooth function with $\mu(f^2) = 1$. By the Li-Yau Harnack inequality (see Theorem 10 in [Bakry and Qian (1999)]),

$$
\begin{aligned}
P_t f^2(x) \le [P_{t+s} f^2(y)]\Big(1+\frac{s}{t}\Big)^{n/2} \exp\bigg[& \frac{(\rho + \sqrt{-nK}\,s)^2}{4s} \\
& + \frac{1}{2}\sqrt{-nK} \min\Big\{ \big(\sqrt{2}-1\big)\rho, \frac{s}{2}\sqrt{-nK}\Big\}\bigg]
\end{aligned}
$$

for all $s, t > 0$, where $\rho := \rho(x, y)$ is the Riemannian distance between x and y. Thus,

$$
\begin{aligned}
(P_t \log P_t f^2)(x) & \le \log P_{2t} f^2(x) \\
& \le \log P_{2t+s} f^2(y) + \frac{n}{2}\log\Big(1+\frac{s}{2t}\Big) + \frac{(\rho+\sqrt{-nK}\,s)^2}{4s} \\
& \quad + \frac{\sqrt{-nK}}{2}\min\Big\{\big(\sqrt{2}-1\big)\rho, \frac{s}{2}\sqrt{-nK}\Big\}.
\end{aligned}
$$

Let $\Pi \in \mathcal{C}(f^2\mu, \mu)$ such that $W_2^2 := W_2(f^2\mu, \mu)^2 = \Pi(\rho^2)$. The existence of the optimal coupling is ensured by Proposition 1.3.2. Integrating both sides of the above inequality w.r.t. Π and applying the symmetry of P_t, we arrive at

$$
\begin{aligned}
\mu((P_t f^2)\log P_t f^2) & = \mu(f^2 P_t \log P_t f^2) \\
& \le \mu(\log P_{2t+s} f^2) + \frac{n}{2}\log\Big(1+\frac{s}{2t}\Big) \\
& \quad + \frac{(W_2+\sqrt{-nK}\,s)^2}{4s} + \frac{-nK}{4}s.
\end{aligned}
\tag{2.7.27}
$$

Since by Jensen's inequality $\mu(\log P_{2t+s} f^2) \le \log \mu(P_{2t+s} f^2) = 0$, combining (2.7.27) with the following semigroup log-Sobolev inequality (Theorem 2.3.1(3))

$$
P_t(f^2 \log f^2) \le \frac{2(1-e^{-2Kt})}{K} P_t|\nabla f|^2 + (P_t f^2)\log P_t f^2,
$$

we obtain

$$
\begin{aligned}
\mu(f^2 \log f^2) \le \ & \frac{2(1-e^{-2Kt})}{K}\mu(|\nabla f|^2) + \frac{n}{2}\log\Big(1+\frac{s}{2t}\Big) \\
& + \frac{(W_2+\sqrt{-nK}\,s)^2}{4s} + \frac{-nK}{4}s.
\end{aligned}
\tag{2.7.28}
$$

Set $s = RW_2$ and $\frac{s}{2t} = \delta\mu(|\nabla f|^2) + r\sqrt{-K}W_2$, where $R, r > 0$ are to be determined. We have

$$\mu(|\nabla f|^2) \le \frac{s}{2\delta t} = \frac{RW_2}{2\delta t} \quad \text{and} \quad t \le \frac{s}{2r\sqrt{-K}\,W_2} = \frac{R}{2r\sqrt{-K}}.$$

Thus,

$$\begin{aligned}\frac{2(1-\mathrm{e}^{-2Kt})}{K}\mu(|\nabla f|^2) &\le 4t\mathrm{e}^{-2Kt}\mu(|\nabla f|^2) \le \frac{2}{\delta}RW_2\mathrm{e}^{\sqrt{-K}R/r},\\ \log(1+s/2t) &= \log(1+\delta\mu(|\nabla f|^2) + r\sqrt{-K}W_2)\\ &\le \log(1+\delta\mu(|\nabla f|^2)) + r\sqrt{-K}W_2.\end{aligned}$$

Combining these with (2.7.28) we obtain

$$\begin{aligned}\mu(f^2\log f^2) \le &\Big\{\frac{2}{\delta}R\mathrm{e}^{\sqrt{-K}\,R/r} + \frac{(1+\sqrt{-nK}\,R)^2}{4R} + \frac{n}{2}r\sqrt{-K}\\ &+ \frac{-nK}{4}R\Big\}W_2 + \frac{n}{2}\log(1+\delta\mu(|\nabla f|^2)) \qquad (2.7.29)\\ =:&\ F(R,r)W_2 + \frac{n}{2}\log(1+\delta\mu(|\nabla f|^2)).\end{aligned}$$

To drop the exponential term of $\sqrt{-K}$ and to make $\mathrm{e}^{\sqrt{-K}\,R/r}$ equal to 1 as it should be when $K=0$, we take $r = (\sqrt{-K} + un^{-1/2})R$, where $u > 0$ is to be determined, so that

$$\begin{aligned}F(R,r) &= R\Big[\frac{2}{\delta}\mathrm{e}^{\sqrt{-K}/(\sqrt{-K}+un^{-1/2})} - nK + \frac{u}{2}\sqrt{-nK}\Big] + \frac{1}{4R} + \frac{\sqrt{-nK}}{2}\\ &=: H(R,u), \quad R,u>0.\end{aligned}$$

Minimizing $H(R,u)$ in $R,u>0$ we arrive at

$$\inf_{R,r>0} F(R,r) \le \inf_{u>0}\sqrt{\frac{2}{\delta}\mathrm{e}^{\sqrt{-K}/(\sqrt{-K}+un^{-1/2})} - nK + \frac{u}{2}\sqrt{-nK}} + \frac{1}{2}\sqrt{-nK}.$$

Since $\mathrm{e}^{\sqrt{-K}/(\sqrt{-K}+un^{-1/2})} \le 1 + \frac{1}{u}\sqrt{-nK}$, it follows that

$$\begin{aligned}\inf_{R,r>0} F(R,r) &\le \inf_{u>0}\sqrt{\frac{2}{\delta} + \big(u/2 + 2/(\delta u)\big)\sqrt{-nK} - nK} + \frac{1}{2}\sqrt{-nK}\\ &= \sqrt{\frac{2}{\delta} + 2\sqrt{-nK/\delta} - nK} + \frac{1}{2}\sqrt{-nK} \le \frac{\sqrt{2}}{\sqrt{\delta}} + \frac{3}{2}\sqrt{-nK}.\end{aligned}$$

Hence, the proof is completed by (2.7.29). $\square$

Finally, corresponding to Theorem 2.3.3(2), we consider the transportation inequality of P_t deduced from (2.7.1).

Proposition 2.7.7. *Assume that* (2.7.1) *holds and let*

$$\bar\rho(x,y)=\begin{cases}\frac{2}{\sqrt{K/(n-1)}}\sin\left[\frac{\rho(x,y)}{2}\sqrt{K/(n-1)}\right], & \text{if } K>0,\\ \rho(x,y), & \text{if } K=0,\\ \frac{2}{\sqrt{-K/(n-1)}}\sinh\left[\frac{\rho(x,y)}{2}\sqrt{-K/(n-1)}\right], & \text{if } K<0.\end{cases}$$

Then for any $p\in[1,\infty)$,

$$W_p^{\bar\rho}(\mu_1P_t,\mu_2P_t)\le \mathrm{e}^{-Kt}W_p^{\bar\rho}(\mu_1,\mu_2),\quad t\ge 0,\mu_1,\mu_2\in\mathcal{P}(M). \tag{2.7.30}$$

If $K>0$ *then*

$$W_1^{\bar\rho}(\mu_1P_t,\mu_2P_t)\le \exp\left[-\frac{nK}{n-1}t\right]W_1^{\bar\rho}(\mu_1,\mu_2) \tag{2.7.31}$$

for all $t\ge 0,\ \mu_1,\mu_2\in\mathcal{P}(M)$, *and hence,*

$$\|\nabla P_tf\|_\infty\le\pi\exp\left[-\frac{nKt}{n-1}\right]\|\nabla f\|_\infty,\quad t\ge 0, f\in C_b^1(M). \tag{2.7.32}$$

Proof. Since the assertion for $K=0$ follows from that for $K>0$ by letting $K\to 0$, below we only prove the desired inequality for $K<0$ and $K>0$ respectively.

(a) Proof of (2.7.31). Let $K>0$. Take $\Pi\in\mathcal{C}(\mu_1,\mu_2)$ such that $W_1^{\bar\rho}(\mu_1,\mu_2)=\Pi(\bar\rho)$. and let (X_0,Y_0) be an $M\times M$-valued random variable with distribution Π. Let (X_t,Y_t) be the coupling by reflection of the L-diffusion process with initial data (X_0,Y_0). We have (see Theorem 2.3.2 for $U=0$)

$$\mathrm{d}\rho(X_t,Y_t)\le 2\sqrt{2}\,\mathrm{d}b_t+I_Z(X_t,Y_t)\mathrm{d}t \tag{2.7.33}$$

for a one-dimensional Brownian motion b_t and

$$I_Z(x,y):=I(x,y)+\langle Z,\nabla\rho(\cdot,y)\rangle(x)+\langle Z,\nabla\rho(x,\cdot)\rangle(y), \tag{2.7.34}$$

where letting $\gamma:[0,\rho(x,y)]\to M$ be the minimal geodesic from x to y and $\{J_i\}_{i=1}^{d-1}$ the Jacobi fields along γ such that at points x,y they together with $\dot\gamma$ consist of an orthonormal basis of the tangent space, we have

$$I(x,y)=\sum_{i=1}^{d-1}\int_0^{\rho(x,y)}\left(|\nabla_{\dot\gamma}J_i|^2-\langle\mathcal{R}(\dot\gamma,J_i)\dot\gamma,J_i\rangle\right)_s\mathrm{d}s,$$

where $\mathcal{R}$ is the curvature tensor on M.

To calculate $I(x,y)$, let us fix points $x\ne y$ and simply denote $\rho=\rho(x,y)$. Let $\{U_i\}_{i=1}^{d-1}$ be constant vector fields along γ such that $\{\dot\gamma,U_i:$

$1 \le i \le d-1\}$ is an orthonormal basis. By the index lemma, for any $f \in C^1([0,\rho])$ with $f(0) = f(\rho) = 1$, we have

$$\begin{aligned} I(x,y) &\le \sum_{i=1}^{d-1} \int_0^\rho \big(|\nabla_{\dot\gamma} f U_i|^2 - f^2 \langle \mathcal{R}(U_i,\dot\gamma)\dot\gamma, U_i\rangle\big)_s \mathrm{d}s \\ &= \int_0^\rho \big\{(d-1)f'(s)^2 - f(s)^2 \mathrm{Ric}(\dot\gamma,\dot\gamma)_s\big\}\mathrm{d}s. \end{aligned} \tag{2.7.35}$$

On the other hand, since $f(0) = f(\rho) = 1$,

$$\langle Z, \nabla\rho(\cdot,y)\rangle(x) + \langle Z, \nabla\rho(x,\cdot)\rangle(y) = \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}s}\{f(s)^2\langle\dot\gamma, Z\circ\gamma\rangle_s\}\mathrm{d}s$$

$$= \int_0^\rho \big\{2(ff')(s)\langle\dot\gamma, Z\circ\gamma\rangle_s + f(s)^2\langle\nabla_{\dot\gamma}Z\circ\gamma,\dot\gamma\rangle_s\big\}\mathrm{d}s$$

$$\le \int_0^\rho \left\{\frac{f(s)^2\langle\dot\gamma, Z\circ\gamma\rangle_s^2}{n-d} + (n-d)f'(s)^2 + f(s)^2\langle\nabla_{\dot\gamma}Z\circ\gamma,\dot\gamma\rangle_s\right\}\mathrm{d}s.$$

Combining this with (2.7.35), (2.7.34) and (2.7.2), we obtain

$$I_Z(x,y) \le \int_0^\rho \big[(n-1)f'(s)^2 - Kf(s)^2\big]\mathrm{d}s. \tag{2.7.36}$$

Taking

$$f(s) = \tan\Big(\frac{\rho}{2}\sqrt{K/(n-1)}\Big)\sin\Big(\sqrt{K/(n-1)}\,s\Big) + \cos\Big(\sqrt{K/(n-1)}\,s\Big)$$

for $s \in [0,\rho]$, we obtain

$$I_Z(x,y) \le -2\sqrt{K(n-1)}\tan\Big(\frac{\rho}{2}\sqrt{K/(n-1)}\Big). \tag{2.7.37}$$

Therefore, it follows from (2.7.33) and the Itô formula that

$$\mathrm{d}\bar\rho(X_t,Y_t) \le \mathrm{d}M_t - \frac{nK}{n-1}\bar\rho(X_t,Y_t)\mathrm{d}t$$

holds for some martingale M_t. Thus,

$$\begin{aligned} W_1^{\bar\rho}(\mu_1 P_t, \mu_2 P_t) &\le \mathbb{E}\bar\rho(X_t,Y_t) \le \exp\Big[-\frac{nK}{n-1}t\Big]\mathbb{E}\bar\rho(X_0,Y_0) \\ &= \exp\Big[-\frac{nK}{n-1}t\Big]W_1^{\bar\rho}(\mu_1,\mu_2). \end{aligned}$$

(b) Proof of (2.7.32). Taking $\mu_1 = \delta_x, \mu_2 = \delta_y$ and noting that (see [Bakry and Ledoux (1996b); Kuwada (2013)]) $\rho\sqrt{-K/(n-1)} \le \pi$, we obtain from (2.7.31) that

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\le \sup_{x'\ne y'} \frac{|f(x') - f(y')|}{\bar\rho(x',y')} \inf_{\tilde\pi\in\mathcal{C}(\delta_x P_t,\delta_y P_t)} \bar\pi(\rho') \\ &\le \pi\|\nabla f\|_\infty W_1^{\bar\rho}(\delta_x P_t, \delta_y P_t) \\ &\le \pi\|\nabla f\|_\infty \exp\Big[-\frac{nKt}{n-1}\Big]\bar\rho(x,y). \end{aligned}$$

This implies

$$|\nabla P_t f(x)| = \limsup_{y\to x} \frac{|P_t f(x) - P_t f(y)|}{\rho(x,y)} \le \pi \exp\left[-\frac{nKt}{n-1}\right] \|\nabla f\|_\infty.$$

(c) Proof of (2.7.30). When $K < 0$, we take

$$\begin{aligned} f(s) =& \cosh\left(\frac{\rho}{2}\sqrt{-K/(n-1)}\right)\sinh\left(\sqrt{-K/(n-1)}\, s\right) \\ &+ \frac{1-\cosh(\rho\sqrt{-K/(n-1)})}{\sinh(\rho\sqrt{-K/(n-1)})}\sinh\left(s\sqrt{-K/(n-1)}\right), \quad s \in [0,\rho]. \end{aligned}$$

It follows from (2.7.36) that

$$I_Z(x,y) \le 2\sqrt{-K(n-1)}\tanh\left(\frac{\rho(x,y)}{2}\sqrt{-K/(n-1)}\right).$$

Combining this with (2.7.37), we obtain

$$I_Z(x,y) = \begin{cases} 2\sqrt{-K(n-1)}\tanh\left(\frac{\rho(x,y)}{2}\sqrt{-\frac{K}{n-1}}\right), & \text{if } K<0; \\ -2\sqrt{K(n-1)}\tan\left(\frac{\rho(x,y)}{2}\sqrt{\frac{K}{n-1}}\right), & \text{if } K>0. \end{cases} \tag{2.7.38}$$

Now, let (X_0, Y_0) have distribution Π such that $\Pi(\bar\rho^p) = W_p^{\bar\rho}(\mu_1,\mu_2)^p$. Using the coupling by parallel displacement rather than by reflection, we have

$$\mathrm{d}\rho(X_t, Y_t) \le I_Z(X_t, Y_t)\mathrm{d}t.$$

Combining this with (2.7.38) we conclude that

$$\mathrm{d}\bar\rho(X_t, Y_t) \le \mathrm{e}^{-Kt}\bar\rho(X_t, Y_t).$$

Therefore,

$$\begin{aligned} W_p^{\bar\rho}(\mu_1 P_t, \mu_2 P_t) &\le (\mathbb{E}\bar\rho(X_t,Y_t)^p)^{1/p} \\ &\le \mathrm{e}^{-Kt}(\mathbb{E}\bar\rho(X_0,Y_0)^p)^{1/p} = \mathrm{e}^{-Kt} W_p^{\bar\rho}(\mu_1,\mu_2). \end{aligned}$$ □

2.8 Intrinsic ultracontractivity on non-compact manifolds

When μ is finite, Theorem 2.4.2(1) provides a criterion for the ultracontractivity of P_t. In this section we consider $L := \Delta + \nabla V$ for infinite $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$. In this case, an important property of P_t is the intrinsic ultracontractivity.

Definition 2.8.1. Let $(E, \mathcal{F}, \mu)$ be a σ-finite measure space, and $(L, \mathcal{D}(L))$ a negative self-adjoint operator generating a (sub-)Markov semigroup $P_t := \mathrm{e}^{tL}$ on $L^2(\mu)$.

(1) If $\lambda_0(-L) := \inf \sigma(-L)$ is a simple eigenvalue with positive unit eigenfunction φ_0, where $\sigma(-L)$ is the spectrum of $-L$, we call φ_0 the *ground state* of L.
(2) If L has a ground state $\varphi_0 > 0$ and the Markov semigroup $P_t^{\varphi_0} := \varphi_0^{-1} e^{\lambda_0 t} P_t(\varphi_0 \cdot)$ is ultracontractive with respect to the probability measure $\mu_{\varphi_0} := \varphi_0^2 \mu$, i.e. $\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \to L^\infty(\mu_{\varphi_0})} < \infty$ for all $t > 0$, then the semigroup P_t is called intrinsically ultracontractive.

Obviously, if P_t is intrinsically ultracontractive, then it has a heat kernel $p_t(x,y)$ w.r.t. μ satisfying

$$p_t(x,y) \le e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y) \|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \to L^\infty(\mu_{\varphi_0})}, \quad t > 0, x, y \in E.$$

The intrinsic ultracontractivity has been well studied in the framework of Dirichlet heat semigroups on (in particular, bounded) domains in $\mathbb{R}^d$. For instance, the Dirichlet heat semigroup on a bounded Hölder domain of order 0 is intrinsically ultracontractive (see [Ciprina (1994); Chen and Song (2000)]). See the recent work [Ouhabaz and Wang (2007)] and references within for sharp estimates on $\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0}) \to L^\infty(\mu_{\varphi_0})}$, and [Kim and Song (2009)] and references within for the study of the intrinsic ultracontractivity of Lévy (in particular, stable) processes on domains.

On the other hand, however, when a non-compact Riemannian manifold with infinite volume is concerned, these results are no longer valid due to the lack of global intrinsic functional inequalities and characterization of the ground state. This section is taken from [Wang (2010c)], where sufficient curvature conditions were derived for the intrinsic ultracontractivity on non-compact complete manifolds. In order to study the intrinsic ultracontractivity of P_t, we make use of the following *intrinsic super Poincaré inequality* introduced in [Wang (2002a)] (see also [Ouhabaz and Wang (2007)]):

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(\varphi_0|f|)^2, \quad r > 0, f \in C_0^1(M), \tag{2.8.1}$$

where $\beta : (0,\infty) \to (0,\infty)$ is a decreasing function.

The intrinsic ultracontractivity of P_t implies (2.8.1) for some β (see Theorem 3.1 in [Wang (2002a)]), and (2.8.1) holds for some β if and only if $\sigma_{ess}(L) = \emptyset$ (see Theorem 2.2 in [Wang (2002a)]), where $\sigma_{ess}(L)$ is the essential spectrum of L. On the other hand, if

$$\Psi(t) := \int_t^\infty \frac{\beta^{-1}(s)}{s} ds < \infty, \quad t > \inf_{r>0} \beta(r), \tag{2.8.2}$$

where $\beta^{-1}(s) := \inf\{r > 0 : \beta(r) \le s\}$ for a positive decreasing function β, then (2.8.1) implies the intrinsic ultracontractivity of P_t with (see Theorem 3.3 in [Wang (2000b)])

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} \le \max\left\{\varepsilon^{-1}\inf\beta,\ \Psi^{-1}((1-\varepsilon)t)\right\}^2 < \infty, \quad (2.8.3)$$

for all $\varepsilon \in (0,1), t > 0$. We refer to [Davies and Simon (1984)] for the study of intrinsic ultracontractivity using the log-Sobolev inequality with parameters.

We will first establish the intrinsic super Poincaré inequality (2.8.1) using the following curvature-dimension operator $\mathrm{Ric}_{m,L}$. Assume that for some $m > 0$ and positive increasing function K one has, instead of the second condition in (2.8.7),

$$\mathrm{Ric}_{L,m} := \mathrm{Ric} - \mathrm{Hess}_V - \frac{\nabla V \otimes \nabla V}{m} \ge -\Phi \circ \rho_o. \qquad (2.8.4)$$

When Φ is a constant, this condition goes back to (2.7.2).

2.8.1 *The intrinsic super Poincaré inequality*

As explained in the last section that due to Theorem2.2 in [Wang (2002a)], (2.8.1) holds for some β if and only if $\sigma_{ess}(L) = \emptyset$. According to the Donnelly-Li decomposition principle (see [Donnelly and Li (1979)]), they are also equivalent to

$$\lambda_0(R) := \inf\{\mu(|\nabla f|^2) :\ \mu(f^2) = 1, f \in C_0^1(M), f|_{B(o,R)} = 0\} \uparrow \infty$$

as $R \uparrow \infty$. The purpose of this section is to estimate β in (2.8.1) by using $\lambda_0(R)$ and the curvature condition. To this end, we will make use of the following *super Poincaré inequality*:

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta_0(r)\mu(|f|)^2, \quad r > 0, f \in C_0^1(M) \qquad (2.8.5)$$

for some decreasing function $\beta_0 : (0,\infty) \to (0,\infty)$. In particular, by Corollary 1.1 (2) in [Wang (2000b)], (2.8.5) with $\beta_0(r) = c(1 + r^{-p/2})$ for some constant $c > 0$ and $p > 2$ is equivalent to the classical Sobolev inequality

$$\mu(|f|^{2p/(p-2)})^{(p-2)/p} \le C(\mu(|\nabla f|^2) + \mu(f^2)), \quad f \in C_0^1(M)$$

for some constant $C > 0$. The latter inequality holds for a large class of non-compact manifolds. For instance, according to [Croke (1980)], it holds true for $V = 0$ provided either the injectivity radius of M is infinite, or the injectivity radius is positive and the Ricci curvature is bounded below.

To derive explicit intrinsic super Poincaré inequality, we first estimate the ground state.

Lemma 2.8.1. *If* (2.8.4) *holds then for the positive ground state* φ_0*, there exists a constant* $C > 0$ *such that*

$$\varphi_0 \geq \frac{1}{C} \exp\left[-C\rho_o\sqrt{\Phi(2\rho_o)}\right].$$

Proof. Since φ_0 is bounded below by a positive constant on a compact set, it suffices to prove for $\rho_o \geq 1$. Let $x \in M$ with $\rho_o(x) \geq 1$. Applying Theorem 5.2 in [Li (2005)] to $\alpha = 2$ and $R = \rho_o(x)$, we obtain

$$\begin{aligned}
&\mathrm{e}^{-\lambda_0}\varphi_0(o) = P_1\varphi_0(o)\\
&\leq (P_{1+s}\varphi_0(x))(1+s)^{m+d}\exp\left[c_1^2 s\Phi(2\rho_o(x)) + \frac{\rho_o(x)^2}{2s}\right]\\
&= \varphi_0(x)\mathrm{e}^{-\lambda_0(1+s)}(1+s)^{m+d}\exp\left[c_1^2 s\Phi(2\rho_o(x)) + \frac{\rho_o(x)^2}{2s}\right], \quad s > 0,
\end{aligned}$$

for some constant $c_1 > 0$. Then the proof is completed by taking $s = \rho_o(x)/\sqrt{\Phi(2\rho_o(x))}$. □

Theorem 2.8.2. *Assume* (2.8.5). *Let* Φ *be positive increasing function on* $[0,\infty)$ *such that* (2.8.4) *holds. If* $\lambda_0(R) \uparrow \infty$ *as* $R \uparrow \infty$*, then* (2.8.1) *holds with*

$$\beta(r) = C\beta_0(r/8)\exp\left[C\lambda_0^{-1}(8/r)\Phi(2 + 2\lambda_0^{-1}(8/r))\right], \quad r > 0.$$

Proof. Since one may always take decreasing β, it suffices to prove for $r \leq 1$. Let $f \in C_0^1(M)$ be fixed. Let $h_R = (\rho_o - R)^+ \wedge 1$, $R > 0$. Then h_R is Lipschitz continuous so that (2.8.5) applies to $f(1-h_R)$ instead of f:

$$\begin{aligned}
\mu(f^2(1-h_R)^2) &\leq 2s\mu(|\nabla f|^2) + 2s\mu(f^2)\\
&\quad + \beta_0(s)\mu(|f|1_{B(o,R+1)})^2, \quad s > 0.
\end{aligned} \tag{2.8.6}$$

Next, since $h_R f = 0$ on $B(o,R)$, we have

$$\mu(f^2h_R^2) \leq \frac{\mu(|\nabla(fh_R)|^2)}{\lambda_0(R)} \leq \frac{2}{\lambda_0(R)}\mu(|\nabla f|^2) + \frac{2}{\lambda_0(R)}\mu(f^2).$$

Combining this with (2.8.6) we obtain

$$\begin{aligned}
\mu(f^2) &\leq 2\mu(f^2h_R^2) + 2\mu(f^2(1-h_R)^2)\\
&\leq \left(4s + \frac{4}{\lambda_0(R)}\right)(\mu(|\nabla f|^2) + \mu(f^2)) + \frac{2\beta_0(s)}{\inf_{B(o,R+1)}\varphi_0^2}\mu(|f|\varphi_0)^2.
\end{aligned}$$

Thus, if $4s + \frac{4}{\lambda_0(R)} \leq \frac{1}{2}$ then

$$\mu(f^2) \leq \left(8s + \frac{8}{\lambda_0(R)}\right)\mu(|\nabla f|^2) + \frac{4\beta_0(s)}{\inf_{B(o,R+1)}\varphi_0^2}\mu(|f|\varphi_0)^2.$$

Hence, (2.8.1) holds for

$$\beta(r) := \inf\left\{\frac{4\beta_0(s)}{\inf_{B(o,R+1)}\varphi_0^2} : \ 8s + \frac{8}{\lambda_0(R)} \le r\right\}, \quad r \le 1.$$

Combining this with Lemma 2.8.1, there exists a constant $c > 0$ such that (2.8.1) holds for

$$\beta(r) := \inf\left\{c\beta_0(s)\exp\left[c(R+1)\sqrt{\Phi(2+2R)}\right] : \ 8s + \frac{8}{\lambda_0(R)} \le r\right\}, \quad r \le 1.$$

This completes the proof by taken $s = r/8$ and $R = \lambda_0^{-1}(8/r)$. □

2.8.2 *Curvature conditions for intrinsic ultracontractivity*

Let k and Φ be two positive increasing functions on $[0, \infty)$ such that

$$\mathrm{Sect} \le -k \circ \rho_o, \quad \mathrm{Ric} \ge -\Phi \circ \rho_o, \quad \rho_o >> 1 \tag{2.8.7}$$

holds on M. Next, for a positive increasing function h on $(0, \infty)$, let

$$h^{-1}(r) := \inf\{s > 0 : \ h(s) \ge r\}, \qquad r \ge 0.$$

The following result provides a sufficient condition for the intrinsic ultracontractivity of P_t with $Z := \nabla V = 0$.

Theorem 2.8.3. *Let M be a Cartan-Hadamard manifold with $d \ge 2$ and let $L = \Delta$. Assume that* (2.8.7) *holds for some positive increasing functions k and Φ with $k(\infty) = \infty$. We have:*

(1) (2.8.1) *holds with*

$$\beta(r) := \theta r^{-d/2}\exp\left[\theta k^{-1}(\theta/r)\sqrt{\Phi(4 + 2k^{-1}(\theta/r))}\right], \quad r > 0$$

for some constant $\theta > 0$.

(2) *If*

$$k^{-1}(R)\sqrt{\Phi(4 + 2k^{-1}(R))} \le cR^{\varepsilon}, \quad R >> 1 \tag{2.8.8}$$

holds for some constants $c > 0$ and $\varepsilon \in (0, 1)$, then P_t is intrinsically ultracontractive with

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} \le \exp\left[C(1 + t^{-\varepsilon/(1-\varepsilon)})\right], \quad t > 0 \tag{2.8.9}$$

for some constant $C > 0$, or equivalently

$$p_t(x, y) \le \mathrm{e}^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)\exp\left[C(1 + t^{-\varepsilon/(1-\varepsilon)})\right] \tag{2.8.10}$$

for all $x, y \in M, t > 0$.

(3) *If* (2.8.8) *holds for some* $c > 0$ *and* $\varepsilon = 1$, *then* P_t *is intrinsically hypercontractive.*

Proof. (a) Since $V = 0$, (2.8.7) implies (2.8.4). Moreover, since M is a Cartan-Hadamard manifold, its injectivity radius is infinite. Hence, by [Croke (1980)], one has $\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} \le ct^{-d/2}$ for some $c > 0$ and all $t > 0$. By Theorem 4.5(b) in [Wang (2000b)], this implies (2.8.5) with $\beta_0(r) = c(1 + r^{-d/2})$ for some constant $c > 0$.

(b) Since M is a Cartan-Hadamard manifold, $B(o, R)^c$ is concave. Let $R_0 > 0$ be such that (2.8.7) holds for $\rho_o \ge R_0$. Then for any $R \ge R_0$, we have Sect $\le -k(R)$ on $B(o, R)^c$. To make use of the Laplacian comparison theorem, we note that the distance to the boundary of $B(o, R)^c$ is $\rho_o - R$ for $\rho_o \ge R$, and the boundary of $B(o, R)^c$ is concave so that $\mathbb{I} \le 0$. So, Theorem 1.2.3(1) holds for

$$h(s) = \cosh\left(\sqrt{k(R)}\, s\right), \quad s \ge 0,$$

i.e.

$$\Delta\rho_o \ge \frac{(d-1)h'(\rho_o - R)}{h(\rho_o - R)} \ge c_0\sqrt{k(R)}, \quad \rho_o \ge R + 1 \tag{2.8.11}$$

holds for some constant $c_0 > 0$ which is independent of R. By the Green formula (Theorem 1.1.6), for any smooth domain $D \subset B(o, R + 1)^c$, it follows from (2.8.11) that

$$c_0\sqrt{k(R)}\mu(D) \le \int_D \Delta\rho_o \mathrm{d}\mu \le \int_{\partial D} |N\rho_o| \mathrm{d}\mu_\partial \le \mu_\partial(\partial D),$$

where N is the unit normal vector field on ∂D. Thus, the following Lemma 2.8.4 yields

$$\lambda_0(R+1) \ge \frac{c_0^2 k(R)}{4}, \quad R \ge R_0.$$

Moreover,

$$\begin{aligned}\lambda_0^{-1}(8/r) &\le \inf\left\{R + 1 : R \ge R_0, \frac{c_0^2}{4}k(R) \ge \frac{8}{r}\right\}\\ &= 1 + R_0 \vee k^{-1}(32/c_0^2 r), \quad r > 0.\end{aligned}$$

Then by Theorem 2.8.2 with $\beta_0(r) = c(1 + r^{-d/2})$, we obtain the desired $\beta(r)$ for some $\theta > 0$.

(c) If (2.8.8) holds then by (1), (2.8.1) holds for

$$\beta(r) = \exp[\theta(1 + r^{-\varepsilon})]$$

for some constant $\theta > 0$. If $\varepsilon \in (0,1)$ then (2.8.9) follows from Corollary 3.4(1) in [Wang (2002a)]. If $\varepsilon = 1$ then

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \exp[\theta(1+r^{-1})]\mu(\varphi_0|f|)^2, \quad r > 0, f \in C_0^1(M).$$

Applying this to $f\varphi_0$ and noting that

$$\frac{1}{2}\mu(\langle\nabla f^2, \nabla\varphi_0^2\rangle) = -\frac{1}{2}\mu(f^2 L\varphi_0^2) = \lambda_0\mu_{\varphi_0}(f^2) - \mu(f^2|\nabla\varphi_0|^2),$$

we arrive at

$$\begin{aligned}\mu_{\varphi_0}(f^2) &\le r\mu_{\varphi_0}(|\nabla f|^2) + r\mu(f^2|\nabla\varphi_0|^2) \\ &\quad + \frac{r}{2}\mu(\langle\nabla f^2, \nabla\varphi_0^2\rangle) + e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2 \\ &= r\mu_{\varphi_0}(|\nabla f|^2) + r\lambda_0\mu_{\varphi_0}(f^2) + e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r > 0.\end{aligned}$$

This implies

$$\mu_{\varphi_0}(f^2) \le 2r\mu_{\varphi_0}(|\nabla f|^2) + 2e^{\theta(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r \in (0, 1/(2\lambda_0)).$$

Hence, there exists a constant $\theta' > 0$ such that

$$\mu_{\varphi_0}(f^2) \le r\mu_{\varphi_0}(|\nabla f|^2) + e^{\theta'(1+r^{-1})}\mu_{\varphi_0}(|f|)^2, \quad r > 0. \tag{2.8.12}$$

By Corollary 1.1(1) in [Wang (2000b)], this is equivalent to the defective log-Sobolev inequality

$$\mu_{\varphi_0}(f^2\log f^2) \le C_1\mu_{\varphi_0}(|\nabla f|^2) + C_2, \quad f \in C_b^1(M), \mu_{\varphi_0}(f^2) = 1 \tag{2.8.13}$$

for some $C_1, C_2 > 0$. On the other hand, by Proposition 1.6.13, (2.8.12) and the weak Poincaré inequality due to Theorem 3.1 in [Röckner and Wang (2001)] imply the Poincaré inequality

$$\mu_{\varphi_0}(f^2) \le C\mu_{\varphi_0}(|\nabla f|^2) + \mu_{\varphi_0}(f)^2, \quad f \in C_b^1(M)$$

for some constant $C > 0$. Combining this and (2.8.13) we obtain the log-Sobolev inequality, namely, (2.8.13) with $C_2 = 0$ and some possibly different $C_1 > 0$. Therefore, due to [Gross (1976)], $P_t^{\varphi_0}$ is hypercontractive since it is associated to the Dirichlet form $\mu_{\varphi_0}(\langle\nabla\cdot, \nabla\cdot\rangle)$ on $H^{2,1}(\mu_{\varphi_0})$, the completion of $C_0^\infty(M)$ with respect to the Sobolev norm

$$\|f\|_{2,1,\mu_{\varphi_0}} := \sqrt{\mu_{\varphi_0}(f^2 + |\nabla f|^2)}. \tag{2.8.14}$$

We remark that the implication of the hypercontractivity from the defective log-Sobolev inequality can also be deduced by using the uniformly positivity improving property of the diffusion semigroup, see e.g. [Aida (1998)] for details. Then the proof is finished. □

Remark 2.8.1. (a) If $\mathrm{Ric} \geq -K$ for some constant $K \geq 0$, then $\sigma_{ess}(\Delta) \neq \emptyset$. Since M is non-compact and complete, this follows from a comparison theorem by Cheng [Cheng (1975)] for the first Dirichlet eigenvalue and the Donnelly-Li decomposition principle [Donnelly and Li (1979)] for the essential spectrum:

$$\inf \sigma_{ess}(-\Delta) \leq \sup_{x \in M} \lambda_0(B(x,1)) \leq \lambda_0(K),$$

where $\lambda_0(B(x,1))$ is the first Dirichlet eigenvalue of $-\Delta$ on D and $\lambda_0(K)$ is the one on the unit geodesic ball in the d-dimensional parabolic space with Ricci curvature equal to K. Thus, the assumption $\Phi(\infty) = \infty$ in Theorem 2.8.3 is necessary for (2.8.1) to hold. Correspondingly, the assumption that $k(\infty) = \infty$ is also reasonable.

(b) The upper bound given in (2.8.9), which is sharp due to Example 2.8.1 below, is quite different from the known one on bounded domains. Indeed, for P_t the Dirichlet heat semigroup on a bounded $C^{1,\alpha}(\alpha > 0)$ domain in $\mathbb{R}^d$, the short time behavior of the intrinsic heat kernel is algebraic rather than exponential (see [Ouhabaz and Wang (2007)]):

$$\sup_{x,y} \frac{p_t(x,y)}{\varphi_0(x)\varphi_0(y)} = \mathrm{O}\big(t^{-(d+2)/2}\big).$$

The following lemma used above is known as Cheeger's inequality [Cheeger (1970)].

Lemma 2.8.4. *Let D be a domain in M, and let*

$$\lambda_0(D) = \inf\big\{\mu(|\nabla f|^2) : \ f \in C_b^1(M), f|_{D^c} = 0, \mu(f^2) = 1\big\},$$
$$c(D) = \inf_{A \subset D, \mu(A) > 0} \frac{\mu_\partial(\partial A)}{\mu(A)}.$$

Then $\lambda_0(D) \geq \frac{c(D)^2}{4}$.

Proof. Let $f \in C_b^1(M)$ with $\mu(f^2) = 1$ and $f|_{D^c} = 0$. Let $A_r = \{f^2 \geq r\}, r \geq 0$. By the coarea formula Theorem 1.1.5 and the Fubini Theorem, we have

$$\mu(|\nabla f^2|) = \int_0^\infty \mu(\partial A_r)\mathrm{d}r \geq c(D)\int_0^\infty \mu(f^2 \geq r)\mathrm{d}r = c(D)\mu(f^2) = c(D).$$

This completes the proof by noting that the Schwarz inequality and $\mu(f^2) = 1$ yield

$$\mu(|\nabla f^2|)^2 \leq 4\mu(f^2)\mu(|\nabla f|^2) = 4\mu(|\nabla f|^2).$$

□

For the case $\nabla V \neq 0$, we let ϑ be a positive increasing function on $[0, \infty)$ such that

$$L\rho_o \geq \sqrt{\vartheta \circ \rho_o}, \quad \rho_o >> 1. \tag{2.8.15}$$

Theorem 2.8.5. *Let o be a pole in M such that* (2.8.4) *and* (2.8.15) *hold for some increasing positive functions Φ and ϑ with $\vartheta(\infty) = \infty$. Then $\sigma_{ess}(L) = \emptyset$. Moreover, assuming*

$$\lim_{\rho_o(x)\to\infty} \frac{\sqrt{\Phi(2 + 2\rho_o(x))}}{\log^+ \mu(B(x,1))} = 0, \tag{2.8.16}$$

where $B(x,1)$ is the unit geodesic ball at x, we have:

(1) (2.8.1) *holds with*

$$\beta(r) = \theta r^{-(m+d+1)/2} \exp\left[\theta\vartheta^{-1}(32/r)\sqrt{\Phi(2 + 2\vartheta^{-1}(32/r))}\right], \quad r > 0$$

for some constant $\theta > 0$.

(2) *If there exist $c > 0$ and $\varepsilon \in (0,1)$ such that*

$$\vartheta^{-1}(R)\sqrt{\Phi(2 + 2\vartheta^{-1}(R))} \leq cR^{\varepsilon}, \quad R >> 1, \tag{2.8.17}$$

then P_t is intrinsically ultracontractive with (2.8.9) *and* (2.8.10) *holding for some constant $C > 0$.*

(3) *If* (2.8.17) *holds for some $c > 0$ and $\varepsilon = 1$, then P_t is intrinsically hypercontractive.*

To prove Theorem 2.8.5, we first establish the super Poincaré inequality (2.8.5) for a concrete β_0.

Lemma 2.8.6. *In the situation of Theorem* 2.8.5. (2.8.16) *implies* (2.8.5) *with $\beta_0(r) = c(1 + r^{-(m+d+1)/2})$ for some constant $c > 0$.*

Proof. By Theorem 5.2 in [Li (2005)] with $\alpha = (m+d+1)/(m+d)$, for any measurable function $f \geq 0$ with $\mu(f) = 1$, we have

$$P_t f(x) \leq (P_{t+s}f(y))\Big(1 + \frac{s}{t}\Big)^{\frac{m+d+1}{2}} \exp\left[C\Phi(2[\rho_o(x) \vee \rho_o(y)])s + \frac{\alpha\rho(x,y)^2}{4s}\right]$$

for some constant $C > 0$ and all $s, t > 0$. This implies

$$\begin{aligned} 1 &= \int_M P_{t+s}f(y)\mu(\mathrm{d}y) \\ &\geq (P_t f(x))\Big(1 + \frac{s}{t}\Big)^{-(m+d+1)/2} \int_{B(x,1)} \mathrm{e}^{-C\Phi(2[\rho_o(x)\vee\rho_o(y)])s - \alpha/[4s]}\mu(\mathrm{d}y). \end{aligned}$$

Taking $s = 1/\sqrt{\Phi(2+2\rho_o(x))}$, we obtain

$$P_t f(x) \le c_0(1+t^{-1})^{(m+d+1)/2} \cdot \frac{\exp[c_0\sqrt{\Phi(2+2\rho_o(x))}]}{\mu(B(x,1))}$$

for some constant $c_0 > 0$ and all $t > 0, x \in M$. Combining this with (2.8.16) we obtain

$$\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} \le c_1(1+t^{-1})^{(m+d+1)/2}, \quad t > 0$$

for some constant $c_1 > 0$. According to Theorem 4.5(b) in [Wang (2000b)], this is equivalent to (2.8.5) with $\beta_0(r) = c(1 + r^{-(m+d+1)/2})$ for some constant $c > 0$. □

Proof. [Proof of Theorem 2.8.5] By (2.8.15) and Cheeger's inequality Lemma 2.8.4, we have

$$\lambda_0(R) \ge \frac{\vartheta(R)}{4}, \quad R >> 1.$$

Since $\vartheta(R) \to \infty$ as $R \to \infty$, the essential spectrum of L is empty and the desired β follows from Theorem 2.8.2 and Lemma 2.8.6. The remainder of the proof is then similar to that of Theorem 2.8.3. □

2.8.3 *Some examples*

The following two examples show that conditions in Theorems 2.8.3 and 2.8.5 can be sharp.

Example 2.8.1. Let M be a Cartan-Hadamard manifold with

$$-c_1\rho_o^\delta \le \mathrm{Sect} \le -c_2\rho_o^\delta, \quad \rho_o >> 1$$

for some constants $c_1, c_2, \delta > 0$. Then $\sigma_{ess}(\Delta) = \emptyset$ and for $L = \Delta$, (2.8.1) holds with

$$\beta(r) = \exp[c(1 + r^{-(2+\delta)/[2\delta]})]$$

for some constant $c > 0$. Consequently:

(1) P_t is intrinsically ultracontractive if and only if $\delta > 2$, and when $\delta > 2$ one has

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} \le \theta_1 \exp\left[\theta_2 t^{-(\delta+2)/(\delta-2)}\right], \quad t > 0$$

for some constants $\theta_1, \theta_2 > 0$, which is sharp in the sense that the constant θ_2 cannot be replaced by any positive function $\theta_2(t)$ with $\theta_2(t) \downarrow 0$ as $t \downarrow 0$.

(2) P_t is intrinsically hypercontractive if and only if $\delta \geq 2$.

Proof. Since Sect $\leq -c_2\rho_o^\delta$ for some $c_2, \delta > 0$ and large ρ_o, Theorem 2.8.3 implies $\sigma_{ess}(\Delta) = \emptyset$. Moreover, one may take $\Phi(r) = (d-1)c_1 r^\delta$ and $k(r) = c_2 r^\delta$ for large r, so that

$$k^{-1}(R)\sqrt{\Phi(4 + 2k^{-1}(R))} \leq cR^{\frac{1}{2}+\frac{1}{\delta}}$$

for some constant $c > 0$ and large R. Then the sufficiency and the desired upper bound of $\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})}$ follow from Theorem 2.8.3.

Next, by the concrete Φ and Lemma 2.8.1 below we have

$$\varphi_0 \geq \frac{1}{C}\exp\left[-C\rho_o^{1+\delta/2}\right] \tag{2.8.18}$$

for some constant $C > 0$. If P_t is intrinsically ultracontractive, i.e. $P_t^{\varphi_0}$ is ultracontractive by definition, then, according to Theorem 2.2.4 in [Davies (1989)] (see also [Gross (1976)] and [Davies and Simon (1984)]), there exists a function $\beta : (0,\infty) \to (0,\infty)$ such that

$$\mu_{\varphi_0}(f^2\log f^2) \leq r\mu_{\varphi_0}(|\nabla f|^2) + \beta(r), \quad f \in C_b^1(M), \mu_{\varphi_0}(f^2) = 1.$$

By the concentration of reference measures induced by super log-Sobolev inequalities (see e.g. Corollary 6.3 in [Röckner and Wang (2003a)]), the above log-Sobloev inequality implies $\mu_{\varphi_0}(e^{\lambda\rho_o^2}) < \infty$ for any $\lambda > 0$. Combining this with (2.8.18) and noting that the Riemannian volume of a Cartan-Hadamard manifold is infinite, we conclude that $\delta > 2$. Similarly, if P_t is intrinsically hypercontractive, then $\mu_{\varphi_0}(e^{\lambda\rho_o^2}) < \infty$ for some $\lambda > 0$, so that $\delta \geq 2$.

Finally, let $\delta > 2$. If there exists $\theta_1 > 0$ and a positive function h with $h(t) \downarrow 0$ as $t \downarrow 0$ such that

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} \leq \theta_1 \exp\left[\theta_2 t^{-(\delta+2)/(\delta-2)}\right], \quad t > 0,$$

then Theorem 4.5 in [Wang (2000a)] implies (2.8.1) for

$$\begin{aligned}\beta(r) &= \inf_{s\leq r, t>0} \frac{s}{t}\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} e^{t/s-1} \\ &= \theta_1 \inf_{s\leq r, t>0} \frac{s}{t}\exp\left[h(t)t^{-(\delta+2)/(\delta-2)} + \frac{t}{s} - 1\right], \quad r > 0.\end{aligned}$$

Taking $s = r \wedge 1$ and $t = \big(r^{(\delta-2)/(2\delta)} \wedge 1\big) h(r^{(\delta-2)/(2\delta)} \wedge 1)^{(\delta-2)/(2\delta)}$, we obtain

$$\beta(r) \leq \theta_2 \exp\left[\tilde{h}(r) r^{-(\delta+2)/(2\delta)}\right], \quad r > 0 \tag{2.8.19}$$

for some constant $\theta_2 > 0$ and positive function $\tilde{h}$ with $\tilde{h}(r) \downarrow 0$ as $r \downarrow 0$.

Finally, we aim to deduce from (2.8.19) that

$$\mu(e^{\lambda\rho_o^{1+\delta/2}}) < \infty, \quad \lambda > 0, \tag{2.8.20}$$

which is contradictive to (2.8.18). To this end, we apply Theorem 6.2 in [Wang (2000b)], which says that

$$\mu(\exp[c_1\rho_o\xi(c_2\rho_o)]) < \infty \tag{2.8.21}$$

holds for some constants $c_1, c_2 > 0$ and

$$\xi(\lambda) := \inf\left\{s \geq 1 : \int_1^s \frac{1}{t^2}\log\beta(1/(2t^2))\mathrm{d}t \geq \lambda\right\}, \quad \lambda > 0.$$

Since (2.8.19) implies

$$\begin{aligned}\int_1^s \frac{1}{t^2}\log\beta(1/(2t^2))\mathrm{d}t &\leq \theta_3 + \theta_3\int_1^s t^{-(\delta-2)/\delta}\tilde{h}(1/(2t^{(\delta-2)/\delta}))\mathrm{d}t\\ &\leq \theta_3 + \varepsilon(s)s^{2/\delta}, \quad s > 1\end{aligned}$$

for some constant $\theta_3 > 0$ and some positive function ε with $\varepsilon(s) \downarrow 0$ as $s \uparrow \infty$, one has $\xi(\lambda)\lambda^{-\delta/2} \to \infty$ as $\lambda \to \infty$. Therefore, (2.8.20) follows from (2.8.21). □

Example 2.8.2. Let M be a Cartan-Hadamard manifold with

$$\mathrm{Ric} \geq -c(\rho_o^{2(\delta-1)} + 1)$$

for some constants $c > 0$ and $\delta > 1$. Let $V = \theta\rho_o^\delta$ for some constant $\theta > 0$ and $\rho_o >> 1$. Then $\sigma_{ess}(L) = \emptyset$ and (2.8.1) holds with

$$\beta(r) = \exp[c(1 + r^{-\delta/[2(\delta-1)]})]$$

for some constant $c > 0$. Consequently:

(1) P_t is intrinsically ultracontractive if and only if $\delta > 2$, and when $\delta > 2$ one has

$$\|P_t^{\varphi_0}\|_{L^1(\mu_{\varphi_0})\to L^\infty(\mu_{\varphi_0})} \leq \theta_1 \exp\left[\theta_2 t^{-\delta/(\delta-2)}\right], \quad t > 0$$

for some constants $\theta_1, \theta_2 > 0$, which is sharp in the sense that the constant θ_2 cannot be replaced by any positive function $\theta_2(t)$ with $\theta_2(t) \downarrow 0$ as $t \downarrow 0$.

(2) P_t is intrinsically hypercontractive if and only if $\delta \geq 2$.

Proof. Since M is a Cartan-Hadamard manifold and $\delta > 1$, by Theorem 1.1.10

$$L\rho_o \geq \delta\rho_o^{\delta-1} =: \sqrt{\vartheta \circ \rho_o}, \quad \rho_o >> 1.$$

In particular, $\vartheta(\infty) = \infty$ so that $\sigma_{ess}(L) = \emptyset$. Moreover, since

$$\mathrm{Ric} \geq -c(1+\rho_o^{2(\delta-1)}), \quad |\nabla V|^2 = \theta^2\delta^2\rho_o^{2(\delta-1)}$$

and $\mathrm{Hess}_V = \theta \mathrm{Hess}_{\rho_o^\delta} \geq 0$ for large ρ_o as M is Cartan-Hadamard, we may take $\Phi(r) = c_1(1+r^{2(\delta-1)})$ for some constant $c_1 > 0$. Therefore, (2.8.17) holds for some $c > 0$ and $\varepsilon = \frac{1}{2} + \frac{1}{2(\delta-1)}$. Then the sufficiency follows from Theorem 2.8.5 as (2.8.16) follows from

$$\mu(B(x,1)) \geq c(d)\exp\Big[\inf_{B(x,1)} V\Big] \geq c(d)\exp[\theta(\rho_o(x)-1)^\delta], \quad \rho_o(x) \geq 1,$$

where $c(d)$ is the volume of the unit ball in $\mathbb{R}^d$.

On the other hand, by Lemma 2.8.1 below and the concrete K, we have

$$\varphi_0 \geq \frac{1}{C}\exp\big[-C\rho_o^\delta\big]$$

for some constant $C > 0$. Then the remainder of the proof is as same as that in the proof of Example 2.8.1. □

Chapter 3

Reflecting Diffusion Processes on Manifolds with Boundary

In this chapter we intend to extend results derived in Chapter 2 to reflecting diffusion processes on manifolds with boundary. Due to the reflection, besides the curvature operator, the geometry of the boundary (the second fundamental form) will be involved in the study.

Let M be a d-dimensional complete connected Riemannian manifold with boundary ∂M and the inward pointing unit normal vector field N. We will study the reflecting diffusion process generated by $L := \Delta + Z$ for some C^1-smooth vector field Z. As in Chapter 2, we first construct the corresponding *horizontal reflecting diffusion process* generated by $\Delta_{O(M)} + \mathbf{H}_Z$ on $O(M)$ by solving the Stratonovich stochastic differential equation (SDE)

$$\mathrm{d}u_t = \sqrt{2}\sum_{i=1}^{d} H_{e_i}(u_t)\circ \mathrm{d}B_t^i + \mathbf{H}_Z(u_t)\mathrm{d}t + \mathbf{H}_N(u_t)\mathrm{d}l_t, \quad u_0 = u \in O(M),$$

where $B_t := (B_t^1, \dots, B_t^d)$ is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and l_t is an increasing process supported on $\{t \geq 0 : X_t := \mathbf{p}u_t \in \partial M\}$. Since $\mathbf{H}_Z$ is C^1, it is well known that (see e.g. [Ikeda and Watanabe (1989); Elworthy (1982)]) the equation has a unique solution (u_t, l_t) up to the life time $\zeta := \lim_{n\to\infty}\zeta_n$, where

$$\zeta_n := \inf\{t \geq 0 : \rho_o(X_t) := \rho(X_t, o) \geq n\}, \quad n \geq 1$$

for a fixed point $o \in M$. It is easy to see that X_t solves the equation

$$\mathrm{d}X_t = \sqrt{2}\, u_t \circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \quad X_0 = x := \mathbf{p}u_0, \tag{3.0.1}$$

up to the life time ζ. By the Itô formula, for any $f \in C_0^2(M)$ with $Nf := Nf|_{\partial M} = 0$,

$$f(X_t) - f(x) - \int_0^t Lf(X_s)\mathrm{d}s = \sqrt{2}\int_0^t \langle u_s^{-1}\nabla f(X_s), \mathrm{d}B_s\rangle$$

is a martingale up to the life time ζ. So, we call X_t the *reflecting diffusion process* generated by L. When $Z = 0$, then $\tilde{X}_t := X_{t/2}$ is generated by $\frac{1}{2}\Delta$ and is called the reflecting Brownian motion on M.

When ∂M is convex, the Riemannian distance can be reached by the minimal geodesic in M and $N\rho_o \leq 0$, so that Theorem 2.1.1 and Corollary 2.1.2 remain true for the reflecting diffusion process. When ∂M is non-convex, let

$$\mathcal{D} = \big\{\phi \in C_b^2(M) : \inf \phi = 1, \mathbb{I} \geq -N \log \phi\big\}. \tag{3.0.2}$$

If there is $\phi \in \mathcal{D}$, then by Theorem 1.2.5 ∂M becomes convex under the new metric $\langle \cdot, \cdot \rangle' = \phi^{-2}\langle \cdot, \cdot \rangle$. Let ρ' be the corresponding Riemannian distance, then with ρ'_o replacing ρ_o, Theorem 2.1.1 and Corollary 2.1.2 work for the reflecting diffusion process generated by L. From now on, we will only consider the case where the reflecting diffusion process is non-explosive.

3.1 Kolmogorov equations and the Neumann problem

In this section we introduce the Kolmogorov equations for P_t, the semigroup of the reflecting diffusion process generated by L. Consequently, letting

$$\mathcal{C}_N(L) = \{f \in C^\infty(M), Nf|_{\partial M} = 0, Lf \in \mathcal{B}_b(M)\},$$

$F(t, x) := P_t f(x)$ is the unique solution to the Neumann heat equation

$$\partial_t F = LF, \ \ NF(t, \cdot) = 0 \text{ for } t > 0, \ \ F(0, \cdot) = f. \tag{3.1.1}$$

To this end, we need the following two lemmas, where the first extends Lemma 2.1.4 to manifolds with boundary, and the second is essentially due to [Wang (2009c)].

Lemma 3.1.1. *For any $x \in M$ and $r_0 > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{P}(\sigma_r \leq t) \leq \mathrm{e}^{-cr^2/t}, \quad r \in [0, r_0], \ t > 0$$

holds, where $\sigma_r = \inf\{s \geq 0 : \rho(X_s, x) \geq r\}$ and X_s is the reflecting diffusion process generated by L with $X_0 = x$.

Proof. Let $\phi \in C_b^\infty(M)$ such that $\phi \geq 1$ in $B(x, r_0)$ and ∂M is convex under the metric $\langle \cdot, \cdot \rangle' := \phi^{-2}\langle \cdot, \cdot \rangle$. Let Δ' and Ric' be the Laplacian and the Ricci curvature for the metric $\langle \cdot, \cdot \rangle'$. We have

$$\phi^2 L = \Delta' + (d-2)\phi\nabla\phi + \phi^2 Z =: \Delta' + Z'.$$

Let ρ' be the Riemannian distance function to x induced by the metric $\langle\cdot,\cdot\rangle'$. By taking smaller r_0 we may and do assume that $(\rho')^2 \in C^\infty(B(x, 2r_0))$. By the convexity of the boundary under the new metric and using the Itô formula, we obtain

$$\mathrm{d}\rho'(X_t)^2 \le 2\sqrt{2}\,\phi^{-1}(X_t)\rho'(X_t)\mathrm{d}b_t + c_1\mathrm{d}t, \quad t \le \sigma_{r_0}$$

for some constant $c_1 > 0$ and a one-dimensional Brownian motion b_t. Due to this inequality, the remainder of the proof is completely similar to that of Lemma 2.1.4. □

Lemma 3.1.2. *Let $x \in \partial M$ and let σ_r be in Lemma* 3.1.1 *for a fixed constant $r > 0$. Then:*

(1) $\mathbb{E}^x e^{\lambda l_{t\wedge\sigma_r}} < \infty$ *for any $\lambda > 0$ and there exists a constant $c > 0$ such that $\mathbb{E}^x l_{t\wedge\sigma_r}^2 \le c(t + t^2)$.*
(2) $\mathbb{E}^x l_{t\wedge\sigma_r} = \frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{O}(t^{3/2})$ *holds for small $t > 0$.*

Proof. (1) Let $h \in C_0^\infty(M)$ be non-negative such that $h|_{\partial M} = 0$ and $Nh = 1$ holds on $(\partial M) \cap B(x, r)$. Since ρ_∂ is smooth in a neighborhood of ∂M, h can be constructed such that $h = \rho_\partial$ in a neighborhood of $(\partial M) \cap B(x, r)$. By (3.0.1) and the Itô formula,

$$\begin{aligned}\mathrm{d}h(X_t) &= \sqrt{2}\,\langle \nabla h(X_t), u_t \mathrm{d}B_t\rangle + Lh(X_t)\mathrm{d}t + \mathrm{d}l_t\\ &\ge \sqrt{2}\,\langle \nabla h(X_t), u_t \mathrm{d}B_t\rangle - c\mathrm{d}t + \mathrm{d}l_t, \quad t \le \sigma_r\end{aligned}$$

holds for some constant $c > 0$. This implies that $\mathbb{E}^x e^{\lambda l_{t\wedge\sigma_r}} < \infty$ for any $\lambda > 0$, and

$$\mathbb{E}^x l_{t\wedge\sigma_r}^2 \le c_1\big(t^2 + t + \mathbb{E}^x h^2(X_{t\wedge\sigma_r})\big), \quad t \ge 0 \tag{3.1.2}$$

holds for some constant $c_1 > 0$. Since h^2 satisfies the Neumann boundary condition, by the Itô formula we have

$$\mathbb{E}^x h^2(X_{t\wedge\sigma_r}) = \mathbb{E}^x \int_0^{t\wedge\sigma_r} Lh^2(X_s)\mathrm{d}s \le c_2 t, \quad t \ge 0$$

for some constant $c_2 > 0$. Combining this with (3.1.2) we prove (1).

(2) Let $r_0 \in (0, r)$ be such that ρ_∂ is smooth on $B(x, 2r_0)$. By the Itô formula we have

$$\mathrm{d}\rho_\partial(X_t) = \sqrt{2}\,\mathrm{d}b_t + L\rho_\partial(X_t)\mathrm{d}t + \mathrm{d}l_t, \quad t \le \sigma_{r_0}, \tag{3.1.3}$$

where b_t is a one-dimensional Brownian motion. Let $\tilde{b}_t$ solve

$$\mathrm{d}\tilde{b}_t = \mathrm{sgn}(\tilde{b}_t)\mathrm{d}b_t, \quad \tilde{b}_0 = 0.$$

Then $\tilde{b}_t$ is a one-dimensional Brownian motion such that

$$\mathrm{d}|\tilde{b}_t| = \mathrm{d}b_t + \mathrm{d}\tilde{l}_t,$$

where $\tilde{l}_t$ is the local time of $\tilde{b}_t$ at 0. Combining this with (3.1.3) and noting that $\mathrm{d}l_t$ is supported on $\{\rho_\partial(X_t) = 0\}$ while $\mathrm{d}\tilde{l}_t$ is supported on $\{\tilde{b}_t = 0\}$, we obtain

$$\begin{aligned}
&\mathrm{d}\big(\rho_\partial(X_t) - \sqrt{2}\,|\tilde{b}_t|\big)^2 \\
&= 2\big(\rho_\partial(X_t) - \sqrt{2}\,|\tilde{b}_t|\big)L\rho_\partial(X_t)\mathrm{d}t + 2\big(\rho_\partial(X_t) - \sqrt{2}\,|\tilde{b}_t|\big)(\mathrm{d}l_t - \sqrt{2}\,\mathrm{d}\tilde{l}_t) \\
&\le 2\big(\rho_\partial(X_t) - \sqrt{2}\,|\tilde{b}_t|\big)L\rho_\partial(X_t)\mathrm{d}t \le c_1\big|\rho_\partial(X_t) - \sqrt{2}\,|\tilde{b}_t|\big|\mathrm{d}t, \quad t \le \sigma_{r_0}
\end{aligned}$$

for some constant $c_1 > 0$. This implies

$$\mathbb{E}^x\big(\rho_\partial(X_{t\wedge\sigma_{r_0}}) - \sqrt{2}\,|\tilde{b}_{t\wedge\sigma_{r_0}}|\big)^2 \le \frac{c_1^2}{4}t^2, \quad t \ge 0.$$

Since due to (3.1.3) one has $\big|\mathbb{E}^x l_{t\wedge\sigma_{r_0}} - \mathbb{E}^x\rho_\partial(X_{t\wedge\sigma_{r_0}})\big|^2 \le c_2 t^2$ for some constant $c_2 > 0$, it follows that

$$\big|\mathbb{E}^x l_{t\wedge\sigma_{r_0}} - \sqrt{2}\,\mathbb{E}^x|\tilde{b}_{t\wedge\sigma_{r_0}}|\big| \le c_3 t, \quad t \ge 0$$

holds for some constant $c_3 > 0$. Noting that $\mathbb{E}^x|\tilde{b}_t| = \sqrt{2t/\pi}$ and $\mathbb{E}^x\tilde{b}_t^2 = t$, combining this with Lemma 3.1.1, we arrive at

$$\begin{aligned}
\Big|\mathbb{E}^x l_{t\wedge\sigma_{r_0}} - \frac{2\sqrt{t}}{\sqrt{\pi}}\Big| &= \big|\mathbb{E}^x l_{t\wedge\sigma_{r_0}} - \sqrt{2}\,\mathbb{E}|\tilde{b}_t|\big| \le c_3 t + \sqrt{2}\mathbb{E}^x(|\tilde{b}_t|1_{\{t>\sigma_{r_0}\}}) \\
&\le c_3 t + \sqrt{2t\mathbb{P}^x(t > \sigma_{r_0})} \le c_4 t, \quad t \in [0,1]
\end{aligned} \tag{3.1.4}$$

for some constant $c_4 > 0$. Since $\sigma_{r_0} \le \sigma_r$ so that $l_{t\wedge\sigma_{r_0}} = l_{t\wedge\sigma_r}$ holds for $t \le \sigma_{r_0}$, it follows from (3.1.4) and (1) that

$$\Big|\mathbb{E}^x l_{t\wedge\sigma_r} - \frac{2\sqrt{t}}{\sqrt{\pi}}\Big| \le c_4 t + \mathbb{E}^x(l_{t\wedge\sigma_r}1_{\{t>\sigma_{r_0}\}}) \le c_4 t + \sqrt{2ct\mathbb{P}^x(t > \sigma_{r_0})} \le c_5 t$$

holds for some constant $c_5 > 0$ and all $t \in [0,1]$. □

Theorem 3.1.3. *Let $f \in C_N(L)$. Then:*

(1) $\frac{\mathrm{d}}{\mathrm{d}t}P_t f = P_t L f = L P_t f, \ t \ge 0;$
(2) $NP_t f|_{\partial M} = 0, \ t \ge 0;$
(3) *Let $t > 0$ and $\psi \in C_b^2([\inf f, \sup f])$. If $|\nabla P_\cdot f|$ is bounded on $[0,t] \times M$, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s\psi(P_{t-s}f) = P_s\big(\psi''(P_{t-s}f)|\nabla P_{t-s}f|^2\big), \quad s \in [0,t].$$

Proof. (1) The first equality follows from $P_t f = f + \int_0^t P_s Lf \mathrm{d}s$ implied by the Itô formula. To prove the second equality, it suffices to show that for any $x \in M^\circ := M \setminus \partial M$,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f(x) = LP_t f(x). \tag{3.1.5}$$

Let $r_0 > 0$ be such that $B(x, r_0) \subset M^\circ$, and take $h \in C_0^\infty(M)$ such that $h|_{B(x,r_0/2)} = 1$ and $h|_{B(x,r_0)^c} = 0$. By the Itô formula we have

$$\begin{aligned} P_{t+s}f(x) - P_t f(x) &= \mathbb{E}^x (hP_t f)(X_s) - (hP_t f)(x) + \mathbb{E}^x\{(1-h)P_t f\}(X_s) \\ &= \mathbb{E}^x \int_0^s L(hP_t f)(X_r)\mathrm{d}r + \mathbb{E}^x\{(1-h)P_t f\}(X_s). \end{aligned}$$

Since $L(hP_t f)(X_r)$ is bounded and goes to $LP_t f(x)$ as $r \to 0$, and noting that by Lemma 3.1.1,

$$\mathbb{E}^x|(1-h)P_t f|(X_s) \le \|f\|_\infty \mathrm{e}^{-c/s}, \quad s \in (0,1]$$

holds for some constant $c > 0$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f(x) = \lim_{s \downarrow 0} \frac{P_{t+s}f(x) - P_t f(x)}{s} = LP_t f(x),$$

that is, (3.1.5) holds.

(2) Let $x \in \partial M$. If $NP_t f(x) \neq 0$, for instance $NP_t f(x) > 0$, then there exist two constants $r_0, \varepsilon > 0$ such that $NP_t f \ge \varepsilon$ holds on $B(x, 2r_0)$. Moreover, by using $f + \|f\|_\infty$ in place of f, we may assume that $f \ge 0$. Let $h \in C_0^\infty(M)$ such that $0 \le h \le 1, Nh = 0, h|_{B(x,r_0)} = 1$ and $h|_{B(x,2r_0)^c} = 0$. By the Itô formula and using (1), we obtain

$$\begin{aligned} P_{t+s}f(x) &\ge P_s(hP_t f)(x) \\ &= P_t f(x) + \int_0^s P_r L(hP_t f)(x)\mathrm{d}r + \mathbb{E}^x \int_0^s (hNP_t f)(X_r)\mathrm{d}l_r \\ &\ge P_t f(x) + sLP_t f(x) + \mathrm{o}(s) + \varepsilon \mathbb{E}^x l_{s\wedge\sigma}, \end{aligned}$$

where $\sigma := \inf\{s \ge 0 : X_s \notin B(x, r_0)\}$. Combining this with (1) we arrive at

$$\varepsilon \lim_{s \to 0} \frac{1}{s} \mathbb{E}^x l_{s\wedge\sigma} \le 0,$$

which is impossible according to Lemma 3.1.2.

(3) By (1) and (2) and using the Itô formula, there is a local martingale M_s such that

$$\begin{aligned} \mathrm{d}\psi(P_{t-s}f)(X_s) &= \mathrm{d}M_s + \{L\psi(P_{t-s}f) - \psi'(P_{t-s}f)LP_{t-s}f\}(X_s)\mathrm{d}s \\ &= \mathrm{d}M_s + \{\psi''(P_{t-s}f)|\nabla P_{t-s}f|^2\}(X_s)\mathrm{d}s, \quad s \in [0,t], \end{aligned}$$

where

$$\mathrm{d}M_s = \sqrt{2}\,\langle \nabla\psi(P_{t-s}f)(X_s), u_s\mathrm{d}B_s\rangle, \quad s\in[0,t].$$

Since $|\nabla P_\cdot f|$ is bounded on $[0,t]\times M$ and $\psi\in C_b^2([\inf f,\sup f])$, M_s is a martingale. Therefore,

$$\begin{aligned}P_s\psi(P_{t-s}f) &= \mathbb{E}\psi(P_{t-s}f)(X_s)\\ &= \psi(P_tf) + \int_0^s P_r\{\psi''(P_{t-s}f)|\nabla P_{t-s}f|^2\}\mathrm{d}r.\end{aligned}$$

This completes the proof. □

Corollary 3.1.4. *For any $f\in\mathcal{C}_N(L)$, $F(t,x):=P_tf(x)$ is the unique C^2-solution to* (3.1.1).

Proof. By Theorem 3.1.3, it suffices to prove the uniqueness. Let F be a C^2-solution to (3.1.1). By the maximal principle and the boundedness of f we see that F is bounded. Moreover, by the Itô formula we see that $\{F(t-s,X_s)\}_{s\in[0,t]}$ is a local martingale, so that

$$F(t,x) = \mathbb{E}^xF(t-t\wedge\zeta_n, X_{t\wedge\zeta_n}), \quad n\geq 1.$$

Thus, letting $n\to\infty$ and using the dominated convergence theorem, we obtain $F(t,x)=\mathbb{E}^xF(0,X_t)=P_tf(x)$. □

3.2 Formulae for ∇P_t, Ric_Z and $\mathbb{I}$

3.2.1 *Formula for ∇P_t*

We first present a formula for ∇P_tf. See also Corollary 4.1.3 for an alternative version.

Theorem 3.2.1. *Let $t>0$ and $u_0\in O_x(M)$ be fixed, and let $K\in C(M)$ and $\sigma\in C(\partial M)$ be such that $\mathrm{Ric}_Z\geq K, \mathbb{I}\geq\sigma$. Assume that*

$$\sup_{r\in[0,t]}\mathbb{E}^x\exp\left[-\int_0^r K(X_s)\mathrm{d}r - \int_0^r\sigma(X_s)\mathrm{d}l_s\right]<\infty. \tag{3.2.1}$$

Then there exists a progressively measurable process $\{Q_s\}_{s\in[0,t]}$ on $\mathbb{R}^d\otimes\mathbb{R}^d$ such that

$$Q_0=I, \quad \|Q_s\|\leq\exp\left[-\int_0^s K(X_r)\mathrm{d}s - \int_0^s\sigma(X_r)\mathrm{d}l_r\right], \quad s\in[0,t],$$

and for any $f \in C_b^1(M)$ such that $\nabla P_\cdot f$ is bounded on $[0,t] \times M$, any $h \in C_b^1([0,t])$ with $h(0)=0, h(t)=1$,

$$\begin{aligned} u_0^{-1}\nabla P_t f(x) &= \mathbb{E}^x\{Q_t^* u_t^{-1}\nabla f(X_t)\} \\ &= \mathbb{E}^x\bigg\{\frac{f(X_t)}{\sqrt{2}}\int_0^t h'(s)Q_s^*\mathrm{d}B_s\bigg\}. \end{aligned} \tag{3.2.2}$$

Proof. (a) Construction of Q_s. For any $n \geq 1$, let $Q_s^{(n)}$ solve the equation

$$\begin{aligned} \mathrm{d}Q_s^{(n)} &= -\mathrm{Ric}_Z^\#(u_s)Q_s^{(n)}\mathrm{d}s - \mathbb{I}(u_s)Q_s^{(n)}\mathrm{d}l_s \\ &\quad - \frac{1}{2}(n+2\sigma(X_s)^+)\big((Q_s^{(n)})^* u_s^{-1}N(X_s)\big)\otimes\big(u_s^{-1}N(X_s)\big)\mathrm{d}l_s, \quad Q_0 = I, \end{aligned}$$

where $\mathrm{Ric}_Z^\#(u_s)$ is in (2.2.2) and $\mathbb{I}(u_s)$ for $X_s \in \partial M$ is an $\mathbb{R}^d \otimes \mathbb{R}^d$-valued random variable such that

$$\mathbb{I}(u_s)(a,b) = \mathbb{I}(P_\partial u_s a, P_\partial u_s b), \quad a,b \in \mathbb{R}^d, \tag{3.2.3}$$

where for $z \in \partial M$, $P_\partial : T_zM \to T_z\partial M$ is the projection operator. It is easy to see that for any $a \in \mathbb{R}^d$,

$$\begin{aligned} \mathrm{d}\|Q_s^{(n)}a\|^2 &= -2\mathrm{Ric}_Z(u_sQ_s^{(n)}a, u_sQ_s^{(n)}a)\mathrm{d}s - 2\mathbb{I}(P_\partial u_sQ_s^{(n)}a, P_\partial u_sQ_s^{(n)}a)\mathrm{d}l_s \\ &\quad - (n+2\sigma(X_s)^+)\langle u_sQ_s^{(n)}a, N(X_s)\rangle^2\mathrm{d}l_s \\ &\leq -\|Q_s^{(n)}a\|^2\{2K(X_s)\mathrm{d}s + 2\sigma(X_s)\mathrm{d}l_s\} \\ &\quad - n\langle u_sQ_s^{(n)}a, N(X_s)\rangle^2\mathrm{d}l_s. \end{aligned}$$

Therefore,

$$\|Q_s^{(n)}\|^2 \leq \exp\left[-2\int_0^s K(X_r)\mathrm{d}r - 2\int_0^s \sigma(X_r)\mathrm{d}l_r\right] < \infty, \quad n \geq 1, \tag{3.2.4}$$

and for any $m \geq 1$,

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{E}^x\int_0^{t\wedge\zeta_m}\|(Q_s^{(n)})^*u_s^{-1}N(X_s)\|^2\mathrm{d}l_s \\ &\leq \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n}\mathbb{E}^x\int_0^{t\wedge\zeta_m}\|Q_s^{(n)}\|^2\{2|K|(X_s)\mathrm{d}s + 2|\sigma|(X_s)\mathrm{d}l_s\}\right) \\ &= 0, \end{aligned} \tag{3.2.5}$$

where the second equality follows from Lemma 3.1.2, (3.2.4) and the boundedness of K and σ on $B(x,m)$. Combining (3.2.4) with (3.2.1) we see that

$$\left\{\mathbb{E}^x\int_0^t \sup_{n\geq 1}\|Q_s^{(n)}\|\mathrm{d}s + \mathbb{E}^x\sup_{n\geq 1}\|Q_t^{(n)}\|\right\} < \infty.$$

So, there exist a subsequence $\{Q^{(n_k)}\}$ and a progressively measurable process Q such that for any bounded measurable process $(\xi_s)_{s\in[0,t]}$ on $\mathbb{R}^d$ and any bounded $\mathbb{R}^d$-valued random variable η, one has

$$\lim_{k\to\infty}\left\{\mathbb{E}^x\int_0^t (Q_s^{(n_k)}-Q_s)\xi_s\mathrm{d}s+\mathbb{E}^x(Q_t^{(n_k)}-Q_t)\eta\right\}=0. \tag{3.2.6}$$

(b) Proof of the first equality. As in the proof of Theorem 2.2.1, by the Itô formula we have

$$\begin{aligned}\mathrm{d}(\mathbf{d}P_{t-s}f)(X_s)&=\nabla_{u_s\mathrm{d}B_s}(\mathbf{d}P_{t-s}f)(X_s)+\mathrm{Ric}_Z(\cdot,\nabla P_{t-s}f(X_s))\mathrm{d}s\\&\quad+\nabla_N(\mathbf{d}P_{t-s}f)(X_s)\mathrm{d}l_s.\end{aligned}$$

So, for any $a\in\mathbb{R}^d$ and $n\geq 1$,

$$\begin{aligned}&\mathrm{d}\langle\nabla P_{t-s}f(X_s),u_sQ_s^{(n)}a\rangle\\&=\mathrm{Hess}_{P_{t-s}f}(u_sQ_s^{(n)}a,u_s\mathrm{d}B_s)+\mathrm{Hess}_{P_{t-s}f}(N,u_sQ_s^{(n)}a)(X_s)\mathrm{d}l_s\\&\quad-\mathbb{I}(P_\partial u_sQ_s^{(n)}a,\nabla P_{t-s}f)(X_s)\mathrm{d}l_s.\end{aligned}$$

For any $z\in\partial M$ and $v\in T_z\partial M$, we have

$$0=v\langle N,\nabla P_{t-s}f\rangle(z)=\langle\nabla_vN,\nabla P_{t-s}f\rangle(z)+\mathrm{Hess}_{P_{t-s}f}(v,N).$$

So,

$$\mathrm{Hess}_{P_{t-s}f}(v,N)=\mathbb{I}(v,\nabla P_{t-s}f)(z). \tag{3.2.7}$$

Thus,

$$\begin{aligned}\mathrm{d}\langle\nabla P_{t-s}f(X_s),u_sQ_s^{(n)}a\rangle&=\mathrm{Hess}_{P_{t-s}f}(u_sQ_s^{(n)}a,u_s\mathrm{d}B_s)\\&\quad+\mathrm{Hess}_{P_{t-s}f}(N,N)\langle u_sQ_s^{(n)}a,N(X_s)\rangle\mathrm{d}l_s.\end{aligned} \tag{3.2.8}$$

Combining this with (3.2.5) and the boundedness of $\nabla P_\cdot f$ on $[0,t]\times M$, we obtain

$$\begin{aligned}\langle\nabla P_tf,u_0a\rangle&=\lim_{m\to\infty}\lim_{k\to 0}\mathbb{E}^x\langle\nabla P_{t-t\wedge\zeta_m}f(X_{t\wedge\zeta_m}),u_{t\wedge\zeta_m}Q_{t\wedge\zeta_m}^{(n_k)}a\rangle\\&=\lim_{m\to\infty}\lim_{k\to\infty}\mathbb{E}^x\{1_{\{t\leq\zeta_m\}}\langle\nabla f(X_t),u_tQ_t^{(n_k)}a\rangle\}\\&=\mathbb{E}^x\langle\nabla f(X_t),u_tQ_ta\rangle.\end{aligned}$$

This implies the first equality.

(c) Proof of the second equality. Since by the Itô formula $\mathrm{d}P_{t-s}f(X_s)=\sqrt{2}\,\langle\nabla P_{t-s}f(X_s),u_s\mathrm{d}B_s\rangle$, we have

$$f(X_t)=P_tf(x)+\sqrt{2}\int_0^t\langle\nabla P_{t-s}f(X_s),u_s\mathrm{d}B_s\rangle.$$

So, for any $a \in \mathbb{R}^d$ and $m \geq 1$, it follows from (3.2.5), (3.2.6), (3.2.8) and the boundedness of $\nabla P_\cdot f$ that

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbb{E}^x\bigg\{f(X_t)\int_0^t h'(s)\langle Q_s a, \mathrm{d}B_s\rangle\bigg\}\\
&= \mathbb{E}^x\int_0^t h'(s)\langle u_s Q_s a, \nabla P_{t-s}f(X_s)\rangle \mathrm{d}s\\
&= \lim_{k\to\infty}\mathbb{E}^x\int_0^t h'(s)\langle u_s Q_s^{(n_k)}a, \nabla P_{t-s}f(X_s)\rangle \mathrm{d}s\\
&= \lim_{m\to\infty}\lim_{k\to\infty}\int_0^t h'(s)\mathbb{E}^x\langle u_{s\wedge\zeta_m}Q_{s\wedge\zeta_m}^{(n_k)}a, \nabla P_{t-s\wedge\zeta_m}f(X_{s\wedge\zeta_m})\rangle \mathrm{d}s\\
&= \int_0^t h'(s)\langle u_0 a, \nabla P_t f(x)\rangle \mathrm{d}s = \langle \nabla P_t f(x), u_0 a\rangle.
\end{aligned}$$

Therefore, the proof is completed. □

We would like to indicate that when M is compact, formula (3.2.2) was first found by Hsu in [Hsu (2002b)], where the strong convergence of $\{Q_s^{(n)}\}_{s\in[0,t]}$ to $\{Q_s\}_{s\in[0,t]}$ in $L^2(\mathrm{d}t\times\mathbb{P})$ and that of $Q_t^{(n)}$ to Q_t in $L^2(\mathbb{P})$ were also proved. Next, combining the above argument with the proof of Theorem 2.2.1, we have the following local derivative formula of P_t.

Proposition 3.2.2. *Let $t > 0, x \in M$ and D be a compact domain such that $x \in D^\circ := D \setminus \partial D$. Let τ_D be the first hitting time of X_t to ∂D, where $X_0 = x$. Then there exists a progressively measurable process Q_s on $\mathbb{R}^d \otimes \mathbb{R}^d$ with*

$$\|Q_s\| \leq \exp\bigg[-\int_0^{s\wedge\tau_D} K(X_r)\mathrm{d}r - \int_0^{s\wedge\tau_D}\sigma(X_r)\mathrm{d}l_r\bigg],\quad s \leq t,$$

such that for any adapted $\mathbb{R}_+$-valued process h satisfying $h(0) = 0, h(s) = 1$ for $s \geq t\wedge\tau_D$ and $\mathbb{E}(\int_0^t h'(s)^2\mathrm{d}s)^\alpha < \infty$ for some $\alpha > \frac{1}{2}$, there holds

$$u_0^{-1}\nabla P_t f(x) = \frac{1}{\sqrt{2}}\mathbb{E}\bigg\{f(X_{t\wedge\tau_D})\int_0^t h'(s)Q_s^*\mathrm{d}B_s\bigg\},\quad f \in \mathcal{B}_b(M).$$

3.2.2 *Formulae for* Ric_Z *and* $\mathbb{I}$

Theorem 3.2.3. *Let $x \in M^\circ := M \setminus \partial M$ and $X \in T_xM$ with $|X| = 1$. Let $f \in C_0^\infty(M)$ such that $Nf|_{\partial M} = 0, \mathrm{Hess}_f(x) = 0$ and $\nabla f(x) = X$, and let $f_n = n + f$ for $n \geq 1$. Then assertions of Theorem 2.2.4 hold.*

Proof. Let $r > 0$ be such that $B(x,r) \subset M^\circ$ and $|\nabla f| \geq \frac{1}{2}$ on $B(x,r)$. Due to Lemma 2.1.4, the proof of Theorem 2.2.4 works for the present case by using $t \wedge \sigma_r$ to replace t, so that the boundary condition is avoided. We only present the proof of (1) for instance. By Lemma 2.1.4 and $\mathrm{Hess}_f(x) = 0$ we have, at point x,

$$\begin{aligned} P_t|\nabla f|^p &= \mathbb{E}^x|\nabla f|^p(X_{t\wedge\sigma_r}) + \mathrm{o}(t) = |\nabla f|^p + tL|\nabla f|^p + \mathrm{o}(t) \\ &= |\nabla f|^p + \frac{pt}{2}|\nabla f|^{p-2}L|\nabla f|^2 + \mathrm{o}(t). \end{aligned}$$

Moreover, since $Nf|_{\partial M} = 0$ and $f \in C_0^2(M)$, by the Kolmogorov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla P_t f|^p|_{t=0} = p|\nabla f|^{p-2}\langle \nabla Lf, \nabla f\rangle$$

so that

$$|\nabla P_t f|^p = |\nabla f|^p + pt|\nabla f|^{p-2}\langle \nabla Lf, \nabla f\rangle + \mathrm{o}(t).$$

Thus, (1) holds. □

Next, the following formulae for $\mathbb{I}$ are modified from [Wang (2009c)].

Theorem 3.2.4. *Let $x \in \partial M$ and $X \in T_x\partial M$ with $|X| = 1$. Then for any $f \in \mathcal{C}_N(L)$ such that $\nabla f(x) = X$,*

$$\begin{aligned} \mathbb{I}(X,X) &= \lim_{t\to 0}\frac{\pi}{2p\sqrt{t}}\{P_t|\nabla f|^p - |\nabla f|^p\}(x) \\ &= \lim_{t\to 0}\frac{\pi}{2p\sqrt{t}}\{P_t|\nabla f|^p - |\nabla P_t f|^p\}(x), \quad p > 0. \end{aligned} \tag{3.2.9}$$

If moreover $f > 0$, then for all $p \in [1,2]$,

$$\begin{aligned} \mathbb{I}(X,X) &= -\lim_{t\to 0}\frac{3\sqrt{\pi}}{8\sqrt{t}}\left(|\nabla f|^2 + \frac{p\{(P_t f^{2/p})^p - P_t f^2\}}{4(p-1)t}\right)(x) \\ &= -\lim_{t\to 0}\frac{3\sqrt{\pi}}{8\sqrt{t}}\left(|\nabla P_t f|^2 + \frac{p\{(P_t f^{2/p})^p - P_t f^2\}}{4(p-1)t}\right)(x), \end{aligned} \tag{3.2.10}$$

where when $p = 1$ we set

$$\begin{aligned} \frac{(P_t f^{2/p})^p - P_t f^2}{p-1} &= \lim_{p\to 1}\frac{(P_t f^{2/p})^p - P_t f^2}{p-1} \\ &= (P_t f^2)\log P_t f^2 - P_t(f^2\log f^2). \end{aligned}$$

Proof. (a) Let $r > 0$ such that $|\nabla f| \geq \frac{1}{2}$ holds on $B(x,r)$, and let $\sigma_r = \inf\{t \geq 0 : X_t \notin B(x,r)\}$. As in (3.2.7), $N|\nabla f|^2 = 2\mathbb{I}(\nabla f, \nabla f)$ holds on ∂M. So, by the Itô formula, and using Lemmas 3.1.1 and 3.1.2,

$$\begin{aligned} P_t|\nabla f|^p(x) &= \mathbb{E}^x|\nabla f|^p(X_{t\wedge\sigma_r}) + \mathrm{o}(t) \\ &= |\nabla f|^p(x) + \mathbb{E}^x \int_0^{t\wedge\sigma_r} \Big(L|\nabla f|^p(X_s)\mathrm{d}s \\ &\qquad + p\big\{|\nabla f|^{p-2}\mathbb{I}(\nabla f, \nabla f)\big\}(X_s)\mathrm{d}l_s\Big) + \mathrm{o}(t) \\ &= |\nabla f|^p(x) + \frac{2p\sqrt{t}}{\sqrt{\pi}}\mathbb{I}(X,X) + \mathrm{o}(\sqrt{t}) \end{aligned} \tag{3.2.11}$$

holds for small $t > 0$. This proves the first equality in (3.2.9). On the other hand, by $P_t f = f + \int_0^t P_s Lf \mathrm{d}s$ and noting that $P_s Lf \in C^\infty([0,t] \times B(x,r))$, we have

$$|\nabla P_t f|^p(x) = \left|\nabla f(x) + \int_0^t \nabla P_s Lf(x)\mathrm{d}s\right|^p = |\nabla f|^p(x) + \mathrm{O}(t). \tag{3.2.12}$$

Combining this with (3.2.11) we prove the second equality in (3.2.9).

(b) By (3.2.12), it remains to prove the first equality in (3.2.10). We only consider $p \neq 1$. By Lemmas 3.1.1 and 3.1.2, and noting that $N|\nabla f|^2(x) = 2\mathbb{I}(X,X)$, we have, at point x,

$$\begin{aligned} P_t f^2 &= \mathbb{E} f^2(X_{t\wedge\sigma_r}) + \mathrm{o}(t^2) = f^2 + \mathbb{E}\int_0^{t\wedge\sigma_r} Lf^2(X_s)\mathrm{d}s + \mathrm{o}(t^2) \\ &= f^2 + tLf^2 + \mathbb{E}\int_0^{t\wedge\sigma_r}\mathrm{d}s_1 \int_0^{s_1} NLf^2(X_{s_2})\mathrm{d}l_{s_2} + \mathrm{O}(t^2) \\ &= f^2 + \frac{4}{\sqrt{\pi}}\{fNLf + N|\nabla f|^2\}\int_0^t \sqrt{s}\,\mathrm{d}s \\ &\qquad + 2tfLf + 2t|\nabla f|^2 + \mathrm{o}(t^{3/2}) \\ &= f^2 + \frac{8t^{3/2}}{3\sqrt{\pi}}\{fNLf + 2\mathbb{I}(X,X)\} \\ &\qquad + 2tfLf + 2t|\nabla f|^2 + \mathrm{o}(t^{3/2}). \end{aligned} \tag{3.2.13}$$

Similarly,

$$\begin{aligned} P_t f^{2/p} = f^{2/p} + tLf^{2/p} + \frac{8t^{3/2}}{3\sqrt{\pi}}\Big(&\frac{1}{p}f^{(2-p)/p}NLf \\ &+ \frac{2(2-p)}{p^2}f^{2(1-p)/p}\mathbb{I}(X,X)\Big) + \mathrm{o}(t^{3/2}). \end{aligned}$$

This implies

$$(P_t f^{2/p})^p = f^2 + 2tfLf + \frac{2(2-p)t}{p}|\nabla f|^2 + \frac{8t^{3/2}}{3\sqrt{\pi}}\Big(fNLf + \frac{2(2-p)}{p}\mathbb{I}(X,X)\Big) + \mathrm{o}(t^{3/2}).$$

Combining this with (3.2.13) we prove the first equality in (3.2.10). $\square$

3.2.3 *Gradient estimates*

To apply the derivative formula in Theorem 3.2.1, we have to verify in advance the boundedness of $\nabla P_\cdot f$ on $[0,t] \times M$. So, we first present a sufficient condition for this boundedness. To this end, we shall make use of the coupling by parallel displacement for the reflecting diffusion process. If ∂M is convex, by Theorem 1.2.1 the distance between two different points can be reached by the minimal geodesic in M, so that we have $N\rho_o|_{\partial M} \le 0$ for any $o \in M$. Therefore, Theorem 2.3.2 works also for the reflecting diffusion process generated by L. More precisely, we have the following result.

Theorem 3.2.5. *Assume that ∂M is convex. Let $x \neq y$ and $T > 0$ be fixed. Let $U : [0,T) \times M^2 \to TM^2$ be C^1-smooth in $[0,T) \times (\mathrm{cut} \cup \mathbf{D})^c$.*

(1) *There exist two Brownian motions B_t and $\tilde{B}_t$ on a completed filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ such that*

$$1_{\{(X_t,Y_t)\notin \mathrm{cut}\}}\mathrm{d}\tilde{B}_t = 1_{\{(X_t,Y_t)\notin \mathrm{cut}\}}\tilde{u}_t^{-1}P_{X_t,Y_t}u_t\mathrm{d}B_t$$

holds, where X_t with lift u_t and local time l_t, and Y_t with lift $\tilde{u}_t$ and local time $\tilde{l}_t$ solve the equation

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}\,u_t \circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \quad X_0 = x \\ \mathrm{d}Y_t = \sqrt{2}\,\tilde{u}_t \circ \mathrm{d}\tilde{B}_t + \{Z(Y_t) + U(t,X_t,Y_t)1_{\{X_t\neq Y_t\}}\}\mathrm{d}t \\ \qquad + N(Y_t)\mathrm{d}\tilde{l}_t, \quad \tilde{X}_0 = y. \end{cases}$$

Moreover,

$$\mathrm{d}\rho(X_t,Y_t) \le \{I_Z(X_t,Y_t) + \langle U(t,X_t,Y_t), \nabla\rho(X_t,\cdot)(Y_t)\rangle 1_{\{X_t\neq Y_t\}}\}\mathrm{d}t.$$

(2) *The first assertion in (1) holds by using M_{X_t,Y_t} to replace P_{X_t,Y_t}. In this case*

$$\begin{aligned}&\mathrm{d}\rho(X_t,Y_t)\\ &\le 2\sqrt{2}\,\mathrm{d}b_t + \{I_Z(X_t,Y_t) + \langle U(t,X_t,Y_t), \nabla\rho(X_t,\cdot)(Y_t)\rangle 1_{\{X_t\neq Y_t\}}\}\mathrm{d}t\end{aligned}$$

holds for some one-dimensional Brownian motion b_t.

The following consequence of Theorem 3.2.5 can be found in e.g. [Qian, Z. (1997); Wang (1997a)].

Corollary 3.2.6. *Assume that ∂M is convex and* $\mathrm{Ric}_Z \geq K$ *for some constant* $K \in \mathbb{R}$. *Then*

$$|\nabla P_t f| \leq \mathrm{e}^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M), t \geq 0.$$

Proof. To apply Theorem 3.2.5, we first observe that

$$I_Z(x,y) \leq -\int_0^{\rho(x,y)} \mathrm{Ric}_Z(\dot\gamma,\dot\gamma)(s)\mathrm{d}s, \quad x,y \in M, \tag{3.2.14}$$

where $\gamma : [0, \rho(x,y)] \to M$ is the minimal geodesic from x to y. Indeed, letting $\{v_i\}_{i=1}^{d-1}$ be constant vector fields along γ such that $\{v_i, \dot\gamma : 1 \leq i \leq d-1\}$ is an orthonormal basis, the index lemma (Theorem 1.1.11) implies

$$\begin{aligned} I_Z(x,y) &\leq -\int_0^{\rho(x,y)} \Big(\sum_{i=1}^{d-1} \langle \mathcal{R}(\dot\gamma, v_i)\dot\gamma, v_i\rangle + \langle \nabla_{\dot\gamma} Z(\gamma), \dot\gamma\rangle\Big)(s)\mathrm{d}s \\ &= -\int_0^{\rho(x,y)} \mathrm{Ric}_Z(\dot\gamma,\dot\gamma)(s)\mathrm{d}s. \end{aligned}$$

Now, let $U = 0$ and $(X_t, \bar X_t)$ be the coupling by parallel displacement for $X_0 = x, \bar X_0 = y$. By Theorem 3.2.5 for $U = 0$ and using (3.2.14), we obtain

$$\begin{aligned} \mathrm{d}\rho(X_t, \bar X_t) &\leq I_Z(X_t, \bar X_t)\mathrm{d}t \\ &\leq -\Big\{\int_0^{\rho(X_t,\bar X_t)} \mathrm{Ric}_Z(\dot\gamma,\dot\gamma)(s)\mathrm{d}s\Big\}\mathrm{d}t \\ &\leq -K\rho(X_t, \bar X_t)\mathrm{d}t. \end{aligned}$$

Thus, $\rho(X_t, \bar X_t) \leq \mathrm{e}^{-Kt}\rho(x,y)$, so that by the dominated convergence theorem,

$$\begin{aligned} |\nabla P_t f(x)| &\leq \limsup_{y \to x} \frac{\mathbb{E}|f(X_t) - f(\bar X_t)|}{\rho(x,y)} \\ &\leq \mathrm{e}^{-Kt} \limsup_{y\to x} \mathbb{E}\frac{|f(X_t) - f(\bar X_t)|}{\rho(X_t, \bar X_t)} \\ &= \mathrm{e}^{-Kt} P_t |\nabla f|(x). \end{aligned}$$

□

By combining Corollary 3.2.6 with conformal change of metric to make the boundary from concave to convex, we have the following result. Since for $d = 1$ a connected manifold with boundary must be an interval, which is thus convex, in the following result we only consider $d \geq 2$.

Proposition 3.2.7. *Let $d \geq 2$ and let* $\mathrm{Ric}_Z \geq K$ *for some* $K \in C(M)$. *If there exists* $\phi \in \mathcal{D}$ *in* (3.0.2) *such that*

$$K_\phi := \inf_M \left\{\phi^2 K + \frac{1}{2} L\phi^2 - |\nabla \phi^2| \cdot |Z| - (d-2)|\nabla\phi|^2\right\} > -\infty, \quad (3.2.15)$$

then for any $f \in \mathcal{C}_N(L)$,

$$|\nabla P_t f| \leq \|\phi\|_\infty \|\nabla f\|_\infty \mathrm{e}^{-K_\phi t} + \|\phi^2 L f\|_\infty \|\nabla \phi^2\|_\infty \|\phi\|_\infty \frac{1 - \mathrm{e}^{-K_\phi t}}{K_\phi}, \quad t \geq 0.$$

Proof. By an approximation argument, it suffices to prove for $f \in C_0^\infty(M)$ with $Nf = 0$. Let Δ' and ∇' be associated to the metric $\langle\cdot,\cdot\rangle' = \phi^{-2}\langle\cdot,\cdot\rangle$, under which ∂M is convex according to Theorem 1.2.5. Then (see e.g. (2.2) in [Thalmaier and Wang (1998)])

$$\phi^2 L = \Delta' + Z', \quad Z' := \phi^2 Z + \frac{d-2}{2}\nabla\phi^2.$$

By Theorem 1.2.4, for any $X \in TM$ such that $\langle X, X\rangle' = 1$, i.e. $|X| = \phi$, we have

$$\mathrm{Ric}'(X,X) = \mathrm{Ric}(X,X) + (d-2)\phi^{-1}\mathrm{Hess}_\phi(X,X) + \frac{1}{2}\Delta\phi^2 - (d-2)|\nabla\phi|^2,$$

and

$$\begin{aligned}\langle \nabla'_X Z', X\rangle' &= \phi^{-2}\langle \nabla_X Z', X\rangle - \phi^{-2}\langle X, \nabla \log\phi\rangle\langle Z', X\rangle - \langle Z', \nabla\log\phi\rangle \\ &\quad + \phi^{-2}\langle X, Z'\rangle\langle X, \nabla\log\phi\rangle \\ &= \langle \nabla_X Z, X\rangle + 2\langle\nabla\log\phi, X\rangle\langle Z, X\rangle + (d-2)\phi^{-1}\mathrm{Hess}_\phi(X,X) \\ &\quad + (d-2)\langle X, \nabla\log\phi\rangle^2 - \phi\langle Z, \nabla\phi\rangle - (d-2)|\nabla\phi|^2.\end{aligned}$$

Therefore, noting that $|X| = \phi$.

$$\begin{aligned}\mathrm{Ric}'_{Z'}(X,X) &= \mathrm{Ric}'(X,X) - \langle\nabla'_X Z', X\rangle' \\ &= \mathrm{Ric}_Z(X,X) - 2\langle\nabla\log\phi, X\rangle\langle Z, X\rangle \\ &\quad + \frac{1}{2}L\phi^2 - (d-2)\langle X, \nabla\log\phi\rangle^2 \\ &\geq K\phi^2 + \frac{1}{2}L\phi^2 - |\nabla\phi^2|\cdot|Z| - (d-2)|\nabla\phi|^2 \\ &\geq K_\phi.\end{aligned} \quad (3.2.16)$$

Let P'_s be the semigroup of the reflecting diffusion process generated by $L' := \Delta' + Z'$. Since ∂M is convex under $\langle\cdot,\cdot\rangle'$, it follows from (3.2.16) and the proof of Corollary 3.2.6 that

$$\rho'(X'_t, \tilde{X}'_t) \leq \mathrm{e}^{-K_\phi t}\rho'(x,y), \quad x, y \in M,$$

where ρ' is the Riemannian distance induced by $\langle\cdot,\cdot\rangle'$ and $(X_t', \tilde X_t')$ is the coupling by parallel displacement for the L'-reflecting diffusion process starting from (x,y). Since $1 \le \phi$, we have $\|\phi\|_\infty^{-1}\rho \le \rho' \le \rho$, so that the above inequality implies

$$\rho(X_t', \tilde X_t') \le \|\phi\|_\infty \mathrm{e}^{-K_\phi t}\rho(x,y), \quad t \ge 0. \tag{3.2.17}$$

To derive the gradient estimate of P_t, we shall make time changes

$$\xi_x(t) = \int_0^t \phi^2(X_s')\mathrm{d}s, \quad \xi_y(t) = \int_0^t \phi^2(\tilde X_s')\mathrm{d}s.$$

Since $L' = \phi^2 L$, we see that $X_t := X'_{\xi_x^{-1}(t)}$ and $\tilde X_t := \tilde X'_{\xi_y^{-1}(t)}$ are generated by L with reflecting boundary. Again by $1 \le \phi$ we have

$$\|\phi\|_\infty^{-2} t \le \xi_x^{-1}(t),\ \xi_y^{-1}(t) \le t, \quad t \ge 0.$$

Combining this with (3.2.17) we arrive at

$$\begin{aligned}
|\xi_x^{-1}(t) - \xi_y^{-1}(t)| &\le \int_{\xi_x^{-1}(t)\wedge\xi_y^{-1}(t)}^{\xi_x^{-1}(t)\vee\xi_y^{-1}(t)} \phi^2(\tilde X_s')\mathrm{d}s \\
&= |\xi_y \circ \xi_y^{-1}(t) - \xi_y \circ \xi_x^{-1}(t)| \\
&= |\xi_x \circ \xi_x^{-1}(t) - \xi_y \circ \xi_x^{-1}(t)| \\
&\le \int_0^{\xi_x^{-1}(t)} |\phi^2(X_s') - \phi^2(\tilde X_s')|\mathrm{d}s \\
&\le \|\nabla\phi^2\|_\infty \|\phi\|_\infty \rho(x,y) \int_0^t \mathrm{e}^{-K_\phi s}\mathrm{d}s \\
&\le \frac{\|\nabla\phi^2\|_\infty\|\phi\|_\infty(1-\mathrm{e}^{-K_\phi t})}{K_\phi}\rho(x,y).
\end{aligned} \tag{3.2.18}$$

Therefore,

$$\begin{aligned}
&|P_t f(x) - P_t f(y)| = |\mathbb{E}\{f(X'_{\xi_x^{-1}(t)}) - f(\tilde X'_{\xi_y^{-1}(t)})\}| \\
&\le \mathbb{E}|f(X'_{\xi_y^{-1}(t)}) - f(\tilde X'_{\xi_y^{-1}(t)})| + |\mathbb{E}\{f(X'_{\xi_x^{-1}(t)}) - f(X'_{\xi_y^{-1}(t)})\}| \\
&=: I_1 + I_2.
\end{aligned} \tag{3.2.19}$$

By (3.2.17) and $\xi_y^{-1}(t) \le t$ we obtain

$$I_1 \le \|\nabla f\|_\infty \mathrm{e}^{-K_\phi t}\|\phi\|_\infty \rho(x,y). \tag{3.2.20}$$

Moreover, since $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$, it follows from the Itô formula and (3.2.18) that

$$\begin{aligned}
I_2 &= \left|\mathbb{E}\int_{\xi_x^{-1}(t)\wedge\xi_y^{-1}(t)}^{\xi_x^{-1}(t)\vee\xi_y^{-1}(t)} L'f(X_s')\mathrm{d}s\right| \le \|L'f\|_\infty \mathbb{E}|\xi_x^{-1}(t) - \xi_y^{-1}(t)| \\
&\le \|\phi^2 Lf\|_\infty \|\nabla\phi^2\|_\infty \|\phi\|_\infty \frac{1-\mathrm{e}^{-K_\phi t}}{K_\phi}\rho(x,y).
\end{aligned}$$

Combining this with (3.2.19) and (3.2.20) we complete the proof. □

Combining Proposition 3.2.7 with Theorem 3.2.1, we obtain the following result.

Corollary 3.2.8. *Let* $\mathrm{Ric}_Z \geq K$ *for some constant* $K \in \mathbb{R}$. *If there exists* $\phi \in \mathcal{D}$ *in* (3.0.2) *such that* $K_\phi > -\infty$, *then*

$$|\nabla P_t f| \leq \|\phi\|_\infty (P_t|\nabla f|^{p/(p-1)})^{(p-1)/p}(x)\mathrm{e}^{-(K+K_\phi^{(p)})t} \tag{3.2.21}$$

holds for $p \in [1,\infty)$, $f \in C_b^1(M)$ *and* $K_\phi^{(p)} := \inf\{\phi^{-1}L\phi - (p+1)|\nabla \log \phi|^2\}$. *Moreover,*

$$|\nabla P_t f|^2 \leq \frac{(K + K_\phi^{(2)})\|\phi\|_\infty^2}{\exp[2(K + K_\phi^{(2)})t] - 1} P_t f^2, \quad t > 0, f \in \mathcal{B}_b(M). \tag{3.2.22}$$

Proof. (a) Since (3.2.21) is equivalent to

$$|P_t f(x) - P_t f(y)| \leq \|\phi\|_\infty \mathrm{e}^{-(K+K_\phi^{(p)})t} \int_0^1 (P_t|\nabla f|^{p/(p-1)})^{(p-1)/p}(\gamma_s)|\dot\gamma_s|\mathrm{d}s$$

for any $x, y \in M$ and any smooth curve $\gamma : [0,1] \to M$ linking x and y, by an approximation argument it suffices to prove for $f \in \mathcal{C}_N(L)$. By the Itô formula, we have

$$\begin{aligned}
\mathrm{d}\phi^{-p}(X_t) &= \langle \nabla \phi^{-p}(X_t), u_t \mathrm{d}B_t\rangle + L\phi^{-p}(X_t)\mathrm{d}t + N\phi^{-p}(X_t)\mathrm{d}l_t \\
&\leq \langle \nabla \phi^{-p}(X_t), u_t \mathrm{d}B_t\rangle - p\phi^{-p}(X_t)\big\{K_\phi^{(p)}\mathrm{d}t + N\log\phi(X_t)\mathrm{d}l_t\big\}.
\end{aligned}$$

So,

$$M_t := \phi^{-p}(X_t)\exp\left[pK_\phi^{(p)}t + p\int_0^t N\log\phi(X_s)\mathrm{d}l_s\right]$$

is a local sup-martingale. Thus, by the Fatou lemma, and noting that $\phi \geq 1$,

$$\begin{aligned}
&\mathbb{E}^x\left\{\phi^{-p}(X_t)\exp\left[pK_\phi^{(p)}t + p\int_0^t N\log\phi(X_s)\mathrm{d}l_s\right]\right\} \\
&\leq \liminf_{n\to\infty} \mathbb{E}^x\left\{\phi^{-p}(X_{t\wedge\zeta_n})\exp\left[pK_\phi^{(p)}(t\wedge\zeta_n) + p\int_0^{t\wedge\zeta_n} N\log\phi(X_s)\mathrm{d}l_s\right]\right\} \\
&\leq \phi^{-p}(x) \leq 1.
\end{aligned}$$

Therefore,

$$\mathbb{E}^x \exp\left[p\int_0^t N\log\phi(X_s)\mathrm{d}l_s\right] \leq \|\phi\|_\infty^p \mathrm{e}^{-pK_\phi^{(p)}t}, \quad t \geq 0. \tag{3.2.23}$$

Since $\mathbb{I} \geq -N \log \phi$, by combining this with Theorem 3.2.1 for $\sigma = -N \log \phi$ and Proposition 3.2.7, we obtain

$$\begin{aligned}|\nabla P_t f(x)|^p &\leq (P_t|\nabla f|^{p/(p-1)}(x))^{p-1}\mathbb{E}^x\|Q_t\|^p \\ &\leq (P_t|\nabla f|^{p/(p-1)}(x))^{p-1}\mathbb{E}^x \exp\left[-pKt + p\int_0^t N\log\phi(X_s)\mathrm{d}l_s\right] \\ &\leq \|\phi\|_\infty^p (P_t|\nabla f|^{p/(p-1)}(x))^{p-1}\mathrm{e}^{-p(K+K_\phi^{(p)})t}.\end{aligned}$$

Therefore, the first inequality holds.

(b) Since (3.2.22) is equivalent to

$$\begin{aligned}&P_t f(x) - P_t f(y) \\ &\leq \|\phi\|_\infty \left(\frac{K + K_\phi^{(2)}}{\exp[2(K + K_\phi^{(2)})t] - 1}\right)^{1/2} \int_0^1 |\dot\gamma_s|(P_t f^2)^{1/2}(\gamma_s)\mathrm{d}s\end{aligned}$$

for any $x, y \in M$ and any smooth curve $\gamma : [0,1] \to M$ linking x and y, by the monotone class theorem, it suffices to prove for $f \in \mathcal{C}_N(L)$. Take

$$h(s) = \frac{\mathrm{e}^{2(K+K_\phi^{(2)})s} - 1}{\mathrm{e}^{2(K+K_\phi^{(2)})t} - 1}, \quad s \in [0,t].$$

Then the second formula in (3.2.2) and (3.2.23) for $p = 2$ imply

$$\begin{aligned}|\nabla P_t f|^2 &\leq \frac{P_t f^2}{2}\mathbb{E}\int_0^t h'(s)^2\|Q_s\|^2\mathrm{d}s \\ &\leq \frac{P_t f^2}{2}\mathbb{E}\int_0^t h'(s)^2 \exp\left[-2Ks + 2\int_0^s N\log\phi(X_r)\mathrm{d}l_r\right]\mathrm{d}s \\ &\leq \|\phi\|_\infty^2 \frac{P_t f^2}{2}\mathbb{E}\int_0^t h'(s)^2 \mathrm{e}^{-2(K+K_\phi^{(2)})s}\mathrm{d}s \\ &= \frac{(K + K_\phi^{(2)})\|\phi\|_\infty^2}{\exp[2(K + K_\phi^{(2)})t] - 1} P_t f^2.\end{aligned}$$

□

Finally, to conclude this section, we present an explicit construction of ϕ under the following assumption, which is trivial when M is compact.

(A3.2.1) At least one of the following holds:

(i) ∂M is convex;

(ii) $\mathbb{I}$ is bounded and there exists $r_0 > 0$ such that on the set $\partial_{r_0} M := \{x \in M : \rho_\partial(x) \leq r_0\}$ ρ_∂ is smooth, Z is bounded, and Sect is bounded above.

Under this assumption, we will be able to construct the desired function ϕ by using ρ_∂. Thus, to calculate K_ϕ and $K_\phi^{(p)}$, we shall make use of the Laplacian comparison Theorem 1.2.3. To this end, for any $\theta, k \geq 0$, let

$$h(s) = \cos\sqrt{k}\,s - \frac{\theta}{\sqrt{k}}\sin\sqrt{k}\,s, \quad s \geq 0.$$

Then $h^{-1}(0) = k^{-1/2}\arcsin\frac{\sqrt{k}}{\sqrt{k+\theta^2}}$. Moreover, let

$$\delta = \delta(r_0, \sigma, k, \theta) = \frac{-\sigma(1-h(r_0))^{d-1}}{\int_0^{r_0}(h(s)-h(r_0))^{d-1}\mathrm{d}s}.$$

Theorem 3.2.9. *Assume* **(A3.2.1)** *and let $K \in C_b(M)$ and $\sigma \in C_b(\partial M)$ such that $\mathrm{Ric}_Z \geq K$ and $\mathbb{I} \geq \sigma$. Then for any $t > 0$:*

(1) *There exists a progressively measurable process $\{Q_s\}_{s\in[0,t]}$ on $\mathbb{R}^d \otimes \mathbb{R}^d$ such that*

$$Q_0 = I, \quad \|Q_s\| \leq \exp\left[-\int_0^s K(X_r)\mathrm{d}s - \int_0^s \sigma(X_r)\mathrm{d}l_r\right], \quad s \in [0,t],$$

and for any $h \in C^1([0,t])$ such that $h(0) = 0, h(t) = 1$,

$$\begin{aligned} u_0^{-1}\nabla P_t f(x) &= \mathbb{E}^x\{Q_t^* u_t^{-1}\nabla f(X_t)\} \\ &= \mathbb{E}^x\left\{\frac{f(X_t)}{\sqrt{2}}\int_0^t h'(s)Q_s^*\mathrm{d}B_s\right\}, \quad f \in C_b^1(M). \end{aligned}$$

(2) *Let K and σ be constant functions, and let $\mathbb{I} \leq \theta, \mathrm{Sect}|_{\partial_{r_0}M} \leq k$ hold for some constants $k, \theta \geq 0$. Then* (3.2.22) *and*

$$\mathbb{E}^x \mathrm{e}^{-p\sigma l_t} \leq \|\phi\|_\infty^p \mathrm{e}^{-pK_\phi^{(p)}t}, \quad x \in M, t \geq 0 \qquad (3.2.24)$$

hold for $K_\phi^{(p)} = 0$ if ∂M is convex, and for

$$K_\phi^{(p)} = -\delta - \sigma^- \sup_{\partial_{r_0}M}|Z| - (p+1)(\sigma^-)^2$$

and

$$\begin{aligned} \|\phi\|_\infty &= 1 + \delta\int_0^{r_0}(h(s)-h(r_0))^{1-d}\mathrm{d}s\int_s^{r_0}(h(r)-h(r_0))^{d-1}\mathrm{d}r \\ &\leq 1 + \frac{r_0^2}{\delta} \end{aligned}$$

if (ii) in **(A3.2.1)** *holds with $r_0 \leq k^{-1/2}\arcsin\frac{\sqrt{k}}{\sqrt{k+\theta^2}}$.*

Proof. According to Theorem 3.2.1, it suffices to prove (2). Moreover, by Corollary 3.2.6, when ∂M is convex the desired assertions follow immediately by taking $\phi = 1$. So, it remains to prove (2) for $\sigma < 0$ by using (ii) in **(A3.2.1)**. The key point of the proof is to construct a suitable function ϕ by using ρ_∂.

Let $\phi = \varphi \circ \rho_\partial$, where

$$\varphi(r) = 1 + \delta \int_0^r (h(s) - h(r_0))^{1-d} \mathrm{d}s \int_{s \wedge r_0}^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u, \quad r \geq 0.$$

By an approximation argument we may regard ϕ as C^∞-smooth (cf. page 1436 in [Wang (2007a)]). Obviously, $\phi \geq 1, N \log \phi = -\sigma \geq -\mathbb{I}$. Since $\varphi' \geq 0$, according to Theorem 1.2.3(1), we have

$$\Delta \varphi \circ \rho_\partial \geq \Big(\frac{(d-1)\varphi' h'}{h} + \varphi''\Big)(\rho_\partial) \geq -\delta, \quad \rho_\partial \leq r_0. \tag{3.2.25}$$

Since $|\nabla \log \phi|$ and $|Z|$ are bounded on $\partial_{r_0} M$, this implies that $K_\phi > -\infty$. Noting that $\varphi' \geq 0, \varphi'' \leq 0$ so that $\frac{\varphi'}{\varphi}$ is decreasing, we have

$$\begin{aligned}\frac{1}{\phi} L\phi - \frac{p+1}{\phi^2} |\nabla \phi|^2 &\geq -\delta - \varphi'(0) \sup_{\partial_{r_0} M} |Z| - (p+1)\varphi'(0)^2 \\ &= -\delta + \sigma \sup_{\partial_{r_0} M} |Z| - (p+1)\sigma^2.\end{aligned}$$

Therefore, the proof is complete by (3.2.22) and (3.2.23). □

3.3 Equivalent semigroup inequalities for curvature condition and lower bound of $\mathbb{I}$

We first introduce equivalent semigroup inequalities for the lower bounds of Ric_Z and $\mathbb{I}$, which are corresponding to those in Theorem 2.3.1 for manifolds without boundary, then extend Theorem 2.7.1 to manifolds with boundary using the curvature-dimension condition and lower bound of $\mathbb{I}$. The first part is mainly based on [Wang (2010b,d)], and the second part is new.

3.3.1 *Equivalent statements for lower bounds of* Ric_Z *and* $\mathbb{I}$

Theorem 3.3.1. *Assume* **(A3.2.1)** *and let* $p \in [1, \infty), \bar{p} = p \wedge 2$. *Then for any* $K \in C_b(M)$ *and* $\sigma \in C_b(\partial M)$, *the following statements are equivalent to each other:*

(1) $\mathrm{Ric}_Z \geq K$ *and* $\mathbb{I} \geq \sigma$.

(2) *For any* $t \geq 0,\ x \in M,\ f \in C_b^1(M)$,

$$|\nabla P_t f(x)|^p \leq \mathbb{E}^x\Big\{|\nabla f|^p(X_t)\exp\big[-p\int_0^t K(X_s)\mathrm{d}s - p\int_0^t \sigma(X_s)\mathrm{d}l_s\big]\Big\}.$$

(3) *For any* $t \geq 0, x \in M$ *and positive* $f \in C_b^1(M)$,

$$\frac{\tilde{p}[P_t f^2 - (P_t f^{2/\tilde{p}})^{\tilde{p}}](x)}{4(\tilde{p}-1)}$$
$$\leq \mathbb{E}^x\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{-2\int_s^t K(X_r)\mathrm{d}r - 2\int_s^t \sigma(X_r)\mathrm{d}l_r}\mathrm{d}s\Big\},$$

where when $p = 1$ *the inequality is understood as its limit as* $p \downarrow 1$:

$$P_t(f^2\log f^2)(x) - (P_t f^2(x))\log P_t f^2(x)$$
$$\leq 4\mathbb{E}^x\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{-2\int_s^t K(X_r)\mathrm{d}r - 2\int_s^t \sigma(X_r)\mathrm{d}l_r}\mathrm{d}s\Big\}.$$

(4) *For any* $t \geq 0, x \in M$ *and positive* $f \in C_b^1(M)$,

$$|\nabla P_t f|^2(x) \leq \frac{P_t f^{\tilde{p}} - (P_t f)^{\tilde{p}}}{\tilde{p}(\tilde{p}-1)} \times$$
$$\frac{1}{\int_0^t(\mathbb{E}^x\{(P_{t-s}f)^{2-\tilde{p}}(X_s)\exp[-2\int_0^s K(X_r)\mathrm{d}r - 2\int_0^s \sigma(X_r)\mathrm{d}l_r]\})^{-1}\mathrm{d}s},$$

where when $p = 1$ *the inequality is understood as its limit as* $p \downarrow 1$:

$$|\nabla P_t f|^2(x)$$
$$\leq \frac{[P_t(f\log f) - (P_t f)\log P_t f](x)}{\int_0^t(\mathbb{E}^x\{P_{t-s}f(X_s)\exp[-2\int_0^s K(X_r)\mathrm{d}r - 2\int_0^s \sigma(X_r)\mathrm{d}l_r]\})^{-1}\mathrm{d}s}.$$

Proof. By Theorem 3.2.9, it is easy to derive (2) from (1). Moreover, according to Theorems 3.2.3 and 3.2.4, we see that each of (2)-(4) implies (1). Finally, taking $f \in C^\infty(M)$ such that $Nf = 0$ and f is constant outside a compact set, similarly to the proof of Theorem 2.3.1 we derive (3) and (4) from (2). □

Next, the following is an extension of Theorem 2.3.3 to manifolds with convex boundary. See Theorem 4.4.2 in Chapter 4 for seven more equivalent transportation-cost inequalities.

Theorem 3.3.2. *Let* $p \in [1,\infty)$ *and* $K \in \mathbb{R}$ *be constants, and let* $p_t(x,y)$ *be the heat kernel of* P_t *w.r.t. a measure* μ *equivalent to the volume measure. Then the following assertions are equivalent to each other:*

(1) ∂M *is convex and* $\mathrm{Ric}_Z \geq K$.

(2) *For any* $x, y \in M$ *and* $t \geq 0$, $W_p^\rho(\delta_x P_t, \delta_y P_t) \leq \rho(x,y)\mathrm{e}^{-Kt}$ *holds.*

(2′) *For any* $\nu_1, \nu_2 \in \mathcal{P}(M)$ *and* $t \geq 0$,

$$W_p^\rho(\nu_1 P_t, \nu_2 P_t) \leq \mathrm{e}^{-Kt} W_p^\rho(\nu_1, \nu_2)$$

holds.

(3) *When* $p > 1$, *for any* $f \in \mathcal{B}_b^+(M)$,

$$(P_t f)^p(x) \leq P_t f^p(y) \exp\left[\frac{Kp\rho(x,y)^2}{2(p-1)(\mathrm{e}^{2Kt}-1)}\right], \quad x, y \in M, t > 0.$$

(4) *For any* $f \in \mathcal{B}_b(M)$ *with* $f \geq 1$,

$$P_t \log f(x) \leq \log P_t f(y) + \frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)}, \quad x, y \in M, t > 0.$$

(5) *When* $p > 1$, *for any* $t > 0$ *and* $x, y \in M$,

$$\int_M p_t(x,z)\left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{\frac{1}{p-1}} \mu(\mathrm{d}z) \leq \exp\left[\frac{Kp\rho(x,y)^2}{2(p-1)^2(\mathrm{e}^{2Kt}-1)}\right].$$

(6) *For any* $t > 0$ *and* $x, y \in M$,

$$\int_M p_t(x,z) \log \frac{p_t(x,z)}{p_t(y,z)} \mu(\mathrm{d}z) \leq \frac{K\rho(x,y)^2}{2(\mathrm{e}^{2Kt}-1)}.$$

(7) *For any* $0 < s \leq t$ *and* $1 < q_1 \leq q_2$ *satisfying* (2.3.1),

$$\{P_s(P_{t-s}f)^{q_2}\}^{\frac{1}{q_2}} \leq (P_t f^{q_1})^{\frac{1}{q_1}}, \quad f \geq 0, f \in \mathcal{B}_b(M).$$

(8) *For any* $0 < s \leq t$ *and* $0 < q_2 \leq q_1$ *or* $q_2 \leq q_1 < 0$ *such that* (2.3.1) *holds,*

$$(P_t f^{q_1})^{\frac{1}{q_1}} \leq \{P_s(P_{t-s}f)^{q_2}\}^{\frac{1}{q_2}}, \quad f > 0, f \in \mathcal{B}_b(M).$$

(9) $|\nabla P_t f|^p \leq \mathrm{e}^{-pKt} P_t |\nabla f|^p, \quad f \in C_b^1(M), t \geq 0.$

(10) *For any* $t \geq 0$ *and positive* $f \in C_b^1(M)$,

$$\frac{(p \wedge 2)\{P_t f^2 - (P_t |f|^{\frac{2}{p\wedge 2}})^{p\wedge 2}\}}{4(p \wedge 2 - 1)} \leq \frac{1 - \mathrm{e}^{-2Kt}}{2K} P_t |\nabla f|^2.$$

When $p = 1$ *the inequality reduces to the log-Sobolev inequality*

$$P_t(f^2 \log f^2) - (P_t f^2) \log P_t f^2 \leq \frac{2(1 - \mathrm{e}^{-2Kt})}{K} P_t |\nabla f|^2.$$

(11) *For any* $t > 0$ *and positive* $f \in C_b^1(M)$,

$$|\nabla P_t f|^2 \leq \frac{2K\{P_t f^{(p\wedge 2)} - (P_t f)^{p\wedge 2}\}(P_t f)^{(2-p)^+}}{(p \wedge 2)(p \wedge 2 - 1)(\mathrm{e}^{2Kt}-1)}.$$

When $p = 1$ *the inequality reduces to*

$$|\nabla P_t f|^2 \leq \frac{2K\{P_t(f \log f) - (P_t f) \log P_t f\} P_t f}{\mathrm{e}^{2Kt} - 1}.$$

(12) *For any $f \in \mathcal{B}_b(M)$ and $t > 0$,*

$$|\nabla P_t f|^2 \leq \frac{K}{e^{2Kt}-1}\{(I_G(P_t f))^2 - (P_t I_G(f))^2\}.$$

(13) *For any $f \in C_b^1(M)$ and $t \geq 0$,*

$$I_G(P_t f) \leq P_t \sqrt{I_G(f)^2 + \frac{1-e^{-2Kt}}{K}|\nabla f|^2}.$$

(14) *For any $f \in \mathcal{B}_b(M)$ and $t > 0$,*

$$\Phi_G^{-1}(P_t f)(x) \leq \Phi_G^{-1}(P_t f)(y) + \rho(x,y)\sqrt{\frac{K}{e^{2Kt}-1}}, \quad x, y \in M.$$

(15) *For any smooth domain $A \subset M$ and $A(r) := \{z \in M : \rho(z, A) \leq r\}$ for $r \geq 0$,*

$$P_t 1_A(x) \leq P_t 1_{A(e^{-Kt}\rho(x,y))}(y), \quad t \geq 0, x, y \in M.$$

Proof. Due to Theorems 3.1.3, 3.2.5 and 3.3.1, except (4) implying (1) all other implications can be proved as in the proof of Theorem 2.3.3. So, below we assume (4) and prove (1).

For a fixed point $x \in M^\circ$ and $X \in T_x M$, taking $f \in C_0^\infty(M)$ such that $\nabla f(x) = X$. $\mathrm{Hess}_f(x) = 0$ and $f = 0$ in a neighborhood of ∂M, then the argument in (c) in the proof of Theorem 2.3.3 works also for the present case. Thus, (4) implies $\mathrm{Ric}_Z \geq K$.

Next, for $x \in \partial M$ and $X \in T_x\partial M$, let $f \in C^\infty(M)$ be such that $f \geq 1, Nf|_{\partial M} = 0$ and $\nabla f(x) = X$. We may further assume that f is constant outside a compact set (see page 311 in [Wang (2010b)]). Let $\exp_x^\partial : T_x\partial M \to \partial M$ be the exponential map on the Riemannian manifold ∂M with the induced metric, and let

$$\gamma_t = \exp_x^\partial\left[-2t\nabla \log f(x)\right], \quad t \geq 0.$$

Applying (4) to $y = \gamma_t$ we obtain

$$P_t \log f(x) \leq \log P_t f(\gamma_t) + \frac{2Kt^2|\nabla \log f|^2(x)}{1-e^{-2Kt}}, \quad t \geq 0. \tag{3.3.1}$$

Since f satisfies the Neumann boundary condition, we have

$$\begin{aligned} P_t \log f(x) &= \log f(x) + \int_0^t P_s L \log f(x)\mathrm{d}s \\ &= \log f(x) + \int_0^t P_s \frac{Lf}{f}(x)\mathrm{d}s - \int_0^t P_s|\nabla \log f|^2(x)\mathrm{d}s. \end{aligned} \tag{3.3.2}$$

On the other hand, let X_s be the L-reflecting diffusion process starting at x with local time l_s on ∂M, and let $\sigma_1 = \inf\{s \geq 0 : \rho(x, X_s) \geq 1\}$. By Lemma 3.1.1 we have

$$\begin{aligned} P_s|\nabla \log f|^2(x) &= \mathbb{E}|\nabla \log f|^2(X_{s\wedge\sigma_1}) + \mathrm{o}(s) \\ &= \mathrm{O}(s) + |\nabla \log f|^2(x) + \mathbb{E}\int_0^{s\wedge\sigma_1} \langle N, \nabla|\nabla \log f|^2\rangle(X_r)\mathrm{d}l_r. \end{aligned}$$

Since f satisfies the Neumann boundary condition so that

$$\langle N, \nabla|\nabla \log f|^2\rangle = 2f^{-2}\mathrm{Hess}_f(N, \nabla f),$$

and since $\langle \nabla f, \nabla\langle N, \nabla f\rangle\rangle = 0$ implies

$$\mathrm{Hess}_f(N, \nabla f) = -\langle \nabla_{\nabla f} N, \nabla f\rangle = \mathbb{I}(\nabla f, \nabla f),$$

it follows that

$$\begin{aligned} P_s|\nabla \log f|^2(x) &= |\nabla \log f|^2(x) + \mathrm{O}(s) \\ &\quad + 2f(x)^{-2}\mathbb{I}(\nabla f, \nabla f)(x)\mathbb{E}l_{s\wedge\sigma_1} + \mathrm{o}(\mathbb{E}l_{s\wedge\sigma_1}). \end{aligned}$$

Noting that due to Lemma 3.1.2 we have $\mathbb{E}l_{s\wedge\sigma_1} = \frac{2\sqrt{s}}{\sqrt{\pi}} + \mathrm{o}(s)$, this and (3.3.2) yield (recall that $\nabla f(x) = X$)

$$\begin{aligned} P_t \log f(x) &= \log f(x) + \int_0^t P_s\frac{Lf}{f}(x)\mathrm{d}s - |\nabla \log f|^2(x)t \\ &\quad - \frac{8t^{3/2}}{3\sqrt{\pi}\, f^2(x)}\mathbb{I}(X, X) + \mathrm{o}(t^{3/2}). \end{aligned} \tag{3.3.3}$$

On the other hand, we have

$$\begin{aligned} P_t f(\gamma_t) &= f(\gamma_t) + \int_0^t P_s Lf(\gamma_t)\mathrm{d}s \\ &= f(x) + t\langle\dot{\gamma}_s, \nabla f(\gamma_s)\rangle|_{s=0} + O(t^2) + \int_0^t P_s Lf(x)\mathrm{d}s \\ &= f(x) - \frac{2t}{f(x)}|\nabla f|^2(x) + \int_0^t P_s Lf(x)\mathrm{d}s + O(t^2). \end{aligned}$$

Thus,

$$\log P_t f(\gamma_t) = \log f(x) + \frac{1}{f(x)}\int_0^t P_s Lf(x)\mathrm{d}s - 2t|\nabla \log f|^2(x) + O(t^2).$$

Combining this with (3.3.1) and (3.3.3) we arrive at

$$\begin{aligned} &\frac{1}{t\sqrt{t}}\int_0^t \Big(P_s\frac{Lf}{f} - \frac{P_sLf}{f}\Big)(x)\mathrm{d}s \\ &+ \frac{1}{\sqrt{t}}\Big(1 - \frac{2Kt}{1 - \mathrm{e}^{-2Kt}}\Big)|\nabla \log f|^2(x) \\ &\leq \frac{8}{3\sqrt{\pi}\, f^2(x)}\mathbb{I}(X, X) + o(1). \end{aligned} \tag{3.3.4}$$

Obviously,

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\Big(1-\frac{2Kt}{1-e^{-2Kt}}\Big)=0.$$

So, to derive $\mathbb{I}(X,X)\geq 0$ from (3.3.4) it suffices to verify

$$\lim_{t\to 0}\frac{1}{t\sqrt{t}}\int_0^t\Big(P_s\frac{Lf}{f}-\frac{P_sLf}{f}\Big)(x)\mathrm{d}s=0. \tag{3.3.5}$$

Noting that since $f^{-1}\in C_b^2(M)$ with Lf^{-1} bounded and $Nf^{-1}=0$, we have

$$\begin{aligned}
&\Big|P_s\frac{Lf}{f}-\frac{P_sLf}{f}\Big|(x)\\
&=\Big|\mathbb{E}\Big\{(Lf(X_s)-Lf(x))\Big(\frac{1}{f(X_s)}-\frac{1}{f(x)}\Big)+Lf(x)\Big(\frac{1}{f(X_s)}-\frac{1}{f(x)}\Big)\Big\}\Big|\\
&\leq\big(\mathbb{E}(Lf(X_s)-Lf(x))^2\big)^{1/2}\big(\mathbb{E}(f(X_s)^{-1}-f(x)^{-1})^2\big)^{1/2}\\
&\quad+\Big|Lf(x)\,\mathbb{E}\int_0^s Lf(X_r)^{-1}\mathrm{d}r\Big|\\
&=\mathrm{o}(1)\big(\mathbb{E}(f(X_s)^{-1}-f(x)^{-1})^2\big)^{1/2}+\mathrm{O}(s).
\end{aligned}$$

Since the bounded ness of Lf^{-1} and $Nf^{-1}=0$ imply

$$\begin{aligned}
&\mathbb{E}(f(X_s)^{-1}-f(x)^{-1})^2\\
&=\mathbb{E}\Big\{\sqrt{2}\int_0^s\langle\nabla f^{-1}(X_r),u_r\mathrm{d}B_r\rangle+\int_0^s Lf^{-1}(X_r)\mathrm{d}r\Big\}^2\\
&=\mathrm{O}(s),
\end{aligned}$$

we conclude that

$$\Big|P_s\frac{Lf}{f}-\frac{P_sLf}{f}\Big|(x)=\mathrm{o}\big(\sqrt{s}\big).$$

Therefore, (3.3.5) holds. □

According to Theorems 3.3.1 and 3.3.2, the argument in §2.3 also work for manifolds with convex boundary. Therefore, Theorems 2.4.1, 2.4.2 and 2.4.4 hold for the reflecting diffusion process provided $\mathbb{I}\geq 0$. When ∂M is non-convex, the situation is however very different.

3.3.2 *Equivalent inequalities for curvature-dimension condition and lower bound of* $\mathbb{I}$

Corresponding to Theorem 2.7.1, we have the following equivalent statements for (2.7.1) and $\mathbb{I} \geq \sigma$.

Theorem 3.3.3. *Assume* **(A3.2.1)**. *Let $K \in \mathbb{R}, \sigma \leq 0$ and $n \geq d$ be constants. Then the following statements are equivalent to each other:*

(1) (2.7.1) *holds and* $\mathbb{I} \geq \sigma$;

(2)

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt}\mathbb{E}\{|\nabla f|^2(X_t)\mathrm{e}^{-2\sigma l_t}\} - \frac{2}{n}\int_0^t \mathrm{e}^{-2Ks}\mathbb{E}\{(P_{t-s}Lf)^2(X_s)\mathrm{e}^{-2\sigma l_s}\}\mathrm{d}s$$

holds for $t \geq 0$ and $f \in C_0^2(M)$ with $Nf|_{\partial M} = 0$;

(3)

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt}\mathbb{E}\{|\nabla f|^2(X_t)\mathrm{e}^{-2\sigma l_t}\} - \frac{1-\mathrm{e}^{-2Kt}}{nK}(P_t Lf)^2$$

holds for $t \geq 0$ and $f \in C_0^2(M)$ with $Nf|_{\partial M} = 0$;

(4)

$$P_t f^2 - (P_t f)^2 \leq 2\mathbb{E}\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{-2K(t-s)-2\sigma(l_t-l_s)}\mathrm{d}s\Big\} - \frac{\mathrm{e}^{-2Kt}-1+2Kt}{nK^2}(P_t Lf)^2$$

holds for $t \geq 0$ and $f \in C_0^2(M)$ with $Nf|_{\partial M} = 0$.

When $\sigma = 0$, i.e. ∂M is convex, they are also equivalent to

(5)

$$P_t f^2 - (P_t f)^2 \geq \frac{\mathrm{e}^{2Kt}-1}{K}|\nabla P_t f|^2 + \frac{\mathrm{e}^{2Kt}-1-2Kt}{K^2 n}(P_t Lf)^2$$

holds for $t \geq 0, f \in C_0^2(M)$ with $Nf|_{\partial M} = 0$.

Proof. **(1) implies (2).** By the Itô formula and (1), there exists a local martingale M_s such that

$$\begin{aligned}
\mathrm{d}|\nabla P_{t-s}f|^2(X_s) &= \mathrm{d}M_s + \{L|\nabla P_{t-s}f|^2 - 2\langle\nabla P_{t-s}f, \nabla LP_{t-s}f\rangle\}(X_s)\mathrm{d}s \\
&\quad + 2\mathbb{I}(\nabla P_{t-s}f, \nabla P_{t-s}f)(X_s)\mathrm{d}l_s \\
&\geq \mathrm{d}M_s + \Big\{2K|\nabla P_{t-s}f|^2(X_s) + \frac{2}{n}(P_{t-s}Lf)^2\Big\}\mathrm{d}s \\
&\quad + 2\sigma|\nabla P_{t-s}f|^2(X_s)\mathrm{d}l_s, \quad s \in [0,t].
\end{aligned}$$

Then

$$[0,t] \ni s \mapsto |\nabla P_{t-s} f|^2(X_s)\mathrm{e}^{-2(Ks+\sigma l_s)} - \frac{2}{n}\int_0^s (P_{t-r}Lf)^2(X_r)\mathrm{e}^{-2(Kr+\sigma l_r)}\mathrm{d}r$$

is a local submartingale. Moreover, by Theorem 3.2.9 this process is square integrable, so that is indeed a submartingale. Thus, (2) holds.

(2) implies (3). It suffices to note that since $\sigma \le 0$,

$$\mathbb{E}\{(P_{t-s}Lf)^2(X_s)\mathrm{e}^{-2\sigma l_s}\} \ge P_s(P_{t-s}Lf)^2 \ge (P_tLf)^2.$$

(3) implies (4)/(5). By (3), the Markov property and the Jensen inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}f)^2 &= 2P_s|\nabla P_{t-s}f|^2\\ &\le 2\mathrm{e}^{-2K(t-s)}\mathbb{E}\{|\nabla f|^2(X_t)\mathrm{e}^{-2\sigma(l_t-l_s)}\}\\ &\quad - \frac{2(1-\mathrm{e}^{-2K(t-s)})}{nK}(P_tLf)^2, \quad s\in[0,t].\end{aligned}$$

Integrating w.r.t. $\mathrm{d}s$ on $[0,t]$ implies (4). When $\sigma = 0$ the proof of (3) implying (5) is similar to the case without boundary (see the proof of Theorem 2.7.1).

(4)/(5) implies (1). Let $x \in M \setminus \partial M$. By Lemma 3.1.1 we have $\mathbb{P}(l_t > 0) \le \mathrm{e}^{-c/t}$ for some constant $c > 0$ and small $t > 0$. Then, similarly to the proof of Theorem 2.7.1, part "(3) implies (2.7.1)", it is easy to see that instead of (3) therein the present (4) implies (2.7.1) as well. Moreover, since (4) is stronger than Theorem 3.3.1(3) for $p = 2$, it also implies $\mathbb{I} \ge \sigma$. Similarly, when (5) implies (1) for $\sigma = 0$. □

Moreover, when ∂M is convex we have $N|\nabla f|^2|_{\partial M} \ge 0$ for f satisfying the Neumann boundary condition. Then repeating the proof of Corollary 2.7.3 and using Theorem 3.3.3 for $K = \sigma = 0$ in place of Theorem 2.7.1, we obtain the following result.

Corollary 3.3.4. *Let $n \ge d$ and $K = 0$. Then each of the following inequalities is equivalent to* (2.7.1) *and* $\mathbb{I} \ge 0$.

(1) $(P_tf)L(\log P_tf) \ge P_t(f\log f)\big(1+\frac{2t}{n}L\log P_tf\big)$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*
(2) $tLP_tf - \frac{n}{2}(P_tf)\log\big(1+\frac{2t}{n}L\log P_tf\big) \le P_t(f\log f) - (P_tf)\log P_tf$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*
(3) $P_t(f\log f) - (P_tf)\log P_tf \le tLP_tf + \frac{n}{2}(P_tf)\log\big(1-\frac{2tP_t(fL\log f)}{nP_tf}\big)$ *holds for all strictly positive $f \in C_b^\infty(M)$ and $t \ge 0$.*

(4) *For any* $q_1 > q_2 > 0$ *and* $t_1, t_2 > 0$ *such that* $t := 2(t_1q_1 - t_2q_2) > 0$,

$$(P_{t_1}\mathrm{e}^{q_1 Q_t f})^{\frac{1}{q_1}} \le (P_{t_2}\mathrm{e}^{q_2 f})^{\frac{1}{q_2}} t_1^{\frac{n}{2q_2}} t_2^{-\frac{n}{2q_1}} \Big(\frac{2(q_1-q_2)}{t}\Big)^{\frac{n(q_1-q_2)}{2q_1q_2}}$$

holds for all $f \in \mathcal{B}_b(M)$.

(5) *For any* $o \in M, t_1, t_2 > 0$ *and positive* $f \in \mathcal{B}_b(M)$ *with* $P_{t_1}f(o) = 1$,

$$W_2^\rho(fP_{t_1}(o,\cdot), P_{t_2}(o,\cdot))^2 \le 4t_1\Big\{P_{t_1}(f\log f)(o) + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big)\Big\}.$$

(6) *For any* $o \in M, t_1, t_2 > 0$ *and strictly positive* $f \in C_b^\infty(M)$ *with* $P_{t_1}f(o) = 1$,

$$\begin{aligned} &W_2^\rho(fP_{t_1}(o,\cdot), P_{t_2}(o,\cdot))^2 \\ &\le 4t_1\Big\{t_1P_{t_1}Lf + \frac{n}{2}(P_{t_1}f)\log\Big(1 - \frac{2t_1P_{t_1}fL\log f}{nP_{t_1}f}\Big) \\ &\quad + \frac{n}{2}\Big(\frac{t_2}{t_1} - 1 - \log\frac{t_2}{t_1}\Big)\Big\}. \end{aligned}$$

3.4 Harnack inequalities for SDEs on $\mathbb{R}^d$ and extension to non-convex manifolds

The purpose of this section is to establish Harnack inequalities on manifolds with non-convex boundary. As we do not have effective coupling argument for the reflecting diffusion processes on non-convex manifolds, we will take a conformal change of metric $\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle$ as in Theorem 1.2.5 to make the boundary convex. According to the proof of Proposition 3.2.7, under the new metric the generator $L := \Delta + Z$ becomes to $\phi^{-2}(\Delta' + Z')$ for some vector field Z', where Δ' is the Laplacian induced by the new metric. This suggests us to investigate the reflecting diffusion processes with non-constant diffusion coefficients on convex manifolds. In order to make our argument easy to follow, we start from a stochastic differential equation on $\mathbb{R}^d$ with non-constant diffusion coefficient, which is interesting by itself.

Consider the following SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(t, X_t)\mathrm{d}B_t + b(t, X_t)\mathrm{d}t, \tag{3.4.1}$$

where B_t is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, and

$$\sigma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$$

are measurable and continuous in the second variable. Throughout the paper we assume that for any $X_0 \in \mathbb{R}^d$ the equation (3.4.1) has a unique strong solution which is non-explosive and continuous in t.

Let X_t^x be the solution to (3.4.1) for $X_0 = x$. We aim to establish the Harnack inequality for the operator P_t:

$$P_t f(x) := \mathbb{E} f(X_t^x), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b^+(\mathbb{R}^d),$$

where $\mathcal{B}_b^+(\mathbb{R}^d)$ is the class of all bounded non-negative measurable functions on $\mathbb{R}^d$. To this end, we shall make use of the following assumptions.

(A3.4.1) There exists an increasing function $K : [0, \infty) \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d, t \geq 0$,

$$\|\sigma(t,x) - \sigma(t,y)\|_{HS}^2 + 2\langle b(t,x) - b(t,y), x - y\rangle \leq K_t |x-y|^2.$$

(A3.4.2) There exists a decreasing function $\lambda : [0, \infty) \to (0, \infty)$ such that

$$\sigma(t,x)^* \sigma(t,x) \geq \lambda_t^2 I, \quad x \in \mathbb{R}^d, t \geq 0.$$

(A3.4.3) There exists an increasing function $\delta : [0, \infty) \to (0, \infty)$ such that

$$|(\sigma(t,x) - \sigma(t,y))^*(x-y)| \leq \delta_t |x-y|, \quad x, y \in \mathbb{R}^d, t \geq 0.$$

(A3.4.4) For $n \geq 1$ there exists a constant $c_n > 0$ such that

$$\|\sigma(t,x) - \sigma(t,y)\|_{HS} + |b(t,x) - b(t,y)| \leq c_n |x-y|, \quad |x|, |y|, t \leq n.$$

It is well known that **(A3.4.1)** ensures the uniqueness of the solution to (3.4.1) while **(A3.4.4)** implies the existence and the uniqueness of the strong solution. On the other hand, if b and σ depend only on the variable $x \in \mathbb{R}^d$, then their continuity in x implies the existence of weak solutions (see Theorem 2.3 in [Ikeda and Watanabe (1989)]), so that by the Yamada-Watanabe principle [Yamada and Watanabe (1971)], the uniqueness ensured by **(A3.4.1)** implies the existence and uniqueness of the strong solution.

Note that if $\sigma(t,x)$ and $b(t,x)$ are deterministic and independent of t, then the solution is a time-homogeneous Markov process generated by

$$L := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j + \sum_{i=1}^d b_i \partial_i,$$

where $a := \sigma\sigma^*$. If further more σ and b are smooth, we may consider the Bakry-Emery curvature condition:

$$\Gamma_2(f,f) \geq -K\Gamma(f,f), \quad f \in C^\infty(\mathbb{R}^d) \tag{3.4.2}$$

for some constant $K \in \mathbb{R}$, where

$$\Gamma(f,g) := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\partial_i f)(\partial_j g), \quad f,g \in C^1(\mathbb{R}^d),$$

$$\Gamma_2(f,f) := \frac{1}{2}L\Gamma(f,f) - \Gamma(f,Lf), \quad f \in C^\infty(\mathbb{R}^d).$$

Then by Theorem 2.3.3(3), the dimension-free Harnack inequality

$$(P_t f(x))^p \le (P_t f^p(y)) \exp\left[\frac{p\rho_a(x,y)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right]$$

holds for $t \ge 0, p > 1, f \in \mathcal{B}_b^+(\mathbb{R}^d), x,y \in \mathbb{R}^d$, and

$$\rho_a(x,y) := \sup\left\{|f(x) - f(y)| : \ f \in C^1(\mathbb{R}^d), \Gamma(f,f) \le 1\right\}, \quad x,y \in \mathbb{R}^d.$$

On the other hand, however, in high dimensions it is very hard to verify the curvature condition (3.4.2) since it depends on second order derivatives of a^{-1}, the inverse matrix of a. This is the main reason why existing results on the dimension-free Harnack inequality for SPDEs are only proved for the additive noise case (i.e. σ is constant).

To handle the non-constant diffusion coefficient case, we first construct the coupling by change of measure required for the Harnack inequality according to Theorem 1.3.7.

3.4.1 *Construction of the coupling*

Let $x,y \in \mathbb{R}^d, T > 0$ and $p > (1+\delta_T/\lambda_T)^2$ be fixed such that $x \ne y$. We have

$$\theta_T := \frac{2\delta_T}{(\sqrt{p}-1)\lambda_T} \in (0,2). \tag{3.4.3}$$

For $\theta \in (0,2)$, let

$$\xi_t = \frac{2-\theta}{K_T}(1 - \mathrm{e}^{K_T(t-T)}), \quad t \in [0,T].$$

Then ξ is smooth and strictly positive on $[0,T)$ such that

$$2 - K_T\xi_t + \xi_t' = \theta, \quad t \in [0,T]. \tag{3.4.4}$$

Consider the coupling

$$\begin{cases} \mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}B_t + b(t,X_t)\mathrm{d}t, \ X_0 = x, \\ \mathrm{d}Y_t = \sigma(t,Y_t)\mathrm{d}B_t + b(t,Y_t)\mathrm{d}t \\ \qquad\quad + \frac{1}{\xi_t}\sigma(t,Y_t)\sigma(t,X_t)^{-1}(X_t - Y_t)\mathrm{d}t, \ Y_0 = y. \end{cases} \tag{3.4.5}$$

Since the additional drift term $\xi_t^{-1}\sigma(t,y)\sigma(t,x)^{-1}(x-y)$ is locally Lipschitzian in y if **(A3.4.4)** holds, and continuous in y when σ and b are deterministic and time independent, the coupling (X_t, Y_t) is a well defined continuous process for $t < T\wedge\zeta$, where ζ is the explosion time of Y_t; namely, $\zeta = \lim_{n\to\infty}\zeta_n$ for

$$\zeta_n := \inf\{t\in[0,T) : |Y_t| \geq n\},$$

where we set $\inf\emptyset = T$. Let

$$\mathrm{d}\bar{B}_t = \mathrm{d}B_t + \frac{1}{\xi_t}\sigma(t,X_t)^{-1}(X_t - Y_t)\mathrm{d}t, \quad t < T\wedge\zeta.$$

If $\zeta = T$ and

$$\begin{aligned} R_s := \exp\bigg[& -\int_0^s \xi_t^{-1}\langle\sigma(t,X_t)^{-1}(X_t-Y_t), \mathrm{d}B_t\rangle \\ & -\frac{1}{2}\int_0^s \xi_t^{-2}|\sigma(t,X_t)^{-1}(X_t-Y_t)|^2\mathrm{d}t\bigg] \end{aligned}$$

is a uniformly integrable martingale for $s\in[0,T)$, then by the martingale convergence theorem, $R_T := \lim_{t\uparrow T} R_t$ exists and $\{R_t\}_{t\in[0,T]}$ is a martingale. In this case, by the Girsanov theorem $\{\bar{B}_t\}_{t\in[0,T)}$ is a d-dimensional Brownian motion under the probability $R_T\mathbb{P}$. Rewrite (3.4.5) as

$$\begin{cases} \mathrm{d}X_t = \sigma(t,X_t)\mathrm{d}\bar{B}_t + b(t,X_t)\mathrm{d}t - \frac{X_t - Y_t}{\xi_t}\mathrm{d}t, \quad X_0 = x, \\ \mathrm{d}Y_t = \sigma(t,Y_t)\mathrm{d}\bar{B}_t + b(t,Y_t)\mathrm{d}t, \quad Y_0 = y. \end{cases} \tag{3.4.6}$$

Since $\int_0^T \xi_t^{-1}\mathrm{d}t = \infty$, we will see that the additional drift $-\frac{X_t-Y_t}{\xi_t}\mathrm{d}t$ is strong enough to force the coupling to be successful up to time T. So, we first prove the uniform integrability of $\{R_{s\wedge\zeta}\}_{s\in[0,T)}$ w.r.t. $\mathbb{P}$ so that $R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}$ exists, then prove that $\zeta = T$ $\mathbb{Q}$-a.s. for $\mathbb{Q} := R_{T\wedge\zeta}\mathbb{P}$ so that $\mathbb{Q} = R_T\mathbb{P}$.

Let

$$\tau_n = \inf\{t\in[0,T) : |X_t| + |Y_t| \geq n\}.$$

Since X_t is non-explosive as assumed, we have $\tau_n\uparrow\zeta$ as $n\uparrow\infty$.

Lemma 3.4.1. *Assume* **(A3.4.1)** *and* **(A3.4.2)**. *Let $\theta\in(0,2), x,y\in\mathbb{R}^d$ and $T>0$ be fixed.*

(1) *There holds*

$$\sup_{s\in[0,T),n\geq 1}\mathbb{E}\big\{R_{s\wedge\tau_n}\log R_{s\wedge\tau_n}\big\} \leq \frac{K_T|x-y|^2}{2\lambda_T^2\theta(2-\theta)(1-\mathrm{e}^{-K_T T})}.$$

Consequently,

$$R_{s\wedge\zeta} := \lim_{n\uparrow\infty} R_{s\wedge\tau_n\wedge(T-1/n)}, \ s\in[0,T], \quad R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}$$

exist such that $\{R_{s\wedge\zeta}\}_{s\in[0,T]}$ is a uniformly integrable martingale.

(2) *Let* $\mathbb{Q} = R_{T\wedge\zeta}\mathbb{P}$. *Then* $\mathbb{Q}(\zeta = T) = 1$ *so that* $\mathbb{Q} = R_T\mathbb{P}$.

Proof. (1) Let $s \in [0,T)$ be fixed. By (3.4.6), **(A3.4.1)** and the Itô formula,

$$\mathrm{d}|X_t - Y_t|^2 \le 2\langle(\sigma(t,X_t) - \sigma(t,Y_t))(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle + K_T|X_t - Y_t|^2\mathrm{d}t - \frac{2}{\xi_t}|X_t - Y_t|^2\mathrm{d}t$$

holds for $t \le s \wedge \tau_n$. Combining this with (3.4.4) we obtain

$$\begin{aligned}\mathrm{d}\frac{|X_t - Y_t|^2}{\xi_t} &\le \frac{2}{\xi_t}\langle(\sigma(t,X_t) - \sigma(t,Y_t))(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle \\ &\quad - \frac{|X_t - Y_t|^2}{\xi_t^2}(2 - K_T\xi_t + \xi_t')\mathrm{d}t \\ &= \frac{2}{\xi_t}\langle(\sigma(t,X_t) - \sigma(t,Y_t))(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle \\ &\quad - \frac{\theta}{\xi_t^2}|X_t - Y_t|^2\mathrm{d}t, \quad t \le s \wedge \tau_n.\end{aligned} \tag{3.4.7}$$

Multiplying by $\frac{1}{\theta}$ and integrating from 0 to $s \wedge \tau_n$, we obtain

$$\int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\mathrm{d}t \le \int_0^{s\wedge\tau_n} \frac{2}{\theta\xi_t}\langle(\sigma(t,X_t) - \sigma(t,Y_t))(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle - \frac{|X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}|^2}{\theta\xi_{s\wedge\tau_n}} + \frac{|x-y|^2}{\theta\xi_0}.$$

By the Girsanov theorem, $\{\bar{B}_t\}_{t\le\tau_n\wedge s}$ is the d-dimensional Brownian motion under the probability measure $R_{s\wedge\tau_n}\mathbb{P}$. So, taking expectation $\mathbb{E}_{s,n}$ with respect to $R_{s\wedge\tau_n}\mathbb{P}$, we arrive at

$$\mathbb{E}_{s,n}\int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\mathrm{d}t \le \frac{|x-y|^2}{\theta\xi_0}, \quad s \in [0,T), n \ge 1. \tag{3.4.8}$$

By **(A3.4.2)** and the definitions of R_t and $\bar{B}_t$, we have

$$\begin{aligned}&\log R_r \\ &= -\int_0^r \frac{1}{\xi_t}\langle\sigma(t,X_t)^{-1}(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle + \frac{1}{2}\int_0^r \frac{|\sigma(t,X_t)^{-1}(X_t - Y_t)|^2}{\xi_t^2}\mathrm{d}t \\ &\le -\int_0^r \frac{1}{\xi_t}\langle\sigma(t,X_t)^{-1}(X_t - Y_t), \mathrm{d}\bar{B}_t\rangle + \frac{1}{2\lambda_T^2}\int_0^r \frac{|X_t - Y_t|^2}{\xi_t^2}\mathrm{d}t, \ r \le s \wedge \tau_n.\end{aligned}$$

Since $\{\bar{B}_t\}$ is the d-dimensional Brownian motion under $R_{s\wedge\tau_n}\mathbb{P}$ up to $s\wedge\tau_n$, combining this with (3.4.8) we obtain

$$\mathbb{E}R_{s\wedge\tau_n}\log R_{s\wedge\tau_n} = \mathbb{E}_{s,n}\log R_{s\wedge\tau_n} \le \frac{|x-y|^2}{2\lambda_T^2\theta\xi_0}, \quad s \in [0,T), n \ge 1.$$

By the martingale convergence theorem and the Fatou lemma, $\{R_{s\wedge\zeta}:\ s\in[0,T]\}$ is a well-defined martingale with

$$\mathbb{E}R_{s\wedge\zeta}\log R_{s\wedge\zeta}\le\frac{|x-y|^2}{2\lambda_T^2\theta\xi_0}=\frac{K_T|x-y|^2}{2\lambda_T^2\theta(2-\theta)(1-\mathrm{e}^{-K_TT})},\quad s\in[0,T].$$

To see that $\{R_{s\wedge\zeta}:\ s\in[0,T]\}$ is a martingale, let $0\le s<t\le T$. By the dominated convergence theorem and the martingale property of $\{R_{s\wedge\tau_n}:\ s\in[0,T)\}$, we have

$$\begin{aligned}\mathbb{E}(R_{t\wedge\zeta}|\mathcal{F}_s)&=\mathbb{E}\Big(\lim_{n\to\infty}R_{t\wedge\tau_n\wedge(T-1/n)}|\mathcal{F}_s\Big)=\lim_{n\to\infty}\mathbb{E}(R_{t\wedge\tau_n\wedge(T-1/n)}|\mathcal{F}_s)\\&=\lim_{n\to\infty}R_{s\wedge\tau_n}=R_{s\wedge\zeta}.\end{aligned}$$

(2) Let $\sigma_n=\inf\{t\ge0:\ |X_t|\ge n\}$. We have $\sigma_n\uparrow\infty$ $\mathbb{P}$-a.s and hence, also $\mathbb{Q}$-a.s. Since $\{\tilde B_t\}$ is a $\mathbb{Q}$-Brownian motion up to $T\wedge\zeta$, it follows from (3.4.7) that

$$\frac{(n-m)^2}{\xi_0}\mathbb{Q}(\sigma_m>t,\zeta_n\le t)\le\mathbb{E}_{\mathbb{Q}}\frac{|X_{t\wedge\sigma_m\wedge\zeta_n}-Y_{t\wedge\sigma_m\wedge\zeta_n}|^2}{\xi_{t\wedge\sigma_m\wedge\zeta_n}}\le\frac{|x-y|^2}{\xi_0}$$

holds for all $n>m>0$ and $t\in[0,T)$. By letting first $n\uparrow\infty$ then $m\uparrow\infty$, we obtain $\mathbb{Q}(\zeta\le t)=0$ for all $t\in[0,T)$. This is equivalent to $\mathbb{Q}(\zeta=T)=1$ according to the definition of ζ. □

Lemma 3.4.1 ensures that under $\mathbb{Q}:=R_{T\wedge\zeta}\mathbb{P}$, $\{\tilde B_t\}_{t\in[0,T]}$ is a Brownian motion. Then by (3.4.6), the coupling (X_t,Y_t) is well-constructed under $\mathbb{Q}$ for $t\in[0,T]$. Since $\int_0^T\xi_t^{-1}\mathrm{d}t=\infty$, we shall see that the coupling is successful up to time T, so that $X_T=Y_T$ holds $\mathbb{Q}$-a.s. (see the proof of Theorem 3.4.3 below). This will provide the desired Harnack inequality for P_t according to Theorem 1.3.7 provided $R_{T\wedge\zeta}$ has finite $p/(p-1)$-moment. The next lemma provides an explicit upper bound on moments of $R_{T\wedge\zeta}$.

Lemma 3.4.2. *Assume* **(A3.4.1)-(A3.4.3)**. *Let R_t and ξ_t be fixed for $\theta=\theta_T$. We have*

$$\begin{aligned}&\sup_{s\in[0,T]}\mathbb{E}\bigg\{R_{s\wedge\zeta}\exp\bigg[\frac{\theta_T^2}{8\delta_T^2}\int_0^{s\wedge\zeta}\frac{|X_t-Y_t|^2}{\xi_t^2}\mathrm{d}t\bigg]\bigg\}\\&\le\exp\bigg[\frac{\theta_TK_T|x-y|^2}{4\delta_T^2(2-\theta_T)(1-\mathrm{e}^{-K_TT})}\bigg].\end{aligned}\tag{3.4.9}$$

Consequently,

$$\sup_{s\in[0,T]}\mathbb{E}R_{s\wedge\zeta}^{1+r_T}\le\exp\bigg[\frac{\theta_TK_T(2\delta_T+\theta_T\lambda_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T+\theta_T\lambda_T)(1-\mathrm{e}^{-K_TT})}\bigg]\tag{3.4.10}$$

holds for

$$r_T=\frac{\lambda_T^2\theta_T^2}{4\delta_T^2+4\theta_T\lambda_T\delta_T}.$$

Proof. Let $\theta = \theta_T$. By (3.4.7), for any $r > 0$ we have

$$\begin{aligned}
&\mathbb{E}_{s,n} \exp\left[r \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} \mathrm{d}t\right] \\
&\leq \exp\left[\frac{r|x-y|^2}{\theta_T \xi_0}\right] \\
&\quad \times \mathbb{E}_{s,n} \exp\left[\frac{2r}{\theta_T} \int_0^{s\wedge\tau_n} \frac{1}{\xi_t}\langle(\sigma(t,X_t) - \sigma(t,Y_t))(X_t - Y_t), \mathrm{d}\tilde{B}_t\rangle\right] \\
&\leq \exp\left[\frac{rK_T|x-y|^2}{\theta_T(2-\theta_T)(1-\mathrm{e}^{-K_T T})}\right] \\
&\quad \times \left(\mathbb{E}_{s,n} \exp\left[\frac{8r^2\delta_T^2}{\theta_T^2} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} \mathrm{d}t\right]\right)^{1/2},
\end{aligned}$$

where the last step is due to **(A3.4.3)** and the fact that

$$\mathbb{E}\mathrm{e}^{M_t} \leq (\mathbb{E}\mathrm{e}^{2\langle M\rangle_t})^{1/2}$$

for a continuous exponential integrable martingale M_t. Taking $r = \frac{\theta_T^2}{8\delta_T^2}$, we arrive at, for all $n \geq 1$,

$$\mathbb{E}_{s,n} \exp\left[\frac{\theta_T^2}{8\delta_T^2} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2} \mathrm{d}t\right] \leq \exp\left[\frac{\theta_T K_T |x-y|^2}{4\delta_T^2(2-\theta_T)(1-\mathrm{e}^{-K_T T})}\right].$$

This implies (3.4.9) by letting $n \to \infty$.

Next, by **(A3.4.2)** and the definition of R_s, we have

$$\begin{aligned}
\mathbb{E}R_{s\wedge\tau_n}^{1+r_T} &= \mathbb{E}_{s,n} R_{s\wedge\tau_n}^{r_T} \\
&= \mathbb{E}_{s,n} \exp\left[-r_T \int_0^{s\wedge\tau_n} \frac{1}{\xi_t}\langle\sigma(t,X_t)^{-1}(X_t - Y_t), \mathrm{d}\tilde{B}_t\rangle\right. \\
&\qquad\qquad \left. + \frac{r_T}{2} \int_0^{s\wedge\tau_n} \frac{|\sigma(t,X_t)^{-1}(X_t - Y_t)|^2}{\xi_t^2} \mathrm{d}t\right].
\end{aligned} \tag{3.4.11}$$

Noting that for any exponential integrable martingale M_t w.r.t. $R_{s\wedge\tau_n}\mathbb{P}$, one has

$$\begin{aligned}
&\mathbb{E}_{s,n} \exp[r_T M_t + r_T\langle M\rangle_t/2] \\
&= \mathbb{E}_{s,n} \exp[r_T M_t - r_T^2 q\langle M\rangle_t/2 + r_T(qr_T + 1)\langle M\rangle_t/2] \\
&\leq \left(\mathbb{E}_{s,n} \exp[r_T q M_t - r_T^2 q^2\langle M\rangle_t/2]\right)^{1/q} \\
&\quad \times \left(\mathbb{E}_{s,n} \exp\left[\frac{r_T q(r_T q + 1)}{2(q-1)}\langle M\rangle_t\right]\right)^{(q-1)/q} \\
&= \left(\mathbb{E}_{s,n} \exp\left[\frac{r_T q(r_T q + 1)}{2(q-1)}\langle M\rangle_t\right]\right)^{(q-1)/q}, \quad q > 1,
\end{aligned}$$

it follows from (3.4.11) that

$$\mathbb{E}R_{s\wedge\tau_n}^{1+r_T} \le \left(\mathbb{E}_{s,n}\exp\left[\frac{qr_T(qr_T+1)}{2(q-1)\lambda_T^2}\int_0^{s\wedge\tau_n}\frac{|X_t-Y_t|^2}{\xi_t^2}\mathrm{d}t\right]\right)^{(q-1)/q}. \tag{3.4.12}$$

Take

$$q = 1+\sqrt{1+r_T^{-1}}, \tag{3.4.13}$$

which minimizes $q(qr_T+1)/(q-1)$ such that

$$\begin{aligned}\frac{qr_T(qr_T+1)}{2\lambda_T^2(q-1)} &= \frac{r_T+\sqrt{r_T(r_T+1)}}{2\lambda_T^2\sqrt{1+r_T^{-1}}}\left(r_T+1+\sqrt{r_T(r_T+1)}\right)\\ &= \frac{(r_T+\sqrt{r_T^2+r_T})^2}{2\lambda_T^2} = \frac{\theta_T^2}{8\delta_T^2}.\end{aligned} \tag{3.4.14}$$

Combining (3.4.12) with (3.4.9) and (3.4.14), and noting that due to (3.4.13) and the definition of r_T

$$\frac{q-1}{q} = \frac{\sqrt{1+r_T^{-1}}}{1+\sqrt{1+r_T^{-1}}} = \frac{2\delta_T+\theta_T\lambda_T}{2\delta_T+2\theta_T\lambda_T},$$

we obtain

$$\mathbb{E}R_{s\wedge\tau_n}^{1+r_T} \le \exp\left[\frac{\theta_T K_T(2\delta_T+\theta_T\lambda_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T+\theta_T\lambda_T)(1-\mathrm{e}^{-K_T T})}\right].$$

According to the Fatou lemma, the proof is then completed by letting $n\to\infty$. □

3.4.2 *Harnack inequality on $\mathbb{R}^d$*

Theorem 3.4.3. *Let $\sigma(t,x)$ and $b(t,x)$ either be deterministic and independent of t, or satisfy* **(A3.4.4)***.*

(1) *If* **(A3.4.1)** *and* **(A3.4.2)** *hold then*

$$P_T\log f(y) \le \log P_T f(x) + \frac{K_T|x-y|^2}{2\lambda_T^2(1-\mathrm{e}^{-K_T T})},\quad f\ge 1, x,y\in\mathbb{R}^d, T>0.$$

(2) *If* **(A3.4.1)**, **(A3.4.2)** *and* **(A3.4.3)** *hold, then for $p>(1+\frac{\delta_T}{\lambda_T})^2$ and $\delta_{p,T} := \max\{\delta_T, \frac{\lambda_T}{2}(\sqrt{p}-1)\}$, the Harnack inequality*

$$(P_T f(y))^p \le (P_T f^p(x))\exp\left[\frac{K_T\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta_{p,T}[(\sqrt{p}-1)\lambda_T-\delta_{p,T}](1-\mathrm{e}^{-K_T T})}\right]$$

holds for all $T>0, x,y\in\mathbb{R}^d$ and $f\in\mathcal{B}_b^+(\mathbb{R}^d)$.

Proof. (1) By Lemma 3.4.1, $\{R_{s\wedge\zeta}\}_{s\in[0,T]}$ is a uniformly integrable martingale and $\{\tilde{B}_t\}_{t\le T}$ is a d-dimensional Brownian motion under the probability $\mathbb{Q}$. Thus, Y_t can be solved up to time T. Let

$$\tau = \inf\{t \in [0,T]:\ X_t = Y_t\}$$

and set $\inf\emptyset = \infty$ by convention. We claim that $\tau \le T$ and thus, $X_T = Y_T$, $\mathbb{Q}$-a.s. Indeed, if for some $\omega \in \Omega$ such that $\tau(\omega) > T$, by the continuity of the processes we have

$$\inf_{t\in[0,T]} |X_t - Y_t|^2(\omega) > 0.$$

So,

$$\int_0^T \frac{|X_t - Y_t|^2}{\xi_t^2}\mathrm{d}t = \infty$$

holds on the set $\{\tau > T\}$. But according to Lemma 3.4.2 we have

$$\mathbb{E}_{\mathbb{Q}}\int_0^T \frac{|X_t - Y_t|^2}{\xi_t^2}\mathrm{d}t < \infty,$$

we conclude that $\mathbb{Q}(\tau > T) = 0$. Therefore, $X_T = Y_T$ $\mathbb{Q}$-a.s.

Now, combining Lemma 3.4.1 with $X_T = Y_T$ and using the Young inequality, for $f \ge 1$ we have

$$\begin{aligned} P_T \log f(y) &= \mathbb{E}_{\mathbb{Q}}[\log f(Y_T)] = \mathbb{E}[R_{T\wedge\zeta}\log f(X_T)] \\ &\le \mathbb{E}R_{T\wedge\zeta}\log R_{T\wedge\zeta} + \log \mathbb{E}f(X_T) \\ &\le \log P_T f(x) + \frac{K_T|x-y|^2}{2\lambda_T^2\theta(2-\theta)(1-\mathrm{e}^{-K_T T})}. \end{aligned}$$

This completes the proof of (1) by taking $\theta = 1$.

(2) Since **(A3.4.3)** holds for $\delta_{T,p}$ in place of δ_T, it suffices to prove the Harnack inequality for δ_T in place of $\delta_{T,p}$. Let $\theta = \theta_T$. Since $X_T = Y_T$ and $\{\tilde{B}_t\}_{t\in[0,T]}$ is the d-dimensional Brownian motion under $\mathbb{Q}$, we have

$$\begin{aligned} (P_T f(y))^p &= (\mathbb{E}_{\mathbb{Q}}[f(Y_T)])^p = (\mathbb{E}[R_{T\wedge\zeta}f(X_T)])^p \\ &\le (P_T f^p(x))(\mathbb{E}R_{T\wedge\zeta}^{p/(p-1)})^{p-1}. \end{aligned} \tag{3.4.15}$$

Due to (3.4.3) we see that

$$\frac{p}{p-1} = 1 + \frac{\lambda_T^2\theta_T^2}{4\delta_T(\delta_T + \theta_T\lambda_T)}.$$

So, it follows from Lemma 3.4.2 and (3.4.3) that

$$\begin{aligned} (\mathbb{E}R_{T\wedge\zeta}^{p/(p-1)})^{p-1} &= (\mathbb{E}R_{T\wedge\zeta}^{1+r_T})^{p-1} \\ &\le \exp\left[\frac{(p-1)\theta_T K_T(2\delta_T + \theta_T\lambda_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T+\theta_T\lambda_T)(1-\mathrm{e}^{-K_T T})}\right] \\ &= \exp\left[\frac{K_T\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta_T[(\sqrt{p}-1)\lambda_T - \delta_T](1-\mathrm{e}^{-K_T T})}\right]. \end{aligned}$$

Then the proof is finished by combining this with (3.4.15). $\square$

Let $p_t(x,y)$ be the density of P_t w.r.t. a Radon measure μ. Then according to Proposition 1.4.4, the above log-Harnack inequality and Harnack inequality are equivalent to the following heat kernel inequalities respectively:

$$\int_{\mathbb{R}^d} p_T(x,z)\log\frac{p_T(x,z)}{p_T(y,z)}\mu(\mathrm{d}z) \le \frac{K_T|x-y|^2}{2\lambda_T^2(1-\mathrm{e}^{-K_T T})}, \tag{3.4.16}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d} p_T(x,z)\Big(\frac{p_T(x,z)}{p_T(y,z)}\Big)^{1/(p-1)}\mu(\mathrm{d}z)\\ &\le \exp\Big[\frac{K_T\sqrt{p}\,|x-y|^2}{4\delta_{p,T}(\sqrt{p}+1)[(\sqrt{p}-1)\lambda_T-\delta_{p,T}](1-\mathrm{e}^{-K_T T})}\Big],\end{aligned} \tag{3.4.17}$$

for $x,y\in\mathbb{R}^d$, $T>0$. So, the following is a direct consequence of Theorem 3.4.3.

Corollary 3.4.4. *Let $\sigma(t,x)$ and $b(t,x)$ either be deterministic and independent of t, or satisfy* **(A3.4.4)**. *Let P_t have a strictly positive density $p_t(x,y)$ w.r.t. a Radon measure μ. Then* **(A3.4.1)** *and* **(A3.4.2)** *imply* (3.4.16), *while* **(A3.4.1)**-**(A3.4.3)** *imply* (3.4.17).

Finally, according to the proof of Theorem 2.4.2, the Harnack inequality with power in Theorem 3.4.3(2) implies the following contractivity properties of P_t.

Corollary 3.4.5. *Let $\sigma(t,x)$ and $b(t.x)$ be deterministic and independent of t, such that* **(A3.4.1)**-**(A3.4.3)** *hold for constant K,λ and δ. Let P_t have an invariant probability measure μ.*

(1) *If there exists $r>K^+/\lambda^2$ such that $\mu(\mathrm{e}^{r|\cdot|^2})<\infty$, then P_t is hypercontractive, i.e. $\|P_t\|_{L^2(\mu)\to L^4(\mu)}=1$ holds for some $t>0$.*
(2) *If $\mu(\mathrm{e}^{r|\cdot|^2})<\infty$ holds for all $r>0$, then P_t is supercontractive, i.e. $\|P_t\|_{L^2(\mu)\to L^4(\mu)}<\infty$ holds for all $t>0$.*
(3) *If $P_t\mathrm{e}^{r|\cdot|^2}$ is bounded for any $t,r>0$, then P_t is ultracontractive, i.e. $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)}<\infty$ for any $t>0$.*

3.4.3 *Extension to manifolds with convex boundary*

Let P_t be the (Neumann) semigroup generated by

$$L:=\psi^2(\Delta+Z)$$

on M with boundary ∂M, where $\psi \in C_b^1(M)$ and Z is a C^1 vector field on M. Assume that ψ is bounded and

$$\mathrm{Ric}_Z \geq K \tag{3.4.18}$$

holds for some constant K. Let

$$\kappa_\psi = K^-\|\psi\|_\infty^2 + 2\|Z\|_\infty\|\nabla\psi\|_\infty\|\psi\|_\infty + (d-1)\|\nabla\psi\|_\infty^2. \tag{3.4.19}$$

Then the (reflecting) diffusion process generated by L is non-explosive.

To formulate P_t as the semigroup associated to a SDE like (3.4.1), we set

$$\sigma = \sqrt{2}\,\psi, \quad b = \psi^2 Z. \tag{3.4.20}$$

Let d_I denote the Itô differential on M. In local coordinates the Itô differential for a continuous semi-martingale X_t on M is given by (see [Emery (1989)] or [Arnaudon *et al* (2006)])

$$(\mathrm{d}_I X_t)^k = \mathrm{d}X_t^k + \frac{1}{2}\sum_{i,j=1}^d \Gamma_{ij}^k(X_t)\mathrm{d}\langle X^i, X^j\rangle_t, \quad 1 \leq k \leq d.$$

Then P_t is the semigroup for the solution to the SDE

$$\mathrm{d}_I X_t = \sigma(X_t)u_t\mathrm{d}B_t + b(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \tag{3.4.21}$$

where B_t is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, u_t is the horizontal lift of X_t onto the frame bundle $O(M)$, and l_t is the local time of X_t on ∂M. When $\partial M = \emptyset$, we simply set $l_t = 0$.

To derive the Harnack inequality as in Section 2, we assume that

$$\lambda := \inf \sigma > 0, \quad \delta := \sup \sigma - \inf \sigma < \infty. \tag{3.4.22}$$

Now, let $x, y \in M$ and $T > 0$ be fixed. Let ρ be the Riemannian distance on M, i.e. $\rho(x,y)$ is the length of the minimal geodesic on M linking x and y, which exits if ∂M is either convex or empty.

Let X_t solve (3.4.21) with $X_0 = x$. Next, for any strictly positive function $\xi \in C([0,T))$, let Y_t solve

$$\begin{aligned}\mathrm{d}_I Y_t &= \sigma(Y_t)P_{X_t,Y_t}u_t\mathrm{d}B_t + b(Y_t)\mathrm{d}t \\ &\quad - \frac{\sigma(Y_t)\rho(X_t,Y_t)}{\sigma(X_t)\xi_t}\nabla\rho(X_t,\cdot)(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t\end{aligned}$$

for $Y_0 = y$, where $\tilde{l}_t$ is the local time of Y_t on ∂M. In the spirit of Theorem 3.2.5, we may assume that the cut-locus of M is empty such that the parallel displacement is smooth. Otherwise, one only needs to replace $\bar{u}_t^{-1} P_{X_t,Y_t} u_t \mathrm{d}B_t$ by a new Brownian motion $\bar{B}_t$ satisfying

$$1_{\{(X_t,Y_t)\notin \mathrm{cut}(M)\}} \mathrm{d}\bar{B}_t = 1_{\{(X_t,Y_t)\notin \mathrm{cut}(M)\}} \bar{u}_t^{-1} P_{X_t,Y_t} u_t \mathrm{d}B_t,$$

where $\bar{u}_t$ is the horizontal lift of Y_t.

Let

$$\mathrm{d}\tilde{B}_t = \mathrm{d}B_t + \frac{\rho(X_t,Y_t)}{\xi_t \sigma(X_t)} u_t^{-1} \nabla \rho(\cdot, Y_t)(X_t) \mathrm{d}t, \quad t < T.$$

By the Girsanov theorem, for any $s \in (0,T)$ the process $\{\tilde{B}_t\}_{t\in[0,s]}$ is the d-dimensional Brownian motion under the changed probability measure $R_s\mathbb{P}$, where

$$\begin{aligned} R_s := \exp\bigg[-\int_0^s \frac{\rho(X_t,Y_t)}{\xi_t \sigma(X_t)} \langle \nabla \rho(\cdot,Y_t)(X_t), u_t \mathrm{d}B_t \rangle \\ - \frac{1}{2} \int_0^s \frac{\rho(X_t,Y_t)^2}{\xi_t^2 \sigma(X_t)^2} \mathrm{d}t \bigg]. \end{aligned} \tag{3.4.23}$$

Thus, by (3.4.21) we have

$$\begin{aligned} \mathrm{d}_I X_t &= \sigma(X_t) u_t \mathrm{d}\tilde{B}_t + b(X_t)\mathrm{d}t \\ &\quad - \frac{\rho(X_t,Y_t)}{\xi_t} \nabla \rho(\cdot,Y_t)(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \\ \mathrm{d}_I Y_t &= \sigma(Y_t) P_{X_t,Y_t} u_t \mathrm{d}\tilde{B}_t + b(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t. \end{aligned}$$

By Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}\rho(X_t,Y_t) \le \sigma(X_t) \langle \nabla \rho(\cdot,Y_t)(X_t), u_t \mathrm{d}\tilde{B}_t \rangle \\ + \sigma(Y_t) \langle \nabla \rho(X_t,\cdot)(Y_t), P_{X_t,Y_t} u_t \mathrm{d}\tilde{B}_t \rangle \\ + \Big\{ \langle b, \nabla \rho(\cdot,Y_t) \rangle (X_t) + \langle b, \nabla \rho(X_t,\cdot) \rangle (Y_t) \\ + \sum_{i=1}^{d-1} U_i^2 \rho(X_t,Y_t) - \frac{\rho(X_t,Y_t)}{\xi_t} \Big\} \mathrm{d}t, \end{aligned} \tag{3.4.24}$$

where $\{U_i\}_{i=1}^{d-1}$ are vector fields on $M \times M$ such that $\nabla U_i(X_t,Y_t) = 0$ and

$$U_i(X_t,Y_t) = \psi(X_t)V_i + \psi(Y_t) P_{X_t,Y_t} V_i, \quad 1 \le i \le d-1$$

for $\{V_i\}_{i=1}^d$ an ONB of $T_{X_t}M$ with $V_d = \nabla \rho(\cdot,Y_t)(X_t)$.

In order to calculate $U_i^2 \rho(X_t,Y_t)$, we adopt the second variational formula for the distance. Let $\rho_t = \rho(X_t,Y_t)$ and let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic $\gamma : [0,\rho_t] \to M$ from X_t to Y_t such that $J_i(0) = \psi(X_t)V_i$ and $J_i(\rho_t) = \psi(Y_t) P_{X_t,Y_t} V_i, 1 \le i \le d-1$. Note that the existence of γ is ensured by the convexity of ∂M. Then, by the second variational formula and noting that $\nabla U_i(X_t,Y_t)=0$, we have

$$I:=\sum_{i=1}^{d-1}U_i^2\rho(X_t,Y_t)=\sum_{i=1}^{d-1}\int_0^{\rho_t}\big\{|\nabla_{\dot\gamma}J_i|^2-\langle\mathcal{R}(\dot\gamma,J_i)J_i,\dot\gamma\rangle\big\}(s)\mathrm{d}s,\quad(3.4.25)$$

where $\mathcal{R}$ is the curvature tensor. Let

$$\tilde J_i(s)=\Big(\frac{s}{\rho_t}\psi(Y_t)+\frac{\rho_t-s}{\rho_t}\psi(X_t)\Big)P_{\gamma(0),\gamma(s)}V_i,\quad 1\le i\le d-1.$$

We have $\tilde J_i(0)=J_i(0)$ and $\tilde J_i(\rho_t)=J_i(\rho_t), 1\le i\le d-1$. By the index lemma,

$$\begin{aligned}I&\le\sum_{i=1}^{d-1}\int_0^{\rho_t}\big\{|\nabla_{\dot\gamma}\tilde J_i|^2-\langle\mathcal{R}(\dot\gamma,\tilde J_i)\tilde J_i,\dot\gamma\rangle\big\}(s)\mathrm{d}s\\&\le(d-1)\|\nabla\psi\|_\infty^2\rho_t\\&\quad-\frac{1}{\rho_t^2}\int_0^{\rho_t}\big\{s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big\}^2\mathrm{Ric}\big(\dot\gamma(s),\dot\gamma(s)\big)\mathrm{d}s.\end{aligned}\quad(3.4.26)$$

Moreover,

$$\begin{aligned}&\langle b,\nabla\rho(\cdot,Y_t)\rangle(X_t)+\langle b,\nabla\rho(X_t,\cdot)\rangle(Y_t)\\&=\frac{1}{\rho_t^2}\int_0^{\rho_t}\frac{\mathrm{d}}{\mathrm{d}s}\Big\{\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle Z(\gamma(s)),\dot\gamma(s)\rangle\Big\}\mathrm{d}s\\&=\frac{2}{\rho_t^2}\Big[\frac12\int_0^{\rho_t}\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle(\nabla_{\dot\gamma}Z)\circ\gamma,\dot\gamma\rangle(s)\mathrm{d}s\\&+\int_0^{\rho_t}\langle Z\circ\gamma,\dot\gamma\rangle(s)(\psi(Y_t)-\psi(X_t))\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)\mathrm{d}s\Big]\\&\le\frac{1}{\rho_t^2}\int_0^{\rho_t}\big(s\psi(Y_t)+(\rho_t-s)\psi(X_t)\big)^2\langle(\nabla_{\dot\gamma}Z)\circ\gamma,\dot\gamma\rangle(s)\mathrm{d}s\\&\quad+2\|Z\|_\infty\|\psi\|_\infty\|\nabla\psi\|_\infty\rho_t.\end{aligned}\quad(3.4.27)$$

Finally, we have

$$\begin{aligned}\langle\nabla\rho(X_t,\cdot)(Y_t),P_{X_t,Y_t}u_t\mathrm{d}\tilde B_t\rangle&=\langle P_{Y_t,X_t}\nabla\rho(X_t,\cdot)(Y_t),u_t\mathrm{d}\tilde B_t\rangle\\&=-\langle\nabla\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde B_t\rangle.\end{aligned}$$

Combining this with (3.4.24) - (3.4.27), we arrive at

$$\begin{aligned}\mathrm{d}\rho(X_t,Y_t)&\le(\sigma(X_t)-\sigma(Y_t))\langle\nabla\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde B_t\rangle\\&\quad+\kappa_\psi\rho(X_t,Y_t)\mathrm{d}t-\frac{\rho(X_t,Y_t)}{\xi_t}\mathrm{d}t,\quad t<T.\end{aligned}$$

Then this implies that

$$\begin{aligned} \mathrm{d}\frac{\rho(X_t,Y_t)^2}{\xi_t} \le & \frac{2}{\xi_t}\rho(X_t,Y_t)(\sigma(X_t)-\sigma(Y_t))\langle\nabla\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde{B}_t\rangle \\ & -\frac{\rho(X_t,Y_t)^2}{\xi_t^2}\big(2-\tilde{\kappa}_\psi\xi_t+\xi_t'\big)\mathrm{d}t \end{aligned} \tag{3.4.28}$$

holds for $t<T$ and

$$\tilde{\kappa}_\psi := 2\kappa_\psi + \|\nabla\sigma\|_\infty^2 = 2\big(\kappa_\psi + \|\nabla\psi\|_\infty^2\big). \tag{3.4.29}$$

In particular, letting

$$\xi_t = \frac{2-\theta}{\tilde{\kappa}_\psi}\big(1-\mathrm{e}^{\tilde{\kappa}_\psi(t-T)}\big), \quad t\in[0,T], \theta\in(0,2),$$

we have

$$2-\tilde{\kappa}_\psi\xi_t+\xi_t'=\theta.$$

Therefore, the following result follows immediately by repeating calculations in the last subsection.

Theorem 3.4.6. *Assume that ∂M is either empty or convex. Let* (3.4.18) *hold for some constant K and let and $Z,\psi,\nabla\psi$ be bounded. Let $\tilde{\kappa}_\psi$ be given by* (3.4.19) *and* (3.4.29). *Then all assertions in Theorem* 3.4.3 *and Corollaries* 3.4.4 *and* 3.4.5 *hold for P_t the (Neumann) semigroup generated by $L=\psi^2(\Delta+Z)$ on M, and for constant functions $K_\cdot := \tilde{\kappa}_\psi, \delta_\cdot :=$* $\sup\psi-\inf\psi$ *and $\lambda_\cdot := \inf|\psi|$, and for $\rho(x,y)$ in place of $|x-y|$.*

3.4.4 *Neumann semigroup on non-convex manifolds*

Assume that $\mathcal{D}\neq\emptyset$ and for some constant $K_0\in\mathbb{R}$ such that (3.4.18) holds. To make the boundary convex, let $\phi\in\mathcal{D}$. By Theorem 1.2.5, ∂M is convex under the metric

$$\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle.$$

Let Δ' and ∇' be the Laplacian and gradient induced by the new metric. Since $\phi\ge 1$, $\rho(x,y)$ is larger than $\rho'(x,y)$, the Riemannian distance between x and y induced by $\langle\cdot,\cdot\rangle'$. Moreover, according to the proof of Proposition 3.2.7 we have

$$L=\phi^{-2}(\Delta'+Z'), \quad \mathrm{Ric}'_{Z'}\ge -K_\phi\langle\cdot,\cdot\rangle',$$

where K_ϕ is given in (3.2.15). Applying Theorem 3.4.6 to the convex manifold $(M,\langle\cdot,\cdot\rangle')$, $\psi=\phi^{-1}$ and (note that $\|\phi^{-1}\|_\infty=1$)

$$\begin{aligned} \tilde{\kappa}_\psi &= 2K_\phi^- + 4\|Z'\|_\infty\|\nabla'\phi^{-1}\|_\infty' + 2d(\|\nabla'\phi^{-1}\|_\infty')^2 \\ &= 2K_\phi^- + 4\|\phi Z+(d-2)\nabla\phi\|_\infty\|\nabla\log\phi\|_\infty + 2d\|\nabla\log\phi\|_\infty^2. \end{aligned} \tag{3.4.30}$$

where $\|\cdot\|'$ is the norm induced by $\langle\cdot,\cdot\rangle'$ and we have used that $f \geq 1$, we obtain the following result.

Theorem 3.4.7. *Let* (3.4.18) *hold. For any* $\phi \in \mathcal{D}$ *in* (3.0.2), *let* $\tilde{\kappa}_\psi$ *be fixed by* (3.2.15) *and* (3.4.30). *Then all assertions in Theorem* 3.4.3 *and Corollaries* 3.4.4 *and* 3.4.5 *hold for constant functions* $K_\cdot = \tilde{\kappa}_\psi, \delta_\cdot := 1 - \inf \phi^{-1}$, *and* $\lambda_\cdot := \inf \phi^{-1}$, *and for* $\rho(x,y)$ *in place of* $|x-y|$.

3.5 Functional inequalities

In this section we intend to investigate functional inequalities for the reflecting diffusion processes on non-convex manifolds. We first present explicit estimates for the spectral gap and the log-Sobolev constants on compact manifolds, then present sufficient and necessary conditions for the log-Sobolev inequality to hold on non-compact manifolds. Stronger inequalities implying the supercontractivity and the ultracontractivity properties are also considered. This section is mainly based on [Wang (2005b, 2007a, 2009a)].

Throughout this section, let $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure. Thus, P_t is symmetric in $L^2(\mu)$ with Dirichlet form

$$\mathcal{E}(f,g) = \mu(\langle \nabla f, \nabla g\rangle), \quad f,g \in \mathcal{D}(\mathcal{E}),$$

where $\mathcal{D}(\mathcal{E}) = \mathbf{H}^{2,1}(\mu)$ is the completion of $C_0^\infty(M)$ under the Sobolev norm $\|f\|_{2,1,\mu} := \sqrt{\mu(f^2) + \mu(|\nabla f|^2)}$.

3.5.1 *Estimates for inequality constants on compact manifolds*

Let $\Phi : I \to \mathbb{R}$ be a convex C^2-function, where I is a (not necessarily bounded) interval. We define the Φ-*entropy* w.r.t. μ by

$$\mathrm{Ent}_\mu^\Phi(f) := \mu(\Phi(f)) - \Phi(\mu(f)), \quad f \in \mathcal{B}_b^+(M), f(M) \subset I.$$

Since Φ is convex, the Jensen inequality implies that $\mathrm{Ent}_\mu^\Phi(f) \geq 0$. For instance, if $\Phi(r) := r^2$ and $I = \mathbb{R}$ then $\mathrm{Ent}_\mu^\Phi(f)$ reduces to the variance $\mathrm{Var}_\mu(f) := \mu(f^2) - \mu(f)^2$, while if $\Phi(r) := r\log r$ and $I = [0,\infty)$, then $\mathrm{Ent}_\mu^\Phi(f)$ coincides with the relative entropy $\mathrm{Ent}_\mu(f) := \mu(f\log[f/\mu(f)])$. Moreover, the convexity of Φ also implies $\Psi(a,f) := \Phi(f) - \Phi(a) - \Phi'(a)(f-a) \geq 0$ for $a \in I$ and

$$\mathrm{Ent}_\mu^\Phi(f) = \inf_{a\in I}\int_M \Psi(a,f)\mathrm{d}\mu. \tag{3.5.1}$$

See e.g. [Chafai (2004)] for further discussions.

Let α_Φ denote the biggest positive constant such that the Φ-*entropy inequality*

$$\alpha_\Phi \mathrm{Ent}_\mu^\Phi(f) \le \mu(\Phi''(f)|\nabla f|^2), \quad f \in C^1(M; I).$$

To estimate α_Φ, we make use of Theorem 1.2.5 to reduce to the convex boundary case, for which the spectral gap and the log-Sobolev constants have been well estimated (cf. [Chen and Wang (1997a); Wang (1999)] and references within). To this end, for any $D > 0$ and $K \in \mathbb{R}$, let $\alpha(d, K, D)$ be the biggest constant such that for any d-dimensional connected compact Riemmannian manifold M^* with convex boundary and diameter less than D, and for any $V^* \in C^2(M^*)$ with $\mathrm{Ric}^{M^*} - \mathrm{Hess}_{V^*}^{M^*} \ge K$, one has

$$\alpha(d, K, D)\mathrm{Ent}_{\mu^*}^\Phi(f) \le \mu^*(\Phi''(f)|\nabla_{M^*} f|_{M^*}^2), \quad f \in C^1(M^*; I), \tag{3.5.2}$$

where $\mu^* := Z^* \mathrm{e}^{V^*(x)} \nu_{M^*}(\mathrm{d}x)$ with ν_{M^*} the volume measure on M^* and $Z^* > 0$ such that μ^* is a probability measure.

Theorem 3.5.1. *Let D be the diameter of M, let $\phi \in \mathcal{D}$ for $\mathcal{D}$ defined by* (3.2.15), *and let K_ϕ be given by* (3.2.16). *Then*

$$\alpha_\Phi \ge \frac{\alpha(d, K_\phi, D)}{\|\phi\|_\infty^4}.$$

Proof. According to the proof of Proposition 3.2.7, we have $\phi^2 L = \Delta' + Z' =: L'$ for some vector field Z' such that

$$\mathrm{Ric}'_{Z'} \ge K_\phi \langle \cdot, \cdot \rangle', \tag{3.5.3}$$

where $\langle \cdot, \cdot \rangle' := \phi^{-2} \langle \cdot, \cdot \rangle$ under which ∂M is convex, and Ric', Δ' are the associated Ricci curvature and Laplacian. It is easy to see that L' is symmetric w.r.t. the probability measure $\mu_\phi := \phi^{-2}\mu/\mu(\phi^{-2})$ and

$$\mu(\Phi''(f)|\nabla f|^2) = \mu(\phi^{-2})\mu_\phi(\Phi''(f)\langle \nabla' f, \nabla' f\rangle'), \quad f \in C^1(M), \tag{3.5.4}$$

where ∇' is the gradient induced by the metric $\langle \cdot, \cdot \rangle'$. Moreover, $\phi \ge 1$ implies that the diameter of M under the new metric is less than D. Combining this with (3.5.1), (3.5.3), $\phi \ge 1$, (3.5.4) and the definition of $\alpha(d, K, D)$, we arrive at

$$\begin{aligned}
\mathrm{Ent}_\mu^\Phi(f) &= \inf_{a \in I} \int_M \Psi(a, f)\mathrm{d}\mu = \inf_{a \in I} \int_M \Psi(a, f)\phi^2 \mu(\phi^{-2})\mathrm{d}\mu_\phi \\
&\le \|\phi\|_\infty^2 \mathrm{Ent}_{\mu_\phi}^\Phi(f) \le \frac{\|\phi\|_\infty^2}{\alpha(d, K_\phi, D)} \mu_\phi(\Phi''(f)\langle \nabla' f, \nabla' f\rangle') \\
&\le \frac{\|\phi\|_\infty^4}{\alpha(d, K_\phi, D)} \mu(\Phi''(f)|\nabla f|^2).
\end{aligned}$$

This completes the proof. $\square$

Combining Theorem 3.5.1 with known estimates on convex manifolds, we have the following results for the spectral gap and the log-Sobolev constant. Let λ_1 be the first Neumann eigenvalue of L. We have $\lambda_1 = \alpha_\Phi$ for $\Phi(r) := r^2$. For any $K \in \mathbb{R}$, let

$$\lambda_1(K, D) = 4 \inf \bigg\{ \int_0^D f'(s)^2 \mathrm{e}^{-\frac{Ks^2}{8}} \mathrm{d}s : f \in C^1([0, D]),$$
$$f(0) = 0, \int_0^D f(s)^2 \mathrm{e}^{-\frac{Ks^2}{8}} \mathrm{d}s = 1 \bigg\},$$

which is the first mixed eigenvalue of $4\frac{\mathrm{d}^2}{\mathrm{d}r^2} - Kr\frac{\mathrm{d}}{\mathrm{d}r}$ on $[0, D]$ with Dirichlet condition at 0 and Neumann condition at D.

Corollary 3.5.2. *Let $\phi \in \mathcal{D}$. Then*

$$\lambda_1 \geq \frac{\lambda_1(K_\phi, D)}{\|\phi\|_\infty^4}$$
$$= \frac{4}{\|\phi\|_\infty^4} \sup_{0<f\in C([0,D])} \inf_{r\in[0,D]} f(r) \bigg\{ \int_0^r \mathrm{e}^{\frac{K_\phi s^2}{8}} \mathrm{d}s \int_s^D \mathrm{e}^{-\frac{K_\phi r^2}{8}} f(r) \mathrm{d}r \bigg\}^{-1}.$$

Consequently,

$$\lambda_1 \geq \frac{1}{\|\phi\|_\infty^4} \bigg(\frac{\pi^2}{D^2} + \max\bigg\{ \frac{K_\phi}{2}, \frac{\pi - 2}{\pi} K_\phi^+ - \frac{\pi^2}{16} K_\phi^- \bigg\} \bigg),$$

where $r^+ := 0 \vee r, r^- := (-r)^+$ for $r \in \mathbb{R}$.

Proof. The first assertion follows from Theorem 3.5.1 and the variational formula for the first eigenvalue presented in [Chen and Wang (1997a)] (see also (2.2.1) in [Wang (2005a)]), while the second assertion follows from the first by taking specific choices of f, see Corollaries 1 and 2 in [Chen and Wang (1997a)] or Corollary 2.2.2 in [Wang (2005a)]. □

Next, we consider the log-Sobolev constant

$$\alpha := \inf\{2\mu(|\nabla f|^2) : \mu(f^2 \log[f^2/\mu(f^2)]) = 1\}.$$

Obviously, letting $\Phi(r) := r \log r$ and $I = [0, \infty)$, we have $\alpha = \frac{1}{2}\alpha_\Phi$. Then the following result follows from Theorem 3.5.1 and Theorem 3.1 in [Wang (1999)].

Corollary 3.5.3. *Let $\phi \in \mathcal{D}$. Then*

$$\alpha \geq \frac{\sqrt{(1 + D^2 K_\phi^-)^2 + 4D^2\lambda_1(K_\phi, D)} - 1 - D^2 K_\phi^-}{2D^2\|\phi\|_\infty^4}. \tag{3.5.5}$$

3.5.2 *A counterexample for Bakry-Emery criterion*

According to Theorem 3.3.1(3) with $t \to \infty$, when $\mathbb{I} \geq 0$ the curvature

$$\mathrm{Ric} - \mathrm{Hess}_V \geq K \tag{3.5.6}$$

for some constant $K > 0$ implies the log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq \frac{2}{K}\mu(|\nabla f|^2), \quad f \in C_b^1(M), \mu(f^2) = 1.$$

This assertion is known as the Bakry-Emery criterion due to [Bakry and Emery (1984)]. From this one might hope that when $\mathbb{I}$ is bounded below by a slightly negative constant, a larger enough curvature lower bound will imply the log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq C\mu(|\nabla f|^2), \quad f \in C_b^1(M), \mu(f^2) = 1 \tag{3.5.7}$$

for some constant $C > 0$. Since the log-Sobolev inequality is stronger than the Poincaré inequality, this is not true according to the following result. Although this result is stated only for $d = 2$, one can construct counterexamples in high dimensions by simply taking product spaces. With e.g. $V(x) = c - R|x|^2$ for constants $c \in \mathbb{R}$ and $R > 0$, for which $\mathrm{Ric} - \mathrm{Hess}_V = -\mathrm{Hess}_V = 2R$, Theorem 3.5.4 disproves the Poincaré inequality for arbitrarily large curvature lower bound and arbitrarily weak concavity of the boundary.

Theorem 3.5.4. *For any $\varepsilon > 0$ and any probability measure μ_0 on $\mathbb{R}^2$ with full support, there exists a smooth connected domain $M \subset \mathbb{R}^2$ with connected ∂M such that $\mathbb{I} \geq -\varepsilon$ but for any $C > 0$ the Poincaré inequality*

$$\mu(f^2) - \mu(f)^2 \leq C\mu(|\nabla f|^2), \quad f \in C_b^1(M) \tag{3.5.8}$$

does not hold.

Proof. (a) Construction of M. We first construct a smooth curve which will produce the main part of ∂M. Let $\varphi_0 \in C^\infty(\mathbb{R})$ be decreasing on $(-\infty, 2]$ and increasing on $[3, \infty)$ such that

$$\varphi_0(x) = \begin{cases} 0, & \text{if } x \in [2,3], \\ 1, & \text{if } x \in (-\infty, 1] \cup [4, \infty). \end{cases}$$

Next, for a sequence $\{r_n \in (0,1)\}$ with $r_n \downarrow 0$ as $n \uparrow \infty$, which will be fixed later on in order to disprove the Poincaré inequality, define

$$\varphi = 21_{(-\infty,5]} + 2\sum_{n=1}^\infty 1_{(5n,5(n+1)]} \frac{r_n + \varphi_0(\cdot - 5n)}{1 + r_n}.$$

Obviously, $\varphi \in C_b^\infty(\mathbb{R})$ with derivatives uniformly bounded in the choice of $\{r_n\}$.

Now, let $D \subset [0,\infty) \times [0,2]$ be a connected smooth domain such that

$$D \cap [1,\infty) \times [0,\infty) = \{(x,y) :\ 0 \le y \le \varphi(x)\}.$$

Therefore,

$$\partial D \cap (\mathbb{R}^2 \setminus [0,1] \times [0,2]) = [1,\infty) \times \{0\} \cup \{(x,\varphi(x)) :\ x \ge 1\}.$$

Obviously, the second fundamental form $\mathbb{I}_{\partial D}$ of ∂D is bounded below in a compact set, the part $[1,\infty) \times \{0\}$ is flat, and since the derivatives of $\varphi \in C_b^\infty(\mathbb{R})$ are bounded uniformly in the choice of $\{r_n\}$, $\mathbb{I}_{\partial D}$ on the part $\{(x,\varphi(x)) :\ x \ge 1\}$ is bounded below uniformly in $\{r_n\}$. So,

$$\mathbb{I}_{\partial D} \ge -\delta \tag{3.5.9}$$

holds for some constant $\delta > 0$ independent of $\{r_n\}$. To make the second fundamental form bounded below by $-\varepsilon$, let

$$M = RD := \{(Rx, Ry) :\ (x,y) \in D\}$$

for sufficient large $R > 1$ such that due to (3.5.9) the second fundamental form $\mathbb{I}$ of ∂M satisfies

$$\mathbb{I} \ge -\frac{\delta}{R} \ge -\varepsilon.$$

(b) Choices of $\{r_n\}$ destroying the Poincaré inequality. Let μ_0 be a probability measure on $\mathbb{R}^2$ with full support. Let $\mu = 1_M \mu_0/\mu_0(M)$. Since $D \supset [1,5] \times [0,2]$, we have

$$\mu_0(M) \ge \mu_0([R,5R] \times [0,2R]) := \delta_0 > 0. \tag{3.5.10}$$

Note that δ_0 is independent of the choice of $\{r_n\}$. Now, for any $n \ge 1$, take

$$f_n(x,y) = (x - 5nR - 2R)^+ \wedge 1, \quad (x,y) \in M.$$

Since for $\tilde{r}_n := 2r_n/(1+r_n)$,

$$([5nR+2R, 5nR+2R+1] \times \mathbb{R}) \cap M = [5nR+2R, 5nR+2R+1] \times [0, R\tilde{r}_n],$$

by (3.5.10) we have

$$\begin{aligned} \mu(|\nabla f_n|^2) &= \mu([5nR+2R, 5nR+2R+1] \times [0, R\tilde{r}_n]) \\ &\le \frac{\mu_0(\mathbb{R} \times [0, \tilde{r}_n R])}{\delta_0}. \end{aligned} \tag{3.5.11}$$

Moreover, since

$$\{f_n = 0\} \supset [R, 5R] \times [0, 2R], \quad \{f_n = 1\} \supset [5nR+4R, 5(n+1)R] \times [0, 2R],$$

we have

$$\begin{aligned}\mu(f_n^2) - \mu(f_n)^2 &\geq \mu(f_n^2)\mu([R, 5R] \times [0, 2R]) \\ &\geq \delta_0\mu_0([5nR + 4R, 5(n+1)R] \times [0, 2R]) \\ &=: u_n.\end{aligned} \tag{3.5.12}$$

Now, for each $n \geq 1$ one may take $r_n \in (0, 1)$ small enough such that

$$\mu_0(\mathbb{R} \times [0, \tilde{r}_n R]) \leq \frac{\delta_0 u_n}{n}.$$

Combining this with (3.5.11) and (3.5.12) we conclude that

$$\lim_{n\to\infty} \frac{\mu(|\nabla f_n|^2)}{\mu(f_n^2) - \mu(f_n)^2} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, the Poincaré inequality is not available. □

3.5.3 *Log-Sobolev inequality on locally concave manifolds*

Since the log-Sobolev inequality holds on any compact smooth domains, it would be possible to extend Theorem 2.4.1 to the case that ∂M is merely concave on a bounded domain. Although this sounds quite natural, a complete proof is however far from trivial. The main point is that, in general, it is not clear how can one split M into a bounded part and an unbounded but convex part.

Theorem 3.5.5. *Assume that for some compact set $M_0 \subset M$ one has $\mathbb{I} \geq 0$ on $(\partial M) \setminus M_0$ and $\mathrm{Ric}_Z \geq K$ on $M \setminus M_0$ for some $K \in \mathbb{R}$.*

(1) *If $\mu(e^{\lambda\rho_o^2}) < \infty$ for some $\lambda > -\frac{K}{2}$, then the log-Sobolev inequality* (3.5.7) *holds for some $C > 0$. In particular, if $K > 0$ then* (3.5.7) *holds.*
(2) *P_t is supercontractive if and only if $\beta(\lambda) := \mu(e^{\lambda\rho_o^2}) < \infty$ for any $\lambda > 0$.*
(3) *P_t is ultracontractive if and only if $P_t e^{\lambda\rho_o^2}$ is bounded for any $\lambda, t > 0$.*

Proof. By Lemma 3.5.8 below, the class $\mathcal{D}$ in (3.0.2) is non-empty. So, according to the proof of Theorem 2.4.2, (2) and (3) follows from the Harnack inequality ensured by Theorem 3.4.7. To prove (1), let $\phi \in \mathcal{D}$ be constructed in the proof of Lemma 3.5.8. Then the volume measure induced by $\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle$ is

$$\nu(\mathrm{d}x) = \phi^{-d}(x)\mathrm{d}x.$$

So,

$$e^{V(x)}dx = e^{\tilde V(x)}\nu(dx), \quad \tilde V = V - d\log\phi. \tag{3.5.13}$$

Let $\mathrm{Ric}', \mathrm{Hess}', \rho_o', \nabla', |\cdot|'$ be induced by $\langle\cdot,\cdot\rangle'$ corresponding to Ric, Hess, $\rho_0, \nabla, |\cdot|$ respectively. Since $\phi = 1$ and (3.5.6) hold outside a compact set M_0, we have

$$\mathrm{Ric}' - \mathrm{Hess}'_{\tilde V} = \mathrm{Ric} - \mathrm{Hess}_V \geq -K \text{ outside } M_0.$$

Moreover, there exists a constant $c > 0$ such that

$$|\rho_o' - \rho_o| \leq c.$$

Therefore, by Lemma 3.5.7 below for the convex manifold $(M, \langle\cdot,\cdot\rangle')$, the conditions in Theorem 3.5.5(1) imply the log-Sobolev inequality

$$\mu(f^2\log f^2) \leq C_1\mu(|\nabla' f|'^2), \quad f \in C_b^1(M), \mu(f^2) = 1$$

for some constant $C_1 > 0$. Since

$$|\nabla' f|'^2 = \phi^2|\nabla f|^2 \leq C_2|\nabla f|^2$$

holds for some constant $C_2 > 0$, we derive the desired log-Sobolev inequality for some constant $C > 0$. □

Lemma 3.5.6. *Let ∂M be convex and*

$$\mathrm{Ric}_Z \geq K \text{ outside a compact set } M_0 \tag{3.5.14}$$

holds for some constant K. Then there exists a constant $c > 0$ such that

$$\begin{aligned}&(P_t f)^\alpha(x)\\&\leq (P_t f^\alpha(y))\exp\left[\frac{\alpha c(t+\rho(x,y))}{\alpha-1} + \frac{\alpha K\rho(x,y)^2}{2(\alpha-1)(e^{2Kt}-1)}\right]\end{aligned} \tag{3.5.15}$$

holds for any bounded positive measurable function f on $M, t > 0, \alpha > 1$ and $x, y \in M$.

Proof. We will use the argument of coupling by change of measure proposed in Theorem 1.3.7. Let us fix two points $x \neq y$ in M and $t_0 > 0$. For a positive function $\xi \in C^1([0,\infty))$, let (X_t, Y_t) be the coupling by parallel displacement in Theorem 3.2.5(1) for

$$U(t, X_t, Y_t) = -\xi_t\nabla\rho(X_t,\cdot)(Y_t).$$

Then

$$d\rho(X_t, Y_t) \leq I_Z(X_t, Y_t)dt - \xi_t dt, \quad t < \tau, \tag{3.5.16}$$

where $\tau := \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time. Let γ be the minimal geodesic linking X_t and Y_t, by (3.2.14) we have

$$I_Z(X_t, Y_t) \leq -\int_0^{\rho(X_t,Y_t)} \mathrm{Ric}_Z(\dot\gamma, \dot\gamma)\mathrm{d}s. \tag{3.5.17}$$

Since (3.5.14) holds outside a compact set M_0 and there exists a constant $c_1 > 0$ such that $\mathrm{Ric}_Z \geq -c_1$ on the compact set M_0, (3.5.17) implies

$$I_Z(X_t, Y_t) \leq -K\rho(X_t, Y_t) + c_2$$

for some constant $c_2 > 0$. Substituting this into (3.5.16) we arrive at

$$\mathrm{d}\rho(X_t, Y_t) \leq \big\{c_2 - K\rho(X_t, Y_t) - \xi_t\big\}\mathrm{d}t, \quad t \leq \tau.$$

Equivalently,

$$\mathrm{d}\big(\mathrm{e}^{Kt}\rho(X_t, Y_t)\big) \leq (c_2 - \xi_t)\mathrm{e}^{Kt}\mathrm{d}t, \quad t \leq \tau. \tag{3.5.18}$$

Taking

$$\xi_t = c_2 + \frac{2K\rho(x,y)\mathrm{e}^{Kt}}{\mathrm{e}^{2Kt_0} - 1},$$

we have

$$\int_0^{t_0} (c_2 - \xi_t)\mathrm{e}^{Kt}\mathrm{d}t = -\rho(x,y).$$

From this and (3.5.18) it is easy to see that $\tau \leq t_0$ and hence, $X_{t_0} = Y_{t_0}$.

Now, due to the Girsanov theorem $\{Y_t\}$ is generated by L under the weighted probability $R\mathbb{P}$, where

$$R := \exp\left[\frac{1}{\sqrt{2}}\int_0^\tau \langle \xi_t \nabla\rho(X_t, \cdot)(Y_t), P_{X_t,Y_t}u_t\mathrm{d}B_t\rangle - \frac{1}{4}\int_0^\tau \xi_t^2\mathrm{d}t\right].$$

Then by Theorem 1.3.7,

$$(P_{t_0}f(y))^\alpha \leq (P_{t_0}f^\alpha(x))(\mathbb{E}R^{\alpha/(\alpha-1)})^{\alpha-1}. \tag{3.5.19}$$

Since $\tau \leq t_0$ and

$$\begin{aligned} s \mapsto N_s := \exp\bigg[&\frac{\alpha}{\sqrt{2}\,(\alpha-1)}\int_0^s \langle \xi_t\nabla\rho(X_t,\cdot)(Y_t), P_{X_t,Y_t}u_t\mathrm{d}B_t\rangle \\ &- \frac{\alpha^2}{4(\alpha-1)^2}\int_0^s \xi_t^2\mathrm{d}t\bigg] \end{aligned}$$

is a martingale, we have $\mathbb{E}N_\tau = 1$ and hence,

$$\begin{aligned} \mathbb{E}R^{\alpha/(\alpha-1)} &= \mathbb{E}\left(N_\tau \exp\left[\frac{\alpha}{4(\alpha-1)^2}\int_0^\tau \xi_t^2\mathrm{d}t\right]\right) \\ &\leq \exp\left[\frac{\alpha}{4(\alpha-1)^2}\int_0^{t_0} \xi_t^2\mathrm{d}t\right] \\ &= \exp\left[\frac{\alpha}{4(\alpha-1)^2}\left(c_2^2 t_0 + \frac{4c_2\rho(x,y)(\mathrm{e}^{Kt_0}-1)}{\mathrm{e}^{2Kt_0}-1} + \frac{2K\rho(x,y)^2}{\mathrm{e}^{2Kt_0}-1}\right)\right]. \end{aligned}$$

Combining this with (3.5.19) we complete the proof. □

Lemma 3.5.7. *Let ∂M be convex and* $\mathrm{Ric}_Z \geq K$ *hold for some* $K \in \mathbb{R}$ *outside a compact set. Then* $\mu(e^{\lambda\rho_o^2}) < \infty$ *for some* $\lambda > -\frac{K}{2}$ *implies the log-Sobolev inequality* (3.5.7) *for some constant* $C > 0$.

Proof. Since ∂M is convex, by the Bakry-Emery criterion, the log-Sobolev inequality holds provided $K > 0$. So, we only consider the case that $K \leq 0$. Let $\delta = \lambda + \frac{K}{2} > 0$. Since

$$\lim_{t\to\infty}\lim_{\alpha\to\infty}\frac{\alpha K}{2(\alpha-1)(e^{2Kt}-1)} = -\frac{K}{2},$$

we may take $t, \alpha > 1$ and a constant $c(t,\alpha) > 0$ such that

$$\frac{\alpha ct}{\alpha-1} + \frac{c\alpha}{\alpha-1}(r+1) + \frac{\alpha K(1+r)^2}{2(\alpha-1)(e^{2Kt}-1)} \leq c(t,\alpha) + \frac{\delta-K}{2}r^2, \quad r \geq 0$$

holds. Therefore, for any bounded measurable f with $\mu(|f|^\alpha) = 1$, by the Harnack inequality (3.5.15) and noting that μ is a P_t-invariant probability measure, we obtain

$$\begin{aligned}
1 &= \int_M P_t|f|^\alpha(y)\mu(dy)\\
&\geq |P_tf|^\alpha(x)\int_M \exp\left[-\frac{\alpha ct}{\alpha-1} - \frac{c\alpha}{\alpha-1}\rho(x,y) - \frac{\alpha K\rho(x,y)^2}{2(\alpha-1)(e^{2Kt}-1)}\right]\mu(dy)\\
&\geq |P_tf|^\alpha(x)\mu(B(o,1))\exp\left[-c(t,\alpha) - \frac{\delta-K}{2}\rho_o(x)^2\right],
\end{aligned}$$

where $B(o,1) = \{x \in M : \rho(o,x) \leq 1\}$ is the unit geodesic ball at o. Since $\lambda = \frac{2\delta-K}{2}$, this implies that

$$\begin{aligned}
\mu(|P_tf|^{\alpha(2\delta-K)/(\delta-K)}) &\leq \frac{\exp[c(t,\alpha)(2\delta-K)/(\delta-K)]}{\mu(B(o,1))^{(2\delta-K)/(\delta-K)}}\mu(e^{\lambda\rho_o^2})\\
&< \infty, \quad \mu(|f|^\alpha) = 1.
\end{aligned}$$

Therefore, $\|P_t\|_{L^\alpha(\mu)\to L^\beta(\mu)} < \infty$ holds for $\beta := \alpha(2\delta-K)/(\delta-K) > \alpha > 1$, so that the desired log-Sobolev inequality holds as explained in the proof of Theorem 2.4.2. □

Lemma 3.5.8. *Assume that ∂M is convex outside a compact set. Then there exists $\phi \in \mathcal{D}$ in* (3.0.2) *such that $\phi = 1$ holds outside a compact set.*

Proof. To construct such a ϕ, let ρ_∂ be the Riemannian distance function to ∂M. Let $R > 1$ be such that ∂M is convex outside $B(o,R)$. Since ρ_∂ is smooth in a neighborhood of ∂M, there exists $r_0 > 0$ such that ρ_∂ is smooth on the compact domain

$$M_1 := B(o,R+1) \cap \{\rho_\partial \leq r_0\}.$$

Now, let $h \in C^\infty([0,\infty))$ be increasing such that $h'(0) \geq \delta, h(0) = 1$ and $h|_{[r_0,\infty)}$ is constant. Then $\phi_1 := h \circ \rho_\partial$ is smooth on M_1 and

$$N \log \phi_1 \geq \delta \quad \text{on } M_1 \cap \partial M. \tag{3.5.20}$$

To extend ϕ_1 to a global smooth function ϕ satisfying our conditions, we take a cut-off function $g \in C^\infty(\partial M)$ such that $0 \leq g \leq 1$, $g = 1$ on $B(o,R) \cap \partial M$ and $g = 0$ on $B(o, R+1)^c \cap \partial M$. It is easy to extend g to a smooth function $\bar{g}$ on M such that $0 \leq \bar{g} \leq 1$ and the Neumann boundary condition $N\bar{g}|_{\partial M} = 0$ holds. This can be done by using the polar coordinates around ∂M. Noting that there exists $r_1 \in (0,1)$ such that the exponential map

$$(B(o, R+3) \cap \partial M) \times [0, r_1] \ni (\theta, r) \mapsto \exp[rN_\theta] \in M$$

is smooth and one-to-one, we define

$$\bar{g}(\exp[rN_\theta]) = g(\theta)h_1(r), \quad \theta \in (\partial M) \cap B(o, R+3), r \in [0, r_1]$$

for some function nonnegative $h_1 \in C^\infty([0,\infty); [0,1])$ with $h_1|_{[0,r_1/2]} = 1$ and $h_1|_{[r_1,\infty)} = 0$. Obviously, $\bar{g}$ is smooth and well-defined on $B(o, R+2)$ with support contained by $B(o, R+1)$. Then $\bar{g}$ extends to a smooth function on M by letting $\bar{g} = 0$ on $M \setminus B(o, R+1)$. Now, let

$$\phi = \bar{g}\phi_1 + 1 - \bar{g}.$$

We have $\phi \in C^\infty(M), \phi \geq 1, \phi|_{B(o,R+1)^c} = 1$. Then $\inf \phi = \phi|_{\partial M} = 1$, so that $\nabla \phi|_{\partial M} \parallel N$. Moreover, since $N\bar{g} = 0$ and $\bar{g}|_{B(o,R)} = 1$, from (3.5.20) we obtain

$$N \log \phi = \frac{\bar{g} N \phi_1}{\bar{g}\phi_1 + 1 - \bar{g}} \geq \delta \geq -\mathbb{I}.$$

□

3.5.4 *Log-Sobolev inequality on non-convex manifolds*

When ∂M is empty, a perturbation argument is proposed in §2.5.4 to establish the super Poincaré inequality. In this subsection, we aim to extend this argument to establish the log-Sobolev inequality on manifolds with boundary. Note that according to its proof, Theorem 2.5.9 works also for the present setting, so that it applies to the situation of Proposition 3.5.10 as it ensures the desired Nash inequality for $p = n$ according to [Bakry *et al* (1995)]. So, below we only consider the log-Sobolev inequality rather than the more general super Poincaré inequality.

According to the perturbation argument in §2.5.4, the key point is to establish a Nash inequality for a weighted volume measure $\mu_W(\mathrm{d}x) :=$

$e^W dx$, where $W \in C^2(M)$. Since the Nash inequality is equivalent to a heat kernel upper bounds (see [Davies (1989)]), we shall make use of the Harnack inequality for the corresponding Neumann semigroup and the following volume comparison theorem, which is well known in geometry analysis when $W = 0$ and $\partial M = \emptyset$.

Let $L_W = \Delta + \nabla W$ and

$$\Gamma_2^W(f,f) = \frac{1}{2} L_W |\nabla f|^2 - \langle \nabla L_W f, \nabla f \rangle, \quad f \in C^\infty(M).$$

Proposition 3.5.9. *Let ∂M be convex and $L_W = \Delta + \nabla W$ satisfy the curvature dimension condition*

$$\Gamma_2^W(f,f) \geq -K|\nabla f|^2 + \frac{1}{n}(L_W f)^2, \quad f \in C^\infty(M) \tag{3.5.21}$$

for some constants $K \geq 0, n > 1$. Then $\mu_W(dx) := e^{W(x)}dx$ satisfies

$$\frac{\mu_W(B(x,\alpha r))}{\mu_W(B(x,r))} \leq \alpha^n \exp\left[\sqrt{(n-1)K}\,(\alpha-1)r\right], \quad r > 0, x \in M, \alpha > 1.$$

Proof. Let ρ_x be the Riemannian distance function to point x. By the Laplacian comparison theorem (see [Qian, Z. (1998)]), (3.5.21) implies that

$$L_W \rho_x \leq \sqrt{(n-1)K} \coth\left[\sqrt{\frac{K}{n-1}}\,\rho_x\right] \tag{3.5.22}$$

holds outside $\mathrm{cut}(x)$. Let

$$(\theta, r) \mapsto \exp[r\theta]$$

be the polar coordinates at x, where $\theta \in \mathbb{S}_x^{d-1} := \{X \in T_xM : |X| = 1\}$ and $r \in [0, r_\theta]$ for

$$r_\theta := \inf\{r \geq 0 : \exp[r\theta] \in \mathrm{cut}(x) \cup \partial M\}.$$

Since M is convex and connected, we have (cf. Proposition 2.1.5 in [Wang (2005a)])

$$M = \{\exp[r\theta] : \theta \in \mathbb{S}_x^{d-1}, r \in [0, r_\theta]\}.$$

Due to this and (3.5.22) we complete the proof by repeating the argument in the proof of Lemma 2.2 in [Gong and Wang (2001)]. □

By Proposition 3.5.9 and the Harnack inequality for the Neumann semigroup P_t^W generated by L_W on M, we obtain the following log-Sobolev inequality for a weighted volume measure.

Proposition 3.5.10. *Let ∂M be convex and* (3.5.21) *hold. Let W_0 be such that $|\nabla W_0| \le \sigma$ for some constant $\sigma > 0$ and*
$$\liminf_{\rho_o \to \infty} \frac{\exp[W_0 - \sqrt{(n-1)K}\,\rho_o]}{\rho_o^n} > 0.$$
Then the following log-Sobolev inequality holds for some $c > 0$ and $\mu_{\tilde W}$ with $\tilde W := W + W_0$:
$$\mu_{\tilde W}(f^2 \log f^2) \le \frac{n}{2} \log\left\{c(\mu_{\tilde W}(|\nabla f|^2) + 1)\right\}, \quad \mu_{\tilde W}(f^2) = 1.$$

Proof. By Proposition 3.5.9 for $\alpha = (1 + \rho_o(x))/r$, we have
$$\begin{aligned} &\mu_W(B(x,r)) \\ &\ge \frac{\mu_W(B(x, 1+\rho_o(x)))r^n}{(1+\rho_o(x))^n} \exp\left[-\sqrt{(n-1)K}\,(\rho_o(x)+1)\right] \\ &\ge \frac{\mu_W(B(o,1))r^n}{(1+\rho_o(x))^n} \exp\left[-\sqrt{(n-1)K}\,(\rho_o(x)+1)\right] \\ &=: c_0 \frac{r^n}{(1+\rho_o(x))^n} \exp\left[-\sqrt{(n-1)K}\,(\rho_o(x)+1)\right], \end{aligned} \tag{3.5.23}$$
for all $r \in [0,1], x \in M$. Next, by (3.5.21) and Theorem 3.3.2(3),
$$(P_t^W f(x))^\alpha \le (P_t^W f^\alpha(y)) \exp\left[\frac{\alpha K \rho(x,y)^2}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})}\right], \tag{3.5.24}$$
holds for $\alpha > 1, t > 0, x, y \in M$ and all bounded nonnegative measurable functions f. Since $|\nabla W_0| \le \sigma$, we have
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} P_{t-s}^W (P_s^{\tilde W} f)^\alpha &= \alpha P_{t-s}^W \{(P_s^{\tilde W} f)^{\alpha-1} \langle \nabla W_0, \nabla P_s^{\tilde W} f\rangle\} \\ &\quad - \alpha(\alpha-1) P_{t-s}^W \{(P_s^{\tilde W} f)^{\alpha-2} |\nabla P_s^{\tilde W} f|^2\} \\ &\le \alpha P_{t-s}^W \left\{(P_s^{\tilde W} f)^\alpha \left(\frac{\sigma|\nabla P_s^{\tilde W} f|}{P_s^{\tilde W} f} - (\alpha-1)\frac{|\nabla P_s^{\tilde W} f|^2}{(P_s^{\tilde W} f)^2}\right)\right\} \\ &\le \frac{\sigma^2 \alpha}{4(\alpha-1)} P_{t-s}^W (P_s^{\tilde W} f)^\alpha. \end{aligned}$$
This implies
$$(P_t^{\tilde W} f)^\alpha \le (P_t^W f^\alpha) \exp\left[\frac{\sigma^2 \alpha}{4(\alpha-1)} t\right], \quad \alpha > 1, t > 0.$$
Similarly, this inequality remains true by exchanging the positions of P_t^W and $P_t^{\tilde W}$. Combining this fact with (3.5.24) and taking $\alpha = 2^{1/3}$, we obtain
$$\begin{aligned} &(P_t^{\tilde W} f)^2(x) \le (P_t^W f^\alpha)^{\alpha^2}(x) \exp\left[\frac{\sigma^2}{2(\alpha-1)} t\right] \\ &\le (P_t^W f^{\alpha^2}(y))^\alpha \exp\left[\frac{\sigma^2}{2(\alpha-1)} t + \frac{\alpha^2 K \rho(x,y)^2}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})}\right] \\ &\le (P_t^{\tilde W} f^2(y)) \exp\left[\frac{\sigma^2}{2(\alpha-1)} t + \frac{\alpha^2 K \rho(x,y)^2}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})} + \frac{\sigma^2 \alpha}{4(\alpha-1)} t\right]. \end{aligned}$$

Thus, for any f with $\mu_{\tilde W}(f^2)=1$ we have

$$\begin{aligned}(P_t^{\tilde W} f(x))^2 &\le \bigg\{\int_M \exp\Big[-\frac{\sigma^2(\alpha+2)}{4(\alpha-1)}t \\ &\qquad -\frac{\alpha^2 K\rho(x,y)^2}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})}\Big]\mu_{\tilde W}(\mathrm{d}y)\bigg\}^{-1} \\ &\le \bigg\{\int_{B(x,\sqrt{t})} \exp\Big[-\frac{\sigma^2(\alpha+2)}{4(\alpha-1)}t \\ &\qquad -\frac{\alpha^2 Kt}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})}\Big]\mu_{\tilde W}(\mathrm{d}y)\bigg\}^{-1} \\ &\le \frac{\exp[c_1(t+1)]}{\mu_{\tilde W}(B(x,\sqrt{t}))},\quad t>0, x\in M\end{aligned} \tag{3.5.25}$$

for some constant $c_1>0$. By (3.5.23) and the conditions on W_0, we have

$$\mu_{\tilde W}(B(x,\sqrt{t}))\ge c_2 t^{n/2},\quad x\in M, t\in(0,1]$$

for some constant $c_2>0$. It follows from (3.5.25) that

$$(P_t^{\tilde W} f)^2\le c_3 t^{-n/2},\quad t\in(0,1]$$

holds for some $c_3>0$ and all f with $\mu_{\tilde W}(f^2)=1$. This is equivalent to the Sobolev inequality of dimension n for $\mu_{\tilde W}$ (see [Davies (1989)]) which, according to [Bakry *et al* (1995)], is also equivalent to the desired log-Sobolev inequality. □

Combining Proposition 3.5.10 with a conformal change of metric, we are able to prove a log-Sobolev inequality for non-convex M.

Proposition 3.5.11. *Assume that* $\mathbb{I}$ *is bounded,* $\mathrm{Ric}\ge -K$ *for some* $K\ge 0$*, the sectional curvature of* M *is bounded above, and* ρ_∂ *is smooth on* $\{\rho_\partial\le r_0\}$ *for some* $r_0>0$*. Then for any* W_0 *with* $\|\nabla W_0\|_\infty<\infty$ *and* $\liminf_{\rho_o\to\infty}\rho_o^{-d}\exp[W_0]>0$*, the log-Sobolev inequality*

$$\mu_{W_0}(f^2\log f^2)\le\frac{d}{2}\log\big\{c(\mu_{W_0}(|\nabla f|^2)+1)\big\},\quad \mu_{W_0}(f^2)=1$$

holds for some $c>0$.

Proof. Let $\mathbb{I}\ge\sigma$. When $\sigma=0$ (i.e. ∂M is convex), let $h\equiv 1$; otherwise, according to the proof of Theorem 3.2.9, there exists an increasing function $h\in C_b^\infty([0,\infty))$ such that $h(0)=1, h'(0)=-\sigma$ and $h|_{[r_0,\infty)}$ is constant. Then $\phi:=h\circ\rho_\partial\in C^\infty(M)$ with $1\le\phi\le R$ for some constant $R>0$ and ∂M is convex under the metric $\langle\cdot,\cdot\rangle':=\phi^{-2}\langle\cdot,\cdot\rangle$.

Let ν be the volume measure induced by $\langle\cdot,\cdot\rangle'$ and let $\nu_{W_0} = e^{W_0}\nu$. Applying Proposition 3.5.10 to $(M, \langle\cdot,\cdot\rangle')$ and $W = 0$, for which (3.5.21) holds with $n = d$, we obtain

$$\nu_{W_0}(f^2 \log f^2) \leq \frac{d}{2} \log\left\{c'(\nu_{W_0}(|\nabla' f|'^2) + 1)\right\}, \quad \nu_{W_0}(f^2) = 1$$

for some constant $c' > 0$. Since ϕ takes values in $[1, R]$ so that

$$|\nabla' f|'^2 \leq R^2 |\nabla f|^2, \quad 1 \leq \frac{d\mu_{W_0}}{d\nu_{W_0}} \leq R^d,$$

we obtain the desired inequality by a standard perturbation argument. □

Having the above preparations, we are able to prove the following main result of this subsection.

Theorem 3.5.12. *Assume that* $\mathbb{I}$ *and* Ric *are bounded, the sectional curvature of* M *is bounded above, and* ρ_∂ *is smooth on* $\{\rho_\partial \leq r_0\}$ *for some* $r_0 > 0$. *If* $\langle N, \nabla V\rangle$ *is bounded below,* $\varepsilon, \varepsilon' > 0$ *such that* $\varepsilon > \varepsilon'\sqrt{(n-1)K}$, *and the function*

$$-\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V - \varepsilon' V + \varepsilon\rho_o$$

is bounded above on M, *then the log-Sobolev inequality* (3.5.7) *holds for some* $C > 0$. *If furthermore*

$$-r\frac{1}{4}|\nabla V|^2 - \frac{1}{2}r\Delta V - V + \rho_o \leq \varphi(r), \quad r > 0 \tag{3.5.26}$$

holds for some positive function φ *on* $(0, \infty)$, *then the super log-Sobolev inequality*

$$\mu(f^2 \log f^2) \leq r\mu(|\nabla f|^2) + c + c\varphi(r) + \frac{d}{2}\log(c(r^{-1} + 1)) \tag{3.5.27}$$

holds for all $r > 0,\ \mu(f^2) = 1$.

Proof. Let Ric $\geq -K$ and $\mathbb{I} \geq \sigma$ for some constants $K \geq 0, \sigma < 0$. Then (3.5.21) holds for $W = 0$. Let $\eta > 0$ be a constant such that $\langle N, \nabla V\rangle \geq -\eta$. Let h be in the proof of Proposition 3.5.11 such that $\Delta\phi$ is bounded. Let $k > \varepsilon_0 := \varepsilon/\varepsilon'\ (> \sqrt{(n-1)K})$ be such that $-\sigma k - \varepsilon_0 \geq \eta$ and let $W_0 = \varepsilon_0\sqrt{1 + \rho_o^2} - k\phi$. Then $\|\nabla W_0\|_\infty < \infty$ and

$$\langle N, \nabla(V - W_0)\rangle \geq -\sigma k - \eta - \varepsilon_0 \geq 0. \tag{3.5.28}$$

By Proposition 3.5.11 we have

$$\mu_{W_0}(f^2 \log f^2) \leq r\mu_{W_0}(|\nabla f|^2) + \frac{d}{2}\log(c_1(1 + r^{-1})), \quad r > 0, \mu_{W_0}(f^2) = 1$$

for some constant $c_1 > 0$. For f with $\mu(f^2) = 1$, we apply this inequality to $fe^{(V-W_0)/2}$ to deduce for $r > 0$

$$\begin{aligned}\mu(f^2 \log f^2) \le r\mu(|\nabla f|^2) + \mu\Big(f^2\Big\{\frac{r}{4}|\nabla(V-W_0)|^2 + W_0 - V\Big\}\Big) \\ + \frac{1}{2}r\mu(\langle\nabla f^2, \nabla(V-W_0)\rangle) + \frac{d}{2}\log(c_1(r^{-1}+1)).\end{aligned} \tag{3.5.29}$$

By (3.5.28) and the Green formula, we have (recall that N is the unit inward vector of ∂M)

$$\begin{aligned}\mu(\langle\nabla f^2, \nabla(V-W_0)\rangle) &= -\int_M f^2 L(V-W_0)\mathrm{d}\mu - \int_{\partial M} f^2\langle N, \nabla(V-W_0)\rangle \mathrm{d}\mu_\partial \\ &\le -\mu(f^2\{\Delta V + |\nabla V|^2 - \Delta W_0 - \langle \nabla V, \nabla W_0\rangle\}).\end{aligned}$$

Since $\|\nabla W_0\|_\infty < \infty$ and by the Laplacian comparison theorem ΔW_0 is bounded above, combining this with (3.5.29) we obtain

$$\begin{aligned}&\mu(f^2\log f^2) \le r\mu(|\nabla f|^2) + \frac{d}{2}\log(c_1(r^{-1}+1)) \\ &+ r\mu\Big(f^2\Big\{-\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V + \frac{\varepsilon_0}{r}\rho_o + \frac{c_2}{r} - \frac{V}{r}\Big\}\Big), \quad r > 0\end{aligned} \tag{3.5.30}$$

for some $c_2 > 0$. Taking $r = 1/\varepsilon'$ and noting that $\varepsilon_0 = \varepsilon/\varepsilon' = \varepsilon r$, we conclude that

$$-\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V + \frac{\varepsilon_0}{r}\rho_o - \frac{V}{r} \le -\frac{1}{4}|\nabla V|^2 + \varepsilon\rho_o - \varepsilon' V - \frac{1}{2}\Delta V.$$

According to our condition this is bounded above. Therefore, (3.5.30) implies the defective log-Sobolev inequality

$$\mu(f^2\log f^2) \le C_1\mu(|\nabla f|)^2 + C_2, \quad \mu(f^2) = 1$$

for some constants $C_1, C_2 > 0$, and hence (3.5.7) holds for some $C > 0$ as M is connected. Finally, (3.5.27) follows immediately from (3.5.30) and (3.5.26). □

3.6 Modified curvature tensors and applications

Let P_t be the semigroup of the reflecting diffusion process generated by $L = \Delta + Z$ for some C^1-smooth vector field Z on M. In the previous sections the curvature tensor Ric_Z and the second fundamental form $\mathbb{I}$ have been used to investigate the reflecting diffusion processes. As shown in the last section, these two tensors play essentially different roles in the study. Moreover, from the derivative formula in e.g. Theorem 3.2.9 we see that the second fundamental form appears in the study as integrals w.r.t. the local time of the process on the boundary. This is because of the fact that the geometry of the boundary affects the process only when the process hits the boundary. To avoid using the local time which is in general less explicit, we aim to derive explicit results for the reflecting diffusion processes by using modified curvature tensors consisting of both Ric_Z and information from the boundary.

3.6.1 *Equivalent semigroup inequalities for the modified curvature lower bound*

For any strictly positive $\phi \in C^2(M)$, we introduce a family of modified curvature tensors

$$\mathrm{Ric}_Z^{\phi,p} := \mathrm{Ric} - \nabla Z - \frac{1}{p}(\phi^p L \phi^{-p})\langle \cdot, \cdot \rangle, \quad p > 0.$$

To ensure that these tensors contain also information from the boundary, the function ϕ will be taken from the class $\mathcal{D}$ defined in (3.0.2). Note that for a vector X and a function f we write $Xf = \langle X, \nabla f \rangle$, and conditions on N and $\mathbb{I}$ are automatically restricted to ∂M and $T\partial M$. According to the proof of Theorem 3.2.9, if **(A3.2.1)** holds then $\mathcal{D} \neq \emptyset$. We also remark that the condition $\inf \phi = 1$ in the definition of class $\mathcal{D}$ is not essential but for convenience, since our main result (see Theorem 3.6.1 below) do not change if one replaces ϕ by $c\phi$ for a constant $c > 0$.

Let X_t^ϕ be the reflecting diffusion process generated by

$$L^\phi := L - 2\nabla \log \phi.$$

Since X_t is non-explosive, so is X_t^ϕ provided $\nabla \log \phi$ is bounded.

Theorem 3.6.1. *Let $\phi \in \mathcal{D}$ in* (3.0.2) *such that* **(A3.2.1)** *holds. Then for any $K \in C_b(M)$, the following statements are equivalent to each other:*

(1) $\mathrm{Ric}_Z^{\phi,1} \geq K$;

(2) *For any $f \in C_b^1(M)$,*

$$\phi(x)|\nabla P_t f(x)|$$
$$\leq \mathbb{E}^x\Big\{(\phi|\nabla f|)(X_t)\mathrm{e}^{-\sqrt{2}\int_0^t \langle u_s^{-1}\nabla \log \phi(X_s),\, \mathrm{d}B_s\rangle - \int_0^t (K + |\nabla \log \phi|^2)(X_s)\mathrm{d}s}\Big\}$$

holds for $t \geq 0$ and $x \in M$;

(3) *For any $f \in C_b^1(M), x \in M$ and $t \geq 0$,*

$$|\nabla P_t f(x)| \leq \frac{1}{\phi(x)}\mathbb{E}^x\Big\{(\phi|\nabla f|)(X_t^\phi)\mathrm{e}^{-\int_0^t K(X_s^\phi)\mathrm{d}s}\Big\}.$$

Proof. According to Proposition 3.6.2 below with $\tilde{Z} = -\sqrt{2}\,\nabla \log\phi$, (2) and (3) are equivalent. Moreover, (3) implying (1) follows from Proposition 3.6.3 below. Therefore, it remains to prove that (1) implies (2). Let $\phi \in \mathcal{D}$ such that (1) holds. Since (1) implies $\mathrm{Ric}_Z \geq K + \phi L\phi^{-1}$ while $\phi \in \mathcal{D}$ ensures $\mathbb{I} \geq -N\log\phi$, according to Theorem 3.2.9 we have

$$|\nabla P_t f(x)| \leq \mathbb{E}^x\Big\{|\nabla f|(X_t)\mathrm{e}^{-\int_0^t (K+\phi L\phi^{-1})(X_s)\mathrm{d}s+\int_0^t N\log\phi(X_s)\mathrm{d}l_s}\Big\}. \quad (3.6.1)$$

On the other hand, by the Itô formula, we have

$$\begin{aligned}\mathrm{d}\log\phi(X_s) = &\sqrt{2}\langle u_s^{-1}\nabla\log\phi(X_s), \mathrm{d}B_s\rangle\\ &+ L\log\phi(X_s)\mathrm{d}s + N\log\phi(X_s)\mathrm{d}l_s.\end{aligned}$$

Let $\tau_n = \inf\{t \geq 0 : \rho(x, X_t) \geq n\},\ n \geq 1$. Then

$$\begin{aligned}\int_0^{t\wedge\tau_n} N\log\phi(X_s)\mathrm{d}l_s = &\log\frac{\phi(X_{t\wedge\tau_n})}{\phi(x)} - \sqrt{2}\int_0^{t\wedge\tau_n}\langle u_s^{-1}\nabla\log\phi(X_s), \mathrm{d}B_s\rangle\\ &- \int_0^{t\wedge\tau_n} L\log\phi(X_s)\mathrm{d}s.\end{aligned}$$

Combining this with (3.6.1) and noting that

$$\phi L\phi^{-1} + L\log\phi = |\nabla\log\phi|^2,$$

we obtain

$$\begin{aligned}|\nabla P_t f(x)| \leq &\frac{1}{\phi(x)}\mathbb{E}^x\Big\{(\phi|\nabla P_{(t-\tau_n)^+}f|)(X_{t\wedge\tau_n})\\ &\times \mathrm{e}^{-\sqrt{2}\int_0^{t\wedge\tau_n}\langle u_s^{-1}\nabla\log\phi(X_s),\, \mathrm{d}B_s\rangle - \int_0^{t\wedge\tau_n}(K+|\nabla\log\phi|^2)(X_s)\mathrm{d}s}\Big\}.\end{aligned}$$

Since $K, |\nabla\log\phi|$ are bounded and due to Theorem 3.2.9 (2) $|\nabla P_\cdot f|$ is bounded on $[0,t]\times M$, by the dominated convergence theorem we complete the proof by letting $n \to \infty$. □

Proposition 3.6.2. *Let $\tilde{Z}$ be a bounded C^1-smooth vector field on M, and let Y_t be the reflecting diffusion process generated by $L + \sqrt{2}\,\tilde{Z}$ starting at x. Then for any bound measurable function F of $X_{[0,t]} := \{X_s\}_{s\in[0,t]}$,*

$$\mathbb{E}^x\Big\{F(X_{[0,t]})\mathrm{e}^{\int_0^t\langle u_s^{-1}\tilde{Z}(X_s),\,\mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t|\tilde{Z}|^2(X_s)\mathrm{d}s}\Big\} = \mathbb{E}^x F(Y_{[0,t]}).$$

Proof. Let

$$R = \exp\bigg[\int_0^t\langle u_s^{-1}\tilde{Z}(X_s),\, \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t|\tilde{Z}|^2(X_s)\mathrm{d}s\bigg].$$

By the Girsanov theorem, under the probability measure $R\mathrm{d}\mathbb{P}$ the process

$$\tilde{B}_s := B_s - \int_0^s\langle u_r^{-1}\tilde{Z}(X_r), \mathrm{d}B_r\rangle,\quad s \in [0,t]$$

is a d-dimensional Brownian motion. Obviously, the equation (3.0.1) can be reformulated as

$$\mathrm{d}X_s = \sqrt{2}\,u_s \circ \mathrm{d}\bar{B}_s + \big(Z + \sqrt{2}\,\bar{Z}\big)(X_s)\mathrm{d}s + N(X_s)\mathrm{d}l_s.$$

Therefore, under the new probability measure, $X_{[0,t]}$ is the reflecting diffusion process generated by $L + \sqrt{2}\,\bar{Z}$. Hence,

$$\mathbb{E}^x\{RF(X_{[0,t]})\} = \mathbb{E}^x F(Y_{[0,t]}).$$

□

Proposition 3.6.3. *For any strictly positive function $\phi \in C_b^2(M)$, the gradient inequality in Theorem* 3.6.1(3) *implies* $\mathrm{Ric}_Z^{\phi,1} \geq K$. *If there exists $r_0 > 0$ such that on $\{\rho_\partial \leq r_0\}$ the distance function ρ_∂ to the boundary is smooth with bounded $L\rho_\partial$, then Theorem* 3.6.1(3) *also implies* $\mathbb{I} \geq -N \log \phi$.

Proof. (a) Let $x \in M \setminus \partial M$ and $X \in T_xM$ with $|X| = 1$, we aim to prove $\mathrm{Ric}_Z^{\phi,1}(X,X) \geq K$ from Theorem 3.6.1(3). Let $f \in C_0^\infty(M)$ with $\mathrm{supp} f \subset M \setminus \partial M$ be such that $\nabla f(x) = X$ and $\mathrm{Hess}_f(x) = 0$. Let $\varepsilon > 0$ such that $|\nabla f| \geq \frac{1}{2}$ on $B(x,\varepsilon)$, the geodesic ball at x with radius ε. Let X_t^ϕ be the reflecting diffusion process generated by L^ϕ with $X_0^\phi = x$, and let

$$\sigma_\varepsilon = \inf\{t \geq 0 : \rho(X_t^\phi, x) \geq \varepsilon\}.$$

By Lemma 3.1.1,

$$\mathbb{P}(\sigma_\varepsilon \leq t) \leq \mathrm{e}^{-c\varepsilon^2/t}, \quad t \in (0,1]$$

holds for some constant $c > 0$. Since $l_s = 0$ for $s \leq \sigma_\varepsilon$, this and $|\nabla f|(x) = 1$ imply that

$$\begin{aligned}
&\mathbb{E}^x\Big\{(\phi|\nabla f|)^2(X_t^\phi)\mathrm{e}^{-2\int_0^t K(X_s^\phi)\mathrm{d}s}\Big\}\\
&= \mathbb{E}^x\Big\{(\phi|\nabla f|)^2(X_{t\wedge\sigma_\varepsilon}^\phi)\mathrm{e}^{-2\int_0^{t\wedge\sigma_\varepsilon} K(X_s^\phi)\mathrm{d}s}\Big\} + \mathrm{o}(t) \qquad (3.6.2)\\
&= \phi^2(x) + t\big\{L^\phi(\phi|\nabla f|)^2 - 2K\phi^2\big\}(x) + \mathrm{o}(t),
\end{aligned}$$

where $\mathrm{o}(t)$ stands for a t-dependent quantity such that $\mathrm{o}(t)/t \to 0$ as $t \to 0$. On the other hand, since $\mathrm{supp} f \subset M \setminus \partial M$ so that $Nf = 0$, we have

$$P_t f = f + \int_0^t P_s L f \mathrm{d}s.$$

This and $|\nabla f(x)| = |X| = 1$ imply that

$$\begin{aligned}
|\nabla P_t f(x)|^2 &= |\nabla f + t(\nabla L f)|^2(x) + \mathrm{o}(t)\\
&= 1 + 2\langle \nabla L f, \nabla f\rangle(x)t + \mathrm{o}(t).
\end{aligned}$$

Combining this with (3.6.2) and the gradient inequality in Theorem 3.6.1(3), we arrive at

$$\frac{L^\phi(\phi|\nabla f|)^2}{\phi^2}(x) - 2\langle \nabla Lf, \nabla f\rangle(x) \geq 2K(x). \tag{3.6.3}$$

Noting that $\mathrm{Hess}_f(x) = 0$ and $|\nabla f(x)| = 1$ imply

$$\frac{L^\phi(\phi|\nabla f|)^2}{\phi^2}(x) = L|\nabla f|^2(x) - 2(\phi L\phi^{-1})(x),$$

combining (3.6.3) with Theorem 1.1.4, we obtain

$$\mathrm{Ric}_Z^{\phi,1}(X,X) = \mathrm{Ric}_Z^{\phi,1}(\nabla f, \nabla f)(x) \geq K(x).$$

(b) Let $x \in \partial M$ and $X \in T_x\partial M$ with $|X| = 1$. Let $f \in C_0^\infty(M)$ be such that $Nf = 0$ and $\nabla f(x) = X$. We have

$$P_t f = f + \int_0^t P_s Lf \mathrm{d}s.$$

Consequently, for small t,

$$|\nabla P_t f(x)|^2 = |\nabla f(x)|^2 + \mathrm{o}(t^{1/2}) = 1 + \mathrm{o}(t^{1/2}). \tag{3.6.4}$$

On the other hand, according to Lemma 3.1.2, and Proposition 4.1 in [Wang (2011e)],

$$\mathbb{E}^x l^\phi_{t\wedge\sigma_1} = \frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{o}(t^{1/2}),$$

where l_t^ϕ is the local time of X_t^ϕ on ∂M and $\sigma_1 := \inf\{s \geq 0 : \rho(x, X_s^\phi) \geq 1\}$. Combining this with Lemma 3.1.1 and noting that since $|\nabla f(x)| = 1$ and

$$\lim_{s\to 0} N(\phi^2|\nabla f|^2)(X_s^\phi) = N(\phi^2|\nabla f|^2)(x),$$

we obtain

$$\begin{aligned}
&\mathbb{E}^x\Big\{(\phi|\nabla f|)(X_t^\phi)\mathrm{e}^{-\int_0^t K(X_s^\phi)\mathrm{d}s}\Big\}^2 = \mathbb{E}^x(\phi|\nabla f|)^2(X^\phi_{t\wedge\sigma_1}) + \mathrm{o}(t^{1/2})\\
&= (\phi^2|\nabla f|^2)(x) + \mathbb{E}^x\int_0^{t\wedge\sigma_1} L^\phi(\phi^2|\nabla f|^2)(X_s^\phi)\mathrm{d}s\\
&\quad + \mathbb{E}^x\int_0^{t\wedge\sigma_1} N(\phi^2|\nabla f|^2)(X_s^\phi)\mathrm{d}l_s^\phi\\
&= \phi^2(x) + \frac{2\sqrt{t}}{\sqrt{\pi}}N(\phi^2|\nabla f|^2)(x) + \mathrm{o}(t^{1/2}).
\end{aligned}$$

Combining this with (3.6.4) and the gradient inequality in Theorem 3.6.1(3), we conclude that

$$N(\phi^2|\nabla f|^2)(x) \geq 0.$$

This implies $\mathbb{I}(X,X) \geq -N\log\phi(x)$, since $X = \nabla f(x)$ and $N|\nabla f|^2 = 2\mathbb{I}(\nabla f, \nabla f)$ due to (3.2.7). $\square$

3.6.2 *Applications of Theorem 3.6.1*

In this subsection, we aim to derive explicit gradient/Poincaré/Harnack type inequalities for P_t by using Theorem 3.6.1. To this end, we first present the following lemma, where the proof of (3.6.5) is standard according to Bakry and Ledoux [Bakry (1997); Ledoux (2000)] (see also the proofs of (3) and (4) in Theorem 2.3.1), while that of (3.6.6) is essentially due to [Röckner and Wang (2010)].

Lemma 3.6.4. *If $|\nabla P_t f|^2 \le \xi_t P_t|\nabla f|^2$ holds for some strictly positive $\xi \in C([0,\infty))$ and all $t \ge 0$ and $f \in C_b^1(M)$, then for all $t \ge 0, f \in C_b^1(M)$,*

$$2|\nabla P_t f|^2 \int_0^t \frac{\mathrm{d}s}{\xi_s} \le P_t f^2 - (P_t f)^2 \le 2(P_t|\nabla f|^2) \int_0^t \xi_s \mathrm{d}s, \qquad (3.6.5)$$

and for any measurable function f with $f \ge 1$,

$$P_t \log f(y) \le \log P_t f(x) + \frac{\rho(x,y)^2}{4\int_0^t \xi_s^{-1}\mathrm{d}s}, \quad t > 0. \qquad (3.6.6)$$

Proof. It suffices to prove for $f \in \mathcal{C}_N(L)$ such that $f \ge 1$. For any $\varepsilon > 0$, let $\gamma : [0,1] \to M$ be the minimal curve such that $\gamma(0) = x, \gamma(1) = y$ and $|\dot\gamma| \le \rho(x,y) + \varepsilon$. Let

$$h(s) = \frac{\int_0^s \xi_r^{-1}\mathrm{d}r}{\int_0^t \xi_r^{-1}\mathrm{d}r}, \quad s \in [0,t].$$

By the Kolmogorov equations (Theorem 3.1.3), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(P_s \log P_{t-s} f)(\gamma \circ h(s)) \\
&\le -P_s|\nabla \log P_{t-s} f|^2(\gamma \circ h(s)) + (\varepsilon + \rho(x,y))\dot h(s)|\nabla P_s \log P_{t-s} f|(\gamma \circ h(s)) \\
&\le \big\{-P_s|\nabla \log P_{t-s} f|^2 + (\varepsilon + \rho(x,y))\dot h(s)\sqrt{\xi_s P_s|\nabla \log P_{t-s} f|^2}\big\}(\gamma \circ h(s)) \\
&\le \frac{1}{4}(\varepsilon + \rho(x,y))^2 \xi_s \dot h(s)^2 = \frac{(\varepsilon + \rho(x,y))^2}{4\xi_s(\int_0^t \xi_r^{-1}\mathrm{d}r)^2}, \quad s \in [0,t].
\end{aligned}$$

Integrating over $[0,t]$ and letting $\varepsilon \downarrow 0$, we obtain (3.6.6).

Next, noting that

$$\frac{\mathrm{d}}{\mathrm{d}s} P_s (P_{t-s} f)^2 = 2P_s|\nabla P_{t-s} f|^2 \begin{cases} \le 2\xi_{t-s} P_t|\nabla f|^2, \\ \ge 2\xi_s^{-1}|\nabla P_t f|^2, \end{cases} \qquad s \in (0,t),$$

we prove (3.6.5). □

Corollary 3.6.5. *Let $\phi \in \mathcal{D}$ such that* **(A3.2.1)** *and $\mathrm{Ric}_Z^{\phi,2} \ge K^{\phi,2}$ hold for some constant $K^{\phi,2}$. Then:*

(1) $\phi^2|\nabla P_t f|^2 \le e^{-2K^{\phi,2}t} P_t(\phi|\nabla f|)^2$ *holds for any* $f \in C_b^1(M)$ *and* $t \ge 0$.
(2) *For any measurable function* $f \ge 1$, *the log-Harnack inequality*

$$P_t \log f(y) \le \log P_t f(x) + \frac{\|\phi\|_\infty^2 K^{\phi,2} \rho(x,y)^2}{2(e^{2K^{\phi,2}t} - 1)}, \quad t \ge 0, x, y \in M$$

holds.
(3) $P_t f^2 \le (P_t f)^2 + \frac{\|\phi\|_\infty^2 (1-e^{-2K^{\phi,2}t})}{K^{\phi,2}} P_t|\nabla f|^2$ *holds for any* $f \in C_b^1(M)$ *and* $t \ge 0$. *Consequently, if* P_t *has an invariant probability measure* μ *and* $K^{\phi,2} > 0$, *then the Poincaré inequality*

$$\mu(f^2) \le \mu(f)^2 + \frac{\|\phi\|_\infty^2}{K^{\phi,2}} \mu(|\nabla f|^2), \quad f \in C_b^1(M)$$

holds.
(4) $P_t f^2 \ge (P_t f)^2 + \frac{e^{2K^{\phi,2}t}-1}{\|\phi\|_\infty^2 K^{\phi,2}} |\nabla P_t f|^2$ *holds for any* $f \in C_b^1(M)$ *and* $t \ge 0$.

Proof. Noting that when P_t has an invariant probability measure μ then $\mu(dx) = e^{V(x)}dx$ holds for some $V \in C(M)$ (see e.g. [Bogachev, Röckner and Wang (2001)]) such that the weak Poincaré inequality holds (see Theorem 3.1 in [Röckner and Wang (2001)]), the second assertion in (3) follows from the first by letting $t \to \infty$. Thus, because of Lemma 3.6.4 and $\phi \ge 1$, it suffices to prove the first assertion. Obviously, $\mathrm{Ric}_Z^{\phi,2} \ge K^{\phi,2}$ implies that $\mathrm{Ric}_Z^{\phi,1} \ge K := K^{\phi,2} + |\nabla \log \phi|^2$. Let

$$R_t = \exp\left[-\sqrt{2} \int_0^t \langle u_s^{-1} \nabla \log \phi(X_s), dB_s \rangle - \int_0^t |\nabla \log \phi(X_s)|^2 ds \right].$$

By Theorem 3.6.1 and $\phi \ge 1$, we obtain

$$\begin{aligned}
(\phi|\nabla P_t f|)^2(x) &\le \left(\mathbb{E}^x \left\{ R_t(|\nabla f|\phi)(X_t) e^{-\int_0^t K(X_s)ds} \right\} \right)^2 \\
&\le \{P_t(\phi|\nabla f|)^2(x)\} \mathbb{E}^x \left(R_t^2 e^{-2\int_0^t K(X_s)ds} \right) \\
&= \{P_t(\phi|\nabla f|)^2(x)\} e^{-2K^{\phi,2}t} \\
&\quad \times \mathbb{E}^x e^{-2\sqrt{2}\int_0^t \langle u_s^{-1} \nabla \log \phi(X_s), dB_s \rangle - 4\int_0^t |\nabla \log \phi(X_s)|^2 ds} \\
&= e^{-2K^{\phi,2}t} P_t(\phi|\nabla f|)^2(x).
\end{aligned}$$

□

Next, we have the following results on the log-Sobolev and HWI inequalities which extend the corresponding ones presented in §2.4 for manifolds without boundary. In particular, if ∂M is convex we may take $\phi \equiv 1$ so that Corollary 3.6.6(1) goes back to the log-Sobolev inequality in Theorem 2.4.1(1) and Corollary 3.6.6(2) reduces to the HWI inequality in Theorem 2.4.1(3).

Corollary 3.6.6. *Let $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure. Let $\phi \in \mathcal{D}$ such that* **(A3.2.1)** *and $\mathrm{Ric}_Z^{\phi,2} \geq K^{\phi,2}$ hold for some constant $K^{\phi,2}$.*

(1) *If $K^{\phi,2} > 0$ then*

$$\mu(f^2 \log f^2) \leq \frac{2\|\phi\|_\infty^6}{K^{\phi,2}}\mu(|\nabla f|^2), \quad f \in C_b^1(M), \mu(f^2) = 1.$$

(2) *If $K^{\phi,2} \leq 0$ then*

$$\mu(f^2 \log f^2) \leq 2\|\phi\|_\infty^4 \sqrt{\mu(|\nabla f|^2)}W_2^\rho(f^2\mu, \mu) - \frac{\|\phi\|_\infty^2 K^{\phi,2}}{2} W_2^\rho(f^2\mu, \mu)^2$$

holds for any $f \in C_b^1(M)$ with $\mu(f^2) = 1$.

(3) *Let $p_t(x, y)$ be the heat kernel of P_t w.r.t. μ. Then*

$$p_t(x, y) \geq \exp\left[-\frac{\|\phi\|_\infty^2 K^{\phi,2}\rho(x, y)^2}{2(\mathrm{e}^{K^{\phi,2}t} - 1)}\right]$$

and

$$\int_M p_t(x, z) \log \frac{p_t(x, z)}{p_t(y, z)}\mu(\mathrm{d}z) \leq \frac{\|\phi\|_\infty^2 K^{\phi,2}\rho(x, y)^2}{2(\mathrm{e}^{2K^{\phi,2}t} - 1)}$$

hold for all $t > 0$ and $x, y \in M$.

To prove Corollary 3.6.6, we present a log-Sobolev inequality which generalizes the corresponding known one on manifolds without boundary (see Theorem 2.3.1(3)).

Lemma 3.6.7. *Let $\phi \in \mathcal{D}$ such that $\mathrm{Ric}_Z^{\phi,2} \geq K^{\phi,2}$ holds for some constant $K^{\phi,2}$. Let $\bar{P}_t^\phi$ be the semigroup of the reflecting diffusion process generated by $\bar{L}^\phi := L - 4\nabla \log \phi$. Then*

$$P_t(f^2 \log f^2) \leq (P_t f^2) \log P_t f^2 + 4\|\phi\|_\infty^2 \int_0^t \mathrm{e}^{-2K^{\phi,2}(t-s)} P_s \bar{P}_{t-s}^\phi |\nabla f|^2 \mathrm{d}s \tag{3.6.7}$$

holds for all $t \geq 0$ and $f \in C_b^1(M)$.

Proof. It suffices to prove for $f \in \mathcal{C}_N(L)$ with $\inf f^2 > 0$. Let R_t be in the proof of Corollary 3.6.5. Since $\mathrm{Ric}_Z^{\phi,2} \geq K^{\phi,2}$ implies that $\mathrm{Ric}_Z^{\phi,1} \geq$

$K := K^{\phi,2} + |\nabla \log \phi|^2$, by Theorem 3.6.1 and $\phi \geq 1$ we have

$$\begin{aligned}
&|\nabla P_t f^2(x)|^2 \\
&\leq \Big(\mathbb{E}^x\Big\{R_t(\phi|\nabla f^2|)(X_t)\mathrm{e}^{-\int_0^t K(X_s)\mathrm{d}s}\Big\}\Big)^2 \\
&\leq 4\|\phi\|_\infty^2 (P_t f^2(x))\mathbb{E}^x\Big\{R_t^2|\nabla f|^2(X_t)\mathrm{e}^{-2\int_0^t K(X_s)\mathrm{d}s}\Big\} \\
&= 4\|\phi\|_\infty^2 (P_t f^2(x))\mathbb{E}^x\Big\{|\nabla f|^2(X_t) \\
&\qquad \times \mathrm{e}^{-2\sqrt{2}\int_0^t \langle u_s^{-1}\nabla \log \phi(X_s), \mathrm{d}B_s\rangle - 2\int_0^t(|\nabla \log \phi(X_s)|^2 + K(X_s))\mathrm{d}s}\Big\} \\
&= 4\|\phi\|_\infty^2 (P_t f^2(x))\mathrm{e}^{-2K^{\phi,2}t}\mathbb{E}^x\{\bar{R}_t|\nabla f|^2(X_t)\},
\end{aligned}$$

where

$$\bar{R}_t := \mathrm{e}^{-2\sqrt{2}\int_0^t \langle u_s^{-1}\nabla \log \phi(X_s), \mathrm{d}B_s\rangle - 4\int_0^t |\nabla \log \phi(X_s)|^2 \mathrm{d}s}.$$

Combining this with Proposition 3.6.2 for $\bar{Z} = -2\sqrt{2}\,\nabla \log \phi$, we obtain

$$|\nabla P_t f^2|^2 \leq 4\|\phi\|_\infty^2 (P_t f^2)\mathrm{e}^{-2K^{\phi,2}t}\bar{P}_t^\phi|\nabla f|^2, \quad t \geq 0.$$

Therefore, by the Kolmogorov equations (Theorem 3.1.3),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}P_s\{(P_{t-s}f^2)\log P_{t-s}f^2\} &= P_s\frac{|\nabla P_{t-s}f^2|^2}{P_{t-s}f^2} \\
&\leq 4\|\phi\|_\infty^2 \mathrm{e}^{-2K^{\phi,2}(t-s)}P_s\bar{P}_{t-s}^\phi|\nabla f|^2.
\end{aligned}$$

Then the proof is completed by integrating over $[0, t]$. □

Proof. [Proof of Corollary 3.6.6] Let $f \in C_b^1(M)$ such that $\mu(f^2) = 1$ and $\mu(|\nabla f|^2) > 0$. Since μ is P_t-invariant while $\phi^{-4}\mathrm{d}\mu$ is $\bar{P}_t^\phi$-invariant, integrating (3.6.7) w.r.t. μ gives

$$\begin{aligned}
&\mu(f^2 \log f^2) \\
&\leq \mu((P_t f^2)\log P_t f^2) + 4\|\phi\|_\infty^2 \int_0^t \mathrm{e}^{-2K^{\phi,2}s}\mu(\bar{P}_s^\phi|\nabla f|^2)\mathrm{d}s \\
&\leq \mu((P_t f^2)\log P_t f^2) + 4\|\phi\|_\infty^6 \int_0^t \mathrm{e}^{-2K^{\phi,2}s}\mu(\phi^{-4}|\nabla f|^2)\mathrm{d}s \\
&\leq \mu((P_t f^2)\log P_t f^2) + \frac{2\|\phi\|_\infty^6(1-\mathrm{e}^{-2K^{\phi,2}t})}{K^{\phi,2}}\mu(|\nabla f|^2).
\end{aligned} \tag{3.6.8}$$

If $K^{\phi,2} > 0$, then letting $t \to \infty$ we prove Corollary 3.6.6(1).

The proof of the second assertion can be done as in the proof of Theorem 2.4.1(3). Applying Corollary 3.6.5(2) for $P_t f^2$ in place of f, we find

$$P_t \log P_t f^2(y) \leq \log P_{2t}f^2(x) + \frac{\|\phi\|_\infty^2 K^{\phi,2}\rho(x,y)^2}{2(\mathrm{e}^{2K^{\phi,2}t}-1)}, \quad x, y \in M, t > 0.$$

Integrating w.r.t. the optimal coupling of $f^2\mu$ and μ, which reaches the inf in the definition of $W_2^\rho(f^2\mu,\mu)$, and noting that P_t is symmetric in $L^2(\mu)$, we obtain

$$\mu((P_tf^2)\log P_tf^2) \le \frac{\|\phi\|_\infty^2 K^{\phi,2} W_2^\rho(f^2\mu,\mu)^2}{2(\mathrm{e}^{2K^{\phi,2}t}-1)}.$$

Combining this with the first inequality in (3.6.8), we arrive at

$$\begin{aligned}\mu(f^2\log f^2) \le \|\phi\|_\infty^6\mu(|\nabla f|^2)r_t + \frac{\|\phi\|_\infty^2}{r_t}W_2^\rho(f^2\mu,\mu)^2 \\ - \frac{\|\phi\|_\infty^2 K^{\phi,2}}{2}W_2^\rho(f^2\mu,\mu)^2,\end{aligned} \tag{3.6.9}$$

where

$$r_t := \frac{2(1-\mathrm{e}^{-2K^{\phi,2}t})}{K^{\phi,2}}, \quad t \ge 0.$$

If $K^{\phi,2} \le 0$, then $\{r_t:\ t\in[0,\infty]\} = [0,\infty]$. So, there exists $t\in[0,\infty]$ such that

$$r_t = \frac{W_2^\rho(f^2\mu,\mu)}{\|\phi\|_\infty^2\sqrt{\mu(|\nabla f|^2)}}.$$

Therefore, the desired HWI inequality follows from (3.6.9).

Finally, the third assertion follows from Corollary 3.6.5(2) according to Proposition 1.4.4(2) and the proof of Theorem 2.4.4(2). □

3.7 Generalized maximum principle and Li-Yau's Harnack inequality

Let M be a d-dimensional connected complete Riemannian manifold and $L=\Delta+Z$ for a C^1-smooth vector field Z satisfying the following curvature-dimension condition:

$$\begin{aligned}\Gamma_2(f,f) &:= \frac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \\ &\ge \frac{(Lf)^2}{m} - K|\nabla f|^2, \quad f\in C^\infty(M)\end{aligned} \tag{3.7.1}$$

for some constants $K\ge 0$ and $m\ge d$. Note that, for convenience in the sequel, in (3.7.1) we use $-K$ rather than K to stand for the curvature lower bound, and use m rather than n to stand for the dimension. When $Z=0$ and M is either without boundary or compact with a convex boundary

∂M, Li and Yau [Li and Yau (1986)] found the following famous gradient estimate for the (Neumann) semigroup P_t generated by L:

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \le \frac{d\alpha^2}{2t} + \frac{d\alpha^2 K}{4(\alpha-1)}, \quad t>0, \alpha>1 \tag{3.7.2}$$

for all positive $f \in C_b(M)$. We note that in [Li and Yau (1986)] the second term in the right hand side of (3.7.2) is $\frac{d\alpha^2 K}{\sqrt{2}(\alpha-1)}$, but $\sqrt{2}$ here can be replaced by 4 according to a refined calculation, see e.g. [Davies (1989)].

As an application, (3.7.2) implies the following parabolic Harnack inequality for P_t:

$$P_t f(x) \le \Big(\frac{t+s}{t}\Big)^{d\alpha/2} (P_{t+s} f(y)) \exp\left[\frac{\alpha \rho(x,y)^2}{4s} + \frac{\alpha K d s}{4(\alpha-1)}\right], \tag{3.7.3}$$

for all $t>0$, $x, y \in M$, where $\alpha > 1$ and $f \in C_b(M)$ is positive. From this Harnack inequality one obtains Gaussian type heat kernel bounds for P_t, see [Li and Yau (1986); Davies (1989)] for details.

The gradient estimate (3.7.2) has been extended and improved in several papers. See e.g. [Bakry and Qian (2000)] for an improved version for $\alpha = 1$ with $Z \neq 0$ and $\partial M = \emptyset$, and see [Wang, J. (1997)] for an extension to a compact manifold with nonconvex boundary. The aim of this section is to investigate the gradient and Harnack inequalities for P_t on noncompact manifolds with (non-convex) boundary.

Recall that the key step of Li-Yau's argument for the gradient estimate (3.7.2) is to apply the maximum principle to the reference function

$$G(t,x) := t(|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f)(x), \quad t \in [0,T], x \in M.$$

When M is compact without boundary, the maximum principle says that for any smooth function G on $[0,T] \times M$ with $G(0,\cdot) \le 0$ and $\sup G > 0$, there exists a maximal point of G such that at this point one has $\nabla G = 0, \partial_t G \ge 0$ and $\Delta G \le 0$. When M is compact with a convex boundary, the same assertion holds for the above specified function G as observed in Proof of Theorem 1.1 in [Li and Yau (1986)]. In 1997, J. Wang [Wang, J. (1997)] was able to extend this maximum principle on a compact manifold with nonconvex boundary by taking

$$G(t,x) = t(\phi |\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f)(x), \quad t \in [0,T], x \in M$$

for a nice function ϕ compensating the concavity of the boundary.

As for a noncompact manifold without boundary, the gradient estimate was established in [Li and Yau (1986)] by applying the maximal principle to a sequence of functions with compact support which approximate the original function G. An alternative way is to apply directly the following generalized maximum principle (see [Yau (1975a)]): for any bounded smooth function G on $[0,T]\times M$ with $G(0,\cdot)\le 0$ and $\sup G>0$, there exists a sequence $\{(t_n,x_n)\}_{n\ge 1}\subset[0,T]\times M$ such that

(i) $0<G(t_n,x_n)\uparrow \sup G$ as $n\uparrow\infty$;
(ii) $LG(t_n,\cdot)(x_n)\le \frac{1}{n}$, $|\nabla G(t_n,\cdot)(x_n)|\le\frac{1}{n}$ and $\partial_t G(\cdot,x_n)(t_n)\ge 0$ for any $n\ge 1$.

To apply this generalized maximal principle for the gradient estimate, one has to first confirm the boundedness of $G(t,\cdot):=t(|\nabla\log P_t f|^2-\alpha\partial_t\log P_t f)$ on $[0,T]\times M$ for $T>0$.

Since the boundedness of this type of reference function is unknown when M is noncompact with a nonconvex boundary, we shall establish a generalized maximum principle on a class of noncompact manifolds with boundary for not necessarily bounded functions. Applying this principle to a careful choice of reference function G, we derive the Li-Yau type gradient and Harnack inequalities for Neumann semigroups. To establish such a maximum principle, we adopt a localization argument so that the classical maximum principle can be applied. The main results of this section first appeared in [Wang (2010a)].

3.7.1 *A generalized maximum principle*

Theorem 3.7.1. *Assume that* (3.7.1) *and* **(A3.2.1)** *hold. Let $T>0$ and G be a smooth function on $[0,T]\times M$ such that $NG|_{\partial M}\ge 0, G(0,\cdot)\le 0$ and $\sup G>0$. Then for any $\varepsilon>0$ there exists a sequence $\{(t_n,x_n)\}_{n\ge1}\subset(0,T]\times M$ such that* (i) *holds and for any $n\ge 1$,*

$$LG(t_n,\cdot)(x_n)\le\frac{G(t_n,x_n)^{1+\varepsilon}}{n},\quad |\nabla G(t_n,\cdot)(x_n)|\le\frac{G(t_n,x_n)^{1+\varepsilon}}{n}$$

and $\partial_t G(\cdot,x_n)(t_n)\ge 0$.

Proof. We first consider the convex case then pass to the nonconvex case by using Theorem 1.2.5. Without loss of generality, we shall assume that $\sup G:=\sup_{[0,T]\times M} G>1$. Otherwise, we may use lG to replace G for a sufficiently large $l>0$.

(a) **The convex case.** Let $h\in C_0^\infty([0,\infty))$ be decreasing such that

$$h(r)=\begin{cases}1, & \text{if } r\le 1,\\ \exp[-(3-r)^{-1}], & \text{if } r\in[2,3),\\ 0, & \text{if } r\ge 3.\end{cases}$$

Obviously, for any $\varepsilon > 0$ we have (note that $0 \cdot \infty = 0$ by convention)

$$\sup_{[0,\infty)} \left\{|h^{\varepsilon-1}h''| + |h^{\varepsilon-1}h'|\right\} < \infty. \tag{3.7.4}$$

Let $W = \sqrt{1+\rho_o^2}$ and take $\varphi_n = h(W/n),\ n \geq 1$. Then

$$\{\varphi_n = 1\} \uparrow M \text{ as } n \uparrow \infty. \tag{3.7.5}$$

So, according to (3.7.9) and (3.7.4),

$$\begin{aligned} \varphi_n^{-1} L\varphi_n &= \frac{h'(W/n)}{nh(W/n)} LW + \frac{h''(W/n)}{n^2 h(W/n)} |\nabla W|^2 \\ &\geq -\frac{c}{n\varphi_n^\varepsilon}, \\ |\nabla \log \varphi_n| &\leq \frac{c}{n\varphi_n^\varepsilon} \end{aligned} \tag{3.7.6}$$

holds for some constant $c > 0$ and all $n \geq 1$.

Let

$$G_n(t,x) = \varphi_n(x) G(t,x), \quad t \in [0,T], x \in M.$$

Since G_n is continuous with compact support, there exists $(t_n, x_n) \in [0,T] \times M$ such that

$$G_n(t_n, x_n) = \max_{[0,T]\times M} G_n.$$

By (3.7.5) and $\sup G > 1$, we have $\lim_{n\to\infty} G(t_n, x_n) = \sup G > 1$. By renumbering from a sufficient large n_0 we may assume that $G_n(t_n, x_n) > 1$ and is increasing in n. In particular, (i) in the beginning of this section holds and

$$\varphi_n(x_n) \geq \frac{1}{G(t_n, x_n)}, \quad n \geq 1. \tag{3.7.7}$$

Moreover, since $G_n(0,\cdot) \leq 0$, we have $t_n > 0$ and

$$\partial_t G(\cdot, x_n)(t_n) \geq 0, \quad n \geq 1.$$

Thus, it remains to confirm

$$\begin{aligned} |\nabla G(t_n,\cdot)(x_n)| &\leq \frac{cG(t_n,x_n)^{1+\varepsilon}}{n}, \\ LG(t_n,\cdot)(x_n) &\leq \frac{cG(t_n,x_n)^{1+\varepsilon}}{n}, \quad n \geq 1 \end{aligned} \tag{3.7.8}$$

for some constant $c > 0$. Indeed, by using a subsequence $\{(t_{mn}, x_{mn})\}_{n\geq 1}$ for $m \geq c$ to replace $\{(t_n, x_n)\}_{n\geq 1}$, one may reduce (3.7.8) with some $c > 0$ to that with $c = 1$.

Since x_n is the maximal point of G_n, we have $\nabla G_n(t_n,\cdot)(x_n) = 0$ if $x_n \in M \setminus \partial M$. If $x_n \in \partial M$, we have $NG_n(t_n,\cdot)(x_n) \le 0$. Since $NG(t_n,\cdot) \ge 0, G(t_n,x_n) > 0$ and noting that $N\rho_0 \le 0$ together with $h' \le 0$ implies $N\varphi_n \ge 0$, we conclude that $NG_n(t_n,\cdot)(x_n) \ge 0$. Hence, $NG_n(t_n,\cdot)(x_n) = 0$. Moreover, since x_n is the maximal point of $G_n(t_n,\cdot)$ on the closed manifold ∂M, we have $UG_n(t_n,\cdot)(x_n) = 0$ for all $U \in T\partial M$. Therefore, $\nabla G_n(t_n,\cdot)(x_n) = 0$ also holds for $x_n \in \partial M$. Combining this with (3.7.6) and (3.7.7) we obtain

$$|\nabla G(t_n,\cdot)(x_n)| \le \frac{G(t_n,x_n)}{\varphi_n(x_n)}|\nabla\varphi_n| \le \frac{cG(t_n,x_n)^{1+\varepsilon}}{n}.$$

Thus, the first inequality in (3.7.8) holds.

Finally, by (3.7.6) one has

$$\varphi_n LG + GL\varphi_n + 2\langle\nabla G, \nabla\varphi_n\rangle \ge \varphi_n LG - \frac{c\varphi_n^{1-\varepsilon}}{n}G - \frac{2c\varphi_n^{1-\varepsilon}}{n}|\nabla G| =: \Phi$$

holds on $\{G_n > 0\} \setminus \mathrm{cut}(o)$, by Lemma 3.7.2 below we obtain at point (t_n,x_n) that

$$LG \le \frac{c}{n\varphi_n^\varepsilon}G + \frac{2c}{n\varphi_n^\varepsilon}|\nabla G|.$$

Combining this with (3.7.7) and the first inequality in (3.7.8) we get

$$LG(t_n,\cdot)(x_n) \le \frac{c}{n}G^{1+2\varepsilon}(t_n,x_n)$$

for some constant $c > 0$ and all $n \ge 1$. Since $\varepsilon > 0$ is arbitrary so that we may use $\varepsilon/2$ to replace ε (recall that $G(t_n,x_n) \ge 1$), we prove the second inequality in (3.7.8).

(b) **The non-convex case.** Under **(A3.2.1)**, there exists $\phi \in \mathcal{D}$ in (3.0.2) such that $N\log\phi|_{\partial M} \ge \sigma$. By Theorem 1.2.5, the boundary ∂M is convex under the new metric

$$\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle,$$

and $L = \phi^{-2}(\Delta' + Z')$ for some C^1-smooth vector Z' such that $\mathrm{Ric}'_{Z'}$ is bounded from below. Therefore, we are able to apply Lemma 3.7.2 below to $L' := \Delta' + Z'$ on the convex Riemannian manifold $(M, \langle\cdot,\cdot\rangle')$ to conclude the existence of the desired sequence $\{(t_n,x_n)\}$. □

Lemma 3.7.2. *Assume that ∂M is convex and* (3.7.1) *holds. Let $\varphi_n, G_n, (t_n,x_n)$ be in the proof of Theorem 3.7.1. Then the reflecting L-diffusion process is nonexplosive, and for any $\Phi \in C_b(M)$ such that*

$$\Phi \le LG_n = GL\varphi_n + \varphi_n LG + 2\langle\nabla\varphi_n, \nabla G\rangle$$

holds on $\{G_n > 0\} \setminus \mathrm{cut}(o)$, we have $\Phi(t_n,x_n) \le 0$ for all $n \ge 1$.

Proof. Let $o \in M$ be fixed and let ρ_o be the Riemannian distance to the point o. Recall that since ∂M is convex, for any $x, y \in M$ there exists a minimal geodesic in M of length $\rho(x,y)$ which links x and y, see e.g. Proposition 2.1.5 in [Wang (2005a)]. So, by (3.7.1) and a comparison theorem (see [Qian, Z. (1998)])

$$L\rho_o \le \sqrt{K(m-1)}\,\coth\left[\sqrt{K/(m-1)}\,\rho_o\right]$$

holds outside $\{o\} \cup \mathrm{cut}(o)$. In the sequel we shall set $L\rho_o = 0$ on $\mathrm{cut}(o)$ such that this implies

$$L\sqrt{1+\rho_o^2} \le c_1 \text{ on } M \tag{3.7.9}$$

for some constant $c_1 > 0$.

Next, let X_t be the reflecting L-diffusion process generated by L, and u_t be its horizontal lift on the frame bundle $O(M)$. By the Itô formula for $\rho_o(X_t)$ established by Kendall [Kendall (1987)] for $\partial M = \emptyset$ and noting that $N\rho_o|_{\partial M} \le 0$ when ∂M is nonempty but convex, we have

$$\mathrm{d}\rho_o(X_t) = \sqrt{2}\,\langle \nabla\rho_o(X_t), u_t\mathrm{d}B_t\rangle + L\rho_o(X_t)\mathrm{d}t - \mathrm{d}l_t + \mathrm{d}l_t', \tag{3.7.10}$$

where B_t is the d-dimensional Brownian motion, $L\rho_o$ is taken to be zero on $\{o\} \cup \mathrm{cut}(o)$, l_t and l_t' are two increasing processes such that l_t' increases only when $X_t = o$ while l_t increases only when $X_t \in \mathrm{cut}(o) \cup \partial M$ (note that $l_t' = 0$ for $d \ge 2$). Combining this with (3.7.9) we obtain

$$\mathrm{d}\sqrt{1+\rho_o^2(X_t)} \le \mathrm{d}M_t + L\sqrt{1+\rho_o^2(X_t)}\,\mathrm{d}t \le \mathrm{d}M_t + c_1\mathrm{d}t$$

for some martingale M_t. This implies immediately the nonexplosion of X_t.

Now, let us take $X_0 = x_n$. Since $h' \le 0$, it follows from (3.7.10) that

$$\mathrm{d}\varphi_n(X_t) \ge \sqrt{2}\,\langle \nabla\varphi_n(X_t), u_t\mathrm{d}B_t\rangle + L\varphi_n(X_t)\mathrm{d}t, \tag{3.7.11}$$

where we set $L\varphi_n = 0$ on $\mathrm{cut}(o)$ as above.

On the other hand, since $NG(t_n, \cdot) \ge 0$, applying the Itô formula to $G(t_n, X_t)$ we obtain

$$\mathrm{d}G(t_n, X_t) \ge \sqrt{2}\,\langle \nabla G(t_n,\cdot)(X_t), u_t\mathrm{d}B_t\rangle + LG(t_n,\cdot)(X_t)\mathrm{d}t. \tag{3.7.12}$$

Due to $G_n(t_n, x_n) > 0$, there exists $r > 0$ such that $G_n > 0$ on $B(x_n, r)$, the geodesic ball in M centered at x_n with radius r. Let

$$\tau = \inf\{t \ge 0 :\ X_t \notin B(x_n, r)\}.$$

Then (3.7.11) and (3.7.12) imply

$$\mathrm{d}G_n(t_n, X_t) \ge \mathrm{d}M_t + LG_n(t_n,\cdot)(X_t)\mathrm{d}t \ge \mathrm{d}M_t + \Phi(t_n, X_t)\mathrm{d}t, \quad t \le \tau$$

for some martingale M_t. Since $G_n(t_n. X_t) \le G_n(t_n, x_n)$ and $X_0 = x_n$, this implies that

$$0 \ge \mathbb{E}G_n(t_n, X_{t\wedge\tau}) - G_n(t_n, x_n) \ge \mathbb{E}\int_0^{t\wedge\tau} \Phi(t_n, X_s)\mathrm{d}s.$$

Therefore, the continuity of Φ implies

$$\Phi(t_n, x_n) = \lim_{t\to 0} \frac{1}{\mathbb{E}(t\wedge\tau)}\mathbb{E}\int_0^{t\wedge\tau} \Phi(t_n, X_s)\mathrm{d}s \le 0. \qquad \square$$

3.7.2 *Li-Yau type gradient estimate and Harnack inequality*

By using the generalized maximum principle, we are able to prove the following Li-Yau type gradient estimate. When M is compact with a convex boundary, the first assertion is well known due to [Li and Yau (1986)] by using the classical maximum principle on compact manifolds, while when M is compact with a non-convex boundary, a similar inequality to (3.7.13) was proved in [Wang, J. (1997)] by using the "interior rolling R-ball" condition.

Theorem 3.7.3. *Let M satisfy* **(A3.2.1)** *and L satisfy* (3.7.1). *Then the reflecting L-diffusion process on M is nonexplosive and the corresponding Neumann semigroup P_t satisfies the following assertions:*

(1) *If ∂M is convex then* (3.7.2) *holds for m in place of d.*
(2) *If ∂M is non-convex, then for any bounded $\phi \in \mathcal{D}$, the gradient inequality*

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \le \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K(\phi,\varepsilon,\alpha)}{4(\alpha - \|\phi^2\|_\infty)} \tag{3.7.13}$$

holds for all positive $f \in C_b(M), \alpha > \|\phi^2\|_\infty, t > 0, \varepsilon \in (0,1)$ and

$$K(\phi,\varepsilon,\alpha) := \frac{1+\varepsilon}{1-\varepsilon}\bigg(K + \frac{1}{\varepsilon}\|\nabla \log \phi^2\|_\infty^2 + \frac{1}{2}\sup(-\phi^{-2}L\phi^2) + \frac{m\alpha^2\|\nabla \log \phi^2\|_\infty^2(1+\varepsilon)}{8(\alpha - \|\phi^2\|_\infty)^2\varepsilon(1-\varepsilon)}\bigg).$$

Proof. When ∂M is convex the nonexplosion of X_t is ensured by Lemma 3.7.2. If ∂M is non-convex, this can be confirmed by a time change of the process. More precisely, let X_t' be the reflecting diffusion process on M generated by $L' := \phi^2 L$, where $L' = \Delta' + Z'$ is given in (b) of the proof of Theorem 3.7.1 on the convex manifold $(M, \langle\cdot,\cdot\rangle')$. By Lemma 3.7.2 the process X_t' generated by L' is nonexplosive. Since $X_t = X'_{\xi^{-1}(t)}$ for ξ^{-1} the inverse of

$$t \mapsto \xi(t) = \int_0^t \phi^2(X_s')\mathrm{d}s$$

so that $t\|\phi\|_\infty^{-2} \le \xi^{-1}(t) \le t$, the process X_t is nonexplosive as well.

Let $f \in C_b^1(M)$ be strictly positive, and let $u(t,x) = \log P_t f(x)$. For a fixed number $T > 0$, we shall apply Theorem 3.7.1 to the reference function

$$G(t,x) = t\{\phi^2(x)|\nabla u|^2(t,x) - \alpha u_t(t,x)\}, \quad t \in [0,T], x \in M.$$

Since $\phi \in \mathcal{D}$, we have

$$NG = t\Big\{(N\phi^2)|\nabla u|^2 + \frac{\phi^2}{(P_t f)^2} N|\nabla P_t f|^2\Big\} \ge 0$$

holds on ∂M.

According to (1.14) in [Ledoux (2000)], (3.7.1) implies

$$L|\nabla u|^2 - 2\langle \nabla Lu, \nabla u\rangle \geq -2K|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}. \tag{3.7.14}$$

Multiplying this inequality by ε and (3.7.1) by $2(1-\varepsilon)$ then combining together, we obtain

$$L|\nabla u|^2 \geq 2\langle \nabla Lu, \nabla u\rangle - 2K|\nabla u|^2 + \frac{2(1-\varepsilon)(Lu)^2}{m} + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}.$$

Moreover, it is easy to check that

$$Lu = u_t - |\nabla u|^2, \quad \partial_t|\nabla u|^2 = 2\langle \nabla u, \nabla u_t\rangle.$$

Then we arrive at

$$\begin{aligned}(L-\partial_t)|\nabla u|^2 \geq\ & \frac{2(1-\varepsilon)}{m}(|\nabla u|^2 - u_t)^2 + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} \\ & - 2\langle \nabla u, \nabla|\nabla u|^2\rangle - 2K|\nabla u|^2.\end{aligned} \tag{3.7.15}$$

On the other hand,

$$\begin{aligned}-\alpha(L-\partial_t)u_t &= 2\alpha\langle \nabla u, \nabla u_t\rangle = 2\langle \nabla u, \nabla(\phi^2|\nabla u|^2 - t^{-1}G)\rangle \\ &= 2\phi^2\langle \nabla u, \nabla|\nabla u|^2\rangle + 2|\nabla u|^2\langle \nabla u, \nabla\phi^2\rangle - 2t^{-1}\langle \nabla u, \nabla G\rangle.\end{aligned}$$

Combining this with (3.7.15) we obtain

$$\begin{aligned}(L-\partial_t)G =& -\frac{G}{t} + t\{\phi^2(L-\partial_t)|\nabla u|^2 + |\nabla u|^2L\phi^2 + 2\langle \nabla\phi^2, \nabla|\nabla u|^2\rangle\} \\ & + t\{2\phi^2\langle \nabla u, \nabla|\nabla u|^2\rangle + 2|\nabla u|^2\langle \nabla u, \nabla\phi^2\rangle - 2t^{-1}\langle \nabla u, \nabla G\rangle\} \\ \geq & -\frac{G}{t} + \frac{2(1-\varepsilon)\phi^2 t}{m}(|\nabla u|^2 - u_t)^2 + \frac{\varepsilon\phi^2 t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} \\ & - 2K\phi^2 t|\nabla u|^2 - 2|\nabla u|\cdot|\nabla G| - 2t|\nabla u|^3|\nabla\phi^2| \\ & - 2t|\nabla\phi^2|\cdot|\nabla|\nabla u|^2| + t|\nabla u|^2L\phi^2.\end{aligned}$$

Noting that

$$\frac{\varepsilon\phi^2 t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} - 2t|\nabla\phi^2|\cdot|\nabla|\nabla u|^2| \geq -\frac{2t|\nabla\phi^2|^2|\nabla u|^2}{\varepsilon\phi^2},$$

we get

$$\begin{aligned}(L-\partial_t)G \geq & -\frac{G}{t} + \frac{2(1-\varepsilon)\phi^2 t}{m}(|\nabla u|^2 - u_t)^2 - 2K\phi^2 t|\nabla u|^2 \\ & - 2|\nabla u|\cdot|\nabla G| - 2t|\nabla u|^3|\nabla\phi^2| \\ & + t|\nabla u|^2L\phi^2 - \frac{2t|\nabla\phi^2|^2|\nabla u|^2}{\varepsilon\phi^2}.\end{aligned} \tag{3.7.16}$$

We assume that $\sup G > 0$, otherwise the proof is done. Since $G(0,\cdot) = 0$ and $NG|_{\partial M} \geq 0$, we can apply Theorem 3.7.1. Let $\{(t_n, x_n)\}$ be fixed in Theorem 3.7.1 for e.g. $\varepsilon = \frac{1}{2}$. So,

$$(L-\partial_t)G(t_n,x_n) \leq \frac{G^{3/2}(t_n,x_n)}{n}, \quad |\nabla G|(t_n,x_n) \leq \frac{G^{3/2}(t_n,x_n)}{n}. \tag{3.7.17}$$

From now on, the value of functions are taken at a fixed point (t_n, x_n), so that $t = t_n$ in the sequel.

Let $\chi = |\nabla u|^2/G$. We have

$$|\nabla u|^2 - u_t = \Big(\chi - \frac{\chi t\phi^2 - 1}{\alpha t}\Big)G = \frac{\chi t(\alpha - \phi^2) + 1}{\alpha t} G.$$

Combining this with (3.7.16) and (3.7.17), we arrive at

$$\begin{aligned}
&\frac{2(1-\varepsilon)\phi^2(\chi t(\alpha-\phi^2)+1)^2}{m\alpha^2 t} G^2 \\
&\leq \frac{G^{3/2}}{n} + \frac{G}{t} + \frac{2\sqrt{\chi}\, G^2}{n} + 2t|\nabla\phi^2|(\chi G)^{3/2} \\
&\quad + \big\{2K\phi^2 + 2\varepsilon^{-1}\phi^{-2}|\nabla\phi^2|^2 - L\phi^2\big\}\chi t G.
\end{aligned} \tag{3.7.18}$$

Since it is easy to see that

$$(\chi t(\alpha-\phi^2)+1)^2 \geq \max\big\{1,\ 4\chi t(\alpha-\phi^2),\ \big(2t(\alpha-\phi^2)\big)^{3/2}\chi^{3/2}\big\},$$

multiplying both sides of (3.7.18) by $t(\chi t(\alpha-\phi^2)+1)^{-2}G^{-2}$, we obtain

$$\begin{aligned}
\frac{2(1-\varepsilon)\phi^2}{m\alpha^2} &\leq \frac{c't}{n(1\wedge\sqrt{G})} + \frac{1}{G} + \frac{2K\phi^2 + 2\varepsilon^{-1}|\nabla\log\phi^2|^2\phi^2 - L\phi^2}{4(\alpha-\phi^2)G} t \\
&\quad + \frac{|\nabla\log\phi^2|\phi^2}{(\alpha-\phi^2)^{3/2}\sqrt{2G}}\sqrt{t} \\
&\leq \frac{c't}{n(1\wedge\sqrt{G})} + \frac{1}{G} + \frac{2K\phi^2 + 2\varepsilon^{-1}|\nabla\log\phi^2|^2\phi^2 - L\phi^2}{4(\alpha-\phi^2)G} t \\
&\quad + \frac{|\nabla\log\phi^2|^2 m\alpha^2(1+\varepsilon)\phi^2 t}{16(\alpha-\phi^2)^3\varepsilon(1-\varepsilon)G} + \frac{2(1-\varepsilon)\varepsilon\phi^2}{m\alpha^2(1+\varepsilon)}
\end{aligned}$$

for some constant $c' > 0$. Taking $n \to \infty$ and noting that $\phi \geq 1$, we conclude that $\theta := \sup G$ satisfies

$$\begin{aligned}
\frac{2(1-\varepsilon)}{m\alpha^2(1+\varepsilon)} &\leq \frac{1}{\theta}\Big(1 + \frac{2K + 2\varepsilon^{-1}\|\nabla\log\phi^2\|_\infty^2 + \sup(-\phi^{-2}L\phi^2)}{4(\alpha - \|\phi^2\|_\infty)} T \\
&\quad + \frac{\|\nabla\log\phi^2\|_\infty^2 m\alpha^2(1+\varepsilon)T}{16(\alpha-\|\phi^2\|_\infty)^3\varepsilon(1-\varepsilon)}\Big).
\end{aligned}$$

Combining this with

$$\theta \geq G(T,x) = T(\phi^2(x)|\nabla u|^2(T,x) - \alpha u_t(T,x)), \quad x \in M,$$

we obtain

$$\begin{aligned}
&\phi^2(x)|\nabla u|^2(T,x) - \alpha u_t(T,x)\\
&\le \frac{m\alpha^2(1+\varepsilon)}{2(1-\varepsilon)}\Big(\frac{1}{T} + \frac{2K + 2\varepsilon^{-1}\|\nabla \log \phi^2\|_\infty^2 + \sup(-\phi^{-2}L\phi^2)}{4(\alpha - \|\phi^2\|_\infty)}\\
&\quad + \frac{\|\nabla \log \phi^2\|_\infty^2 m\alpha^2(1+\varepsilon)}{16(\alpha - \|\phi^2\|_\infty)^3\varepsilon(1-\varepsilon)}\Big)
\end{aligned}$$

for all $x \in M$. Then the proof is completed since $T > 0$ is arbitrary. □

By a standard argument due to Li and Yau [Li and Yau (1986)], the gradient estimate (3.7.13) implies the following result on Harnack inequality.

Corollary 3.7.4. *Let M satisfy* **(A3.2.1)** *and L satisfy* (3.7.1), *and let $\phi \in \mathcal{D}$ in* (3.0.2). *Then*

$$\begin{aligned}
P_t f(x) \le \Big(\frac{t+s}{t}\Big)^{\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)}} &(P_{t+s}f(y))\\
&\times \exp\Big[\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha m K(\phi,\varepsilon,\alpha)s}{4(\alpha - \|\phi^2\|_\infty)}\Big]
\end{aligned} \tag{3.7.19}$$

for all positive $f \in C_b(M), t, \varepsilon \in (0,1), \alpha > \|\phi^2\|_\infty$ and $x, y \in M$. In particular, if ∂M is convex then (3.7.3) *holds for m in place of d and all $\alpha > 1$.*

Proof. Due to Theorem 3.7.3, the proof is standard according to [Li and Yau (1986)]. For $x, y \in M$, let $\gamma : [0,1] \to M$ be the shortest curve in M linking x and y such that $|\dot\gamma| = \rho(x,y)$. Then, for any $s, t > 0$ and $f \in C_b^\infty(M)$, it follows from (3.7.13) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r} \log P_{t+rs}f(\gamma_r) = s\partial_u \log P_u f(\gamma_r)|_{u=t+rs} + \langle \dot\gamma_r, \nabla \log P_{t+rs}f(\gamma_r)\rangle\\
&\ge \frac{s}{\alpha}|\nabla \log P_{t+rs}f|^2(\gamma_r) - \rho(x,y)|\nabla \log P_{t+rs}f|(\gamma_r)\\
&\quad - s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha - \|\phi^2\|_\infty)}\Big)\\
&\ge -\frac{\alpha}{4s}\rho(x,y)^2 - s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)} + \frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha - \|\phi^2\|_\infty)}\Big).
\end{aligned}$$

This completes the proof by integrating w.r.t. $\mathrm{d}r$ over $[0,1]$. □

3.8 Robin semigroup and applications

In this section we consider the Robin semigroup with application to the HWI inequality on non-convex manifolds. The key point of the study is to identify the semigroup by using the local time of the reflecting diffusion process and the underlying function Q on the boundary. Throughout the section we assume that $Z = \nabla V$ for some $V \in C^2(M)$, so that $L := \Delta + \nabla V$ is symmetric w.r.t. $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$.

For a non-negative measurable function Q on ∂M and a bounded measurable function W on M, we will consider the operator $L^W := L - W$ with the Robin boundary condition

$$\langle N, \nabla f\rangle = Qf \text{ on } \partial M, \tag{3.8.1}$$

where N is the inward pointing unit normal vector field of ∂M. Let $\mathcal{D}_0$ be the set of functions $f \in C_0^\infty(M)$ satisfying (3.8.1), by Theorem 1.1.6(4) we have

$$\begin{aligned}\mathcal{E}_{Q,W}(f,g) &:= \int_M \{\langle \nabla f, \nabla g\rangle + Wfg\}\,\mathrm{d}\mu + \int_{\partial M} Qfg\,\mathrm{d}\mu_\partial \\ &= -\int_M fL^W g\,\mathrm{d}\mu, \quad f,g \in \mathcal{D}_0,\end{aligned} \tag{3.8.2}$$

where μ_∂ is the area measure on ∂M induced by μ. It is easy to see that

$$\mathcal{D}_0 \supset \{f \in C_0^\infty(M) : \ \mathrm{supp} f \subset M \setminus \partial M\},$$

which is dense in $L^2(\mu)$. So, $(\mathcal{E}_{Q,W}, \mathcal{D}_0)$ is symmetric, bounded below, densely defined on $L^2(\mu)$. By (3.8.2), $(\mathcal{E}_{Q,W}, \mathcal{D}_0)$ is closable and its closure $(\mathcal{E}_{Q,W}, \mathcal{D}(\mathcal{E}_{Q,W}))$ is associated to a symmetric C_0-semigroup $P_t^{Q,W}$ on $L^2(\mu)$. Let $(L_{Q,W}, \mathcal{D}(L_{Q,W}))$ be its generator, which thus extends $(L^W, C_0^\infty(M))$ due to (3.8.2). When $W = 0$ we simply denote $\mathcal{E}_Q = \mathcal{E}_{Q,W}$ and $P_t^Q = P_t^{Q,W}$. In this section we aim to study the Poincaré inequality

$$\mu(f^2) \le C\mathcal{E}_Q(f,f), \quad f \in \mathcal{D}(\mathcal{E}_Q) \tag{3.8.3}$$

and estimate the first Robin eigenvalue

$$\lambda_Q := \inf\{\mathcal{E}_Q(f,f) : \ f \in \mathcal{D}(\mathcal{E}_Q), \mu(f^2) = 1\}.$$

Let P_t^Q be the associate (sub-)Markov semigroup, which is called the Robin semigroup on M generated by L and boundary condition (3.8.1). We have

$$\|P_t^Q f\|_2 \le \mathrm{e}^{-\lambda_Q t}\|f\|_2, \quad f \in L^2(\mu).$$

Recall that for any $p \ge 1$, $\|\cdot\|_p$ stands for the L^p-norm w.r.t. μ.

Unlike for the study of the first Neumann and Dirichlet eigenvalues, known results on the first Robin eigenvalue are very rare: there is no any non-trivial explicit estimate of λ_Q for general Q. Nevertheless, it is easy to see that λ_Q is bounded above by the first Dirichlet eigenvalue

$$\lambda_D := \inf\{\mu(|\nabla f|^2) : \mu(f^2) = 1, f \in C_0^1(M),\ f|_{\partial M} = 0\}.$$

To describe λ_Q, we shall first present a probability representation of the Robin semigroup P_t^Q and characterize the domain $\mathcal{D}(\mathcal{E}_Q)$.

The remainder of this section consists of three parts. In the first part we characterize the Robin semigroup $P_t^{Q,W}$ and the associated quadratic form, in the second part we investigate the first eigenvalue λ_Q, and finally, in the last part we use the Robin semigroup to establish the HWI inequality on non-convex manifolds.

3.8.1 *Characterization of $P_t^{Q,W}$ and $\mathcal{D}(\mathcal{E}_Q)$*

Proposition 3.8.1. *Let $Q \geq 0$. For any $f \in L^2(\mu)$,*

$$P_t^{Q,W} f(x) = \mathbb{E}^x\left\{f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}\right\},\quad x \in M, \tag{3.8.4}$$

where X_t is the L-reflecting diffusion process on M with local time l_t on ∂M, and $\mathbb{E}^x$ is the expectation for the process starting at point x.

Proof. Let us denote

$$\tilde{P}_t^{Q,W} f(x) = \mathbb{E}^x\left\{f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}\right\}.$$

We aim to prove that $P_t^{Q,W} = \tilde{P}_t^{Q,W}$ holds on $L^2(\mu)$. To this end, we first consider $f \in \mathcal{D}_0$. In this case, by (3.0.1), the Itô formula and the Robin boundary condition,

$$\mathrm{d}f(X_t) = \sqrt{2}\,\langle \nabla f(X_t), u_t \mathrm{d}B_t\rangle + Lf(X_t)\mathrm{d}t + \{fQ\}(X_t)\mathrm{d}l_t.$$

This implies

$$\begin{aligned}
&\mathrm{d}\left\{f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}\right\}\\
&= \mathrm{d}M_t + \left\{(L^W f(X_t))\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}\right\}\mathrm{d}t
\end{aligned}$$

for some martingale M_t. Therefore,

$$\tilde{P}_t^{Q,W} f(x) = f(x) + \int_0^t \tilde{P}_s^{Q,W} L^W f(x)\mathrm{d}s,\quad x \in M, t \geq 0. \tag{3.8.5}$$

Now, for any $f \in \mathcal{D}(L_{Q,W})$, there exists $\{f_n\}_{n\geq 1} \subset \mathcal{D}_0$ such that $L^W f_n = L_{Q,W} f_n \to L_{Q,W} f$ and $f_n \to f$ in $L^2(\mu)$. Since $\tilde{P}_t^{Q,W}$ is bounded in $L^2(\mu)$, (3.8.5) implies that

$$\tilde{P}_t^{Q,W} f = f + \int_0^t \tilde{P}_s^{Q,W} L_{Q,W} f \mathrm{d}s, \quad f \in \mathcal{D}(L_{Q,W}).$$

Thus,

$$\frac{\mathrm{d}\tilde{P}_t^{Q,W} f}{\mathrm{d}t} = \tilde{P}_t^{Q,W} L_{Q,W} f, \quad f \in \mathcal{D}(L_{Q,W}), \quad t \geq 0 \tag{3.8.6}$$

holds in $L^2(\mu)$. Since $P_s^{Q,W} f \in \mathcal{D}(L_{Q,W})$ for $s \geq 0$, combining this with the fact that

$$\frac{\mathrm{d}P_s^{Q,W} f}{\mathrm{d}s} = L_{Q,W} P_s^{Q,W} f = P_s^{Q,W} L_{Q,W} f$$

holds on $\mathcal{D}(L_{Q,W})$ uniformly in $s \in [0,t]$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}s} \tilde{P}_{t-s}^{Q,W} P_s^{Q,W} f = 0, \quad s \in [0,t].$$

Therefore, $\tilde{P}_t^{Q,W} f = P_t^{Q,W} f$ holds in $L^2(\mu)$ for $f \in \mathcal{D}(L_{Q,W})$. Since $\mathcal{D}(L_{Q,W})$ is dense in $L^2(\mu)$, $P_t^{Q,W} = \tilde{P}_t^{Q,W}$ holds in $L^2(\mu)$. □

Proposition 3.8.2. *Let $Q \geq 0$ and $W_Q^{2,1}$ be the completion of $\mathcal{C}_Q$ with respect to the $\mathcal{E}_Q^1$-norm, where*

$$\mathcal{C}_Q := \big\{ f \in C_b^1(M) : \ \mathcal{E}_Q(f,f) := \mu(|\nabla f|^2) + \mu_\partial(Qf^2) < \infty \big\}.$$

Then $\mathcal{D}(\mathcal{E}_Q) = W_Q^{2,1}$.

Proof. It suffices to show that $\mathcal{D}(\mathcal{E}_Q) \supset \mathcal{C}_Q$.

(a) We first prove $\mathcal{D}(\mathcal{E}_Q) \supset C_0^1(M)$. Since $C_0^\infty(M)$ is dense in $C_0^1(M)$ under the uniform norm up to the first-order derivatives, we only need to consider $f \in C_0^\infty(M)$. Let $x_0 \in M$ be fixed. For any $f \in C_0^\infty(M)$ there exists $R > 0$ such that

$$\mathrm{supp}\, f \subset B(x_0, R) := \{x \in M : \rho(x_0, x) \leq R\},$$

where ρ is the Riemannian distance on M, i.e. for any $x, y \in M$, $\rho(x,y)$ is the length (induced by the Riemannian metric) of the shortest continuous curve linking x and y. Let $r_0 \in (0,1)$ such that the exponential map

$$\varphi_N : \ \big\{(\partial M) \cap B(x_0, R+1)\big\} \times [0, r_0] \ni (\theta, r) \mapsto \exp_\theta[rN] \in \mathbf{K}$$

is C^∞-smooth and invertible, where $\mathbf{K}$ is the image of φ_N. Let ρ_∂ be the Riemannian distance to the boundary ∂M. It is easy to see that

$$\begin{aligned}\mathbf{K} = \big\{ x \in M : & \text{ there exists } \theta \in (\partial M) \cap B(x_0, R+1) \\ & \text{ such that } \rho_\partial(x) = \rho(\theta, x) \leq r_0 \big\}.\end{aligned}$$

Then the polar coordinate $(\theta, r)(x) := \varphi_N^{-1}(x)$ for $x \in \mathbf{K}$ is smooth in x. Let

$$\tilde{f}(\theta, r) = \big(1 + Q(\theta)r\big)f(\theta, 0).$$

We see that $\tilde{f} \in C^\infty(\mathbf{K})$ and satisfies the Robin condition on $(\partial M) \cap \mathbf{K}$. Since $f(x) = 0$ for $\rho(x_0, x) \geq R$, we have $\tilde{f}(x) = 0$ for $x \in \mathbf{K}$ with $\rho(x_0, x) \geq R + r_0$. Noting that $r_0 < 1$, by letting $\tilde{f}(x) = 0$ for $x \notin \mathbf{K}$ we extend $\tilde{f}$ as a function on M which is C^∞-smooth on

$$\partial_{r_0} M := \big\{x \in M : \rho_\partial(x) \leq r_0\big\}$$

and satisfies the Robin condition on ∂M. Now, let $h \in C^\infty(\mathbb{R})$ such that $0 \leq h \leq 1, h|_{[0,1]} = 0$ and $h|_{[2,\infty)} = 1$. Then for any natural number $n \geq 1 + \frac{2}{r_0}$,

$$f_n := \big\{1 - h(n\rho_\partial)\big\}\tilde{f} + h(n\rho_\partial)f \in C_0^\infty(M).$$

Moreover, since $f_n = \tilde{f}$ in a neighborhood of ∂M, f_n satisfies the Robin boundary condition. Therefore, $f_n \in \mathcal{D}_0$. Obviously, $f_n \to f$ in $L^2(\mu)$ as $n \to \infty$. It remains to observe that

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{E}_Q(f_n - f, f_n - f)\\
&= \lim_{n\to\infty} \mathcal{E}_Q\big(\{1 - h(n\rho_\partial)\}(\tilde{f} - f), \{1 - h(n\rho_\partial)\}(\tilde{f} - f)\big)\\
&\leq 4(\|\tilde{f}\|_\infty^2 + \|\nabla \tilde{f}\|_\infty^2) \lim_{n\to\infty} \mu(B(x_0, R+1) \cap \{\rho_\partial \leq 2/n\}) = 0.
\end{aligned}$$

(b) Let $f \in \mathcal{C}_Q$. For any $n \geq 1$ let $g_n \in C_0^\infty(M)$ such that $0 \leq g_n \leq 1, g_n|_{B(x_0,n)} = 1$, $g_n|_{B(x_0,2n)^c} = 0$ and $|\nabla g_n| \leq \frac{2}{n}$. Then $fg_n \to f$ in $L^2(\mu)$ and $\mathcal{E}_Q((g_n - 1)f, (g_n - 1)f) \to 0$ as $n \to \infty$. Combining this with (a), we conclude that $f \in \mathcal{D}(\mathcal{E}_Q)$. □

Finally, let P_t^N and P_t^D be the Neumann and Dirichlet semigroups generated by L on M respectively. We have

$$P_t^N f(x) = \mathbb{E}^x f(X_t), \quad P_t^D f(x) = \mathbb{E}^x\{f(X_t)1_{\{t<\tau\}}\}, \quad x \in M, f \in \mathcal{B}_b(M),$$

where τ is the hitting time of X_t to the boundary ∂M. As a consequence of Proposition 3.8.1, the following result says that P_t^Q interpolates the Neumann and Dirichlet semigroups.

Corollary 3.8.3. *For any non-negative measurable function Q on ∂M,*

$$\lim_{r\downarrow 0} P_t^{rQ} f(x) = P_t^N f(x), \quad x \in M, f \in \mathcal{B}_b(M). \tag{3.8.7}$$

If $Q > 0$, then

$$\lim_{r\uparrow\infty} P_t^{rQ} f(x) = P_t^D f(x), \quad x \in M, f \in \mathcal{B}_b(M). \tag{3.8.8}$$

Proof. By the dominated convergence theorem, (3.8.7) follows from (3.8.4) immediately. Next, by (3.8.4) we have

$$\begin{aligned} P_t^{rQ} f(x) &= \mathbb{E}^x\Big(f(X_t)\mathrm{e}^{-r\int_0^t Q(X_s)\mathrm{d}l_s}\Big) \\ &= P_t^D f(x) + \mathbb{E}^x\Big(1_{\{\tau\le t\}} f(X_t)\mathrm{e}^{-r\int_0^t Q(X_s)\mathrm{d}l_s}\Big). \end{aligned} \tag{3.8.9}$$

Let $\pi_x(\mathrm{d}s, \mathrm{d}\theta)$ be the distribution of (τ, X_τ) restricted on $\{\tau < \infty\}$ given $X_0 = x$. By the strong Markov property of the reflecting L-diffusion process and the fact that $\tau \ne t$ a.s., we have

$$\begin{aligned} &\mathbb{E}^x\Big(1_{\{\tau\le t\}} f(X_t)\mathrm{e}^{-r\int_0^t Q(X_s)\mathrm{d}l_s}\Big) \\ &= \int_{[0,t)\times\partial M} \Big\{\mathbb{E}^\theta f(X_{t-s})\mathrm{e}^{-r\int_0^{t-s} Q(X_r)\mathrm{d}l_r}\Big\}\pi_x(\mathrm{d}s, \mathrm{d}\theta). \end{aligned} \tag{3.8.10}$$

Since $Q > 0$ and $l_{t-s} > 0$ a.s. for $t > s$ (cf. Theorem 7.2 in [Sato and Ueno (1965)]), we have $\int_0^{t-s} Q(X_r)\mathrm{d}l_r > 0$ a.s. for $X_0 \in \partial M$ and $t > s$. Combining this with (3.8.9) and (3.8.10), we prove (3.8.8). □

3.8.2 *Some criteria on λ_Q for $\mu(M) = 1$*

Throughout this subsection, we assume that $\mu(M) = 1$. We first present a probability characterization of λ_Q.

Theorem 3.8.4. *For any measurable function $Q \ge 0$ on ∂M,*

$$\lambda_Q = \liminf_{t\to\infty} \frac{-1}{2t}\log\int_M \Big(\mathbb{E}^x \mathrm{e}^{-\int_0^t Q(X_s)\mathrm{d}l_s}\Big)^2 \mu(\mathrm{d}x).$$

Consequently,

$$\liminf_{t\to\infty}\frac{-1}{2t}\log \mathbb{E}^\mu \mathrm{e}^{-2\int_0^t Q(X_s)\mathrm{d}l_s} \le \lambda_Q \le \frac{-1}{t}\log\mathbb{E}^\mu\mathrm{e}^{-\int_0^t Q(X_s)\mathrm{d}l_s}.$$

Proof. By the Jensen inequality, the lower bound in second assertion follows from the first one immediately, while the upper bound follows from the fact that

$$\mathbb{E}^\mu \mathrm{e}^{-\int_0^t Q(X_s)\mathrm{d}l_s} = \mu(P_t^Q 1) \le \mu((P_t^Q 1)^2)^{1/2} \le \mathrm{e}^{-\lambda_Q t}.$$

So, it remains to prove the first assertion. Let

$$\delta = \liminf_{t\to\infty}\frac{-1}{2t}\log\int_M\Big(\mathbb{E}^x\mathrm{e}^{-\int_0^t Q(X_s)\mathrm{d}l_s}\Big)^2\mu(\mathrm{d}x).$$

By Proposition 3.8.1,

$$\begin{aligned} &\int_M \Big(\mathbb{E}^x \mathrm{e}^{-\int_0^t Q(X_s)\mathrm{d}l_s}\Big)^2 \mu(\mathrm{d}x) \\ &= \int_M (P_t^Q 1(x))^2 \mu(\mathrm{d}x) \le \mathrm{e}^{-2\lambda_Q t}\mu(M) = \mathrm{e}^{-2\lambda_Q t}, \quad t > 0. \end{aligned}$$

This implies that $\lambda_Q \le \delta$.

On the other hand, for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\frac{-1}{2t}\log\int_M\left(\mathbb{E}^x e^{-\int_0^t Q(X_s)\mathrm{d}l_s}\right)^2\mu(\mathrm{d}x) \ge \delta\wedge\varepsilon^{-1}-\varepsilon,\quad t\ge t_\varepsilon.$$

So, for any $f\in L^\infty(\mu)$, we have

$$\begin{aligned}\int_M (P_t^Q f)^2\mathrm{d}\mu &\le \|f\|_\infty^2\int_M\left(\mathbb{E}^x e^{-\int_0^t Q(X_s)\mathrm{d}l_s}\right)^2\mu(\mathrm{d}x)\\ &\le \|f\|_\infty^2 e^{-2(\delta\wedge\varepsilon^{-1}-\varepsilon)^+ t},\quad t\ge t_\varepsilon.\end{aligned}$$

Combining this with Lemma 2.2 in [Röckner and Wang (2001)], we obtain

$$\begin{aligned}\mu((P_s^Q f)^2) &\le \mu((P_t^Q f)^2)^{s/t}\mu(f^2)^{1-s/t}\\ &\le \|f\|_\infty^{2s/t}\mu(f^2)^{1-s/t}e^{-2(\delta\wedge\varepsilon^{-1}-\varepsilon)^+ s},\quad t\ge t_\varepsilon, s\in[0,t].\end{aligned}$$

Letting $t\to\infty$, we arrive at

$$\mu((P_s^Q f)^2)\le \mu(f^2)\,e^{-2(\delta\wedge\varepsilon^{-1}-\varepsilon)^+ s},\quad s\ge 0, f\in L^\infty(\mu).$$

Since $L^\infty(\mu)$ is dense in $L^2(\mu)$, this implies that $\lambda_Q\ge\delta\wedge\varepsilon^{-1}-\varepsilon$ for all $\varepsilon>0$. Therefore, $\lambda_Q\ge\delta$. □

By Theorem 3.8.4, $\mu_\partial(Q)>0$ is necessary to ensure $\lambda_Q>0$. The next result provides some equivalent statements for $\lambda_Q>0$ for all non-trivual Q.

Theorem 3.8.5. *Let M be non-compact. Then the following statements are equivalent to each other:*

(*i*) *For any non-negative measurable function Q on ∂M with $\mu_\partial(Q)>0$, there holds $\lambda_Q>0$.*

(*ii*) *There exists a non-negative measurable function Q on ∂M with compact support such that $\lambda_Q>0$.*

(*iii*) *There exist two constants $C_1, C_2>0$ such that the defective Poincaré inequality*

$$\mu(f^2)\le C_1\mu(|\nabla f|^2)+C_2\mu(|f|)^2,\quad f\in C_0^1(M)$$

holds.

(*iv*) *There exists a constant $C>0$ such that the Poincaré inequality*

$$\mu(f^2)\le C\mu(|\nabla f|^2)+\mu(f)^2,\quad f\in C_0^1(M)$$

holds.

Proof. Since the Neumann semigroup has strictly positive density, according to [Aida (1998)] it is uniformly positivity improving and hence the Neumann Dirichlet form satisfies the weak spectral gap property. So, according to Proposition 1.2 in [Röckner and Wang (2001)], the weak Poincaré inequality

$$\mu(f^2) \le \alpha(r)\mu(|\nabla f|^2) + r\|f\|_\infty, \quad r > 0, f \in C_b^1(M), \mu(f) = 0$$

holds for some $\alpha : (0,\infty) \to (0,\infty)$. Thus, due to Proposition 1.3 in [Röckner and Wang (2001)], the statements (iii) and (iv) are equivalent. Moreover, it is clear that (i) implies (ii). Therefore, it suffices to prove that (ii) implies (iii) while (iii) implies (i).

(a) Let $\lambda_Q > 0$ for some non-negative Q with compact support $\mathbf{K}$. Take $h \in C_0^\infty(M)$ such that $|\nabla h| \le 1, h|_{\mathbf{K}} = 0$ and $h|_{\mathbf{K}_1^c} = 1$ for some compact smooth domain $\mathbf{K}_1 \supset \mathbf{K}$. Then by Proposition 3.8.2, for any $f \in C_0^1(M)$,

$$\begin{aligned}\mu((fh)^2) &\le \frac{1}{\lambda_Q}\mu(|\nabla(fh)|^2) + \frac{1}{\lambda_Q}\mu_\partial(Q(fh)^2) = \frac{1}{\lambda_Q}\mu(|\nabla(fh)|^2)\\ &\le \frac{2}{\lambda_Q}\mu(|\nabla f|^2) + \frac{2}{\lambda_Q}\mu(f^2 1_{\mathbf{K}_1}).\end{aligned}$$

This implies

$$\mu(f^2) \le \mu((fh)^2) + \mu(f^2 1_{\mathbf{K}_1}) \le \frac{2}{\lambda_Q}\mu(|\nabla f|^2) + \Big(1 + \frac{2}{\lambda_Q}\Big)\mu(f^2 1_{\mathbf{K}_1}).$$

By the local (defective in case $\mathbf{K}_1$ is non-connected) Poincaré inequality,

$$\mu(f^2 1_{\mathbf{K}_1}) \le A\mu(|\nabla f|^2) + B\mu(|f|)^2$$

holds for some constants $A, B > 0$. Therefore, (iii) holds.

(b) (iii) implies (i). Let $\mu_\partial(Q) > 0$. Then it is easy to see that $\mathcal{E}_Q(f,f) = 0$ implies that $f = 0$. Then according to Theorem 1.6.17, the weak Poincaré inequality

$$\mu(f^2) \le \alpha(r)\mathcal{E}_Q(f,f) + r\|f\|_\infty^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_Q)$$

holds for some $\alpha : (0,\infty) \to (0,\infty)$. On the other hand, (iii) implies that $(\mathcal{E}_Q, \mathcal{D}(\mathcal{E}_Q))$ satisfies the defective Poincaré inequality. Therefore, by [Wang (2003)], we have $\lambda_Q > 0$. $\square$

Obviously, the above proof of (iii) implying (i) indeed gives the following stronger assertion.

Theorem 3.8.6. *If there exist two constants $C_1, C_2 > 0$ such that*

$$\mu(f^2) \le C_1\mathcal{E}_Q(f,f) + C_2\mu(|f|)^2, \quad f \in C_0^1(M)$$

holds, then $\lambda_Q > 0$.

As a consequence of Theorem 3.8.6, we have the following drift conditions for $\lambda_Q > 0$.

Corollary 3.8.7. *If there exists $W \in C^2(M)$ and a compact set $\mathbf{K} \subset M$ such that $W \geq 1$ and*

$$LW \leq -\lambda W + b1_{\mathbf{K}}, \qquad NW|_{\partial M} \leq \alpha WQ|_{\partial M} \tag{3.8.11}$$

holds for some constants $\lambda, b, \alpha > 0$. Then $\lambda_Q > 0$.

Proof. Without loss of generality, we assume that $\mathbf{K}$ is a smooth compact domain such that the Poincaré inequality

$$\int_{\mathbf{K}} f^2 \mathrm{d}\mu \leq C \int_{\mathbf{K}} |\nabla f|^2 \mathrm{d}\mu + \left(\int_{\mathbf{K}} f \mathrm{d}\mu \right)^2, \quad f \in C^1(\mathbf{K})$$

holds for some constant $C > 0$. By (3.8.11) we have $1 \leq \frac{-LW}{W\lambda} + \frac{b}{\lambda} 1_{\mathbf{K}}$, so that

$$\begin{aligned}
\int_M f^2 \mathrm{d}\mu &\leq \frac{1}{\lambda} \int_M f^2 \left(\frac{-LW}{W} \right) \mathrm{d}\mu + \frac{b}{\lambda} \int_{\mathbf{K}} f^2 \mathrm{d}\mu \\
&= \frac{1}{\lambda} \int_M \left\langle \nabla \left(\frac{f^2}{W} \right), \nabla W \right\rangle \mathrm{d}\mu + \frac{1}{\lambda} \int_{\partial M} (NW) \frac{f^2}{W} \mathrm{d}\mu_\partial + \frac{b}{\lambda} \int_{\mathbf{K}} f^2 \mathrm{d}\mu \\
&\leq \frac{1 + Cb}{\lambda} \int_M |\nabla f|^2 \mathrm{d}\mu + \frac{\alpha}{\lambda} \int_{\partial M} Q f^2 \mathrm{d}\mu_\partial + \frac{b}{\lambda} \mu(|f|)^2 \\
&\leq \frac{1 + Cb + \alpha}{\lambda} \mathcal{E}_Q(f, f) + \frac{b}{\lambda} \mu(|f|)^2.
\end{aligned}$$

This implies $\lambda_Q > 0$ according to Theorem 3.8.6. □

In applications, a standard choice of W is $\mathrm{e}^{\varepsilon \rho_o}$ for $\varepsilon > 0$ and the Riemannian distance ρ_o to a fixed point $o \in M$. More precisely, if

$$\limsup_{\rho_o \to \infty} L\rho_o < 0$$

holds outside the cut-locus of o, which can be verified by using curvature conditions due to the second variational formula of the Riemannian distance (cf. [Wang (2005a)]), one has for $\mathbf{K} = \{\rho_o \leq R\}$ for sufficiently large $R > 0$,

$$L\rho_o \leq -\delta$$

holds outside $\mathbf{K}$ for some constant $\delta > 0$. Then

$$L\mathrm{e}^{\varepsilon \rho_0} = \varepsilon \mathrm{e}^{\varepsilon \rho_0} (L\rho_0 + \varepsilon) \leq -\varepsilon(\delta - \varepsilon) \mathrm{e}^{\varepsilon \rho_0}$$

holds outside $\mathbf{K}$. Thus, by letting e.g. $W = \mathrm{e}^{\varepsilon \rho_o}$ for small ε and large ρ_o (note that by an approximation argument we may assume that ρ_o is smooth, see e.g. [Wang (2005a)]), then $LW \leq -\lambda W + b1_{\mathbf{K}}$ holds for some $\lambda, b > 0$. Next, the boundary condition holds provided either ∂M is convex such that $N\rho_o \leq 0$, or $\inf Q > 0$.

3.8.3 *Application to HWI inequality*

As observed in the beginning of this section, when $Q \geq 0$ the Robin semigroup $P_t^{Q,W}$ is symmetric in $L^2(\mu)$. Since according to e.g. Theorem 3.3.3 when ∂M is non-convex we have to treat P_t^Q for negative Q, we first consider the symmetry of $P_t^{Q,W}$ for possibly negative Q.

Lemma 3.8.8. *Assume* **(A3.2.1)(ii)**. *Let W be a bounded measurable function on M and $Q \in C_b(\partial M)$. Then for any $t \geq 0$,*

$$P_t^{Q,W} f := \mathbb{E}^{\cdot}\left\{f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}\right\}$$

is a symmetric bounded operator on $L^2(\mu)$.

Proof. Since both Q and W are bounded, and by Theorem 3.2.9(2) $\sup_{x\in M}\mathbb{E}^x\mathrm{e}^{\lambda l_t} < \infty$ holds for all $\lambda > 0$, it is easy to see that $P_t^{Q,W}$ is bounded in $L^2(\mu)$ since the Neumann semigroup P_t is contractive.

To describe $\int_0^t Q(X_s)\mathrm{d}l_s$ we shall apply the Itô formula to a proper reference function of X_s. To this end, we first extend Q to a smooth function on M. By assumption **(A3.2.1)**(ii), one may find a function $\bar{Q} \in C^\infty(M)$ such that $\bar{Q}|_{\partial M} = Q, N\bar{Q}|_{\partial M} = 0$ and $|\nabla\bar{Q}| + |L\bar{Q}|$ is bounded. This can be realized by using the polar coordinates

$$\partial M \times [0, r_0) \ni (\theta, s) \mapsto \exp[sN_\theta],$$

for $r_0 > 0$ given in **(A3.2.1)**(ii). From this one may take $\bar{Q}(\theta, s) = Q(\theta)h(s)$ on $\partial_{r_0}M$ for some $h \in C^\infty([0,\infty))$, such that $h(0) = 1, h'(0) = 0$ and $h(s) = 0$ for $s \geq r_0$, and let $\bar{Q} = 0$ outside $\partial_{r_0}M$. This $\bar{Q}$ meets our requirements since $L\rho_\partial$ is bounded on $\partial_{r_0}M$.

Let $\Phi \in C_0^\infty([0,\infty))$ be such that $0 \leq \Phi \leq 1, \Phi(s) = 1$ for $s \in [0,1]$ and $\Phi(s) = 0$ for $s \geq 2$. Let

$$\psi_n = \frac{1}{n}\int_0^{n\rho_\partial}\Phi(s)\mathrm{d}s.$$

Then $0 \leq \psi_n \leq 2n^{-1}$. $\psi_n = \rho_\partial$ for $\rho_\partial \leq n^{-1}, \psi_n$ is constant for $\rho_\partial \geq 2n^{-1}$ and $|\nabla\psi_n| \leq 1$. Moreover, $\psi_n \in C^\infty(M)$ for large n. Since $\nabla\psi_n = N$ and $N\bar{Q} = 0$ on ∂M, by the Itô formula we have

$$(\bar{Q}\psi_n)(X_t) = M_n(t) + \int_0^t L(\psi_n\bar{Q})(X_s)\mathrm{d}s + \int_0^t Q(X_s)\mathrm{d}l_s,$$

where $M_n(t)$ is a martingale with quadratic variational process

$$\langle M_n\rangle(t) = 2\int_0^t |\nabla(\bar{Q}\psi_n)|^2(X_s)\mathrm{d}s. \tag{3.8.12}$$

Note that $L(\bar{Q}\psi_n)$ and $|\nabla\bar{Q}| + |L\bar{Q}| + 1_{\partial_r M}|L\rho_\partial|$ are bounded. So,

$$\begin{aligned}&\mathbb{E}^x[f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}]\\ &= \mathbb{E}^x[f(X_t)\mathrm{e}^{\int_0^t \{L(\bar{Q}\psi_n)-W\}(X_s)\mathrm{d}s}] + \varepsilon_n,\end{aligned} \tag{3.8.13}$$

where

$$\varepsilon_n := \mathbb{E}^x[f(X_t)\mathrm{e}^{-\int_0^t W(X_s)\mathrm{d}s - \int_0^t Q(X_s)\mathrm{d}l_s}(1 - \mathrm{e}^{(\bar{Q}\psi_n)(X_t) - M_n(t)})],$$

which goes to zero uniformly in x as $n \to \infty$ according to (3.8.12) and the above mentioned properties of $\bar{Q}$ and ψ_n. Therefore, letting

$$P_t^{(n)} f(x) = \mathbb{E}^x[f(X_t)\mathrm{e}^{\int_0^t \{L(\bar{Q}\psi_n)-W\}(X_s)\mathrm{d}s}],$$

we have

$$\lim_{n\to\infty} \mu(|P_t^{Q,W} f - P_t^{(n)} f|^2) = 0, \quad f \in L^2(\mu).$$

Noting that Proposition 3.8.1 for $Q = 0$ implies that $P_t^{(n)}$ is symmetric in $L^2(\mu)$ for any $n \geq 1$, so is $P_t^{Q,W}$. □

Theorem 3.8.9. *Let $Z = \nabla V$ for some $V \in C^2(M)$ such that μ is a probability measure. Assume* **(A3.2.1)** *and let $\mathrm{Ric}_Z \geq K$ and $\mathbb{I} \geq \sigma$ hold for some $K, \sigma \in \mathbb{R}$. Let*

$$\eta_\lambda(s) := \sup_{x \in M} \mathbb{E}^x \mathrm{e}^{\lambda l_s}, \quad s, \lambda \in \mathbb{R}.$$

Then for any $t > 0$ and $f \in C^1(M)$ with $\mu(f^2) = 1$,

$$\begin{aligned}\mu(f^2 \log f^2) \leq\,& 4\Big(\int_0^t \mathrm{e}^{-2Ks}\eta_{-2\sigma}(s)\mathrm{d}s\Big)\mu(|\nabla f|^2)\\ &+ \frac{W_2(f^2\mu, \mu)^2}{4\int_0^t \mathrm{e}^{2Ks}\eta_{-2\sigma}(s)^{-1}\mathrm{d}s}.\end{aligned} \tag{3.8.14}$$

Proof. When ∂M is convex, the desired HWI inequality follows from the log-Sobolev inequality in Theorem 3.3.1(3) for $\sigma = 0$ and the log-Harnack inequality in Theorem 3.3.2(4) (see the proof of Theorem 2.4.1(3)). So, we only consider the non-convex case under assumption **(A3.2.1)**(ii).

By Theorem 3.2.9(2), **(A3.2.1)**(ii) implies $\eta_\lambda < \infty$. Let $f \in C_b^1(M)$ and $t > 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}s} P_s\{(P_{t-s}f^2)\log P_{t-s}f^2\} = P_s \frac{|\nabla P_{t-s}f^2|^2}{P_{t-s}f^2}, \quad s \in [0, t], \tag{3.8.15}$$

where P_t is the semigroup of the reflecting diffusion process generated by $L = \Delta + \nabla V$ on M. By Theorem 3.3.1(2) for $p = 1$, (3.8.4) and using the Schwartz inequality, we obtain

$$\begin{aligned}
\frac{|\nabla P_{t-s} f^2|^2}{P_{t-s} f^2}(y) &\leq \mathrm{e}^{-2K(t-s)} \frac{(\mathbb{E}^y\{|\nabla f^2|(X_{t-s})\mathrm{e}^{-\sigma l_{t-s}}\})^2}{P_{t-s} f^2(y)} \\
&\leq 4\mathrm{e}^{-2K(t-s)}\mathbb{E}^y\{|\nabla f|^2(X_{t-s})\mathrm{e}^{-2\sigma l_{t-s}}\} \\
&= 4\mathrm{e}^{-2K(t-s)} P_{t-s}^{2\sigma}|\nabla f|^2(y), \quad s \in [0,t], y \in M.
\end{aligned}$$

Combining this with (3.8.15) we obtain

$$P_t(f^2 \log f^2) \leq (P_t f^2) \log P_t f^2 + 4\int_0^t \mathrm{e}^{-2K(t-s)} P_s P_{t-s}^{2\sigma}|\nabla f|^2 \mathrm{d}s.$$

Since μ is an invariant measure of P_s and $P_{t-s}^{2\sigma}$ is symmetric in $L^2(\mu)$ according to Lemma 3.8.8, taking integral for both sides with respect to μ, we arrive at

$$\begin{aligned}
\mu(f^2 \log f^2) \leq\ & \mu((P_t f^2) \log P_t f^2) \\
& + 4\mu(|\nabla f|^2) \int_0^t \mathrm{e}^{-2Ks}\eta_{-2\sigma}(s)\mathrm{d}s.
\end{aligned} \tag{3.8.16}$$

On the other hand, for any $x, y \in M$, let $x_\cdot : [0,1] \to M$ be the minimal curve from x to y with constant speed. We have $|\dot{x}_s| = \rho(x,y)$. Let $h \in C^1([0,t])$ be such that $h(0) = 1, h(t) = 0$. According to Theorem 3.3.1(2) for $p = 1$, we have

$$\begin{aligned}
&P_t \log f^2(x) - \log P_t f^2(y) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} P_s(\log P_{t-s} f^2)(x_{h(s)})\mathrm{d}s \\
&\leq \int_0^t \Big\{|h'(s)|\rho(x,y)|\nabla P_s(\log P_{t-s} f^2)|(x_{h(s)}) \\
&\qquad - \mathbb{E}^{x_{h(s)}} \frac{|\nabla P_{t-s} f^2|^2}{(P_{t-s} f^2)^2}(X_s)\Big\}\mathrm{d}s \\
&\leq \int_0^t \mathbb{E}^{x_{h(s)}}\Big\{|h'(s)|\rho(x,y)\frac{|\nabla P_{t-s} f^2|}{P_{t-s} f^2}(X_s)\mathrm{e}^{-Ks-\sigma l_s} \\
&\qquad - \frac{|\nabla P_{t-s} f^2|^2}{(P_{t-s} f^2)^2}(X_s)\Big\}\mathrm{d}s \\
&\leq \frac{\rho(x,y)^2}{4} \int_0^t |h'(s)|^2 \mathrm{e}^{-2Ks}\eta_{-2\sigma}(s)\mathrm{d}s =: c(t)\rho(x,y)^2.
\end{aligned} \tag{3.8.17}$$

Now, let $\mu(f^2) = 1$ and $\pi \in \mathcal{C}(f^2\mu, \mu)$ be the optimal coupling for $W_2(f^2\mu, \mu)$. It follows from the symmetry of P_t and (3.8.17) that

$$\begin{aligned}\mu((P_t f^2)\log P_t f^2) &= \mu(f^2 P_t \log P_t f^2) = \int_{M\times M} P_t(\log P_t f^2)(x)\pi(\mathrm{d}x, \mathrm{d}y)\\ &\le \int_{M\times M} \big\{\log P_{2t} f^2(y) + c(t)\rho(x,y)^2\big\}\pi(\mathrm{d}x,\mathrm{d}y)\\ &= \mu(\log P_{2t} f^2) + c(t) W_2(f^2\mu,\mu)^2 \le c(t) W_2(f^2\mu,\mu)^2,\end{aligned}$$

where in the last step we have used the Jensen inequality that

$$\mu(\log P_{2t} f^2) \le \log \mu(P_{2t} f^2) = 0.$$

Combining this with (3.8.16) we obtain

$$\begin{aligned}\mu(f^2\log f^2) &\le 4\mu(|\nabla f|^2)\int_0^t \mathrm{e}^{-2Ks}\eta_{-2\sigma}(s)\mathrm{d}s\\ &\quad + \frac{W_2(f^2\mu,\mu)^2}{4}\int_0^t |h'(s)|^2 \mathrm{e}^{-2Ks}\eta_{-2\sigma}(s)\mathrm{d}s.\end{aligned}$$

Then the proof is completed by taking

$$h(s) = \frac{\int_s^t \mathrm{e}^{2Ku}\eta_{-2\sigma}(u)^{-1}\mathrm{d}u}{\int_0^t \mathrm{e}^{2Ku}\eta_{-2\sigma}(u)^{-1}\mathrm{d}u}, \quad s\in[0,t].$$

□

Chapter 4

Stochastic Analysis on Path Space over Manifolds with Boundary

Stochastic analysis on the path space over a complete Riemannian manifold without boundary has been well developed since 1992 when B. K. Driver [Driver (1992)] proved the quasi-invariance theorem for the Brownian motion on compact Riemannian manifolds. A key point of the study is to first establish an integration by parts formula for the associated gradient operator induced by the quasi-invariant flows, then prove functional inequalities for the corresponding Dirichlet form (see e.g. [Fang (1994); Hsu (1997); Capitaine *et al* (1997)] and references within). For more analysis on Riemannian path spaces we refer to [Elworthy and Li (2008); Malliavin (1997); Stroock (2000)] and references within. On the other hand, the Talagrand type transportation-cost inequality has been established in [Wang (2004b); Fang *et al* (2008)] on the path space with respect to the intrinsic distance induced by the Malliavin gradient and the uniform distance respectively, see also [Feyel and Üstünel (2002); Wu and Zhang (2004)] for the study of transportation-cost inequality on Wiener space and the path space of diffusion processes on $\mathbb{R}^d$.

The aim of this chapter is to establish the corresponding theory on the path space for the reflecting diffusion process on manifolds with boundary. In Section §4.1 we introduce an alternative construction of Hsu's the multiplicative functional initiated in [Hsu (2002b)], then define the corresponding damped gradient operator in §4.2, which satisfies an integration by parts formula induced by intrinsic quasi-invariant flows. In §4.3 we establish the log-Sobolev inequality for the associated Dirichlet form. These three sections are mainly modified from [Wang (2011a)]. Moreover, some transportation-cost inequalities, which are equivalent to the curvature condition and the convexity of the manifold, will be addressed in §4.4 and then partly extended in §4.5 to the non-convex case.

4.1 Multiplicative functional

In this section, we aim to construct a modified version of Hsu's Multiplicative functional introduced in [Hsu (2002b)] for the reflecting diffusion processes. Let M be a d-dimensional connected Riemannian manifold with boundary ∂M. Let $T > 0$ be fixed. The path space for the reflecting diffusion process on M with time-interval $[0,T]$ is

$$W^T := C([0,T]; M).$$

For each point $x \in M$, let $W_x^T = \{\gamma \in W^T : \gamma_0 = x\}$. Let B_t be the d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For any $x \in M$, let $O_x M$ be the set of all orthonormal bases for the tangent space $T_x M$ at point x, and let $O(M) := \bigcup_{x\in M} O_x(M)$ be the frame bundle. Let Z be a C^1-smooth vector field. Then for any $x \in M$, the reflecting diffusion process generated by $L := \Delta + Z$ starting at x can be constructed by solving the SDE (see (3.0.1))

$$\mathrm{d}X_t^x = \sqrt{2}\, u_t^x \circ \mathrm{d}B_t + Z(X_t^x)\mathrm{d}t + N(X_t^x)\mathrm{d}l_t^x, \tag{4.1.1}$$

where $u_t^x \in O_{X_t^x}(M)$ is the horizontal lift of X_t^x on the frame bundle $O(M)$, N is the inward unit normal vector field on ∂M, and l_t^x is the local time of X_t^x on the boundary ∂M. Let $X_{[0,T]}^x = \{X_t^x : 0 \leq t \leq T\}$.

To construct the desired continuous multiplicative functional, we need the following assumptions.

(A4.1.1) There exist two constants $K, \sigma \in \mathbb{R}$ such that $\mathrm{Ric}_Z := \mathrm{Ric} - \nabla Z \geq K$ and $\mathbb{I} \geq \sigma$; and $\mathbb{E}e^{\lambda\sigma^- l_t^x} < \infty$ holds for $\lambda, t > 0, x \in M$, where $\sigma^- := 0 \vee (-\sigma)$.

Due to Theorem 3.2.9, **(A4.1.1)** follows from **(A3.2.1)**. To introduce Hsu's discontinuous multiplicative functional, we need the lift operators $\mathrm{Ric}_Z(u)$ and $\mathbb{I}(u)$ defined by (2.2.2) and (3.2.3) for $u \in O(M)$. Moreover, for $u \in \partial O(M) := \{u \in O(M) : \mathbf{p}u \in \partial M\}$, let

$$P_u(a,b) = \langle ua, N\rangle\langle ub, N\rangle, \quad a, b \in \mathbb{R}^d.$$

For any $\varepsilon > 0$ and $r \geq 0$, let $Q_{r,t}^{x,\varepsilon}$ solve the following SDE on $\mathbb{R}^d \bigotimes \mathbb{R}^d$:

$$\mathrm{d}Q_{r,t}^{x,\varepsilon} = -Q_{r,t}^{x,\varepsilon}\Big\{\mathrm{Ric}_Z^{\#}(u_t^x)\mathrm{d}t + \big(\varepsilon^{-1}P_{u_t^x} + \mathbb{I}_{u_t^x}\big)\mathrm{d}l_t^x\Big\}, \tag{4.1.2}$$

with $Q_{r,r}^{x,\varepsilon} = I$, for all $t \geq r$. According to Theorem 3.4 in [Hsu (2002b)] for compact M, when $\varepsilon \downarrow 0$ the process $Q_{r,t}^{x,\varepsilon}$ converges in $L^2(\mathbb{P})$ to an adapted right-continuous process $\mathbb{Q}_{r,t}^x$ with left limit, such that $\mathbb{Q}_{r,t}^x P_{u_t^x} = 0$

if $X_t^x \in \partial M$. Here, we introduce a slightly different but simpler construction of the *multiplicative functional* by solving a random integral equation on $\mathbb{R}^d \otimes \mathbb{R}^d$.

Theorem 4.1.1. *Assume* **(A4.1.1)**.

(1) *Let $r \geq 0$. For any $x \in M$ and $u_0^x \in O_x(M)$, the equation*

$$Q_{r,t}^x = \left(I - \int_r^t Q_{r,s}^x \mathrm{Ric}_Z^\#(u_s^x)\mathrm{d}s - \int_r^t Q_{r,s}^x \mathbb{I}_{u_s^x}\mathrm{d}l_s^x\right)\left(I - 1_{\{X_t^x \in \partial M\}} P_{u_t^x}\right)$$

has a unique solution for $t \geq r$.

(2) *For any $0 \leq r \leq t$, $\|Q_{r,t}^x\| \leq \mathrm{e}^{-K(t-r)-\sigma(l_t-l_r)}$ a.s., where $\|\cdot\|$ is the operator norm for $d \times d$-matrices.*

(3) *For any $0 \leq r \leq s \leq t$, $Q_{r,t}^x = Q_{r,s}^x Q_{s,t}^x$ a.s.*

Proof. The uniqueness of solution is obvious. It remains to construct a solution up to an arbitrarily given time $T > r$. For simplicity, we will drop the superscript x. By **(A4.1.1)** and (4.1.2), we have, for $t \geq r$,

$$\|Q_{r,t}^\varepsilon\|^2 \leq 1 - 2K\int_r^t \|Q_{r,s}^\varepsilon\|^2\mathrm{d}s - 2\sigma\int_r^t \|Q_{r,s}^\varepsilon\|^2\mathrm{d}l_s - \frac{2}{\varepsilon}\int_r^t \|Q_{r,s}^\varepsilon P_{u_s}\|^2\mathrm{d}l_s.$$

In particular,

$$\begin{aligned}&\|Q_{r,t}^\varepsilon\|^2 \leq \mathrm{e}^{-2K(t-r)-2\sigma(l_t-l_r)}, \quad t \geq r;\\ &\int_r^T \|Q_{r,s}^\varepsilon P_{u_s}\|^2\mathrm{d}l_s \leq \frac{\varepsilon}{2}\Big(1 + \big(2K^-T + 2\sigma^- l_T\big)\mathrm{e}^{2K^-T+2\sigma^- l_T}\Big).\end{aligned} \tag{4.1.3}$$

Combining this with **(A4.1.1)**, we obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_r^T \|Q_{r,s}^\varepsilon P_{u_s}\|^2\mathrm{d}l_s = 0 \tag{4.1.4}$$

and

$$\sup_{\varepsilon\in(0,1)}\mathbb{E}\int_r^T \|Q_{r,t}^\varepsilon\|^2(\mathrm{d}t + \mathrm{d}l_t) < \infty.$$

Because of the latter we may find a sequence $\varepsilon_n \downarrow 0$ and an adapted process $\bar{Q}_{r,\cdot} \in L^2(\Omega \times [r,T] \to \mathbb{R}^d \otimes \mathbb{R}^d; \mathbb{P} \times (\mathrm{d}t + \mathrm{d}l_t))$, such that for any $g \in L^2(\Omega \times [r,T] \to \mathbb{R}^d; \mathbb{P} \times (\mathrm{d}t + \mathrm{d}l_t))$,

$$\lim_{n\to\infty}\mathbb{E}\int_r^T Q_{r,t}^{\varepsilon_n} g_t(\mathrm{d}t + \mathrm{d}l_t) = \mathbb{E}\int_r^T \bar{Q}_{r,t} g_t(\mathrm{d}t + \mathrm{d}l_t).$$

Noting that $\mathrm{d}t$ and $\mathrm{d}l_t$ are singular to each other, replacing g_t by $g_t 1_{\{X_t\in\partial M\}}$ and $g_t 1_{\{X_t\notin\partial M\}}$ respectively, we arrive at

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\int_r^T Q_{r,t}^{\varepsilon_n} g_t\mathrm{d}t &= \mathbb{E}\int_r^T \tilde{Q}_{r,t}g_t\mathrm{d}t,\\ \lim_{n\to\infty}\mathbb{E}\int_r^T Q_{r,t}^{\varepsilon_n} g_t\mathrm{d}l_t &= \mathbb{E}\int_r^T \tilde{Q}_{r,t}g_t\mathrm{d}l_t\end{aligned} \tag{4.1.5}$$

for all $g\in L^2(\Omega\times[r,T]\to\mathbb{R}^d;\mathbb{P}\times(\mathrm{d}t+\mathrm{d}l_t))$. In particular, it follows from (4.1.4) that

$$\mathbb{E}\int_r^T \|\tilde{Q}_{r,t}P_{u_t}\|^2\mathrm{d}l_t = 0. \tag{4.1.6}$$

Now, for bounded g, let

$$\tilde{g}_t = (I - 1_{\{X_t\in\partial M\}}P_{u_t})g_t.$$

It follows from (4.1.2) that

$$\begin{aligned}&\mathbb{E}\int_r^T Q_{r,t}^{\varepsilon_n}\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t)\\ &=\mathbb{E}\int_r^T\left(\tilde{g}_t - \int_r^t Q_{r,s}^{\varepsilon_n}\mathrm{Ric}_Z^\#(u_s)\tilde{g}_t\mathrm{d}s - \int_r^t Q_{r,s}^{\varepsilon_n}\mathbb{I}_{u_s}\tilde{g}_t\mathrm{d}l_s\right)(\mathrm{d}t+\mathrm{d}l_t)\\ &=\mathbb{E}\int_r^T\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t) - \mathbb{E}\int_r^T\left(Q_{r,s}^{\varepsilon_n}\mathrm{Ric}_Z^\#(u_s)\int_s^T\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t)\right)\mathrm{d}s\\ &\quad-\mathbb{E}\int_r^T\left(Q_{r,s}^{\varepsilon_n}\mathbb{I}_{u_s}\int_s^T\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t)\right)\mathrm{d}l_s.\end{aligned}$$

Letting $n\uparrow\infty$ and using (4.1.5), we obtain

$$\begin{aligned}&\mathbb{E}\int_r^T\tilde{Q}_{r,t}\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t)\\ &=\mathbb{E}\int_r^T\left(I-\int_r^t\tilde{Q}_{r,s}\mathrm{Ric}_Z^\#(u_s)\mathrm{d}s - \int_r^t\tilde{Q}_{r,s}\mathbb{I}_{u_s}\mathrm{d}l_s\right)\tilde{g}_t(\mathrm{d}t+\mathrm{d}l_t).\end{aligned}$$

Combining this with (4.1.6) we conclude that

$$\tilde{Q}_{r,t} = \left(I-\int_r^t\tilde{Q}_{r,s}\mathrm{Ric}_Z^\#(u_s)\mathrm{d}s - \int_r^t\tilde{Q}_{r,s}\mathbb{I}_{u_s}\mathrm{d}l_s\right)(I-1_{\{X_t\in\partial M\}}P_{u_t})$$

holds for $\mathbb{P}\times(\mathrm{d}t+\mathrm{d}l_t)$-a.e. So, letting

$$Q_{r,t} = \left(I-\int_r^t\tilde{Q}_{r,s}\mathrm{Ric}_Z^\#(u_s)\mathrm{d}s - \int_r^t\tilde{Q}_{r,s}\mathbb{I}_{u_s}\mathrm{d}l_s\right)(I-1_{\{X_t\in\partial M\}}P_{u_t})$$

for all $t \in [r, T]$, we have $Q = \tilde{Q}$, $\mathbb{P} \times (\mathrm{d}t + \mathrm{d}l_t)$-a.e and thus,

$$Q_{r,t} = \left(I - \int_r^t Q_{r,s}\mathrm{Ric}_Z^\#(u_s)\mathrm{d}s - \int_r^t Q_{r,s}\mathbb{I}_{u_s}\mathrm{d}l_s\right)(I - 1_{\{X_t \in \partial M\}}P_{u_t})$$

holds for $t \in [r, T]$.

Next, by the first inequality in (5.5.32) and the weak convergence of $Q^{\varepsilon_n}_{r,\cdot}$ to $\tilde{Q}_{r,\cdot}$, we have $\|\tilde{Q}_{r,t}\| \le \mathrm{e}^{-K(t-r)-\sigma(l_t-l_r)}$, $\mathbb{P} \times (\mathrm{d}t + \mathrm{d}l_t)$-a.e. Thus, $Q_{r,t}$ satisfies the same inequality since $Q_{r,\cdot} = \tilde{Q}_{r,\cdot}$, $\mathbb{P} \times (\mathrm{d}t + \mathrm{d}l_t)$-a.e. Noting that $\mathbb{P}(X_t \in \partial M) = 0$ holds for any $t > 0$, and when $X_t \notin \partial M$, $Q_{r,\cdot}$ is right continuous at t, we conclude that $\|Q_{r,t}\| \le \mathrm{e}^{-K(t-r)-\sigma(l_t-l_r)}$ a.s. So, (2) holds. Finally, since $Q^{\varepsilon_n}_{r,t} = Q^{\varepsilon_n}_{r,s}Q^{\varepsilon_n}_{s,t}$ holds for all $n \ge 1$ and all $0 \le r \le s \le t$, we prove (3) by a similar argument. □

We remark that our multiplicative functional Q^x is slightly different from Hsu's $\mathbb{Q}^x$, since the latter is right-continuous but the former is not. As $Q^x_{0,\cdot}$ is continuous on $\{t : X^x_t \notin \partial M\}$ which is dense in $[0, \infty)$, and both functional are weak limits of Q^{x,ε_n} as $n \to \infty$, we conclude that they are equivalent, i.e. for any $r \le t$, $Q^x_{r,t} = \mathbb{Q}^x_{r,t}$, a.s.

Let $Q^x_t = Q^x_{0,t}, t \ge 0$. The following property of Q^x_t will be useful in the sequel.

Proposition 4.1.2. *Assume* **(A4.1.1)**. *For any $\mathbb{R}^d$-valued continuous semi-martingale g_t with $1_{\{X^x_t \in \partial M\}}P_{u_t}g_t = 0$,*

$$\mathrm{d}Q^x_{r,t}g_t = \tilde{Q}^x_{r,t}\mathrm{d}g_t - Q^x_{r,t}\mathrm{Ric}_Z^\#(u^x_t)g_t\mathrm{d}t - Q^x_{r,t}\mathbb{I}_{u_t}g_t\mathrm{d}l^x_t, \quad t \ge r,$$

where

$$\tilde{Q}^x_{r,t} := \left(I - \int_r^t Q^x_s\mathrm{Ric}_Z^\#(u^x_s)\mathrm{d}s - \int_r^t Q^x_s\mathbb{I}_{u^x_s}\mathrm{d}l^x_s\right).$$

Proof. For simplicity, we only consider $r = 0$. By Theorem 4.1.1 and $1_{\partial M}(X_t)P_{u_t}g_t = 0$, we have

$$Q^x_t g_t = \left(I - \int_0^t Q^x_s\mathrm{Ric}_Z^\#(u^x_s)\mathrm{d}s - \int_0^t Q^x_s\mathbb{I}_{u^x_s}\mathrm{d}l^x_s\right)g_t = \tilde{Q}^x_t g_t.$$

Then the proof is completed by using Itô's formula. □

Let P_t be the Neumann semigroup generated by L, i.e.

$$P_t f(x) = \mathbb{E}f(X^x_t), \quad x \in M, t \ge 0, f \in \mathcal{B}_b(M),$$

where $\mathcal{B}_b(M)$ is the set of all bounded measurable functions on M. The following is a consequence of Proposition 4.1.2.

Corollary 4.1.3. *Assume* **(A3.2.1)**. *Then for any* $f \in C_b^\infty(M)$ *and* $t > r \geq 0$,

$$[r,t] \ni s \mapsto Q^x_{r,s}(u^x_s)^{-1}\nabla P_{t-s}f(X^x_s)$$

is a martingale. Consequently,

$$(u^x_r)^{-1}\nabla P_{t-r}f(X^x_r) = \mathbb{E}\big(Q^x_{r,t}(u^x_t)^{-1}\nabla f(X_t)\big|\mathcal{F}_r\big), \tag{4.1.7}$$

and for any non-negative adapted process h *such that* $\mathbb{E}\int_0^t |h'(s)|^2\mathrm{d}s < \infty$ *and* $h(r) = 0, h(t) = 1$,

$$(u^x_r)^{-1}\nabla P_{t-r}f(X^x_r) = \frac{1}{\sqrt{2}}\mathbb{E}\bigg(f(X^x_t)\int_r^t h'(s)Q^x_{r,s}\mathrm{d}B_s\bigg|\mathcal{F}_r\bigg). \tag{4.1.8}$$

Proof. Again we only consider $r = 0$ and drop the superscript x for simplicity.

(a) Let $g_s = (u_s)^{-1}\nabla P_{t-s}f(X_s)$. Since on ∂M the vector field $\nabla P_{t-s}f$ is vertical to N, we have $1_{\{X_s\in\partial M\}}P_{u_s}g_s = 0$. Then, by Proposition 4.1.2, we have

$$\mathrm{d}Q^x_s g_s = Q^x_s\mathrm{d}g_s - Q^x_s\mathrm{Ric}^\#_Z(u_s)g_s\mathrm{d}s - Q^x_s\mathbb{I}_{u_s}g_s\mathrm{d}l_s, \quad s \in [0,t]. \tag{4.1.9}$$

To calculate $\mathrm{d}g_s$, let

$$F(u,t-s) = u^{-1}\nabla P_{t-s}f(\mathbf{p}u), \quad u \in O(M).$$

Let $\{e_i\}_{i=1}^d$ be the canonical ONB on $\mathbb{R}^d$ and $\{H_{e_i}\}_{i=1}^d$ the corresponding family of horizontal vector fields. For any vector field U on M, let $\mathbf{H}_U$ be its horizontal lift. Then the horizontal Laplacian is $\Delta_{O(M)} = \sum_{i=1}^d H^2_{e_i}$, and the generator of u_t, the horizontal lift of X_t, is

$$L_{O(M)} := \Delta_{O(M)} + \mathbf{H}_Z.$$

By the Bochner-Weitzenböck formula and noting that $\frac{\mathrm{d}}{\mathrm{d}s}P_{t-s}f = -LP_{t-s}f$, we obtain (see also (b) in the proof of Theorem 3.2.1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}F(u,t-s) &= -u^{-1}\nabla(LP_{t-s}f)(\mathbf{p}u)\\ &= -L_{O(M)}F(\cdot,t-s)(u) + \mathrm{Ric}^\#_Z(u)F(u,t-s),\end{aligned} \tag{4.1.10}$$

for all $s \in [0,t]$. On the other hand, noting that

$$\mathrm{d}u_t = \sqrt{2}\sum_{i=1}^d H_{e_i}\circ\mathrm{d}B^i_t + \mathbf{H}_Z(u_t)\mathrm{d}t + \mathbf{H}_N(u_t)\mathrm{d}l_t,$$

by Itô's formula, for any fixed $t_0 \in [0,t]$ we have

$$\mathrm{d}F(u_s,t_0) = \mathrm{d}M_s + L_{O(M)}F(\cdot,t_0)(u_s)\mathrm{d}s + \mathbf{H}_NF(\cdot,t_0)(u_s)\mathrm{d}l_s,$$

where

$$\mathrm{d}M_s := \sqrt{2}\sum_{i=1}^d (H_{e_i}F(\cdot,t_0))(u_s)\mathrm{d}B_s^i.$$

Therefore,

$$\mathrm{d}g_s = \mathrm{d}M_s + \mathrm{Ric}_Z^\#(u_s)g_s\mathrm{d}s + \mathbf{H}_N F(\cdot,t-s)(u_s)\mathrm{d}l_s.$$

Since $1_{\{X_s\in\partial M\}}Q_s^x P_{u_s} = 0$, combining this with (4.1.9) we obtain

$$\mathrm{d}Q_s^x g_s = Q_s^x \mathrm{d}M_s + Q_s^x(I-P_{u_s})\big\{\mathbf{H}_N F(\cdot,t-s)(u_s) - \mathbb{I}_{u_s}F(u_s,t-s)\big\}\mathrm{d}l_s.$$

Noting that for any $e\in\mathbb{R}^d$, it follows from (3.2.7) that when $X_s\in\partial M$,

$$\begin{aligned}&\langle (I-P_{u_s})\mathbf{H}_N F(\cdot,t-s)(u_s), e\rangle\\ &= \mathrm{Hess}_{P_{t-s}f}(N,\mathbf{p}_\partial u_s e) = \mathbb{I}(\nabla P_{t-s}f(X_s),\mathbf{p}_\partial u_s e)\\ &= \mathbb{I}_{u_s}(F(u_s,t-s),e) = \langle \mathbb{I}_{u_s}F(u_s,t-s),e\rangle\end{aligned}$$

we conclude that

$$(I-P_{u_s})\big\{\mathbf{H}_N F(\cdot,t-s)(u_s) - \mathbb{I}_{u_s}F(u_s,t-s)\big\}\mathrm{d}l_s = 0.$$

Therefore, $Q_s^x g_s$ is a local martingale. Since **(A3.2.1)** implies **(A4.1.1)**, $Q_s^x g_s$ is indeed a martingale according to Theorems 3.3.1 and 4.1.1.

(b) (4.1.7) follows immediately from the first assertion as $P_t f$ satisfies the Neumann boundary condition. The proof of (4.1.8) is similar to that of (3.2.2). Indeed, as shown in step (c) in the proof of Theorem 3.2.1, we have

$$f(X_t) = P_t f(x) + \sqrt{2}\int_0^t \langle u_s^{-1}\nabla P_{t-s}f(X_s),\mathrm{d}B_s\rangle. \tag{4.1.11}$$

Next, since $Q_s^x u_s^{-1}\nabla P_{t-s}f(X_s)$ is a martingale,

$$Q_s^x u_s^{-1}\nabla P_{t-s}f(X_s) = \mathbb{E}(Q_t^x u_t^{-1}\nabla f(X_t)|\mathcal{F}_s),\quad s\in[0,t].$$

Combining these with (4.1.7) for $r=0$ we arrive at

$$\begin{aligned}&\frac{1}{\sqrt{2}}\mathbb{E}\Big\{f(X_t)\int_0^t h'(s)Q_s^x\mathrm{d}B_s\Big\} = \mathbb{E}\int_0^t \big\{h'(s)Q_s^x u_s^{-1}\nabla P_{t-s}f(X_s)\big\}\mathrm{d}s\\ &= \mathbb{E}\int_0^t \big\{h'(s)Q_t^x u_t^{-1}\nabla f(X_t)\big\}\mathrm{d}s = \mathbb{E}\big\{Q_t^x u_t^{-1}\nabla f(X_t)\big\} = u_0^{-1}\nabla P_t f(x).\end{aligned}$$

This completes the proof. □

4.2 Damped gradient, quasi-invariant flows and integration by parts

It is well-known that for diffusion on manifolds without boundary, the Malliavin derivative can be realized by quasi-invariant flows. In this section, by using the multiplicative functional constructed in the last section, we first introduce the damped gradient operator as in [Fang and Malliavin (1993)] for manifolds without boundary, then introduce quasi-invariant flows induced by SDEs with reflection, and finally link them by establishing an integration by parts formula.

Before moving on, let us mention that the existing study in this direction is very limited. To see this, let us recall [Zambotti (2005)] where an integration by parts formula was established on the path space of the one-dimensional reflecting Brownian motion. Let e.g. $X_t = |b_t|$, where b_t is the one-dimensional Brownian motion. For $h \in C([0,T];\mathbb{R})$ with $h(0) = 0$ and $\int_0^T |h'(t)|^2 dt < \infty$, let ∂_h be the derivative operator induced by the flow $X + \varepsilon h$, i.e.

$$\partial_h F = \sum_{i=1}^n h_{t_i} \nabla_i f(X_{t_1}, \dots, X_{t_n}),$$

where $n \in \mathbb{N}, 0 < t_1 < \dots < t_n \le T$ and $F(X_{[0,T]}) = f(X_{t_1}, \dots, X_{t_n})$ for some $f \in C^\infty(M^n)$. As the main result of [Zambotti (2005)], when $h \in C_0^2((0,T))$, Theorem 2.3 in [Zambotti (2005)] provides an integration by parts formula for ∂_h by using an infinite-dimensional generalized functional in the sense of Schwarz. Since for a non-trivial function h, $X + \varepsilon h$ is not quasi-invariant, this integration by parts formula cannot be formulated by using the distribution of X with a density function, and the induced gradient operator does not provide a Dirichlet form on the L^2-space of the distribution of $X_{[0,T]}$. In this section, we shall establish an essentially different integration by parts formula using quasi-invariant flows.

4.2.1 *Damped gradient operator and quasi-invariant flows*

We shall use multiplicative functionals $\{Q^x_{r,t} : 0 \le r \le t \le T\}$ to define the damped gradient operator for functionals of X^x (see [Fang and Malliavin (1993)] for the damped gradient operator for manifolds without boundary).

Let

$$\mathbb{H}_0 = \left\{ h \in C([0,T];\mathbb{R}^d) : \ h(0) = 0, \|h\|_{\mathbb{H}_0}^2 := \int_0^T |h'(t)|^2 dt < \infty \right\},$$

which is a Hilbert space with respect to the inner product

$$\langle h_1, h_2\rangle_{\mathbb{H}_0} := \int_0^T \langle h_1'(t), h_2'(t)\rangle \mathrm{d}t.$$

Consider the following class of smooth cylindrical functions on W^T:

$$\begin{aligned}\mathcal{F}C_0^\infty = \big\{&W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : n \geq 1,\\ &0 < t_1 < \ldots < t_n \leq T, f \in C_0^\infty(M^n)\big\}.\end{aligned}$$

For any $F \in \mathcal{F}C_0^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define the *damped gradient* $D^0F(X^x_{[0,T]})$ as an $\mathbb{H}_0$-valued random variable by setting $(D^0F(X^x_{[0,T]}))(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(D^0F(X^x_{[0,T]}))(t) = \sum_{i=1}^n 1_{\{t<t_i\}} Q^x_{t,t_i}(u^x_{t_i})^{-1}\nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n}), \;\; t \in [0,T],$$

where ∇_i denotes the gradient operator w.r.t. the i-th component. Then, for any $\mathbb{H}_0$-valued random variable h, let

$$\begin{aligned}D^0_hF(X^x_{[0,T]}) &= \langle D^0F(X^x_{[0,T]}), h\rangle_{\mathbb{H}_0}\\ &= \sum_{i=1}^n \int_0^{t_i} \big\langle (u^x_{t_i})^{-1}\nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n}), (Q^x_{t,t_i})^* h'(t)\big\rangle \mathrm{d}t.\end{aligned} \tag{4.2.1}$$

Note that when $\partial M = \emptyset$, we may let $l_t \equiv 0$ in (4.1.2), so that our formulation of D^0_hF goes back to the known one presented in [Fang and Malliavin (1993)] for manifolds without boundary.

Now, we intend to link D^0_hF to the directional derivative induced by a quasi-invariant flow. The idea comes from §4(a) in [Bismut (1984)] where quasi-invariant flows are constructed for M being a half-space of $\mathbb{R}^d$, which essentially reduces to the one-dimensional setting, by solving SDEs with reflecting boundary. Let $\tilde{\mathbb{H}}_0$ be the set of all adapted elements in $L^2(\Omega \to \mathbb{H}_0; \mathbb{P})$; i.e.

$$\tilde{\mathbb{H}}_0 = \big\{h \in L^2(\Omega \to \mathbb{H}_0; \mathbb{P}) : \;\; h(t) \text{ is } \mathcal{F}_t\text{-measurable}, \;\; t \in [0,T]\big\}.$$

Then $\tilde{\mathbb{H}}_0$ is a Hilbert space with inner product

$$\langle h, \tilde{h}\rangle_{\tilde{\mathbb{H}}_0} := \mathbb{E}\langle h, \tilde{h}\rangle_{\mathbb{H}_0} = \mathbb{E}\int_0^T \langle h'(t), \tilde{h}'(t)\rangle \mathrm{d}t, \;\;\; h, \tilde{h} \in \tilde{\mathbb{H}}_0.$$

To describe D^0F by using a quasi-invariant flow, for $h \in \tilde{\mathbb{H}}_0$ and $\varepsilon > 0$ let $X^{\varepsilon,h}_t$ solve the SDE

$$\begin{aligned}\mathrm{d}X^{\varepsilon,h}_t &= \sqrt{2}\, u^{\varepsilon,h}_t \circ \mathrm{d}B_t + Z(X^{\varepsilon,h}_t)\mathrm{d}t + N(X^{\varepsilon,h}_t)\mathrm{d}l^{\varepsilon,h}_t\\ &\quad + \varepsilon\sqrt{2}\, u^{\varepsilon,h}_t h'(t)\mathrm{d}t, \;\;\; X^{\varepsilon,h}_0 = x = \mathbf{p}u^x_0,\end{aligned} \tag{4.2.2}$$

where $l_t^{\varepsilon,h}$ and $u_t^{\varepsilon,h}$ are, respectively, the local time on ∂M and the horizontal lift on $O(M)$ for $X_t^{\varepsilon,h}$.

Moreover, let us explain that the flow is quasi-invariant, i.e. for each $\varepsilon \geq 0$, the distribution of $X_{[0,T]}^{\varepsilon,h}$ is absolutely continuous w.r.t. that of $X_{[0,T]}^x$. Let

$$R^{\varepsilon,h} = \exp\left[\varepsilon \int_0^T \langle h'(t), \mathrm{d}B_t\rangle - \frac{\varepsilon^2}{2}\int_0^T |h'(t)|^2 \mathrm{d}t\right].$$

By the Girsanov theorem,

$$B_t^{\varepsilon,h} := B_t - \varepsilon h(t)$$

is the d-dimensional Brownian motion under the probability $R^{\varepsilon,h}\mathbb{P}$. Thus, the distribution of $X_{[0,T]}^x$ under $R^{\varepsilon,h}\mathbb{P}$ coincides with that of $X_{[0,T]}^{\varepsilon,h}$ under $\mathbb{P}$. Therefore, the map $X_{[0,T]}^x \mapsto X_{[0,T]}^{\varepsilon,h}$ is quasi-invariant.

4.2.2 *Integration by parts formula*

The following result provides an integration by parts formula for $D_h^0 F$ and a link to the derivative induced by the flow $\{X_{[0,T]}^{\varepsilon,h}\}_{\varepsilon\geq 0}$.

Theorem 4.2.1. *Assume* **(A4.1.1)**. *For any $x \in M$ and $F \in \mathcal{F}C_0^\infty$,*

$$\begin{aligned}\sqrt{2}\,\mathbb{E}\{D_h^0 F\}(X_{[0,T]}^x) &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\frac{F(X_{[0,T]}^{\varepsilon,h}) - F(X_{[0,T]}^x)}{\varepsilon}\\ &= \mathbb{E}\left\{F(X_{[0,T]}^x)\int_0^T \langle h'(t), \mathrm{d}B_t\rangle\right\}\end{aligned}$$

holds for all $h \in \bar{\mathbf{H}}_{0,b}$, the set of all elements in $\bar{\mathbb{H}}_0$ with bounded $\|h\|_{\mathbb{H}_0}$.

Since $\bar{\mathbf{H}}_{0,b}$ is dense in $\bar{\mathbb{H}}_0$, the above result implies that the projection of D^0 onto $\bar{\mathbb{H}}_0$ can be determined by the flows $X^{\varepsilon,h}, h \in \bar{\mathbf{H}}_{0,b}$. But it is not clear whether

$$\sqrt{2}\, D_h^0 F(X_{[0,T]}^x) = \lim_{\varepsilon\downarrow 0}\frac{F(X_{[0,T]}^{\varepsilon,h}) - F(X_{[0,T]}^x)}{\varepsilon},\quad h \in \bar{\mathbb{H}}_0 \tag{4.2.3}$$

holds or not.

To prove Theorem 4.2.1, we need some preparations. In particular, we shall use (4.1.7) and a conducting argument as in [Hsu (1997)] for the case without boundary.

Lemma 4.2.2. *Let $x \in M$ and $F \in \mathcal{F}C_0^\infty$. Then*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\frac{F(X^{\varepsilon,h}_{[0,T]}) - F(X^x_{[0,T]})}{\varepsilon} = \mathbb{E}\Big\{F(X^x_{[0,T]})\int_0^T \langle h'(t), \mathrm{d}B_t\rangle\Big\}$$

holds for all $h \in \tilde{\mathbf{H}}_{0,b}$.

Proof. Let $B_t^{\varepsilon,h} = B_t - \varepsilon h(t)$, which is the d-dimensional Brownian motion under $R^{\varepsilon,h}\mathbb{P}$. Reformulate (4.1.1) as

$$\mathrm{d}X_t^x = \sqrt{2}\,u_t^x \circ \mathrm{d}B_t^{\varepsilon,h} + Z(X_t^x)\mathrm{d}t + N(X_t^x)\mathrm{d}l_t^x + \varepsilon\sqrt{2}\,u_t h'(t)\mathrm{d}t.$$

By the weak uniqueness of (4.2.2), we conclude that the distribution of X^x under $R^{\varepsilon,h}\mathbb{P}$ coincides with that of $X^{\varepsilon,h}$ under $\mathbb{P}$. In particular, $\mathbb{E}F(X^{\varepsilon,h}_{[0,T]}) = \mathbb{E}[R^{\varepsilon,h}F(X^x_{[0,T]})]$. Thus,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\mathbb{E}\frac{F(X^{\varepsilon,h}_{[0,T]}) - F(X^x_{[0,T]})}{\varepsilon} &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\Big\{F(X^x_{[0,T]})\cdot\frac{R^{\varepsilon,h}-1}{\varepsilon}\Big\}\\ &= \mathbb{E}\Big\{F(X^x_{[0,T]})\int_0^T\langle h'(t), \mathrm{d}B_t\rangle\Big\},\end{aligned}$$

where the last step is due to the dominated convergence theorem since $\{R^{\varepsilon,h}\}_{\varepsilon\in[0,1]}$ is uniformly integrable for $h \in \tilde{\mathbf{H}}_{0,b}$. □

Lemma 4.2.3. *For any $n \ge 1, 0 < t_1 < \ldots < t_n \le T$, and $f \in C_0^\infty(M^n)$,*

$$(u_0^x)^{-1}\nabla_x\mathbb{E}f(X_{t_1}^x,\ldots,X_{t_n}^x) = \sum_{i=1}^n \mathbb{E}\{Q_{t_i}^x(u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x,\ldots,X_{t_n}^x)\}$$

holds for all $x \in M$ and $u_0^x \in O_x(M)$, where ∇_x denotes the gradient w.r.t. x.

Proof. By (4.1.7), the desired assertion holds for $n = 1$. Assume that it holds for $n = k$ for some natural number $k \ge 1$. It remains to prove the assertion for $n = k+1$. To this end, set

$$g(x) = \mathbb{E}f(x, X^x_{t_2-t_1},\ldots,X^x_{t_{k+1}-t_1}),\quad x \in M.$$

By the assumption for $n = k$ we have

$$(u_0^x)^{-1}\nabla g(x) = \sum_{i=1}^{k+1}\mathbb{E}\{Q^x_{t_i-t_1}(u^x_{t_i-t_1})^{-1}\nabla_i f(x, X^x_{t_2-t_1},\ldots,X^x_{t_{k+1}-t_1})\}$$

for all $x \in M, u_0 \in O_x(M)$. Combining this with the assertion for $k = 1$ and using the Markov property, we obtain

$$\begin{aligned}&(u_0^x)^{-1}\nabla_x\mathbb{E}f(X_{t_1}^x,\ldots,X_{t_{k+1}}^x) = (u_0^x)^{-1}\nabla_x\mathbb{E}g(X_{t_1}^x)\\ &= \mathbb{E}\{Q_{t_1}^x(u_{t_1}^x)^{-1}\nabla g(X_{t_1}^x)\} = \sum_{i=1}^{k+1}\mathbb{E}\{Q_{t_i}^x(u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x,\ldots,X_{t_{k+1}}^x)\}.\end{aligned}$$

□

The next lemma is a direct consequence of (4.1.7) and (4.1.11).

Lemma 4.2.4. *Let $f \in C^\infty(M)$. Then for any $x \in M$ and $t > 0$,*

$$\mathbb{E}\Big\{f(X_t^x)\int_0^t \langle h_s', \mathrm{d}B_s\rangle\Big\} = \mathbb{E}\int_0^t \langle (u_t^x)^{-1}\nabla f(X_t^x), (Q_{s,t}^x)^* h_s'\rangle \mathrm{d}s$$

holds for all $h \in \tilde{\mathbb{H}}_0, t \in [0,T]$.

Proof. [Proof of Theorem 4.2.1] By Lemma 4.2.2, it suffices to prove

$$\sqrt{2}\,\mathbb{E}\{D_h^0 F\}(X_{[0,T]}^x) = \mathbb{E}\Big\{F(X_{[0,T]}^x)\int_0^T \langle h'(t), \mathrm{d}B_t\rangle\Big\},\quad h \in \tilde{\mathbb{H}}_0 \tag{4.2.4}$$

for $F(X_{[0,T]}^x) = f(X_{t_1}^x, \ldots, X_{t_n}^x)$ with $f \in C^\infty(M^n)$, where $n \geq 1, 0 < t_1 < \ldots < t_n \leq T$. According to Lemma 4.2.4, (4.2.4) holds for $n = 1$. Assuming (4.2.4) holds for $n = k$ for some $k \geq 1$, we aim to prove it for $n = k+1$. To this end, let

$$g(x) = \mathbb{E} f(x, X_{t_2 - t_1}^x, \ldots, X_{t_{k+1}-t_1}^x),\quad x \in M.$$

By the result for $n = 1$ and the Markov property,

$$\begin{aligned}
&\sqrt{2}\int_0^{t_1} \mathbb{E}\langle (u_{t_1}^x)^{-1}\nabla g(X_{t_1}^x), (Q_{t,t_1}^x)^* h_t'\rangle \mathrm{d}t\\
&= \mathbb{E}\Big\{\mathbb{E}(F(X_{[0,T]}^x)|\mathcal{F}_{t_1})\int_0^{t_1}\langle h'(t), \mathrm{d}B_t\rangle\Big\}\\
&= \mathbb{E}\Big\{F(X_{[0,T]}^x)\int_0^{t_1}\langle h'(t), \mathrm{d}B_t\rangle\Big\}.
\end{aligned}\tag{4.2.5}$$

On the other hand, by Theorem 4.1.1(3), Lemma 4.2.3 and the Markov property,

$$\begin{aligned}
&\int_0^{t_1} \mathbb{E}\langle (u_{t_1}^x)^{-1}\nabla g(X_{t_1}^x), (Q_{t,t_1}^x)^* h_t'\rangle \mathrm{d}t\\
&= \int_0^{t_1}\mathbb{E}\Big\langle \mathbb{E}\Big(\sum_{i=1}^{k+1} Q_{t_1,t_i}^x (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \ldots, X_{t_{k+1}}^x)\Big|\mathcal{F}_{t_1}\Big), (Q_{t,t_1}^x)^* h'(t)\Big\rangle \mathrm{d}t\\
&= \mathbb{E}\sum_{i=1}^{k+1}\int_0^{t_1}\langle (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \ldots, X_{t_{k+1}}^x), (Q_{t,t_i}^x)^* h'(t)\rangle \mathrm{d}t.
\end{aligned}$$

Combining this with (4.2.1) and (4.2.5) we obtain

$$\begin{aligned}
\mathbb{E}\{D_h^0 F(X_{[0,T]}^x)\} &= \frac{1}{\sqrt{2}}\mathbb{E}\Big\{F(X_{[0,T]}^x)\int_0^{t_1}\langle h'(t), \mathrm{d}B_t\rangle\Big\}\\
&\quad + \mathbb{E}\sum_{i=2}^{k+1}\int_{t_1}^{t_i}\langle (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \ldots, X_{t_{k+1}}^x), (Q_{t,t_i}^x)^* h'(t)\rangle \mathrm{d}t.
\end{aligned}\tag{4.2.6}$$

By the Markov property and the assumption for $n = k$, we have

$$\begin{aligned}&\sum_{i=2}^{k+1}\mathbb{E}\int_{t_1}^{t_i}\langle(u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x,\ldots,X_{t_{k+1}}^x),(Q_{t,t_i}^x)^*h'(t)\rangle\mathrm{d}t\\&=\frac{1}{\sqrt{2}}\mathbb{E}\bigg\{F(X_{[0,T]}^x)\int_{t_1}^T\langle h'(t),\mathrm{d}B_t\rangle\bigg\}.\end{aligned}$$

Combining this with (4.2.6) we complete the proof. □

4.3 The log-Sobolev inequality

We first consider the path space with a fixed initial point, then move to the free path space following an idea of [Fang and Wang (2005)], where the (non-damped) gradient operator is studied on the free path space over manifolds without boundary.

4.3.1 *Log-Sobolev inequality on* W_x^T

Let Π_x^T be the distribution of $X_{[0,T]}^x$. Let

$$\mathcal{E}^x(F,G)=\mathbb{E}\big\{\langle D^0F,D^0G\rangle_{\mathbb{H}_0}(X_{[0,T]}^x)\big\},\quad F,G\in\mathcal{F}C_0^\infty.$$

Since both D^0F and D^0G are functionals of X, $(\mathcal{E}^x,\mathcal{F}C_0^\infty)$ is a positive bilinear form on $L^2(W_x^T;\Pi_x^T)$. It is standard that the integration by parts formula (4.2.4) implies the closability of the form (see Lemma 4.3.1). We shall use $(\mathcal{E}^x,\mathcal{D}(\mathcal{E}^x))$ to denote the closure of $(\mathcal{E}^x,\mathcal{F}C_0^\infty)$. Moreover, (4.2.4) also implies the Clark-Ocone type martingale representation formula (see Lemma 4.3.2), which leads to the standard Gross [Gross (1976)] log-Sobolev inequality. It is well known that the log-Sobolev inequality implies that the associated Markov semigroup is hypercontractive and converges exponentially to Π_x^T in the sense of relative entropy.

Lemma 4.3.1. *Assume* **(A4.1.1)**. $(\mathcal{E}^x,\mathcal{F}C_0^\infty)$ *is closable in* $L^2(W_x^T;\Pi_x^T)$.

Proof. Let $\{F_n\}_{n\geq1}\subset\mathcal{F}C_0^\infty$ such that $\mathcal{E}^x(F_n,F_n)\leq1$ for all $n\geq0$ and $\Pi_x^T(F_n^2)+\mathcal{E}^x(F_n-F_m,F_n-F_m)\to0$ as $n,m\to\infty$. We aim to prove that $\mathcal{E}^x(F_n,F_n)\to0$ as $n\to\infty$. Since

$$\begin{aligned}\mathcal{E}^x(F_n,F_n)&=\mathcal{E}^x(F_n,F_n-F_m)+\mathcal{E}^x(F_n,F_m)\\&\leq\sqrt{\mathcal{E}^x(F_n-F_m,F_n-F_m)}+\mathcal{E}^x(F_n,F_m),\end{aligned}$$

it suffices to show that for any $G \in \mathcal{F}C_0^\infty$, one has $\mathcal{E}^x(F_n, G) \to 0$ as $n \to \infty$. To this end, let $\{h^i\}_{i\geq 1}$ be an ONB on $\mathbb{H}_0$. For any $\varepsilon > 0$ there exists $k \geq 1$ such that

$$\mathbb{E}\Big\|D^0 G - \sum_{i=1}^k (D^0_{h_i} G) h_i\Big\|^2_{\mathbb{H}_0} < \varepsilon^2.$$

Then

$$\Big|\mathcal{E}^x(F_n, G) - \sum_{i=1}^k \mathbb{E}\{(D^0_{h^i} F_n)(D^0_{h^i} G)\}(X^x_{[0,T]})\Big| < \varepsilon, \quad n \geq 1.$$

Since $\mathcal{F}C_0^\infty$ is dense in $L^2(W_x^T; \Pi_x^T)$, there exists $G_i \in \mathcal{F}C_0^\infty$ such that

$$\mathbb{E}\big\{|D^0_{h^i} G - G_i|^2(X^x_{[0,T]})\big\} < \varepsilon, \quad 1 \leq i \leq k.$$

Therefore,

$$|\mathcal{E}^x(F_n, G)| \leq 2\varepsilon + \sum_{i=1}^k \big|\mathbb{E}\langle (G_i D^0 F_n)(X^x_{[0,T]}), h_i\rangle_{\mathbb{H}_0}\big|.$$

Noting that $G_i D^0 F_n = D^0(F_n G_i) - F_n D^0 G_i$, by (4.2.4), we obtain

$$\begin{aligned}
&|\mathcal{E}^x(F_n, G)| \\
&\leq 2\varepsilon + \sum_{i=1}^k \bigg|\mathbb{E}\bigg[F_n(X^x_{[0,T]})\bigg\{G_i(X^x_{[0,T]}) \int_0^T \langle \dot h^i_t, \mathrm{d}B_t\rangle - D^0_{h^i} G_i(X^x_{[0,T]})\bigg\}\bigg]\bigg|.
\end{aligned}$$

Since $\Pi_x^T(F_n^2) \to 0$ as $n \to \infty$, by letting first $n \to \infty$ then $\varepsilon \to 0$ we complete the proof. □

Lemma 4.3.2. *Assume* **(A4.1.1)**. *For any $F \in \mathcal{F}C_0^\infty$, let $\tilde D^0 F(X^x_{[0,T]})$ be the projection of $D^0 F(X^x_{[0,T]})$ on $\tilde{\mathbb{H}}_0$, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde D^0 F(X^x_{[0,T]}))(t) = \mathbb{E}\Big(\frac{\mathrm{d}}{\mathrm{d}t}(D^0 F(X^x_{[0,T]}))(t)\Big|\mathcal{F}_t\Big)$$

for $t \in [0,T]$, $(\tilde D^0 F)(0) = 0$. Then

$$F(X^x_{[0,T]}) = \mathbb{E}F(X^x_{[0,T]}) + \sqrt{2}\int_0^T \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}(\tilde D^0 F(X^x_{[0,T]}))(t), \mathrm{d}B_t\Big\rangle.$$

Proof. By Theorem 4.2.1, we have

$$\mathbb{E}\langle h, \tilde D^0 F\rangle_{\mathbb{H}_0}(X^x_{[0,T]}) = \frac{1}{\sqrt{2}}\mathbb{E}\Big\{F(X^x_{[0,T]})\int_0^T \langle \dot h_t, \mathrm{d}B_t\rangle\Big\}, \quad h \in \tilde{\mathbb{H}}_0. \quad (4.3.1)$$

On the other hand, by the martingale representation, there exists a predictable process β_t such that

$$\mathbb{E}(F(X^x_{[0,T]})|\mathcal{F}_t) = \mathbb{E}F(X^x_{[0,T]}) + \int_0^t \langle \beta_s, \mathrm{d}B_s \rangle, \quad t \in [0,T]. \tag{4.3.2}$$

Let

$$\varphi_t = \int_0^t \beta_s \mathrm{d}s, \quad t \in [0,T].$$

We have $\varphi \in \bar{\mathbb{H}}_0$ and by (4.3.2),

$$\mathbb{E}\langle h, \varphi \rangle_{\mathbb{H}_0} = \mathbb{E}\int_0^T \langle \dot{h}_t, \beta_t \rangle \mathrm{d}t = \mathbb{E}\bigg\{ F(X^x_{[0,T]}) \int_0^T \langle \dot{h}_t, \mathrm{d}B_t \rangle \bigg\}$$

holds for all $h \in \bar{\mathbb{H}}_0$. Combining this with (4.3.1) we conclude that $\sqrt{2}\,\tilde{D}^0 F(X^x_{[0,T]}) = \varphi$. Therefore, the desired formula follows from (4.3.2). □

It is standard that the martingale representation in Lemma 4.3.2 implies the following log-Sobolev inequality. Since the parameter T has been properly contained in the Dirichlet form $\mathcal{E}$ just as in the case without boundary (see [Fang and Malliavin (1993)]), the resulting log-Sobolev constant is independent of T. Moreover, since it is well-known that the constant 2 in the inequality is sharp for $M = \mathbb{R}^d$, it is also sharp as a universal constant for compact manifolds with boundary as $\mathbb{R}^d$ can be approximated by bounded balls.

Theorem 4.3.3. *Assume* **(A4.1.1)**. *For any $T > 0$ and $x \in M$, there holds the following log-Sobolev inequality*

$$\Pi^T_x(F^2 \log F^2) \le 4\mathcal{E}^x(F,F), \quad F \in \mathcal{D}(\mathcal{E}^x),\ \Pi^T_x(F^2) = 1.$$

Proof. It suffices to prove the inequality for $F \in \mathcal{F}C_0^\infty$. Let $m_t = \mathbb{E}(F(X^x_{[0,T]})^2|\mathcal{F}_t)$, $t \in [0,T]$. By Lemma 4.3.2 and the Itô formula,

$$\mathrm{d}m_t \log m_t = (1 + \log m_t)\mathrm{d}m_t + \frac{|\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}^0 F^2(X^x_{[0,T]}))(t)|^2}{m_t}\mathrm{d}t.$$

Thus,

$$\begin{aligned}
\Pi^T_x(F^2 \log F^2) &= \mathbb{E}m_T \log m_T \\
&= \int_0^T \frac{4\mathbb{E}(F(X^x_{[0,T]})\frac{\mathrm{d}}{\mathrm{d}t}(D^0 F(X^x_{[0,T]}))(t)|\mathcal{F}_t)^2}{\mathbb{E}(F(X^x_{[0,T]})^2|\mathcal{F}_t)}\mathrm{d}t \\
&\le 4\int_0^T \mathbb{E}\Big|\frac{\mathrm{d}}{\mathrm{d}t}(D^0 F(X^x_{[0,T]}))(t)\Big|^2 \mathrm{d}t \\
&= 4\mathbb{E}\|D^0 F(X^x_{[0,T]})\|^2_{\mathbb{H}_0} = 4\mathcal{E}^x(F,F).
\end{aligned}$$

□

4.3.2 *Log-Sobolev inequality on the free path space*

Let Π_μ^T be the distribution of the reflecting diffusion process on the time-interval $[0,T]$ generated by $L := \Delta + Z$ with initial distribution μ. Let

$$\mathbb{H} = \left\{ h \in C([0,T];\mathbb{R}^d) : \int_0^T |h'(t)|^2 \mathrm{d}t < \infty \right\}.$$

Then $\mathbb{H}$ is a Hilbert space under the inner product

$$\langle h_1, h_2 \rangle_{\mathbb{H}} = \langle h_1(0), h_2(0) \rangle + \int_0^T \langle h_1'(t), h_2'(t) \rangle \mathrm{d}t.$$

To define the damped gradient operator on the free path space, let $\bar\Omega = M \times \Omega, \bar{\mathcal{F}}_t = \mathcal{B}(M) \times \mathcal{F}_t$, and $\bar{\mathbb{P}} = \mu \times \mathbb{P}$. Let $u_0 : M \to TM$ be measurable, and let $X_0(x,w) = x$ for $(x,w) \in M \times \Omega$. Then, under the filtered probability space $(\bar\Omega, \bar{\mathcal{F}}_t, \bar{\mathbb{P}})$, $X_t(x,w) := X_t^x(w)$ is the reflecting diffusion process generated by L with initial distribution μ, and $u_t(x,w) := u_t^x(w)$ is its horizontal lift. Moreover, let $Q_{r,t}(x,w) = Q_{r,t}^x(w)$ for $0 \le r \le t$.

Now, for any $F \in \mathcal{F}C_0^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \dots, \gamma_{t_n})$, let

$$DF(X) = D^0F(X) + \sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i f(X_{t_1}, \dots, X_{t_n}), \tag{4.3.3}$$

where

$$D^0F(X) := \sum_{i=1}^n \int_0^{t_i} Q_{t,t_i} u_{t_i}^{-1} \nabla_i f(X_{t_1}, \dots, X_{t_n}) \mathrm{d}t$$

is the damped gradient on the path space with fixed initial point. Obviously, $DF(X) \in L^2(\bar\Omega \to \mathbb{H}; \bar{\mathbb{P}})$. Define

$$\mathcal{E}^\mu(F,G) = \mathbb{E}_{\bar{\mathbb{P}}} \langle DF, DG \rangle_{\mathbb{H}}, \quad F, G \in \mathcal{F}C_0^\infty.$$

We aim to prove that $(\mathcal{E}^\mu, \mathcal{F}C_0^\infty)$ is closable in $L^2(W^T; \Pi_\mu^T)$ and to establish the log-Sobolev inequality for its closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$. To prove the closability, we need the following two lemmas modified from [Fang and Wang (2005)]. Let $\mathcal{X}_0(M)$ be the class of all smooth vector fields on M with compact support.

Lemma 4.3.4. *Assume* **(A4.1.1)**. *For any* $f \in C_0^\infty(M^n)$, *there exist* $\{\xi_i\}_{i \ge 1} \subset L^2(\bar\Omega; \bar{\mathbb{P}})$ *and* $\{U_i\}_{i \ge 1} \subset \mathcal{X}_0(M)$ *such that*

$$\sum_{i=1}^n Q_{t_i} u_{t_i}^{-1} \nabla_i f(X_{t_1}, \dots, X_{t_n}) = \sum_{j=1}^\infty \xi_j u_0^{-1} U_j(X_0)$$

holds in $L^2(\bar\Omega \to \mathbb{R}^d; \bar{\mathbb{P}})$.

Proof. By the decomposition of identity, it suffices to prove for X_0 restricting on an open set $\mathcal{O}$ obeying a smooth ONB $\{U_j\}_{j=1}^d$ for the tangent space. Then the desired formula holds on $\{X_0 \in \mathcal{O}\}$ for

$$\xi_j = \sum_{i=1}^n \langle u_0 Q_{t_i} u_{t_i}^{-1} \nabla_i f(X_{t_1}, \dots, X_{t_n}), U_j(X_0)\rangle, \quad j = 1, \dots, d.$$

□

To introduce the integration by parts formula for DF, we need the divergence operator div_μ w.r.t. μ, which is the minus adjoint of ∇ in $L^2(\mu)$; that is, for any smooth vector field U,

$$\int_M (Uf)\mathrm{d}\mu = -\int_M f(\mathrm{div}_\mu U)\mathrm{d}\mu, \quad f \in C_0^1(M).$$

Lemma 4.3.5. *Assume* **(A4.1.1)**. *For any $F \in \mathcal{F}C_0^\infty$, $U \in \mathcal{X}_0(M)$, and $\bar{\mathcal{F}}_t$-adapted $h \in L^2(\bar\Omega \to \mathbb{H}; \bar{\mathbb{P}})$,*

$$\begin{aligned}&\mathbb{E}_{\bar{\mathbb{P}}}\langle DF(X), h + u_0^{-1}U(X_0)\rangle_{\mathbb{H}}\\ &= \mathbb{E}_{\bar{\mathbb{P}}}\bigg\{F(X)\bigg(\int_0^T \langle h'(t), \mathrm{d}B_t\rangle - (\mathrm{div}_\mu U)(X_0)\bigg)\bigg\}.\end{aligned}$$

Proof. Let $\{h^x(t)\}(w) = \{h(t)\}(x,w),\ (x,w) \in \bar\Omega,\ t \in [0,T]$. By Theorem 4.2.1 and (4.3.3), we have

$$\begin{aligned}\mathbb{E}_{\bar{\mathbb{P}}}\langle DF(X), h\rangle_{\mathbb{H}} &= \int_M \Big(\mathbb{E}^x\langle D^0F(X^x), h^x\rangle_{\mathbb{H}_0}\Big)\mu(\mathrm{d}x)\\ &= \mathbb{E}_{\bar{\mathbb{P}}}\bigg\{F(X)\int_0^T \langle h'(t), \mathrm{d}B_t\rangle\bigg\}.\end{aligned}$$

On the other hand, by Lemma 4.2.3 and the definition of div_μ, we obtain, for $F(X) = f(X_{t_1}, \dots, X_{t_n})$, that

$$\begin{aligned}&\mathbb{E}_{\bar{\mathbb{P}}}\langle DF(X), u_0^{-1}U(X_0)\rangle_{\mathbb{H}}\\ &= \mathbb{E}_{\bar{\mathbb{P}}}\Big\langle \sum_{i=1}^n Q_{t_i}u_{t_i}^{-1}\nabla_i f(X_{t_1}, \dots, X_{t_n}), u_0^{-1}U(X_0)\Big\rangle\\ &= \int_M \Big\langle \mathbb{E}^{\cdot}\Big\{\sum_{i=1}^n Q_{t_i}u_{t_i}^{-1}\nabla_i f(X_{t_1}, \dots, X_{t_n})\Big\}, u_0^{-1}U(\cdot)\Big\rangle \mathrm{d}\mu\\ &= \int_M \langle \nabla \mathbb{E}^{\cdot} F(X), U\rangle \mathrm{d}\mu = -\int_M (\mathbb{E}^{\cdot} F(X))\mathrm{div}_\mu(U)\,\mathrm{d}\mu\\ &= -\mathbb{E}_{\bar{\mathbb{P}}}\Big\{F(X)(\mathrm{div}_\mu U)(X_0)\Big\}.\end{aligned}$$

Then the proof is finished. □

Theorem 4.3.6. *Assume* **(A4.1.1)**. *Then the form* $(\mathcal{E}^\mu, \mathcal{F}C_0^\infty)$ *is closable in* $L^2(W^T; \Pi_\mu^T)$, *and its closure is a symmetric Dirichlet form.*

Proof. It suffices to prove the closability. Let $\{F_n\}_{n\geq 1} \subset \mathcal{F}C_0^\infty$ such that $\lim_{n\to\infty} F_n = 0$ in $L^2(W^T; \Pi_\mu^T)$ and $\tilde{h} := \lim_{n\to\infty} DF_n(X)$ exists in $L^2(\bar{\Omega} \to H; \bar{\mathbb{P}})$. We intend to prove that $\tilde{h} = 0$. By (4.3.3) and Lemma 4.3.4, it suffices to prove $\mathbb{E}_{\bar{\mathbb{P}}}\langle \tilde{h}, h + \xi u_0^{-1} U(X_0)\rangle_{\mathbb{H}} = 0$ for $\bar{\mathcal{F}}_t$-adapted $h \in L^2(\bar{\Omega} \to \mathbb{H}_0; \bar{\mathbb{P}})$, $\xi \in L^2(\bar{\Omega}; \bar{\mathbb{P}})$ and $U \in \mathcal{X}_0(M)$. According to Theorem 4.2.1 and noting that $F_n(X) \to 0$ in $L^2(\bar{\mathbb{P}})$, we have

$$\begin{aligned}
\mathbb{E}_{\bar{\mathbb{P}}}\langle \tilde{h}, h\rangle_{\mathbb{H}} &= \lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{P}}}\langle DF_n(X), h\rangle_{\mathbb{H}_0} = \lim_{n\to\infty} \int_M \big(\mathbb{E} D_h^0 F_n(X^x)\big)\mu(\mathrm{d}x) \\
&= \lim_{n\to\infty} \int_M \mathbb{E}\left\{F_n(X) \int_0^T \langle h'(t), \mathrm{d}B_t\rangle\right\}\mathrm{d}\mu \\
&= \lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{P}}}\left\{F_n(X) \int_0^T \langle h'(t), \mathrm{d}B_t\rangle\right\} = 0.
\end{aligned}$$

So, we need only to prove

$$\mathbb{E}_{\bar{\mathbb{P}}}\Big\{\xi\big\langle \tilde{h}(0), u_0^{-1}U(X_0)\big\rangle\Big\} = 0.$$

Since $\mathcal{F}C_0^\infty$ is dense in $L^2(W^T; \Pi_\mu^T)$, we may assume that $\xi = G(X)$ for some $G \in \mathcal{F}C_0^\infty$. In this case, it follows from Lemma 4.3.5 that

$$\begin{aligned}
&\mathbb{E}_{\bar{\mathbb{P}}}\Big\{\xi\big\langle \tilde{h}(0), u_0^{-1}U(X_0)\big\rangle\Big\} = \lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{P}}}\big\langle DF_n(X), G(X)u_0^{-1}U(X_0)\big\rangle_{\mathbb{H}} \\
&= \lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{P}}}\Big\{\big\langle \{D(F_nG)(X)\}(0), u_0^{-1}U(X_0)\big\rangle \\
&\qquad\qquad - F_n(X)\big\langle (DG(X))(0), u_0^{-1}U(X_0)\big\rangle\Big\} \\
&= -\lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{P}}}\Big\{F_n(X)\big(G(X)(\mathrm{div}_\mu U)(X_0) + \big\langle (DG(X))(0), u_0^{-1}U(X_0)\big\rangle\big)\Big\} \\
&= 0.
\end{aligned}$$

□

Theorem 4.3.7. *Assume* **(A4.1.1)**. *If the log-Sobolev inequality*

$$\mu(g^2 \log g^2) \leq C\mu(|\nabla g|^2), \quad g \in C_b^1(M), \mu(g^2) = 1 \tag{4.3.4}$$

holds for some constant $C > 0$, *then*

$$\Pi_\mu^T(F^2 \log F^2) \leq (4 \vee C)\mathcal{E}^\mu(F, F), \quad F \in \mathcal{D}(\mathcal{E}^\mu), \Pi_\mu^T(F^2) = 1.$$

Proof. It suffices to prove for $F \in \mathcal{F}C_0^\infty$. By Theorem 4.3.3 and (4.3.4) we obtain

$$\begin{aligned}
\Pi_\mu^T(F^2 \log F^2) &= \int_M \Pi_x^T(F^2 \log F^2)\mu(\mathrm{d}x) \\
&\le 4\int_M \mathcal{E}^x(F,F)\mu(\mathrm{d}x) + \int_M \Pi_x^T(F^2)\log \Pi_x^T(F^2)\mu(\mathrm{d}x) \qquad (4.3.5)\\
&\le 4\mathbb{E}_{\bar{\mathbb{P}}}\|D^0F(X)\|_{\mathbb{H}_0}^2 + C\int_M \left|\nabla\sqrt{\mathbb{E}^{\cdot} F^2(X)}\right|^2 \mathrm{d}\mu.
\end{aligned}$$

On the other hand, letting $F(X) = f(X_{t_1}, \dots, X_{t_n})$, it follows from Lemma 4.2.3 that

$$\begin{aligned}
\left|\nabla\sqrt{\mathbb{E}^{\cdot} F^2(X)}\right|^2 &= \frac{\left|\mathbb{E}^{\cdot} F(X)\sum_{i=1}^n Q_{t_i}u_{t_i}^{-1}\nabla_i f(X_{t_1},\dots,X_{t_n})\right|^2}{\mathbb{E}^{\cdot} F^2(X)} \\
&\le \mathbb{E}^{\cdot}\left|\sum_{i=1}^n Q_{t_i}u_{t_i}^{-1}\nabla_i f(X_{t_1},\dots,X_{t_n})\right|^2.
\end{aligned}$$

Combining this with (4.3.5) we complete the proof. □

4.4 Transportation-cost inequalities on path spaces over convex manifolds

In 1996, Talagrand [Talagrand (1996)] established an inequality to bound from above the L^2-Wasserstein distance of a probability measure to the standard Gaussian measure by the relative entropy. This inequality is called (Talagrand) transportation-cost inequality, and has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of diffusion processes with respect to several different distances (i.e. cost functions): see e.g. [Feyel and Üstünel (2002)] for the study on the Wiener space with the Cameron-Martin distance, [Wang (2000b); Djellout *et al* (2004)] on the path space of diffusions with the L^2-distance, [Wang (2004c)] on the Riemannian path space with intrinsic distance induced by the Malliavin gradient operator, and [Fang *et al* (2008); Wu and Zhang (2004)] on the path space of diffusions with the uniform distance.

The main purpose of this section is to investigate the Talagrand inequality on the path space W^T of the (reflecting) diffusion processes on a convex manifold.

Let M be a connected complete Riemannian manifold possibly with a boundary ∂M. Let $L = \Delta + Z$ for a C^1-smooth vector field Z on M. Let

X_t be the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by L with initial distribution $\mu \in \mathcal{P}(M)$. Assume that X_t is non-explosive, which is the case if ∂M is convex and $\mathrm{Ric}_Z \geq K$ holds for some constant $K \in \mathbb{R}$. Recall that Π_μ^T is the distribution of $X_{[0,T]} := \{X_t : \ t \in [0,T]\}$, which is a probability measure on the (free) path space $W^T := C([0,T]; M)$. When $\mu = \delta_x$, we denote $\Pi_{\delta_x}^T = \Pi_x^T$. For any nonnegative measurable function F on W^T such that $\Pi_\mu^T(F) = 1$, one has

$$\mu_F^T(\mathrm{d}x) := \Pi_x^T(F)\mu(\mathrm{d}x) \in \mathcal{P}(M). \tag{4.4.1}$$

Consider the uniform distance on W^T:

$$\rho_\infty(\gamma, \eta) := \sup_{t\in[0,T]} \rho(\gamma_t, \eta_t), \quad \gamma, \eta \in W^T.$$

Let $W_2^{\rho_\infty}$ be the L^2-Wasserstein distance (or L^2-transportation cost) induced by ρ_∞. In general, for any $p \in [1, \infty)$ and for two probability measures Π_1, Π_2 on W^T,

$$W_p^{\rho_\infty}(\Pi_1, \Pi_2) := \inf_{\pi\in\mathcal{C}(\Pi_1,\Pi_2)} \left\{ \int_{W^T\times W^T} \rho_\infty(\gamma, \eta)^p \pi(\mathrm{d}\gamma, \mathrm{d}\eta) \right\}^{1/p}$$

is the L^p-Warsserstein distance (or L^p-transportation cost) of Π_1 and Π_2 induced by the uniform norm, where $\mathcal{C}(\Pi_1, \Pi_2)$ is the set of all couplings for Π_1 and Π_2.

Additional to Theorems 3.3.1 and 3.3.2, the following Theorem 4.4.2 provides 7 more equivalent transportation-cost inequalities for $\mathrm{Ric}_Z \geq K$ and the convexity of ∂M (when $\partial M \neq \emptyset$). To prove this result, we need the following inequality due to [Otto and Villani (2000)].

Lemma 4.4.1. *Let μ be a probability measure on M and $f \in C_b^2(M)$ such that $\mu(f) = 0$. For small enough $\varepsilon > 0$ such that $f_\varepsilon := 1 + \varepsilon f \geq 0$, there holds*

$$\mu(f^2) \leq \frac{1}{\varepsilon}\sqrt{\mu(|\nabla f|^2)}\, W_2^\rho(f_\varepsilon\mu, \mu) + \frac{\|\mathrm{Hess}_f\|_\infty}{2\varepsilon} W_2^\rho(f_\varepsilon\mu, \mu)^2,$$

where $\|\mathrm{Hess}_f\|_\infty = \sup_{x\in M} \|\mathrm{Hess}_f\|$ for $\|\cdot\|$ the operator norm in $\mathbb{R}^d$.

Proof. Let $\pi_\varepsilon \in \mathcal{C}(f_\varepsilon\mu, \mu)$ reach $W_2^\rho(f_\varepsilon\mu, \mu)$. Then the Taylor expansion and the Schwarz inequality imply

$$\begin{aligned}
\mu(f^2) &= \frac{\mu(f_\varepsilon f) - \mu(f)}{\varepsilon} = \frac{1}{\varepsilon}\int_{M\times M} \{f(x) - f(y)\}\pi_\varepsilon(\mathrm{d}x, \mathrm{d}y) \\
&\leq \frac{1}{\varepsilon}\int_{M\times M} |\nabla f(y)|\rho(x,y)\pi_\varepsilon(\mathrm{d}x, \mathrm{d}y) \\
&\qquad + \frac{\|\mathrm{Hess}_f\|_\infty}{2\varepsilon}\int_{M\times M} \rho(x,y)^2\pi_\varepsilon(\mathrm{d}x, \mathrm{d}y) \\
&\leq \frac{1}{\varepsilon}\sqrt{\mu(|\nabla f|^2)}\, W_2^\rho(f_\varepsilon\mu, \mu) + \frac{\|\mathrm{Hess}_f\|_\infty}{2\varepsilon} W_2^\rho(f_\varepsilon\mu, \mu)^2. \qquad \square
\end{aligned}$$

Theorem 4.4.2. *Let $P_T(o,\cdot)$ be the distribution of X_T with $X_0 = o$, and let P_T be the corresponding semigroup. For any $K \in \mathbb{R}$ and any $p \in [1,\infty)$, the following statements are equivalent to each other:*

(1) *∂M is either convex or empty, and $\mathrm{Ric}_Z \geq K$.*
(2) *For any $T > 0, \mu \in \mathcal{P}(M)$ and nonnegative F with $\Pi_\mu^T(F) = 1$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \leq \frac{2}{K}(1-\mathrm{e}^{-2KT})\Pi_\mu^T(F\log F)$$

holds, where $\mu_F^T \in \mathcal{P}(M)$ is fixed by (4.4.1).
(3) *For any $o \in M$ and $T > 0$,*

$$W_2^{\rho_\infty}(F\Pi_o^T, \Pi_o^T)^2 \leq \frac{2}{K}(1-\mathrm{e}^{-2KT})\Pi_o^T(F\log F)$$

holds for all $F \geq 0$, $\Pi_o^T(F) = 1$.
(4) *For any $o \in M$ and $T > 0$,*

$$W_2^{\rho}\big(P_T(o,\cdot), fP_T(o,\cdot)\big)^2 \leq \frac{2}{K}(1-\mathrm{e}^{-2KT})P_T(f\log f)(o)$$

holds for all $f \geq 0$, $P_Tf(o) = 1$.
(5) *For any $o \in M$ and $T > 0$,*

$$W_2^{\rho}\big(P_T(o,\cdot), fP_T(o,\cdot)\big)^2 \leq \Big(\frac{1-\mathrm{e}^{-2KT}}{K}\Big)^2 P_T\frac{|\nabla f|^2}{f}(o)$$

holds for all $f \geq 0$, $P_Tf(o) = 1$.
(6) *For any $T > 0$, and $\mu, \nu \in \mathcal{P}(M)$,*

$$W_p^{\rho_\infty}(\Pi_\mu^T, \Pi_\nu^T) \leq \mathrm{e}^{-KT}W_p^{\rho}(\mu,\nu).$$

(7) *For any $T > 0$, $\mu \in \mathcal{P}(M)$, and $F \geq 0$ with $\Pi_\mu^T(F) = 1$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_\mu^T) \leq \Big\{\frac{2}{K}(1-\mathrm{e}^{-2KT})\Pi_\mu^T(F\log F)\Big\}^{\frac{1}{2}} + \mathrm{e}^{-KT}W_2^{\rho}(\mu_F^T, \mu).$$

(8) *For any $\mu \in \mathcal{P}(M)$ and $C \geq 0$ such that*

$$W_2^{\rho}(f\mu,\mu)^2 \leq C\mu(f\log f), \quad f \geq 0, \mu(f) = 1,$$

there holds

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_\mu^T)^2 \leq \bigg(\sqrt{\frac{2}{K}(1-\mathrm{e}^{-2KT})} + \mathrm{e}^{-KT}\sqrt{C}\bigg)^2 \Pi_\mu^T(F\log F)$$

holds for all $F \geq 0, \Pi_\mu^T(F) = 1$.

Proof. By taking $\mu = \delta_o$, we have $\mu_F^T = \Pi_o^T(F)\delta_o = \delta_o$. So, (3) follows from each of (2), (7) and (8). Next, (4) follows from (3) by taking $F(X_{[0,T]}) = f(X_T)$, and (6) implies Theorem 3.3.2(2) and thus implies (1). Moreover, it is clear that (8) follows from (7) while (7) is implied by each of (2) and (6). So, it suffices to prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (2), each of (4) and (5)$\Rightarrow$ (1), (1) $\Rightarrow$ (5), and (1) $\Rightarrow$ (6), where " $\Rightarrow$" stands for "implies".

(a) (1) $\Rightarrow$ (3). We shall only consider the case where ∂M is non-empty and convex. For the case without boundary, the following argument works well by taking $l_t = 0$ and $N = 0$. Simply denote $X_{[0,T]} = X^o_{[0,T]}$. Let F be a positive bounded measurable function on W^T such that $\inf F > 0$ and $\Pi_x^T(F) = 1$. Then

$$m_t := \mathbb{E}(F(X_{[0,T]})|\mathcal{F}_t) \quad \text{and} \quad L_t := \int_0^t \frac{\mathrm{d}m_s}{m_s}, \quad t \in [0,T]$$

are square-integrable $\mathcal{F}_t$-martingales. Obviously, we have

$$m_t = \mathrm{e}^{L_t - \frac{1}{2}\langle L\rangle_t}, \quad t \in [0,T]. \tag{4.4.2}$$

Moreover, by the martingale representation theorem (cf. Theorem 6.6 in Chapter 1 of [Ikeda and Watanabe (1989)]), there exists a unique $\mathcal{F}_t$-predictable process β_t on $\mathbb{R}^d$ such that

$$L_t = \int_0^t \langle \beta_s, \mathrm{d}B_s\rangle, \quad t \in [0,T]. \tag{4.4.3}$$

Let $\mathrm{d}\mathbb{Q} = F(X_{[0,T]})\mathrm{d}\mathbb{P}$. Since $\mathbb{E}F(X_{[0,T]}) = \Pi_\mu^T(F) = 1$, $\mathbb{Q}$ is a probability measure on Ω. Due to (4.4.2) and (4.4.3) we have

$$F(X_{[0,T]}) = m_T = \mathrm{e}^{\int_0^T \langle \beta_s, \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^T |\beta_s|^2 \mathrm{d}s}.$$

Moreover, by the Girsanov theorem,

$$\bar{B}_t := B_t - \int_0^t \beta_s \mathrm{d}s, \quad t \in [0,T] \tag{4.4.4}$$

is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}$.

Let Y_t solve the SDE

$$\mathrm{d}Y_t = \sqrt{2}\, P_{X_t,Y_t} u_t \circ \mathrm{d}\bar{B}_t + Z(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = o, \tag{4.4.5}$$

where P_{X_t,Y_t} is the parallel displacement along the minimal geodesic from X_t to Y_t and $\tilde{l}_t$ is the local time of Y_t on ∂M. According to Theorem 2.3.2, we may simply consider the case that $P_{x,y}$ is smooth in $x, y \in M$. Since, under $\mathbb{Q}$, $\bar{B}_t$ is a d-dimensional Brownian motion, the distribution of $Y_{[0,T]}$ is Π_o^T.

On the other hand, by (4.4.4) we have

$$\mathrm{d}X_t = \sqrt{2}\, u_t \circ \mathrm{d}\tilde{B}_t + Z(X_t)\mathrm{d}t + \sqrt{2}\, u_t\beta_t \mathrm{d}t + N(X_t)\mathrm{d}l_t. \tag{4.4.6}$$

Since for any bounded measurable function G on W^T

$$\mathbb{E}_{\mathbb{Q}} G(X_{[0,T]}) := \mathbb{E}(FG)(X_{[0,T]}) = \Pi_o^T(FG),$$

we conclude that under $\mathbb{Q}$ the distribution of $X_{[0,T]}$ coincides with $F\Pi_o^T$. Therefore,

$$W_2^{\rho_\infty}(F\Pi_o^T, \Pi_o^T)^2 \le \mathbb{E}_{\mathbb{Q}} \rho_\infty(X_{[0,T]}, Y_{[0,T]})^2 = \mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t, Y_t)^2. \tag{4.4.7}$$

By the convexity of ∂M we have

$$\langle N(x), \nabla \rho(y,\cdot)(x)\rangle = \langle N(x), \nabla\rho(\cdot, y)(x)\rangle \le 0, \quad x \in \partial M.$$

Combining this with the Itô formula for (X_t, Y_t) given by (4.4.5) and (4.4.6), we obtain from $\mathrm{Ric}_Z \ge K$ that

$$\begin{aligned}\mathrm{d}\rho(X_t, Y_t) &\le -K\rho(X_t, Y_t)\mathrm{d}t + \sqrt{2}\,\langle u_t\beta_t, \nabla\rho(\cdot, Y_t)(X_t)\rangle \mathrm{d}t\\ &\le \Big(\sqrt{2}\,|\beta_t| - K\rho(X_t, Y_t)\Big)\mathrm{d}t,\end{aligned}$$

see Theorem 2.3.2. Since $X_0 = Y_0$, this implies

$$\begin{aligned}\rho(X_t, Y_t)^2 &\le \mathrm{e}^{-2Kt}\bigg(\sqrt{2}\int_0^t \mathrm{e}^{Ks}|\beta_s|\,\mathrm{d}s\bigg)^2\\ &\le \frac{1-\mathrm{e}^{-2Kt}}{K}\int_0^t |\beta_s|^2\mathrm{d}s, \quad t\in[0,T].\end{aligned}$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t, Y_t)^2 \le \frac{1-\mathrm{e}^{-2KT}}{K}\int_0^T \mathbb{E}_{\mathbb{Q}}|\beta_s|^2\mathrm{d}s. \tag{4.4.8}$$

It is clear that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}|\beta_s|^2 &= \mathbb{E}\big(m_T|\beta_s|^2\big)\\ &= \mathbb{E}\big(|\beta_s|^2\mathbb{E}(m_T|\mathcal{F}_s)\big) = \mathbb{E}\big(m_s|\beta_s|^2\big), \quad s\in[0,T].\end{aligned} \tag{4.4.9}$$

Finally, since (4.4.2) and (4.4.3) yield

$$\mathrm{d}\langle m\rangle_t = m_t^2\mathrm{d}\langle L\rangle_t = m_t^2|\beta_t|^2\mathrm{d}t,$$

we have

$$\begin{aligned}\mathrm{d}m_t\log m_t &= (1+\log m_t)\mathrm{d}m_t + \frac{\mathrm{d}\langle m\rangle_t}{2m_t}\\ &= (1+\log m_t)\mathrm{d}m_t + \frac{m_t}{2}|\beta_t|^2\mathrm{d}t.\end{aligned}$$

As m_t is a $\mathbb{P}$-martingale, combining this with (4.4.9) we obtain

$$\int_0^T \mathbb{E}_{\mathbb{Q}}|\beta_s|^2 \mathrm{d}s = 2\mathbb{E}F(X_{[0,T]})\log F(X_{[0,T]}). \tag{4.4.10}$$

Therefore, (3) follows from (4.4.7), (4.4.8) and (4.4.10).

(b) (3) $\Rightarrow$ (2). By (3), for each $x \in M$, there exists

$$\pi_x \in \mathcal{C}\Big(\frac{F}{\Pi_x^T(F)}\Pi_x^T, \Pi_x^T\Big)$$

such that

$$\begin{aligned}&\int_{W^T\times W^T} \rho_\infty(\gamma,\eta)^2 \pi_x(\mathrm{d}\gamma,\mathrm{d}\eta)\\ &\le \frac{2}{K}(1-\mathrm{e}^{-2KT})\Pi_x^T\Big(\frac{F}{\Pi_x^T(F)}\log\frac{F}{\Pi_x^T(F)}\Big).\end{aligned} \tag{4.4.11}$$

If $x \mapsto \pi_x(G)$ is measurable for bounded continuous functions G on $W^T \times W^T$, then

$$\pi := \int_M \pi_x \mu_F^T(\mathrm{d}x) \in \mathcal{C}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)$$

is well defined and by (4.4.11)

$$\begin{aligned}\int_{W^T\times W^T} \rho_\infty^2 \mathrm{d}\pi &\le \frac{2}{K}(1-\mathrm{e}^{-2KT})\int_M \Pi_x^T\Big(F\log\frac{F}{\Pi_x^T(F)}\Big)\mu(\mathrm{d}x)\\ &\le \frac{2}{K}(1-\mathrm{e}^{-2KT})\Pi_\mu^T(F\log F).\end{aligned}$$

This implies the inequality in (2).

To confirm the measurability of $x \mapsto \pi_x$, we first consider discrete μ, i.e. $\mu = \sum_{n=1}^\infty \varepsilon_n \delta_{x_n}$ for some $\{x_n\} \subset M$ and $\varepsilon_n \ge 0$ with $\sum_{n=1}^\infty \varepsilon_n = 1$. In this case

$$\pi_x = \sum_{n=1}^\infty 1_{\{x=x_n\}}\pi_{x_n}, \quad \mu\text{-a.e.}$$

is measurable in x and $\pi = \sum_{n=1}^\infty \mu_F^T(\{x_n\})\pi_{x_n}$. Hence, the inequality in (2) holds. Then, for general μ, the desired inequality can be derived by approximating μ with discrete distributions in a standard way, see (b) in the proof of Theorem 4.1 in [Fang *et al* (2008)].

(c) (4) $\Rightarrow$ (1). Let $f \in C_b^2(M)$ such that $P_T f(o) = 0$. Then, for small $\varepsilon > 0$ such that $f_\varepsilon := 1 + \varepsilon f \ge 0$, we have

$$\begin{aligned}P_T(f_\varepsilon \log f_\varepsilon)(o) &= P_T\Big\{(1+\varepsilon f)\Big(\varepsilon f - \frac{1}{2}(\varepsilon f)^2 + \mathrm{o}(\varepsilon^2)\Big)\Big\}(o)\\ &= \frac{\varepsilon^2}{2}P_T f^2(o) + \mathrm{o}(\varepsilon^2).\end{aligned}$$

Combining this with Lemma 4.4.1 and (4), we obtain

$$\begin{aligned}(P_T f^2)^2(o) &\le \frac{2(1-\mathrm{e}^{-2KT})}{K} P_T|\nabla f|^2(o) \lim_{\varepsilon\to 0} \frac{P_T f_\varepsilon \log f_\varepsilon(o)}{\varepsilon^2}\\ &= \frac{1-\mathrm{e}^{-2KT}}{K}\big(P_T|\nabla f|^2(o)\big) P_T f^2(o).\end{aligned}$$

This is equivalent to Theorem 3.3.1(3) for $\sigma = 0, p = 2$ and constant K. Therefore, by Theorem 3.3.1, (1) holds.

(d) (5) $\Rightarrow$ (1). Similarly to (c), combining (5) with Lemma 4.4.1 we obatin

$$\begin{aligned}P_T f^2(o) \le &\frac{1-\mathrm{e}^{-2KT}}{K}\sqrt{P_T|\nabla f|^2(o)} \lim_{\varepsilon\to 0}\sqrt{P_T\frac{|\nabla f_\varepsilon|^2}{f_\varepsilon \varepsilon^2}(o)}\\ &= \frac{1-\mathrm{e}^{-2KT}}{K} P_T|\nabla f|^2(o).\end{aligned}$$

Hence, (1) holds.

(e) (1) $\Rightarrow$ (5). Since (1) implies (4) and, due to Theorem 3.3.1 for $\sigma = 0$ and constant K,

$$P_T(f\log f)(o) \le \frac{1-\mathrm{e}^{-2KT}}{2K} P_T\frac{|\nabla f|^2}{f}(o), \quad f \ge 0, P_T f(o) = 1,$$

we conclude that (1) implies (5).

(f) (1) $\Rightarrow$ (6). According to Theorem 3.3.2, (1) implies Theorem 3.3.2(2). So, for any $x, y \in M$, there exists $\pi_{x,y} \in \mathcal{C}(\Pi_x^T, \Pi_y^T)$ such that

$$\int_{W^T\times W^T} \rho_\infty^p \mathrm{d}\pi_{x,y} \le \mathrm{e}^{-pKT}\rho(x,y)^p.$$

As explained in (b), we assume that μ and ν are discrete, so that for any $\pi^0 \in \mathcal{C}(\mu,\nu)$, $\pi_{x,y}$ has a π^0-version measurable in (x,y). Thus,

$$\pi := \int_{M\times M} \pi_{x,y}\pi^0(\mathrm{d}x,\mathrm{d}y) \in \mathcal{C}(\Pi_\mu^T, \Pi_\nu^T)$$

satisfies

$$\int_{W^T\times W^T} \rho_\infty^p \mathrm{d}\pi \le \mathrm{e}^{-pKT}\int_{M\times M} \rho(x,y)^p\pi^0(\mathrm{d}x,\mathrm{d}y).$$

This implies the desired inequality. □

4.5 Transportation-cost inequality on the path space over non-convex manifolds

Similarly to §3.4.4 where the Harnack inequality is investigated on non-convex manifolds, we first consider the operator $\psi^2(\Delta + Z)$ as in §3.4.3 on convex manifolds.

4.5.1 *The case with a diffusion coefficient*

Let $\psi > 0$ be a smooth function on M, and let $\Pi_{\mu,\psi}^T$ be the distribution of the (reflecting if $\partial M \neq \emptyset$) diffusion process generated by $\psi^2(\Delta + Z)$ on time interval $[0, T]$ with initial distribution μ, and let $\Pi_{x,\psi}^T = \Pi_{\delta_x,\psi}^T$ for $x \in M$. Moreover, for $F \geq 0$ with $\Pi_{\mu,\psi}^T(F) = 1$, let

$$\mu_{F,\psi}^T(\mathrm{d}x) = \Pi_{x,\psi}^T(F)\mu(\mathrm{d}x).$$

Theorem 4.5.1. *Assume that ∂M is either empty or convex, and* $\mathrm{Ric}_Z \geq K$ *for some constant K. Let $\psi \in C_b^\infty(M)$ be strictly positive. Let*

$$\kappa_\psi = K^- \|\psi\|_\infty^2 + 2\|Z\|_\infty \|\nabla\psi\|_\infty \|\psi\|_\infty + (d-1)\|\nabla\psi\|_\infty^2.$$

Then

$$W_2^{\rho_\infty}(F\Pi_{\mu,\psi}^T, \Pi_{\mu_{F,\psi}^T,\psi}^T)^2 \leq 2C(T,\psi)\Pi_{\mu,\psi}^T(F\log F)$$

holds for $\mu \in \mathcal{P}(M)$, $F \geq 0$, $\Pi_{\mu,\psi}^T(F) = 1$ and

$$C(T,\psi) := \inf_{R>0} \left\{(1+R^{-1})\|\psi\|_\infty^2 \frac{\mathrm{e}^{2\kappa_\psi T} - 1}{\kappa_\psi} \exp\left[4(1+R)\|\nabla\psi\|_\infty^2 \frac{\mathrm{e}^{2\kappa_\psi T} - 1}{\kappa_\psi}\right]\right\}.$$

Proof. As explained in (a) of the proof of Theorem 4.4.2, we shall only consider the case that ∂M is non-empty and convex. According to the proof of "(3) $\Rightarrow$ (2)", it suffices to prove for $\mu = \delta_o, o \in M$. In this case the desired inequality reduces to

$$W_2^{\rho_\infty}(F\Pi_{o,\psi}^T, \Pi_{o,\psi}^T) \leq 2C(T,\psi)\Pi_{o,\psi}^T(F\log F), \tag{4.5.1}$$

for all $F \geq 0$, $\Pi_{o,\psi}^T(F) = 1$. Since the diffusion coefficient is non-constant, it is convenient to adopt the Itô differential d_I for the Girsanov transformation. So, the reflecting diffusion process generated by $\psi^2(\Delta + Z)$ can be constructed by solving the Itô SDE

$$\mathrm{d}_I X_t = \sqrt{2}\,\psi(X_t)u_t \mathrm{d}B_t + \psi^2(X_t)Z(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t, \tag{4.5.2}$$

where $X_0 = o$ and B_t is the d-dimensional Brownian motion with natural filtration $\mathcal{F}_t$. Let $\beta_t, \mathbb{Q}$ and $\tilde{B}_t$ be fixed in the proof of Theorem 4.4.2. Then

$$\begin{aligned}\mathrm{d}_I X_t = {} & \sqrt{2}\,\psi(X_t)u_t \mathrm{d}\tilde{B}_t + N(X_t)\mathrm{d}l_t \\ & + \left\{\psi^2(X_t)Z(X_t) + \sqrt{2}\,\psi(X_t)u_t\beta_t\right\}\mathrm{d}t.\end{aligned} \tag{4.5.3}$$

Let Y_t solve

$$\mathrm{d}_I Y_t = \sqrt{2}\,\psi(Y_t)P_{X_t,Y_t}u_t \mathrm{d}\tilde{B}_t + \psi^2(Y_t)Z(Y_t)\mathrm{d}t + N(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = o, \tag{4.5.4}$$

where $\bar{l}_t$ is the local time of Y_t on ∂M. As in (a), under $\mathbb{Q}$, the distributions of $Y_{[0,T]}$ and $X_{[0,T]}$ are $\Pi^T_{o,\psi}$ and $F\Pi^T_{o,\psi}$ respectively. So,

$$W_2^{\rho_\infty}(F\Pi^T_{o,\psi},\Pi^T_{o,\psi})^2 \le \mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t,Y_t)^2. \tag{4.5.5}$$

Noting that due to the convexity of ∂M

$$\langle N(x),\nabla\rho(y,\cdot)(x)\rangle = \langle N(x),\nabla\rho(\cdot,y)(x)\rangle \le 0, \quad x\in\partial M,$$

by (4.5.3), (4.5.4) and the Itô formula, we obtain

$$\begin{aligned} \mathrm{d}\rho(X_t,Y_t) \le \sqrt{2}\,&\{\psi(X_t)\langle\nabla\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}\bar{B}_t\rangle \\ &+\psi(Y_t)\langle\nabla\rho(X_t,\cdot)(Y_t),P_{X_t,Y_t}u_t\mathrm{d}\bar{B}_t\rangle\} \\ &+\Big\{\langle\psi(X_t)^2Z(X_t)+\sqrt{2}\,\psi(X_t)u_t\beta_t,\nabla\rho(\cdot,Y_t)(X_t)\rangle \\ &+\sum_{i=1}^{d-1}U_i^2\rho(X_t,Y_t)+\psi(Y_t)^2\langle Z(Y_t),\nabla\rho(X_t,\cdot)(Y_t)\rangle\Big\}\mathrm{d}t, \end{aligned} \tag{4.5.6}$$

where $\{U_i\}_{i=1}^{d-1}$ are vector fields on $M\times M$ such that $\nabla U_i(X_t,Y_t)=0$ and

$$U_i(X_t,Y_t)=\psi(X_t)V_i+\psi(Y_t)P_{X_t,Y_t}V_i, \quad 1\le i\le d-1$$

for $\{V_i\}_{i=1}^d$ an ONB of $T_{X_t}M$ with $V_d=\nabla\rho(\cdot,Y_t)(X_t)$. By the calculations leading to (3.4.28), we obtain

$$\begin{aligned} \mathrm{d}\rho(X_t,Y_t) \le&\sqrt{2}\,(\psi(X_t)-\psi(Y_t))\langle\nabla\rho(\cdot,Y_t)(X_t),u_t\mathrm{d}\bar{B}_t\rangle \\ &+\kappa_\psi\rho(X_t,Y_t)\mathrm{d}t+\sqrt{2}\,\|\psi\|_\infty|\beta_t|\mathrm{d}t. \end{aligned}$$

Then

$$M_t:=\sqrt{2}\int_0^t \mathrm{e}^{-\kappa_\psi s}(\psi(X_s)-\psi(Y_s))\langle\nabla\rho(\cdot,Y_s)(X_s),u_s\mathrm{d}\bar{B}_s\rangle$$

is a $\mathbb{Q}$-martingale such that

$$\rho(X_t,Y_t)\le \mathrm{e}^{\kappa_\psi t}M_t+\sqrt{2}\,\mathrm{e}^{\kappa_\psi t}\int_0^t \mathrm{e}^{-\kappa_\psi s}\|\psi\|_\infty|\beta_s|\mathrm{d}s, \quad t\in[0,T]. \tag{4.5.7}$$

So, by the Doob inequality we obtain

$$\begin{aligned} \ell_t:=&\,\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\rho(X_s,Y_s)^2 \\ \le&\,(1+R)\mathrm{e}^{2\kappa_\psi t}\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}M_s^2 \\ &+2\|\psi\|_\infty^2(1+R^{-1})\mathrm{e}^{2\kappa_\psi t}\mathbb{E}_{\mathbb{Q}}\Big(\int_0^t \mathrm{e}^{-\kappa_\psi s}|\beta_s|\mathrm{d}s\Big)^2 \\ \le&\,4(1+R)\mathrm{e}^{2\kappa_\psi t}\,\mathbb{E}_{\mathbb{Q}}M_t^2+(1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2\kappa_\psi t}-1}{\kappa_\psi}\int_0^t\mathbb{E}_{\mathbb{Q}}|\beta_s|^2\mathrm{d}s \\ \le&\,8(1+R)\|\nabla\psi\|_\infty^2\mathrm{e}^{2\kappa_\psi t}\int_0^t \mathrm{e}^{-2\kappa_\psi s}\ell_s\mathrm{d}s \\ &+(1+R^{-1})\|\psi\|_\infty^2\frac{\mathrm{e}^{2\kappa_\psi T}-1}{\kappa_\psi}\int_0^t\mathbb{E}_{\mathbb{Q}}|\beta_s|^2\mathrm{d}s \end{aligned}$$

for any $R > 0$. Since $e^{-2\kappa_\psi s}$ is decreasing in s while ℓ_s is increasing in s, by the FKG inequality we have

$$\int_0^t e^{-2\kappa_\psi s}\ell_s ds \le \left(\frac{1}{t}\int_0^t e^{-2\kappa_\psi s}ds\right)\int_0^t \ell_s ds = \frac{1-e^{-2\kappa_\psi t}}{2\kappa_\psi t}\int_0^t \ell_s ds.$$

Therefore,

$$\ell_t \le 4(1+R)\|\nabla\psi\|_\infty^2 \frac{e^{2\kappa_\psi T}-1}{\kappa_\psi T}\int_0^t \ell_s ds + (1+R^{-1})\|\psi\|_\infty^2 \frac{e^{2\kappa_\psi T}-1}{\kappa_\psi}\int_0^t \mathbb{E}_{\mathbb{Q}}|\beta_s|^2 ds$$

holds for $t \in [0,T]$. Since $\ell_0 = 0$, this implies that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho(X_t,Y_t)^2 = \ell_T\\ &\le (1+R^{-1})\|\psi\|_\infty^2 \frac{e^{2\kappa_\psi T}-1}{\kappa_\psi}\\ &\quad\times \exp\left[4(1+R)\|\nabla\psi\|_\infty^2 \frac{e^{2\kappa_\psi T}-1}{\kappa_\psi}\right]\int_0^T \mathbb{E}_{\mathbb{Q}}|\beta_s|^2 ds.\end{aligned}$$

Combining this with the (4.5.5) and (4.4.10), we complete the proof. □

Theorem 4.5.2. *In the situation of Theorem* 4.5.1,

$$W_2^{\rho_\infty}(\Pi_{\mu,\psi}^T, \Pi_{\nu,\psi}^T) \le 2e^{(\kappa_\psi + \|\nabla\psi\|_\infty^2)T} W_2^\rho(\mu,\nu), \quad \mu,\nu \in \mathcal{P}(M), T > 0.$$

Proof. As explained in the proof of "(6) ⇒ (5)", we only consider $\mu = \delta_x$ and $\nu = \delta_y$. Let X_t solve (4.5.2) with $X_0 = x$, and let Y_t solve, instead of (4.5.4),

$$d_I Y_t = \sqrt{2}\,\psi(Y_t)P_{X_t,Y_t}u_t dB_t + \psi^2(Y_t)Z(Y_t)dt + N(Y_t)d\tilde{l}_t, \ \ Y_0 = y.$$

Then, repeating the proof of Theorem 4.5.1, we have, instead of (4.5.7),

$$\rho(X_t,Y_t) \le e^{\kappa_\psi t}(M_t + \rho(x,y)), \quad t \ge 0 \tag{4.5.8}$$

for

$$M_t := \sqrt{2}\int_0^t e^{-\kappa_\psi s}(\psi(X_s)-\psi(Y_s))\langle \nabla\rho(\cdot,Y_s)(X_s), \Phi_s dB_s\rangle.$$

So,

$$\mathbb{E}\rho(X_t,Y_t)^2 \le e^{2\kappa_\psi t}\left\{\rho(x,y)^2 + 2\|\nabla\psi\|_\infty^2 \int_0^t e^{-2\kappa_\psi s}\mathbb{E}\rho(X_s,Y_s)^2 ds\right\},$$

which implies

$$\mathbb{E}\rho(X_t,Y_t)^2 \le e^{2(\kappa_\psi + \|\nabla\psi\|_\infty^2)t}\rho(x,y)^2.$$

Combining this with (4.5.8) and the Doob inequality, we arrive at

$$W_2^{\rho_\infty}(\Pi_{x,\psi}^T, \Pi_{y,\psi}^T)^2 \le \mathbb{E} \max_{t\in[0,T]} \rho(X_t, Y_t)^2 \le \mathrm{e}^{2\kappa_\psi T}\mathbb{E} \max_{t\in[0,T]} (M_t + \rho(x,y))^2$$
$$\le 4\mathrm{e}^{2\kappa_\psi T}\mathbb{E}(M_T + \rho(x,y))^2 = 4\mathrm{e}^{2\kappa_\psi T}\big(\mathbb{E}M_T^2 + \rho(x,y)^2\big)$$
$$= 4\mathrm{e}^{2\kappa_\psi T}\left(\rho(x,y)^2 + 2\|\nabla\psi\|_\infty^2 \int_0^T \mathrm{e}^{-2\kappa_\psi t}\mathbb{E}\rho(X_t,Y_t)^2 \mathrm{d}t\right)$$
$$\le 4\mathrm{e}^{2(\kappa_\psi + \|\nabla\psi\|_\infty^2)T}\rho(x,y)^2.$$

This implies the desired inequality for $\mu = \delta_x$ and $\nu = \delta_y$. □

4.5.2 *Non-convex manifolds*

As in §3.4.4, by combining Theorem 4.5.1 with a proper conformal change of metric, we are able to establish the following transportation-cost inequality on a class of manifolds with non-convex boundary. Let K_ϕ be in (3.2.15) and $K_\phi^- = 0 \vee (-K_\phi)$.

Theorem 4.5.3. *Let $\partial M \neq \emptyset$ with $\mathbb{I} \ge -\sigma$ for some constant $\sigma > 0$, and let $\mathrm{Ric}_Z \ge K$ hold for some $K \in \mathbb{R}$. For $\phi \in C_b^\infty(M)$ with $\phi \ge 1$, and $N \log\phi|_{\partial M} \ge \sigma$, let K_ϕ be in* (3.2.15). *Then for any $\mu \in \mathcal{P}(M)$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \le 2\|\phi\|_\infty^2 c(T,\phi)\Pi_\mu^T(F\log F), \quad F \ge 0, \Pi_\mu^T(F) = 1$$

holds for

$$c(T,\phi) = \inf_{R>0}\left\{(1+R^{-1})\frac{\mathrm{e}^{2\bar\kappa_\phi T} - 1}{\bar\kappa_\phi} \exp\left[4(1+R)\|\nabla\phi\|_\infty^2 \frac{\mathrm{e}^{2\bar\kappa_\phi T} - 1}{\bar\kappa_\phi}\right]\right\},$$

where

$$\bar\kappa_\phi := K_\phi^-\|\phi\|_\infty^2 + 2\|\phi Z + (d-2)\nabla\phi\|_\infty\|\nabla\phi\|_\infty\|\phi\|_\infty + (d-1)\|\nabla\phi\|_\infty.$$

In particular,

$$W_2^{\rho_\infty}(F\Pi_o^T, \Pi_o^T)^2 \le 2\|\phi\|_\infty^2 c(T,\phi)\Pi_o^T(F\log F), \quad o \in M, F \ge 0, \Pi_o^T(F) = 1.$$

Proof. Let $\langle\cdot,\cdot\rangle' = \phi^{-2}\langle\cdot,\cdot\rangle$. By Theorem 1.2.5, $(M, \langle\cdot,\cdot\rangle')$ is convex. According to the proof of Proposition 3.2.7, we have $L = \phi^{-2}(\Delta' + Z')$ and $\mathrm{Ric}'_{Z'} \ge K_\phi\langle\cdot,\cdot\rangle'$, where $Z' = \phi^2 Z + \frac{d-2}{2}\nabla\phi^2$. Letting κ_ψ be defined in Theorem 4.5.1 for the manifold $(M, \langle\cdot,\cdot\rangle')$ and $L = \psi^2(\Delta' + Z')$ with $\psi = \phi^{-1} \le 1$, we see that $\bar\kappa_\phi \le \kappa_\psi$ since

$$\|Z'\|' = \phi^{-1}\|Z'\|, \quad \|\nabla'\psi\|' = \phi^{-1}\|\nabla\phi\| \le \|\nabla\phi\|.$$

So, $C(T,\psi) \le c(T,\phi)$. Therefore, Theorem 4.5.1 yields

$$W_2^{\rho'_\infty}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \le 2c(T,\phi)\Pi_\mu^T(F\log F), \quad F \ge 0, \Pi_\mu^T(F) = 1,$$

where ρ'_∞ is the uniform distance on W^T induced by the metric $\langle\cdot,\cdot\rangle'$. The proof is completed by noting that $\rho_\infty \le \|\phi\|_\infty\rho'_\infty$. □

Similarly, since $\bar{\kappa}_\phi \le \kappa_\psi$ and

$$\rho' \le \rho \le \|\phi\|_\infty \rho',$$

the following result follows from Theorem 4.5.2 by taking $\psi = \phi^{-1}$.

Theorem 4.5.4. *In the situation of Theorem* 4.5.3,

$$W_2^{\rho_\infty}(\Pi_\mu^T, \Pi_\nu^T) \le 2\|\phi\|_\infty e^{(\bar{\kappa}_\phi + \|\nabla\phi^{-1}\|_\infty^2)T} W_2^\rho(\mu,\nu), \quad \mu,\nu \in \mathcal{P}(M), T > 0.$$

Chapter 5

Subelliptic Diffusion Processes

In this chapter we investigate hypoelliptic diffusion processes. §5.1 is devoted to functional inequalities, including super/weak Poincaré inequalities (§5.1.1) and Nash/log-Sobolev inequalities (§5.1.2). In §5.2 we introduce and apply the generalized curvature-dimension condition to the study of functional/Harnack/HWI inequalities. Finally, in §5.3-§5.5 we use Malliavin calculus and coupling arguments to derive explicit Bismut type formulae and Harnack inequalities.

Let M be a connected complete d-dimensional differentiable manifold without boundary. Consider the following second order differential operator on M:

$$L = \sum_{i=1}^{n} X_i^2 + X_0,$$

where $X_0, \dots, X_n$ are smooth vector fields on M. The associated square field of L is

$$\Gamma(f,g) := \sum_{i=1}^{n} (X_i f)(X_i g), \quad f, g \in C^1(M).$$

Throughout this chapter, we assume that L is subelliptic (also called hypoelliptic in references), i.e. the Lie algebra induced by the family $\{X_i, [X_0, X_i] : 1 \le i \le n\}$ equals to the whole tangent space at any point. This condition is known as the Hörmander condition due to the pioneering paper [Hörmander (1967)], where it is proved that this condition implies the existence of smooth heat kernel of the associated diffusion semigroup.

When the functional inequality is concerned, we assume that L is symmetric w.r.t. a probability measure μ and $\mathrm{Lie}\{X_i : 1 \le i \le n\} = TM$. In this case for any $x \in M$, there exists $k \ge 1$ such that the commutators of $\mathcal{H}_0 := \{X_i : 1 \le i \le n\}$ up to order k

$$\mathcal{H}_k := \big\{X_{i_0}, \dots, [X_{i_0}, [X_{i_1}, \dots, [X_{i_{k-1}}, X_{i_k}] \dots]] : 1 \le i_0, \dots, i_k \le n\big\}$$

spans T_xM. Let μ have a strictly positive and C^1-smooth density w.r.t. a Riemannian volume measure. Then the symmetry of L in $L^2(\mu)$ is equivalent to

$$L = \sum_{i=1}^{n} \{X_i^2 + (\mathrm{div}_\mu X_i) X_i\}, \tag{5.0.1}$$

where $\mathrm{div}_\mu X_i$ is the unique continuous function such that the integration by parts formula

$$\int_M (\mathrm{div}_\mu X_i) f \mathrm{d}\mu = -\int_M (X_i f) \mathrm{d}\mu, \quad f \in C_0^\infty(M)$$

holds. Then

$$-\int_M gLf \mathrm{d}\mu = \mu(\Gamma(f,g)), \quad f, g \in C_0^\infty(M)$$

and thus, the form

$$\mathcal{E}(f,g) := \mu(\Gamma(f,g)), \quad f,g \in C_0^\infty(M)$$

is closable in $L^2(\mu)$ and its closure is a symmetric Dirichlet form.

Before moving on, let us introduce some typical examples.

Example 5.0.1. (Gruschin operator) Let $M = \mathbb{R}^2$ and take

$$X = \frac{\partial}{\partial x}, \quad Y = x^k \frac{\partial}{\partial y},$$

where $k \in \mathbb{N}$. Obviously, the Hörmander condition holds for $\mathcal{H}_0 = \{X, Y\}$ with commutators up to order k. Then the Gruschin semigroup of order k is generated by $X^2 + Y^2$. Let $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ be a probability measure for some $V \in C^2(\mathbb{R}^2)$. Then the associated symmetric subelliptic diffusion operator is

$$L = X^2 + Y^2 + (XV)X + (YV)Y.$$

Example 5.0.2. (Kohn-Laplacian) Consider the three-dimensional Heisenberg group realized as $\mathbb{R}^3$ equipped with the group multiplication

$$(x,y,z)(x',y',z') := (x + x', y + y', z + z' + (xy' - x'y)/2),$$

which is a Lie group with left-invariant orthonormal frame $\{X, Y, Z\}$, where

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad Z = [X,Y] = \frac{\partial}{\partial z}.$$

Then the Kohn-Laplacian is $\Delta_H := X^2 + Y^2$. Let $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ be a probability measure for some $V \in C^2(\mathbb{R}^3)$. Then the associated symmetric subelliptic diffusion operator is $L = X^2 + Y^2 + (XV)X + (YV)Y$.

Example 5.0.3. (Stochastic Hamiltonian system) Let $m, d \geq 1$, $A \in \mathbb{R}^m \otimes \mathbb{R}^m$ and $B \in \mathbb{R}^m \otimes \mathbb{R}^d$ such that the Kalman rank condition (see [Kalman *et al* (1969)])

$$\mathrm{Rank}[B, AB, \ldots, A^k B] = m$$

holds for some $0 \leq k \leq m-1$. Then the operator

$$L = \frac{1}{2} \sum_{i=1}^{d} \left\{ \frac{\partial^2}{\partial y_i^2} - Z(x,y)\frac{\partial}{\partial y_i} + \big((Ax)_i + (By)_i\big)\frac{\partial}{\partial x_i} \right\}$$

generates a stochastic Hamiltonian system. See §5.3.1 and §5.4.1 for more details.

5.1 Functional inequalities

§5.1.1 is devoted to the weak and super Poincaré inequalities, while the Nash and log-Sobolev inequalities are investigated in §5.1.2. Main results presented in these two parts are illustrated by the Gruschin type and Kohn-Laplacian type operators in §5.1.3 and §5.1.4 respectively. Throughout this section, we use vol instead of $\mathrm{d}x$ to stand for a reference Riemannian volume measure on M. Assume that $\mathrm{d}\mu = \mathrm{e}^V \mathrm{dvol}$ is a probability measure on M for some $V \in C^2(M)$. Let L be given in (5.0.1) and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the associated Dirichlet form.

5.1.1 *Super and weak Poincaré inequalities*

In order to describe the essential spectrum $\sigma_{ess}(L)$ of $(L, \mathcal{D}(L))$, we shall establish the following Poincaré type inequality:

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(\phi|f|)^2, \quad r > r_0, f \in \mathcal{D}(\mathcal{E}), \tag{5.1.1}$$

where $r_0 \geq 0$ is a constant, $\phi > 0$ is in $L^2(\mu)$ and $\beta : (r_0, \infty) \to (0, \infty)$ is a positive (decreasing) function. Since the corresponding semigroup P_t has transition density with respect to μ, according to Corollary 1.6.5, $\sigma_{ess}(-L) \subset [r_0^{-1}, \infty)$ if and only if (5.1.1) holds for some ϕ and β specified above. In particular, $\sigma_{ess}(L) = \emptyset$ if and only if the super Poincaré inequality

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(\phi|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}), \tag{5.1.2}$$

holds for some positive function $\beta : (0, \infty) \to (0, \infty)$.

We will adopt a split argument, that is, the desired functional inequality follows from a local inequality and a Lyapunov type condition. We first prove the local Nash inequality. Recall that since $\mathcal{H}_0$ satisfies the Hörmander condition, for any compact domain K in M there exists $k \geq 1$ such that $\mathcal{H}_k(x)$ spans T_xM for any $x \in K$. In this case, we have the following the Hörmander inequality (see e.g. [Jacob (2002); Rothschild and Stein (1977)])

$$\int_K \left((1-\Delta)^{1/(2k)} f\right)^2 \mathrm{dvol} \leq C_K \int_K \left(\Gamma(f,f) + f^2\right) \mathrm{d\,vol}, \quad f \in C_0^\infty(K), \tag{5.1.3}$$

where $C_0^\infty(K)$ is the set of all smooth functions on M with supports contained in the interior of K.

Proposition 5.1.1. *Let K be a compact domain in M and $k \geq 1$ be such that $\mathcal{H}_k(x)$ spans T_xM for any $x \in K$. Then there exists a constant $C_K > 0$ such that*

$$\mu(f^2) \leq C_K \mathcal{E}(f,f)^{dk/(2+dk)}, \quad f \in C_0^\infty(K), \mu(|f|) = 1.$$

Proof. Since V is bounded on K, it suffices to prove for $V = 0$. By the classical Nash inequality on compact domains there exists a constant $c_0 > 0$ such that

$$\int_K f^2 \mathrm{d\,vol} \leq c_0 \left\{ \int_K \left((1-\Delta)^{1/2} f\right)^2 \mathrm{d\,vol} \right\}^{d/(2+d)}$$

holds for all $f \in C_0^\infty(K)$, $\int_M |f| \mathrm{d\,vol} = 1$. According to Theorem 1.3 in [Bendikov and Maheaux (2007)] for fractional Dirichlet forms, this implies

$$\int_K f^2 \mathrm{d\,vol} \leq c_1 \left\{ \int_K \left((1-\Delta)^{1/(2k)} f\right)^2 \mathrm{d\,vol} \right\}^{dk/(2+dk)}$$

holds for all $f \in C_0^\infty(K)$, $\int_M |f| \mathrm{d\,vol} = 1$ for some constant $c_1 > 0$. Combining this with (5.1.3) we obtain

$$\int_K f^2 \mathrm{d\,vol} \leq c_2 \left\{ \int_K \left(\Gamma(f,f) + f^2\right) \mathrm{d\,vol} \right\}^{dk/(2+dk)} \tag{5.1.4}$$

holds for all $f \in C_0^\infty(K)$, $\int_K |f| \mathrm{d\,vol} = 1$ for some $c_2 > 0$. So, to complete the proof, it remains to confirm the following local Poincaré inequality:

$$\int_K f^2 \mathrm{d\,vol} \leq c_3 \int_K \Gamma(f,f) \mathrm{d\,vol}, \quad f \in C_0^\infty(K) \tag{5.1.5}$$

for some constant $c_3 > 0$. To this end, let P_t^K be the Dirichlet heat semigroup generated by

$$L_0 := \sum_{i=1}^n \left\{X_i^2 + (\mathrm{div} X_i)X_i\right\} \tag{5.1.6}$$

on K. Then P_t^K is symmetric in $L^2(K;\mathrm{vol})$. Let $p_t > 0$ be the heat kernel of L_0 on M, for any $f \in L^\infty(K;\mathrm{vol})$ we have

$$\|P_1^K f\|_\infty \le \varepsilon \|f\|_\infty$$

for

$$\varepsilon := 1 - \inf_{x\in K} \int_{K^c} p_1(x,y)\mathrm{vol}(\mathrm{d}y) < 1.$$

This implies that P_t^K decays exponentially fast in $L^\infty(K;\mathrm{vol})$ as $t \to \infty$, and thus, so is in $L^2(K;\mathrm{vol})$. Therefore, (5.1.5) follows from e.g. the proof of Theorem 2.3 in [Röckner and Wang (2001)]. □

The next result is an extension of a classical estimate on the first Dirichlet eigenvalue for elliptic operators.

Lemma 5.1.2. *Let Ω be an open domain in M. If there exists a smooth function $\tilde\rho$ such that $\Gamma(\tilde\rho,\tilde\rho) \le 1$ and $|L\tilde\rho| \ge \theta > 0$ hold on Ω, then*

$$\mu(f^2) \le \frac{4}{\theta^2}\mathcal{E}(f,f), \quad f \in C_0^\infty(\Omega).$$

Proof. Without loss of generality, we assume that $L\tilde\rho \le -\theta$. Otherwise, just use $-\tilde\rho$ to replace $\tilde\rho$. So,

$$L\exp\left[\theta\tilde\rho/2\right] \le -\frac{\theta^2}{4}\exp\left[\theta\tilde\rho/2\right] \tag{5.1.7}$$

holds on Ω. Let $h := \exp\left[\theta\tilde\rho/2\right]$. Since

$$-\int_\Omega f_1 L f_2 \mathrm{d}\mu = \int_\Omega \Gamma(f_1,f_2)\mathrm{d}\mu, \quad f_1, f_2 \in C_0^\infty(\Omega),$$

it follows from (5.1.7) that

$$\begin{aligned}
\mu(\Gamma(f,f)) &= -\int_\Omega fLf\mathrm{d}\mu = -\int_\Omega fL\Big(\frac{f}{h}h\Big)\mathrm{d}\mu \\
&= -\int_\Omega \Big[\frac{f^2}{h}Lh + hfL\frac{f}{h} + 2f\Gamma\Big(\frac{f}{h},h\Big)\Big]\mathrm{d}\mu \\
&\ge \frac{\theta^2}{4}\mu(f^2) + \int_\Omega \Big[\Gamma\Big(hf,\frac{f}{h}\Big) - 2f\Gamma\Big(\frac{f}{h},h\Big)\Big]\mathrm{d}\mu \\
&= \frac{\theta^2}{4}\mu(f^2) + \int_\Omega h^2\Gamma\Big(\frac{f}{h},\frac{f}{h}\Big)\mathrm{d}\mu \ge \frac{\theta^2}{4}\mu(f^2), \quad f \in C_0^\infty(\Omega).
\end{aligned}$$

□

Recall that a function ϱ on M is called compact, if it has compact level sets; i.e. $\{\varrho \le r\}$ is compact for any $r \in \mathbb{R}$.

Theorem 5.1.3. *Let L be hypoelliptic. If there exists a smooth compact function ϱ such that $\Gamma(\varrho,\varrho) \le 1$ and*

$$\delta := \liminf_{\varrho\to\infty} |L\varrho| > 0,$$

then $\sigma_{ess}(-L) \subset [\delta^2/4, \infty)$. If in particular $\delta = \infty$ then $\sigma_{ess}(-L) = \emptyset$.

Proof. Let $\phi \in L^2(\mu)$ be locally uniformly positive. By Corollary 1.6.5, we only need to prove (5.1.1) for $r_0 = 4/\delta^2$ and some $\beta : (r_0, \infty) \to (0,\infty)$. For any $r > 4/\delta^2$ and any $\varepsilon \in (0,\delta)$ such that $r > 4/(\delta-\varepsilon)^2$, let $R_\varepsilon > 0$ be such that $|L\varrho| \ge \delta - \varepsilon$ on $\{\varrho > R_\varepsilon\}$. Then, by Lemma 5.1.2, one has

$$\mu(f^2) \le \frac{4}{(\delta-\varepsilon)^2}\mathcal{E}(f,f), \quad f \in C_0^\infty(\{\varrho > R_\varepsilon\}). \tag{5.1.8}$$

For any $N \ge 1$ and $f \in C_0^\infty(M)$, applying (5.1.8) to $f_1 := f\big(\frac{(\varrho-R_\varepsilon)^+}{N} \wedge 1\big)$ in place of f, we obtain

$$\mu(f_1^2) \le \frac{4(1+\varepsilon)}{(\delta-\varepsilon)^2}\mathcal{E}(f,f) + \frac{4(1+\varepsilon^{-1})}{N^2(\delta-\varepsilon)^2}\mu(f^2). \tag{5.1.9}$$

On the other hand, since $\{\varrho \le R_\varepsilon + N + 1\}$ is compact, by Proposition 5.1.1 there exists $c_0(\varepsilon, N), c_1(\varepsilon, N) > 0$ such that for any $f \in C_0^\infty(M)$, the function $f_2 := f\big((R_\varepsilon + N + 1 - \varrho)^+ \wedge 1\big)$ satisfies

$$\begin{aligned}\mu(f_2^2) &\le s\mathcal{E}(f_2, f_2) + c_1(\varepsilon,N)s^{-c_0(\varepsilon,N)}\mu(|f_2|)^2\\ &\le 2s\mathcal{E}(f,f) + 2s\mu(f^2) + c_2(\varepsilon,N)s^{c_0(\varepsilon,N)}\mu(|f|\phi)^2, \quad s>0\end{aligned}$$

for $c_2(\varepsilon, N) := c_1(\varepsilon,N)\sup_{\varrho\le R_\varepsilon+N+1}\phi^{-2} < \infty$. Combining this with (5.1.9) and noting that $f^2 \le f_1^2 + f_2^2$, we arrive at

$$\begin{aligned}\mu(f^2) \le {} & \Big(\frac{4(1+\varepsilon)}{(\delta-\varepsilon)^2} + 2s\Big)\mathcal{E}(f,f) + \Big(\frac{4(1+\varepsilon^{-1})}{N^2(\delta-\varepsilon)^2} + 2s\Big)\mu(f^2)\\ & + c_2(\varepsilon,N)s^{-c_0(\varepsilon,N)}\mu(|f|\phi)^2, \quad s>0.\end{aligned}$$

Taking N large enough and s small enough such that

$$u(\varepsilon,N,s) := \frac{4(1+\varepsilon^{-1})}{N^2(\delta-\varepsilon)^2} + 2s < 1,$$

we obtain

$$\begin{aligned}\mu(f^2) \le {} & \frac{1}{1-u(\varepsilon,N,s)}\Big(\frac{4(1+\varepsilon)}{(\delta-\varepsilon)^2} + 2s\Big)\mathcal{E}(f,f)\\ & + \frac{c_2(\varepsilon,N)}{1-u(\varepsilon,N,s)}s^{-c_0(\varepsilon,N)}\mu(|f|\phi)^2, \quad s>0.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0}\lim_{s\to 0}\lim_{N\to\infty}\frac{1}{1-u(\varepsilon,N,s)}\Big(\frac{4(1+\varepsilon)}{(\delta-\varepsilon)^2}+2s\Big)=\frac{4}{\delta^2}<r,$$

the set

$$A(r):=\Big\{(\varepsilon,N,s)\in(0,\delta)\times[1,\infty)\times(0,1):\ 2s+\frac{4(1+\varepsilon)}{(\delta-\varepsilon)^2}\le r(1-u(\varepsilon,N,s))\Big\}$$

is nonempty. So, (5.1.1) holds for

$$\beta(r):=\inf\Big\{\frac{c_2(\varepsilon,N)}{1-u(\varepsilon,N,s)}s^{-c_0(\varepsilon,N)}:\ (\varepsilon,N,s)\in A(r)\Big\}<\infty,\ r>\frac{4}{\delta^2}.\ \square$$

Let ϱ be a smooth compact function such that $a\le|\nabla\varrho|\le b$ for two constants $b>a>0$ and large ϱ. A very simple example for Theorem 5.1.3 to apply is that $\{X_1,\dots,X_n\}$ satisfies the Hörmander condition with $X_1=\nabla\varrho$. In this case, let

$$\psi(s)=\sup_{\varrho\le s}\Big|\sum_{i=1}^n\{X_i^2\varrho+(\mathrm{div}X_i)(X_i\varrho)\}\Big|,\quad s\ge 0.$$

Then the condition in Theorem 5.1.3 holds for $V:=\varphi\circ\varrho$ with

$$\liminf_{s\to\infty}\big\{a|\varphi'(s)|-\psi(s)\big\}>0.$$

When $L\varrho<0$ for large ϱ, we are able to extend Theorem 5.1.3 to the case where $\Gamma(\varrho,\varrho)$ is possibly unbounded.

Theorem 5.1.4. *Let L be hypoelliptic and let $\Lambda(s)=\sup_{\varrho\le s}\Gamma(\varrho,\varrho),\ s>0$, for ϱ be a smooth compact function. If*

$$\delta:=\liminf_{\varrho\to\infty}\frac{-L\varrho}{\sqrt{\Lambda(\varrho)}}>0,$$

then $\sigma_{ess}(-L)\subset[\delta^2/4,\infty)$.

Proof. Since ϱ is a smooth compact function, Λ is continuous. Moreover, Λ is nonnegative and increasing. So, for any $\varepsilon\in(0,\delta)$, by a classical approximation theorem, there exists an increasing smooth function Λ_ε such that $|\Lambda-\Lambda_\varepsilon|<\varepsilon$. Let

$$\tilde\varrho=\int_0^\rho\frac{1}{\sqrt{\Lambda_\varepsilon(s)+\varepsilon}}\mathrm{d}s.$$

We have $\Gamma(\tilde\varrho,\tilde\varrho)\le 1$ and there exists $R_\varepsilon>0$ such that

$$L\tilde\varrho=\frac{L\varrho}{\sqrt{\Lambda_\varepsilon(\varrho)+\varepsilon}}+\Gamma(\varrho,\varrho)\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\sqrt{\Lambda_\varepsilon(s)+\varepsilon}}\Big|_{s=\varrho}\le-(\delta-\varepsilon),\quad \rho\ge R_\varepsilon.$$

According to Lemma 5.1.2, this implies (5.1.8). The remainder of the proof is completely similar to the proof of Theorem 5.1.3. $\square$

Next, we consider the weak Poincaré inequality for μ being a probability measure:

$$\mu(f^2) \le \alpha(r)\mathcal{E}(f,f) + r\|f\|_\infty^2, \quad r>0, \mu(f)=0, \tag{5.1.10}$$

where $\alpha : (0,\infty) \to (0,\infty)$ is corresponding to the convergence rate of the associated semigroup (see §1.6.4). To estimate the function α, we consider below a special class of hypoelliptic operators on $\mathbb{R}^d$ with algebraic growth, more precisely:

(A5.1.1) There exist $r_0 \ge 0$ and $r_1, \dots, r_d > 0$ such that

$$\Gamma(f\circ\varphi_s, f\circ\varphi_s) = s^{r_0}\Gamma(f,f)\circ\varphi_s, \quad s>0, f\in C^1(\mathbb{R}^d) \tag{5.1.11}$$

holds for $\varphi_s(x_1,\dots,x_d) := (s^{r_1}x_1, \dots, s^{r_d}x_d)$.

Theorem 5.1.5. *Let $M=\mathbb{R}^d$ and $\mathrm{d}\mu = \mathrm{e}^{V(x)}\mathrm{d}x$ be a probability measure. Assume that Γ satisfies* **(A5.1.1)**. *Let $D_s = \{|x_i| \le s^{r_i} : 1\le i\le d\}$ and*

$$\delta_s(V) = \sup_{D_s} V - \inf_{D_{2s}} V, \quad s>0.$$

Then there exists a constant $c_0>0$ such that (5.1.10) *holds for*

$$\alpha(r) := c_0 \inf\left\{s^{r_0}\mathrm{e}^{\delta_s(V)} :\ 2\mu(D_s^c) \le r\wedge 1\right\}, \quad r>0.$$

Proof. To establish the weak Poincaré inequality, we need to estimate the local Poincaré constant. Let

$$\langle f\rangle_{D_s} = \frac{1}{|D_s|}\int_{D_s} f(x)\mathrm{d}x,$$

where $|D_s|$ is the volume of D_s. For any $s>0$, let $\gamma(s)>0$ be the smallest positive constant such that

$$\int_{D_s}\left(f(x)-\langle f\rangle_{D_s}\right)^2\mathrm{d}x \le \gamma(s)\int_{D_{2s}}\Gamma(f,f)(x)\mathrm{d}x, \quad f\in C^1(D_{2s}) \tag{5.1.12}$$

holds.

By the local Poincaré inequality implied by (5.1.3), we have $c_0 := \gamma(1) \in (0,\infty)$. Combining this with **(A5.1.1)**, we obtain

$$\int_{D_1}\left(f\circ\varphi_s(x)\right)^2\mathrm{d}x \le c_0 s^{r_0}\int_{D_2}\Gamma(f,f)\circ\varphi_s(x)\mathrm{d}x, \quad f\in C^1(D_{2s}), \langle f\rangle_{D_s}=0.$$

Therefore, (5.1.12) holds for $\gamma(s) = c_0 s^{r_0}$. Combining this with a simple perturbation argument, we obtain

$$\mu(f^2 1_{D_s}) \le c_0 s^{r_0}\mathrm{e}^{\delta_s(V)}\mu(\Gamma(f,f)) + \frac{\mu(f1_{D_s})^2}{\mu(D_s)}, \quad s>0.$$

Since $\mu(f) = 0$ implies

$$\frac{\mu(f1_{D_s})^2}{\mu(D_s)} = \frac{\mu(f1_{D_s^c})^2}{\mu(D_s)} \leq \|f\|_\infty^2 \mu(D_s^c)$$

provided $\mu(D_s^c) \leq \frac{1}{2}$, we obtain

$$\mu(f^2 1_{D_s}) \leq c_0 s^{r_0} e^{\delta_s(V)} \mu(\Gamma(f,f)) + \|f\|_\infty^2 \mu(D_s^c), \quad \mu(D_s^c) \leq \frac{1}{2}.$$

Thus,

$$\mu(f^2) \leq c_0 s^{r_0} e^{\delta_s(V)} \mu(\Gamma(f,f)) + 2\|f\|_\infty^2 \mu(D_s^c), \quad \mu(D_s^c) \leq \frac{1}{2}.$$

This implies the weak Poincaré inequality for the desired function α. □

5.1.2 *Nash and log-Sobolev inequalities*

We first establish the Nash inequality for $V = 0$, then derive the log-Sobolev inequality by a perturbation argument as in the elliptic case. To establish the Nash inequality, we will estimate the intrinsic distance induced by Γ and apply heat kernel upper bounds for the associated diffusion semigroup. To this end, we assume that the square field has an algebraic growth in the sense of **(A5.1.1)** and **(A5.1.2)** below.

(A5.1.2) $M = \mathbb{R}^d$ and there exists $\{m_j \geq 0 : \ 1 \leq j \leq d\}$ such that $m_{i_0} = 0$ for some $1 \leq i_0 \leq d$, and

$$\Gamma(f,f) \geq \theta_1^2 \sum_{j=1}^d |x_{i_0}|^{2m_j} (\partial_{x_j} f)^2, \quad |x|_\infty := \max_{1 \leq i \leq d} |x_i| \leq \varepsilon \tag{5.1.13}$$

holds for some constants $\theta_1, \varepsilon > 0$.

Next, for any $f \in C^1(\mathbb{R}^d)$, let

$$f_i(x) = f(i^{m_1+1}x_1, \ldots, i^{m_d+1}x_d), \quad i \geq 1, x = (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{5.1.14}$$

(A5.1.3) $M = \mathbb{R}^d$ and there exists a constant $\theta_2 > 0$ such that $\Gamma(f_i, f_i)(x) \leq \theta_2 i^2 \Gamma(f,f)(i^{m_1+1}x_1, \ldots, i^{m_d+1}x_d)$ holds for all $x \in \mathbb{R}^d, i \geq 1$, and $f \in C^1(\mathbb{R}^d)$.

5.1.2.1 *Heat kernel estimate*

Let us first recall a known heat kernel upper bound for hypoelliptic diffusions on a compact connected Riemannian manifold M. Let $\{X_1, \dots, X_n\}$ be a family of smooth vector fields on M satisfying the Hörmander condition.

For any $x \in M$ and $v \in T_xM$, let

$$|v|_\Gamma = \sup\{|vf|(x) : \ f \in C^1(M), \Gamma(f,f)(x) \le 1\}.$$

For any smooth curve $\gamma : [0,r] \to M$ linking two points x, y, the intrinsic length of γ induced by these vector fields is

$$\ell_\Gamma(\gamma) := \int_0^r |\dot\gamma_s|_\Gamma \mathrm{d}s.$$

The intrinsic distance $\rho_\Gamma(x,y)$ between x and y is defined as the infimum over the intrinsic lengths of all smooth curves linking x and y. Recall that the Hörmander condition implies $\rho_\Gamma < \infty$.

An equivalent definition of ρ_Γ is given by using subunit curves. A C^1-curve $\gamma : [0,r] \to M$ is called subunit, if

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \le \sqrt{\Gamma(f,f)(\gamma_t)}, \quad f \in C^1(M), t \in [0,r].$$

Then

$$\rho_\Gamma(x,y) = \inf\big\{r > 0 : \gamma : [0,r] \to M \text{ is subunit}, \gamma_0 = x, \gamma_r = y\big\}.$$

Moreover, we have

$$\rho_\Gamma(x,y) = \sup\big\{|f(x) - f(y)| : \ f \in C^1(M), \Gamma(f,f) \le 1\big\}.$$

Let $p_t(x,y)$ be the heat kernel of the operator L_0 given in (5.1.6) on M. If M is compact, then due to Lemma 8 in [Fefferman and Sánchez-Calle (1986)] (see also [Jerison and Sánchez-Calle (1986)]), there exists a constant $C > 0$ such that

$$p_t(x,x) \le \frac{C}{\mathrm{vol}(B_\Gamma(x, t^{1/2}))}, \quad t > 0, x \in M, \tag{5.1.15}$$

where vol is the Riemannian volume measure and $B_\Gamma(x,r) := \{\rho_\Gamma(x,\cdot) < r\}$.

We intend to extend this estimate to the Dirichlet heat kernel on a bounded domain when M is non-compact. For any open domain $\Omega \subset M$, let p_t^Ω be the Dirichlet heat kernel of L_0 on Ω.

Lemma 5.1.6. *Let $\Omega \subset \Omega_1$ be two bounded open domains in M such that $\bar\Omega \subset \Omega_1$ and Ω_1 is diffeomorphic to the unit open ball in $\mathbb{R}^d$. Then there exists a constant $C > 0$ and $t_0 > 0$ such that*

$$p_t^\Omega(x,x) \le \frac{C}{\mathrm{vol}(B_\Gamma(x, t^{1/2}))}, \quad t \in (0, t_0], x \in \Omega.$$

Proof. Let $r_0 > 0$ such that

$$\Omega' := \Big\{ y \in M : \inf_{x \in \Omega} \rho_\Gamma(x, y) \le r_0 \Big\} \subset \Omega_1.$$

Let $\tilde{\Omega}_1$ be an open geodesic ball in $\mathbb{S}^d$. Then there exists a diffeomorphism

$$\varphi : \Omega_1 \to \tilde{\Omega}_1.$$

In particular, we take a Riemannian metric $\tilde{g}$ on $\mathbb{S}^d$ such that φ is indeed isoperimetric. Let $\widetilde{\text{vol}}$ be the associated Riemannian volume measure. So, the vector fields $\{\varphi^*(X_i)\}$ satisfies the Hörmander condition on $\tilde{\Omega}_1$. Let $h \in C^\infty(\mathbb{S}^d)$ such that

$$0 \le h \le 1, \quad h|_{\varphi(\Omega')} = 1, \quad h|_{\tilde{\Omega}_1^c} = 0.$$

Moreover, let $\{Y_1, \dots, Y_m\}$ be vector fields on $\mathbb{S}^d$ which span the tangent space at any point. Then

$$\mathcal{H} := \{h\varphi^*(X_i), (1 - h)Y_j : 1 \le i \le n, 1 \le j \le m\}$$

satisfies the Hörmander condition on $\mathbb{S}^d$. Let $\tilde{\rho}$ be the corresponding intrinsic distance. It is easy to see that

$$\rho_\Gamma(x, y) = \tilde{\rho}(\varphi(x), \varphi(y)), \quad x \in \Omega, y \in \Omega'. \tag{5.1.16}$$

Let $\tilde{p}_t$ be the heat kernel of the self-adjoint operator

$$\tilde{L} := \sum_{X \in \mathcal{H}} \{X^2 + (\text{div}X)X\}$$

on $(\mathbb{S}^d, \tilde{g})$. Due to (5.1.15) one has

$$\tilde{p}_t(\tilde{x}, \tilde{x}) \le \frac{C}{\widetilde{\text{vol}}(B_{\tilde{\rho}}(\tilde{x}, t^{1/2}))}, \quad t > 0, \tilde{x} \in \mathbb{S}^d, \tag{5.1.17}$$

where $B_{\tilde{\rho}}(\tilde{x}, r) := \{\tilde{\rho}(\tilde{x}, \cdot) < r\}$ for $r > 0$. Since

$$\tilde{L}f = \{L(f \circ \varphi)\} \circ \varphi^{-1}, \quad f \in C_0^\infty(\varphi(\Omega)),$$

one has

$$p_t^\Omega(x, y) = \tilde{p}_t^{\varphi(\Omega)}(\varphi(x), \varphi(y)),$$

where $\tilde{p}_t^{\varphi(\Omega)}$ is the Dirichlet heat kernel of $\tilde{L}$ on $\varphi(\Omega)$, which is smaller than $\tilde{p}_t$. Thus, the desired assertion follows from (5.1.16) and (5.1.17) for $t_0 = r_0^2$ by noting that φ is isoperimetric. $\square$

According to Lemma 5.1.6, to obtain an upper bound of p_t^Ω depending only on t, we need to estimate the intrinsic distance.

Lemma 5.1.7. *Let $\{X_i\}$ satisfy* (**A5.1.2**) *with $m_1 = 0$. Then*

$$\rho_\Gamma(x,y) \le \frac{|x_1 - y_1|}{\theta_1} + \frac{1}{\theta_1} \inf_{r\in(0,\varepsilon)} \Big\{2r + \sum_{i=2}^d \frac{|x_i - y_i|}{r^{m_i}}\Big\}, \quad |x|_\infty, |y|_\infty < \varepsilon.$$

Proof. For fixed x, y with $|x|_\infty, |y|_\infty < \varepsilon$, let

$$x^{(i)} = (y_1, \dots, y_{i-1}, x_i, \dots, x_d), \quad 1 \le i \le d+1.$$

In particular, $x^{(1)} = x, x^{(d+1)} = y$. Taking $\gamma_s = (sx_1 + (1-s)y_1, x_2, \dots, x_d)$ for $s \in [0,1]$, by (**A5.1.2**) with $m_1 = 0$ we obtain

$$\Gamma(f,f) \ge \theta_1^2 |\partial_{x_1} f|^2 = \frac{\theta_1^2}{|x_1 - y_1|^2} \Big|\frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_s)\Big|^2.$$

Thus,

$$\rho_\Gamma(x, x^{(2)}) \le \frac{|x_1 - y_1|}{\theta_1}. \tag{5.1.18}$$

Next, for any $x_1 \ne 0$, let

$$\gamma_i(s) = (1-s)x^{(i)} + s x^{(i+1)}, \quad s \in [0,1], \quad i \ge 2.$$

Similarly, we have

$$\Gamma(f,f) \ge \theta_1^2 |x_1|^{2m_i} |\partial_{x_i} f|^2 = \frac{\theta_1^2 |x_1|^{2m_i}}{|x_i - y_i|^2} \Big|\frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_i(s))\Big|^2.$$

Then

$$\rho_\Gamma(x^{(i)}, x^{(i+1)}) \le \frac{|x_i - y_i|}{\theta_1 |x_1|^{m_i}}, \quad 2 \le i \le d.$$

If $x_1 \ne 0$, this and (5.1.18) yield

$$\rho_\Gamma(x,y) \le \frac{|x_1 - y_1|}{\theta_1} + \frac{1}{\theta_1} \sum_{i=2}^d \frac{|x_i - y_i|}{|x_1|^{m_i}}. \tag{5.1.19}$$

Moreover, for any $x_1 \in \mathbb{R}$ and $r \in (0,\varepsilon)$, let $\bar{x}_1 \in \mathbb{R}$ be such that $|\bar{x}_1 - x_1| \le r, |\bar{x}_1| \ge r$. Let $\bar{x} = (\bar{x}_1, x_2, \dots, x_d), \bar{y} = (\bar{x}_1, y_2, \dots, y_d)$. It follows from (5.1.19) that

$$\begin{aligned}\rho_\Gamma(x,y) &\le \rho_\Gamma(x,\bar{x}) + \rho_\Gamma(\bar{x},\bar{y}) + \rho_\Gamma(\bar{y},y) \\ &\le \frac{2r + |x_1 - y_1|}{\theta_1} + \frac{1}{\theta_1} \sum_{i=2}^d \frac{|x_i - y_i|}{r^{m_i}}.\end{aligned}$$

So, the proof is completed. □

Combining Lemmas 5.1.6 and 5.1.7 we obtain the following main result of this section.

Proposition 5.1.8. *Let $\{X_i\}_{i=1}^n$ satisfy the Hörmander condition and* **(A5.1.2), (A5.1.3)**. *Then for any open domain $\Omega \subset \{x \in \mathbb{R}^d : |x|_\infty < \frac{\varepsilon}{2}\}$, there exists a constant $C > 0$ such that*

$$\sup_{x\in\Omega} p_t^\Omega(x,x) \le Ct^{-(d+m_1+\ldots+m_d)/2}, \quad t > 0, x \in \Omega.$$

Proof. Without loss of generality, we assume that $m_1 = 0$. Let t_0, λ be in Lemmas 5.1.6 and 5.1.7. Let $c_1 = \max_{2\le i\le d}\theta_1^{-(1+m_i)}$ and take $t_1 \in (0, t_0 \wedge 1]$ such that

$$(\theta_1 + c_1^{-1})\sqrt{t} < \frac{\varepsilon}{2}.$$

Since $|x|_\infty < \frac{\varepsilon}{2}$, Lemma 5.1.7 with $r := \theta_1 t^{1/2}/4$ implies that for any $t \in (0, t_1]$,

$$\begin{aligned} B_\Gamma(x, t^{1/2}) &\supset \left\{y \in \mathbb{R}^d : \ \frac{|x_1 - y_1|}{\theta_1} + c_1 \sum_{i=2}^d \frac{|y_i - x_i|}{t^{m_i/2}} < \frac{t^{1/2}}{2}\right\} \\ &\supset \left\{y \in \mathbb{R}^d : \ |x_1 - y_1| < \frac{\theta_1 t^{1/2}}{2d}, \ |x_i - y_i| \le \frac{t^{(m_i+1)/2}}{2dc_1}, \quad 2 \le i \le d\right\}. \end{aligned}$$

Thus, there exists a constant $c_2 > 0$ such that

$$\mathrm{vol}\big(B_\Gamma(x, t^{1/2})\big) \ge c_2 t^{(d+m_1+\ldots+m_d)/2}, \ \ t \in (0, t_1].$$

Hence, the desired estimate holds for $t \in (0, t_1]$. To complete the proof, we only need to show that

$$\sup_{x\in\Omega} p_t^\Omega(x,x) \le c_3 \mathrm{e}^{-\lambda_0 t}, \quad t \ge t_1 \tag{5.1.20}$$

holds for some constants $c_3, \lambda_0 > 0$. To this end, let P_t^Ω be the Dirichlet semigroup of L on Ω. Since the semigroup generated by L on $\mathbb{R}^d$ has a positive heat kernel $p_t(x, y)$,

$$\varepsilon_1 := \inf_{\Omega} \int_{\Omega^c} p_{t_1/2}(x, y)\mathrm{vol}(\mathrm{d}y) > 0.$$

Since $p_{t_1}^\Omega \le p_{t_1}$, this implies that

$$\|P_{t_1/2}^\Omega\|_{L^\infty(\Omega)\to L^\infty(\Omega)} = \sup_{x\in\Omega}\int_\Omega p_{t_1/2}^\Omega(x,y)\mathrm{vol}(\mathrm{d}y) \le 1 - \varepsilon_1 < 1.$$

Therefore, by the semigroup property,

$$\|P_t^\Omega\|_{L^\infty(\Omega)\to L^\infty(\Omega)} \le c_1 \mathrm{e}^{-\lambda_0 t}, \quad t \ge \frac{t_1}{2} \tag{5.1.21}$$

holds for some $c_1, \lambda_0 > 0$. Moreover, by the local Nash inequality in Proposition 5.1.1, one has $\|P_t^\Omega\|_{L^1(\Omega)\to L^\infty(\Omega)} < \infty$ for all $t > 0$. Combining this with (5.1.21) we obtain

$$\begin{aligned}\|P_t^\Omega\|_{L^1(\Omega)\to L^\infty(\Omega)} &\le \|P_{t_1/2}^\Omega\|_{L^1(\Omega)\to L^\infty(\Omega)}\|P_{t-t_1/2}^\Omega\|_{L^\infty(\Omega)\to L^\infty(\Omega)}\\ &\le c\mathrm{e}^{-\lambda_0 t}, \quad t \ge t_1\end{aligned}$$

for some constant $c > 0$. Thus, (5.1.20) holds. □

5.1.2.2 *Nash and log-Sobolev inequalities*

It is well known that the uniform heat kernel upper bound implies a Nash inequality. To derive the log-Sobolev inequality from the Nash inequality, we present below a perturbation result for Hörmander diffusions on manifolds.

Proposition 5.1.9. *Let Γ be the square field associated to vector fields $\{X_i\}_{i=1}^n$ on a connected complete Riemannian manifold M satisfying the Hörmander condition. Let $\mathrm{d}\mu_0 = \mathrm{e}^{V_0}\mathrm{d}\,\mathrm{vol}$ for some $V_0 \in C^2(M)$ such that*

$$\mu_0(f^2) \le C\mu_0(\Gamma(f,f) + f^2)^{m/(m+2)}, \quad f \in C_0^1(M), \mu_0(|f|) = 1 \quad (5.1.22)$$

holds for some $C, m > 0$. Let $V \in C^2(M)$ such that $\mathrm{d}\mu := \mathrm{e}^V\mathrm{d}\,\mu_0$ is a probability measure.

(1) *If there exists $\delta > 0$ such that*

$$\mu_0(\exp[\delta(\Gamma(V,V) - 2LV) - V]) < \infty,$$

then there exists a constant $C > 0$ such that

$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f,f)), \quad f \in C_0^1(M), \mu(f^2) = 1.$$

(2) *If for any $s > 0$*

$$U(s) := \mu_0\big(\exp[s(\Gamma(V,V) - 2LV) - V]\big) < \infty,$$

then there exists a constant $c_0 > 0$ such that

$$\mu(f^2 \log f^2) \le r\mu(\Gamma(f,f)) + c_0 + m \log \frac{1}{r \wedge 1} + \log U(r/4)$$

holds for all $r > 0$ and $f \in C_0^1(M)$ with $\mu(f^2) = 1$.

Proof. By [Bakry *et al* (1995)], the Nash inequality (5.1.22) implies

$$\mu_0(f^2 \log f^2) \le \frac{m}{2} \log\big\{a\mu_0(\Gamma(f,f)) + b\big\}, \quad f \in C_0^1(M), \mu_0(f^2) = 1$$

for some constants $a, b > 0$. Thus, there exists $c > 0$ such that

$$\mu_0(f^2 \log f^2) \leq \frac{r}{2}\mu_0(\Gamma(f,f)) + c + \frac{m}{2}\log(r^{-1} \vee 1)$$

for all $r > 0$, $f \in C_0^1(M)$, $\mu_0(f^2) = 1$. Replacing f by $f\mathrm{e}^{V/2}$ for $f \in C_0^1(M)$ with $\mu(f^2) = 1$, and noting that

$$\begin{aligned}
&\mu_0(f^2\mathrm{e}^V \log(f^2\mathrm{e}^V)) = \mu(f^2 \log f^2) + \mu(f^2 V),\\
&\mu_0(\Gamma(f\mathrm{e}^{V/2}, f\mathrm{e}^{V/2})) = \mu_0\Big(\mathrm{e}^V\Big\{\Gamma(f,f) + f\Gamma(f,V) + \frac{1}{4}f^2\Gamma(V,V)\Big\}\Big)\\
&= \mu(\Gamma(f,f)) + \frac{1}{2}\mu(\Gamma(f^2,V)) + \frac{1}{4}\mu(f^2\Gamma(V,V))\\
&= \mu(\Gamma(f,f)) + \frac{1}{4}\mu\big(f^2\{\Gamma(V,V) - 2LV\}\big),
\end{aligned}$$

we arrive at

$$\begin{aligned}
\mu(f^2 \log f^2) &\leq \frac{r}{2}\mu(\Gamma(f,f)) + \mu\big(f^2\big\{\frac{r}{8}[\Gamma(V,V) - 2LV] - V\big\}\big)\\
&\qquad + c + \frac{m}{2}\log(r^{-1} \vee 1)\\
&\leq \frac{1}{2}\mu(f^2 \log f^2) + \frac{r}{2}\mu(\Gamma(f,f)) + c + \frac{m}{2}\log(r^{-1} \vee 1)\\
&\qquad + \frac{1}{2}\log \mu_0\big(\mathrm{e}^{\frac{r}{4}[\Gamma(V,V) - 2LV] - V}\big).
\end{aligned}$$

This implies (2).

If the condition in (1) holds, taking $r = 4\delta$ in the above display we obtain the defective log-Sobolev inequality:

$$\mu(f^2 \log f^2) \leq C_1\mu(\Gamma(f,f)) + C_2, \quad f \in C_0^1(M), \mu(f^2) = 1$$

for some $C_1, C_2 > 0$. Since due to the Hörmander theorem, the operator

$$\sum_{i=1}^n \{X_i^2 + (X_i(V + V_0))X_i\}$$

has positive heat kernel so that the corresponding semigroup is uniformly positivity improving, according to [Aida (1998)], this defective log-Sobolev inequality implies the exact one. □

Theorem 5.1.10. *Let $\{X_i\}_{i=1}^n$ on $\mathbb{R}^d$ satisfy the Hörmander condition such that* **(A5.1.2)** *and* **(A5.1.3)** *hold.*

(1) *Let $m = d + \sum_{i=1}^d m_i$. There exists a constant $C > 0$ such that*

$$\mathrm{vol}(f^2) \leq C\mathrm{vol}(\Gamma(f,f))^{m/(2+m)}\mathrm{vol}(|f|)^{4/(2+m)} \tag{5.1.23}$$

holds for all $f \in C_0^1(\mathbb{R}^d)$.

(2) *If there exists $\delta > 0$ such that*

$$\mathrm{vol}(\exp[-V - \delta(\Gamma(V,V) + 2L_0V)]) < \infty,$$

then there exists a constant $C > 0$ such that

$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f,f)), \quad f \in C_0^1(\mathbb{R}^d), \mu(f^2) = 1. \tag{5.1.24}$$

(3) *If for any $s > 0$*

$$U(s) := \mathrm{vol}\big(\exp[-V - s(\Gamma(V,V) + 2L_0V)]\big) < \infty,$$

then there exists a constant $c_0 > 0$ such that

$$\mu(f^2 \log f^2) \le r\mu(\Gamma(f,f)) + c_0 + \Big(d + \sum_{j=1}^d m_j\Big) \log \frac{1}{r \wedge 1} + \log U(r/4)$$

holds for all $r > 0$ and $f \in C_0^1(\mathbb{R}^d)$ with $\mu(f^2) = 1$.

Proof. By Proposition 5.1.9 with $V_0 = 0$ (i.e. $\mu_0 = \mathrm{vol}$), and noting that $LV = L_0V + \Gamma(V,V)$, it suffices to prove the first assertion. Let

$$B_s = \big\{x \in \mathbb{R}^d : \ 2|x_j| < (\varepsilon s)^{m_j+1}, 1 \le j \le d\big\}, \quad s > 0.$$

By Proposition 5.1.8, there exists a constant $C > 0$ such that (cf. Theorem 2.4.6 in [Davies (1989)])

$$\mathrm{vol}(f^2) \le C\mathrm{vol}(\Gamma(f,f))^{m/(2+m)}\mathrm{vol}(|f|)^{4/(2+m)}, \quad f \in C_0^1(B_1). \tag{5.1.25}$$

Now, for any $f \in C_0^1(\mathbb{R}^d)$, there exists $i \ge 1$ such that $f \in C_0^1(B_i)$. We have $f_i \in C_0^1(B_1)$ and

$$\mathrm{vol}(f_i^2) = i^{-m}\mathrm{vol}(f^2), \quad \mathrm{vol}(|f_i|) = i^{-m}\mathrm{vol}(|f|). \tag{5.1.26}$$

Combining this with **(A5.1.3)** and (5.1.25), we obtain

$$\begin{aligned}\mathrm{vol}(f^2) &\le Ci^m\big(i^{2-m}\mathrm{vol}(\Gamma(f,f))\big)^{m/(2+m)}\big(i^{-m}\mathrm{vol}(|f|)\big)^{4/(2+m)}\\ &= C\mathrm{vol}(\Gamma(f,f))^{m/(2+m)}\mathrm{vol}(|f|)^{4/(2+m)}.\end{aligned}$$

This implies (5.1.23). Hence, the proof of (1) is completed. □

To see that Theorem 5.1.10 applies to a reasonable class of Hörmander type operators, we consider below a specific class of vector fields. Let $d = d_1 + d_2$ and at point $(x,y) \in \mathbb{R}^{d_1+d_2}$,

$$X_i = \sum_{j=1}^{d_1} \sigma_{ij}\partial_{x_j} + \sum_{j=1}^{d_2} h_{ij}(x)\partial_{y_j}, \quad i = 1, \dots, n, \tag{5.1.27}$$

where $\sigma := (\sigma_{ij})_{n\times d_1}$ is a matrix such that $a := \sigma^*\sigma$ is strictly positive definite, and $\{h_{ij}\}$ are homogenous functions on $\mathbb{R}^{d_1}$ such that

$$h_{ij}(sx) = s^{l_j}h_{ij}(x), \quad s>0, x\in\mathbb{R}^{d_1}, 1\le i\le n, 1\le j\le d_2 \tag{5.1.28}$$

holds for some constants $\{l_j\ge 0\}$ and

$$\inf_{|x|\ge 1, |v|\ge 1}\sum_{i=1}^n\sum_{j,l=1}^{d_2} h_{ij}(x)h_{il}(x)v_jv_l > 0. \tag{5.1.29}$$

Corollary 5.1.11. *Let $\{X_i\}_{i=1}^n$ be in* (5.1.27) *satisfying the Hörmander condition such that $\sigma^*\sigma$ is strictly positive definite and* (5.1.28) *and* (5.1.29) *hold. Then* (**A5.1.2**) *and* (**A5.1.3**) *hold with $d = d_1+d_2$, $m_i = 0$ for $1\le i\le d_1$ and $m_{d_1+i} = l_i$ for $1\le i\le d_2$. In particular,* (5.1.23) *holds for some constant $C>0$ if and only if $m = d_1+d_2+\sum_{i=1}^{d_2} l_i$.*

Proof. Obviously, **(A5.1.2)** follows from (5.1.28) and (5.1.29). Next, let f_N be in (5.1.14). By (5.1.28)

$$\begin{aligned}
&\Gamma(f_N,f_N)(x,y)\\
&=\sum_{i=1}^n\Big\{\sum_{j=1}^{d_1} N\sigma_{ij}\partial_{x_j}f + Nh_{ij}\partial_{y_j}f\Big\}^2(Nx, N^{l_1+1}y_1,\dots,N^{l_{d_2}+1}y_{d_2})\\
&= N^2\Gamma(f,f)_N(x,y), \quad (x,y)\in\mathbb{R}^{d_1+d_2}.
\end{aligned}$$

Thus, **(A5.1.3)** holds. Moreover, assume that (5.1.23) holds for some $m>0$, it suffices to prove that $m = d_1+d_2+\sum_{i=1}^{d_2} l_i$. By an approximation argument we are able to apply (5.1.23) to the function

$$f_{(s)}(x,y) := (s-|x|)^+(s^{l_1+1}-|y_1|)^+\dots(s^{l_{d_2}+1}-|y_{d_2}|)^+, \quad (x,y)\in\mathbb{R}^{d_1+d_2}$$

for $s>0$. Obviously, there exist $c_1,c_2>0$ such that for $s>0$,

$$\begin{aligned}
&\mathrm{vol}(f_{(s)}^2)\ge c_1 s^{2+d_1+3d_2+3l_1+\dots+3l_{d_2}},\\
&\mathrm{vol}(|f_{(s)}|)\le c_2 s^{1+d_1+2d_2+2l_1+\dots+2l_{d_2}}.
\end{aligned}\tag{5.1.30}$$

Finally, by (5.1.27) and (5.1.28) there exists $c_3>0$ such that

$$\begin{aligned}
\Gamma(f_{(s)},f_{(s)}) &\le c_3\big\{1_{\{|x|<s\}}(s^{l_1+1}-|y_1|)^+\dots(s^{l_{d_2}+1}-|y_{d_2}|)^+\big\}^2\\
&\quad + c_3\Big\{\sum_{i=1}^{d_2} 1_{\{|y_i|<s^{l_i+1}\}}\frac{f_{(s)}s^{l_i}}{(s^{l_i+1}-|y_i|)^+}\Big\}^2.
\end{aligned}$$

So, there exists $c_4>0$ such that

$$\mathrm{vol}(\Gamma(f_{(s)},f_{(s)}))\le c_4 s^{d_1+3d_2+3l_1+\dots+3l_{d_2}}, \quad s>0.$$

Therefore, it follows from (5.1.23) that

$$s^{2+d_1+3d_2+3l_1+\ldots+3l_{d_2}}$$
$$\leq C' s^{(d_1+3d_2+3l_1+\ldots+3l_{d_2})m/(2+m)+(1+d_1+2d_2+2l_1+\ldots+2l_{d_2})4/(2+m)}, \quad s>0$$

holds for some $C'>0$. Therefore,

$$2+d_1+3d_2+3l_1+\ldots+3l_{d_2}$$
$$=\frac{m(d_1+3d_2+3l_1+\ldots+3l_{d_2})+4(1+d_1+2d_2+2l_1+\ldots+2l_{d_2})}{2+m},$$

which implies $m=d_1+d_2+l_1+\ldots+l_{d_2}$. □

As a generalization to the known log-Sobolev inequality for $V=-c\rho^2$ on a Riemannian manifold with curvature bounded below, where ρ is the Riemannian distance function to a fixed point (cf. Corollary 1.6 in [Wang (2001)]), we present below a corollary for hypoelliptic operators.

Corollary 5.1.12. *Let* $\{X_i\}_{i=1}^n$ *satisfy the conditions of Theorem* 5.1.10. *Let* $\rho\in C^2(\mathbb{R}^d)$ *be nonnegative such that* $\mathrm{vol}(\exp[-\varepsilon\rho^2])<\infty$ *for any* $\varepsilon>0$, *and*

$$\Gamma(\rho,\rho)\geq\theta_1,\quad L_0\rho\leq\theta_2(1+\rho^{-1}) \tag{5.1.31}$$

for some constants $\theta_1,\theta_2>0$. *Let* $V=c(\delta)-\rho^\delta$ *for some constants* $\delta, c(\delta)>0$ *such that* μ *is a probability measure.*

(1) *If* $\delta\geq 2$ *then there exists a constant* $C>0$ *such that* (5.1.24) *holds.*
(2) *If* $\delta>2$ *then*

$$\mu(f^2\log f^2)\leq r\mu(\Gamma(f,f))+cr^{-\delta/(\delta-2)},\quad r>0, f\in C_0^1(\mathbb{R}^d) \tag{5.1.32}$$

holds for some $c>0$. *Consequently, the associated semigroup* P_t *is ultracontractive with*

$$\|P_t\|_{L^1(\mu)\to L^\infty(\mu)}\leq\exp[c'(1+t^{-\delta/(\delta-1)})],\quad t>0$$

for some $c'>0$.

Proof. We may assume that $\rho\geq 1$ by using $\rho+1$ to replace ρ. Obviously, (5.1.31) implies

$$\Gamma(V,V)+2L_0V\geq\delta_1\delta^2\rho^{2(\delta-1)}-\delta_2(\rho^{\delta-2}+\rho^{\delta-1}) \tag{5.1.33}$$

for some constants $\delta_1,\delta_2>0$. So, if $\delta\geq 2$ then $-[\Gamma(V,V)+2L_0V]\leq c_1+c_2V$ holds for some $c_1,c_2>0$. Thus, (5.1.24) holds according to Theorem 5.1.10(2) and the assumption that $\mathrm{vol}(\exp[-\varepsilon\rho^2])<\infty$ for any $\varepsilon>0$.

Next, if $\delta>2$ then (5.1.33) implies

$$s(\Gamma(V,V)+2L_0V)\geq -2V-c_3s^{-\delta/(\delta-2)}-c_4,\quad s>0$$

for some constants $c_3,c_4>0$. This implies $U(s)\leq\exp[c_3s^{-\delta/(\delta-2)}+c_5]$ for some $c_5>0$. Therefore, (5.1.32) follows from Theorem 5.1.10(3) and (5.1.24). □

5.1.3 *Gruschin type operator*

In this part we consider the Gruschin type operator L given in Example 5.0.1.

5.1.3.1 *Weak Poincaré inequality*

Obviously, **(A5.1.1)** holds for $r_0 = 2, r_1 = 1, r_2 = k+1$. So, Theorem 5.1.5 applies for $r_0 = 2$ and

$$D_s := \{|x| \le s, |y| \le s^{k+1}\}, \quad s > 0.$$

In particular, we obtain explicit algebraic convergence rate for the following example. Let

$$V(x,y) = c_0 - \delta \log(x^2+1) - \frac{k+2\delta}{2(k+1)} \log(1+y^2)$$

for some $\delta > 1/2$. Then for some $c_0 \in \mathbb{R}$

$$\mathrm{d}\mu = \mathrm{e}^{c_0}(1+x^2)^{-\delta}(1+y^2)^{-(k+2\delta)/2(k+1)}\,\mathrm{d}x\,\mathrm{d}y$$

is a probability measure. Next, there exists a constant $c_1 > 0$ such that

$$\mu(D_s^c) \le \mu(|x| > s) + \mu(|y| > s^{k+1}) \le c_1 s^{1-2\delta}, \quad s > 0.$$

Moreover, there exists a constant $c_2 > 0$ such that

$$\mathrm{e}^{\delta_s(V)} \le c_2 s^{k+4\delta}, \quad s \ge 1.$$

Then by Theorem 5.1.5 the weak Poincaré inequality holds for

$$\alpha(r) = c_3\big(1 + r^{-(2+k+4\delta)/(2\delta-1)}\big), \quad r > 0$$

for some constant $c_3 > 0$. Therefore, it follows from Theorem 1.6.14 for $\mathbb{H} = \{f \in L^2(\mu) : \mu(f) = 0\}$ that

$$\|P_t - \mu\|_{\infty\to 2} \le ct^{-(2\delta-1)/(2+k+4\delta)}, \quad t > 0.$$

5.1.3.2 *Super Poincaré inequality*

For simplicity, we only consider $k = 1$ so that $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$. Let $V(x,y) = \xi(a + (c+x)^2)^l(b+y)^m$ for some constants $a, b, l, m > 0$ and $\xi, c \ne 0$. Then $\sigma_{ess}(L) = \emptyset$, i.e. the super Poincaré inequality (5.1.2) holds for some $\phi \in L^2(\mu)$ and some function β, provided either (a) $\xi > 0, l > 1/2, m > 5/4$; or (b) $\xi < 0, l > 1, m > 5/4$.

Proof. Let

$$\rho(x,y) = \sqrt{(c+x)^2 + \sqrt{1+y^2}}, \quad x, y \in \mathbb{R},$$

which is a smooth compact function. We have

$$\Gamma(\rho,\rho)(x,y) = \{(X\rho)^2 + (Y\rho)^2\}(x,y) = \frac{(x+c)^2 + x^2y^2/[4(1+y^2)]}{(c+x)^2 + \sqrt{y^2+1}} \le c_1$$

for some constant $c_1 > 0$ and all $x, y \in \mathbb{R}$. By Theorem 5.1.3, it suffices to show that

$$\lim_{\rho\to\infty} |L\rho| = \infty. \tag{5.1.34}$$

Noting that

$$\begin{aligned} X^2\rho(x,y) &= \frac{1}{\rho(x,y)} - \frac{(c+x)^2}{\rho^3(x,y)}, \\ Y^2\rho(x,y) &= \frac{x^2}{2\rho(x,y)(1+y^2)^{3/2}} - \frac{x^2y^2}{4\rho^3(x,y)(1+y^2)}. \end{aligned}$$

So,

$$\lim_{\rho\to\infty} \left| X^2\rho + Y^2\rho - \frac{x^2}{2\rho(x,y)(1+y^2)^{3/2}} \right| = 0. \tag{5.1.35}$$

Moreover, since $c \neq 0$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\frac{(XV)(X\rho) + (YV)(Y\rho)}{\xi}(x,y) \\ &= \frac{2l(a+(c+x)^2)^{l-1}(b+y^2)^m(c+x)^2}{\rho(x,y)} \\ &\qquad + \frac{mx^2(a+(c+x)^2)^l(b+y^2)^{m-1}y^2}{\rho\sqrt{y^2+1}} \\ &\ge \frac{\varepsilon(x^{2l}+y^{2(m-1)})}{\rho(x,y)}. \end{aligned} \tag{5.1.36}$$

Now, we are able to prove (5.1.34) for cases (a) and (b) respectively.

(a) Let $\xi > 0$. It follows from (5.1.35) and (5.1.36) that $\lim_{\rho\to\infty} L\rho = \infty$ provided $l > 1/2$ and $m > 5/4$.

(b) Let $\xi < 0$. By (5.1.35) and (5.1.36)

$$\liminf_{\rho\to\infty}(-L\rho) \ge \liminf_{\rho\to\infty} \frac{\varepsilon(x^{2l}+y^{2(m-1)}) - x^2}{\rho} = \infty$$

provided $l > 1$ and $m > 5/4$. □

5.1.3.3 *Nash inequality*

Obviously, (5.1.28) and (5.1.29) hold for $a = I, d_1 = d_2 = 1, l_1 = k$. By Corollary 5.1.11, (5.1.23) holds for some $C > 0$ if and only if $m = 2 + k$.

5.1.3.4 *Log-Sobolev inequality*

Let $\phi(x,y) = |x-c|^{k+1} + ay^2$ for some constants $c \neq 0, a > 0$. Let $V = C_0 - \delta\phi^m$ for some $C_0 \in \mathbb{R}$ and $\delta, m > 0$ such that μ is a probability measure. If $m \geq 2$ then there exists $C > 0$ such that (5.1.24) holds. If $m > 2$ then there exists $c > 0$ such that

$$\mu(f^2 \log f^2) \leq r\mu(\Gamma(f,f)) + cr^{-m/(m-2)} \tag{5.1.37}$$

holds for all $r > 0,\ f \in C_0^1(\mathbb{R}^2)$ with $\mu(f^2) = 1$.

Proof. It is easy to check that

$$\Gamma(\phi,\phi) := (X\phi)^2 + (Y\phi)^2 = (k+1)^2(x-c)^{2k} + 4a^2x^{2k}y^2$$
$$L_0\phi := k(k+1)|x-c|^{k-1} + 2ax^{2k}.$$

So, for sufficiently large ϕ,

$$\begin{aligned}&\Gamma(V,V) + 2L_0V\\ &= (m^2\delta^2\phi^{2(m-1)} - 2\delta m(m-1)\phi^{m-2})\Gamma(\phi,\phi) - 2\delta m\phi^{m-1}L_0\phi\\ &\geq c_1\phi^{2(m-1)}\end{aligned}$$

for some constant $c_1 > 0$. Thus, there exists $c_2 > 0$ such that

$$\Gamma(V,V) + 2L_0V \geq c_1\phi^{2(m-1)} - c_2. \tag{5.1.38}$$

In particular, if $m \geq 2$ then the condition of Theorem 5.1.10(2) holds for $\delta = 1$ and $\varepsilon = c_1/2$. Moreover, if $m > 2$ then (5.1.38) implies

$$\begin{aligned}s[\Gamma(V,V) + 2L_0V] + V &\geq c_1s\phi^{2(m-1)} - c_2s - \delta\phi^m + C_0\\ &\geq -c_3(1 + s^{-m/(m-2)}),\quad s \in (0,1]\end{aligned}$$

for some $c_3 > 0$. Therefore, by Theorem 5.1.10(3), the desired log-Sobolev inequality holds for some $c > 0$ and all $r \in (0,1]$, hence it also holds for all $r > 0$ and a possibly larger $c > 0$, since the weak Poincaré inequality and the defective log-Sobolev inequality imply the strict log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq C\mu(\Gamma(f,f)),\quad f \in C_0^1(\mathbb{R}^2), \mu(f^2) = 1$$

for some constant $C > 0$. □

5.1.4 ***Kohn-Laplacian type operator***

Corresponding to the last subsection, we consider in this part the Kohn-Laplacian type operator L given in Example 5.0.2.

5.1.4.1 *Weak Poincaré inequality*

Obviously, **(A5.1.1)** holds for $r_0 = r_3 = 2, r_1 = r_2 = 1$. So, Theorem 5.1.5 applies to $r_0 = 2$ and

$$D_s := \{|x| \le s, |y| \le s, |z| \le s^2\}, \quad s > 0.$$

Let

$$V(x,y,z) = c_0 - \delta \log\left[(x^2+1)(1+y^2)\right] - \frac{1+2\delta}{4}\log(1+z^2)$$

for some $\delta > 1/2$. Then for some $c_0 \in \mathbb{R}$

$$\mathrm{d}\mu = \mathrm{e}^{c_0}(1+x^2)^{-\delta}(1+y^2)^{-\delta}(1+z^2)^{-(1+2\delta)/4}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

is a probability measure. Next, there exists a constant $c_1 > 0$ such that

$$\mu(D_s^c) \le \mu(|x| > s) + \mu(|y| > s) \le c_1 s^{1-2\delta}, \quad s > 0.$$

Moreover, there exists a constant $c_2 > 0$ such that

$$\mathrm{e}^{\delta_s(V)} \le c_2 s^{1+6\delta}, \quad s \ge 1.$$

Then by Theorem 5.1.5 the weak Poincaré inequality holds for

$$\alpha(r) = c_3\big(1 + r^{-(3+6\delta)/(2\delta-1)}\big), \quad r > 0$$

for some constant $c_3 > 0$. Therefore, it follows from Theorem 1.6.14 for $\mathbb{H} = \{f \in L^2(\mu) : \mu(f) = 0\}$ that

$$\|P_t - \mu\|_{\infty\to 2} \le ct^{-(2\delta-1)/(3+6\delta)}, \quad t > 0.$$

5.1.4.2 *Super Poincaré inequality*

Let $V(x,y,z) = c(1+x^2+y^2)^l + (1+z^2)^m, \ x,y,z \in \mathbb{R}$.

Proposition 5.1.13. *If $c > 0, l > 1$ and $m > 3/4$ then $\sigma_{ess}(L) = \emptyset$, or equivalently the super Poincaré inequality (5.1.2) holds for some $\phi \in L^2(\mu)$ and some function β.*

Proof. For any $\varepsilon \in (0,1)$ let

$$\rho_\varepsilon(x,y,z) = \sqrt{x^2+y^2+\sqrt{z^2+1}} + \sqrt{\varepsilon + x^2 + y^2}, \quad x,y,z \in \mathbb{R}.$$

We have

$$\begin{aligned}
&\Gamma(\rho_\varepsilon,\rho_\varepsilon)(x,y,z)\\
&=\left(\frac{2x-yz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}+\frac{x}{\sqrt{x^2+y^2+\varepsilon}}\right)^2\\
&\quad+\left(\frac{2y+xz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}+\frac{y}{\sqrt{x^2+y^2+\varepsilon}}\right)^2 \qquad (5.1.39)\\
&=(x^2+y^2)\left(\frac{1}{\sqrt{x^2+y^2+\sqrt{z^2+1}}}+\frac{1}{\sqrt{x^2+y^2+\varepsilon}}\right)^2\\
&\quad+\frac{z^2(x^2+y^2)}{16(z^2+1)(x^2+y^2+\sqrt{z^2+1})}\le 5.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&(X^2+Y^2)\rho_\varepsilon(x,y,z)\\
&=\frac{1}{\sqrt{x^2+y^2+\varepsilon}}+X\left(\frac{2x-yz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}\right)\\
&\quad+Y\left(\frac{2y+xz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}\right)\\
&=\frac{1}{\sqrt{x^2+y^2+\varepsilon}}+\frac{4+(x^2+y^2)/(4(z^2+1)^{3/2})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}} \qquad (5.1.40)\\
&\quad-\frac{(2x-yz/(2\sqrt{z^2+1}))^2+(2y+xz/(2\sqrt{z^2+1}))^2}{4(x^2+y^2+\sqrt{z^2+1})^{3/2}}\\
&\ge\frac{1}{\sqrt{x^2+y^2+\varepsilon}}-\frac{(x^2+y^2)z^2/(z^2+1)}{16(x^2+y^2+\sqrt{z^2+1})^{3/2}}\\
&\ge\frac{1}{\sqrt{x^2+y^2+\varepsilon}}-\frac{1}{16}.
\end{aligned}$$

Finally, since

$$\begin{aligned}
&\big\{(XV)(X\rho_\varepsilon)+(YV)(Y\rho_\varepsilon)\big\}(x,y,z)\\
&=\left(\frac{2x-yz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}+\frac{x}{\sqrt{x^2+y^2+\varepsilon}}\right)\\
&\qquad\times\Big(2cl(1+x^2+y^2)^{l-1}x-mzy(z^2+1)^{m-1}\Big)\\
&\quad+\left(\frac{2y+xz/(2\sqrt{z^2+1})}{2\sqrt{x^2+y^2+\sqrt{z^2+1}}}+\frac{y}{\sqrt{x^2+y^2+\varepsilon}}\right)\\
&\qquad\times\Big(2cl(1+x^2+y^2)^{l-1}y+mxz(z^2+1)^{m-1}\Big),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\big\{(XV)(X\rho_\varepsilon)+(YV)(Y\rho_\varepsilon)\big\}(x,y,z)\\
&\geq \frac{2cl(x^2+y^2)(1+x^2+y^2)^{l-1}}{\sqrt{x^2+y^2+\varepsilon}}\\
&\qquad+\frac{mz^2(x^2+y^2)(1+z^2)^{m-1}}{4\sqrt{(z^2+1)(x^2+y^2+\sqrt{z^2+1})}}.
\end{aligned}\tag{5.1.41}$$

This implies

$$\begin{aligned}
&\lim_{\rho_\varepsilon\to\infty}\inf_{x^2+y^2\geq\varepsilon}(L\rho_\varepsilon)(x,y,z)\\
&\geq \lim_{\rho_\varepsilon\to\infty}\bigg\{\frac{2cl(x^2+y^2)(1+x^2+y^2)^{l-1}}{\sqrt{x^2+y^2+\varepsilon}}\\
&\qquad+\frac{m\varepsilon z^2(1+z^2)^{m-1}}{4\sqrt{(z^2+1)(\varepsilon+\sqrt{z^2+1})}}\bigg\}\\
&=\infty.
\end{aligned}\tag{5.1.42}$$

On the other hand, (5.1.40) and (5.1.41) imply

$$\inf_{x^2+y^2\leq\varepsilon} L\rho_\varepsilon\geq\frac{1}{\sqrt{2\varepsilon}}-\frac{1}{16}.$$

Combining this with (5.1.42) we obtain

$$\liminf_{\rho_\varepsilon\to\infty} L\rho_\varepsilon\geq\frac{1}{\sqrt{2\varepsilon}}-\frac{1}{16}.$$

Since (5.1.39) implies $\Gamma(\rho_\varepsilon/\sqrt{5},\rho_\varepsilon/\sqrt{5})\leq 1$, it then follows from Theorem 5.1.3 that

$$\inf\sigma_{ess}(-L)\geq\frac{1}{20}\Big(\frac{1}{\sqrt{2\varepsilon}}-\frac{1}{16}\Big)^2.$$

Letting $\varepsilon\to 0$ we complete the proof. □

5.1.4.3 *Nash inequality*

(5.1.28) and (5.1.29) hold for $a=I, d_1=2, l_1=d_2=1$. By Corollary 5.1.11, (5.1.23) holds for some $C>0$ if and only if $m=4$.

5.1.4.4 *Log-Sobolev inequality*

We consider the high dimensional case. Let

$$X_i = \partial_{x_i} - \frac{y_i}{2}\partial_z, \quad Y_i = \partial_{y_i} + \frac{x_i}{2}\partial_z, \quad 1 \le i \le n.$$

Let $L = \sum_{i=1}^n (X_i^2 + Y_i^2)$ and

$$\Gamma(f,f) = \sum_{i=1}^n \{(X_i f)^2 + (Y_i f)^2\}, \quad f \in C^1(\mathbb{R}^{2n+1}).$$

Let

$$\rho(x,y,z) = \sqrt{|x|^2 + |y|^2 + z^2/(|x|^2+|y|^2)}, \quad x,y \in R^n, z \in \mathbb{R}.$$

Then $\mathrm{vol}(\exp[-\varepsilon\rho^2]) < \infty$ for any $\varepsilon > 0$.

Proposition 5.1.14. *Let $n > 2$ and $V = c(\delta) - \rho^\delta$ for $\delta \ge 2$ and some $c(\delta) \in \mathbb{R}$ such that μ is a probability measure. Then* (5.1.24) *holds for some $C > 0$. If $\delta > 2$ then* (5.1.32) *holds for some $c > 0$.*

Proof. Since V is smooth only on the set $\Omega := \{(x,y,z) \in \mathbb{R}^{n+n+1} : |x| + |y| > 0\}$, we shall first restrict everything on Ω. Obviously,

$$\begin{aligned}
(X_i\rho)(x,y,z) &= \frac{1}{\rho(x,y,z)}\Big\{x_i - \frac{x_i z^2}{(|x|^2+|y|^2)^2} - \frac{y_i z}{2(|x|^2+|y|^2)}\Big\}\\
(Y_i\rho)(x,y,z) &= \frac{1}{\rho(x,y,z)}\Big\{y_i - \frac{y_i z^2}{(|x|^2+|y|^2)^2} + \frac{x_i z}{2(|x|^2+|y|^2)}\Big\}.
\end{aligned} \tag{5.1.43}$$

So,

$$\begin{aligned}
&\Gamma(\rho,\rho)(x,y,z)\\
&= \frac{1}{\rho(x,y,z)^2}\Big\{(|x|^2+|y|^2)\Big(1 - \frac{z^2}{(|x|^2+|y|^2)^2}\Big)^2 + \frac{z^2}{4(|x|^2+|y|^2)}\Big\}.
\end{aligned}$$

Thus, when $z^2 \le \frac{1}{2}(|x|^2+|y|^2)^2$ we have $\Gamma(\rho,\rho)(x,y,z) \ge \frac{1}{4}$; while when $z^2 > \frac{1}{2}(|x|^2+|y|^2)^2$ we have

$$\frac{z^2}{2(|x|^2+|y|^2)} \ge \frac{z^2}{8(|x|^2+|y|^2)} + \frac{|x|^2+|y|^2}{8} = \frac{\rho(x,y,z)^2}{8},$$

so that $\Gamma(\rho,\rho)(x,y,z) \ge \frac{1}{8}$. In conclusion, we have

$$\Gamma(\rho,\rho) \ge \frac{1}{16} \quad \text{on } \Omega. \tag{5.1.44}$$

Next, by (5.1.43), we have

$$
\begin{aligned}
&(X_i^2\rho)(x,y,z)\\
&\quad\le \frac{1}{\rho(x,y,z)}\Big\{1-\frac{z^2-2x_iy_iz}{(|x|^2+|y|^2)^2}+\frac{4x_i^2z^2}{(|x|^2+|y|^2)^3}+\frac{y_i^2}{4(|x|^2+|y|^2)}\Big\},\\
&(Y_i^2\rho)(x,y,z)\\
&\quad\le \frac{1}{\rho(x,y,z)}\Big\{1-\frac{z^2-2x_iy_iz}{(|x|^2+|y|^2)^2}+\frac{4y_i^2z^2}{(|x|^2+|y|^2)^3}+\frac{x_i^2}{4(|x|^2+|y|^2)}\Big\}.
\end{aligned}
$$

Combining this with the fact that $2|x_iy_i|\le 2(x_i^2+y_i^2)$, we arrive at

$$
\begin{aligned}
(L\rho)(x,y,z) &\le \frac{1}{\rho(x,y,z)}\Big\{2n-\frac{(2n-4)z^2}{(|x|^2+|y|^2)^2}+\frac{2|z|}{(|x|^2+|y|^2)}+\frac{1}{4}\Big\}\\
&\le \frac{c_0}{\rho(x,y,z)}
\end{aligned}
$$

for some $c_0>0$. Due to this and (5.1.44), the proofs of Theorem 5.1.10 and Corollary 5.1.12 with $\mathbb{R}^d$ replaced by Ω lead to

$$
\mu(f^2\log f^2)\le C\mu(\Gamma(f,f)),\quad f\in C_0^1(\Omega),\mu(f^2)=1 \tag{5.1.45}
$$

provided $\delta\ge 2$. This implies (5.1.24) by an approximation argument and the proof of (5.1.32) for $\delta>2$ is similar. More precisely, for any $f\in C_0^1(\mathbb{R}^{2n+1})$ with $\mu(f^2)=1$, let $f_\varepsilon=fh_\varepsilon$, where

$$
h_\varepsilon(x,y,z):=\Big(\frac{1}{\varepsilon}\sqrt{|x|^2+|y|^2}-1\Big)^+\wedge 1,\quad \varepsilon>0.
$$

We have

$$
\mu(\Gamma(f_\varepsilon,f_\varepsilon))\le(1+r)\mu(\Gamma(f,f))+(1+r^{-1})\varepsilon^{-2}\|f\|_\infty^2\mu(\Omega_\varepsilon),\ \ r>0, \tag{5.1.46}
$$

where $\Omega_\varepsilon:=\{(x,y,z)\in\Omega:\ |x|^2+|y|^2<\varepsilon^2\}$. Since

$$
\mu(\Omega_\varepsilon)\le c_1\int_{\{|x|^2+|y|^2<\varepsilon^2\}}\mathrm{d}x\,\mathrm{d}y\int_{\mathbb{R}}\mathrm{e}^{-\varepsilon^{-2}z^2}\mathrm{d}z\le c_2\varepsilon^{2n+1}
$$

for some $c_1,c_2>0$, by first letting $\varepsilon\to0$ then $r\to0$ in (5.1.46), we obtain

$$
\limsup_{\varepsilon\to0}\mu(\Gamma(f_\varepsilon,f_\varepsilon))\le\mu(\Gamma(f,f)).
$$

Thus, (5.1.24) follows by first applying (5.1.45) to $f_\varepsilon\mu(f_\varepsilon^2)^{-1/2}$ in place of f then letting $\varepsilon\to0$. $\square$

5.2 Generalized curvature and applications

As shown in the previous chapters that the Bakry-Emery curvature condition has played a crucial role in the study of elliptic diffusion processes. When the diffusion operator is merely subelliptic, this condition is however no longer available. Recently, in order to study subelliptic diffusion processes, a generalized curvature-dimension condition was introduced and applied in [Baudoin and Bonnefont (2012); Baudoin *et al* (2010); Baudoin and Garofalo (2011)], so that many important results derived in the elliptic setting have been extended to subelliptic diffusion processes with generators of type

$$L := \sum_{i=1}^{n} X_i^2 + X_0$$

for smooth vector fields $\{X_i : 0 \le i \le n\}$ on a differentiable manifold such that $\{X_i, \nabla_{X_i} X_j : 1 \le i, j \le n\}$ spans the tangent space (see [Wang (2012c)] for details). In this section we aim to introduce a general version of curvature condition to study more general subelliptic diffusion semigroups.

Let M be a connected differentiable manifold, and let L be given above for some C^2-smooth vector fields $\{X_i\}_{i=1}^n$ and a C^1-smooth vector field X_0. The square field for L is a symmetric bilinear differential form given by

$$\Gamma(f,g) = \sum_{i=1}^{n} (X_i f)(X_i g), \quad f, g \in C^1(M).$$

Obviously, Γ satisfies

$$\begin{aligned} &\Gamma(f) := \Gamma(f,f) \ge 0, \\ &\Gamma(fg,h) = g\Gamma(f,h) + f\Gamma(g,h), \\ &\Gamma(\phi \circ f, g) = (\phi' \circ f)\Gamma(f,g) \end{aligned}$$

for any $f, g, h \in C^1(M)$ and $\phi \in C^1(\mathbb{R})$. From now on, a symmetric bilinear differential form $\bar{\Gamma}$ satisfying these properties is called a diffusion square field. If moreover for any $x \in M$ and $f \in C^1(M)$, $\bar{\Gamma}(f)(x) = 0$ implies $(\mathrm{d}f)(x) = 0$, we call $\bar{\Gamma}$ elliptic or non-degenerate.

For any C^2-diffusion square field $\bar{\Gamma}$ (i.e. $\bar{\Gamma}(f,g) \in C^2(M)$ for $f, g \in C^\infty(M)$), we define the associated Bakry-Emery curvature operator w.r.t. L by

$$\bar{\Gamma}_2(f) = \frac{1}{2} L\bar{\Gamma}(f) - \bar{\Gamma}(f, Lf), \quad f \in C^3(M).$$

Then the generalized curvature-dimension condition introduced in [Baudoin and Garofalo (2011)] reads

$$\Gamma_2(f) + r\Gamma_2^Z(f) \geq \frac{(Lf)^2}{d} + \Big(\rho_1 - \frac{\kappa}{r}\Big)\Gamma(f) + \rho_2\Gamma^Z(f), \tag{5.2.1}$$

for all $f \in C^2(M)$, $r > 0$, where $\rho_2 > 0, \kappa \geq 0, \rho_1 \in \mathbb{R}$ and $d \in (0, \infty]$ are constants, and Γ^Z is a C^2-diffusion square field such that $\Gamma + \Gamma^Z$ is elliptic and

$$\Gamma(\Gamma^Z(f), f) = \Gamma^Z(\Gamma(f), f), \quad f \in C^\infty(M) \tag{5.2.2}$$

holds. When $\Gamma^Z = 0$, (5.2.1) reduces back to the Bakry-Emery curvature-dimension condition [Bakry and Emery (1984)], and when $d = \infty$ it becomes the following generalized curvature condition

$$\Gamma_2(f) + r\Gamma_2^Z(f) \geq \Big(\rho_1 - \frac{\kappa}{r}\Big)\Gamma(f) + \rho_2\Gamma^Z(f), \quad f \in C^2(M), r > 0. \tag{5.2.3}$$

Using (5.2.2) and (5.2.3) for symmetric subelliptic operators, the Poincaré inequality for the associated Dirichlet form, the Harnack inequality and the log-Sobolev inequality (for, however, an enlarged Dirichlet form given by $\Gamma + \Gamma^Z$) for the associated diffusion semigroup, and the HWI inequality (where the energy part is given by the enlarged Dirichlet form) are investigated in [Baudoin and Bonnefont (2012)].

The generalized curvature-dimension condition we proposed is

$$\Gamma_2(f) + \sum_{i=1}^{l} r_i\Gamma_2^{(i)}(f) \geq \frac{(Lf)^2}{d} + \sum_{i=0}^{l} K_i(r_1, \dots, r_l)\Gamma^{(i)}(f),$$

for all $f \in C^3(M), r_1, \dots, r_l > 0$, where $d \in (0, \infty]$ is a constant, $\Gamma^{(0)} := \Gamma, \{\Gamma^{(i)}\}_{1 \leq i \leq l}$ are some C^2-diffusion square fields, and $\{K_i\}_{0 \leq i \leq l}$ are some continuous functions on $(0, \infty)^l$. We will only consider the condition with $d = \infty$, i.e.

$$\Gamma_2(f) + \sum_{i=1}^{l} r_i\Gamma_2^{(i)}(f) \geq \sum_{i=0}^{l} K_i(r_1, \dots, r_l)\Gamma^{(i)}(f), \tag{5.2.4}$$

for $f \in C^3(M), r_1, \dots, r_l > 0$, but the condition with finite d will be useful for other purposes as in [Baudoin *et al* (2010); Baudoin and Garofalo (2011)]. In fact, we will make use of the following assumption.

(A5.2.1) (5.2.4) holds for some C^2-diffusion square fields $\{\Gamma^{(i)}\}_{i=0}^l$ and $\{K_i\}_{0\le i\le l} \subset C((0,\infty)^l)$, where $\Gamma^{(0)} = \Gamma$. There exists a smooth compact function $W \geq 1$ on M and a constant $C > 0$ such that $LW \leq CW$ and $\tilde\Gamma(W) \leq CW^2$, where $\tilde\Gamma = \sum_{i=0}^l \Gamma^{(i)}$.

Recall that W is called a compact function if $\{W \leq r\}$ is compact for any constant r. The condition $LW \leq CW$ is standard to ensure the non-explosion of the L-diffusion process, and the condition $\tilde\Gamma(W) \leq CW^2$ is used to prove the boundedness of $\tilde\Gamma(P_t f)$ for $f \in \mathcal{C}$, where

$$\mathcal{C} := \Big\{ f \in C^\infty(M) \cap \mathcal{B}_b(M): \quad \tilde\Gamma(f) \text{ is bounded} \Big\}.$$

5.2.1 *Derivative inequalities*

The main result in this subsection is the following theorem.

Theorem 5.2.1. *Assume* **(A5.2.1)**. *For fixed $t > 0$, let $\{b_i\}_{0\le i\le l} \subset C^1([0,t])$ be strictly positive on $(0,t)$ such that*

$$b_i'(s) + 2\Big\{ b_0 K_i\Big(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\Big)\Big\}(s) \geq 0, \quad s \in (0,t), 1 \leq i \leq l \tag{5.2.5}$$

and

$$c_b := -\inf_{(0,t)} \Big\{ b_0' + 2b_0 K_0\Big(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0}\Big)\Big\} < \infty.$$

Then:

(1) *For any $f \in \mathcal{C}$,*

$$2\sum_{i=0}^l \big\{ b_i(0)\Gamma^{(i)}(P_t f) - b_i(t) P_t \Gamma^{(i)}(f)\big\} \leq c_b \{P_t f^2 - (P_t f)^2\}.$$

(2) *If*

$$\Gamma^{(i)}\big(\Gamma(f), f\big) = \Gamma\big(\Gamma^{(i)}(f), f\big), \quad 1 \leq i \leq l, f \in C^\infty(M), \tag{5.2.6}$$

then for any positive $f \in \mathcal{C}$,

$$\sum_{i=0}^l \Big\{ b_i(0)\frac{\Gamma^{(i)}(P_t f)}{P_t f} - b_i(t) P_t \frac{\Gamma^{(i)}(f)}{f}\Big\} \leq c_b \{P_t(f\log f) - (P_t f)\log P_t f\}.$$

To prove this theorem using a modified Bakry-Emery semigroup argument as in [Baudoin and Bonnefont (2012)], we need to first confirm that $P_t\mathcal{C} \subset \mathcal{C}$, which follows immediately from the following lemma.

Lemma 5.2.2. *Assume* **(A5.2.1)** *and let $K = \min_{0\le i\le l} K_i(1,\dots,1)$. Then*

$$\tilde\Gamma(P_t f) \leq e^{-2Kt} P_t \tilde\Gamma(f), \quad t \geq 0, f \in \mathcal{C}. \tag{5.2.7}$$

Proof. (i) We first prove for any $f \in C_0^2(M)$ and $t > 0$, $\tilde{\Gamma}(P_\cdot f)$ is bounded on $[0,t]\times M$. To this end, we approximate the generator L by using operators with compact support, so that the approximating diffusion processes stay in compact sets. Take $h \in C_0^\infty([0,\infty))$ such that $h' \le 0, h|_{[0,1]} = 1$ and $\mathrm{supp} h = [0,2]$. For any $m \ge 1$, let $\varphi_m = h(W/m)$ and $L_m = \varphi_m^2 L$. Then L_m has compact support $B_m := \{W \le 2m\}$. Let $x \in \{W \le m\}$ and X_s^m be the L_m-diffusion process starting at x. Let

$$\tau_m = \inf\{s \ge 0 : W(X_s^m) \ge 2m\}.$$

Since $LW \le CW, \Gamma(W) \le \tilde{\Gamma}(W) \le CW^2, h' \le 0, 0 \le h \le 1$ and $h'(W/m) = 0$ for $W \ge 2m$, we have

$$\begin{aligned} L_m\frac{1}{\varphi_m^2} &= -\frac{2L\varphi_m}{\varphi_m} + \frac{6\Gamma(\varphi_m)}{\varphi_m^2} \\ &= -\frac{2h'(W/m)LW}{m\varphi_m} - \frac{2h''(W/m)\Gamma(W)}{m^2\varphi_m} + \frac{6h'(W/m)^2\Gamma(W)}{m^2\varphi_m^2} \le \frac{C_1}{\varphi_m^2} \end{aligned}$$

for some constant $C_1 > 0$ independent of m. By a standard argument, this implies that $\tau_m = \infty$ and

$$\mathbb{E}\Big(\frac{1}{\varphi_m^2}\Big)(X_s^m) \le \frac{e^{C_1 s}}{\varphi_m^2(x)} \le e^{C_1 s}, \quad s \ge 0. \tag{5.2.8}$$

Now, let P_s^m be the diffusion semigroup generated by L_m. By the Itô formula and $\tilde{\Gamma}_2 \ge K\tilde{\Gamma}$ implied by **(A5.2.1)** we obtain

$$\begin{aligned} &d\tilde{\Gamma}(P_{t-s}^m f)(X_s^m) - dM_s^m \\ &= dM_s^m + \{\varphi_m^2 L\tilde{\Gamma}(P_{t-s}^m f) - 2\tilde{\Gamma}(P_{t-s}^m f, \varphi_m^2 L P_{t-s}^m f)\}(X_s^m)ds \\ &\ge \{2\varphi_m^2\tilde{\Gamma}_2(P_{t-s}^m f) - 4\tilde{\Gamma}(\log\varphi_m, P_{t-s}^m f)P_{t-s}^m L_m f\}(X_s^m)ds \\ &\ge \Big\{2|K|\tilde{\Gamma}(P_{t-s}^m f) + 4\|Lf\|_\infty\sqrt{\tilde{\Gamma}(\log\varphi_m)\tilde{\Gamma}(P_{t-s}^m f)}\Big\}(X_s^m)ds \\ &\ge -C_2\tilde{\Gamma}(P_{t-s}^m f)(X_s^m)ds - \tilde{\Gamma}(\log\varphi_m)(X_s^m)ds, \quad s \in [0,t] \end{aligned} \tag{5.2.9}$$

for some martingale M_s^m and some constant $C_2 > 0$ independent of m. Since $h'(W/m) = 0$ for $W \ge 2m$ and $\tilde{\Gamma}(W) \le CW^2$,

$$\tilde{\Gamma}(\log\varphi_m) = \frac{h'(W/m)^2\tilde{\Gamma}(W)}{m^2\varphi_m^2} \le \frac{C_3}{\varphi_m^2}$$

holds for some constant $C_3 > 0$ independent of m. Combining this with (5.2.8) and (5.2.9) we conclude that

$$\begin{aligned} \tilde{\Gamma}(P_t^m f) &\le e^{C_2 t}P_t^m\tilde{\Gamma}(f) + C_3\int_0^t \mathbb{E}\Big(\frac{1}{\varphi_m^2}\Big)(X_s^m)ds \\ &\le e^{C_2 t}\|\tilde{\Gamma}(f)\|_\infty + C_3 t e^{C_1 t} \end{aligned} \tag{5.2.10}$$

holds on $\{W \le m\}$. Letting $\tilde\rho$ be the intrinsic distance induced by $\tilde\Gamma$, i.e.

$$\tilde\rho(z,y) := \sup\{|g(z)-g(y)| : \ \tilde\Gamma(g) \le 1\}, \quad z,y \in M,$$

we deduce from (5.2.10) that for any $z,y \in M$,

$$|P_t^m f(z) - P_t^m f(y)|^2 \le \tilde\rho(z,y)^2 \Big(\mathrm{e}^{C_2 t}\|\tilde\Gamma(f)\|_\infty + C_3 t \mathrm{e}^{C_1 t}\Big), \quad t>0 \quad (5.2.11)$$

holds for large enough m. Noting that the L-diffusion process is non-explosive and X_s^m is indeed generated by L before time $\sigma_m := \inf\{s \ge 0 : W(X_s^m) \ge m\}$ which increases to ∞ as $m \to \infty$, we conclude that $\lim_{m\to\infty} P_t^m f = P_t f$ holds point-wisely. Therefore, letting $m \to \infty$ in (5.2.11) we obtain

$$|P_t f(z) - P_t f(y)|^2 \le \tilde\rho(z,y)^2 \Big(\mathrm{e}^{C_2 t}\|\tilde\Gamma(f)\|_\infty + C_3 t \mathrm{e}^{C_1 t}\Big), \quad t \ge 0, y,z \in M.$$

This implies that $\tilde\Gamma(P_\cdot f)$ is bounded on $[0,t]\times M$ for any $t>0$.

(ii) By an approximation argument, it suffices to prove (5.2.7) for $f \in C_0^2(M)$. By the Itô formula and (5.2.4), there exists a local martingale M_s such that

$$\begin{aligned}\mathrm{d}\tilde\Gamma(P_{t-s}f)(X_s) &= \mathrm{d}M_s + 2\tilde\Gamma_2(P_{t-s}f)(X_s)\mathrm{d}s\\ &\ge \mathrm{d}M_s + 2K\tilde\Gamma(P_{t-s}f)(X_s)\mathrm{d}s, \quad s\in[0,t].\end{aligned}$$

Thus,

$$[0,t] \ni s \mapsto \mathrm{e}^{-2Ks}\tilde\Gamma(P_{t-s}f)(X_s)$$

is a local submartingale. Since due to (i) this process is bounded, so that it is indeed a submartingale. Therefore, (5.2.7) holds. □

Proof. [Proof of Theorem 5.2.1] (1) It suffices to prove for $f \in C^\infty(M)$ which is constant outside a compact set. In this case we have $\frac{\mathrm{d}}{\mathrm{d}s}P_s f = LP_s f = P_s Lf$. Since X_s is non-explosive, by the Itô formula for any $0 \le i \le l$ there exists a local martingale $M_s^{(i)}$ such that

$$\begin{aligned}&\mathrm{d}\Gamma^{(i)}(P_{t-s}f)(X_s)\\ &= \mathrm{d}M_s^{(i)} + \big\{L\Gamma^{(i)}(P_{t-s}f) - 2\Gamma^{(i)}(P_{t-s}f, LP_{t-s}f)\big\}(X_s)\mathrm{d}s\\ &= \mathrm{d}M_s^{(i)} + 2\Gamma_2^{(i)}(P_{t-s}f)(X_s)\mathrm{d}s, \quad s\in[0,t].\end{aligned}$$

Therefore, due to (5.2.4) and (5.2.5), there exists a local martingale M_s such that

$$\begin{aligned}
&\mathrm{d}\Big\{\sum_{i=0}^{l} b_i(s)\Gamma^{(i)}(P_{t-s}f)(X_s)\Big\}\\
&\geq \mathrm{d}M_s + \Big\{\sum_{i=0}^{l}\Big(2b_i(s)\Gamma_2^{(i)}(P_{t-s}f) + b_i'(s)\Gamma^{(i)}(P_{t-s}f)\Big)\Big\}(X_s)\mathrm{d}s\\
&\geq \mathrm{d}M_s + \Big\{\sum_{i=0}^{l}\Big(b_i'(s) + 2b_0(s)K_i\Big(\frac{b_1}{b_0},\dots,\frac{b_l}{b_0}\Big)(s)\Big)\Gamma^{(i)}(P_{t-s}f)\Big\}(X_s)\mathrm{d}s\\
&\geq \mathrm{d}M_s + \Big\{b_0'(s) + 2b_0(s)K_0\Big(\frac{b_1}{b_0},\dots,\frac{b_l}{b_0}\Big)(s)\Big\}\Gamma(P_{t-s}f)(X_s)\mathrm{d}s.
\end{aligned}$$

So, if $c_b < \infty$ then

$$\sum_{i=0}^{l} b_i(s)\Gamma^{(i)}(P_{t-s}f)(X_s) + c_b\int_0^s \Gamma(P_{t-r}f)(X_r)\mathrm{d}r$$

is a local submartingale for $s \in [0,t]$. Since, due to Lemma 5.2.2, $\{\Gamma^{(i)}(P_{t-s}f)\}_{0\leq i\leq l}$ are bounded, it is indeed a submartingale. In particular,

$$\sum_{i=0}^{l}\big\{b_i(0)\Gamma^{(i)}(P_tf) - b_i(t)P_t\Gamma^{(i)}(f)\big\} \leq c_b\int_0^t P_s\Gamma(P_{t-s}f)\mathrm{d}s.$$

Then the proof is finished by noting that

$$P_s\Gamma(P_{t-s}f) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}P_s(P_{t-s}f)^2.$$

(2) Let f be strictly positive and be constant outside a compact set. Let

$$\phi^{(i)}(s,x) = \big\{(P_{t-s}f)\Gamma^{(i)}(\log P_{t-s}f)\big\}(x), \quad 0\leq i\leq l, s\in[0,t], x\in M.$$

It is easy to see that (5.2.6) implies (cf. [Baudoin and Garofalo (2011)])

$$L\phi^{(i)} + \frac{\partial}{\partial s}\phi^{(i)} = 2(P_{t-s}f)\Gamma_2^{(i)}(\log P_{t-s}f),\ \ 0\leq i\leq l.$$

So, for each $0\leq i\leq l$, there exists a local martingale $M_s^{(i)}$ such that

$$\mathrm{d}\phi^{(i)}(s,X_s) = \mathrm{d}M_s^{(i)} + 2\big\{(P_{t-s}f)\Gamma_2^{(i)}(\log P_{t-s}f)\big\}(X_s)\mathrm{d}s,\quad s\in[0,t].$$

The remainder of the proof is then completely similar to (1); that is,

$$\begin{aligned}
&\sum_{i=0}^{l} b_i(s)\big\{(P_{t-s}f)\Gamma^{(i)}(\log P_{t-s}f)\big\}(X_s)\\
&+ c_b\int_0^s \big\{(P_{t-r}f)\Gamma(\log P_{t-r}f)\big\}(X_r)\mathrm{d}r
\end{aligned}$$

is a submartingale for $s\in[0,t]$, so that the desired inequality follows by noting that

$$P_s\big\{(P_{t-s}f)\Gamma(\log P_{t-s}f)\big\} = \frac{\mathrm{d}}{\mathrm{d}s}P_s\big\{(P_{t-s}f)\log P_{t-s}f\big\}. \qquad \square$$

5.2.2 *Applications of Theorem 5.2.1*

For any non-negative symmetric measurable functions $\bar\rho$ on $M \times M$, let $W_2^{\bar\rho}$ be the L^2-transportation-cost with cost function $\bar\rho$; i.e. for any two probability measures μ_1, μ_2 on M,

$$W_2^{\bar\rho}(\mu_1,\mu_2) := \inf_{\pi\in\mathcal{C}(\mu_1,\mu_2)} \pi(\bar\rho^2)^{1/2},$$

where $\pi(\bar\rho)$ stands for the integral of $\bar\rho$ w.r.t. π, and $\mathcal{C}(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2.

5.2.2.1 *L^2-derivative estimate and applications*

Proposition 5.2.3. *Assume* **(A5.2.1)**. *Let $t > 0$ and $\{b_i\}_{0\le i\le l} \subset C^1([0,t])$ be strictly positive in $(0,t)$ such that (5.2.5) holds. If $b_i(t) = 0, 0 \le i \le l$ and $c_b < \infty$, then:*

(1) *For any $f \in \mathcal{B}_b(M)$,*

$$2\sum_{i=0}^{l} b_i(0)\Gamma^{(i)}(P_t f) \le c_b\{P_t f^2 - (P_t f)^2\}. \qquad (5.2.12)$$

(2) *For any non-negative $f \in \mathcal{B}_b(M)$, the Harnack type inequality*

$$P_t f(x) \le P_t f(y) + \frac{\sqrt{c_b}}{\sqrt{2}}\rho_b(x,y)\sqrt{P_t f^2(x)}, \quad x,y\in M \qquad (5.2.13)$$

holds for ρ_b being the intrinsic distance induced by $\Gamma_b := \sum_{i=0}^l b_i(0)\Gamma^{(i)}$.

(3) *If P_t has an invariant probability measure μ, then for any $f \ge 0$ with $\mu(f) = 1$, the variance-cost inequality*

$$\mathrm{Var}_\mu(P_t^* f) \le \frac{\sqrt{c_b}}{\sqrt{2}} W_2^{\rho_b}(f\mu,\mu)\sqrt{\mu((P_t^* f)^3)} \qquad (5.2.14)$$

holds, where P_t^* is the adjoint operator of P_t in $L^2(\mu)$, and

$$\mathrm{Var}_\mu(P_t^* f) := \mu((P_t^* f)^2) - \mu(P_t^* f)^2 = \mu((P_t^* f)^2) - 1.$$

Proof. By an approximation argument, it suffices to prove for $f \in \mathcal{C}$. The first assertion is a direct consequence of Theorem 5.2.1(1), while according to Proposition 1.5.3, (5.2.12) implies (5.2.13). Finally, (5.2.14) follows from (5.2.13) according to the following Lemma 5.2.4.

□

Lemma 5.2.4. *Let P be a Markov operator on $\mathcal{B}_b(E)$ for a measurable space $(E.\mathcal{B})$. Let μ be an invariant probability measure of P. If*

$$Pf(x) \le Pf(y) + C\rho(x,y)\sqrt{Pf^2(x)}, \quad f \in \mathcal{B}_b^+(E) \tag{5.2.15}$$

holds for some constant $C > 0$ and non-negative symmetric function ρ on $E \times E$, then

$$\mathrm{Var}_\mu(P^* f) \le CW_2^\rho(f\mu,\mu)\sqrt{\mu((P^*f)^3)}, \quad f \ge 0, \mu(f) = 1.$$

Proof. Let $f \ge 0$ with $\mu(f) = 1$. For any $\pi \in \mathcal{C}(f\mu,\mu)$, (5.2.15) implies

$$\mu((P^*f)^2) = \mu(fPP^*f) = \int_{E\times E} P(P^*f)(x)\pi(\mathrm{d}x,\mathrm{d}y)$$
$$\le \int_{E\times E} P(P^*f)(y)\pi(\mathrm{d}x,\mathrm{d}y) + C\int_{E\times E}\rho(x,y)\sqrt{P(P^*f)^2(x)}\,\pi(\mathrm{d}x,\mathrm{d}y)$$
$$\le \mu(PP^*f) + C\sqrt{\pi(\rho^2)\mu(fP(P^*f)^2)} = 1 + C\sqrt{\pi(\rho^2)\mu((P^*f)^3)}.$$

This completes the proof. □

5.2.2.2 *Entropy-derivative estimate and applications*

Proposition 5.2.5. *Assume* **(A5.2.1)** *and* (5.2.6). *Let $t > 0$ and $\{b_i\}_{0\le i\le l} \subset C^1([0,t])$ be strictly positive in $(0,t)$ such that* (5.2.5) *holds. If $b_i(t) = 0, 0 \le i \le l$ and $c_b < \infty$, then:*

(1) *For any strictly positive $f \in \mathcal{B}_b(M)$,*

$$\sum_{i=0}^{l} b_i(0)\Gamma^{(i)}(P_t f) \le c_b(P_t f)\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}. \tag{5.2.16}$$

(2) *For any non-negative $f \in \mathcal{B}_b(M)$ and $\alpha > 1$, the Harnack type inequality*

$$(P_t f)^\alpha(x) \le P_t f^\alpha(y)\exp\left[\frac{\alpha c_b \rho_b(x,y)^2}{4(\alpha-1)}\right], \quad x,y \in M \tag{5.2.17}$$

holds for ρ_b being the intrinsic distance induced by $\Gamma_b := \sum_{i=0}^{l} b_i(0)\Gamma^{(i)}$. Consequently, the log-Harnack inequality

$$P_t \log f(x) \le \log P_t f(y) + \frac{c_b\rho_b(x,y)^2}{4} \tag{5.2.18}$$

holds for strictly positive $f \in \mathcal{B}_b(M)$.

(3) *If P_t has an invariant probability measure μ, then for any $f \ge 0$ with $\mu(f) = 1$, the entropy-cost inequality*

$$\mu\big((P_t^* f)\log P_t^* f\big) \le \frac{c_b}{4}W_2^{\rho_b}(f\mu,\mu)^2. \tag{5.2.19}$$

Proof. By an approximation argument, it suffices to prove for $f \in \mathcal{C}$. The first assertion is a direct consequence of Theorem 5.2.1(2), (5.2.17) follows from (1) in the spirit of Proposition 1.5.2 (see also Lemma 3.4 in [Wang (2012c)]), (5.2.18) follows from (5.2.17) according to Corollary 1.4.3, and finally, (5.2.19) follows from (5.2.18) and Proposition 1.4.4. $\square$

5.2.2.3 *Exponential decay and Poincaré inequality*

Proposition 5.2.6. *Assume* **(A5.2.1)**. *For $r_i > 0, 1 \le i \le l$, let*

$$\lambda(r_1, \ldots, r_l) = \min_{0 \le i \le l} \frac{K_i(r_1, \ldots, r_l)}{r_i},$$

where $r_0 := 1$. Then

$$\sum_{i=0}^{l} r_i \Gamma^{(i)}(P_t f) \le \mathrm{e}^{-2\lambda(r_1,\ldots,r_l)t} \sum_{i=0}^{l} r_i P_t \Gamma^{(i)}(f), \quad t \ge 0, f \in C_b^1(M).$$

Consequently, if P_t is symmetric with respect to a probability measure μ and

$$\lambda := \sup_{r_1,\ldots,r_l>0} \lambda(r_1, \ldots, r_l) > 0,$$

then the Poincaré inequality

$$\mu(f^2) \le \frac{1}{\lambda}\mu(\Gamma(f)) + \mu(f)^2, \quad f \in C_0^1(M) \tag{5.2.20}$$

holds.

Proof. By a standard spectral theory (cf. the proof of Corollary 2.4 in [Baudoin and Bonnefont (2012)]), the Poincaré inequality follows immediately from the desired derivative inequality. To prove the derivative inequality, we take

$$b_0(s) = \mathrm{e}^{-2\lambda(r_1,\ldots,r_l)s}, \quad b_i(s) = r_i b_0(s), \quad 1 \le i \le l, s \ge 0.$$

Then

$$b_i' + 2b_0 K_i\Big(\frac{b_1}{b_0}, \ldots, \frac{b_l}{b_0}\Big) = -2r_i\lambda(r_1, \ldots, r_l)b_0 + 2b_0 K_i(r_1, \ldots, r_l) \ge 0$$

for all $0 \le i \le l$. Therefore, the desired gradient inequality follows from Theorem 5.2.1(1). $\square$

5.2.2.4 *Derivative inequalities by (5.2.3)*

Coming back to condition (5.2.3), Theorem 5.2.1 implies the following exact extensions of sharp gradient estimates in the elliptic setting (see Theorem 2.3.1 for constant K).

Proposition 5.2.7. *Assume* (5.2.3) *for some constants* $\rho_2 > 0, \kappa \ge 0$ *and* $\rho_1 \in \mathbb{R}$. *Assume there exist a smooth compact function* $W \ge 1$ *and a constant* $C > 0$ *such that* $LW \le CW$ *and* $\tilde{\Gamma}(W) \le CW^2$, *where* $\tilde{\Gamma} := \Gamma + \Gamma^Z$.

(1) *For any* $t > 0$ *and* $f \in \mathcal{B}_b(M)$,

$$\begin{aligned}&\Gamma(P_t f) + \frac{\rho_2(e^{2\rho_1 t} - 1 - 2\rho_1 t)}{\rho_1(e^{2\rho_1 t} - 1)}\Gamma^Z(P_t f)\\&\le \Big(1 + \frac{\kappa(e^{2\rho_1^+ t} - 1)^2}{\rho_2(e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t)}\Big)\frac{\rho_1}{e^{2\rho_1 t} - 1}\{P_t f^2 - (P_t f)^2\},\end{aligned}$$

where when $\rho_1 \le 0$,

$$\frac{(e^{2\rho_1^+ t} - 1)^2}{e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t} := \lim_{r\downarrow 0}\frac{(e^r - 1)^2}{e^r - 1 - r} = 2.$$

Consequently, if $\rho_1 > 0$ *and* P_t *is symmetric w.r.t. a probability measure* μ, *then the Poincaré inequality*

$$\mu(f^2) \le \frac{1}{\rho_1}\mu(\Gamma(f)) + \mu(f)^2, \quad f \in C_0^1(M) \tag{5.2.21}$$

holds.

(2) *If* (5.2.2) *holds, then for any* $t > 0$ *and positive* $f \in \mathcal{B}_b(M)$,

$$\begin{aligned}&\Gamma(P_t f) + \frac{\rho_2(e^{2\rho_1 t} - 1 - 2\rho_1 t)}{\rho_1(e^{2\rho_1 t} - 1)}\Gamma^Z(P_t f)\\&\le \Big(1 + \frac{\kappa(e^{2\rho_1^+ t} - 1)^2}{\rho_2(e^{2\rho_1^+ t} - 1 - 2\rho_1^+ t)}\Big)\frac{2\rho_1(P_t f)\{P_t(f\log f) - (P_t f)\log P_t f\}}{e^{2\rho_1 t} - 1}.\end{aligned}$$

Proof. By an approximation argument, it suffices to prove for $f \in \mathcal{C}$. Let

$$\begin{aligned}&b_0(s) = \frac{e^{2\rho_1(t-s)} - 1}{2\rho_1},\\&b_1(s) = 2\rho_2\int_s^t b_0(r)\mathrm{d}r = \frac{\rho_2(e^{2\rho_1(t-s)} - 1 - 2\rho_1(t-s))}{2\rho_1^2}, \quad s \in [0,t].\end{aligned}$$

Then it is easy to see that $(b_1' + 2b_0\rho_2)(t) = 0$ and

$$\Big\{b_0' + 2b_0\Big(\rho_1 - \frac{\kappa b_0}{b_1}\Big)\Big\}(s) = -1 - \frac{\kappa(\mathrm{e}^{2\rho_1(t-s)}-1)^2}{\rho_2(\mathrm{e}^{2\rho_1(t-s)}-1-2\rho_1(t-s))}$$
$$\geq -1 - \frac{\kappa(\mathrm{e}^{2\rho_1^+ t}-1)^2}{\rho_2(\mathrm{e}^{2\rho_1^+ t}-1-2\rho_1^+ t)}.$$

Since (5.2.3) implies (5.2.4) for $l = 1, \Gamma^{(1)} = \Gamma^Z, K_0(r) = \rho_1 - \frac{\kappa}{r}$ and $K_1(r) = \rho_2$, the desired derivative inequalities follow from (5.2.12) and (5.2.16). □

5.2.3 *Examples*

We present some concrete examples to illustrate results derived in this section. In the first example the Poincaré and log-Sobolev inequalities are confirmed in the symmetric setting. The second example is the Kohn-Laplacian on the Heisenberg group for which condition (5.2.3) holds (see [Baudoin and Garofalo (2011)] for more examples satisfying this condition). In the last two examples (5.2.6) does not hold so that we are only able to derive results in Proposition 5.2.3. For simplicity, we make use of the notion $f_{x_{i_1}\dots x_{i_k}} := \partial_{x_{i_1}}\dots\partial_{x_{i_k}} f$ for a smooth function f on $\mathbb{R}^d$ and $1 \leq i_1, \dots, i_k \leq d, k \geq 1$.

Example 5.2.1. Let $M = \mathbb{R} \times \bar{M}$, where $\bar{M}$ is a complete connected Riemannian manifold. Let $\bar{L}$ be an elliptic differential operator on $\bar{M}$ satisfying the curvature-dimension condition

$$\bar{\Gamma}_2(f) \geq K\bar{\Gamma}(f) + \frac{(\bar{L}f)^2}{m}, \quad f \in C^\infty(\bar{M}), \tag{5.2.22}$$

for some constant $K \geq 0$ and $m \in (1,\infty)$, where $\bar{\Gamma}$ is the square field of $\bar{L}$ and $\bar{\Gamma}_2$ is the associated curvature operator, i.e. $\bar{\Gamma}_2(f) = \frac{1}{2}\bar{L}\bar{\Gamma}(f) - \bar{\Gamma}(f, \bar{L}f)$. Consider

$$Lf(x,y) = f_{xx}(x,y) - r_0 x f_x(x,y) + x^2\bar{L}f(x,\cdot)(y), \quad f \in C^\infty(M), (x,y) \in M$$

for some constant $r_0 \in \mathbb{R}$, where and in the sequel, we set

$$f_{x_1\dots x_k} = \frac{\partial^k}{\partial x_1 \dots \partial x_k} f, \ \ k \geq 1.$$

Then

$$\Gamma^{(0)}(f,g)(x,y) := \Gamma(f,g)(x,y) = (f_x g_x)(x,y) + x^2\bar{\Gamma}(f(x,\cdot), g(x,\cdot))(y).$$

Let

$$\Gamma^{(1)}(f,g)(x,y) = \bar{\Gamma}(f(x,\cdot),g(x,\cdot))(y), \quad f,g \in C^\infty(M),(x,y) \in M.$$

According to (5.2.22), there exists a positive smooth compact function $\bar{W}$ on $\bar{M}$ such that $\bar{L}\bar{W}, \bar{\Gamma}(\bar{W}) \leq 1$. In fact, let $\bar{\rho}$ be the intrinsic distance to a fixed point induced by $\bar{\Gamma}$, by (5.2.22) for $K \geq 0$ and the comparison theorem, one has (see [Qian, Z. (1998)])

$$\bar{L}\bar{\rho} \leq \frac{m-1}{\bar{\rho}}$$

outside the fixed point and the cut-locus of this point. By Greene-Wu's approximation theorem (see [Greene and Wu (1979)]), we may assume that $\bar{\rho}^2$ is smooth so that $\bar{L}\sqrt{1+\bar{\rho}^2} \leq c_1$ holds for some constant $c_1 > 0$. Noting that $\bar{\Gamma}(\bar{\rho}) = 1$, we may take $\bar{W} = \varepsilon\sqrt{1+\bar{\rho}^2}$ for small enough constant $\varepsilon > 0$.

Now, let $W(x,y) = 1 + x^2 + \bar{W}(y)$, which is a smooth compact function on M. It is easy to see that

$$\begin{aligned} LW(x,y) &\leq 2(1+r_0^-)W(x,y), \\ \tilde{\Gamma}(W)(x,y) &= 4x^2 + (1+x^2)\bar{\Gamma}(\bar{W})(y) \leq 5W(x,y), \end{aligned} \tag{5.2.23}$$

where $\tilde{\Gamma} = \Gamma + \Gamma^{(1)}$.

Proposition 5.2.8. *In* Example 5.2.1 *the generalized curvature condition* (5.2.4) *holds for* $l = 1$ *and*

$$K_1(r) = 1, \quad K_0(r) = \Big(r_0 - \frac{m}{r}\Big) \wedge \Big(Kr - r_0 - \frac{4}{r}\Big), \quad r > 0,$$

and (5.2.6) *holds. Consequently:*

(1) *Propositions 5.2.3 and 5.2.5 hold for* $b_0(0) = t, b_1(0) = t^2$ *and*

$$c_b = 1 + 2 \sup_{r\in(0,t)} \big\{(m - r_0 r) \vee (r_0 r + 4 - Kr^2)\big\}.$$

(2) *If* $K, r_0 > 0$ *and* $\bar{L}$ *is symmetric w.r.t. a probability measure* $\bar{\mu}$ *on* $\bar{M}$, *then* P_t *is symmetric w.r.t.*

$$\mu(\mathrm{d}x,\mathrm{d}y) := \Big(\frac{\sqrt{r_0}\exp[-\frac{r_0}{2}x^2]}{\sqrt{2\pi}}\,\mathrm{d}x\Big)\bar{\mu}(\mathrm{d}y),$$

and the Poincaré inequality (5.2.20) *holds for*

$$\lambda = \min\Big\{\frac{2K}{r_0 + \sqrt{r_0^2 + 20K}}, \frac{r_0}{m+1}\Big\} > 0.$$

Moreover, the log-Sobolev inequality

$$\mu(f^2\log f^2) \leq c\mu(\Gamma(f)), \quad f \in C_0^1(M), \mu(f^2) = 1$$

holds for some constant $c > 0$.

Proof. (i) The proof of (5.2.6) is trivial. Below we intend to prove (5.2.4) for the desired K_0 and K_1; that is,

$$\Gamma_2(f) + r\Gamma_2^{(1)}(f) \geq \Gamma^{(1)}(f) + \Big\{\Big(r_0 - \frac{m}{r}\Big) \wedge \Big(Kr - r_0 - \frac{4}{r}\Big)\Big\}\Gamma(f) \quad (5.2.24)$$

holds for all $f \in C^\infty(M)$. It is easy to see that at point (x, y),

$$\begin{aligned}\Gamma_2(f) &= f_{xx}^2 + (1 - r_0x^2)\Gamma^{(1)}(f) + 4x\Gamma^{(1)}(f, f_x)\\ &\quad + 2x^2\Gamma^{(1)}(f_x) + x^4\bar{\Gamma}_2(f(x,\cdot))(y) - 2xf_x\bar{L}f(x,\cdot)(y) + r_0f_x^2,\\ \Gamma_2^{(1)}(f) &= x^2\bar{\Gamma}_2(f(x,\cdot))(y) + \Gamma^{(1)}(f_x).\end{aligned}$$

Combining these with (5.2.22) we obtain

$$\begin{aligned}&\Gamma_2(f) + r\Gamma_2^{(1)}(f)\\ &\geq \Gamma^{(1)}(f) - r_0x^2\Gamma^{(1)}(f) + \big\{(2x^2 + r)\Gamma^{(1)}(f_x) + 4x\Gamma^{(1)}(f, f_x)\big\} + r_0f_x^2\\ &\quad + \Big\{\frac{(x^4 + rx^2)(\bar{L}f(x,\cdot)(y))^2}{m} - 2xf_x\bar{L}f(x,\cdot)(y)\Big\} + (x^4 + rx^2)K\Gamma^{(1)}(f)\\ &\geq \Gamma^{(1)}(f) + \Big(K(x^2 + r) - r_0 - \frac{4}{2x^2 + r}\Big)x^2\Gamma^{(1)}(f) - \frac{mx^2}{x^4 + rx^2}f_x^2 + r_0f_x^2\\ &\geq \Gamma^{(1)}(f) + \Big(Kr - r_0 - \frac{4}{r}\Big)x^2\Gamma^{(1)}(f) + \Big(r_0 - \frac{m}{r}\Big)f_x^2\\ &\geq \Gamma^{(1)}(f) + \Big\{\Big(r_0 - \frac{m}{r}\Big) \wedge \Big(Kr - r_0 - \frac{4}{r}\Big)\Big\}\Gamma(f).\end{aligned}$$

Therefore, (5.2.24) holds.

(ii) Whence (5.2.6) and (5.2.4) are confirmed for the desired K_0 and K_1, due to (5.2.23) the assumption **(A5.2.1)** holds. Then (1) follows immediately by taking

$$b_0(s) = t - s, \quad b_1(s) = (t - s)^2, \quad s \in [0, t].$$

It remains to prove the Poincaré inequality and the log-Sobolev inequality for $K, r_0 > 0$ in the symmetric setting. By Proposition 5.2.6, the Poincaré inequality holds for

$$\lambda = \sup_{r>0}\Big\{K_0(r) \wedge \frac{K_1(r)}{r}\Big\} = \sup_{r>0}\Big\{\frac{1}{r} \wedge \Big(r_0 - \frac{m}{r}\Big) \wedge \Big(Kr - r_0 - \frac{4}{r}\Big)\Big\}.$$

Since $\frac{1}{r}$ is decreasing in $r > 0$ with range $(0, \infty)$ while $\Big(r_0 - \frac{m}{r}\Big) \wedge \Big(Kr - r_0 - \frac{4}{r}\Big)$ is increasing in $r > 0$ with range $(-\infty, r_0)$, λ is reached by a unique number $r_1 > 0$ such that

$$\frac{1}{r_1} = \Big(r_0 - \frac{m}{r_1}\Big) \wedge \Big(Kr_1 - r_0 - \frac{4}{r_1}\Big).$$

Then the value of λ can be fixed by considering the following two situations:

A. If $r_0 - \frac{m}{r_1} \le Kr_1 - r_0 - \frac{4}{r_1}$, we have $\frac{1}{r_1} = r_0 - \frac{m}{r_1}$ so that $r_1 = \frac{m+1}{r_0}$ and hence, $\lambda = \frac{r_0}{m+1}$.
B. If $Kr_1 - r_0 - \frac{4}{r_1} < r_0 - \frac{m}{r_1}$, then $\frac{1}{r_1} = Kr_1 - r_0 - \frac{4}{r_1}$ so that $r_1 = \frac{r_0 + \sqrt{r_0^2 + 20K}}{2K}$ and $\lambda = \frac{2K}{r_0 + \sqrt{r_0^2+20K}}$.

To prove the validity of the log-Sobolev inequality, we observe that

$$c_b \le 1 + 2\Big\{m \wedge \Big(\frac{r_0^2}{4K} + 4\Big)\Big\} =: c_0.$$

Moreover, by the Meyer diameter theorem (see [Bakry and Ledoux (1996a)] and references within), (5.2.22) with $K > 0$ implies that the intrinsic distance induced by $\bar\Gamma$ is bounded by a constant $D > 0$. Noting that

$$\Gamma_b(f)(x,y) := tf_x^2(x,y) + t^2\bar\Gamma(f(x,\cdot))(y),$$

the associated distance satisfies

$$\rho_b^2((x,y),(x',y')) \le \frac{|x-x'|^2}{t} + \frac{D^2}{t^2}, \quad t>0, (x,y),(x',y') \in M.$$

Thus, for any $\lambda > \frac{c_0}{4}\big(\ge \frac{c_b}{4}\big)$, $\mu(e^{\lambda \rho_b(o,\cdot)^2}) < \infty$ holds for $o \in M$ and large $t > 0$. Combining this with the Harnack inequality (5.2.17), we see that $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ holds for some $t > 0$, so that according to [Gross (1976)], the defective log-Sobolev inequality holds. As explained in the proof of Proposition 5.1.9, in the subelliptic setting the defective log-Sobolev inequality holds if and only if so does the exact one. □

Example 5.2.2. Consider the Kohn-Laplacian operator $\mathbf{L} = X^2 + Y^2$ in Example 5.0.2. We have $[X,Y] = Z := \partial_z$. Let $\Gamma^Z(f,g) = (Zf)(Zg)$. Then (5.2.2) holds and

$$\begin{aligned}\Gamma_2(f) &= (X^2f)^2 + (XYf)^2 + (YXf)^2 + (Y^2f)^2 \\ &\quad + (Xf)\{(Y^2X - XY^2)f\} + (Yf)\{(X^2Y - YX^2)f\} \\ &\ge \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf), \\ \Gamma_2^Z(f) &= (XZf)^2 + (YZf)^2.\end{aligned}$$

Therefore, (5.2.3) holds for $\rho_1 = 0, \rho_2 = \frac{1}{2}$ and $\kappa = 1$. Therefore, all assertions in Propositions 5.2.3 and 5.2.5 hold.

Example 5.2.3. Consider the Gruschin operator $Lf = f_{xx} + x^{2l} f_{yy}$ on $M := \mathbb{R}^2$, where $l \in \mathbb{N}$. We have

$$\Gamma^{(0)}(f,g)(x,y) := \Gamma(f,g)(x,y) = (f_x g_x)(x,y) + x^{2l}(f_y g_y)(x,y)$$

and $L = X^2 + Y^2$ for $X = \frac{\partial}{\partial x}, Y = x^l \frac{\partial}{\partial y}$. When $l \geq 2$, $\{X, Y, \nabla_X Y = lx^{l-1}\frac{\partial}{\partial y}, \nabla_Y X = 0\}$ does not span the whole space for $x = 0$. Let

$$\Gamma^{(i)}(f,g)(x,y) = x^{2(l-i)}(f_y g_y)(x,y), \quad 1 \leq i \leq l.$$

It is easy to see that $W(x,y) := 1 + x^2 + \frac{y^2}{1+x^{2l}}$ is a smooth compact function such that

$$LW \leq CW, \quad \tilde{\Gamma}(W) \leq CW^2 \tag{5.2.25}$$

holds for some constant $C > 0$.

Proposition 5.2.9. *In* Example 5.2.3 *there exist two constants $\alpha, \beta > 0$ depending only on l such that* (5.2.4) *holds for*

$$K_0(r_1, \ldots, r_l) = -\alpha \sum_{i=1}^{l} \frac{r_{i-1}^{i-1}}{r_i^i},$$

$$K_i(r_1, \ldots, r_l) = \beta r_{i-1}, \quad 1 \leq i \leq l, r_0 = 1, r_i > 0.$$

Consequently, Proposition 5.2.3 holds for $b_i(0) = c_i t^{2l-1+i}$, where

$$c_0 = 1, \ c_i = \frac{2\beta c_{i-1}}{2l-1+i}, \quad 1 \leq i \leq l,$$

and

$$c_b = \sup_{r\in(0,t)} \left\{(2l-1)r^{2(l-1)} + 2\alpha \sum_{i=1}^{l} \frac{c_{i-1}^{i-1}}{c_i^i} r^{2(l-i)}\right\} \leq C_0(1 + t^{l-1}), \quad t > 0$$

for some constant $C_0 > 0$.

Proof. According to Proposition 5.2.3 for $b_i(s) = c_i(t-s)^{2l-1+i}$, $s \in [0,t], 0 \leq i \leq l$, it suffices to verify (5.2.4) for the desired $\{K_i\}_{0 \leq i \leq l}$, which satisfy

$$b_i'(s) + 2b_0(s) K_i\Big(\frac{b_1}{b_0}, \ldots, \frac{b_l}{b_0}\Big)(s) = 0, \quad 1 \leq i \leq l, s \in [0,t]$$

and

$$\begin{aligned} &-\Big\{b_0' + 2b_0 K_0\Big(\frac{b_1}{b_0}, \ldots, \frac{b_l}{b_0}\Big)\Big\}(s) \\ &= (2l-1)(t-s)^{2(l-1)} + 2\alpha \sum_{i=1}^{l} \frac{c_{i-1}^{i-1}}{c_i^i}(t-s)^{2(l-i)}, \quad s \in [0,t]. \end{aligned}$$

It is easy to see that at point $(x,y)\in\mathbb{R}^2$ and for $1\le i\le l$,

$$\begin{aligned}\Gamma_2(f) &= f_{xx}^2 + l(2l-1)x^{2(l-1)}f_y^2 + x^{4l}f_{yy}^2 + 2x^{2l}f_{xy}^2\\ &\quad + 4lx^{2l-1}f_yf_{xy} - 2lx^{2l-1}f_xf_{yy},\\ \Gamma_2^{(i)}(f) &= (l-i)(2l-2i-1)x^{2(l-i-1)}f_y^2 + 4(l-i)x^{2l-2i-1}f_yf_{xy}\\ &\quad + x^{2(l-i)}f_{xy}^2 + x^{2(2l-i)}f_{yy}^2.\end{aligned}$$

So, for $r_0=1$ and $r_i>0, 1\le i\le l$,

$$\begin{aligned}&\Gamma_2(f)+\sum_{i=1}^l r_i\Gamma_2^{(i)}(f)\\ &\ge f_y^2\sum_{i=0}^l r_i(l-i)(2l-2i-1)x^{2(l-i-1)} + f_{yy}^2\sum_{i=0}^l r_ix^{2(2l-i)} - 2lx^{2l-1}f_xf_{yy}\\ &\qquad + f_{xy}^2\Big\{2x^{2l}+\sum_{i=1}^l r_ix^{2(l-i)}\Big\} + f_yf_{xy}\sum_{i=0}^l 4r_i(l-i)x^{2l-2i-1}\\ &\ge f_y^2\sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)}\\ &\qquad -\frac{l^2}{r_1}f_x^2 - 4\sum_{i=0}^{l-1}\frac{r_i^2(l-i)^2x^{4l-4i-2}}{r_{i+1}x^{2(l-i-1)}}f_y^2\\ &\ge \frac{f_y^2}{2}\sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)} - \frac{l^2}{r_1}f_x^2\\ &\qquad -\frac12\sum_{i=1}^l\Big\{\frac{8r_{i-1}^2(l+1-i)^2x^{2(l+1-i)}}{r_i}\\ &\qquad\qquad - r_{i-1}(l+1-i)(2l-2i+1)x^{2(l-i)}\Big\}f_y^2\\ &\qquad\ge \frac12\sum_{i=1}^l r_{i-1}(l+1-i)(2l-2i+1)\Gamma^{(i)}(f)\\ &\qquad\qquad -\frac{l^2}{r_1}f_x^2 - \sum_{i=1}^l\frac{\alpha_ir_{i-1}^{i-1}}{r_i^i}x^{2l}f_y^2\end{aligned}$$

holds for some constants $\alpha_i>0, 1\le i\le l$, where the last step is due to the fact that for constants $A_i, B_i>0$,

$$A_ix^{-2(i-1)} - B_ix^{-2i} \le \sup_{s>0}\big\{A_is^{i-1}-B_is^i\big\} = \frac{(i-1)^{i-1}A_i^i}{i^iB_i^{i-1}}. \qquad \square$$

Example 5.2.4. Consider $L = X_1^2 + X_2^2 + X_3^2$ on $\mathbb{R}^3$, where $X_1 = \frac{\partial}{\partial x}, X_2 = x\frac{\partial}{\partial y}, X_3 = y\frac{\partial}{\partial z}$. We have

$$\begin{aligned}\Gamma^{(0)}(f,g)(x,y,z) &:= \Gamma(f,g)(x,y,z)\\ &= (f_x g_x)(x,y,z) + x^2(f_y g_y)(x,y,z) + y^2(f_z g_z)(x,y,z).\end{aligned}$$

Let

$$\begin{aligned}\Gamma^{(1)}(f,g)(x,y,z) &= (f_y g_y)(x,y,z) + x^2(f_z g_z)(x,y,z),\\ \Gamma^{(2)}(f,g)(x,y,z) &= (f_z g_z)(x,y,z).\end{aligned}$$

It is easy to see that $W(x,y,z) := 1 + x^2 + y^2$ is a smooth compact function on $\mathbb{R}^3$ such that (5.2.25) holds for some constant $C > 0$.

Proposition 5.2.10. *In* Example 5.2.4 (5.2.4) *holds for*

$$K_0(r_1,r_2) = -\Big(\frac{5}{r_1} + \frac{2r_1}{r_2}\Big),\ K_1(r_1,r_2) = 1 - \frac{4r_1^2}{r_2},\ K_2(r_1,r_2) = r_1,$$

here $r_1, r_2 > 0$. *Consequently, Proposition 5.2.3 holds for*

$$b_0(0) = t,\quad b_1(0) = \frac{t^2}{7},\quad b_2(0) = \frac{2t^3}{21},$$

and $c_b = 77$.

Proof. We first prove (5.2.4) for the desired $K_i, 0 \le i \le 2$. It is easy to see that at point (x,y,z),

$$\begin{aligned}\Gamma_2(f) &= f_{xx}^2 + f_y^2 + x^2 f_z^2 + 2x^2 f_{xy}^2 + 2y^2 f_{xz}^2 + x^4 f_{yy}^2 + 2x^2y^2 f_{yz}^2 + y^4 f_{zz}^2\\ &\quad + 4x f_y f_{xy} + 4x^2 y f_z f_{yz} - 2x f_x f_{yy} - 2x^2 y f_y f_{zz},\\ \Gamma_2^{(1)}(f) &= f_z^2 + f_{xy}^2 + x^2 f_{yy}^2 + (y^2 + x^4) f_{yz}^2 + x^2 f_{xz}^2 + x^2y^2 f_{zz}^2\\ &\quad + 4x f_z f_{xz} - 2y f_y f_{zz},\\ \Gamma_2^{(2)}(f) &= f_{xz}^2 + x^2 f_{yz}^2 + y^2 f_{zz}^2.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\Gamma_2(f)+r_1\Gamma_2^{(1)}(f)+r_2\Gamma^{(2)}(f)\\
&\ge \Gamma^{(1)}(f)+r_1\Gamma^{(2)}(f)+\big\{(2x^2+r_1)f_{xy}^2+4xf_yf_{xy}\big\}\\
&\quad+\big\{(2y^2+r_1x^2+r_2)f_{xz}^2+4r_1xf_zf_{xz}\big\}+\big\{(x^4+r_1x^2)f_{yy}^2-2xf_xf_{yy}\big\}\\
&\quad+\big\{(2x^2y^2+r_1y^2+r_1x^4+r_2x^2)f_{yz}^2+4x^2yf_zf_{yz}\big\}\\
&\quad+\big\{(y^4+r_1x^2y^2+r_2y^2)f_{zz}^2-2x^2yf_yf_{zz}-2r_1yf_yf_{zz}\big\}\\
&\ge \Gamma^{(1)}(f)+r_1\Gamma^{(2)}(f)-\frac{4x^2}{2x^2+r_1}f_y^2-\frac{4r_1^2x^2}{r_2+r_1x^2+2y^2}f_z^2-\frac{x^2}{x^4+r_1x^2}f_x^2\\
&\quad-\frac{4x^4y^2}{2x^2y^2+r_1y^2+r_1x^4+r_2x^2}f_z^2-\frac{(r_1+x^2)^2}{y^2+r_1x^2+r_2}f_y^2\\
&\ge \Gamma^{(1)}(f)+r_1\Gamma^{(2)}(f)-\frac{4x^2}{r_1}f_y^2-\frac{4r_1^2x^2}{r_2}f_z^2-\frac{1}{r_1}f_x^2\\
&\quad-\frac{4y^2}{r_1}f_z^2-\frac{x^2}{r_1}f_y^2-\frac{2r_1x^2}{r_2}f_y^2-\frac{r_1^2}{r_2}f_y^2\\
&\ge\Big(1-\frac{4r_1^2}{r_2}\Big)\Gamma^{(1)}(f)+r_1\Gamma^{(2)}(f)-\Big(\frac{5}{r_1}+\frac{2r_1}{r_2}\Big)\Gamma(f).
\end{aligned}
$$

This implies (5.2.4) for the claimed $K_i, 0\le i\le 2$.

Next, take

$$b_0(s)=t-s,\quad b_1(s)=\frac{1}{7}(t-s)^2,\quad b_2(s)=\frac{2}{21}(t-s)^3,\quad s\in[0,t].$$

Then

$$
\begin{aligned}
&\Big\{b_1'+2b_0K_1\Big(\frac{b_1}{b_0},\frac{b_2}{b_0}\Big)\Big\}(s)=-\frac{2(t-s)}{7}+2(t-s)\Big(1-\frac{6}{7}\Big)=0,\\
&\Big\{b_2'+2b_0K_2\Big(\frac{b_1}{b_0},\frac{b_2}{b_0}\Big)\Big\}(s)=-\frac{6(t-s)^2}{21}+\frac{2(t-s)^2}{7}=0,\\
&\Big\{b_0'+2b_0K_0\Big(\frac{b_1}{b_0},\frac{b_2}{b_0}\Big)\Big\}(s)=-1-2(t-s)\Big(\frac{35}{t-s}+\frac{3}{t-s}\Big)=-77.
\end{aligned}
$$

Therefore, the second assertion holds. □

5.2.4 *An extension of Theorem 5.2.1*

If $\mathcal{H}_0$ does not satisfy the Hörmander condition, (5.2.4) may only hold for non-positively definite differential forms $\Gamma^{(i)}$ and some (not all) $r_1,\dots,r_l>0$. For instance, when $L=\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial y}$, one has $\Gamma(f)=f_x^2$ and $\Gamma_2(f)=f_{xx}^2-f_xf_y$. So, to verify (5.2.4), it is natural to take $\Gamma^{(1)}(f,g)=-\frac{1}{2}(f_xg_y+f_yg_x)$, which is however not positively definite. See Example 5.2.5 below for details.

To investigate such operators, we make use of the following weaker version of assumption **(A5.2.1)**. We call bilinear symmetric form $\bar{\Gamma}$: $C^3(M) \times C^3(M) \to C^2(M)$ a C^2 symmetric differential form, if

$$\bar{\Gamma}(fg, h) = f\bar{\Gamma}(g, h) + g\bar{\Gamma}(f, h), \quad \bar{\Gamma}(f, \phi \circ g) = (\phi' \circ g)\bar{\Gamma}(f, g)$$

holds for all $f, g, h \in C^3(M)$, $\phi \in C^1(\mathbb{R})$.

(A5.2.2) There exist some C^2 symmetric differential forms $\{\Gamma^{(i)}\}_{1\le i\le l}$, a non-empty set $\Omega \subset (0,\infty)^l$, a smooth compact function $W \ge 1$, and some functions $\{K_i\}_{0\le i\le l} \subset C(\Omega)$ such that

$(B1)$ $\Gamma_2(f) + \sum_{i=1}^l r_i \Gamma_2^{(i)}(f) \ge \sum_{i=0}^l K_i(r_1, \dots, r_l)\Gamma^{(i)}(f)$ holds for all $f \in C^3(M)$ and $(r_1, \dots, r_l) \in \Omega$, where $\Gamma^{(0)} = \Gamma$.
$(B2)$ $LW \le CW$ and $\sum_{i=0}^l |\Gamma^{(i)}(W)| \le CW^2$ hold for some constant $C > 0$.
$(B3)$ There exist $\varepsilon > 0$ and $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_l) \in \Omega$ such that

$$\tilde{\Gamma}(f) := \Gamma(f) + \sum_{i=1}^l \tilde{r}_i \Gamma^{(i)}(f) \ge \varepsilon \sum_{i=0}^l |\Gamma^{(i)}(f)|, \quad f \in C^1(M).$$

Theorem 5.2.11. *Assume* **(A5.2.2)**. *For fixed* $t > 0$, *let* $\{b_i\}_{0\le i\le l} \subset C^1([0,t])$ *be strictly positive on* $(0,t)$ *such that*

(*i*) $(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0})(s) \in \Omega$ *holds for all* $s \in (0,t)$;
(*ii*) $b_i'(s) + 2\{b_0 K_i(\frac{b_1}{b_0}, \dots, \frac{b_l}{b_0})\}(s) = 0, \quad s \in (0,t), 1 \le i \le l.$

Then assertions in (1) *and* (2) *of Theorem* 5.2.1 *hold.*

Proof. By $(B1)$ and $(B3)$, $\tilde{\Gamma}_2 \ge K\tilde{\Gamma}$ and $\tilde{\Gamma} \ge \varepsilon \sum_{i=0}^l |\Gamma^{(i)}|$ hold for some $K \in \mathbb{R}$ and $\varepsilon > 0$. Combining these with $(B2)$ and repeating the proof of Lemma 5.2.2, we conclude that $\{\Gamma^{(i)}(P_\cdot f)\}_{0\le i\le l}$ are bounded on $[0,t] \times M$. Therefore, due to (*i*) and (*ii*) the proof of Theorem 5.2.1 works also for the present case. Since $\{\Gamma^{(i)}\}_{1\le i\le l}$ might be not positively definite, the equality in (*ii*) cannot be replaced by $\ge$. □

To illustrate this result, we consider the following example which was also mentioned in the beginning of this section, where the resulting gradient and Harnack inequalities have the same time behaviors as the corresponding ones presented in Corollaries 3.2 and 4.2 in [Gong and Wang (2002)] by using coupling methods. In this example, it is easy to find correct choices of W, $\Gamma^{(i)}$, K_i and Ω such that assumption **(A5.2.2)** and condition (i) in Theorem 5.2.11 hold. The technical (also difficult) point is to construct functions $\{b_i\}_{i=0}^l$ such that condition (ii) holds and $\sum_{i=0}^l b_i(0)\Gamma^{(i)}$ is an elliptic square field.

Example 5.2.5. Consider $L = \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. We have

$$\Gamma^{(0)}(f,g) := \Gamma(f,g) = f_x g_x.$$

Let

$$\Gamma^{(1)}(f,g) = -\frac{1}{2}(f_x g_y + f_y g_x), \quad \Gamma^{(2)}(f,g) = f_y g_y.$$

Then **(A5.2.2)** holds for $W(x,y) = 1 + x^2 + y^2, \Omega = \{(r_1, r_2) : r_1, r_2 > 0, r_1^2 \le 4r_2\}$, and

$$K_0(r_1,r_2) = 0, \quad K_1(r_1,r_2) = 1, \quad K_2(r_1,r_2) = \frac{r_1}{2}.$$

Moreover, (5.2.6) holds. Then

$$\begin{aligned}&\frac{2-\sqrt{3}}{2}\Big\{t(P_t f)_x^2 + \frac{t^3}{3}(P_t f)_y^2\Big\}\\ &\le (P_t f)\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}, \quad t>0,\end{aligned} \tag{5.2.26}$$

and hence,

$$\begin{aligned}&(P_t f)^\alpha((x,y))\\ &\le (P_t f^\alpha(x',y'))\exp\left[\frac{\alpha}{2(2-\sqrt{3})(\alpha-1)}\Big(\frac{|x-x'|^2}{t} + \frac{3|y-y'|^2}{t^3}\Big)\right]\end{aligned} \tag{5.2.27}$$

holds for all $\alpha > 1, t > 0, (x,y), (x',y') \in \mathbb{R}^2$ and positive $f \in \mathcal{B}_b(\mathbb{R}^2)$.

Proof. Obviously, $(B2)$ holds for the given W and $(B3)$ holds for $\tilde{r}_1 = \tilde{r}_2 = 1$ (hence $(\tilde{r}_1, \tilde{r}_2) \in \Omega$) and $\varepsilon = \frac{1}{4}$. Next, it is easy to see that (5.2.6) holds and

$$\Gamma_2(f) = f_{xx}^2 - f_x f_y, \quad \Gamma_2^{(1)}(f) = \frac{1}{2}f_y^2 - f_{xx}f_{xy}, \quad \Gamma_2^{(2)}(f) = f_{xy}^2.$$

Then, for $r_1, r_2 > 0$ with $r_1^2 \le 4r_2$, we have

$$\Gamma_2(f) + r_1\Gamma_2^{(1)}(f) + r_2\Gamma_2^{(2)}(f) \ge \Gamma^{(1)}(f) + \frac{r_1}{2}\Gamma^{(2)}(f).$$

Therefore, $(B1)$ holds.

To prove (5.2.26) and (5.2.27), we take $l = 2$ and

$$b_0(s) = t - s, \; b_1(s) = (t-s)^2, \; b_2(s) = \frac{(t-s)^3}{3}, \quad s \in [0,t].$$

Then it is easy to see that (i) and (ii) in Theorem 5.2.11 hold. Noting that $-b_0' = 1, K_0 = 0$ and $b_i(t) = 0$ for $0 \le i \le 2$, it follows from Theorem 5.2.11 that

$$\begin{aligned}
&(P_t f)\big\{P_t(f\log f) - (P_t f)\log P_t f\big\} \ge \sum_{i=0}^{2} b_i(0)\Gamma^{(i)}(P_t f)\\
&= t(P_t f)_x^2 - t^2(P_t f)_x(P_t f)_y + \frac{t^3}{3}(P_t f)_y^2\\
&\ge \frac{2-\sqrt{3}}{2}\Big\{t(P_t f)_x^2 + \frac{t^3}{3}(P_t f)_y^2\Big\}.
\end{aligned}$$

Therefore, (5.2.26) holds.

Finally, letting $\bar\rho$ be the intrinsic distance induced by the square field

$$\bar\Gamma(f) := \frac{2-\sqrt{3}}{2}\Big\{(t\wedge t_\theta)f_x^2 + \frac{t^3}{3}f_y^2\Big\},$$

we have

$$\bar\rho((x,y),(x',y'))^2 = \frac{2}{2-\sqrt{3}}\Big(\frac{|x-x'|^2}{t\wedge t_\theta} + \frac{3|y-y'|^2}{(t\wedge t_\theta)^3}\Big).$$

Then the desired Harnack inequality follows in the spirit of Proposition 1.5.1 (see also Lemma 3.4 in [Wang (2012c)]) since (5.2.26) is equivalent to

$$\sqrt{\bar\Gamma(P_t f)} \le \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\} + \frac{P_t f}{4\delta},\quad \delta > 0.$$

□

5.3 Stochastic Hamiltonian system: Coupling method

This section is organized from [Guillin and Wang (2012)], where coupling by change of measure is used to derive Bismut formula and Harnack inequality for generalized stochastic Hamiltonian systems. Let $\sigma_\cdot \in C([0,\infty);\mathbb{R}^d\otimes\mathbb{R}^d)$ be such that σ_t is invertible for every $t \ge 0$, $A \in \mathbb{R}^m\otimes\mathbb{R}^d$ with rank m, $(B_t)_{t\ge0}$ be a d-dimensional Brownian motion, and $Z_\cdot \in C^1(\mathbb{R}^m\times\mathbb{R}^d;\mathbb{R}^d)$. Consider the following degenerate stochastic differential equation on $\mathbb{R}^m\times\mathbb{R}^d$:

$$\begin{cases}
\mathrm{d}X_t^{(1)} = AX_t^{(2)}\mathrm{d}t,\\
\mathrm{d}X_t^{(2)} = \sigma_t\mathrm{d}B_t + Z_t(X_t^{(1)}, X_t^{(2)})\mathrm{d}t.
\end{cases} \tag{5.3.1}$$

We shall use $(X_t^{(1)}(x), X_t^{(2)}(x))$ to denote the solution with initial data $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d} := \mathbb{R}^m\times\mathbb{R}^d$. Let

$$P_t f(x) = \mathbb{E}f(X_t^{(1)}(x), X_t^{(2)}(x)),\quad t\ge 0, x\in\mathbb{R}^{m+d}.$$

We aim to establish the Harnack inequality and derivative formula for P_t. The generator of the solution to (5.3.1) is

$$L_s(x) = \frac{1}{2}\sum_{i,j=m+1}^{d+m}(\sigma_s\sigma_s^*)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}$$
$$+\sum_{j=m+1}^{d+m}(Z_s(x))_j\frac{\partial}{\partial x_j}+\sum_{i=1}^{m}(Ax^{(2)})_i\frac{\partial}{\partial x_i},\quad s\ge 0, x=(x^{(1)},x^{(2)})\in\mathbb{R}^{m+d}.$$

In the case where $m=d$, $\sigma_t=A=I$ and

$$Z_t(x) = -\nabla V(x^{(1)}) - cx^{(2)},$$

this model is known as "stochastic damping Hamiltonian system" in probability see [Wu (2001); Bakry *et al* (2008)] (see also [Soize (1994)] for more general model of stochastic Hamiltonian system).

5.3.1 *Derivative formulae*

Since A has rank m, we have $d\ge m$ and for any $z^{(1)}\in\mathbb{R}^m$, the set

$$A^{-1}z^{(1)} := \{z^{(2)}\in\mathbb{R}^d:\ Az^{(2)}=z^{(1)}\}\neq\emptyset.$$

For any $z^{(1)}\in\mathbb{R}^m$, let

$$|A^{-1}z^{(1)}| = \inf\{|\tilde z^{(2)}|:\ \tilde z^{(2)}\in A^{-1}z^{(1)}\}.$$

Then it is clear that

$$\|A^{-1}\| := \sup\left\{|A^{-1}z^{(1)}|:\ z^{(1)}\in\mathbb{R}^m, |z^{(1)}|\le 1\right\}<\infty.$$

We shall use $|\cdot|$ to denote the absolute value and the norm in Euclidean spaces, and use $\|\cdot\|$ to denote the operator norm of a matrix. For $z\in\mathbb{R}^{m+d}$, we use ∇_z to stand for the directional derivative along z.

Let us introduce now the assumption that we will use in the sequel:

(A5.3.1) There exists a constant $C>0$ such that $L_sW\le CW$ and

$$|Z_s(x)-Z_s(y)|^2\le C|x-y|^2W(y),\quad x,y\in\mathbb{R}^{m+d}, |x-y|\le 1$$

hold for some Lyapunov function W and $s\in[0,t]$.

The main result in this section provides various different versions of derivative formula by making different choices of the pair functions (u,v).

Theorem 5.3.1. *Assume* **(A5.3.1)**. *Then the process* $(X_t^{(1)}, X_t^{(2)})_{t\geq 0}$ *is non-explosive for any initial point in* $\mathbb{R}^{m+d}$. *Moreover, let* $t > 0$ *and* $u, v \in C^2([0,t])$ *be such that*

$$u(t) = v'(0) = 1, \quad u(0) = v(0) = u'(0) = u'(t) = v'(t) = v(t) = 0. \tag{5.3.2}$$

Then for any $z = (z^{(1)}, z^{(2)}) \in \mathbb{R}^m \times \mathbb{R}^d$ *and* $\bar{z}^{(2)} \in A^{-1}z^{(1)}$,

$$\begin{aligned}\nabla_z P_t f = \mathbb{E}\bigg\{ f(X_t^{(1)}, X_t^{(2)}) \int_0^t \Big\langle \sigma_s^{-1}\big\{u''(s)\bar{z}^{(2)} - v''(s)z^{(2)} \\ + (\nabla_{\Theta(z,\bar{z}^{(2)},s)} Z_s)(X_s^{(1)}, X_s^{(2)})\big\}, \mathrm{d}B_s\Big\rangle\bigg\}\end{aligned} \tag{5.3.3}$$

holds for $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$, *where*

$$\Theta(z, \bar{z}^{(2)}, s) = \big(\{1 - u(s)\}z^{(1)} + v(s)Az^{(2)},\ v'(s)z^{(2)} - u'(s)\bar{z}^{(2)}\big).$$

Proof. The non-explosion follows since $L_s W \leq CW$ implies

$$\mathbb{E}W(X_s^{(1)}(x), X_s^{(2)}(x)) \leq W(x)\mathrm{e}^{Cs}, \quad s \in [0,t], x \in \mathbb{R}^{m+d}. \tag{5.3.4}$$

To prove (5.3.3), we make use of the coupling by change of measure. Since the process is now degenerate, the construction of coupling is highly technical: we have to force the coupling to be successful before a fixed time by using a lower dimensional noise.

Let $t > 0, x = (x^{(1)}, x^{(2)}), z = (z^{(1)}, z^{(2)}) \in \mathbb{R}^{m+d}$ and $\bar{z}^{(2)} \in A^{-1}z^{(1)}$ be fixed. Simply denote $(X_s^{(1)}, X_s^{(2)}) = (X_s^{(1)}(x), X_s^{(2)}(x))$. Let

$$\varepsilon_0 = \inf_{s\in[0,t]} \frac{1}{1 \vee |\Theta(z, \bar{z}^{(2)}, s)|} > 0,$$

so that $\varepsilon_0|\Theta(z, \bar{z}^{(2)}, s)| \leq 1$ for $s \in [0,t]$. For any $\varepsilon \in (0, \varepsilon_0)$, let $(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)})$ solve the equation

$$\begin{cases} \mathrm{d}X_s^{(1,\varepsilon)} = AX_s^{(2,\varepsilon)}\mathrm{d}s, \\ \mathrm{d}X_s^{(2,\varepsilon)} = \sigma_s \mathrm{d}B_s + Z_s(X_s^{(1)}, X_s^{(2)})\mathrm{d}s \\ \qquad\qquad + \varepsilon\{v''(s)z^{(2)} - u''(s)\bar{z}^{(2)}\}\mathrm{d}s \end{cases} \tag{5.3.5}$$

with $X_0^{(1,\varepsilon)} = x^{(1)} + \varepsilon z^{(1)}$ and $X_0^{(2,\varepsilon)} = x^{(2)} + \varepsilon z^{(2)}$. By (5.3.2) and noting that $A\bar{z}^{(2)} = z^{(1)}$, we have

$$\begin{cases} X_s^{(2,\varepsilon)} = X_s^{(2)} + \varepsilon v'(s)z^{(2)} - \varepsilon u'(s)\bar{z}^{(2)}, \\ X_s^{(1,\varepsilon)} = x^{(1)} + \varepsilon z^{(1)} + A\int_0^s X_r^{(2,\varepsilon)}\mathrm{d}r \\ \qquad\quad\ = X_s^{(1)} + \varepsilon\{1 - u(s)\}z^{(1)} + \varepsilon v(s)Az^{(2)}. \end{cases} \tag{5.3.6}$$

Due to (5.3.2), this in particular implies

$$(X_t^{(1)}, X_t^{(2)}) = (X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)}), \tag{5.3.7}$$

and also that

$$(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)}) = (X_s^{(1)}, X_s^{(2)}) + \varepsilon\Theta(z, \tilde{z}^{(2)}, s), \quad s \in [0,t]. \tag{5.3.8}$$

On the other hand, let

$$\xi_\varepsilon(s) = Z_s(X_s^{(1)}, X_s^{(2)}) - Z_s(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)}) + \varepsilon v''(s)z^{(2)} - \varepsilon u''(s)\tilde{z}^{(2)}$$

and

$$R_\varepsilon(s) = \exp\left[-\int_0^s \langle \sigma_r^{-1}\xi_\varepsilon(r), \mathrm{d}B_r\rangle - \frac{1}{2}\int_0^s |\sigma_r^{-1}\xi_\varepsilon(r)|^2 \mathrm{d}r\right] \tag{5.3.9}$$

for $s \in [0,t]$. We have

$$\mathrm{d}X_s^{(2,\varepsilon)} = \sigma_s \mathrm{d}B_s^\varepsilon + Z_s(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)})\mathrm{d}s$$

for

$$B_s^\varepsilon := B_s + \int_0^s \sigma_r^{-1}\xi_\varepsilon(r)\mathrm{d}r, \quad s \in [0,t],$$

which is d-dimensional Brownian motion under the probability measure $\mathbb{Q}_\varepsilon := R_\varepsilon(t)\mathbb{P}$ according to Lemma 5.3.2 below and the Girsanov theorem. Thus, due to (5.3.7) we have

$$P_t f(x + \varepsilon z) = \mathbb{E}_{\mathbb{Q}_\varepsilon} f(X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)}) = \mathbb{E}[R_\varepsilon(t) f(X_t^{(1)}, X_t^{(2)})].$$

Since $P_t f(x) = \mathbb{E} f(X_t^{(1)}, X_t^{(2)})$, we arrive at

$$P_t f(x + \varepsilon z) - P_t f(x) = \mathbb{E}[(R_\varepsilon(t) - 1) f(X_t^{(1)}, X_t^{(2)})].$$

The proof is then completed by Lemma 5.3.3. □

Lemma 5.3.2. *If* **(A5.3.1)** *holds, then*

$$\sup_{s\in[0,t],\varepsilon\in(0,\varepsilon_0)} \mathbb{E}\big(R_\varepsilon(s)\log R_\varepsilon(s)\big) < \infty.$$

Consequently, for each $\varepsilon \in (0, \varepsilon_0)$, $(R_\varepsilon(s))_{s\in[0.t]}$ is a uniformly integrable martingale.

Proof. Let

$$\tau_n = \inf\{t \ge 0 : |X_t^{(1)}(x)| + |X_t^{(2)}(x)| \ge n\}, \quad n \ge 1.$$

Then $\tau_n \uparrow \infty$ as $n \uparrow \infty$. By the Girsanov theorem, $(R_\varepsilon(s \wedge \tau_n))_{s\in[0,t]}$ is a martingale and $\{B_s^\varepsilon : 0 \le s \le t \wedge \tau_n\}$ is a Brownian motion under the probability measure $\mathbb{Q}_{\varepsilon,n} := R_\varepsilon(t \wedge \tau_n)\mathbb{P}$. Noting that

$$\log R_\varepsilon(s\wedge\tau_n) = -\int_0^{s\wedge\tau_n} \langle \sigma_r^{-1}\xi_\varepsilon(r), \mathrm{d}B_r^\varepsilon\rangle + \frac{1}{2}\int_0^{s\wedge\tau_n} |\sigma_r^{-1}\xi_\varepsilon(r)|^2 \mathrm{d}r, \quad s \in [0,t],$$

where the stochastic integral is a $\mathbb{Q}_{\varepsilon,n}$-martingale, we have

$$\begin{aligned}\mathbb{E}[R_\varepsilon(s\wedge\tau_n)\log R_\varepsilon(s\wedge\tau_n)] &= \mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}[\log R_\varepsilon(s\wedge\tau_n)]\\ &\le \frac{1}{2}\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{t\wedge\tau_n}|\sigma_r^{-1}\xi_\varepsilon(r)|^2\mathrm{d}r,\quad s\in[0,t].\end{aligned}\tag{5.3.10}$$

Noting that by **(A5.3.1)** and (5.3.8)

$$|\sigma_r^{-1}\xi_\varepsilon(r)|^2 \le c\varepsilon^2 W(X_r^{(1,\varepsilon)},X_r^{(2,\varepsilon)}),\quad r\in[0,t]\tag{5.3.11}$$

holds for some constant $c>0$, and moreover under the probability measure $\mathbb{Q}_{\varepsilon,n}$ the process $(X_s^{(1,\varepsilon)},X_s^{(2,\varepsilon)})_{s\le t\wedge\tau_n}$ is generated by L_s, $L_sW\le CW$ implies

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{s\wedge\tau_n}W(X_r^{(1,\varepsilon)},X_r^{(2,\varepsilon)})\mathrm{d}r &\le \int_0^s \mathbb{E}_{\mathbb{Q}_\varepsilon}W(X_r^{(1,\varepsilon)},X_r^{(2,\varepsilon)})\mathrm{d}r\\ &\le W(X_0^{(1,\varepsilon)},X_0^{(2,\varepsilon)})\int_0^t \mathrm{e}^{Cr}\mathrm{d}r.\end{aligned}\tag{5.3.12}$$

Combining this with (5.3.10) we obtain

$$\mathbb{E}[R_\varepsilon(s\wedge\tau_n)\log R_\varepsilon(s\wedge\tau_n)]\le c,\quad s\in[0,t],\varepsilon\in(0,\varepsilon_0),n\ge 1\tag{5.3.13}$$

for some constant $c>0$. Since for each n the process $(R_\varepsilon(s\wedge\tau_n))_{s\in[0,t]}$ is a martingale, letting $n\to\infty$ in the above inequality we complete the proof. □

Lemma 5.3.3. *If* **(A5.3.1)** *holds then the family* $\left\{\frac{|R_\varepsilon(t)-1|}{\varepsilon}\right\}_{\varepsilon\in(0,\varepsilon_0)}$ *is uniformly integrable w.r.t.* $\mathbb{P}$. *Consequently,*

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{R_\varepsilon(t)-1}{\varepsilon} = \int_0^t \Big\langle \sigma_s^{-1}\{&u''(s)\bar z^{(2)} - v''(s)z^{(2)}\\ &+(\nabla_{\Theta(z,\bar z^{(2)},s)}Z_s)(X_s^{(1)},X_s^{(2)})\},\,\mathrm{d}B_s\Big\rangle\end{aligned}\tag{5.3.14}$$

holds in $L^1(\mathbb{P})$.

Proof. Let τ_n be in the proof of Lemma 5.3.2 and let

$$N_\varepsilon(s)=\sigma_s^{-1}\big\{\nabla_{\Theta(z,\bar z^{(2)},s)}Z_s(X_s^{(1,\varepsilon)},X_s^{(2,\varepsilon)})+u''(s)\bar z^{(2)}-v''(s)z^{(2)}\big\}$$

for $s\in[0,t]$, $\varepsilon\in(0,\varepsilon_0)$. By **(A5.3.1)** and (5.3.11), there exists a constant $c>0$ such that

$$\begin{aligned}\big|\langle N_\varepsilon(s),\sigma_s^{-1}\xi_\varepsilon(s)\rangle\big| &\le \varepsilon|N_\varepsilon(s)|^2+\varepsilon^{-1}|\sigma_s^{-1}\xi_\varepsilon(s)|^2\\ &\le c\varepsilon W(X_s^{(1,\varepsilon)},X_s^{(2,\varepsilon)})\end{aligned}\tag{5.3.15}$$

holds for $\varepsilon \in (0, \varepsilon_0)$ and $s \in [0, t]$. Since ∇Z is locally bounded, it follows from (5.3.8) and (5.3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_\varepsilon(t \wedge \tau_n) = R_\varepsilon(t \wedge \tau_n) \left\{ \int_0^{t \wedge \tau_n} \langle N_\varepsilon(s), \mathrm{d}B_s \rangle + \int_0^{t \wedge \tau_n} \langle N_\varepsilon(s), \sigma_s^{-1} \xi_\varepsilon(s) \rangle \mathrm{d}s \right\}$$

holds for $\varepsilon \in (0, \varepsilon_0)$ and $n \geq 1$. Combining this with (5.3.15) we obtain

$$\begin{aligned} \frac{|R_\varepsilon(t \wedge \tau_n) - 1|}{\varepsilon} &\leq \frac{1}{\varepsilon} \int_0^\varepsilon R_r(t \wedge \tau_n) \mathrm{d}r \int_0^{t \wedge \tau_n} \langle N_r(s), \mathrm{d}B_s \rangle \\ &\quad + c \int_0^{\varepsilon_0} R_r(t \wedge \tau_n) \mathrm{d}r \int_0^{t \wedge \tau_n} W(X_s^{(1,r)}, X_s^{(2,r)}) \mathrm{d}s \end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0), n \geq 1$. Noting that under $\mathbb{Q}_r$ the process $(X_s^{(1,r)}, X_s^{(2,r)})_{s \in [0,t]}$ is generated by L_s, by (5.3.4) we have

$$\begin{aligned} &\mathbb{E} \int_0^{\varepsilon_0} R_r(t) \mathrm{d}r \int_0^t W(X_s^{(1,r)}, X_s^{(2,r)}) \mathrm{d}s \\ &= \int_0^{\varepsilon_0} \mathrm{d}r \int_0^t \mathbb{E}_{\mathbb{Q}_r} W(X_s^{(1,r)}, X_s^{(2,r)}) \mathrm{d}s \end{aligned}$$

is finite. Thus, for the first assertion it remains to show that the family

$$\eta_{\varepsilon,n} := \frac{1}{\varepsilon} \int_0^\varepsilon R_r(t \wedge \tau_n) |\Xi_{t,n}|(r) \mathrm{d}r, \quad \varepsilon \in (0, \varepsilon_0), n \geq 1$$

is uniformly integrable, where

$$\Xi_{t,n}(r) := \int_0^{t \wedge \tau_n} \langle N_r(s), \mathrm{d}B_s \rangle.$$

Since $r \log^{1/2}(\mathrm{e} + r)$ is increasing and convex in $r \geq 0$, by the Jensen inequality,

$$\begin{aligned} &\mathbb{E}\left\{ \eta_{\varepsilon,n} \log^{1/2}(\mathrm{e} + \eta_{\varepsilon,n}) \right\} \\ &\leq \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left\{ R_r(t \wedge \tau_n) |\Xi_{t,n}|(r) \log^{1/2} \left(\mathrm{e} + R_r(t \wedge \tau_n) |\Xi_{t,n}|(r) \right) \right\} \mathrm{d}r \\ &\leq \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\Big\{ R_r(t \wedge \tau_n) |\Xi_{t,n}|(r)^2 \\ &\qquad + R_r(t \wedge \tau_n) \log \left(\mathrm{e} + R_r(t \wedge \tau_n) |\Xi_{t,n}|(r) \right) \Big\} \mathrm{d}r \\ &\leq \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left\{ c + 2R_r(t \wedge \tau_n) |\Xi_{t,n}|(r)^2 + R_r(t \wedge \tau_n) \log R_r(t \wedge \tau_n) \right\} \mathrm{d}r \end{aligned}$$

holds for some constant $c > 0$. Combining this with (5.3.13) and noting that (5.3.15) and (5.3.12) imply

$$\begin{aligned}
&\mathbb{E}\{R_r(t\wedge\tau_n)|\Xi_{t,n}|(r)^2\} = \mathbb{E}_{\mathbb{Q}_{r,n}}\bigg(\int_0^{t\wedge\tau_n}\langle N_r(s),\ \mathrm{d}B_s\rangle\bigg)^2\\
&= \mathbb{E}_{\mathbb{Q}_{r,n}}\int_0^{t\wedge\tau_n}|N_r(s)|^2\mathrm{d}s\\
&\le c\mathbb{E}_{\mathbb{Q}_{r,n}}\int_0^{t\wedge\tau_n}W(X_s^{(1,r)},X_s^{(2,r)})\mathrm{d}s\le c',\quad n\ge 1, r\in(0,\varepsilon_0)
\end{aligned}$$

for some constants $c, c' > 0$, we conclude that $\{\eta_{\varepsilon,n}\}_{\varepsilon\in(0,\varepsilon_0),n\ge1}$ is uniformly integrable. Thus, the proof of the first assertion is finished.

Next, by **(A5.3.1)** and (5.3.8) we have

$$\lim_{\varepsilon\to0}\left|\frac{\xi_\varepsilon(s)}{\varepsilon}+(\nabla_{\Theta(z,\tilde z^{(2)},s)}Z_s)(X_s^{(1)},X_s^{(2)})+u''(s)\tilde z^{(2)}-v''(s)z^{(2)}\right|=0.$$

Moreover, for each $n \ge 1$ this sequence is bounded on $\{\tau_n \ge t\}$. Thus, (5.3.14) holds a.s. on $\{\tau_n \ge t\}$. Since $\tau_n \uparrow \infty$, we conclude that (5.3.14) holds a.s. Therefore, it also holds on $L^1(\mathbb{P})$ since $\{\frac{R_\varepsilon(t)-1}{\varepsilon}\}_{\varepsilon\in(0,\varepsilon_0)}$ is uniformly integrable according to the first assertion. □

To conclude this subsection, we present an example of kinetic Fokker-Planck equation.

Example 5.3.1 (Kinetic Fokker-Planck equation) Let $m = d$ and consider

$$\begin{cases}\mathrm{d}X_t^{(1)} = X_t^{(2)}\mathrm{d}t,\\ \mathrm{d}X_t^{(2)} = \mathrm{d}B_t - \nabla V(X_t^{(1)})\mathrm{d}t - X_t^{(2)}\mathrm{d}t\end{cases}\tag{5.3.16}$$

for some C^2-function $V \ge 0$ with compact level sets. Let $W(x^{(1)}, x^{(2)}) = \exp[2V(x^{(1)}) + |x^{(2)}|^2]$. We easily get that $LW = dW$. Thus, it is easy to see that **(A5.3.1)** holds for e.g. $V(x^{(1)}) = (1 + |x^{(1)}|^2)^l$ or even $V(x^{(1)}) = \mathrm{e}^{(1+|x^{(1)}|^2)^l}$ for some constant $l \ge 0$. Therefore, by Theorem 5.3.1 the derivative formula (5.3.3) holds for (u, v) satisfying (5.3.2).

5.3.2 *Gradient estimates*

In this section we aim to derive gradient estimates from the derivative formula (5.3.3). For simplicity, we only consider the time-homogenous case that σ_t and Z_t are independent of t. In general, we have the following result.

Proposition 5.3.4. *Assume* **(A5.3.1)** *and let* (u,v) *satisfy* (5.3.2). *Then for any* $f \in \mathcal{B}_b(\mathbb{R}^{m+d}), t > 0$ *and* $z = (z^{(1)}, z^{(2)}) \in \mathbb{R}^{m+d}$, $\tilde{z}^{(2)} \in A^{-1}z^{(1)}$,

$$\begin{aligned}|\nabla_z P_t f|^2 \le \|\sigma^{-1}\|^2 (P_t f^2)\mathbb{E}\int_0^t \big|u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} \\ + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}, X_s^{(2)})\big|^2 \mathrm{d}s.\end{aligned} \tag{5.3.17}$$

If $f \ge 0$ *then for any* $\delta > 0$,

$$\begin{aligned}|\nabla_z P_t f| &\le \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\} + \frac{\delta P_t f}{2} \\ &\times \log \mathbb{E}\mathrm{e}^{\frac{2\|\sigma^{-1}\|^2}{\delta^2}\int_0^t |u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}, X_s^{(2)})|^2 \mathrm{d}s}.\end{aligned} \tag{5.3.18}$$

Proof. Let

$$M_t = \int_0^t \big\langle \sigma^{-1}\big\{u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}, X_s^{(2)})\big\},\ \mathrm{d}B_s\big\rangle.$$

By (5.3.3) and the Schwarz inequality we obtain

$$\begin{aligned}|\nabla_z P_t f|^2 \le (P_t f^2)\mathbb{E}M_t^2 \le \|\sigma^{-1}\|^2 (P_t f^2)\mathbb{E}\int_0^t \big|u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} \\ + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}, X_s^{(2)})\big|^2 \mathrm{d}s.\end{aligned}$$

That is, (5.3.17) holds. Similarly, (5.3.18) follows from (5.3.3) and the Young inequality (cf. Lemma 2.4 in [Arnaudon *et al* (2009)]):

$$|\nabla_z P_t f| \le \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\} + \delta(P_t f)\log \mathbb{E}\exp\Big[\frac{M_t}{\delta}\Big]$$

since

$$\begin{aligned}\mathbb{E}\exp\Big[\frac{M_t}{\delta}\Big] &\le \bigg(\mathbb{E}\exp\Big[\frac{2\langle M\rangle_t}{\delta^2}\Big]\bigg)^{1/2} \\ &\le \bigg(\mathbb{E}\exp\bigg[\frac{2\|\sigma^{-1}\|^2}{\delta^2}\int_0^t \big|u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} \\ &\qquad + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}, X_s^{(2)})\big|^2 \mathrm{d}s\bigg]\bigg)^{1/2}.\end{aligned}$$

□

To derive explicit estimates, we will take the following explicit choice of the pair (u,v):

$$u(s) = \frac{s^2(3t-2s)}{t^3}, \quad v(s) = \frac{s(t-s)^2}{t^2}, \quad s \in [0,t], \tag{5.3.19}$$

which satisfies (5.3.2). In this case we have

$$\begin{aligned} u'(s) &= \frac{6s(t-s)}{t^3}, \ u''(s) = \frac{6(t-2s)}{t^3}, \ v'(s) = \frac{(t-s)(t-3s)}{t^2}, \\ v''(s) &= \frac{2(3s-2t)}{t^2}, \ 1-u(s) = \frac{(t-s)^2(t+2s)}{t^3}, \quad s \in [0,t]. \end{aligned} \tag{5.3.20}$$

In this case, Proposition 5.3.4 holds for

$$\begin{aligned} u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} &= \Lambda(z, \tilde{z}^{(2)}, s) \\ &:= \frac{6(t-2s)}{t^3}\tilde{z}^{(2)} + \frac{2(2t-3s)}{t^2}z^{(2)}, \\ \Theta(z, \tilde{z}^{(2)}, s) &= \Big(\frac{(t-s)^2(t+2s)}{t^3}z^{(1)} + \frac{s(t-s)^2}{t^2}Az^{(2)}, \\ &\qquad \frac{(t-s)(t-3s)}{t^2}z^{(2)} - \frac{6s(t-s)}{t^3}\tilde{z}^{(2)}\Big). \end{aligned} \tag{5.3.21}$$

Below we consider the following three cases respectively:

(i) $|\nabla Z|$ is bounded;
(ii) $|\nabla Z|$ has polynomial growth and $\langle Z(x), x^{(2)}\rangle \le C(1+|x|^2)$ holds for some constant $C > 0$ and all $x = (x^{(1)}, x^{(2)})$;
(iii) A more general case including the kinetic Fokker-Planck equation.

5.3.2.1 $|\nabla Z|$ *is bounded*

In this case **(A5.3.1)** holds for e.g. $W(x) = 1 + |x|^2$, so that Proposition 5.3.4 holds for $u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)}$ and $\Theta(z, \tilde{z}^{(2)}, s)$ given in (5.3.21). From this specific choice of $\Theta(z, \tilde{z}^{(2)}, s)$ we see that $\nabla^{(1)}Z$ and $\nabla^{(2)}Z$ will lead to different time behaviors of $\nabla_z P_t f$, where $\nabla^{(1)}$ and $\nabla^{(2)}$ are the gradient operators w.r.t. $x^{(1)} \in \mathbb{R}^m$ and $x^{(2)} \in \mathbb{R}^d$ respectively. So, we adopt the condition

$$|\nabla^{(1)}Z(x)| \le K_1, \quad |\nabla^{(2)}Z(x)| \le K_2, \quad x \in \mathbb{R}^{m+d} \tag{5.3.22}$$

for some constants $K_1, K_2 \ge 0$. Moreover, for $t > 0$ and $r_1, r_2 \ge 0$, let

$$\begin{aligned} \Psi_t(r_1, r_2) = \|\sigma^{-1}\|^2 t \Big\{ & r_1\Big(\frac{6\|A^{-1}\|}{t^2} + K_1 + \frac{3K_2\|A^{-1}\|}{2t}\Big) \\ & + r_2\Big(\frac{4}{t} + \frac{4K_1 t\|A\|}{27} + K_2\Big)\Big\}^2 \end{aligned}$$

and

$$\Phi_t(r_1, r_2) = \inf_{s \in (0,t]} \Psi_s(r_1, r_2). \tag{5.3.23}$$

In the following result the inequality (5.3.24) corresponds to the pointwise estimate of the $H^1 \to L^2$ regularization investigated in Theorem A.8 in [Villani (2009b)], while (5.3.26) corresponds to the pointwise estimate of the regularization "Fisher information to entropy" see Theorem A.18 in [Villani (2009b)].

Corollary 5.3.5. *Let* (5.3.22) *hold for some constants* $K_1, K_2 \geq 0$. *Then for any* $t > 0, z = (z^{(1)}, z^{(2)}) \in \mathbb{R}^{m+d}$,

$$|\nabla_z P_t f|^2 \leq (P_t f^2)\Phi_t(|z^{(1)}|, |z^{(2)}|), \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}). \tag{5.3.24}$$

If $f \geq 0$, *then*

$$|\nabla_z P_t f| \leq \delta\big\{P_t(f\log f) - (P_t f)\log(P_t f)\big\} + \frac{P_t f}{\delta}\Phi_t(|z^{(1)}|, |z^{(2)}|) \tag{5.3.25}$$

holds for all $\delta > 0$, *and consequently*

$$|\nabla_z P_t f|^2 \leq 4\Phi_t(|z^{(1)}|, |z^{(2)}|)\big\{P_t(f\log f) - (P_t f)\log(P_t f)\big\}P_t f. \tag{5.3.26}$$

Proof. Let $\tilde{z}^{(2)}$ be such that $|\tilde{z}^{(2)}| = |A^{-1}z^{(1)}| \leq \|A^{-1}\| \cdot |z^{(1)}|$, and take

$$\eta_s = \Lambda(z, \tilde{z}^{(2)}, s) + \nabla_{\Theta(z,\tilde{z}^{(2)},s)} Z(X_s^{(1)}(x), X_s^{(2)}(x)). \tag{5.3.27}$$

By (5.3.17),

$$|\nabla_z P_t f(x)|^2 \leq \|\sigma^{-1}\|^2 (P_t f^2)(x)\mathbb{E}\int_0^t |\eta_s|^2 \mathrm{d}s. \tag{5.3.28}$$

Since (5.3.22) implies $|\nabla_z Z| \leq K_1|z^{(1)}| + K_2|z^{(2)}|$, it follows that

$$\begin{aligned}
|\eta_s| \leq & \Big|\frac{6(t-2s)}{t^3}\tilde{z}^{(2)} + \frac{2(2t-3s)}{t^2}z^{(2)}\Big| \\
&\quad + K_1\Big|\frac{(t-s)^2(t+2s)}{t^3}z^{(1)} + \frac{s(t-s)^2}{t^2}Az^{(2)}\Big| \\
&\quad + K_2\Big|\frac{(t-s)(t-3s)}{t^2}z^{(2)} - \frac{6s(t-s)}{t^3}\tilde{z}^{(2)}\Big| \\
\leq & |z^{(1)}|\Big(\frac{6\|A^{-1}\|}{t^2} + K_1 + \frac{3K_2\|A^{-1}\|}{2t}\Big) + |z^{(2)}|\Big(\frac{4}{t} + \frac{4K_1 t\|A\|}{27} + K_2\Big).
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t |\eta_s|^2 \mathrm{d}s \leq & t\bigg\{|z^{(1)}|\Big(\frac{6\|A^{-1}\|}{t^2} + K_1 + \frac{3K_2\|A^{-1}\|}{2t}\Big) \\
&\quad + |z^{(2)}|\Big(\frac{4}{t} + \frac{4K_1 t\|A\|}{27} + K_2\Big)\bigg\}^2.
\end{aligned} \tag{5.3.29}$$

Combining this with (5.3.28) we obtain

$$|\nabla_z P_t f|^2 \leq (P_t f^2)\Psi_t(|z^{(1)}|, |z^{(2)}|).$$

Therefore, for any $s \in (0,t]$ by the semigroup property and the Jensen inequality one has

$$\begin{aligned}|\nabla_z P_t f|^2 &= |\nabla_z P_s(P_{t-s}f)|^2 \le \Psi_s(|z^{(1)}|,|z^{(2)}|)P_s(P_{t-s}f)^2\\ &\le \Psi_s(|z^{(1)}|,|z^{(2)}|)P_t f^2.\end{aligned}$$

This proves (5.3.24) according to (5.3.23).

To prove (5.3.25) we let $f \ge 0$ be bounded. By (5.3.18),

$$\begin{aligned}|\nabla_z P_t f| &\le \delta\big\{P_t(f\log f) - (P_t f)\log(P_t f)\big\}\\ &\quad + \frac{\delta P_t f}{2}\log \mathbb{E}\exp\left[\frac{2\|\sigma^{-1}\|^2}{\delta^2}\int_0^t |\eta_s|^2 \mathrm{d}s\right].\end{aligned} \tag{5.3.30}$$

Combining this with (5.3.29) we obtain

$$|\nabla_z P_t f| \le \delta\big\{P_t(f\log f) - (P_t f)\log(P_t f)\big\} + \frac{P_t f}{\delta}\Psi_t(|z^{(1)}|,|z^{(2)}|).$$

As observed above, by the semigroup property and the Jensen inequality, this implies (5.3.25).

Finally, minimizing the right hand side of (5.3.25) in $\delta > 0$, we obtain

$$|\nabla_z P_t f| \le 2\sqrt{\Phi_t(|z^{(1)}|,|z^{(2)}|)\{P_t(f\log f) - (P_t f)\log P_t f\}P_t f}.$$

This is equivalent to (5.3.26). □

5.3.2.2 $|\nabla Z|$ *has polynomial growth*

Assume there exists $l > 0$ such that

(A5.3.2) There exists a constant $C > 0$ such that for any $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}$,

(i) $\langle Z(x), x^{(2)}\rangle \le C(|x|^2+1)$;
(ii) $|\nabla Z|(x) := \sup\{|\nabla_z Z|(x) : \ |z| \le 1\} \le C(1+|x|^2)^l$.

It is easy to see that **(A5.3.2)** implies **(A5.3.1)** for $W(x) = (1+|x|^2)^{2l}$, so that Proposition 5.3.4 holds for $u''(s)\bar{z}^{(2)} - v''(s)z^{(2)}$ and $\Theta(z,\bar{z}^{(2)},s)$ given in (5.3.21).

Corollary 5.3.6. *Let* **(A5.3.2)** *hold.*

(1) *There exists a constant $c > 0$ such that*

$$|\nabla P_t f|^2(x) \le \frac{c}{(t\wedge 1)^3}P_t f^2(x), \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}), \quad t > 0, x \in \mathbb{R}^{m+d}.$$

(2) *If $l < \frac{1}{2}$, then there exists a constant $c > 0$ such that*

$$|\nabla P_t f|(x) \le \delta\big\{P_t(f\log f) - (P_t f)\log(P_t f)\big\}(x) + \frac{cP_t f(x)}{\delta(t\wedge 1)^4}\big\{|x|^{4l} + (\delta(1\wedge t)^2)^{-8l/(1-2l)}\big\}$$

holds for all $\delta > 0$ and positive $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$ and $x \in \mathbb{R}^{m+d}$.

(3) *If $l = \frac{1}{2}$, then there exist two constants $c, c' > 0$ such that for any $t > 0$ and $\delta \ge t^{-2}\mathrm{e}^{c(1+t)}$,*

$$|\nabla P_t f|(x) \le \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}(x) + \frac{c'\mathrm{e}^{c't}P_t f(x)}{t^4\delta}(1+|x|^2)$$

holds for all positive $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$ and $x \in \mathbb{R}^{m+d}$.

Proof. As observed in the proof of Corollary 5.3.5, we only have to prove the results for $t \in (0,1]$.

(1) It is easy to see that η_s in the proof of Corollary 5.3.5 satisfies

$$|\sigma^{-1}\eta_s|^2 \le c_1(t^2 + t^{-4})|z|^2(1 + |X_s^{(1)}(x)|^2 + |X_s^{(2)}(x)|^2)^{2l} \tag{5.3.31}$$

for some constant $c_1 > 0$. Thus, the first assertion follows from (5.3.28) and (5.3.4).

(2) Let **(A5.3.2)** hold for some $l \in (0, 1/2)$. Then

$$L(1+|x|^2)^{2l} \le c_2(1+|x|^2)^{2l}$$

holds for some constant $c_2 > 0$. Let $(X_s^{(1)}, X_s^{(2)}) = (X_s^{(1)}(x), X_s^{(2)}(x))$. By Itô's formula, we have

$$\begin{aligned}
&\mathrm{d}(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l} \\
&\le 4l(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l-1}\langle X_s^{(2)}, \sigma\mathrm{d}B_s\rangle \\
&\quad + c_2(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l}\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{d}\big\{\mathrm{e}^{-(1+c_2)s}(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l}\big\} \\
&\le 4l\mathrm{e}^{-(1+c_2)s}(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l-1}\langle X_s^{(2)}, \sigma\mathrm{d}B_s\rangle \\
&\quad - \mathrm{e}^{-(1+c_2)s}(1 + |X_s^{(1)}|^2 + |X_s^{(2)}|^2)^{2l}\mathrm{d}s.
\end{aligned}$$

Therefore, for any $\lambda > 0$,

$$\begin{aligned}
&\mathrm{e}^{-\lambda(1+|x|^2)^{2l}}\mathbb{E}\mathrm{e}^{\lambda\int_0^t \mathrm{e}^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l}\mathrm{d}s} \\
&\le \mathbb{E}\mathrm{e}^{4\lambda l\int_0^t \mathrm{e}^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l-1}\langle X_s^{(2)},\sigma\mathrm{d}B_s\rangle} \\
&\le \Big\{\mathbb{E}\mathrm{e}^{32\lambda^2 l^2\|\sigma\|^2\int_0^t \mathrm{e}^{-2(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2(2l-1)}|X_s^{(2)}|^2\mathrm{d}s}\Big\}^{1/2} \\
&\le \Big\{\mathbb{E}\mathrm{e}^{32\lambda^2 l^2\|\sigma\|^2\int_0^t \mathrm{e}^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{4l-1}\mathrm{d}s}\Big\}^{1/2}.
\end{aligned} \tag{5.3.32}$$

On the other hand, since $l < \frac{1}{2}$ implies $4l - 1 < 2l$, there exists a constant $c_3 > 0$ such that

$$32\lambda^2 l^2 \|\sigma\|^2 r^{4l-1} \leq \lambda r^{2l} + c_3 \lambda^{1/(1-2l)}, \quad r \geq 0.$$

Combining this with (5.3.32) we arrive at

$$\begin{aligned}
&\mathbb{E}\exp\left[\lambda \int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l} ds\right] \\
&\leq \exp\left[\lambda(1+|x|^2)^{2l} + \frac{c_3}{2}\lambda^{1/(1-2l)}\right] \\
&\quad \times \left(\mathbb{E}\exp\left[\lambda \int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l} ds\right]\right)^{1/2}.
\end{aligned}$$

As the argument works also for $t \wedge \tau_n$ in place of t, we may assume that the left-hand side of the above inequality is finite, so that

$$\begin{aligned}
&\mathbb{E}\exp\left[\lambda \int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l} ds\right] \\
&\leq \exp\left[2\lambda(1+|x|^2)^{2l} + c_3\lambda^{1/(1-2l)}\right].
\end{aligned}$$

Letting

$$\lambda_t(\delta) = \frac{2c_1(t^2+t^{-4})}{\delta^2} e^{(1+c_2)t},$$

and combining the above inequality with (5.3.30) and (5.3.31), we arrive at

$$\begin{aligned}
&\Big(|\nabla P_t f| - \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}\Big)(x) \\
&\leq \frac{\delta P_t f(x)}{2} \log \mathbb{E} e^{\lambda_t(\delta)\int_0^t \exp[-(1+c_2)s](1+|X_s^{(1)}|^2+|X_s^{(2)}|^2)^{2l} ds} \\
&\leq \delta P_t f(x)\Big\{\lambda_t(\delta)(1+|x|^2)^{2l} + \frac{c_3}{2}\lambda_t(\delta)^{1/(1-2l)}\Big\} \\
&\leq \frac{P_t f(x) e^{c(1+t)}}{\delta t^4}\big\{|x|^{4l} + \delta^{4(l-1)/(1-2l)} t^{-8l/(1-2l)}\big\}
\end{aligned} \tag{5.3.33}$$

for some constant $c > 0$. This proves the desired estimate for $t \in (0,1]$, and hence for all $t > 0$ as observed in the proof of Corollary 5.3.5.

(3) Let **(A5.3.2)** hold for $l = \frac{1}{2}$, so that (5.3.32) reduces to

$$\begin{aligned}
&\mathbb{E} e^{\lambda \int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2) ds} \\
&\leq e^{\lambda(1+|x|^2)}\Big\{\mathbb{E} e^{8\lambda^2\|\sigma\|^2\int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2) ds}\Big\}^{1/2}.
\end{aligned}$$

Taking $\lambda = (8\|\sigma\|)^{-2}$, we obtain

$$\mathbb{E}\exp\left[\frac{1}{8\|\sigma\|^2}\int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2+|X_s^{(2)}|^2) ds\right] \leq \exp\left[\frac{1}{4\|\sigma\|^2}(1+|x|^2)\right].$$

Obviously, there exists a constant $c > 0$ such that if $\delta \geq t^{-2}e^{c(1+t)}$ then $\lambda_t(\delta) \leq (8\|\sigma\|)^{-2}$ so that

$$\begin{aligned}
&\Big(|\nabla P_t f| - \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}\Big)(x)\\
&\leq \frac{\delta P_t f(x)}{2}\log\bigg(\mathbb{E}\exp\bigg[\frac{1}{8\|\sigma\|^2}\int_0^t e^{-(1+c_2)s}(1+|X_s^{(1)}|^2\\
&\qquad\qquad + |X_s^{(2)}|^2)^{2l}\mathrm{d}s\bigg]\bigg)^{8\|\sigma\|^2\lambda_t(\delta)}\\
&\leq \lambda_t(\delta)\delta P_t f(x)\big(1+|x|^2\big) \leq \frac{c'e^{c't}P_t f(x)}{t^4\delta}\big(1+|x|^2\big)
\end{aligned}$$

holds for some constant $c' > 0$. □

5.3.2.3 *A general case*

Corollary 5.3.7. *Assume* **(A5.3.1)**. *Then there exists a constant $c > 0$ such that*

$$|\nabla P_t f|^2 \leq c\Big(\frac{1}{(1\wedge t)^3} + \frac{W}{1\wedge t}\Big)P_t f^2, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}). \tag{5.3.34}$$

If moreover there exist constants $\lambda, K > 0$ and a C^2-function $\bar{W} \geq 1$ such that

$$\lambda W \leq K - \frac{L\bar{W}}{\bar{W}}, \tag{5.3.35}$$

then there exist constants $c, \delta_0 > 0$ such that

$$\begin{aligned}
|\nabla P_t f| &\leq \delta\big\{P_t(f\log f) - (P_t f)\log P_t f\big\}\\
&\quad + \frac{c}{\delta}\Big\{\frac{1}{(t\wedge 1)^3} + \frac{\log \bar{W}}{(t\wedge 1)^2}\Big\}P_t f
\end{aligned} \tag{5.3.36}$$

holds for $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$ and $\delta \geq \delta_0/t$.

Proof. Again, it suffices to prove for $t \in (0,1]$. By (5.3.21) and taking $\tilde{z}^{(2)} \in A^{-1}z^{(1)}$ such that $|\tilde{z}^{(2)}| = \|A^{-1}\| \cdot |z^{(1)}|$, there exists a constant $c > 0$ such that

$$|\Lambda(z, \tilde{z}^{(2)}, s)| \leq \frac{c}{t^2}|z|, \quad |\Theta(z, \tilde{z}^{(2)}, s)| \leq \frac{c}{t}|z|.$$

So, by **(A5.3.1)**

$$\begin{aligned}
&\big|u''(s)\tilde{z}^{(2)} - v''(s)z^{(2)} + \nabla_{\Theta(z,\tilde{z}^{(2)},s)}Z(X_s^{(1)}, X_s^{(2)})\big|^2\\
&\leq \frac{c}{t^4} + \frac{c}{t^2}W(X_s^{(1)}, X_s^{(2)})
\end{aligned} \tag{5.3.37}$$

holds for some constant $c > 0$. Since $W \geq 1$ and $\mathbb{E}W(X_s^{(1)}, X_s^{(2)}) \leq e^{Cs}W$, this and (5.3.17) yield that

$$\begin{aligned}
&|\nabla P_t f|^2 \\
&\leq c_1(P_t f^2)\left\{\int_0^t |\Lambda(z, \bar{z}^{(2)}, s)|^2 \mathrm{d}s + \mathbb{E}\int_0^t |\Theta(z, \bar{z}^{(2)}, s)|^2 W(X_s^{(1)}, X_s^{(2)})\mathrm{d}s\right\} \\
&\leq c_2\Big(\frac{1}{t^3} + \frac{W}{t}\Big)P_t f^2
\end{aligned}$$

holds for some constants $c_1, c_2 > 0$.

Next, it is easy to see that the process

$$M_s := \tilde{W}(X_s^{(1)}, X_s^{(2)}) \exp\left[-\int_0^s \frac{L\tilde{W}}{\tilde{W}}(X_r^{(1)}, X_r^{(2)})\mathrm{d}r\right]$$

is a local martingale, and thus a supermartingale due to the Fatou lemma. Combining this with (5.3.35) and noting that $\tilde{W} \geq 1$, we obtain

$$\mathbb{E}e^{\lambda \int_0^t W(X_s^{(1)}, X_s^{(2)})\mathrm{d}s} \leq e^{Kt}\mathbb{E}M_t \leq e^{Kt}\tilde{W}. \tag{5.3.38}$$

Then the second assertion follows from (5.3.18) and (5.3.37) since for any constant $\alpha > 0$ and $\delta t \geq \sqrt{\alpha/\lambda}$,

$$\begin{aligned}
&\mathbb{E}\exp\left[\frac{\alpha}{\delta^2 t^2}\int_0^t W(X_s^{(1)}, X_s^{(2)})\mathrm{d}s\right] \\
&\leq \left(\mathbb{E}\exp\left[\lambda\int_0^t W(X_s^{(1)}, X_s^{(2)})\mathrm{d}s\right]\right)^{\alpha/(\lambda\delta^2 t^2)}.
\end{aligned}$$

□

5.3.3 *Harnack inequality and applications*

The aim of this subsection is to establish the log-Harnack inequality and the Harnack inequality for P_t associated to (5.3.1). We first consider the general case with assumption **(A5.3.1)** then move to the more specific setting with assumption **(A5.3.2)**. Again, we only consider the time-homogenous case.

5.3.3.1 *Harnack inequality under* **(A5.3.1)**

We first introduce a result parallel to Proposition 1.5.2.

Proposition 5.3.8. *Let $\mathbb{H}$ be a Hilbert space and P a Markov operator on $\mathcal{B}_b(\mathbb{H})$. Let $z \in \mathbb{H}$ such that for some $\delta_z \in (0,1)$ and measurable function $\gamma_z : [\delta_z, \infty) \times \mathbb{H} \to (0, \infty)$,*

$$|\nabla_z Pf| \leq \delta\big\{P(f\log f) - (Pf)\log Pf\big\} + \gamma_z(\delta, \cdot)Pf, \quad \delta \geq \delta_z \tag{5.3.39}$$

holds for all positive $f \in \mathcal{B}_b(\mathbb{H})$. *Then for any* $\alpha \geq \frac{1}{1-\delta_z}$, $x \in \mathbb{H}$ *and positive* $f \in \mathcal{B}_b(\mathbb{H})$,

$$(Pf)^\alpha(x) \leq Pf^\alpha(x+z)\exp\left[\int_0^1 \frac{\alpha}{1+(\alpha-1)s}\gamma_z\Big(\frac{\alpha-1}{1+(\alpha-1)s}, x+sz\Big)\mathrm{d}s\right].$$

Proof. Let $\beta(s) = 1+(\alpha-1)s, s \in [0,1]$. We have $\frac{\alpha-1}{\beta(s)} \geq \delta_z$ provided $\alpha \geq \frac{1}{1-\delta_z}$. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\log(Pf^{\beta(s)})^{\alpha/\beta(s)}(x+sz)\\
&= \frac{\alpha(\alpha-1)\{P(f^{\beta(s)}\log f^{\beta(s)}) - (Pf^{\beta(s)})\log Pf^{\beta(s)}\}}{\beta(s)^2 Pf^{\beta(s)}}(x+sz)\\
&\quad + \frac{\alpha\nabla_z Pf^{\beta(s)}}{\beta(s)Pf^{\beta(s)}}(x+sz)\\
&\geq -\frac{\alpha}{\beta(s)}\gamma_z\Big(\frac{\alpha-1}{\beta(s)},\ x+sz\Big), \quad s \in [0,1].
\end{aligned}$$

Then the proof is completed by taking integral over $[0,1]$ w.r.t. $\mathrm{d}s$. □

Below is a consequence of (5.3.36) and Proposition 5.3.8.

Corollary 5.3.9. *Let* **(A5.3.1)** *and* (5.3.35) *hold. Then there exist constants* $\delta_0, c > 0$ *such that for any* $\alpha > 1, t > 0$ *and positive* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$,

$$\begin{aligned}
&(P_t f)^\alpha(x)\\
&\leq P_t f^\alpha(x+z)\exp\left[\frac{\alpha c|z|^2}{\alpha-1}\left(\frac{1}{(1\wedge t)^3} + \frac{\int_0^1 \log\tilde{W}(x+sz)\mathrm{d}s}{(1\wedge t)^2}\right)\right]
\end{aligned} \tag{5.3.40}$$

holds for $x, z \in \mathbb{R}^{m+d}$ *with* $|z| \leq \frac{(\alpha-1)t}{\alpha\delta_0}$.

Proof. By (5.3.36),

$$|\nabla_z P_t f| \leq \delta\{P_t(f\log f) - (P_t f)\log P_t f\} + \frac{c|z|^2}{\delta}\Big\{\frac{1}{(t\wedge 1)^3} + \frac{\log\tilde{W}}{(t\wedge 1)^2}\Big\}P_t f$$

holds for $\delta \geq \delta_0|z|/t$. Thus, (5.3.39) holds for $P = P_t$ and

$$\delta_z = \frac{\delta_0|z|}{t}, \quad \gamma_z(\delta, x) = \frac{c|z|^2}{\delta}\Big(\frac{1}{t^3} + \frac{\log\tilde{W}(x)}{t^2}\Big).$$

Therefore, the desired Harnack inequality follows from Proposition 5.3.8.

□

To derive the log-Harnack inequality, we need the following slightly stronger condition than the second one in **(A5.3.1)**: there exist a constant $\lambda > 0$ and an increasing function U on $[0, \infty)$ such that

$$|Z(x) - Z(y)|^2 \le |x - y|^2 \{U(|x - y|) + \lambda W(y)\}, \quad x, y \in \mathbb{R}^{m+d}. \quad (5.3.41)$$

Theorem 5.3.10. *Assume* **(A5.3.1)** *such that* (5.3.41) *holds. Then there exists a constant* $c > 0$ *such that*

$$P_t \log f(x) - \log P_t f(y) \le c|x-y|^2 \left\{ \frac{1}{(1 \wedge t)^3} + \frac{U((1 \vee t^{-1})|x - y|) + W(y)}{t \wedge 1} \right\}$$

holds for any $t > 0$, *positive function* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$, *and* $x, y \in \mathbb{R}^{m+d}$.

Proof. Again as in the proof of Corollary 5.3.5, it suffices to prove for $t \in (0, 1]$. Let $x = (x^{(1)}, x^{(2)})$ and $y = (y^{(1)}, y^{(2)})$. We will make use of the coupling constructed in the proof of Theorem 5.3.1 for $\varepsilon = 1, z = y - x$ and (u, v) being in (5.3.19). We have $(X_t^{(1)}, X_t^{(2)}) = (X_t^{(1,1)}, X_t^{(2,1)})$, and $(X_s^{(1,1)}, X_s^{(2,1)})_{s \in [0,t]}$ is generated by L under the probability $\mathbb{Q}_1 = R_1(t)\mathbb{P}$. So, by the Young inequality (see Lemma 2.4 in [Arnaudon *et al* (2009)]), we have

$$\begin{aligned} P_t \log f(y) &= \mathbb{E}\big(R_1(t) \log f(X_t^{(1,1)}, X_t^{(2,1)})\big) = \mathbb{E}\big(R_1(t) \log f(X_t^{(1)}, X_t^{(2)})\big) \\ &\le \mathbb{E}(R_1(t) \log R_1(t)) + \log \mathbb{E} f(X_t^{(1)}, X_t^{(2)}) \\ &= \log P_t f(x) + \mathbb{E}(R_1(t) \log R_1(t)). \end{aligned}$$

Combining this with (5.3.10) we arrive at

$$P_t \log f(y) - \log P_t f(x) \le \frac{1}{2} \mathbb{E}_{\mathbb{Q}_1} \int_0^t |\sigma^{-1} \xi_1(s)|^2 \mathrm{d}s. \quad (5.3.42)$$

Taking $\bar{z}^{(2)}$ such that $|\bar{z}^{(2)}| \le \|A^{-1}\| \cdot |z^{(1)}|$, we obtain from (5.3.8), (5.3.41), (5.3.19) and (5.3.20) that for some constants $c_1, c_2 > 0$,

$$\begin{aligned} |\sigma^{-1}\xi_1(s)|^2 &\le c_1 \Big\{ |\Lambda(z, \bar{z}^{(2)}, s)|^2 + |\Theta(z, \bar{z}^{(2)}, s)|^2 \big(U(|\Theta(z, \bar{z}^{(2)}, s)|) \\ &\qquad + \lambda W(X_s^{(1,1)}, X_s^{(2,1)})\big)\Big\} \\ &\le c_2 |z|^2 \left\{ \frac{1}{t^4} + \frac{U(|z|/t) + W(X_s^{(1,1)}, X_s^{(2,1)})}{t^2} \right\}. \end{aligned}$$

Combining this with (5.3.42) and noting that $LW \le CW$ implies $\mathbb{E}_{\mathbb{Q}_1} W(X_s^{(1,1)}, X_s^{(2,1)}) \le \mathrm{e}^{Cs} W(y)$ for $s \in [0, t]$, we complete the proof. $\square$

5.3.3.2 *Harnack inequality under assumption* **(A5.3.2)**

Theorem 5.3.11. *Let* (5.3.22) *hold and let* Φ_t *be in* (5.3.23). *Then for any* $t > 0, \alpha > 1$ *and positive function* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$,

$$(P_t f)^\alpha(x) \le (P_t f^\alpha)(y) \exp\Big[\frac{\alpha}{\alpha-1}\Phi_t(|x^{(1)} - y^{(1)}|, |x^{(2)} - y^{(2)}|)\Big] \quad (5.3.43)$$

holds, here $x = (x^{(1)}, x^{(2)}),\ y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^{m+d}$. *Consequently,*

$$P_t \log f(x) \le \log P_t f(y) + \Phi_t(|x^{(1)} - y^{(1)}|, |x^{(2)} - y^{(2)}|) \quad (5.3.44)$$

holds for all $x, y \in \mathbb{R}^{m+d}$.

Proof. It is easy to see that (5.3.43) follows from (5.3.25) and Proposition 5.3.8. Next, according to Corollary 1.4.3, (5.3.44) follows from (5.3.43) since $\mathbb{R}^{m+d}$ is a length space under the metric

$$\rho(x, y) := \sqrt{\Phi_t(|x^{(1)} - y^{(1)}|, |x^{(2)} - y^{(2)}|)}. \qquad \square$$

The next result extends Theorem 5.3.11 to unbounded ∇Z.

Theorem 5.3.12. *Assume* **(A5.3.2)**. *Then there exists a constant* $c > 0$ *such that for any* $t > 0$ *and positive* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$,

$$\begin{aligned} &P_t \log f(y) - \log P_t f(x) \\ &\le |x-y|^2\Big\{\frac{c}{(1\wedge t)^3} + \frac{c|x-y|^{2l}}{(1\wedge t)^{2l+1}} + \frac{c(1+|y|^{4l})}{t\wedge 1}\Big\} \end{aligned} \quad (5.3.45)$$

holds for $x, y \in \mathbb{R}^{m+d}$. *If* **(A5.3.2)** *holds for some* $l < \frac{1}{2}$, *then there exists a constant* $c > 0$ *such that*

$$\begin{aligned} (P_t f)^\alpha(x) \le (P_t f^\alpha)(y) \exp\Big[&\frac{\alpha c|x-y|^2}{(\alpha-1)(1\wedge t)^4}\Big\{(|x| \vee |y|)^{4l} \\ &+ \big((\alpha-1)(1\wedge t)^2\big)^{4(l-1)/(1-2l)}\Big\}\Big] \end{aligned} \quad (5.3.46)$$

holds for all $t > 0, \alpha > 1, x, y \in \mathbb{R}^{m+d}$ *and positive* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$.

Proof. (5.3.45) follows from Theorem 5.3.10 since in this case **(A5.3.1)** and (5.3.41) hold for $W(x) = (1+|x|^2)^{2l}$ and $U(r) = cr^{2l}$ for some $\lambda, c > 0$; while (5.3.46) follows from Corollary 5.3.6(2) and Proposition 5.3.8. $\square$

According to Proposition 1.4.4, we have the following consequence of Theorems 5.3.11 and 5.3.12.

Corollary 5.3.13. *Let* p_t *be the transition density of* P_t *w.r.t. some* σ-*finite measure* μ *equivalent to the Lebesgue measure on* $\mathbb{R}^{m+d}$. *Let* Φ_t *be in Theorem* 5.3.11.

(1) (5.3.22) *implies*

$$\int_{\mathbb{R}^{m+d}} \left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{1/(\alpha-1)} p_t(x,z)\mu(\mathrm{d}z)$$
$$\leq \exp\left[\frac{\alpha}{(\alpha-1)^2}\Phi_t(|x^{(1)}-y^{(1)}|,|x^{(2)}-y^{(2)}|)\right],$$
$$\int_{\mathbb{R}^{m+d}} p_t(x,z)\log\frac{p_t(x,z)}{p_t(y,z)}\mu(\mathrm{d}z) \leq \Phi_t(|x^{(1)}-y^{(1)}|,|x^{(2)}-y^{(2)}|)$$

for all $t>0$ *and* $x=(x^{(1)},x^{(2)}), y=(y^{(1)},y^{(2)})\in\mathbb{R}^{m+d}$.

(2) *If* **(A5.3.2)** *holds for some* $l\in(0,\frac{1}{2})$, *then there exists a constant* $c>0$ *such that*

$$\int_{\mathbb{R}^{m+d}} \left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{1/(\alpha-1)} p_t(x,z)\mu(\mathrm{d}z)$$
$$\leq \exp\left[\frac{\alpha c|x-y|^2}{(\alpha-1)^2(1\wedge t)^4}\left\{(|x|\vee|y|)^{4l}+((\alpha-1)(1\wedge t)^2)^{4(l-1)/(1-2l)}\right\}\right]$$

holds for all $t>0$ *and* $x,y\in\mathbb{R}^{m+d}$.

(3) *If* **(A5.3.2)** *holds then there exists a constant* $c>0$ *such that*

$$\int_{\mathbb{R}^{m+d}} p_t(x,z)\log\frac{p_t(x,z)}{p_t(y,z)}\mu(\mathrm{d}z)$$
$$\leq |x-y|^2\left\{\frac{c}{(1\wedge t)^3}+\frac{c|x-y|^{2l}}{(1\wedge t)^{2l+1}}+\frac{c(1+|y|^{4l})}{t\wedge 1}\right\}$$

holds for all $t>0$ *and* $x,y\in\mathbb{R}^{m+d}$.

Next, for two probability measures μ and ν, let $\mathcal{C}(\nu,\mu)$ be the class of their couplings, i.e. $\pi\in\mathcal{C}(\nu,\mu)$ if π is a probability measure on $\mathbb{R}^{m+d}\times\mathbb{R}^{m+d}$ such that $\pi(\mathbb{R}^{m+d}\times\cdot)=\mu(\cdot)$ and $\pi(\cdot\times\mathbb{R}^{m+d})=\nu(\cdot)$. In the spirit of Proposition 1.4.4, Theorems 5.3.11 and 5.3.12 also imply the following entropy-cost inequalities. Recall that for any non-negative symmetric measurable function $\mathbf{c}$ on $\mathbb{R}^{m+d}\times\mathbb{R}^{m+d}$, and for any two probability measures μ,ν on $\mathbb{R}^{m+d}$, we call

$$W_{\mathbf{c}}(\nu,\mu):=\inf_{\pi\in\mathcal{C}(\nu,\mu)}\int_{\mathbb{R}^{m+d}\times\mathbb{R}^{m+d}}\mathbf{c}(x,y)\,\pi(\mathrm{d}x,\mathrm{d}y)$$

the transportation-cost between these two distributions induced by the cost function $\mathbf{c}$, where $\mathcal{C}(\nu,\mu)$ is the set of all couplings of ν and μ.

Corollary 5.3.14. *Let* P_t *have an invariant probability measure* μ, *and let* P^* *be the adjoint operator of* P *in* $L^2(\mu)$.

(1) *If* (5.3.22) *holds then*

$$\mu(P_t^* f \log P_t^* f) \le W_{\mathbf{c}_t}(f\mu, \mu), \quad t > 0, f \ge 0, \mu(f) = 1, \tag{5.3.47}$$

where $\mathbf{c}_t(x,y) = \Phi_t(|x^{(1)} - y^{(1)}|, |x^{(2)} - y^{(2)}|)$.

(2) *If* **(A5.3.2)** *holds, then there exists* $c > 0$ *such that* (5.3.47) *holds for*

$$\mathbf{c}_t(x,y) = |x-y|^2\Big\{\frac{c}{(1\wedge t)^3} + \frac{c|x-y|^{2l}}{(1\wedge t)^{2l+1}} + \frac{c(1+|y|^{4l})}{t\wedge 1}\Big\}.$$

5.3.4 *Integration by parts formula and shift Harnack inequality*

In this subsection we aim to establish the integration by parts formula and the corresponding shift Harnack inequality for the Hamiltonian system using coupling (see [Wang (2012d)] for the study of more stochastic equations).

Consider the following degenerate stochastic differential equation on $\mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d (m \ge 0, d \ge 1)$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = \{AX_t^{(1)} + BX_t^{(2)}\}\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = Z_t(X_t^{(1)}, X_t^{(2)})\mathrm{d}t + \sigma_t \mathrm{d}B_t, \end{cases} \tag{5.3.48}$$

where A and B are two matrices of order $m \times m$ and $m \times d$ respectively, $Z : [0,\infty) \times \mathbb{R}^{m+d} \to \mathbb{R}^d$ is measurable with $Z_t \in C^1(\mathbb{R}^{m+d}; \mathbb{R}^d)$ for $t \ge 0$, $\{\sigma_t\}_{t\ge 0}$ are invertible $d\times d$-matrices measurable in t such that the operator norm $\|\sigma_\cdot^{-1}\|$ is locally bounded, and B_t is the d-dimensional Brownian motion.

When $m \ge 1$ this equation is degenerate, and when $m = 0$ we set $\mathbb{R}^m = \{0\}$, so that the first equation disappears and thus, the equation reduces to a non-degenerate equation on $\mathbb{R}^d$. To ensure the existence of the transition density (or heat kernel) of the associated semigroup P_t w.r.t. the Lebesgue measure on $\mathbb{R}^{m+d}$, we make use of the following Kalman rank condition (see [Kalman *et al* (1969)]) which implies that the associated diffusion is subelliptic:

(A5.3.3) There exists $0 \le k \le m-1$ such that $\mathrm{Rank}[B, AB, \dots, A^k B] = m$.

When $m = 0$ this condition is trivial, and for $m = 1$ it means that $\mathrm{Rank}(B) = 1$, i.e. $B \ne 0$. In general, we allow that m is much larger than d, so that the associated diffusion process is highly degenerate (see Example 5.3.1 below).

Let the solution to (5.3.48) be non-explosive, and let

$$P_t f = \mathbb{E} f(X_t^{(1)}, X_t^{(2)}), \quad t \geq 0, f \in \mathcal{B}_b(\mathbb{R}^{m+d}).$$

To state our main results, let us fix $T > 0$. For non-negative $\phi \in C([0,T])$ with $\phi > 0$ in $(0,T)$, define

$$Q_\phi = \int_0^T \phi(t) \mathrm{e}^{(T-t)A} BB^* \mathrm{e}^{(T-t)A^*} \mathrm{d}t.$$

Then Q_ϕ is invertible (cf. [Saloff-Coste (1994)]). For any $z \in \mathbb{R}^{m+d}$ and $r > 0$, let $B(z;r)$ be the ball centered at z with radius r.

Theorem 5.3.15. *Assume* **(A5.3.3)** *and that the solution to* (5.3.48) *is non-explosive with*

$$\sup_{t\in[0,T]} \mathbb{E}\Big\{ \sup_{B(X_t^{(1)}, X_t^{(2)}; r)} |\nabla Z_t|^2 \Big\} < \infty, \quad r > 0. \tag{5.3.49}$$

Let $\phi, \psi \in C^1([0,T])$ such that $\phi(0) = \phi(T) = 0, \phi > 0$ in $(0,T)$, and

$$\psi(T) = 1, \ \psi(0) = 0, \ \int_0^T \psi(t) \mathrm{e}^{(T-t)A} B \mathrm{d}t = 0. \tag{5.3.50}$$

Moreover, for $e = (e_1, e_2) \in \mathbb{R}^{m+d}$, let

$$h(t) = \phi(t) B^* \mathrm{e}^{(T-t)A^*} Q_\phi^{-1} e_1 + \psi(t) e_2 \in \mathbb{R}^d,$$

$$\Theta(t) = \left(\int_0^t \mathrm{e}^{(t-s)A} B h(s) \mathrm{d}s, \ h(t) \right) \in \mathbb{R}^{m+d}, \quad t \in [0,T].$$

(1) *For any $f \in C_b^1(\mathbb{R}^{m+d})$, there holds*

$$P_T(\nabla_e f) = \mathbb{E}\Big\{ f(X_t^{(1)}, X_t^{(2)}) \int_0^T \Big\langle \sigma_t^{-1} \{ h'(t) - \nabla_{\Theta(t)} Z_t(X_t^{(1)}, X_t^{(2)}) \}, \ \mathrm{d}B_t \Big\rangle \Big\}.$$

(2) *Let $(X_0^{(1)}, X_0^{(2)}) = x = (x^{(1)}, x^{(2)})$ and*

$$R = \exp\left[-\int_0^T \langle \sigma_t^{-1} \xi_1(t), \mathrm{d}B_t \rangle - \frac{1}{2} \int_0^T |\sigma_t^{-1} \xi_1(t)|^2 \mathrm{d}t \right],$$

where

$$\xi_1(t) = h'(t) + Z_t(X_t^{(1)}, X_t^{(2)}) - Z_t(X_t^{(1,1)}, X_t^{(2,1)})$$

and $(X_t^{(1,1)}, X_t^{(2,1)})$ solves the equation

$$\begin{cases} \mathrm{d}X_t^{(1,1)} = \{ AX_t^{(1,1)} + BX_t^{(2,1)} \} \mathrm{d}t, & X_0^{(1,1)} = x^{(1)}, \\ \mathrm{d}X_t^{(2,1)} = \sigma_t \mathrm{d}B_t + \{ Z_t(X_t^{(1)}, X_t^{(2)}) + h'(t) \} \mathrm{d}t, & X_0^{(2,1)} = x^{(2)}. \end{cases}$$

Then

$$|P_T f(x)|^p \leq P_T\{|f|^p(e + \cdot)\}(x) \big(\mathbb{E} R^{\frac{p}{p-1}}\big)^{p-1}, \quad p > 1, f \in \mathcal{B}_b(\mathbb{R}^{m+d}),$$

$$P_T \log f(x) \leq \log P_T\{f(e + \cdot)\}(x) + \mathbb{E}(R \log R), \ f \in \mathcal{B}_b(\mathbb{R}^{m+d}), f > 0.$$

Proof. We only prove (1), since (2) follows from Theorem 1.3.9 with the coupling constructed below for $\varepsilon = 1$. Let $(X_t^{(1,0)}, X_t^{(2,0)}) = (X_t^{(1)}, X_t^{(2)})$ solve (5.3.48) with initial data $(x^{(1)}, x^{(2)})$, and for $\varepsilon \in (0,1]$ let $(X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)})$ solve the equation

$$\begin{cases} \mathrm{d}X_t^{(1,\varepsilon)} = \{AX_t^{(1,\varepsilon)} + BX_t^{(2,\varepsilon)}\}\mathrm{d}t, \\ \mathrm{d}X_t^{(2,\varepsilon)} = \sigma_t \mathrm{d}B_t + \{Z_t(X_t^{(1)}, X_t^{(2)}) + \varepsilon h'(t)\}\mathrm{d}t, \end{cases} \tag{5.3.51}$$

with initial data $(x^{(1)}, x^{(2)})$. Then it is easy to see that

$$\begin{cases} X_t^{(2,\varepsilon)} = X_t^{(2)} + \varepsilon h(t), \\ X_t^{(1,\varepsilon)} = X_t^{(1)} + \varepsilon \int_0^t \mathrm{e}^{(t-s)A} Bh(s)\mathrm{d}s. \end{cases} \tag{5.3.52}$$

Combining this with $\phi(0) = \phi(T) = 0$ and (5.3.50), we see that $h(T) = e_2$ and

$$\begin{aligned} &\int_0^T \mathrm{e}^{(T-t)A} Bh(t)\mathrm{d}t \\ &= \int_0^T \phi(t)\mathrm{e}^{(T-t)A} BB^* \mathrm{e}^{(T-t)A^*} Q_\phi^{-1} e_1 \mathrm{d}t + \int_0^T \psi(t)\mathrm{e}^{(T-t)A} Be_2 \mathrm{d}t \\ &= e_1. \end{aligned}$$

Therefore,

$$(X_T^{(1,\varepsilon)}, X_T^{(2,\varepsilon)}) = (X_T^{(1)}, X_T^{(2)}) + \varepsilon e, \quad \varepsilon \in [0,1]. \tag{5.3.53}$$

Next, to see that $\big((X_t^{(1)}, X_t^{(2)}),\ (X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)})\big)$ is a coupling by change of measure for the solution to (5.3.48), reformulate (5.3.51) as

$$\begin{cases} \mathrm{d}X_t^{(1,\varepsilon)} = \{AX_t^{(1,\varepsilon)} + BX_t^{(2,\varepsilon)}\}\mathrm{d}t, \quad X_0^{(1,\varepsilon)} = x^{(1)}, \\ \mathrm{d}X_t^{(2,\varepsilon)} = \sigma_t \mathrm{d}W_t^\varepsilon + Z_t(X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)})\mathrm{d}t, \quad X_0^{(2,\varepsilon)} = x^{(2)}, \end{cases} \tag{5.3.54}$$

where

$$W_t^\varepsilon := B_t + \int_0^t \sigma_s^{-1}\{\varepsilon h'(s) + Z_s(X_s^{(1)}, X_s^{(2)}) - Z_s(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)})\}\mathrm{d}s$$

for $t \in [0,T]$. Let

$$\xi_\varepsilon(s) = \varepsilon h'(s) + Z_s(X_s^{(1)}, X_s^{(2)}) - Z_s(X_s^{(1,\varepsilon)}, X_s^{(2,\varepsilon)}) \tag{5.3.55}$$

and

$$R_\varepsilon = \exp\left[-\int_0^T \langle \sigma_s^{-1}\xi_\varepsilon(s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^T |\sigma_s^{-1}\xi_\varepsilon(s)|^2 \mathrm{d}s\right].$$

By Lemma 5.3.16 below and the Girsanov theorem, W_t^ε is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}_\varepsilon := R_\varepsilon \mathbb{P}$. Therefore,

$((X_t^{(1)}, X_t^{(2)}),\ (X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)}))$ is a coupling by change of measure with changed probability $\mathbb{Q}_\varepsilon$. Moreover, combining (5.3.52) with the definition of R_ε, we see from (5.3.49) that

$$-\frac{\mathrm{d}R_\varepsilon}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = \int_0^T \Big\langle \sigma_s^{-1}\{h'(s) - \nabla_{\Theta(s)} Z_s(X_s^{(1)}, X_s^{(2)})\}, \mathrm{d}B_s\Big\rangle$$

holds in $L^1(\mathbb{P})$. Then the proof is completed by Theorem 1.3.9(2). □

We remark that as shown in the last subsection, condition (5.3.49) and the non-explosion are implied by **(A5.3.1)**.

Lemma 5.3.16. *Let the solution to* (5.3.48) *be non-explosive such that* (5.3.49) *holds, and let* ξ_ε *be in* (5.3.55). *Then for any* $\varepsilon \in [0,1]$ *the process*

$$R_\varepsilon(t) = \exp\left[-\int_0^t \langle \sigma_s^{-1}\xi_\varepsilon(s), \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^t |\sigma_s^{-1}\xi_\varepsilon(s)|^2\mathrm{d}s\right], \quad t \in [0,T]$$

is a uniformly integrable martingale with $\sup_{t\in[0,T]} \mathbb{E}\big\{R_\varepsilon(t)\log R_\varepsilon(t)\big\} < \infty$.

Proof. Let $\tau_n = \inf\{t \geq 0 : |X_t^{(1)}| + |X_t^{(2)}| \geq n\}, n \geq 1$. Then $\tau_n \uparrow \infty$ as $n \uparrow \infty$. It suffices to show that

$$\sup_{t\in[0,T],n\geq 1} \mathbb{E}\big\{R_\varepsilon(t\wedge\tau_n)\log R_\varepsilon(t\wedge\tau_n)\big\} < \infty. \tag{5.3.56}$$

By (5.3.52), there exists $r > 0$ such that

$$(X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)}) \in B(X_t^{(1)}, X_t^{(2)}; r), \quad t \in [0,T], \varepsilon \in [0,1]. \tag{5.3.57}$$

Let $Q_{\varepsilon,n} = R_\varepsilon(T\wedge\tau_n)\mathbb{P}$. By the Girsanov theorem, $\{W_t^\varepsilon\}_{t\in[0,T\wedge\tau_n]}$ is the d-dimensional Brownian motion under the changed probability $\mathbb{Q}_{\varepsilon,n}$. Then, due to (5.3.57),

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\{R_\varepsilon(t\wedge\tau_n)\log R_\varepsilon(t\wedge\tau_n)\} = \frac{1}{2}\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{T\wedge\tau_n} |\sigma_s^{-1}\xi_\varepsilon(s)|^2\mathrm{d}s\\
&\leq C + C\mathbb{E}_{\mathbb{Q}_{\varepsilon,n}}\int_0^{T\wedge\tau_n} \sup_{B(X_t^{(1,\varepsilon)}, X_t^{(2,\varepsilon)}; r)} |\nabla Z_t|^2\mathrm{d}t
\end{aligned}$$

holds for some constant $C > 0$ independent of n. Since the law of $(X_{\cdot\wedge\tau_n}^{(1,\varepsilon)}, X_{\cdot\wedge\tau_n}^{(2,\varepsilon)})$ under $\mathbb{Q}_{\varepsilon,n}$ coincides with that of $(X_{\cdot\wedge\tau_n}^{(1)}, X_{\cdot\wedge\tau_n}^{(2)})$ under $\mathbb{P}$, combining this with (5.3.49) we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\{R_\varepsilon(t\wedge\tau_n)\log R_\varepsilon(t\wedge\tau_n)\} \leq C + C\int_0^T \mathbb{E}\sup_{B(X_t^{(1)}, X_t^{(2)}; r)} |\nabla Z_t|^2\mathrm{d}t < \infty.$$

Therefore, (5.3.56) holds. □

To derive explicit inequalities from Theorem 5.3.15, we consider below a special case where $\|\nabla Z_t\|_\infty$ is bounded and $A^l = 0$ for some natural number $l \geq 1$.

Corollary 5.3.17. *Assume* **(A5.3.3)**. *If $\|\nabla Z_t\|_\infty$ and $\|\sigma_t^{-1}\|$ are bounded in $t \geq 0$, and $A^l = 0$ for some $l \geq 1$. Then there exists a constant $C > 0$ such that for any positive $f \in \mathcal{B}_b(\mathbb{R}^{m+d}), T > 0$ and $e = (e_1, e_2) \in \mathbb{R}^{m+d}$:*

(1) $(P_T f)^p \leq P_T\{f^p(e + \cdot)\} \exp\left[\frac{Cp}{p-1}\Big(\frac{|e_2|^2}{1 \wedge T} + \frac{|e_1|^2}{(1 \wedge T)^{4k+3}}\Big)\right], \quad p > 1;$

(2) $P_T \log f \leq \log P_T\{f(e + \cdot)\} + C\Big(\frac{|e_2|^2}{1 \wedge T} + \frac{|e_1|^2}{(1 \wedge T)^{4k+3}}\Big);$

(3) *For $f \in C_b^1(\mathbb{R}^{m+d})$*, $|P_T \nabla_e f|^2 \leq C|P_T f^2|\Big(\frac{|e_2|^2}{1 \wedge T} + \frac{|e_1|^2}{(1 \wedge T)^{4k+3}}\Big);$

(4) *For strictly positive $f \in C_b^1(\mathbb{R}^{m+d})$,*

$$\begin{aligned}|P_T \nabla_e f| \leq \delta\big\{P_T(f \log f) - (P_T f)\log P_T f\big\}\\ + \frac{C}{\delta}\Big(\frac{|e_2|^2}{1 \wedge T} + \frac{|e_1|^2}{(1 \wedge T)^{4k+3}}\Big)P_T f, \quad \delta > 0.\end{aligned}$$

Proof. By $P_T = P_{T-1}P_{T-1,T}$ and the Jensen inequality, we only need to prove for $T \in (0, 1]$. Let $\phi(t) = \frac{t(T-t)}{T^2}$. Then $\phi(0) = \phi(T) = 0$ and due to Theorem 4.2(1) in [Wang and Zhang (2013)], the rank condition **(A5.3.3)** implies that

$$\|Q_\phi^{-1}\| \leq cT^{-(2k+1)} \tag{5.3.58}$$

for some constant $c > 0$ independent of $T \in (0, 1]$. To fix the other reference function ψ in Theorem 5.3.15, let $\{c_i\}_{1 \leq i \leq l+1} \in \mathbb{R}$ be such that

$$\begin{cases} 1 + \sum_{i=1}^{l+1} c_i = 0, \\ 1 + \sum_{i=1}^{l+1} \frac{j+1}{j+1+i} c_i = 0, \quad 0 \leq j \leq l-1. \end{cases}$$

Take

$$\psi(t) = 1 + \sum_{i=1}^{l+1} c_i \frac{(T-t)^i}{T^i}, \quad t \in [0, T].$$

Then $\psi(0) = 0, \psi(T) = 1$ and $\int_0^T (T-t)^j \psi(t) \mathrm{d}t = 0$ for $0 \leq j \leq l-1$. Since $A^l = 0$, we conclude that $\int_0^T \psi(t) \mathrm{e}^{(T-t)A} \mathrm{d}t = 0$. Therefore, (5.3.50) holds. It is easy to see that

$$|\psi(t)| \leq c, \quad |\psi'(t)| \leq cT^{-1}, \quad t \in [0, T]$$

holds for some constant $c > 0$. Combining this with (5.3.58), (5.3.52) and the boundedness of $\|\nabla Z\|_\infty$ and $\|\sigma^{-1}\|$, we obtain

$$\begin{aligned}&|\xi_1(t)| \le c\big(T^{-2(k+1)}|e_1| + T^{-1}|e_2|\big),\\ &|\Theta(t)| \le c(T^{-(2k+1)}|e_1| + |e_2|)\end{aligned} \tag{5.3.59}$$

for some constant $c > 0$. From this and Theorem 5.3.15, we derive the desired assertions. □

Corollary 5.3.18. *In the situation of Corollary* 5.3.17. *Let* $\|\cdot\|_{p\to q}$ *be the operator norm from* L^p *to* L^q *w.r.t. the Lebesgue measure on* $\mathbb{R}^{m+d}$. *Then there exists a constant* $C > 0$ *such that*

$$\|P_T\|_{p\to\infty} \le C^{\frac{1}{p}}\Big(\frac{p}{p-1}\Big)^{\frac{m+d}{2p}}(1\wedge T)^{-\frac{d+(4k+3)m}{2p}}, \quad p > 1, T > 0. \tag{5.3.60}$$

Consequently, the transition density $p_T((x,y),(x',y'))$ *of* P_T *w.r.t. the Lebesgue measure on* $\mathbb{R}^{m+d}$ *satisfies*

$$\begin{aligned}&\int_{\mathbb{R}^{m+d}} p_T((x,y),(x',y'))^{\frac{p}{p-1}}\,\mathrm{d}x'\mathrm{d}y'\\ &\le C^{\frac{1}{p-1}}\Big(\frac{p}{p-1}\Big)^{\frac{m+d}{2(p-1)}}(1\wedge T)^{-\frac{d+(4k+3)m}{2(p-1)}}\end{aligned} \tag{5.3.61}$$

for all $T > 0$, $(x,y) \in \mathbb{R}^{m+d}$, $p > 1$.

Proof. By Corollary 5.3.17(1), (5.3.60) follows from (1.4.11) for $P_T = P, \Phi(r) = r^p$ and

$$C_\Phi(x,e) = \frac{Cp}{p-1}\Big(\frac{|e_2|^2}{1\wedge T} + \frac{|e_1|^2}{(1\wedge T)^{4k+3}}\Big).$$

Moreover, (5.3.61) follows from (1.4.13). □

Example 5.3.2. A simple example for Corollary 5.3.17 to hold is that $\sigma_t = \sigma$ and $Z_t = Z$ are independent of t with $\|\nabla Z\|_\infty < \infty$, $A = 0$, and $\text{Rank}(B) = m$. In this case we have $d \ge m$, i.e. the dimension of the generate part is controlled by that of the non-degenerate part. In general, our results allow m to be much larger than d. For instance, let $m = ld$ for some $l \ge 2$ and

$$A = \begin{pmatrix} 0 & I_{d\times d} & 0 & \dots & 0 & 0\\ 0 & 0 & I_{d\times d} & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & I_{d\times d}\\ 0 & 0 & 0 & \dots & 0 & 0\end{pmatrix}_{(ld)\times(ld)}, \quad B = \begin{pmatrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\\ I_{d\times d}\end{pmatrix}_{(ld)\times d}.$$

Then $A^l = 0$ and **(A5.3.3)** holds for $k = l-1$. Therefore, assertions in Corollary 5.3.17 hold for $k = l-1$.

5.4 Stochastic Hamiltonian system: Malliavin calculus

This section is due to [Wang and Zhang (2013)] where explicit Bismut formula and Harnack inequality are studied for the stochastic Hamiltonian system by using Malliavin calculus. In general, the formula can be given by a pull-back operator (see e.g. §6 in [Aldous and Thorisson (1993)]), which is normally less explicit in the subelliptic case. Nevertheless, in some concrete degenerate cases the derivative formula can be explicitly established by solving certain control problems.

5.4.1 *A general result*

Consider the following degenerate stochastic differential equation on $\mathbb{R}^m \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = Z^{(1)}(X_t^{(1)}, X_t^{(2)})\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = Z^{(2)}(X_t^{(1)}, X_t^{(2)})\mathrm{d}t + \sigma \mathrm{d}B_t, \end{cases} \tag{5.4.1}$$

where $X_t^{(1)}$ and $X_t^{(2)}$ take values in $\mathbb{R}^m$ and $\mathbb{R}^d$ respectively, σ is an invertible $d \times d$-matrix, B_t is a d-dimensional Brownian motion, $Z^{(1)} \in C^2(\mathbb{R}^{m+d}; \mathbb{R}^m)$ and $Z^{(2)} \in C^1(\mathbb{R}^{m+d}; \mathbb{R}^d)$. Let $X_t = (X_t^{(1)}, X_t^{(2)}), Z = (Z^{(1)}, Z^{(2)})$. Then the equation can be formulated as

$$\mathrm{d}X_t = Z(X_t)\mathrm{d}t + (0, \sigma \mathrm{d}B_t). \tag{5.4.2}$$

We assume that the solution is non-explosive, which is ensured by **(A5.4.2)**(1) below. Our purpose is to establish an explicit derivative formula for the associated Markov semigroup P_t:

$$P_t f(x) = \mathbb{E} f(X_t(x)), \quad t > 0, x \in \mathbb{R}^{m+d}, f \in \mathcal{B}_b(\mathbb{R}^{m+d}),$$

where $X_t(x)$ is the solution of (5.4.2) with $X_0 = x$, and $\mathcal{B}_b(\mathbb{R}^{m+d})$ is the set of all bounded measurable functions on $\mathbb{R}^{m+d}$.

To compare the present equation with that investigated in §5.3 where $Z^{(1)}$ is linear, let us recall some simple notations. Firstly, we write the gradient operator on $\mathbb{R}^{m+d}$ as $\nabla = (\nabla^{(1)}, \nabla^{(2)})$, where $\nabla^{(1)}$ and $\nabla^{(2)}$ stand for the gradient operators for the first and the second components respectively, so that $\nabla f : \mathbb{R}^{m+d} \to \mathbb{R}^{m+d}$ for a differentiable function f on $\mathbb{R}^{m+d}$. Next, for a smooth function $\xi = (\xi_1, \ldots, \xi_k) : \mathbb{R}^{m+d} \to \mathbb{R}^k$, let

$$\nabla \xi = \begin{pmatrix} \nabla \xi_1 \\ \vdots \\ \nabla \xi_k \end{pmatrix}, \quad \nabla^{(i)} \xi = \begin{pmatrix} \nabla^{(i)} \xi_1 \\ \vdots \\ \nabla^{(i)} \xi_k \end{pmatrix}, \quad i = 1, 2.$$

Then $\nabla\xi, \nabla^{(1)}\xi, \nabla^{(2)}\xi$ are matrix-valued functions of orders $k \times (m+d), k \times m, k \times d$ respectively. Moreover, for an $l \times k$-matrix $M = (M_{ij})_{1\le i\le l, 1\le j\le k}$ and $v = (v_i)_{1\le i\le k} \in \mathbb{R}^k$, let $Mv \in \mathbb{R}^l$ with $(Mv)_i = \sum_{j=1}^k M_{ij}v_j$, $1 \le i \le l$. We have

$$\nabla_z\xi = (\nabla\xi)z, \quad \nabla^{(i)}_{z^{(i)}}\xi = (\nabla^{(i)}\xi)z^{(i)}, \quad i = 1, 2.$$

When $Z^{(1)}(x^{(1)}, x^{(2)})$ depends only on $x^{(2)}$ and $\nabla^{(2)}Z^{(1)}$ is a constant matrix with rank m, then equation (5.4.1) reduces back to the one studied in [Guillin and Wang (2012)] (and also in [Zhang, X. (2010)] for $m = d$). In this case we are able to construct very explicit successful couplings with control, which imply the desired derivative formula and Harnack inequalities as in the elliptic case. But when $Z^{(1)}$ is non-linear, it seems very hard to construct such couplings. The idea is to split $Z^{(1)}$ into a linear term and a non-linear term, and to derive an explicit derivative formula by controlling the non-linear part using the linear part in a reasonable way. More precisely, let

$$\nabla^{(2)}Z^{(1)} = B_0 + B,$$

where B_0 is a constant $m \times d$-matrix. We will be able to establish derivative formulae for P_t provided B is dominated by B_0 in the sense that

$$\langle BB_0^*a, a\rangle \ge -\varepsilon|B_0^*a|^2, \quad \forall a \in \mathbb{R}^m \tag{5.4.3}$$

holds for some constant $\varepsilon \in [0, 1)$.

To state our main result, we first briefly recall the integration by parts formula for the Brownian motion. Let $T > 0$ be fixed. For a Hilbert space H, let

$$\mathbb{H}^1(H) = \left\{h \in C([0,T]; H):\ h(0) = 0, \|h\|^2_{\mathbb{H}^1(H)} := \int_0^T |h'(t)|^2_H \mathrm{d}t < \infty\right\}$$

be the Cameron-Martin space over H. Let $\mathbb{H}^1 = \mathbb{H}^1(\mathbb{R}^d)$ and, without confusion in the context, simply denote $\|\cdot\|_{\mathbb{H}^1} = \|\cdot\|_{\mathbb{H}^1(H)}$ for any Hilbert space H.

Let μ be the distribution of $\{B_t\}_{t\in[0,T]}$, which is a probability measure (i.e. Wiener measure) on the path space $\Omega = C([0,T]; \mathbb{R}^d)$. The probability space (Ω, μ) is endowed with the natural filtration of the coordinate process $B_t(w) := w_t, t \in [0,T]$. A function $F \in L^2(\Omega; \mu)$ is called differentiable if for any $h \in \mathbb{H}^1$, the directional derivative

$$D_hF := \lim_{\varepsilon\to 0} \frac{F(\cdot + \varepsilon h) - F(\cdot)}{\varepsilon}$$

exists in $L^2(\Omega;\mu)$. If the map $\mathbb{H}^1 \ni h \mapsto D_hF \in L^2(\Omega;\mu)$ is bounded, then there exists a unique $DF \in L^2(\Omega \to \mathbb{H}^1;\mu)$ such that $\langle DF, h\rangle_{\mathbb{H}^1} = D_hF$ holds in $L^2(\Omega;\mu)$ for all $h \in \mathbb{H}^1$. In this case we write $F \in \mathcal{D}(D)$ and call DF the Malliavin gradient of F. It is well known that $(D, \mathcal{D}(D))$ is a closed operator in $L^2(\Omega;\mu)$, whose adjoint operator $(D^*, \mathcal{D}(D^*))$ is called the divergence operator. That is,

$$\mathbb{E}(D_hF) = \int_\Omega D_hF\mathrm{d}\mu = \int_\Omega FD^*h\mathrm{d}\mu = \mathbb{E}(FD^*h) \tag{5.4.4}$$

for $F \in \mathcal{D}(D)$, $h \in \mathcal{D}(D^*)$.

For any $s \geq 0$, let $\{K(t,s)\}_{t\geq s}$ solve the following random ODE on $\mathbb{R}^m \otimes \mathbb{R}^m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t,s) = (\nabla^{(1)}Z^{(1)})(X_t)K(t,s), \quad K(s,s) = I_{m\times m}. \tag{5.4.5}$$

We will make use of the following assumption.

(A5.4.1) The function

$$U(x) := \mathbb{E}\exp\left[2\int_0^T \|\nabla Z(X_t(x))\|\mathrm{d}t\right], \quad x \in \mathbb{R}^{m+d}$$

is locally bounded.

For any $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^{m+d}$ with $|v| = 1$, we aim to search for $h = h(v) \in \mathcal{D}(D^*)$ such that

$$\nabla_v P_T f(x) = \mathbb{E}\big[f(X_T(x))D^*h\big], \quad f \in C_b^1(\mathbb{R}^{m+d}) \tag{5.4.6}$$

holds. To construct h, for an $\mathbb{H}^1$-valued random variable $\alpha = (\alpha(s))_{s\in[0,T]}$, let

$$\begin{aligned} g(t) &= K(t,0)v^{(1)} + \int_0^t K(t,s)\nabla^{(2)}_{\alpha(s)}Z^{(1)}(X_s(x))\mathrm{d}s, \\ h(t) &= \int_0^t \sigma^{-1}\big(\nabla_{(g_s,\alpha(s))}Z^{(2)}(X_s(x)) - \alpha'(s)\big)\mathrm{d}s, \quad t \in [0,T]. \end{aligned} \tag{5.4.7}$$

We will show that h satisfies (5.4.6) provided it is in $\mathcal{D}(D^*)$ and $\alpha(0) = v^{(2)}, \alpha(T) = 0, g(T) = 0$.

Theorem 5.4.1. *Assume* **(A5.4.1)** *for some* $T > 0$. *For* $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^{m+d}$, *let* $(\alpha(s))_{0\leq s\leq T}$ *be an* $\mathbb{H}^1$*-valued random variable such that* $\alpha(0) = v^{(2)}$ *and* $\alpha(T) = 0$, *and let* $g(t)$ *and* $h(t)$ *be given in* (5.4.7). *If* $g(T) = 0$ *and* $h \in \mathcal{D}(D^*)$, *then* (5.4.6) *holds.*

Proof. For simplicity, we will drop the initial data of the solution by writing $X_t(x) = X_t$. By **(A5.4.1)** and (5.4.2) we have $X_t \in \mathcal{D}(D)$, and due to the chain rule and the definition of $h(t)$,

$$\begin{aligned} D_h X_t &= \int_0^t \nabla_{D_h X_s} Z(X_s)\mathrm{d}s + \int_0^t (0, \sigma h'(s))\mathrm{d}s \\ &= (0, v^{(2)} - \alpha(t)) + \int_0^t \nabla_{D_h X_s} Z(X_s)\mathrm{d}s \qquad (5.4.8) \\ &\quad + \int_0^t \left(0, \nabla_{(g_s, \alpha(s))} Z^{(2)}(X_s)\right) \mathrm{d}s \end{aligned}$$

holds for $t \in [0, T]$. Next, it is easy to see that

$$g(t) = v^{(1)} + \int_0^t \nabla_{(g_s,\alpha(s))} Z^{(1)}(X_s)\mathrm{d}s, \quad t \in [0,T].$$

Combining this with (5.4.8) we obtain

$$D_h X_t + (g(t), \alpha(t)) = v + \int_0^t \nabla_{D_h X_s + (g_s, \alpha(s))} Z(X_s)\mathrm{d}s, \quad t \in [0,T].$$

On the other hand, the directional derivative process

$$\nabla_v X_t := \lim_{\varepsilon \to 0} \frac{X_t(x + \varepsilon v) - X_t(x)}{\varepsilon}$$

satisfies the same equation, i.e.

$$\nabla_v X_t = v + \int_0^t \nabla_{\nabla_v X_s} Z(X_s)\mathrm{d}s, \quad t \in [0,T]. \qquad (5.4.9)$$

Thus, by the uniqueness of the ODE we conclude that

$$D_h X_t + (g(t), \alpha(t)) = \nabla_v X_t, \quad t \in [0,T].$$

In particular, since $(g(T), \alpha(T)) = 0$, we have

$$D_h X_T = \nabla_v X_T \qquad (5.4.10)$$

and due to **(A5.4.1)** and (5.4.9),

$$\mathbb{E}|D_h X_T|^2 = \mathbb{E}|\nabla_v X_T|^2 \le |v|^2 \mathbb{E} \exp\left[2\int_0^T \|\nabla Z\|(X_s)\mathrm{d}s\right]. \qquad (5.4.11)$$

Combining this with (5.4.4) and letting $f \in C_b^1(\mathbb{R}^{m+d})$, we are able to adopt the dominated convergence theorem to obtain

$$\begin{aligned} \nabla_v P_T f &= \mathbb{E}\langle \nabla f(X_T), \nabla_v X_T\rangle = \mathbb{E}\langle \nabla f(X_T), D_h X_T\rangle \\ &= \mathbb{E} D_h f(X_T) = \mathbb{E}[f(X_T) D^* h]. \end{aligned}$$

□

5.4.2 *Explicit formula*

According to Theorem 5.4.1, to derive explicit derivative formula, we need to calculate D^*h for h given by (5.4.7). To this end, we assume

(A5.4.2) The matrix $\sigma \in \mathbb{R}^d \otimes \mathbb{R}^d$ is invertible, and there exists $W \in C^2(\mathbb{R}^{m+d})$ with $W \geq 1$ and $\lim_{|x|\to\infty} W(x) = \infty$ such that for some constants $C, l_2 \geq 0$ and $l_1 \in [0,1]$,

(1) $LW \leq CW$, $|\nabla^{(2)}W|^2 \leq CW$, where $L = \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*\nabla^{(2)}\nabla^{(2)}) + Z \cdot \nabla$;
(2) $\|\nabla Z\| \leq CW^{l_1}$, $\|\nabla^2 Z\| \leq CW^{l_2}$.

Theorem 5.4.2. *Assume* **(A5.4.2)** *and let* $\nabla^{(2)}Z^{(1)} = B_0 + B$ *for some constant matrix* B_0 *such that* (5.4.3) *holds for some constant* $\varepsilon \in [0,1)$. *If there exist an increasing function* $\xi \in C([0,T])$ *and* $\phi \in C^1([0,T])$ *with* $\xi(t) > 0$ *for* $t \in (0,T]$, $\phi(0) = \phi(T) = 0$ *and* $\phi(t) > 0$ *for* $t \in (0,T)$ *such that*

$$\int_0^t \phi(s)K(T,s)B_0B_0^*K(T,s)^*\mathrm{d}s \geq \xi(t)I_{m\times m}, \quad t \in (0,T]. \tag{5.4.12}$$

Then

(1) $Q_t := \int_0^t \phi(s)K(T,s)\nabla^{(2)}Z^{(1)}(X_s)B_0^*K(T,s)^*\mathrm{d}s$ *is invertible for* $t \in (0,T]$ *with*

$$\|Q_t^{-1}\| \leq \frac{1}{(1-\varepsilon)\xi(t)}, \quad t \in [0,T]. \tag{5.4.13}$$

(2) *Let* h *be determined by* (5.4.7) *for*

$$\begin{aligned}\alpha(t) :=\ & \frac{T-t}{T}v^{(2)} \\ & - \frac{\phi(t)B_0^*K(T,t)^*}{\int_0^T \xi(s)^2\mathrm{d}s}\int_t^T \xi(s)^2Q_s^{-1}K(T,0)v^{(1)}\mathrm{d}s \\ & - \phi(t)B_0^*K(T,t)^*Q_T^{-1}\int_0^T \frac{T-s}{T}K(T,s)\nabla^{(2)}_{v^{(2)}}Z^{(1)}(X_s)\mathrm{d}s.\end{aligned} \tag{5.4.14}$$

Then for any $p \geq 2$, *there exists a constant* $T_p \in (0,\infty)$ *if* $l_1 = 1$ *and* $T_p = \infty$ *if* $l_1 < 1$, *such that for any* $T \in (0,T_p)$, (5.4.6) *holds with* $\mathbb{E}|D^*h|^p < \infty$.

(3) *For any* $p > 1$ *there exist constants* $c_1(p), c_2(p) \geq 0$, *where* $c_2(p) = 0$ *if* $l_1 = l_2 = 0$, *such that*

$$\begin{aligned}|\nabla P_T f| \leq\ & c_1(p)(P_T|f|^p)^{1/p} \\ & \times \frac{\sqrt{T\wedge 1}\{(T\wedge 1)^{3/2} + \xi(T\wedge 1)\}\mathrm{e}^{c_2(p)W}}{\int_0^{T\wedge 1}\xi(s)^2\mathrm{d}s}\end{aligned} \tag{5.4.15}$$

holds for all $T > 0$ *and* $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$.

The idea of the proof is to apply Theorem 5.4.1 for the given process $\alpha(s)$. Obviously, **(A5.4.2)**(1) implies that for any $l \geq 1$, there exists a constant C_l such that $LW^l \leq C_l W^l$, so that $\mathbb{E}W(X_t(x))^l \leq e^{C_l t}W(x)^l$ and thus, the process is non-explosive; while **(A5.4.2)**(2) implies that $\|\nabla Z\| + \|\nabla^2 Z\| \leq CW^{l_1\vee l_2}$ holds for some $C > 0$, so that

$$\mathbb{E}\left((\|\nabla Z\|^p + \|\nabla^2 Z\|^p)(X_t)\right) \leq e^{c(p)t}W^{p(l_1\vee l_2)}, \quad t \geq 0 \tag{5.4.16}$$

holds for any $p \geq 1$ with some constant $c(p) > 0$. The following lemma ensures that **(A5.4.2)** implies **(A5.4.1)** for all $T > 0$ if $l_1 < 1$ and for small $T > 0$ if $l_1 = 1$.

Lemma 5.4.3. *If* **(A5.4.2)**(1) *holds, then for any* $T > 0$,

$$\mathbb{E}\exp\left[\frac{2}{T^2C\|\sigma\|^2e^{4+2CT}}\int_0^T W(X_t)\mathrm{d}t\right] \leq \exp\left[\frac{2W}{TC\|\sigma\|^2e^{2+CT}}\right].$$

Consequently, **(A5.4.2)**(2) *implies that* $U := \mathbb{E}\exp[2\int_0^T \|\nabla Z\|(X_t)\mathrm{d}t]$ *is locally bounded on* $\mathbb{R}^{m+d}$ *if either* $l_1 < 1$ *or* $l_1 = 1$ *but* $T^2C^2\|\sigma\|^2e^{4+2CT} \leq 1$.

Proof. It suffices to prove the first assertion. By the Itô formula and **(A5.4.2)**(1), we have

$$\begin{aligned}\mathrm{d}W(X_t) &= \langle\nabla^{(2)}W(X_t), \sigma\mathrm{d}B_t\rangle + LW(X_t)\mathrm{d}t\\ &\leq \langle\nabla^{(2)}W(X_t), \sigma\mathrm{d}B_t\rangle + CW(X_t)\mathrm{d}t.\end{aligned}$$

So, for $t \in [0, T]$,

$$\begin{aligned}\mathrm{d}\{e^{-(C+2/T)t}W(X_t)\} \leq\ & e^{-(C+2/T)t}\langle\nabla^{(2)}W(X_t), \sigma\mathrm{d}B_t\rangle\\ & - \frac{2}{T}e^{-CT-2}W(X_t)\mathrm{d}t.\end{aligned}$$

Thus, letting $\tau_n = \inf\{t \geq 0 : W(X_t) \geq n\}$, for any $n \geq 1$ and $\lambda > 0$ we have

$$\begin{aligned}&\mathbb{E}\exp\left[\frac{2\lambda}{Te^{CT+2}}\int_0^{T\wedge\tau_n} W(X_t)\mathrm{d}t\right]\\ &\leq e^{\lambda W}\mathbb{E}\exp\left[\lambda\int_0^{T\wedge\tau_n} e^{-(C+2/T)t}\langle\nabla^{(2)}W(X_t), \sigma\mathrm{d}B_t\rangle\right]\\ &\leq e^{\lambda W}\left(\mathbb{E}\exp\left[2\lambda^2C\|\sigma\|^2\int_0^{T\wedge\tau_n} W(X_t)\mathrm{d}t\right]\right)^{1/2},\end{aligned}$$

where the second inequality is due to the exponential martingale and **(A5.4.2)**(1). By taking

$$\lambda = \frac{1}{TC\|\sigma\|^2e^{CT+2}},$$

we arrive at

$$\mathbb{E}\exp\left[\frac{2}{T^2C\|\sigma\|^2\mathrm{e}^{4+2CT}}\int_0^{T\wedge\tau_n}W(X_t)\mathrm{d}t\right]\le\exp\left[\frac{2W}{TC\|\sigma\|^2\mathrm{e}^{2+CT}}\right].$$

This completes the proof by letting $n\to\infty$. □

To ensure that $\mathbb{E}|D^*h|^p<\infty$, we need the following two lemmas.

Lemma 5.4.4. *Assume* **(A5.4.2)**. *Then there exists a constant* $c>0$ *such that*

$$\|DX_t\|_{\mathbb{H}^1}\le\sqrt{t}\|\sigma\|\mathrm{e}^{c\int_0^t W^{l_1}(X_s)\mathrm{d}s},\quad t\ge 0. \tag{5.4.17}$$

Consequently, if $l_1<1$, *then for any* $p\ge 1$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|DX_t\|_{\mathbb{H}^1}^p\right)<\infty,\quad T\ge 0;$$

and if $l_1=1$, *then for any* $p\ge 1$ *there exists a constant* $T_p>0$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|DX_t\|_{\mathbb{H}^1}^p\right)<\infty,\quad T\in(0,T_p).$$

Proof. Due to Lemma 5.4.3, it suffices to prove (5.4.17). From (5.4.2) we see that for any $h\in\mathbb{H}^1$, D_hX_t solves the following random ODE:

$$D_hX_t=\int_0^t(\nabla_{D_hX_s}Z)(X_s)\mathrm{d}s+(0,\sigma h(t)).$$

Combining this with **(A5.4.2)**(2) and $|h(t)|\le\sqrt{t}\,\|h\|_{\mathbb{H}^1}$, we obtain

$$|D_hX_t|\le C\int_0^t W^{l_1}(X_s)|D_hX_s|\mathrm{d}s+\sqrt{t}\|\sigma\|\cdot\|h\|_{\mathbb{H}^1},\quad h\in\mathbb{H}^1.$$

Therefore,

$$\|DX_t\|_{\mathbb{H}^1}\le C\int_0^t W^{l_1}(X_s)\|DX_s\|_{\mathbb{H}^1}\mathrm{d}s+\sqrt{t}\|\sigma\|.$$

This implies (5.4.17) by Gronwall's inequality. □

Lemma 5.4.5. *Assume* **(A5.4.2)**. *Then for any* $s\in[0,T]$,

$$\begin{aligned}&\|K(T,s)\|\le\mathrm{e}^{C\int_s^T W^{l_1}(X_r)\mathrm{d}r},\\&\|\partial_sK(T,s)\|\le CW^{l_1}(X_s)\mathrm{e}^{C\int_s^T W^{l_1}(X_r)\mathrm{d}r},\end{aligned} \tag{5.4.18}$$

and

$$\|DK(T,s)\|_{\mathbb{H}^1}\le C\mathrm{e}^{2C\int_s^T W^{l_1}(X_r)\mathrm{d}r}\int_s^T W^{l_2}(X_r)\|DX_r\|_{\mathbb{H}^1}\mathrm{d}r. \tag{5.4.19}$$

Consequently, for any $p>1$ *there exists* $T_p\in(0,\infty)$ *if* $l_1=1$ *and* $T_p=\infty$ *if* $l_1<1$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|DK(T,t)\|_{\mathbb{H}^1}^p\right)<\infty,\quad T\in(0,T_p).$$

Proof. By Lemma 5.4.4 and $\sup_{t\in[0,T]}\mathbb{E}W^l(X_t)<\infty$ for any $l>0$ as observed in the beginning of this section, it suffices to prove (5.4.18) and (5.4.19). First of all, by (5.4.5) and **(A5.4.2)**(2), we have

$$\begin{aligned}\|K(t,s)\| &\le 1+\int_s^t \|\nabla^{(1)}Z^{(1)}(X_r)\|\cdot\|K(r,s)\|\mathrm{d}r\\ &\le 1+C\int_s^t W^{l_1}(X_r)\|K(r,s)\|\mathrm{d}r,\end{aligned}$$

which yields the first estimate in (5.4.18) by Gronwall's inequality. Moreover, noticing that

$$\partial_s K(t,s)=\int_s^t(\nabla^{(1)}Z^{(1)})(X_r)\partial_s K(r,s)\mathrm{d}r-(\nabla^{(1)}Z^{(1)})(X_s),$$

by **(A5.4.2)**(2) we have

$$\|\partial_s K(t,s)\|\le C\int_s^t W^{l_1}(X_r)\|\partial_s K(r,s)\|\mathrm{d}r+CW^{l_1}(X_s).$$

The second estimate in (5.4.18) follows. As for (5.4.19), since

$$\frac{\mathrm{d}}{\mathrm{d}t}DK(t,s)=(\nabla_{DX_t}\nabla^{(1)}Z^{(1)})(X_t)K(t,s)+(\nabla^{(1)}Z^{(1)})(X_t)DK(t,s),$$

with $DK(s,s)=0$, it follows from **(A5.4.2)**(2) and (5.4.18) that

$$\begin{aligned}\|DK(t,s)\|_{\mathbb{H}^1} &\le \int_s^t \|\nabla\nabla^{(1)}Z^{(1)}(X_r)\|\ \|DX_r\|_{\mathbb{H}^1}\|K(r,s)\|\mathrm{d}r\\ &\quad+\int_s^t\|\nabla^{(1)}Z^{(1)}(X_r)\|\ \|DK(r,s)\|_{\mathbb{H}^1}\mathrm{d}r\\ &\le C\mathrm{e}^{C\int_s^T W^{l_1}(X_r)\mathrm{d}r}\int_s^t W^{l_2}(X_r)\|DX_r\|_{\mathbb{H}^1}\mathrm{d}r\\ &\quad+C\int_s^t W^{l_1}(X_r)\|DK(r,s)\|_{\mathbb{H}^1}\mathrm{d}r.\end{aligned}$$

This implies (5.4.19). □

Proof. **(Proof of Theorem 5.4.2)** (1) Let $a\in\mathbb{R}^m$. By (5.4.3), (5.4.12) and $\nabla^{(2)}Z^{(1)}=B_0+B$ we have

$$\begin{aligned}\langle Q_t a,a\rangle &=\int_0^t\phi(s)\Big(\langle K(T,s)B_0B_0^*K(T,s)^*a,a\rangle\\ &\quad+\langle K(T,s)B(X_s)B_0^*K(T,s)^*a,a\rangle\Big)\mathrm{d}s\\ &\ge(1-\varepsilon)\int_0^t\phi(s)|B_0^*K(T,s)^*a|^2\mathrm{d}s\ge(1-\varepsilon)\xi(t)|a|^2.\end{aligned}$$

This implies that Q_t is invertible and (5.4.13) holds.

(2) According to Lemma 5.4.3, **(A5.4.2)** implies **(A5.4.1)** for all $T > 0$ if $l_1 < 1$ and for small $T > 0$ if $l_1 = 1$. Next, we intend to prove that $h \in \mathcal{D}(D^*)$ and $\mathbb{E}|D^*h|^p < \infty$ for small $T > 0$ if $l_1 = 1$ and for all $T > 0$ if $l_1 < 1$. Indeed, by Lemmas 5.4.4, 5.4.5, (5.4.16), and the fact that

$$DQ_t^{-1} = -Q_t^{-1}(DQ_t)Q_t^{-1},$$

there exists $T_p > 0$ if $l_1 = 1$ and $T_p = \infty$ if $l_1 < 1$ such that

$$\sup_{t\in[0,T]} \mathbb{E}|DQ_t|^p < +\infty, \quad T \in (0, T_p),$$

and by (5.4.13),

$$\left(\mathbb{E}\|DQ_t^{-1}\|_{\mathbb{H}^1}^p\right)^{1/p} \le \frac{\left(\mathbb{E}|DQ_t|^p\right)^{1/p}}{[(1-\epsilon)\xi(t)]^2}, \quad t \in (0, T], \tag{5.4.20}$$

$$\sup_{t\in[0,T]} \left(\mathbb{E}\|D\alpha(t)\|_{\mathbb{H}^1}^p + \mathbb{E}\|Dg(t)\|_{\mathbb{H}^1}^p\right)^{1/p} < \infty, \quad T \in (0, T_p). \tag{5.4.21}$$

Since

$$\begin{aligned} &h'(t) = \sigma^{-1}\big\{(\nabla_{(g(t),\alpha(t))}Z^{(2)})(X_t) - \alpha'(t)\big\},\\ &\|Dh'(t)\|_{\mathbb{H}^1} \le \|\sigma^{-1}\|\big\{\|\nabla^2 Z^{(2)}(X_t)\|\ \|DX_t\|_{\mathbb{H}^1}\ |(g(t),\alpha(t))| \\ &\qquad + \|\nabla Z^{(2)}(X_t)\|\ \|(Dg(t), D\alpha(t))\|_{\mathbb{H}^1} + \|D\alpha'(t)\|_{\mathbb{H}^1}\big\}, \end{aligned} \tag{5.4.22}$$

we conclude from **(A5.4.2)**(2), (5.4.16) and (5.4.21) that

$$\mathbb{E}\bigg(\int_0^T \|Dh'(t)\|_{\mathbb{H}^1}^2 \mathrm{d}t\bigg)^{p/2} + \mathbb{E}\|h\|_{\mathbb{H}^1}^p < \infty, \quad T \in (0, T_p).$$

Therefore, according to e.g. Proposition 1.5.8 in [Nualart (1995)], we have $h \in \mathcal{D}(D^*)$ and $\mathbb{E}|D^*h|^p < \infty$ provided $T \in (0, T_p)$.

Now, to prove (5.4.6), it remains to verify the required conditions of Theorem 5.4.1 for $\alpha(t)$ given by (5.4.14). Since $\phi(0) = \phi(T) = 0$, we have $\alpha(0) = v^{(2)}$ and $\alpha(T) = 0$. Moreover, noting that

$$\begin{aligned} I_1 &:= \frac{1}{\int_0^T \xi(t)^2 \mathrm{d}t} \int_0^T \phi(t)K(T,t)\nabla^{(2)}Z^{(1)}(X_t)B_0^*K(T,t)^*\mathrm{d}t \\ &\qquad \cdot \int_t^T \xi(s)^2 Q_s^{-1}K(T,0)v^{(1)}\mathrm{d}s \\ &= \frac{1}{\int_0^T \xi(t)^2 \mathrm{d}t} \int_0^T \dot{Q}_t \mathrm{d}t \int_t^T \xi(s)^2 Q_s^{-1}K(T,0)v^{(1)}\mathrm{d}s \\ &= \frac{1}{\int_0^T \xi(t)^2 \mathrm{d}t} \int_0^T \xi(t)^2 Q_t Q_t^{-1}K(T,0)v^{(1)}\mathrm{d}t = K(T,0)v^{(1)} \end{aligned}$$

and

$$\begin{aligned}I_2 &:= \left(\int_0^T \phi(t) K(T,t) \nabla^{(2)} Z^{(1)}(X_t) B_0^* K(T,t)^* \mathrm{d}t \right) \\ &\quad \cdot Q_T^{-1} \int_0^T \frac{T-s}{T} K(T,s) \nabla^{(2)}_{v^{(2)}} Z^{(1)}(X_s) \mathrm{d}s \\ &= Q_T Q_T^{-1} \int_0^T \frac{T-s}{T} K(T,s) \nabla^{(2)}_{v^{(2)}} Z^{(1)}(X_s) \mathrm{d}s \\ &= \int_0^T \frac{T-s}{T} K(T,s) \nabla^{(2)}_{v^{(2)}} Z^{(1)}(X_s) \mathrm{d}s,\end{aligned}$$

we obtain by (5.4.14)

$$\begin{aligned}g(T) &= K(T,0)v^{(1)} + \int_0^T K(T,t) \nabla^{(2)}_{\alpha(t)} Z^{(1)}(X_t) \mathrm{d}t \\ &= K(T,0)v^{(1)} - I_1 + \int_0^T \frac{T-t}{T} K(T,t) \nabla^{(2)}_{v^{(2)}} Z^{(1)}(X_t) \mathrm{d}t - I_2 \\ &= 0.\end{aligned}$$

(3) By an approximation argument, it suffices to prove the desired gradient estimate for $f \in C_b^1(\mathbb{R}^{m+d})$. Moreover, by the semigroup property and the Jensen inequality, we only have to prove for $p \in (1,2]$ and $T \in (0, T_p \wedge 1)$. In this case we obtain from (5.4.6) that

$$|\nabla P_T f| \le (P_T |f|^p)^{1/p} (\mathbb{E}|D^* h|^q)^{1/q},$$

where $q := \frac{p}{p-1} \ge 2$. Therefore, it remains to find constants $c_1, c_2 \ge 0$, where $c_2 = 0$ if $l_1 = l_2 = 0$, such that

$$(\mathbb{E}|D^* h|^q)^{1/q} \le \frac{c_1 \sqrt{T} (T^{3/2} + \xi(T)) \mathrm{e}^{c_2 W}}{\int_0^T \xi(s)^2 \mathrm{d}s}. \tag{5.4.23}$$

To this end, we take $\phi(t) = \frac{t(T-t)}{T^2}$ such that $0 \le \phi \le 1$ and $|\phi'(t)| \le \frac{1}{T}$ for $t \in [0,T]$. Since ξ is increasing, by (5.4.18) and (5.4.12), we have for some constant $c > 0$,

$$\int_0^t \xi(s)^2 \mathrm{d}s \le \xi(t)^2 \le ct^2, \quad t \in [0,1].$$

Thus, by Lemmas 5.4.3, 5.4.4, 5.4.5 and (5.4.16), it is easy to see that for any $\theta \ge 2$ there exist constants $c_1, c_2 \ge 0$, where $c_2 = 0$ if $l_1 = l_2 = 0$, such that for all $0 < t \le T \le T_p \wedge 1$,

$$\left(\mathbb{E}\|DX_t\|_{\mathbb{H}^1}^\theta\right)^{1/\theta} \le c_1\sqrt{T}\mathrm{e}^{c_2W}, \quad \left(\mathbb{E}\|DK(T,t)\|_{\mathbb{H}^1}^\theta\right)^{1/\theta} \le c_1T^{3/2}\mathrm{e}^{c_2W}$$

$$(\mathbb{E}\|DQ_t^{-1}\|_{\mathbb{H}^1}^\theta)^{1/\theta} \le \left\{\mathbb{E}(\|Q_t^{-1}\|\|DQ_t\|_{\mathbb{H}^1}\|Q_t^{-1}\|)^\theta\right\}^{1/\theta} \le \frac{c_1t\sqrt{T}}{\xi(t)^2}\mathrm{e}^{c_2W},$$

$$(\mathbb{E}\|D\alpha(t)\|_{\mathbb{H}^1}^\theta)^{1/\theta} \le \frac{c_1T^{5/2}\mathrm{e}^{c_2W}}{\int_0^T\xi(s)^2\mathrm{d}s}, \quad (\mathbb{E}\|Dg(t)\|_{\mathbb{H}^1}^\theta)^{1/\theta} \le \frac{c_1T^{7/2}\mathrm{e}^{c_2W}}{\int_0^T\xi(s)^2\mathrm{d}s},$$

$$(\mathbb{E}\|D\alpha'(t)\|_{\mathbb{H}^1}^\theta)^{1/\theta} \le \frac{c_1T^{3/2}\mathrm{e}^{c_2W}}{\int_0^T\xi(s)^2\mathrm{d}s}, \quad (\mathbb{E}|h'(t)|^\theta)^{1/\theta} \le \frac{c_1\xi(T)\mathrm{e}^{c_2W}}{\int_0^T\xi(s)^2\mathrm{d}s}.$$

Combining these with (5.4.22), **(A5.4.2)**(2) and (5.4.16), we obtain

$$\begin{aligned}
\|h\|_{\mathbb{D}^{1,q}} &:= \left(\mathbb{E}\|Dh\|_{\mathbb{H}^1\otimes\mathbb{H}^1}^q\right)^{1/q} + \|\mathbb{E}h\|_{\mathbb{H}^1} \\
&\le \sqrt{T}\left\{\mathbb{E}\left(\frac{1}{T}\int_0^T \|Dh'(t)\|_{\mathbb{H}^1}^2\mathrm{d}t\right)^{q/2}\right\}^{1/q} + \mathbb{E}\|h\|_{\mathbb{H}^1} \\
&\le \sqrt{T}\left(\frac{1}{T}\int_0^T \mathbb{E}\|Dh'(t)\|_{\mathbb{H}^1}^q\mathrm{d}t\right)^{1/q} + \left(\mathbb{E}\int_0^T |h'(t)|^2\mathrm{d}t\right)^{1/2} \\
&\le \frac{c_1\sqrt{T}(T^{3/2}+\xi(T))\mathrm{e}^{c_2W}}{\int_0^T\xi(s)^2\mathrm{d}s}.
\end{aligned}$$

This implies (5.4.23) since $D^*: \mathbb{D}^{1,q} \to L^q$ is bounded, see e.g. Proposition 1.5.8 in [Nualart (1995)]. □

5.4.3 *Two specific cases*

We intend to apply Theorem 5.4.2 with concrete choices of ξ satisfying (5.4.12).

5.4.3.1 Rank$[B_0] = m$

Theorem 5.4.6. *Assume* **(A5.4.2)** *and* (5.4.3) *for some* $\varepsilon \in [0,1)$. *If* Rank$[B_0] = m$, *then there exist constants* $c_1, c_2 > 0$ *such that* (5.4.12) *holds for*

$$\xi(t) = c_1\int_0^t \phi(s)\mathrm{e}^{-c_2(T-s)}\mathrm{d}s, \quad t \in [0,T].$$

Consequently, for any $p > 1$ *there exist two constants* $c_1(p), c_2(p) \ge 0$, *where* $c_2(p) = 0$ *if* $l_1 = l_2 = 0$, *such that*

$$|\nabla P_Tf| \le \frac{c_1(p)(P_T|f|^p)^{1/p}}{(T\wedge 1)^{3/2}}\mathrm{e}^{c_2(p)W}, \quad T > 0.$$

Proof. It is easy to see that the desired gradient estimate follows from (5.4.15) for the claimed ξ with $\phi(t) = \frac{t(T-t)}{T^2}$, we only prove the first assertion. Since $\nabla^{(1)}Z^{(1)}$ is bounded, there exists a constant $C > 0$ such that

$$|K(T,s)^*a| \geq \mathrm{e}^{-C(T-s)}|a|, \quad a \in \mathbb{R}^m.$$

If $\mathrm{Rank}[B_0] = m$, then $|B_0^* a| \geq c'|a|$ holds for some constant $c' > 0$ and all $a \in \mathbb{R}^m$. Therefore,

$$M_t := \int_0^t \phi(s) K(T,s) B_0 B_0^* K(T,s)^* \mathrm{d}s$$

satisfies

$$\langle M_t a, a\rangle = \int_0^t \phi(s)|B_0^* K(T,s)^* a|^2 \mathrm{d}s \geq c'^2 \int_0^t \phi(s) \mathrm{e}^{-2C(T-s)}|a|^2 \mathrm{d}s.$$

This completes the proof. $\square$

5.4.3.2 $A := \nabla^{(1)}Z^{(1)}$ *is constant*

We assume that

(A5.4.3) (Kalman condition) $A := \nabla^{(1)}Z^{(1)}$ is constant and there exists an integer number $0 \leq k \leq m-1$ such that

$$\mathrm{Rank}[B_0, AB_0, \ldots, A^k B_0] = m. \tag{5.4.24}$$

When $k = 0$, (5.4.24) means $\mathrm{Rank}[B_0] = m$ which has been considered in Theorem 5.4.6.

Theorem 5.4.7. *Assume* **(A5.4.2), (A5.4.3)** *and* (5.4.3) *for some* $\varepsilon \in (0,1)$. *Let* $\phi(t) = \frac{t(T-t)}{T^2}$. *Then:*

(1) *There exist constants* $c_1, c_2 > 0$ *such that* (5.4.12) *holds for*

$$\xi(t) = \frac{c_1(t \wedge 1)^{2(k+1)}}{T\mathrm{e}^{c_2 T}}, \quad t \in [0,T].$$

(2) *For any* $p > 1$, *there exist two constants* $c_1(p), c_2(p) \geq 0$, *where* $c_2(p) = 0$ *if* $l_1 = l_2 = 0$, *such that*

$$|\nabla P_T f| \leq \frac{c_1(p)(P_T|f|^p)^{1/p}}{(T \wedge 1)^{(4k-1)\vee 0 + 3/2}} \mathrm{e}^{c_2(p)W}, \quad T > 0.$$

(3) *If* $\nabla^{(2)}Z^{(1)} = B_0$ *is constant and* $l_1 < \frac{1}{2}$, *then there exists a constant* $c > 0$ *such that*

$$|\nabla P_T f| \le \lambda\{P_T(f\log f) - (P_T f)\log P_T f\}$$
$$+\frac{c}{\lambda}\left\{\frac{l_1 W}{(1+\lambda^{-1})^2} + \frac{(1+\lambda^{-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(1\wedge T)^{4k+3}}\right\}P_T f$$

holds for all $\lambda > 0$, $T > 0$ *and* $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$, *the set of positive functions in* $\mathcal{B}_b(\mathbb{R}^{m+d})$.

(4) *If* $\nabla^{(2)}Z^{(1)} = B_0$ *is constant and* $l_1 = \frac{1}{2}$, *then there exist constants* $c, c' > 0$ *such that for any* $T > 0, \lambda \ge \frac{c}{(T\wedge 1)^{2k}}$ *and* $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$,

$$|\nabla P_T f| \le \lambda\{P_T(f\log f) - (P_T f)\log P_T f\} + \frac{c'((1\wedge T)^2 W + 1)}{\lambda(T\wedge 1)^{4k+3}}P_T f.$$

Proof. Since (2) is a direct consequence of (5.4.15) and (1), we only prove (1), (3) and (4).

(1) Let

$$M_t = \int_0^t \frac{s(T-s)}{T^2} \mathrm{e}^{(T-s)A} B_0 B_0^* \mathrm{e}^{(T-s)A^*}\mathrm{d}s,$$
$$U_t = \int_0^t \mathrm{e}^{sA} B_0 B_0^* \mathrm{e}^{sA^*}\mathrm{d}s, \quad t \in [0, T].$$

According to §3 in [Saloff-Coste (1994)], the limit

$$Q := \lim_{t\to 0} t^{-(2k+1)}\gamma_t U_t \gamma_t$$

exists and is an invertible matrix, where $(\gamma_t)_{t>0}$ is a family of projection matrices. Thus, $U_t \ge c(t\wedge 1)^{2k+1} I_{m\times m}$ holds for some constant $c > 0$ and all $t > 0$. Then there exist constants $c_1, c_2 > 0$ such that for any $t \in (0, \frac{T}{2}]$,

$$M_t \ge \frac{t}{4T}\int_{t/2}^t \mathrm{e}^{(T-s)A} B_0 B_0^* \mathrm{e}^{(T-s)A^*}\mathrm{d}s$$
$$\ge \frac{t\mathrm{e}^{-2\|A\|T}}{4T}\int_0^{t/2} \mathrm{e}^{sA} B_0 B_0^* \mathrm{e}^{sA^*}\mathrm{d}s \ge \frac{c_1 (t\wedge 1)^{2(k+1)}}{4T\mathrm{e}^{c_2 T}} I_{m\times m}$$

holds. This proves the first assertion.

(3) By the semigroup property and the Jensen inequality, we assume that $T \in (0, 1]$. Let $\nabla^{(2)}Z^{(1)} = B_0$ be constant. Then h given in Theorem 5.4.2 is adapted such that

$$D^* h = \int_0^T \langle h'(t), \mathrm{d}B_t\rangle.$$

Moreover, it is easy to see that for $\xi(t)$ given in (1) and $T \in (0,1]$,

$$|h'(t)| \le \frac{c_1(TW^{l_1}(X_t)+1)}{T^{2(k+1)}}, \quad t \in [0,T]$$

holds for some constant $c_1 > 0$ independent of T. Thus, for any $\lambda > 0$,

$$\begin{aligned}\mathbb{E}e^{D^*h/\lambda} &= \mathbb{E}\exp\left[\frac{1}{\lambda}\int_0^T \langle h'(t), \mathrm{d}B_t\rangle\right] \\ &\le \left(\mathbb{E}\exp\left[\frac{2}{\lambda^2}\int_0^T |h'(t)|^2 \mathrm{d}t\right]\right)^{1/2} \\ &\le \left(\mathbb{E}\exp\left[\frac{c_2}{\lambda^2}\left(\frac{\int_0^T W^{2l_1}(X_t)\mathrm{d}t}{T^{4k+2}} + \frac{1}{T^{4k+3}}\right)\right]\right)^{1/2}.\end{aligned} \tag{5.4.25}$$

On the other hand, since $l_1 \in [0,1]$, by Lemma 5.4.3 and the Jensen inequality, there exist two constants $c_3, c_4 > 0$ such that

$$\mathbb{E}\exp\left[\frac{c_3 l_1}{T}\int_0^T W(X_t)\mathrm{d}t\right] \le e^{c_4 l_1 W}, \quad T \in (0,1]. \tag{5.4.26}$$

Moreover, since $2l_1 < 1$, there exists a constant $c_5 > 0$ such that

$$\frac{c_2 W^{2l_1}}{\lambda^2 T^{4k+2}} \le \frac{c_3 l_1 W}{(1+\lambda)^2 T} + \frac{c_5(1+\lambda^{-1})^{4l_1/(1-2l_1)}}{\lambda^2 T^{(4k+2-2l_1)/(1-2l_1)}}, \quad \lambda, T > 0.$$

Combining this with (5.4.25) and (5.4.26), we conclude that

$$\log \mathbb{E}e^{D^*h/\lambda} \le \frac{cl_1 W}{(1+\lambda)^2} + \frac{c(1+\lambda^{-1})^{4l_1/(1-2l_1)}}{\lambda^2 T^{(4k+2-2l_1)/(1-2l_1)}} + \frac{c}{\lambda^2 T^{4k+3}}, \; T \in (0,1], \lambda > 0$$

holds for some constant $c > 0$. This completes the proof of (3) by (5.4.6) and the Young inequality (see Lemma 2.4 in [Arnaudon *et al* (2009)])

$$\begin{aligned}|\nabla P_T f| &= |\mathbb{E}[f(X_T)D^*h]| \\ &\le \lambda\{P_T(f\log f) - (P_T f)\log P_T f\} \\ &\quad + \lambda(P_T f)\log \mathbb{E}e^{D^*h/\lambda}.\end{aligned} \tag{5.4.27}$$

(4) Again, we only consider $T \in (0,1]$. Let c_2 and C be in (5.4.25) and Lemma 5.4.3 respectively. Then there exists a constant $c > 0$ such that for any $T \in (0,1]$, $\lambda \ge \frac{c}{T^{2k}}$ implies

$$\frac{c_2}{\lambda^2 T^{4k+2}} \le \frac{2}{T^2 C\|\sigma\|^2 e^{4+2CT}}.$$

Thus, by (5.4.25) and Lemma 5.4.3, if $\lambda \ge \frac{c}{T^{2k}}$ then

$$\begin{aligned}\log \mathbb{E}e^{D^*h/\lambda} &\le \frac{c_2 T^2 C\|\sigma\|^2 e^{4+2CT}}{4\lambda^2 T^{4k+2}} \log \mathbb{E}\exp\left[\frac{2\int_0^T W(X_t)\mathrm{d}t}{T^2 C\|\sigma\|^2 e^{4+2CT}}\right] + \frac{c_2}{\lambda^2 T^{4k+3}} \\ &\le \frac{c'(T^2 W + 1)}{\lambda^2 T^{4k+3}}\end{aligned}$$

holds for some constant $c' > 0$ independent of T. Combining this with (5.4.27) we finish the proof. $\square$

To derive the Harnack inequality of P_T from Theorem 5.4.7 (3) and (4), let us recall a result of [Gong and Wang (2002)]. If there exist a constant $\lambda_0 > 0$ and a positive measurable function $\gamma : [\lambda_0, \infty) \times \mathbb{R}^{m+d} \to [0, \infty)$ such that

$$|\nabla_v P_T f| \le \lambda\{P_T(f \log f) - (P_T f) \log P_T f\} + \gamma(\lambda, \cdot) P_T f, \quad \lambda \ge \lambda_0 \tag{5.4.28}$$

holds for some constant $\lambda_0 \in (0, \infty]$ and all $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$, then by Proposition 4.1 in [Gong and Wang (2002)],

$$P_T f(x) \le (P_T f^p)^{1/p}(x+v) \exp\left[\int_0^1 \frac{\gamma(\frac{p-1}{1+(p-1)s}, x+sv)}{1+(p-1)s} \mathrm{d}s\right] \tag{5.4.29}$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$ and $p \ge 1 + \lambda_0$. Then we have the following consequence of Theorem 5.4.7 (3) and (4).

Corollary 5.4.8. *Let* **(A5.4.2)** *and* **(A5.4.3)** *hold such that* $\nabla^{(2)} Z^{(1)} = B_0$ *is constant.*

(1) *If* $l_1 \in [0, 1/2)$, *then there exists a constant* $c > 0$ *such that*

$$P_T f(x) \le (P_T f^p)^{1/p}(x+v) \exp\left[\frac{c|v|^2}{p-1}\left(\frac{(p-1)l_1 \int_0^1 W(x+sv)\mathrm{d}s}{p-1+|v|} + \frac{(1+\frac{p|v|}{p-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(1\wedge T)^{4k+3}}\right)\right]$$

holds for all $x, v \in \mathbb{R}^{m+d}, T > 0, p > 1$ *and* $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$.

(2) *If* $l_1 = \frac{1}{2}$, *then there exist two constants* $c, c' > 0$ *such that for any* $T > 0, f \in \mathbb{R}^{m+d}$ *and* $x, v \in \mathbb{R}^{m+d}$,

$$P_T f(x) \le (P_T f^p)^{1/p}(x+v) \exp\left[\frac{c'|v|^2\{1 + (T\wedge 1)^2 \int_0^1 W(x+sv)\mathrm{d}s\}}{(p-1)(T\wedge 1)^{4k+3}}\right]$$

holds for $p \ge 1 + \frac{c|v|}{(T\wedge 1)^{2k}}$.

Proof. (1) Let $v \in \mathbb{R}^{m+d}$ with $|v| > 0$. By Theorem 5.4.7(3), we have

$$\begin{aligned}|\nabla_v P_T f| \le& \lambda|v|\{P_T(f\log f) - (P_T f)\log P_T f\} + \frac{c|v|}{\lambda}\bigg\{\frac{l_1 W}{(1+\lambda^{-1})^2} \\ &+ \frac{(1+\lambda^{-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T\wedge 1)^{4k+3}}\bigg\} P_T f, \ \lambda > 0.\end{aligned}$$

Replacing λ by $\frac{\lambda}{|v|}$, we see that (5.4.28) holds for any $\lambda_0 > 0$ and

$$\gamma(\lambda, \cdot) = \frac{c|v|^2}{\lambda}\left\{\frac{l_1 W}{(1+|v|\lambda^{-1})^2} + \frac{(1+|v|\lambda^{-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T\wedge 1)^{4k+3}}\right\}$$

for $\lambda > 0$. Then the desired Harnack inequality follows from (5.4.29) since

$$\begin{aligned}
&\int_0^1 \frac{\gamma(\frac{p-1}{1+(p-1)s}, x+sv)}{1+(p-1)s}\mathrm{d}s \\
&= \frac{c|v|^2}{p-1}\int_0^1 \bigg\{\frac{l_1 W(x+sv)}{1+\frac{|v|(1+(p-1)s)}{p-1}} + \frac{(1+\frac{|v|(1+(p-1)s)}{p-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} \\
&\quad + \frac{1}{(T\wedge 1)^{4k+3}}\bigg\}\mathrm{d}s \\
&\le \frac{c|v|^2}{p-1}\bigg(\frac{l_1(p-1)\int_0^1 W(x+sv)\mathrm{d}s}{p-1+|v|} + \frac{(1+\frac{p|v|}{p-1})^{4l_1/(1-2l_1)}}{(T\wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} \\
&\quad + \frac{1}{(T\wedge 1)^{4k+3}}\bigg).
\end{aligned}$$

(2) Let $v \in \mathbb{R}^{m+d}$ with $|v| > 0$. By Theorem 5.4.7(4),

$$|\nabla_v P_T f| \le |v|\lambda\{P_T(f\log f) - (P_T f)\log P_T f\} + \frac{c'|v|((1\wedge T)^2 W + 1)}{\lambda(T\wedge 1)^{4k+3}} P_T f$$

holds for $\lambda \ge \frac{c}{(T\wedge 1)^{2k}}$. Using $\frac{\lambda}{|v|}$ to replace λ, we see that (5.4.28) holds for $\lambda_0 = \frac{c|v|}{(T\wedge 1)^{2k}}$ and

$$\gamma(\lambda,\cdot) = \frac{c'|v|^2((1\wedge T)^2 W + 1)}{\lambda(T\wedge 1)^{4k+3}}.$$

Then the proof is completed by (5.4.29). □

5.5 Gruschin type semigroups

In this section we investigate the Bismut formula and Harnack type inequalities for Gruschin type semigroups. The first part is organized from [Wang (2012b)] where Bismut type derivative formulae were derived by using Malliavin calculus, and the second part is based on [Wang and Xu (2012)] where the log-Harnack inequality was established using coupling by change of measure.

5.5.1 *Derivative formula*

We will work with the following Gruschin type operators on $\mathbb{R}^{m+d}$:

$$L(x,y) = \frac{1}{2}\bigg\{\sum_{i=1}^m \partial_{x_i}^2 + \sum_{j,k=1}^d (\sigma(x)\sigma(x)^*)_{jk}\partial_{y_j}\partial_{y_k}\bigg\}$$

for $(x,y) \in \mathbb{R}^m \times \mathbb{R}^d = \mathbb{R}^{m+d}$, where $\sigma \in C^1(\mathbb{R}^m; \mathbb{R}^d \otimes \mathbb{R}^d)$ might be degenerate. When $m = d = 1$ and $\sigma(x) = x$, it goes back to the Gruschin operator. To construct the associated diffusion process, we consider the stochastic differential equation on $\mathbb{R}^{m+d}$:

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}B_t, \\ \mathrm{d}Y_t = \sigma(X_t)\mathrm{d}\tilde{B}_t, \end{cases} \tag{5.5.1}$$

where $(B_t, \tilde{B}_t)$ is a Brownian motion on $\mathbb{R}^{m+d}$. It is easy to see that for any initial data the equation has a unique solution and the solution is non-explosive. Let $\mathbb{E}^{x,y}$ stand for the expectation taken for the solution starting at $(x,y) \in \mathbb{R}^{m+d}$. We have

$$P_t f(x,y) = \mathbb{E}^{x,y} f(X_t, Y_t), \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}), (x,y) \in \mathbb{R}^{m+d}, t \geq 0.$$

To establish explicit derivative formula for P_t, we need the following assumption.

(A5.5.1) For any $T > 0$ and $x \in \mathbb{R}^m$, $Q_T := \int_0^T \sigma(x+B_t)\sigma(x+B_t)^* \mathrm{d}t$ is invertible such that

$$\mathbb{E}\left\{ \|Q_T^{-1}\|^2 \int_0^T (\|\nabla\sigma(x+B_t)\|^4 + \|\sigma(x+B_t)\|^4 + 1)\mathrm{d}t \right\} < \infty.$$

Lemma 5.5.1. *For fixed $T > 0$ and $v = (v_1, v_2) \in \mathbb{R}^{m+d}$, let $h_1 \in C^1([0,T]; \mathbb{R}^m)$ with $h_1(0) = 0$ and $h_1(T) = v_1$. If there exists a process $\{h_2(t)\}_{t\in[0,T]}$ on $\mathbb{R}^d$ such that $h_2(0) = 0$, and $h := (h_1, h_2) \in \mathcal{D}(D^*)$ satisfying*

$$\int_0^T \sigma(X_t)h_2'(t)\mathrm{d}t + \int_0^T (\nabla_{h_1(t)-v_1}\sigma)(X_t)\mathrm{d}\tilde{B}_t = v_2, \tag{5.5.2}$$

then

$$\nabla_v P_T f = \mathbb{E}\{f(X_T, Y_T)D^*h\}, \quad f \in C_b^1(\mathbb{R}^{m+d}).$$

Proof. From (5.5.1) it is easy to see that the derivative process $(\nabla_v X_t, \nabla_v Y_t)_{t\geq 0}$ solve the equation

$$\begin{cases} \mathrm{d}\nabla_v X_t = 0, & \nabla_v X_0 = v_1, \\ \mathrm{d}\nabla_v Y_t = (\nabla_{\nabla_v X_t}\sigma)(X_t)\mathrm{d}\tilde{B}_t, & \nabla_v Y_0 = v_2. \end{cases}$$

So,

$$\nabla_v X_t = v_1, \quad \nabla_v Y_t = v_2 + \int_0^t (\nabla_{v_1}\sigma)(X_s)\mathrm{d}\tilde{B}_s. \tag{5.5.3}$$

Next, for h given in the lemma, we have

$$\begin{cases} \mathrm{d}D_hX_t = h_1'(t)\mathrm{d}t, & D_hX_0 = 0, \\ \mathrm{d}D_hY_t = \sigma(X_t)h_2'(t)\mathrm{d}t + (\nabla_{D_hX_t}\sigma)(X_t)\mathrm{d}\tilde{B}_t, & D_hY_0 = 0. \end{cases}$$

Thus,

$$\begin{cases} D_hX_t = h_1(t), \\ D_hY_t = \int_0^t \sigma(X_s)h_2'(s)\mathrm{d}s + \int_0^t (\nabla_{h_1(s)}\sigma)(X_s)\mathrm{d}\tilde{B}_s. \end{cases}$$

Since $h_1(T) = v_1$, combining this with (5.5.2) and (5.5.3) we obtain

$$(\nabla_v X_T, \nabla_v Y_T) = (D_hX_T, D_hY_T).$$

Therefore, for any $f \in C_b^1(\mathbb{R}^{m+d})$,

$$\begin{aligned} \nabla_v P_T f &= \mathbb{E}\langle \nabla f(X_T, Y_T), (\nabla_v X_T, \nabla_v Y_T)\rangle \\ &= \mathbb{E}\langle \nabla f(X_T, Y_T), (D_hX_T, D_hY_T)\rangle \\ &= \mathbb{E}D_h\{f(X_T, Y_T)\} \\ &= \mathbb{E}\{f(X_T, Y_T)D^*h\}. \end{aligned}$$

□

According to Lemma 5.5.1, to derive explicit derivative formula, the key point is to solve the control problem (5.5.2). To this end, we will need the following fundamental lemma which is a direct consequence of Itô's fromula.

Lemma 5.5.2. *Let ρ_t be a predictable process on $\mathbb{R}^d$ with $\mathbb{E}\int_0^T |\rho_t|^q\mathrm{d}t < \infty$ for some $q \geq 2$. Then*

$$\begin{aligned} \mathbb{E}\left|\int_0^T \langle \rho_t, \mathrm{d}\tilde{B}_t\rangle\right|^q &\leq \left\{\frac{q(q-1)}{2}\right\}^{q/2}\left(\int_0^T (\mathbb{E}|\rho_t|^q)^{2/q}\mathrm{d}t\right)^{q/2} \\ &\leq \left\{\frac{q(q-1)}{2}\right\}^{q/2} T^{(q-2)/2}\int_0^T \mathbb{E}|\rho_t|^q\mathrm{d}t. \end{aligned}$$

Proof. It suffices to prove the first inequality since the second follows immediately from Jensen's inequality. Let $N_t = \int_0^t \langle \rho_s, \mathrm{d}\tilde{B}_s\rangle,\ t \geq 0$. Then $\mathrm{d}\langle N\rangle_t = |\rho_t|^2\mathrm{d}t$ and

$$\mathrm{d}N_t^2 = 2N_t\mathrm{d}N_t + |\rho_t|^2\mathrm{d}t.$$

Noting that $|N_t|^q = (N_t^2)^{q/2}$, by Itô's formula we obtain

$$\begin{aligned} \mathrm{d}|N_t|^q &= \frac{q}{2}(N_t^2)^{(q-2)/2}\mathrm{d}N_t^2 + \frac{q(q-2)}{2}(N_t^2)^{(q-4)/2}N_t^2|\rho_t|^2\mathrm{d}t \\ &= qN_t|N_t|^{q-2}\mathrm{d}N_t + \frac{q(q-1)}{2}|N_t|^{q-2}|\rho_t|^2\mathrm{d}t. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}|N_T|^q &= \frac{q(q-1)}{2}\int_0^T \mathbb{E}\{|N_t|^{q-2}|\rho_t|^2\}\mathrm{d}t\\ &\le \frac{q(q-1)}{2}\int_0^T \big(\mathbb{E}|N_t|^q\big)^{(q-2)/q}\big(\mathbb{E}|\rho_t|^q\big)^{2/q}\mathrm{d}t\\ &\le \frac{q(q-1)}{2}\big(\mathbb{E}|N_T|^q\big)^{(q-2)/q}\int_0^T \big(\mathbb{E}|\rho_t|^q\big)^{2/q}\mathrm{d}t.\end{aligned}$$

Up to an approximation argument we may assume that $\mathbb{E}|N_T|^q < \infty$, so that this implies

$$\mathbb{E}|N_T|^q \le \Big\{\frac{q(q-1)}{2}\Big\}^{q/2}\bigg(\int_0^T (\mathbb{E}|\rho_t|^q)^{2/q}\mathrm{d}t\bigg)^{q/2}.$$

□

We are now able to prove our main result in this section.

Theorem 5.5.3. *Assume* **(A5.5.1)**. *For any* $f \in C_b^1(\mathbb{R}^{m+d})$ *and* $v = (v_1, v_2) \in \mathbb{R}^{m+d}$,

$$\nabla_v P_T f(x,y) = \mathbb{E}^{x,y}\{f(X_T,Y_T)M_T\},\quad (x,y)\in\mathbb{R}^{m+d}, T>0$$

holds for

$$\begin{aligned}M_T = &\frac{\langle v_1, B_T\rangle}{T} - \mathrm{Tr}\bigg(Q_T^{-1}\int_0^T \frac{T-t}{T}\{(\nabla_{v_1}\sigma)\sigma^*\}(x+B_t)\mathrm{d}t\bigg)\\ &+\bigg\langle Q_T^{-1}\bigg\{v_2+\int_0^T\frac{T-t}{T}(\nabla_{v_1}\sigma)(x+B_t)\mathrm{d}\tilde{B}_t\bigg\},\int_0^T\sigma(x+B_t)\mathrm{d}\tilde{B}_t\bigg\rangle.\end{aligned}$$

Proof. We assume that $(X_0,Y_0)=(x,y)$ and simply denote $\mathbb{E}^{x,y}$ by $\mathbb{E}$. Let

$$h_1(t) = \frac{tv_1}{T} \tag{5.5.4}$$

and

$$h_2(t) = \bigg(\int_0^t \sigma(X_s)^*\mathrm{d}s\bigg)Q_T^{-1}\bigg(v_2+\int_0^T\frac{T-s}{T}(\nabla_{v_1}\sigma)(X_s)\mathrm{d}\tilde{B}_s\bigg) \tag{5.5.5}$$

for $t\in[0,T]$. Then it is easy to see that (5.5.2) holds. To see that $h := (h_1,h_2)\in\mathcal{D}(D^*)$ and to calculate D^*h, let

$$\begin{aligned}g_i &= \bigg\langle e_i, Q_T^{-1}\bigg(v_2+\int_0^T\frac{T-s}{T}(\nabla_{v_1}\sigma)(X_s)\mathrm{d}\tilde{B}_s\bigg)\bigg\rangle,\\ \tilde{h}_i(t) &= \int_0^t\sigma(X_s)^*e_i\mathrm{d}s,\quad i=1,\dots,d,\end{aligned}$$

where $\{e_i\}_{i=1}^d$ is the canonical ONB on $\mathbb{R}^d$. We have

$$h(t) = (h_1(t), 0) + \sum_{i=1}^d g_i(0, \tilde{h}_i(t)). \tag{5.5.6}$$

It is easy to see that h_1 and $\tilde{h}_i$ are adapted and

$$\begin{aligned} D^*(h_1, 0) &= \int_0^T \langle h_1'(t), \mathrm{d}B_t\rangle = \frac{\langle v_1, B_T\rangle}{T}, \\ D^*(0, \tilde{h}_i) &= \int_0^T \langle \tilde{h}_i'(t), \mathrm{d}\tilde{B}_t\rangle = \int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle. \end{aligned} \tag{5.5.7}$$

Let $\mathcal{F}_T$ be the σ-field induced by $\{B_s : s \in [0, T]\}$. By Lemma 5.5.2 and noting that X_t is measurable w.r.t. $\mathcal{F}_T$ while $\tilde{B}$ is independent of $\mathcal{F}_T$, we have

$$\begin{aligned} &\mathbb{E}\Big(\{g_i D^*(0, \tilde{h}_i)\}^2\Big|\mathcal{F}_T\Big) = \mathbb{E}\bigg(\bigg\{g_i \int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle\bigg\}^2\bigg|\mathcal{F}_T\bigg) \\ &\le 2\|v_2\|^2\|Q_T^{-1}\|^2\mathbb{E}\bigg(\bigg\{\int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle\bigg\}^2\bigg|\mathcal{F}_T\bigg) \\ &\quad + 2\mathbb{E}\bigg(\bigg\{\int_0^T \frac{T-t}{T}\big\langle\big(\nabla_{v_1}\sigma(X_t)\big)^*(Q_T^{-1})^* e_i, \mathrm{d}\tilde{B}_t\big\rangle\bigg\}^2 \\ &\qquad\qquad \times \bigg\{\int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle\bigg\}^2\bigg|\mathcal{F}_T\bigg) \\ &\le c\|Q_T^{-1}\|^2\int_0^T \|\sigma(X_t)\|^2\mathrm{d}t \\ &\qquad + 2\bigg[\mathbb{E}\bigg(\bigg\{\int_0^T \frac{T-t}{T}\big\langle\big(\nabla_{v_1}\sigma(X_t)\big)^*(Q_T^{-1})^* e_i, \mathrm{d}\tilde{B}_t\big\rangle\bigg\}^4\bigg|\mathcal{F}_T\bigg) \\ &\qquad\qquad \times \mathbb{E}\bigg(\bigg\{\int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle\bigg\}^4\bigg|\mathcal{F}_T\bigg)\bigg]^{1/2} \\ &\le c'\|Q_T^{-1}\|^2\int_0^T (\|\nabla\sigma(X_t)\|^4 + \|\sigma(X_t)\|^4 + 1)\mathrm{d}t \end{aligned}$$

for some constants $c, c' > 0$. So, **(A5.5.1)** implies $g_i D^*(0, \tilde{h}_i) \in L^2(\mathbb{P})$ for $i = 1, \ldots, d$. Hence, if for any $i \in \{1, \ldots, d\}$ one has $D_{(0,\tilde{h}_i)} g_i \in L^2(\mathbb{P})$, then $h \in \mathcal{D}(D^*)$ and by (5.5.6) and (5.5.7),

$$D^* h = \frac{\langle v_1, B_T\rangle}{T} + \sum_{i=1}^d \bigg\{g_i \int_0^T \langle \sigma(X_t)^* e_i, \mathrm{d}\tilde{B}_t\rangle - D_{(0,\tilde{h}_i)} g_i\bigg\}. \tag{5.5.8}$$

Noting that $X_t = x + B_t$ is independent of $\bar{B}$, it is easy to see that

$$D_{(0,\bar{h}_i)} g_i = \left\langle e_i, Q_T^{-1} \int_0^T \frac{T-t}{T} (\nabla_{v_1}\sigma)(X_t)\bar{h}_i'(t)\mathrm{d}t \right\rangle$$
$$= \left\langle e_i, Q_T^{-1} \int_0^T \frac{T-t}{T} \{(\nabla_{v_1}\sigma)\sigma^*\}(X_t)e_i\mathrm{d}t \right\rangle,$$

which is in $L^2(\mathbb{P})$ according to **(A5.5.1)**. Combining this with (5.5.8) and noting that $X_t = x + B_t$, we conclude that $h \in \mathcal{D}(D^*)$ and $D^*h = M_T$. Then the proof is finished by Lemma 5.5.1. □

The next result is a consequence of Theorem 5.5.3 for $\sigma(x)$ comparable with $|x|^l I_{d\times d}$ in the sense of (5.5.9) below. We will use $|\cdot|$ and $\|\cdot\|$ to denote the Euclidean norm and the operator norm for matrices respectively.

Corollary 5.5.4. *Let $l \in [1, \infty)$ and assume that*

$$\|\sigma(x)\| \geq a|x|^l, \quad \|\sigma(x)\| + \|\nabla\sigma(x)\| \cdot |x| \leq b|x|^l, \quad x \in \mathbb{R}^m \tag{5.5.9}$$

holds for some constants $a, b > 0$. Then for any $p > 1$ there exists a constant $C_p > 0$ such that for any $v = (v_1, v_2) \in \mathbb{R}^{m+d}$, $T > 0$, $(x, y) \in \mathbb{R}^{m+d}$

$$|\nabla_v P_T f(x,y)| \leq C_p (P_T|f|^p)^{1/p}(x,y)\left(\frac{|v_1|}{\sqrt{T}} + \frac{|v_2|}{\sqrt{T(|x|^2+T)^l}}\right). \tag{5.5.10}$$

Consequently,

$$\Gamma_1(P_T f) \leq \frac{C P_T f^2}{T}, \quad T > 0, f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \tag{5.5.11}$$

holds for some constant $C > 0$, where

$$\Gamma_1(f)(x,y) := |\nabla f(\cdot, y)(x)|^2 + |\sigma(x)^*\nabla f(x,\cdot)(y)|^2$$

for $f \in C^1(\mathbb{R}^{m+d})$, $(x, y) \in \mathbb{R}^{m+d}$.

To verify **(A5.5.1)** for σ given in Corollary 5.5.4, we first present the following lemma.

Lemma 5.5.5. *For any $n \in [1, \infty)$ and $\alpha > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{E}^{x,y}\left(\int_0^T |X_t|^{2n}\mathrm{d}t\right)^{-\alpha} \leq \frac{c}{T^\alpha(|x|^2+T)^{\alpha n}}, \quad T > 0, (x,y) \in \mathbb{R}^{m+d}.$$

Proof. We shall simply denote $\mathbb{E}^{x,y}$ by $\mathbb{E}$. Since $X_t = x + B_t$, for any $\lambda > 0$ we have (see e.g. page 142 in [Borodin and Salminen (1996)])

$$\mathbb{E}e^{-\lambda\int_0^T |X_t|^2 dt} = \prod_{i=1}^m \mathbb{E}e^{-\lambda\int_0^T (x_i+B_t^{(i)})^2 dt} \le \frac{\exp[-\frac{|x|^2\sqrt{\lambda}}{\sqrt{2}}\tanh(\sqrt{2\lambda}\,T)]}{\{\coth\big(\sqrt{2\lambda}T\big)\}^{m/2}}$$

$$\le 2^{m/2}\exp\Big[-\frac{mT\sqrt{\lambda}}{\sqrt{2}} - \frac{|x|^2\sqrt{\lambda}}{2\sqrt{2}}\Big\{\big(\sqrt{2\lambda}\,T\big)\wedge 1\Big\}\Big]$$

$$\le 2^{m/2}\exp\Big[-\frac{(|x|^2+T)\sqrt{\lambda}}{2\sqrt{2}}\Big\{\big(\sqrt{2\lambda}\,T\big)\wedge 1\Big\}\Big]$$

$$\le 2^{m/2}\exp\Big[-\frac{(|x|^2+T)\sqrt{\lambda}}{2\sqrt{2}}\Big] + 2^{m/2}\exp\Big[-\frac{(|x|^2+T)\lambda T}{2}\Big].$$

This implies that for any $r > 0$,

$$\mathbb{E}\exp\Big[-\lambda\int_0^T |X_t|^{2n}dt\Big] = \mathbb{E}\exp\Big[-\int_0^T \big(\lambda^{1/n}|X_t|^2\big)^n dt\Big]$$

$$\le \mathbb{E}\exp\Big[\int_0^T \Big(\frac{n-1}{n^{n/(n-1)}}r^{n/(n-1)} - r\lambda^{1/n}|X_t|^2\Big)dt\Big]$$

$$\le 2^{m/2}\exp\Big[\frac{T(n-1)}{n^{n/(n-1)}}r^{n/(n-1)}\Big]$$

$$\times\Big(\exp\Big[-\frac{(|x|^2+T)\lambda^{1/(2n)}\sqrt{r}}{2\sqrt{2}}\Big] + \exp\Big[-\frac{(|x|^2+T)T\lambda^{1/n}r}{2}\Big]\Big).$$

Taking $r = T^{-(n-1)/n}$ we obtain

$$\mathbb{E}\exp\Big[-\lambda\int_0^T |X_t|^{2n}dt\Big]$$

$$\le c_1\Big(\exp\Big[-\frac{(|x|^2+T)\lambda^{1/(2n)}}{2\sqrt{2}\,T^{(n-1)/2n}}\Big] + \exp\Big[-\frac{(|x|^2+T)(\lambda T)^{1/n}}{2}\Big]\Big)$$

for some constant $c_1 > 0$. Noting that

$$\int_0^\infty \lambda^{\alpha-1}e^{-\theta\lambda^{1/l}}d\lambda = \frac{l}{\theta^{\alpha l}}\int_0^\infty e^{-s}s^{\alpha l-1}ds = \frac{l\Gamma(\alpha l)}{\theta^{\alpha l}}$$

holds for all $l \ge 1$ and $\theta, \alpha > 0$, we conclude that

$$\mathbb{E}\Big(\int_0^T |X_t|^{2n}dt\Big)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-1}\mathbb{E}\exp\Big[-\lambda\int_0^T |X_t|^{2n}dt\Big]d\lambda$$

$$\le \frac{c_1}{\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-1}\Big\{\exp\Big[-\frac{(|x|^2+T)\lambda^{1/(2n)}}{2\sqrt{2}\,T^{(n-1)/2n}}\Big]$$

$$+\exp\Big[-\frac{(|x|^2+T)(\lambda T)^{1/n}}{2}\Big]\Big\}d\lambda$$

$$\le \frac{c_2T^{\alpha(n-1)}}{(|x|^2+T)^{2\alpha n}} + \frac{c_3}{(|x|^2+T)^{\alpha n}T^{\alpha}} \le \frac{c}{(|x|^2+T)^{\alpha n}T^{\alpha}}$$

holds for some constants c_2, c_3 and c. □

Proof. **(Proof of Corollary 5.5.4)** By the Jensen inequality, it suffices to prove for $p \in (1,2]$ so that $q := \frac{p}{p-1} \geq 2$. It is easy to see that (5.5.9) implies

$$Q_T \geq \left(a^2 \int_0^T |X_t|^{2l} \mathrm{d}t\right) I_{d\times d},$$

and hence,

$$\|Q_T^{-1}\| \leq \frac{1}{a^2 \int_0^T |X_t|^{2l} \mathrm{d}t}. \tag{5.5.12}$$

Since $\{X_t\}_{t\in[0,T]}$ is measurable w.r.t. $\mathcal{F}_T$ and due to (5.5.9)

$$\|\{(\nabla_{v_1}\sigma)\sigma^*\}(X_t)\| \leq b^2|v_1|\cdot|X_t|^{2l-1},$$

we obtain

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{\langle v_1, B_T\rangle}{T} - \mathrm{Tr}\left(Q_T^{-1}\int_0^T \frac{T-t}{T}\{(\nabla_{v_1}\sigma)\sigma^*\}(X_t)\mathrm{d}t\right)\right|^q \middle| \mathcal{F}_T\right)\\
&= \left|\frac{\langle v_1, B_T\rangle}{T} - \mathrm{Tr}\left(Q_T^{-1}\int_0^T \frac{T-t}{T}\{(\nabla_{v_1}\sigma)\sigma^*\}(X_t)\mathrm{d}t\right)\right|^q \qquad (5.5.13)\\
&\leq c_1|v_1|^q\left(\frac{|B_T|^q}{T^q} + \frac{T^{q-1}\int_0^T |X_t|^{(2l-1)q}\mathrm{d}t}{(\int_0^T |X_t|^{2l}\mathrm{d}t)^q}\right)
\end{aligned}$$

for some constant $c_1 > 0$. Moreover, since $\bar{B}_t$ is independent of $\mathcal{F}_T$, due to (5.5.12) and Lemma 5.5.2 there exist constants $c_2, c_3 > 0$ such that

$$\mathbb{E}\left(\|Q_T^{-1}\|^q\left|\left\langle v_2, \int_0^T \sigma(X_t)\mathrm{d}\bar{B}_t\right\rangle\right|^q \middle| \mathcal{F}_T\right) \leq \frac{c_2|v_2|^q T^{q/2-1}\int_0^T |X_t|^{lq}\mathrm{d}t}{(\int_0^T |X_t|^{2l}\mathrm{d}t)^q}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\|Q_T^{-1}\|^q\left|\left\langle \int_0^T \frac{T-t}{T}(\nabla_{v_1}\sigma)(X_t)\mathrm{d}\bar{B}_t,\ \int_0^T \sigma(X_t)\mathrm{d}\bar{B}_t\right\rangle\right|^q \middle| \mathcal{F}_T\right)\\
&\leq \frac{c_2}{(\int_0^T |X_t|^{2l}\mathrm{d}t)^q}\left\{\mathbb{E}\left(\left|\int_0^T \frac{T-t}{T}(\nabla_{v_1}\sigma)(X_t)\mathrm{d}\bar{B}_t\right|^{(2l-1)q/(l-1)} \middle| \mathcal{F}_T\right)\right\}^{\frac{l-1}{2l-1}}\\
&\quad\times\left\{\mathbb{E}\left(\left|\int_0^T \sigma(X_t)\mathrm{d}\bar{B}_t\right|^{(2l-1)q)/l} \middle| \mathcal{F}_T\right)\right\}^{l/(2l-1)}\\
&\leq \frac{c_3|v_1|^q T^{q-1}\int_0^T |X_t|^{(2l-1)q}\mathrm{d}t}{(\int_0^T |X_t|^{2l}\mathrm{d}t)^q}.
\end{aligned}$$

hold. Combining these with (5.5.13) we obtain

$$\begin{aligned}\mathbb{E}|M_T|^q &= \mathbb{E}\{\mathbb{E}(|M_T|^q|\mathcal{F}_T)\}\\ &\le c_3\mathbb{E}\bigg\{\frac{|v_1|^q}{T^{q/2}} + \frac{|v_1|^qT^{q-1}\int_0^T|X_t|^{(2l-1)q}\mathrm{d}t}{(\int_0^T|X_t|^{2l}\mathrm{d}t)^q}\\ &\quad + \frac{|v_2|^qT^{\frac{q}{2}-1}\int_0^T|X_t|^{lq}\mathrm{d}t}{(\int_0^T|X_t|^{2l}\mathrm{d}t)^q}\bigg\}.\end{aligned} \tag{5.5.14}$$

By Lemma 5.5.5 and noting that $X_t = x + B_t$, we conclude that for any $\beta \ge 1$,

$$\mathbb{E}\bigg\{\frac{\int_0^T|X_t|^\beta\mathrm{d}t}{(\int_0^T|X_t|^{2l}\mathrm{d}t)^q}\bigg\} \le \bigg\{\mathbb{E}\bigg(\int_0^T|X_t|^\beta\mathrm{d}t\bigg)^2\bigg\}^{1/2}\bigg\{\mathbb{E}\bigg(\int_0^T|X_t|^{2l}\mathrm{d}t\bigg)^{-2q}\bigg\}^{1/2}$$

$$\le \frac{c_3(T\mathbb{E}\int_0^T|X_t|^{2\beta}\mathrm{d}t)^{1/2}}{T^q(|x|^2+T)^{ql}} \le \frac{c_4T(|x|^2+T)^{\beta/2}}{T^q(|x|^2+T)^{ql}} = \frac{c_4}{T^{q-1}(|x|^2+T)^{ql-\beta/2}}$$

holds for some constants $c_3, c_4 > 0$. Substituting this into (5.5.14) we arrive at

$$(\mathbb{E}|M_T|^q)^{1/q} \le c_5\bigg\{\frac{|v_1|^q}{T^{q/2}} + \frac{|v_2|^q}{T^{q/2}(|x|^2+T)^{ql/2}}\bigg\}^{1/q}$$

for some constant $c_5 > 0$. Therefore, the proof is completed since by the first assertion

$$|\nabla_v P_T f(x,y)| = |\mathbb{E}\{f(X_T,Y_T)M_T\}| \le (P_T|f|^p)^{1/p}(\mathbb{E}|M_T|^q)^{1/q}. \qquad \square$$

Finally, we intend to extend Theorem 5.5.3 to a more general model. Consider the following SDE on $\mathbb{R}^{m+d}$:

$$\begin{cases}\mathrm{d}X_t = \sigma_1(X_t)\mathrm{d}B_t + b_1(X_t)\mathrm{d}t,\\ \mathrm{d}Y_t = \sigma_2(X_t)\mathrm{d}\tilde{B}_t + b_2(X_t)\mathrm{d}t,\end{cases} \tag{5.5.15}$$

where $(B_t, \tilde{B}_t)$ is a Brownian motion on $\mathbb{R}^{m+d}$, $\sigma_1 \in C_b^1(\mathbb{R}^m;\mathbb{R}^m\otimes\mathbb{R}^m)$ is invertible with $\|\sigma_1^{-1}\| \le c$ for some constant $c > 0$, $\sigma_2 \in C^1(\mathbb{R}^m;\mathbb{R}^d\otimes\mathbb{R}^d)$ might be degenerate, $b_1 \in C_b^1(\mathbb{R}^m;\mathbb{R}^m)$ and $b_2 \in C^1(\mathbb{R}^m;\mathbb{R}^d)$. It is easy to see that for any initial data the solution exists uniquely and is non-explosive. Let P_t be the associated Markov semigroup. To establish the derivative formula, let $v = (v_1, v_2) \in \mathbb{R}^{m+d}$ and $T > 0$ be fixed, and let ξ_t solve the following SDE on $\mathbb{R}^m$:

$$\mathrm{d}\xi_t = (\nabla_{\xi_t}\sigma_1)(X_t)\mathrm{d}B_t + \Big\{(\nabla_{\xi_t}b_1)(X_t) - \frac{\xi_t}{T-t}\Big\}\mathrm{d}t, \quad \xi_0 = v_1. \tag{5.5.16}$$

Since $\nabla\sigma_1$ and ∇b_1 are bounded, the equation has a unique solution up to time T. It is easy to see from the Itô formula that

$$\begin{aligned}\mathrm{d}\Big\{\frac{|\xi_t|^2}{T-t}\Big\} &= 2\Big\langle\frac{\xi_t}{T-t}, (\nabla_{\xi_t}\sigma_1)(X_t)\mathrm{d}B_t\Big\rangle \\ &\quad + \bigg(\frac{\|(\nabla_{\xi_t}\sigma_1)(X_t)\|^2}{T-t} + \frac{2\langle\xi_t, (\nabla_{\xi_t}b_1)(X_t)\rangle}{T-t} - \frac{|\xi_t|^2}{(T-t)^2}\bigg)\mathrm{d}t \\ &\le 2\Big\langle\frac{\xi_t}{T-t}, (\nabla_{\xi_t}\sigma_1)(X_t)\mathrm{d}B_t\Big\rangle + \Big(\frac{C|\xi_t|^2}{T-t} - \frac{|\xi_t|^2}{(T-t)^2}\Big)\mathrm{d}t\end{aligned}$$

holds for all $t\in[0,T)$ and some constant $C>0$. This implies that for $t\in[0,T)$,

$$\mathbb{E}|\xi_t|^2 \le \frac{(T-t)|v_1|^2}{T}\mathrm{e}^{Ct},\quad \mathbb{E}\int_0^T \frac{|\xi_t|^2}{(T-t)^2}\mathrm{d}t < \infty, \tag{5.5.17}$$

Consequently, we may set $\xi_T=0$ so that ξ_t solves (5.5.16) for $t\in[0,T]$. Moreover, for any $n\ge 1$ we have

$$\mathrm{d}|\xi_t|^{2n} \le 2n|\xi_t|^{2(n-1)}\Big\langle\xi_t, (\nabla_{\xi_t}\sigma_1)(X_t)\mathrm{d}B_t\Big\rangle + c(n)|\xi_t|^{2n}\mathrm{d}t$$

for some constant $c(n)\ge 0$. Therefore,

$$\sup_{t\in[0,T]}\mathbb{E}|\xi_t|^{2n} < \infty,\quad n\ge 1. \tag{5.5.18}$$

We are now able to state the derivative formula for P_t as follows.

Theorem 5.5.6. *Let $Q_T=\int_0^T\sigma_2(X_t)\sigma_2(X_t)^*\mathrm{d}t$ be invertible such that*

$$\mathbb{E}^{x,y}\bigg(\|Q_T^{-1}\|^2\int_0^T\big\{\|\nabla\sigma_2(X_t)\|^4 + \|\nabla b_2(X_t)\|^4 + 1\big\}\mathrm{d}t\bigg) < \infty. \tag{5.5.19}$$

Then

$$\nabla_v P_T f(x,y) = \mathbb{E}^{x,y}\big\{f(X_T,Y_T)M_T\big\}$$

holds for $f\in C_b^1(\mathbb{R}^{m+d})$ and

$$\begin{aligned}M_T &= \int_0^T\Big\langle\frac{\sigma_1(X_t)^{-1}\xi_t}{T-t}, \mathrm{d}B_t\Big\rangle - \mathrm{Tr}\bigg(Q_T^{-1}\int_0^T\frac{T-t}{T}\{(\nabla_{\xi_t}\sigma_2)\sigma_2^*\}(X_t)\mathrm{d}t\bigg) \\ &\quad + \bigg\langle Q_T^{-1}\bigg\{v_2 + \int_0^T\frac{T-t}{T}(\nabla_{\xi_t}\sigma_2)(X_t)\mathrm{d}\bar{B}_t + \int_0^T(\nabla_{\xi_t}b_2)(X_t)\bigg\}, \\ &\qquad \int_0^T\sigma_2(X_t)\mathrm{d}\bar{B}_t\bigg\rangle.\end{aligned}$$

Proof. Let $h = (h_1, h_2)$, where

$$h_1(t) = \int_0^t \frac{\sigma_1(X_s)^{-1}\xi_s}{T-s}\,\mathrm{d}s,\ t \in [0,T],$$

$$h_2(t) = \Big(\int_0^t \sigma_2(X_s)^*\mathrm{d}s\Big)Q_T^{-1}$$

$$\times\Big(v_2 + \int_0^T (\nabla_{\xi_t}\sigma_2)(X_t)\mathrm{d}\tilde{B}_t + \int_0^T (\nabla_{\xi_t}b_2)(X_t)\mathrm{d}t\Big).$$

As in the proof of Theorem 5.5.3, it is easy to see from (5.5.17), (5.5.18), (5.5.19) and $\|\sigma_1^{-1}\| \le c$ that $h \in \mathcal{D}(D^*)$ with $D^*h = M_T$. Therefore, it remains to verify that $(\nabla_v X_T, \nabla_v Y_T) = (D_h X_T, D_h Y_T)$. It is easy to see that both $\nabla_v X_t$ and $D_h X_t + \xi_t$ solve the equation

$$\mathrm{d}V_t = (\nabla_{V_t}\sigma_1)(X_t)\mathrm{d}B_t + (\nabla_{V_t}b_1)(X_t)\mathrm{d}t,\quad t \in [0,T], V_0 = v_1.$$

By the uniqueness of the solution we have $\nabla_v X_t = D_h X_t + \xi_t$ for $t \in [0,T]$. Since $\xi_T = 0$, this implies that $\nabla_v X_T = D_h X_T$. Moreover, we have

$$\begin{cases} \mathrm{d}\nabla_v Y_t = (\nabla_{\nabla_v X_t}\sigma_2)(X_t)\mathrm{d}\tilde{B}_t + (\nabla_{\nabla_v X_t}b_2)(X_t)\mathrm{d}t, & \nabla_v Y_0 = v_2, \\ \mathrm{d}D_h Y_t = (\nabla_{D_h X_t}\sigma_2)(X_t)\mathrm{d}\tilde{B}_t + \sigma_2(X_t)h_2'(t)\mathrm{d}t \\ \qquad\qquad + (\nabla_{D_h X_t}b_2)(X_t)\mathrm{d}t, \quad D_h Y_0 = 0. \end{cases}$$

Combining this with the definition of h_2 and $D_h X_t = \nabla_v X_t - \xi_t$, we obtain

$$\begin{aligned} D_h Y_T &= \nabla_v Y_T - v_2 - \int_0^T (\nabla_{\xi_t}\sigma_2)(X_t)\mathrm{d}\tilde{B}_t \\ &\quad + \int_0^T \sigma_2(X_t)h_2'(t)\mathrm{d}t - \int_0^T (\nabla_{\xi_t}b_2)(X_t)\mathrm{d}t \\ &= \nabla_v Y_T. \end{aligned}$$

Therefore, the proof is finished. □

5.5.2 *Log-Harnack inequality*

Let us start with the classical Gruschin semigroup on $\mathbb{R}^2$ with order $l > 0$, which is generated by

$$L(x^{(1)}, x^{(2)}) := \frac{1}{2}\Big(\frac{\partial^2}{\partial(x^{(1)})^2} + |x^{(1)}|^{2l}\frac{\partial^2}{\partial(x^{(2)})^2}\Big).$$

The corresponding diffusion process can be constructed by solving the SDE

$$\begin{cases} \mathrm{d}X_t^{(1)} = \mathrm{d}B_t^{(1)}, \\ \mathrm{d}X_t^{(2)} = |X_t^{(1)}|^l\,\mathrm{d}B_t^{(2)}, \end{cases}$$

where $B_t := (B_t^{(1)}, B_t^{(2)})$ is a two-dimensional Brownian motion. Clearly, the equation is degenerate, and when $l < 1$ the coefficient in the second equation is non-Lipschitzian. In the simplest case that $l = 1$ (see Example 5.2.5), the Harnack inequality (5.2.27) holds. According to Corollary 1.4.3, in this case the log-Harnack inequality

$$P_t(\log f)(x,y) \le \log P_t f(x',y') + \frac{1}{2(2-\theta)}\Big(\frac{|x-x'|^2}{t\wedge t_\theta} + \frac{3|y-y'|^2}{(t\wedge t_\theta)^3}\Big)$$

holds for all $\theta \in (\frac{3}{2}, 2)$, $t > 0, (x,y), (x',y') \in \mathbb{R}^2$ and strictly positive function $f \in \mathcal{B}_b(\mathbb{R}^2)$, where $t_\theta > 0$ is fixed in Example 5.2.5. But for $l \neq 1$, it is not clear how can one establish the log-Harnack inequality using the generalized curvature condition or Malliavin calculus. In this subsection we aim to establish the log-Harnack inequality of the Gruschin semigroup for all $l > 0$ by using coupling by change of measure. But, our argument does not imply the Harnack inequality with power like (5.2.27).

We consider the following more general SDE for $X_t := (X_t^{(1)}, X_t^{(2)})$ on $\mathbb{R}^m \times \mathbb{R}^d = \mathbb{R}^{m+d} (m, d \ge 1)$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = b^{(1)}(t, X_t^{(1)})\mathrm{d}t + \sigma^{(1)}(t)\,\mathrm{d}B_t^{(1)}, \\ \mathrm{d}X_t^{(2)} = \{AX_t^{(2)} + b^{(2)}(t, X_t^{(1)})\}\mathrm{d}t + \sigma^{(2)}(t, X_t^{(1)})\,\mathrm{d}B_t^{(2)}, \end{cases} \tag{5.5.20}$$

where $B_t := (B_t^{(1)}, B_t^{(2)})$ is the $(m+d)$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$, A is a $(d\times d)$-matrix, and

$$b^{(1)} : [0,\infty)\times\mathbb{R}^m \to \mathbb{R}^m, \quad b^{(2)} : [0,\infty)\times\mathbb{R}^m \to \mathbb{R}^d,$$
$$\sigma^{(1)} : [0,\infty) \to \mathbb{R}^m\otimes\mathbb{R}^m, \quad \sigma^{(2)} : [0,\infty)\times\mathbb{R}^m \to \mathbb{R}^d\otimes\mathbb{R}^d$$

are measurable, and $b^{(1)}, b^{(2)}, \sigma^{(2)}$ are continuous in the second variable. Assume

(A5.5.2) There exists a decreasing function $\lambda : [0,\infty) \to (0,\infty)$ such that $\sigma^{(1)}(t)\sigma^{(1)}(t)^* \ge \lambda_t^2 I_{m\times m}$, $t \ge 0$.

(A5.5.3) There exists an increasing function $K : [0,\infty) \to \mathbb{R}$ such that

$$\langle b^{(1)}(t, x^{(1)}) - b^{(1)}(t, y^{(1)}), x^{(1)} - y^{(1)}\rangle \le K_t|x^{(1)} - y^{(1)}|^2$$

for $t \ge 0$, $x^{(1)}$, $y^{(1)} \in \mathbb{R}^m$.

(A5.5.4) There exist increasing functions $\Theta : [0,\infty) \to \mathbb{R}, h : [0,\infty) \to$

$[1,\infty)$ and $\varphi_\cdot : [0,\infty)^2 \to [0,\infty)$ with $\varphi_\cdot(0) = 0$ such that

$$\begin{aligned}&\langle A(x^{(2)} - y^{(2)}) + b^{(2)}(t, x^{(1)}) - b^{(2)}(t, y^{(1)}), x^{(2)} - y^{(2)}\rangle \\ &\quad + \frac{1}{2}\|\sigma^{(2)}(t, x^{(1)}) - \sigma^{(2)}(t, y^{(1)})\|_{HS}^2 \\ &\le \Theta_t|x^{(2)} - y^{(2)}|^2 + \varphi_t(|x^{(1)} - y^{(1)}|^2)h(|x^{(1)}| \vee |y^{(1)}|)\end{aligned}$$

holds for all $t \ge 0$ and $x = (x^{(1)}, x^{(2)}), y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^{m+d}$.

It is well known that **(A5.5.3)** implies the existence, uniqueness and non-explosion of strong solutions to the first equation in (5.5.20). Once $X_t^{(1)}$ is fixed, then it follows from **(A5.5.4)** that the second equation in (5.5.20) admits a unique global solution. Note that **(A5.5.4)** allows $\sigma^{(2)}(t,\cdot)$ to be merely Hölder continuous when e.g. $\varphi_t(r) = r^\alpha$ for some constant $\alpha \in (0,1)$. For any $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}$, we let $X_t(x) = (X_t^{(1)}(x), X_t^{(2)}(x))$ denote the solution to (5.5.20) with $X_0 = x$. Since $X_t^{(1)}(x)$ does not depend on $x^{(2)}$ we also write $X_t^{(1)}(x) = X_t^{(1)}(x^{(1)})$. We intend to establish Harnack type inequalities for the associated semigroup P_t:

$$P_t f(x) := \mathbb{E} f(X_t(x)), \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}), t \ge 0, x \in \mathbb{R}^{m+d}.$$

Let

$$Q_T = \int_T^{2T} \mathrm{e}^{A(T-t)}\sigma^{(2)}(t, X_t^{(1)})\sigma^{(2)}(t, X_t^{(1)})^*\mathrm{e}^{A^*(T-t)}\mathrm{d}t, \quad T > 0.$$

Theorem 5.5.7. *Assume that* **(A5.5.2)**, **(A5.5.3)** *and* **(A5.5.4)** *hold. Let* $\theta_T = \sup_{t\in[0,T]} \|\mathrm{e}^{-At}\|$. *If* Q_T *is invertible and*

$$\begin{aligned}\Psi_T(x^{(1)}, y^{(1)}) := \mathbb{E}^{y^{(1)}}\bigg\{&\|Q_T^{-1}\|^2\bigg(\int_T^{2T} \|\sigma^{(2)}(t, X_t^{(1)})\|^2\mathrm{d}t\bigg) \\ &\times \sup_{t\in[0,T]} h\big(|X_t^{(1)}| + |x^{(1)} - y^{(1)}|\big)\bigg\}\end{aligned}$$

is finite, then for any strictly positive $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$,

$$\begin{aligned}P_{2T}\log f(y) \le{}& \log P_{2T}f(x) + \frac{K_T|x^{(1)} - y^{(1)}|^2}{\lambda_T^2(1 - \mathrm{e}^{-2K_T T})} \\ &+ \frac{\theta_T \mathrm{e}^{2\Theta_T T}\Psi_T(x^{(1)}, y^{(1)})}{2} \\ &\times \bigg\{|x^{(2)} - y^{(2)}|^2 + \frac{1 - \mathrm{e}^{-2\Theta_T T}}{\Theta_T}\varphi_T(|x^{(1)} - y^{(1)}|^2)\bigg\}.\end{aligned}$$

Because of Theorem 5.5.7, we are now able to present the log-Harnack inequality for the Gruschin semigroup on $\mathbb{R}^{m+d}$ with any $l > 0$.

Corollary 5.5.8 (Gruschin Semigroup). *Let* $A = 0, b^{(1)} = 0, b^{(2)} = 0, \sigma^{(1)} = I_{m\times m}$ *and* $\sigma^{(2)}(x^{(1)}) = |x^{(1)}|^l I_{d\times d}$ *for some constant* $l > 0$. *Then there exists a constant* $c > 0$ *such that*

$$\begin{aligned} P_{2T}\log f(y) \le \log P_{2T}f(x) &+ \frac{|x^{(1)} - y^{(1)}|^2}{2T} \\ &+ \frac{c}{T^{l+1}}\Big(|x^{(1)}|^{2(l-1)^+} + |y^{(1)}|^{2(l-1)^+} + T^{(l-1)^+}\Big) \\ &\qquad\times\Big(|x^{(2)} - y^{(2)}|^2 + 2T|x^{(1)} - y^{(1)}|^{2(l\wedge 1)}\Big) \end{aligned}$$

holds for all $T > 0$ *and* $x, y \in \mathbb{R}^{m+d}$.

Proof. It is easy to see that **(A5.5.2)-(A5.5.4)** hold for $\lambda_t = 1, K_t = \Theta = 0, \varphi_t(r) = c_1 r^{l\wedge 1}$ and $h(r) = c_1 \vee r^{2(l-1)^+}$ for some constant $c_1 \ge 1$. Moreover,

$$Q_T = I_{d\times d}\int_T^{2T} |B_t^{(1)} + x_t^{(1)}|^{2l}\mathrm{d}t$$

is invertible and

$$\|Q_T^{-1}\|^2\int_T^{2T}\|\sigma^{(2)}(X_t^{(1)})\|^2\mathrm{d}t = \frac{1}{\int_T^{2T}|B_t^{(1)} + x^{(1)}|^{2l}\mathrm{d}t}.$$

Then, using the fact that for any $r \ge 0$,

$$\mathbb{E}\sup_{t\in[0,T]}|B_t^{(1)} + x^{(1)}|^{2r} \le c(r)(|x^{(1)}|^{2r} + T^r)$$

holds for some constant $c(r) > 0$, and noting that Lemma 5.5.5 implies

$$\begin{aligned} &\mathbb{E}\bigg(\int_T^{2T}|B_t^{(1)} + x^{(1)}|^{2l}\mathrm{d}t\bigg)^{-2} \\ &= \mathbb{E}\bigg\{\mathbb{E}\bigg(\bigg(\int_0^T\Big|(B_{T+t}^{(1)} - B_T^{(1)}) + (B_T^{(1)} + x^{(1)})\Big|^{2l}\mathrm{d}t\bigg)^{-2}\bigg|B_T^{(1)}\bigg)\bigg\} \\ &\le \frac{C}{T^{2(l+1)}} \end{aligned}$$

for some constant $C > 0$, we conclude that

$$\begin{aligned} \Psi_T(x^{(1)}, y^{(1)}) &\le \bigg(\mathbb{E}\sup_{t\in[0,T]} h(|B_t^{(1)} + x^{(1)}| + |x^{(1)} - y^{(1)}|)^2\bigg)^{\frac{1}{2}} \\ &\qquad\times\bigg(\mathbb{E}\bigg(\int_T^{2T}|B_t^{(1)} + x^{(1)}|^{2l}\mathrm{d}t\bigg)^{-2}\bigg)^{\frac{1}{2}} \\ &\le \frac{c}{T^{l+1}}\Big(|x^{(1)}|^{2(l-1)^+} + |y^{(1)}|^{2(l-1)^+} + T^{(l-1)^+}\Big) \end{aligned}$$

holds for some constant $c > 0$. Therefore, the desired log-Harnack inequality follows from Theorem 5.5.7. □

The remainder of this subsection is devoted to the proof of Theorem 5.5.7. Let $x = (x^{(1)}, x^{(2)}), y = (y^{(1)}, y^{(2)})$ and $T > 0$ be fixed. The idea to establish a Harnack type inequality of P_{2T} using a coupling by change of measure is as follows: construct two processes X_t, Y_t and a probability density function R such that $X_{2T} = Y_{2T}, X_0 = x, Y_0 = y$, and

$$P_{2T}f(x) = \mathbb{E}f(X_{2T}), \quad P_{2T}f(y) = \mathbb{E}\big\{Rf(Y_{2T})\big\}, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}).$$

Then, by e.g. the Young inequality, for strictly positive f one obtains

$$\begin{aligned} P_{2T}\log f(y) &= \mathbb{E}\big\{R\log f(Y_{2T})\big\} = \mathbb{E}\big\{R\log f(X_{2T})\big\} \\ &\le \mathbb{E}(R\log R) + \log P_{2T}f(x). \end{aligned} \tag{5.5.21}$$

This implies the log-Harnack inequality provided $\mathbb{E}(R\log R) < \infty$.

When the SDE is driven by an additive noise, this idea can be easily realized by adding a proper drift to the equation and using the Girsanov theorem. In the non-degenerate multiplicative noise case, the argument has been well modified in §3.4 by constructing a coupling with singular additional drifts. For the present model, as the SDE is driven by a multiplicative noise with a possibly degenerate and singular coefficient, it is hard to follow the known ideas to construct a coupling in one go. We will construct a coupling in two steps, where the second step will be realized under the regular conditional probability given $B^{(1)}$:

(1) We first construct a coupling $(X_t^{(1)}, Y_t^{(1)})$ by change of measure for the first component of the process such that $X_t^{(1)} = Y_t^{(1)}$ for $t \ge T$. This part is now standard as the first equation in (5.5.20) is driven by the non-degenerate additive noise $\sigma^{(1)}(t)\mathrm{d}B_t^{(1)}$.
(2) Once $X_t^{(1)} = Y_t^{(1)}$ holds for $t \ge T$, the equations for $X_t^{(2)}$ and $Y_t^{(2)}$ will have same noise part for $t \ge T$, so that we are able to construct a coupling by change of measure for them such that $X_{2T}^{(2)} = Y_{2T}^{(2)}$.

We first construct the Brownian motion B_t as the coordinate process on the Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = C([0,\infty); \mathbb{R}^{m+d}) = C([0,\infty); \mathbb{R}^m) \times C([0,\infty); \mathbb{R}^d),$$

$\mathcal{F}$ is the Borel σ-field, $\mathbb{P}$ is the Wiener measure (i.e. the distribution of the $(m+d)$-dimensional Brownian motion starting at 0). Let

$$B_t(\omega) = (B_t^{(1)}(\omega), B_t^{(2)}(\omega)) = (\omega_t^{(1)}, \omega_t^{(2)}), \quad \omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega, t \ge 0.$$

Then B_t is the $(m+d)$-dimensional Brownian motion w.r.t. the natural filtration $(\mathcal{F}_t)_{t\ge 0}$. Moreover, let $\mathcal{F}^{(1)} = \sigma(B_t^{(1)} : t \ge 0)$ and $\mathcal{F}_t^{(2)} = \sigma(B_s^{(2)} :$

$0 \le s \le t), t \ge 0$. It is well known that the conditional regular probability $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$ given $\mathcal{F}^{(1)}$ exists. This structure will enable us to first construct a coupling $(X_t^{(1)}, Y_t^{(1)})$ for the first component process up to time T under probability $\mathbb{P}$, then construct a coupling $(X_t^{(2)}, Y_t^{(2)})$ for the second component process from time T on under the regular conditional probability $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$. For any probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$, we denote by $\mathbb{E}_{\tilde{\mathbb{P}}}$ the expectation w.r.t. $\tilde{\mathbb{P}}$. When $\tilde{\mathbb{P}} = \mathbb{P}$, we simply denote the expectation by $\mathbb{E}$ as usual.

Let $X_t = (X_t^{(1)}, X_t^{(2)})$ solve the equation (5.5.20) with $X_0 = x = (x^{(1)}, x^{(2)})$. Given $Y_0 = y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^{m+d}$, we are going to construct $Y_t^{(1)}$ on $\mathbb{R}^m$ and $Y_t^{(2)}$ on $\mathbb{R}^d$ respectively, such that $Y_t^{(1)} = X_t^{(1)}$ for $t \ge T$ and $Y_{2T}^{(2)} = X_{2T}^{(2)}$.

5.5.2.1 *Construction of* $Y_t^{(1)}$

Consider the equation

$$\mathrm{d}Y_t^{(1)} = b^{(1)}(t, Y_t^{(1)})\mathrm{d}t + \sigma^{(1)}(t)\mathrm{d}B_t^{(1)} - v_t^{(1)}\mathrm{d}t, \quad Y_0^{(1)} = y^{(1)}, \tag{5.5.22}$$

where

$$v_t^{(1)} := \frac{2K_T|x^{(1)} - y^{(1)}|\mathrm{e}^{-K_T t}(Y_t^{(1)} - X_t^{(1)})}{(1 - \mathrm{e}^{-2K_T T})|X_t^{(1)} - Y_t^{(1)}|} 1_{\{X_t^{(1)} \ne Y_t^{(1)}\}}, \quad t \ge 0.$$

Obviously, the equation has a unique strong solution before the coupling time

$$\tau_1 := \inf\left\{t \ge 0 : X_t^{(1)} = Y_t^{(1)}\right\}.$$

Then, letting $Y_t^{(1)} = X_t^{(1)}$ for $t \ge \tau_1$, we see that $(Y_t^{(1)})_{t \ge 0}$ is a strong solution to (5.5.22). So, we can reformulate $v_t^{(1)}$ as

$$v_t^{(1)} = \frac{2K_T|x^{(1)} - y^{(1)}|\mathrm{e}^{-K_T t}(Y_t^{(1)} - X_t^{(1)})}{(1 - \mathrm{e}^{-2K_T T})|X_t^{(1)} - Y_t^{(1)}|} 1_{[0,\tau_1)}(t), \quad t \ge 0. \tag{5.5.23}$$

Proposition 5.5.9. *For any* $t \ge 0$,

$$\begin{aligned} |X_t^{(1)} - Y_t^{(1)}| &\le \frac{\mathrm{e}^{-K_T t} - \mathrm{e}^{-K_T(2T-t)}}{1 - \mathrm{e}^{-2K_T T}} |x^{(1)} - y^{(1)}| 1_{[0,T]}(t) \\ &\le |x^{(1)} - y^{(1)}| 1_{[0,T]}(t). \end{aligned} \tag{5.5.24}$$

Consequently, $\tau_1 \le T$ *and* $X_t^{(1)} = Y_t^{(1)}$ *for* $t \ge T$.

Proof. By **(A5.5.3)** and (5.5.23), we have

$$\mathrm{d}|X_t^{(1)} - Y_t^{(1)}| \le \left(K_T|X_t^{(1)} - Y_t^{(1)}| - \frac{2K_T|x^{(1)} - y^{(1)}|\mathrm{e}^{-K_T t}}{1 - \mathrm{e}^{-2K_T T}} \right)\mathrm{d}t$$

for $t \in [0, \tau_1) \cap [0, T]$. Then

$$|X_t^{(1)} - Y_t^{(1)}| \le \frac{\mathrm{e}^{-K_T t} - \mathrm{e}^{-K_T(2T-t)}}{1 - \mathrm{e}^{-2K_T T}}|x^{(1)} - y^{(1)}|$$

for $t \in [0, \tau_1) \cap [0, T]$. This implies that $\tau_1 \le T$ and also (5.5.24) since $X_t^{(1)} = Y_t^{(1)}$ for $t \ge \tau_1$. □

To formulate (5.5.22) as the first equation in (5.5.20), we let

$$\tilde{B}_t^{(1)} = B_t^{(1)} - \int_0^t \xi^{(1)}(s)\mathrm{d}s, \quad \xi^{(1)}(t) = \sigma^{(1)}(t)^{-1}v_t^{(1)}, \quad t \ge 0.$$

From **(A5.5.2)** and (5.5.23) we see that $\xi^{(1)}(s)$ is bounded and adapted. So, by the Girsanov theorem, $\tilde{B}_t$ is an m-dimensional Brownian motion under the probability measure $\mathbb{Q}^{(1)} := R_1(T)\mathbb{P}$, where

$$R_1(t) := \exp\left[\int_0^t \langle \xi^{(1)}(s), \mathrm{d}B_s^{(1)} \rangle - \frac{1}{2}\int_0^t |\xi^{(1)}(s)|^2 \mathrm{d}s \right], \quad t \ge 0$$

is a martingale. Obviously, (5.5.22) can be formulated as

$$\mathrm{d}Y_t^{(1)} = b^{(1)}(t, Y_t^{(1)})\mathrm{d}t + \sigma^{(1)}(t)\mathrm{d}\tilde{B}_t^{(1)}, \quad Y_0^{(1)} = y^{(1)}. \tag{5.5.25}$$

As shown in (5.5.21), for the log-Harnack inequality we need to estimate the entropy of $R_1 := R_1(T)$.

Proposition 5.5.10. *Let $R_1 = R_1(T)$. Then*

$$\mathbb{E}\{R_1 \log R_1\} \le \frac{K_T|x^{(1)} - y^{(1)}|^2}{\lambda_T^2(1 - \mathrm{e}^{-2K_T T})}. \tag{5.5.26}$$

Proof. By $\tau_1 \le T$, **(A5.5.2)**, (5.5.23), we have

$$\int_0^T |\sigma^{(1)}(t)^{-1}v_t^{(1)}|^2 \mathrm{d}t \le \frac{2K_T|x^{(1)} - y^{(1)}|^2}{\lambda_T^2(1 - \mathrm{e}^{-2K_T T})}. \tag{5.5.27}$$

Then, it follows from (5.5.22) and the definition of R_1 that

$$\begin{aligned}\mathbb{E}\{R_1 \log R_1\} &= \mathbb{E}_{\mathbb{Q}^{(1)}} \log R_1 \\ &= \frac{1}{2}\mathbb{E}_{\mathbb{Q}^{(1)}} \int_0^T |\sigma^{(1)}(t)^{-1}v_t^{(1)}|^2 \mathrm{d}t \le \frac{K_T|x^{(1)} - y^{(1)}|^2}{\lambda_T^2(1 - \mathrm{e}^{-2K_T T})}.\end{aligned}$$

□

5.5.2.2 *Construction of* $Y_t^{(2)}$

Let

$$\xi_t^{(2)} = \sigma^{(2)}(t, Y_t^{(1)})^* \mathrm{e}^{A^*(T-t)} Q_T^{-1}(Y_T^{(2)} - X_T^{(2)}) 1_{[T,2T]}(t), \quad t \geq 0.$$

Let $Y_t^{(2)}$ solve the equation

$$\mathrm{d}Y_t^{(2)} = \{AY_t^{(2)} + b^{(2)}(t, Y_t^{(1)})\}\mathrm{d}t + \sigma^{(2)}(t, Y_t^{(1)})\{\mathrm{d}B_t^{(2)} - \xi_t^{(2)}\mathrm{d}t\} \quad (5.5.28)$$

with $Y_0^{(2)} = y^{(2)}$. Since under $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$ the processes $X_t^{(1)}$ and $Y_t^{(1)}$ are fixed and $B_t^{(2)}$ is a d-dimensional Brownian motion, by **(A5.5.4)** this equation has a unique solution. Since $X_t^{(1)} = Y_t^{(1)}$ for $t \geq T$, for the present $b^{(2)}$ we have

$$AX_t^{(2)} + b^{(2)}(t, X_t^{(1)}) - \{AY_t^{(2)} + b^{(2)}(t, Y_t)\} = A(X_t^{(2)} - Y_t^{(2)}), \quad t \geq T.$$

So,

$$X_{2T}^{(2)} - Y_{2T}^{(2)} = \mathrm{e}^{AT}(X_T^{(2)} - Y_T^{(2)}) + \int_T^{2T} \mathrm{e}^{A(2T-t)}\sigma^{(2)}(t, Y_t^{(1)})\xi_t^{(2)}\mathrm{d}t = 0$$

as $Y_t^{(1)} = X_t^{(1)}$ for $t \geq T$. Therefore, $X_{2T} = Y_{2T}$. Moreover, let

$$R_2(t) = \exp\left[\int_T^t \langle \xi_s^{(2)}, \mathrm{d}B_s^{(2)}\rangle - \frac{1}{2}\int_T^t |\xi_s^{(2)}|^2 \mathrm{d}s\right], \quad t \in [T, 2T].$$

Proposition 5.5.11. *Under* $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$, $\{R_2(t)\}_{t\in[T,2T]}$ *is an* $\mathcal{F}_t^{(2)}$-*martingale and* $R_2 := R_2(2T)$ *satisfies*

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\{R_2 \log R_2\}\\ &\leq \frac{\theta_T}{2}\left\{\mathrm{e}^{2\Theta_T T}|x^{(2)} - y^{(2)}|^2 + \frac{\mathrm{e}^{2\Theta_T T} - 1}{\Theta_T}\varphi_T(|x^{(1)} - y^{(1)}|^2)\right\}\\ &\quad \times \sup_{t\in[0,T]} h(|Y_t^{(1)}| + |x^{(1)} - y^{(1)}|)\|Q_T^{-1}\|^2 \int_T^{2T} \|\sigma^{(2)}(t, Y_t^{(1)})\|^2 \mathrm{d}t.\end{aligned}$$

Proof. We make use of an approximation argument. Let $\xi_n^{(2)}(s) = \xi_s^{(2)} 1_{\{|\xi_s^{(2)}| \leq n\}}$, and let

$$R_{2,n}(t) = \exp\left[\int_T^t \langle \xi_n^{(2)}(s), \mathrm{d}B_s^{(2)}\rangle - \frac{1}{2}\int_T^t |\xi_n^{(2)}(s)|^2 \mathrm{d}s\right], \quad n \geq 1, t \in [T, 2T].$$

Then $\{R_{2,n}(t)\}_{t\in[T,2T]}$ is an $\mathcal{F}_t^{(2)}$-martingale under $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$. So it remains to show that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\big\{R_{2,n}\log R_{2,n}\big\}(t)\\
&\le \frac{\theta_T}{2}\bigg\{e^{2\Theta_T T}|x^{(2)}-y^{(2)}|^2+\frac{e^{2\Theta_T T}-1}{\Theta_T}\varphi_T(|x^{(1)}-y^{(1)}|^2)\bigg\}\\
&\quad\times\sup_{t\in[0,T]}h(|Y_t^{(1)}|+|x^{(1)}-y^{(1)}|) \qquad (5.5.29)\\
&\quad\times\|Q_T^{-1}\|^2\int_T^{2T}\|\sigma^{(2)}(t,Y_t^{(1)})\|^2\mathrm{d}t
\end{aligned}$$

holds for all $t\in[T,2T]$ and $n\ge 1$. Let $\mathbb{Q}_{2,n}=R_{2,n}(2T)\mathbb{P}(\cdot|\mathcal{F}^{(1)})$. By the Girsanov theorem, under $\mathbb{Q}_{2,n}$ the process

$$\tilde{B}_t^{(2)}:=B_t^{(2)}-\int_T^{T\vee t}\xi_n^{(2)}(s)\mathrm{d}s,\quad t\in[0,2T]$$

is a d-dimensional Brownian motion. Then, by the definition of $\xi_n^{(2)}(s)$, for any $t\in[T,2T]$ we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\big\{R_{2,n}\log R_{2,n}\big\}(t)\\
&=\mathbb{E}_{\mathbb{Q}_{2,n}}\log R_{2,n}(t)\le\frac{1}{2}\int_T^{2T}\mathbb{E}_{\mathbb{Q}_{2,n}}|\xi_n^{(2)}(s)|^2\mathrm{d}s\\
&\le\frac{\theta_T}{2}\Big(\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}R_{2,n}(2T)|X_T^{(2)}-Y_T^{(2)}|^2\Big) \qquad (5.5.30)\\
&\quad\times\|Q_T^{-1}\|^2\int_T^{2T}\|\sigma^{(2)}(s,Y_s^{(1)})\|^2\mathrm{d}s.
\end{aligned}$$

Since $\{R_{2,n}(t)\}_{t\in[0,T]}$ is an $\mathcal{F}_t^{(2)}$-martingale under $P(\cdot|\mathcal{F}^{(1)})$, and $R_{2,n}(T)=1$, we have

$$\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\big\{R_{2,n}(2T)|X_T^{(2)}-Y_T^{(2)}|^2\big\}=\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}|X_T^{(2)}-Y_T^{(2)}|^2. \qquad (5.5.31)$$

Finally, by **(A5.5.4)**, (5.5.24) and Itô's formula, we obtain

$$\begin{aligned}
&\mathrm{d}|X_t^{(2)}-Y_t^{(2)}|^2\\
&\le 2\big\{\Theta_T|X_t^{(2)}-Y_t^{(2)}|^2+\varphi_T(|x^{(1)}-y^{(1)}|^2)h(|Y_t^{(1)}|+|x^{(1)}-y^{(1)}|)\big\}\mathrm{d}t\\
&\quad+2\big\langle X_t^{(2)}-Y_t^{(2)},\{\sigma^{(2)}(t,X_t^{(1)})-\sigma^{(2)}(t,Y_t^{(1)})\}\mathrm{d}B_t^{(2)}\big\rangle,\quad t\in[T,2T].
\end{aligned}$$

Since $h\ge 1$, this implies

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}|X_T^{(2)}-Y_T^{(2)}|^2\\
&\le\bigg(e^{2\Theta_T T}|x^{(2)}-y^{(2)}|^2+\frac{e^{2\Theta_T T}-1}{\Theta_T}\varphi_T(|x^{(1)}-y^{(1)}|^2)\bigg)\\
&\quad\times\sup_{t\in[0,T]}h(|Y_t^{(1)}|+|x^{(1)}-y^{(1)}|).
\end{aligned}$$

Combining this with (5.5.30) and (5.5.31), we prove (5.5.29). $\square$

Proof. **(Proof of Theorem 5.5.7)** Let $X_t = (X_t^{(1)}, X_t^{(2)})$ and $Y_t = (Y_t^{(1)}, Y_t^{(2)})$ be constructed above. Then $X_{2T} = Y_{2T}$. Let $R = R_1 R_2$. By Propositions 5.5.9, 5.5.10, 5.5.11, and noting that the distribution of $Y^{(1)}$ under $R_1\mathbb{P}$ coincides with that of $X^{(1)}(y^{(1)})$ under $\mathbb{P}$, we have $\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}R_2 = 1$ and

$$\begin{aligned}
\mathbb{E}\{R\log R\} &= \mathbb{E}\Big\{(R_1\log R_1)\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}R_2\Big\} + \mathbb{E}\Big\{R_1\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}(R_2\log R_2)\Big\}\\
&\le \frac{K_T|x^{(1)}-y^{(1)}|^2}{\lambda_T^2(1-\mathrm{e}^{-2K_TT})}\\
&\quad + \frac{\theta_T}{2}\mathbb{E}^{y^{(1)}}\bigg\{\Big(\mathrm{e}^{2\Theta_TT}|x^{(2)}-y^{(2)}|^2 + \frac{\mathrm{e}^{2\Theta_TT}-1}{\Theta_T}\varphi_T(|x^{(1)}-y^{(1)}|^2)\Big)\\
&\quad\times \sup_{t\in[0,T]} h(|X_t^{(1)}|+|x^{(1)}-y^{(1)}|)\|Q_T^{-1}\|^2\int_T^{2T}\|\sigma^{(2)}(t,X_t^{(1)})\|^2\mathrm{d}t\bigg\}.
\end{aligned}$$

Combining this with the definition of Ψ_T, we obtain

$$\begin{aligned}
&\mathbb{E}\{R\log R\}\\
&\le \frac{K_T|x^{(1)}-y^{(1)}|^2}{\lambda_T^2(1-\mathrm{e}^{-2K_TT})} + \frac{\theta_T\mathrm{e}^{2\Theta_TT}\Psi_T(x^{(1)},y^{(1)})}{2}\\
&\quad\times\Big\{|x^{(2)}-y^{(2)}|^2 + \frac{1-\mathrm{e}^{-2\Theta_TT}}{\Theta_T}\varphi_T(|x^{(1)}-y^{(1)}|^2)\Big\}.
\end{aligned}\tag{5.5.32}$$

Since $B_t^{(2)}$ is a d-dimensional Brownian motion under $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$, by the Girsanov theorem, under $R_2\mathbb{P}(\cdot|\mathcal{F}^{(1)})$ the process

$$\tilde{B}_t^{(2)} := B_t^{(2)} - \int_T^t \xi_s^{(2)}\mathrm{d}s,\quad t\in[T,2T]$$

is a d-dimensional Brownian motion. Noting that

$$Y_t^{(2)} = Y_T^{(2)} + \int_T^t\{AY_s^{(2)} + b^{(2)}(s,Y_s^{(1)})\}\mathrm{d}s + \int_T^t\sigma^{(2)}(s,Y_s^{(1)})\mathrm{d}\tilde{B}_s^{(2)}$$

holds for $t\in[T,2T]$, we see that the distribution of $Y_{2T}^{(2)}$ under $R_2\mathbb{P}(\cdot|\mathcal{F}^{(1)})$ coincides with that of $\tilde{Y}_{2T}^{(2)}$ under $\mathbb{P}(\cdot|\mathcal{F}^{(1)})$, where

$$\tilde{Y}_t^{(2)} = Y_t^{(2)},\quad t\in[0,T],$$

and when $t\in[T,2T]$

$$\tilde{Y}_t^{(2)} = Y_T^{(2)} + \int_T^t\{A\tilde{Y}_s^{(2)} + b^{(2)}(s,Y_s^{(1)})\}\mathrm{d}s + \int_T^t\sigma^{(2)}(s,Y_s^{(1)})\mathrm{d}B_s^{(2)}.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\{R_2\log f(Y_{2T})\} = \mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\{\log f(Y_{2T}^{(1)},\tilde{Y}_{2T}^{(2)})\}.$$

Combining this with $X_{2T} = Y_{2T}$, we obtain

$$\begin{aligned}
\mathbb{E}\{R\log f(X_{2T})\} &= \mathbb{E}\{R_1R_2\log f(Y_{2T})\}\\
&= \mathbb{E}\Big(R_1\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\{R_2\log f(Y_{2T})\}\Big)\\
&= \mathbb{E}\Big(R_1\mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}^{(1)})}\{\log f(Y_{2T}^{(1)}, \tilde{Y}_{2T}^{(2)})\}\Big)\\
&= \mathbb{E}\{R_1\log f(Y_{2T}^{(1)}, \tilde{Y}_{2T}^{(2)})\}.
\end{aligned}\tag{5.5.33}$$

Moreover, again by the Girsanov theorem, under $R_1\mathbb{P}$ the process $(\tilde{B}_t^{(1)}, B_t^{(2)})_{t\in[0,2T]}$ is a $(d+m)$-dimensional Brownian motion, recall that

$$\tilde{B}_t^{(1)} = B_t^{(1)} - \int_0^{T\wedge t}\xi^{(1)}(s)\mathrm{d}s, \quad t\in[0,2T].$$

Noting that $(Y_t^{(1)}, \tilde{Y}_t^{(2)})$ solves the equation

$$\begin{cases}\mathrm{d}Y_t^{(1)} = b^{(1)}(t, Y_t^{(1)})\mathrm{d}t + \sigma^{(1)}(t)\,\mathrm{d}\tilde{B}_t^{(1)}, \ Y_0^{(1)} = y^{(1)},\\ \mathrm{d}\tilde{Y}_t^{(2)} = \{A\tilde{Y}_t^{(2)} + b^{(2)}(t, Y_t^{(1)}, \tilde{Y}_t^{(2)})\}\mathrm{d}t + \sigma^{(2)}(t, Y_t^{(1)})\,\mathrm{d}B_t^{(2)}, \ \tilde{Y}_0^{(2)} = y^{(2)},\end{cases}$$

we conclude that the distribution of $(Y_{2T}^{(1)}, \tilde{Y}_{2T}^{(2)})$ under $R_1\mathbb{P}$ coincides with that of $X_{2T}(y)$ under $\mathbb{P}$. Therefore, it follows from (5.5.33) and the Young inequality that

$$\begin{aligned}
P_{2T}\log f(y) &= \mathbb{E}\{R_1\log f(Y_{2T}^{(1)}, \tilde{Y}_{2T}^{(2)})\} = \mathbb{E}\{R\log f(X_{2T})\}\\
&\le \log P_{2T}f(x) + \mathbb{E}\{R\log R\}.
\end{aligned}$$

Combining this with (5.5.32) we complete the proof. $\square$

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Index

$B(x,r)$: geodesic ball at x with radius r, 52
C^∞-compatible, 1
$C^p_\infty(M) = \mathbb{R} \oplus C^p_0(M)$, 83
$H^{2,1}$, 133
$P_u \in \mathbb{R}^d \otimes \mathbb{R}^d,\ u \in O(M)$, 228
$W^T = C([0,T];M),\ W^T_x = \{\gamma \in W^T : \gamma_0 = x\}$, 228
$X_{[0,T]} = \{X_t\}_{t\in[0,T]}$, 228
$\mathcal{C}_N(L) = \{f \in C^\infty(M), Nf = 0, Lf \in \mathcal{B}_b(M)\}$, 142
$\mathcal{D} = \{\phi \in C^2_b(M) : \inf \phi = 1, \mathbb{I} \geq -N\log\phi\}$, 142
Φ-entropy, 181
$\mathrm{Ric}_Z = \mathrm{Ric} - \nabla Z$, 51
$\mathrm{cut} = \{(x,y) : x \in \mathrm{cut}(y)\}$, 62
p-tensor, 7
$C^p(M)$: set of all C^p-smooth real functions on M, 2
$C^p(x)$: set of C^p-smooth real functions defined in a neighborhood of x, 2
$C^p_0(M)$: set of all C^p-smooth real functions on M with compact supports, 2
$C^p_b(M)$: the set of functions in $C^p(M)$ with bounded derivatives up to order p, 9
D^0F: the damped gradient on path space, 235
H_e: the horizontal vector field induced by $e \in \mathbb{R}^d$, 8
$I(X,X)$: index form of X, 11
$M_{x,y}$: mirror reflection from T_xM to T_yM, 62
$O(M)$: orthonormal frame bundle of M, 8
$P_{x,y} : T_xM \to T_yM$: parallel transform, 62
$Q^x_{r,t} \in \mathbb{R}^d \otimes \mathbb{R}^d$: the multiplicative functional, 229
TM: vector bundle on M, 3
T_xM: tangent space at x, 2
W^ρ_p: L^p-transportation-cost or Wasserstein distance induced by ρ, 17
$X^{\varepsilon,h}_{[0,T]}$: the quasi-invariant flow, 235
$\mathcal{B}(E)$: set of all measurable functions on E, 16
$\mathcal{B}^+(E)$: set of all bounded non-negative measurable functions on E, 16
$\mathcal{B}_b(E)$: set of all bounded measurable functions on E, 16
$\mathcal{C}(\mu,\nu)$: set of all couplings of μ and ν, 15
$\Delta_p = -(\mathbf{dd}^* + \mathbf{d}^*\mathbf{d})$: the negative Hodge-de Rham Laplacian on $\Omega^p(M)$, 8
$\Delta_{O(M)}$: horizontal Laplace operator, 9
$\mathbf{d}$: exterior differential, 7
$\Gamma(TM)$: set of all vector fields on M, 3

$\Gamma^p(TM)$: set of all C^p-smooth vector fields on M, 3
$\mathbf{H}_Z$: horizontal lift of $Z \in TM$, 9
$\mathbb{I}$: second fundamental form of ∂M, 12
$\Omega^p(M)$: set of smooth p-forms on M, 7
Π^T_μ: the distribution of the reflecting diffusion process with initial distribution μ and time-interval $[0,T]$, 242
Π^T_x: the distribution of $X^x_{[0,T]}$, 239
Ric: Ricci curvature, 6
Sect: sectional curvature, 6
cut(x): the cut-locus of x, 5
$\mathbf{d}^*$: codifferential, 8
δ_x: Dirac measure at x, 18
div_μ: the divergence operator w.r.t. μ, 243
$\dot\gamma$: tangent vector of curve γ, 4
i_M: injectivity radius of M, 5
$\mathcal{P}(E)$: set of probability measures on E, 15
$\mathcal{R}$: curvature operator, 5
$\mathcal{X}_0(M)$: all smooth vector fields on M with compact support, 242
μ_∂: area measure induced by μ, 10
$\mathbf{p}: O(M) \to M$: canonical projection, 8
ρ_∂: Riemannian distance to the boundary of the manifold, 13
ρ_o: Riemannian distance to point o, 4
$\sigma(L)$: spectrum of L, 39
$\sigma_{ess}(L)$: essential spectrum of L, 39
$\mathrm{vol}(\mathrm{d}x) = \mathrm{d}x$: volume measure, 3
$\mathbf{Q}_t$: Hamilton-Jacob semigroup, 114

boundary, 12
  convex, 12
  injectivity radius, 13

condition
  Hörmander, 257
  Kalman rank, 259
connection, 3
  Levi-Civita, 4
coupling
  by parallel displacement, 63
  by reflection, 63
  for probability measures, 15
  optimal, 18
  Otto-Villani's, 83
  shift, 20
  successful, 19
  Wasserstein, 18
coupling by change of measure, 22
curvature, 5
  Ricci, 6
  sectional, 6
cut-locus, 5

derivative
  covariant , 8
differential structure, 1
distance
  Riemannian, 4
  Wasserstein, 17

equation
  backward Kolmogorov, 52
  forward Kolmogorov, 52
exponential map, 4
exterior differential, 7

field
  Jacobi, 6
  vector, 3
form
  differential, 7
  second fundamental, 12
formula
  Bismut, 23
  Bochner-Weitzenböck, 9
  coarea, 10
  derivative, 54
  first variational, 10
  Green, 10
  integration by parts, 23
  Monge-Kontorovich, 17
  second variational, 10
function
  compact, 262

geodesic, 4
  minimal, 4, 33
ground state, 128

horizontal lift, 9

inequality
  F-Sobolev, 42
  Φ-entropy, 182
  FKG, 16
  Harnack, 22, 25
  HWI, 74
  intrinsic super, 40
  intrinsic super Poincaré, 128
  log-Harnack, 28
  log-Sobolev, 42, 72
  Nash, 43
  Poincaré, 39
  Poincaré type, 39
  shift Harnack, 23
  Sobolev, 43
  super, 129
  super Poincaré, 40
  weak Poincaré, 43
  weighted log-Sobolev, 79
  Young, 30
injectivity radius, 5

lemma
  index, 11
local (coordinate) chart, 2

manifold, 2
  differentiable, 2
  orientable, 7
  Riemannian, 3
measure
  quasi-invariant, 24
  invariant, 24
  volume, 3
multiplicative functional, 229

normal frame, 9

operator
  compact, 40
  damped gradient, 235
  divergence, 6
  Feller, 24
  gradient, 6
  horizontal Laplace, 9
  Laplace, 6
  Markov, 25, 33
  strong Feller, 24
optimal transportation map, 18
orthonormal frame bundle, 8

process
  diffusion, 50
  horizontal diffusion, 49
  horizontal reflecting diffusion, 141
  reflecting diffusion, 142
product
  inner, 3
  alternating, 7
property
  tensorization, 78

semigroup
  diffusion, 50
  Hamilton-Jacob, 114
  hypercontractive, 74
  intrinsically ultracontractive, 128
  supercontractive, 74
  ultracontractive, 74
space
  geodesic, 33

tail norm, 42
tangent space, 2
tangent vector, 4
theorem
  Hessian comparison, 11, 13
  Laplacian comparison, 11, 14
transportation, 16
  cost, 17
  parallel, 4

vector bundle, 3