D.Q. Durdiyev

Xususiy hosilali differensial tenglamalar

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u + G(x, u, \ldots, D^\beta u, \ldots) = 0$$

Mundarija

Qisqacha sharh

Xususiy hosilali differensial tenglamalar fanining qamrovi keng bo‘lib, ushbu darslik universitetlarning matematika, amaliy matematika va informatika ta‘lim yo‘nalishlari talabalari uchun fanning amaldagi dasturi asosida yozilgan.

Kitobda dastlab umumlashgan funksiyalar va ularning Furye almashtirishlari keltirilgan bo‘lib, bunda umumlashgan funksiyalar nazaryasining boshlang‘ich tushunchalari misollar yordamida bayon etilgan. Dirakning delta-funksiyasi xossalarini o‘rganishga alohida e’tibor qaratilgan. Umumlashgan funksiyalar nazariyasi keng bo‘lib, uning elementlari kitob doirasida zarur bo‘ladigan hajmda keltirilgan. Differensial tenglamalarning umumlashgan yechimi tushunchasi kiritilgan, o‘zgarmas koeffitsientli differensial operatorlarning fundamental yechimlari qurilgan, hamda shu asosda to‘lqin va issiqlik o‘tkazuvchanlik tenglamalari uchun qo‘yilgan umumlashgan va klassik Koshi masalalarining yechimlari hosil qilingan.

Shuningdek, kitobdan ikkinchi tartibli xususiy hosilali differensial tenglamalarning uchta tipiga mansub: giperbolik, parabolik va elliptik tipdagi tenglamalar uchun korrekt qo‘yilgan boshlang‘ich (Koshi), boshlang‘ich-chegaraviy va chegaraviy masalalarning yechimlarini topish keng o‘rin olgan. Bunda masalalar yechimini beruvchi formulalar olingan bo‘lib, yechimni yagonaligi va berilganlardan uzluksiz bog‘liqligi o‘rganilgan. Bundan tashqari, xususiy hosilali differensial tenglamalarni yechishda ko‘p qo‘llaniladigan o‘zgaruvchilarni ajratish, integral almashtirishlar usullarida keng ishlatiladigan ayrim maxsus funksiyalar va ularning xossalari keltirilgan.

Darslikni yozishda muallifning qator yillar davomida Buxoro davlat universitetida o‘qigan ma‘ruzalari hamda ushbu soha bo‘yicha taniqli olimlarning rus va ingliz tillaridagi o‘quv adabiyotlaridan foydalanildi.

Аннотация

Учебная дисциплина "Дифференциальные уравнения с частными производными" охватывает широкий спектр знаний. Данный учебник написан на основе действующих учебных программ для студентов бакалавриата по направлениям "математика", "прикладная математика и информатика".

Вначале приведены обобщенные функции и изучены их преобразования Фурье. При помощи примеров подробно изложены первичные понятия теории обобщенных функций. Особое внимание уделено изучению свойств дельта-функции Дирака. Теория обобщенных функций обширна, и в учебнике приведены лишь те понятия этой теории, которые необходимы в пределах этой книги. Введено понятие обобщенных решений для обыкновенных дифференциальных уравнений и уравнений с частными производными, построены фундаментальные решения дифференциальных операторов с постоянными коэффициентами. На этой основе получены решения обобщенной и классической задач Коши для уравнений распространения волн и теплопроводности.

А так же, в учебнике значительное место занимают традиционные разде- лы теории линейных уравнений с частными производными второго порядка гиперболического, параболического и эллиптического типа. Получены фор- мулы, дающие решения различных задач, исследованы вопросы единствен- ности и непрерывной зависимости решения от входных данных. Кроме того в книге приведены некоторые специальные функции и их свойства, которые часто используются при решении дифференциальных уравнений с частными производными с помощью методов разделения переменных и интегрального преобразования

При написании данного учебника использованы учебные литературы известных ученых по этой тематике на русском и английском языках, а также курсы лекций, читавшихся автором в течение ряда лет студентам Бухарского государственного университета.

Annotation

The subject Partial differential equations is so extensive that this textbook is written for the undergraduate students of Mathematics, Applied mathematics and IT directions basing on the current program of this course.

The book deals with the generalized functions and their Fourier transformation, in which the initial concepts of the generalized functions are illustrated using examples. Particular attention is given to the study of the properties of Dirac delta-function. The theory of generalized functions is wide, and its elements are presented in the necessary volume within this book. The concept of generalized solutions for ordinary differential equations and partial differential equations is introduced; fundamental solutions of differential operators with constant coefficients are constructed. On this basis, solutions of the generalized and classical Cauchy problems for the equations of wave propagation and heat conduction are obtained.

And also in the textbook a significant place is occupied by the traditional sections of the theory of linear partial differential equations of the second order of hyperbolic, parabolic and elliptic type. The methods of solving initial (Cauchy), initial-boundary and boundary problems for such equations are described. Formulas are obtained that give solutions to problems, and questions of uniqueness and continuous dependence of a solution on data are investigated. In addition, the book contains some special functions and their properties, which are often used in solving partial differential equations using the methods of separation of variables and the integral transformation.

In the process of writing the textbook, the author used his lectures those are given at Bukhara State University for many years and Russian as well as English textbooks by prominent scientists in this sphere.

So‘z boshi

Ushbu darslik universitetlarning matematika, amaliy matematika va informatika ta’lim yo‘nalishlari talabalari uchun o‘qitiladigan "Xususiy hosilali differensial tenglamalar" fani dasturi asosida yozilgan. Xususiy hosilali differensial tenglamalarning doirasi keng bo‘lib, bu kitobda fizik, texnik, mexanik va boshqa jarayonlarning matematik modellarini tuzishda hosil bo‘ladigan xususiy hosilali tenglamalar uchun masalalar o‘rganiladi.

Kitobning o‘ziga xosligi - bu unda umumlashgan yechimlardan keng foydalanilganligidir. Umumlashgan yechim tushunchasi umumlashgan hosila va, umuman aytganda, umumlashgan funksiyalar tushunchasiga tayanadi. Hozirgi kunda umumlashgan funksiyalar nazariyasi xususiy hosilali differensial tenglamalar uchun qo‘yilgan umumlashgan va klassik masalalarni tekshirishda qulay matematik vosita bo‘lib xizmat qilmoqda. Kvant fizikasining matematik modellarini o‘rganishda matematikaning ushbu yangi sohasidan keng foydalaniladi.

Umumlashgan funksiyalar birinchi bo‘lib P. Dirak tomonidan kiritilgan. U kvant fizikada sistematik ravishda delta-funksiyadan foydalangan. Bu funksiya keyinchalik uning nomi bilan atalib, Dirakning delta-funksiyasi deb yuritiladigan bo‘ldi. Umumlashgan funksiyalar nazariyasi matematik asoslari 1950-yillardan boshlab S.L. Sobolev va L. Shvars ishlarida o‘z aksini topgan. Hozirgi kunda tez rivojlanib borayotgan umumlashgan funksiyalar nazariyasi matematika, fizika va boshqa soha vakillarinining ilmiy tadqiqotlarida asosiy matematik vosita sifatida namoyon bo‘lmoqda. Ushbu darslikning 1-bobi umumlashgan funksiyalar va ularning Furye almashtirishiga bag‘ishlangan bo‘lib, bunda umumlashgan funksiyalar nazariyasining boshlang‘ich tushunchalari misollar asosida keltirilgan. Shuningdek, Dirakning delta-funksiyasini o‘rganishga alohida e’tibor qaratilgan. Qolaversa, bu bobda turli umumlashgan va klassik funksiyalarning Furye almashtirishlarini hisoblash keng o‘rin olgan. Aytish lozim umumlashgan funksiyalar nazariyasi keng bo‘lib, uning elementlari keyingi boblarda qo‘llaniladigan hajmda keltirilgan. 2-bob xususiy hosilali differensial tenglamalarni klassifikatsiyalash va ular uchun asosiy masalalarning qo‘yilishiga bag‘ishlangan. 3-bobda differensial tenglamalarning umumlashgan yechimi tushunchasi, o‘zgarmas koeffitsientli differensial operatorlarning fundamental yechimlarini qurish hamda to‘lqin va issiqlik o‘tkazuvchanlik tenglamalari uchun qo‘yilgan umumlashgan va klassik Koshi masalalari o‘rganiladi. O‘rganishning xususiyati shundan iboratki, umumlashgan Koshi masalasida boshlang‘ich shartlar oniy ta’sir etuvchi manbalarga kiritiladi va bu yechimni manba bilan mos ravishda tanlangan fundamental yechimning yig‘masi ko‘rinishida qurishga imkon beradi. 4-, 5- va 6-boblar uchta klassik: giperbolik, parabolik va elliptik tipdagi tenglamalar uchun qo‘yilgan boshlang‘ich, boshlang‘ich-chegaraviy va chegaraviy masalalarni o‘rganishga qaratilgan. 7-bobda xususiy hosilali differensial tenglamalarni yechishda tez-tez qo‘llaniladigan o‘zgaruvchilarni ajratish, integral almashtirishlar usullari bilan bog‘liq ba’zi maxsus funksiyalar va ularning xossalari keltirilgan.

Darslikning ayrim boblarini yozishda muallif O‘zbekiston Respublikasi "El-Yurt umidi" jamg‘armasi tanlovi g‘olibi sifatida Ispaniyaning Valensiya politexnika universitetida malaka oshirishda bo‘lgan davrida ushbu soha bo‘yicha ingliz tilidagi adabiyotlardan foydalandi. Foydalanilgan adabiyotlar ro‘yxati darslikning oxirida berilgan. Shuningdek, darslikni yozishda muallifning qator yillar davomida Buxoro davlat universitetida o‘qigan ma’ruzalaridan foydalanildi.

Darslik to‘g‘risidagi kitobxonlarning tanqidiy fikr va mulohazalarini muallif mamnuniyat bilan qabul qiladi.

Muallif

1-Bob. Umumlashgan funksiyalar. Furye almashtirishi

1.1 Funksiya berilishining turli xil usullari haqida

Matematik tahlilda, odatda, funksiya analitik usulda beriladi. Masalan, bir o‘zgaruvchili $y = f(x)$ funksiyaning ta’rifini eslatamiz. Agar $X, Y - \mathbb{R}^1$ sonli to‘g‘ri chiziq nuqtalaridan iborat to‘plamlar bo‘lsa, X to‘plamda $y = f(x)$ funksiya berilgan deyiladi, agarda har bir $x \in X$ nuqta bilan yagona $y \in Y$ nuqta o‘rtasida moslik o‘rnatilgan bo‘lsa, boshqacha aytganda, $y = f(x)$ funksiyani berish har bir $x \in X$ uchun uning qiymati $f(x) = y \in Y$ ni ko‘rsatishga teng kuchli. Bunday usulga funksiyani nuqtaviy berish usuli ham deyiladi.

Funksiyani nuqtalar yordamida bermasdan, uning "integral" xarakteristikalaridan foydalanib ham berish mumkin. Masalan, funksiyaning bunday "integral" xarakteristikalari sifatida biror bir $\varphi_k(x)$, $k = 1, 2, \ldots$ funksiyalar sistemasidagi uning Furye koeffitsientlari kelishi mumkin. Ma‘lumki, ixtiyoriy $f(x) \in L_2[a, b]$ funksiya o‘zining Furye koeffitsientlari

$$f_k = (f(x), \varphi_k(x)) = \int\limits_a^b f(x)\varphi_k(x)dx, \ \ k = 1, 2, \ldots$$

yordamida bir qiymatli Furye qatori

$$f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$$

bilan aniqlanadi, bu yerda $\{\varphi_k\}_{k=1}^{\infty} - [a, b]$ kesmada aniqlangan to‘liq ortonormal funksiyalar sinfi.

Xususan, $[0, \pi]$ kesmada uzluksiz $f(x)$ funksiya

$$f_k = \sqrt{\frac{2}{\pi}} \int\limits_0^{\pi} f(x) \sin kx dx, \;\; k = 1, 2, \ldots .$$

sonlar to'plami bilan bir qiymatli aniqlanadi. Demak, $\{f_k\}$ sonlar ketma-ketligining berilishi $[0, \pi]$ kesmaning ixtiyoriy nuqtasida $f(x)$ uzluksiz funksiyaning berilishiga teng kuchli. $\{f_k\}$ sonlarni cheksiz o'lchovli fazodagi $f(x)$ funksiyaning koordinatalari, $\varphi_k(x), k = 1, 2, \ldots$ funksiyalarni esa fazodagi bazis deb qarash mumkin. Geometrik nuqtayi nazardan f_k larni $f(x)$ funksiyaning $\varphi_k(x)$ koordinat funksiyalardagi proeksiyalari deb ham tushunsa bo'ladi.

$f(x)$ funksiyani biror bir usul bilan tanlangan $\{f_k(x)\}$ bazis funksiyalardagi proeksiyalari bilan emas, balki ba'zi kengroq $\{\varphi_k(x)\}$ funksiyalar to'plamidagi proeksiyalarini berish bilan aniqlab bo'lmaydimi, degan savol tug'iladi. Bu savolga keyingi paragraflarda javob beriladi.

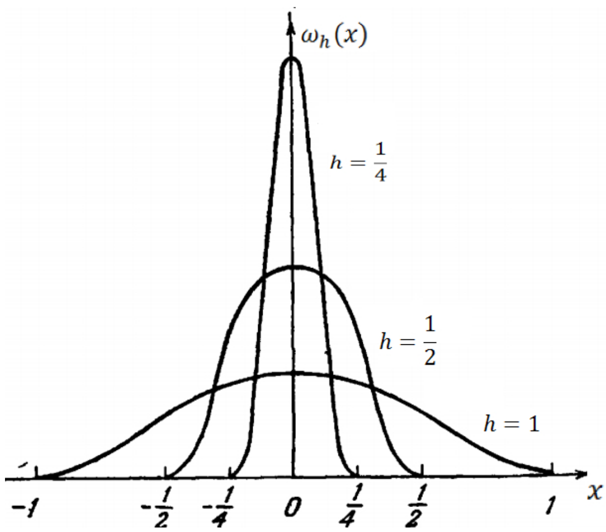
1.2 Asosiy funksiyalar fazosi va uning xossalari

Bir o'lchovli holni qaraylik. Soddalik uchun $\mathbb{R}^1 = \mathbb{R}$ deb olamiz. $C^\infty(\mathbb{R})$ bilan $\mathbb{R}$ da cheksiz marta differensiallanuvchi funksiyalar fazosini belgilaymiz.

T a ' r i f. Biror chegaralangan to'plamdan tashqarida nolga teng bo'lgan funksiya finit deyiladi. Bir o'lchovli funksiya uchun boshqacha aytadigan bo'lsak, agar $\varphi : \mathbb{R} \to \mathbb{R}$ funksiya uchun shunday $[a_\varphi, b_\varphi] \subset \mathbb{R}$ kesma topilib, $x \overline{\in} [a_\varphi, b_\varphi]$ larda $\varphi(x) \equiv 0$ bo'lsa, bu funksiya finit funksiya va $[a_\varphi, b_\varphi]$ kesma uning *tashuv- chisi* deyiladi.

T a ' r i f. Cheksiz marta differensiallanuvchi finit funksiyalar to'plamiga asosiy funksiyalar fazosi deyiladi va u $D(\mathbb{R})$ orqali belgilanadi. Shunday qilib,

$$D(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) | \; [a_\varphi, b_\varphi] \text{ kesmadan tashqarida } \varphi(x) \equiv 0\} .$$



1-chizma. Turli h lar uchun $\omega_h(x)$ funksiyaning grafigi.

M i s o l. Quyidagi funksiyani qaraymiz:

$$\omega_h(x) = \frac{1}{\lambda}\begin{cases} e^{-\frac{h^2}{h^2-x^2}}, & |x| < h, \\ 0, & |x| \geq h, \end{cases} \tag{1}$$

bu yerda o‘zgarmas λ ushbu $\int_{\mathbb{R}} \omega_h(x) = 1$ shartdan tanlanadi, ya’ni

$$\lambda = \left[h \int_{-1}^{1} e^{\frac{1}{t^2-1}} dt \right]^{-1}.$$

Ko‘rinib turibdiki, bu funksiya - finit. Undan tashqari cheksiz marta differen- siallanuvchi (biz buni keyingi paragrafda ko‘rsatib o‘tamiz) bo‘lib, bu funksiya- ning $x = \pm h$ nuqtadagi ixtiyoriy bir tomonlama hosilalari nol ga teng. Shunday qilib, $\omega_h(x)-$ asosiy funksiya. Ba’zan, (1) funksiyaga "shapkacha" funksiyasi deb ham aytiladi. Uning grafigi 1-chizmada keltirilgan.

Asosiy funksiyalar fazosining ba’zi xossalari bilan tanishamiz.

1) ixtiyoriy φ, $\psi \in D(\mathbb{R})$ funksiyalar va α, $\beta \in \mathbb{R}$ sonlarning chiziqli kombinatsiyasi $(\alpha\varphi + \beta\psi) \in D(\mathbb{R})$ bo‘ladi, ya’ni $D(\mathbb{R})-$ chiziqli fazo.

Haqiqatan ham, φ, $\psi \in D(\mathbb{R})$ ekanligidan, $\alpha\varphi + \beta\psi$ funksiya chegaralangan tashuvchiga ega ekanligi hamda bu yig‘indining cheksiz differensiallanuvchi bo‘lishi kelib chiqadi. Quyidagi xossalar ham shunga o‘xshash osongina isbot qilinadi:

2) ixtiyoriy $\varphi \in D(\mathbb{R})$ va $\psi \in C^{\infty}(\mathbb{R})$ funksiyalar uchun, ularning ko‘paytmasi asosiy funksiyadir, ya’ni $\varphi\psi \in D(\mathbb{R})$;

3) ixtiyoriy $\varphi \in D(\mathbb{R})$ funksiya uchun uning $k-$tartibli hosilasi asosiy funksiya bo‘ladi, ya’ni $\varphi^{(k)} \in D(\mathbb{R})$;

4) ixtiyoriy $\varphi \in D(\mathbb{R})$ funksiyaning ixtiyoriy tayin $a \in \mathbb{R}$ nuqtadagi siljishidan hosil bo‘lgan funksiya ham asosiy funksiyadir, ya’ni $\varphi(x \pm a) \in D(\mathbb{R})$.

Yuqoridagi xossalardan foydalanib "shapkacha" funksiya yordamida ko‘plab asosiy funksiyalarni qurish mumkin. Masalan,

$$\varphi(x) = x^2 \sin x\omega_h(x-a) + \frac{d^n(e^x\omega_h(x))}{dx^n}$$

ham asosiy funksiyadir.

Endi, $D(\mathbb{R})$ da yaqinlashish tushunchasini kiritamiz.

T a ’ r i f. Quyidagi shartlar bajarilsin:

1) shunday $[a, b] \subset \mathbb{R}$ kesma topilib, ixtiyoriy $n \in \mathbb{N}$ sonlar uchun $[a, b]$ kesmadan tashqarida $\varphi_n(x) \equiv 0$;

2) ixtiyoriy $k \in \mathbb{N}\cup\{0\}$ sonlar va ixtiyoriy tayin $x \in [a, b]$ nuqtalar uchun, $n \to \infty$ da ushbu $\varphi_n^{(k)}(x) \rightrightarrows \varphi^{(k)}(x)$ ketma-ketliklar tekis yaqinlashadi.

U holda $\{\varphi_n\}_{n=1}^{\infty}$ ketma-ketlik $n \to \infty$ da $\varphi \in D(\mathbb{R})$ funksiyaga yaqinlashadi deyiladi.

M i s o l. Ushbu

$$\varphi_n(x) = \begin{cases} \frac{1}{n}\exp\left(\frac{a^2}{x^2-a^2}\right), & |x| < a \\ 0, & |x| \geq a, \end{cases}$$

funksiyalar ketma-ketligi nolga yaqinlashadi, shuning bilan birgalikda

$$\varphi_n(x) = \begin{cases} \frac{1}{n}\exp\left(\frac{n^2}{x^2-n^2}\right), & |x| < n \\ 0, & |x| \geq n, \end{cases}$$

funksiyalar ketma-ketligi $D(\mathbb{R})$ (asosiy funksiyalar) ma’nosida yaqinlashuvchi emas.

Yuqoridagi ta'rifga o'xshash ravishda, $D(\mathbb{R}^n),\ D(\Omega)$ fazolarda ham yaqinlashish ta'riflari kiritiladi. Bunda $\mathbb{R}^n-$ n o'lchovli haqiqiy sonli evklid fazo va $\Omega-$ unda biror soha.

L e m m a. $[a,b]$ kesmada uzluksiz va finit bo'lgan ixtiyoriy $f(x)$ funksiya ushbu

$$\int_a^b f(x)\varphi(x)dx,\ \ \varphi(x)\in D(\mathbb{R})$$

integral bilan bir qiymatli aniqlanadi.

I s b o t. $f(x)$ funksiyani $[a,b]$ kesmadan tashqarida nolga teng qilib aniqlab, quyidagi funksiyani qaraymiz:

$$f_h(x)=\int_{\mathbb{R}} f(\xi)\omega_h(\xi-x)d\xi,$$

bu yerda $\omega_h(x)-$ 'shapkacha" funksiyasi va $\int_{\mathbb{R}}\omega_h(x)dx=1$. $\omega_h(x)\in\mathbb{R}$ ekanligidan $f_h(x)$ funksiya ixtiyoriy $h>0$ uchun cheksiz marta differensiallanuvchi funksiya va $[a-h,b+h]$ da $f_h(x)\equiv 0$. Shu bilan birga,

$$f_h(x)-f(x)=\int_{-\infty}^{\infty}[f(\xi)-f(x)]\omega_h(\xi-h)d\xi=$$

$$=\int_{-h}^{h}[f(x+t)-f(x)]\omega_h(t)dt,$$

$$|f_h(x)-f(x)|\le\max_{x-h\le\xi\le x+h}|f(\xi)-f(x)|.$$

Bu yerdan, $\{f_h(x)\}$ ning $h\to 0$ da $f(x)$ ga $x\in[a,b]$ lar uchun tekis yaqinlashishi kelib chiqadi.

Lemmani isbotlash uchun ixtiyoriy $\varphi(x)\in D(\mathbb{R})$ uchun

$$\int_a^b f(x)\varphi(x)dx=0$$

tenglikdan $f(x)\equiv 0,\ x\in[a,b]$ kelib chiqishini ko'rsatish yetarli.

$f_h(x) \in D(\mathbb{R})$ ekanligi uchun $\varphi(x)$ funksiya o‘rniga $f_h(x)$ ni olamiz.

$$\int_a^b f(x)f_h(x)dx = 0$$

tenglikda $h \to 0$ da limitga o‘tamiz. $f(x)$ funksiiyaning $[a, b]$ oraliqda uzluksizligidan va oxirgi tenglikdan $f(x) \equiv 0$ ekanligi kelib chiqadi. Lemma isbotlandi.

Shunday qilib, uzluksiz finit $f(x)$ funksiyani nuqtaviy berish va uning $D(\mathbb{R})$ funksiyalar to‘plamidagi "proeksiya" larini berish $f(x)$ funksiyani berishga teng kuchli. Ammo ikkinchi usulda funksiyani berish qaysidir ma’noda qulaydir. Ya’ni u funksiya tushunchasini kengaytirishga, fanga yangi ba’zi nuqtalarda ma’noga ega bo‘lmagan biror funksiyalar to‘plamida o‘zining qiymatlari bilan to‘liq aniqlanadigan funksiyalarni kiritishga imkon beradi.

Bunday funksiyaga misol tariqasida ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ funksiya uchun

$$\int_a^b \delta(x)\varphi(x)dx = \varphi(0)$$

tenglik bilan aniqlanadigan Dirakning $\delta(x)$ delta-funksiyasini keltirish mumkin. $\delta(x)$ funksiya $\omega_h(x)$ ning $h \to 0$ dagi kuchsiz limiti hisoblanadi. Haqiqatdan ham, ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ funksiya uchun

$$\int_{-\infty}^{\infty} \varphi(x)\omega_h(x)dx = \varphi(\theta h)\int_{-\infty}^{\infty} \omega_h(x)dx = \varphi(\theta h),\ -1 < \theta < 1.$$

Bundan,

$$\lim_{h\to 0}\int_{-\infty}^{\infty} \varphi(x)\omega_h(x)dx = \varphi(0) = \int_{-\infty}^{\infty} \delta(x)\varphi(x)dx$$

ekanligi kelib chiqadi.

T a ’ r i f (Uzluksiz funksiyaning tashuvchisi). Uzluksiz φ funksiyaning tashuvchisi deb, ushbu

$$\text{supp}\varphi = \{\overline{x|\ \varphi(x) \neq 0}\}$$

to'plamga aytiladi.

Uzluksiz funksiyaning tashuvchisi tushunchasidan foydalanib, $D(\mathbb{R})$ da yaqinlashish tushunchasini quyidagicha berish ham mumkin.

T a ' r i f ($D(\mathbb{R})$ ma'nosida yaqinlashish). Agar

1) shunday $M > 0$ son topilib, ixtiyoriy $n \in \mathbb{N}$ soni uchun $\text{supp}\varphi_n \subset [-M, M]$ munosabat bajarilsa;

2) ixtiyoriy $p \in \mathbb{N}$ son uchun $\{\varphi_n^{(p)}\}_{n=1}^{\infty}$ funksiyalar ketma-ketligi $\mathbb{R}$ da $\varphi^{(p)}$ ga tekis yaqinlashsa ($\varphi^{(p)}$ - φ ning p tartibli hosilasi), u holda $\{\varphi_n\}_{n=1}^{\infty}$ asosiy funksiyalar ketma-ketligi $n \to \infty$ da φ ga D ma'nosida yaqinlashuvchi deyiladi va qisqacha tarzda quyidagicha belgilanadi:

$$\varphi_n \overset{D}{\to} \varphi.$$

Aynan noldan farqli asosiy funksiyalar mavjudmi, degan savol tug'iladi. Biz yuqorida ko'rgan (1) "shapkacha"

$$\omega_h(x) = \frac{1}{\lambda} \begin{cases} \exp\Big(-\frac{h^2}{h^2-x^2}\Big), & |x| < h \\ 0, & |x| \geq h, \end{cases},$$

$$\lambda = \left[h \int_{-1}^{1} \exp\Big(\frac{1}{t^2-1}\Big) dt \right]^{-1}, \ x \in \mathbb{R}$$

funksiyasi asosiy funksiyaga misol bo'la oladi. Haqiqatan ham, uning finitligi berilishidan ayon va $\text{supp}\omega_h(x) = [-h, h]$. Cheksiz uzluksiz differensiallanuvchi ekanligini ko'rsatamiz. Ravshanki, bu funksiya $x = \pm h$ nuqtalardan tashqari barch nuqtalarda cheksiz differensiallanuvchi bo'ladi. Ixtiyoriy bir tomonlama $x = -h$ da o'ngdan va $x = h$ da chapdan hosilalari nolga teng ekanligiga ishonch hosil qilish uchun $h^2 - x^2 = y^2$ deb belgilab,

$$f(y) = \exp\Big(-\frac{h^2}{y^2}\Big)$$

funksiyaning $y = 0$ nuqtadagi barcha tartibli hosilalarining nolga teng bo'lishini ko'rsatamiz. Haqiqatan ham, $y \neq 0$ da hosilalarni hisoblaymiz:

$$f'(y) = \frac{2h^2}{y^3} \exp\Big(-\frac{h^2}{y^2}\Big),$$

$$f''(y) = -\frac{6h^2}{y^4}\exp\Big(-\frac{h^2}{y^2}\Big) + \frac{4h^4}{y^6}\exp\Big(-\frac{h^2}{y^2}\Big),$$

umuman,

$$f^{(n)}(y) = P_n\Big(\frac{1}{y}\Big)\exp\Big(-\frac{h^2}{y^2}\Big),$$

$P_n\Big(1/y\Big)$– $1/y$ ga nisbatan ko‘phad:

$$P_n\Big(\frac{1}{y}\Big) = \sum_{k=1}^{n}\frac{\lambda_{m_k}}{y^{n+2k}},$$

bu yerda $m_k \in \mathbb{N}$, λ_{m_k} - h ga bog‘liq koeffitsientlar.

Quyidagi $\lim\limits_{y\to 0} f^{(n)}(y)$ limitni hisoblab,

$$\lim_{y\to 0} f^{(n)}(y) = \lim_{y\to 0} P_n\Big(\frac{1}{y}\Big)\exp\Big(-\frac{h^2}{y^2}\Big) = 0$$

bo‘lishini ko‘rish qiyin emas ($y \to 0$ da $\exp\Big(-\frac{h^2}{y^2}\Big)$ nolga tez yaqinlashadi).

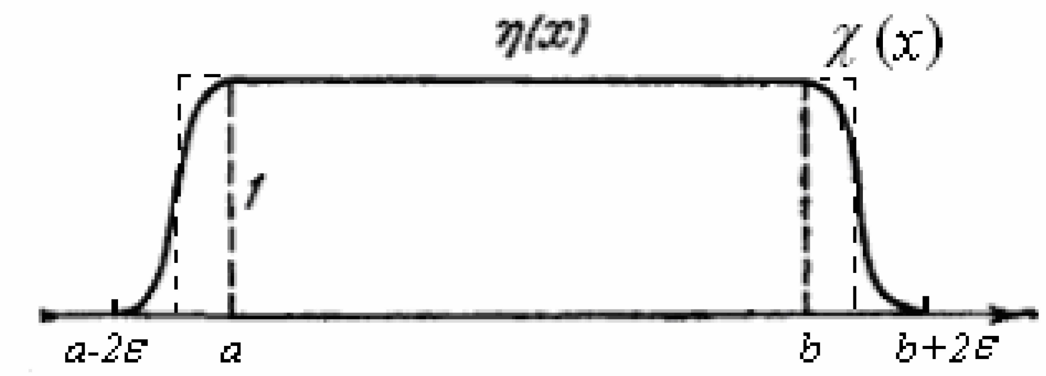
$$\lim_{y\to 0}\exp\Big(-\frac{h^2}{y^2}\Big) = 0 = f(0)$$

tenglikdan $f(y)$ funksiyaning $y = 0$ nuqtada uzluksizligi va yuqoridagilarga asosan bu nuqtada differensiallanuvchanligi hamda $f'(0) = 0$ ekanligi kelib chiqadi. Shunday qilib, f' funksiya $y = 0$ nuqtada uzluksizdir. Xuddi shunga o‘xshash $f''(y)$ funksiya uchun yuqoridagi mulohazalarni takrorlab, $y = 0$ nuqtada $f''(y)$ ning mavjud ekanligi va $f''(0) = 0$ hamda $f''(y)$ ning $y = 0$ da uzluksiz ekanligiga ega bo‘lamiz. Bu jarayonni davom ettirib, ixtiyoriy $n = 1, 2, \ldots$ lar uchun $f^{(n)}(y)$ funksiya yuqoridagi xossalarga ega bo‘lishiga ishonch hosil qilish mumkin. Demak, $\omega_h(x) \in C^\infty(\mathbb{R}),\ \mathrm{supp}\omega_h(x) = [-h, h]$ ekanligidan ω_h funksiyaning finitligi va, umuman, $\omega_h(x) \in D(\mathbb{R})$ kelib chiqadi.

Quyidagi lemma asosiy funksiyalarga ko‘plab misollar keltirish mumkinligini ko‘rsatadi.

L e m m a. Ixtiyoriy $[a, b]$ kesma va $h > 0$ soni uchun $0 \le \eta(x) \le 1;\ \eta(x) = 1,\ x \in [a, b],\ \eta(x) = 0,\ x\overline{\in}[a-2h, b+2h]$ shartlarni qanoatlantiruvchi $\eta(x) \in C^\infty(\mathbb{R})$ funksiya mavjud.

$\eta(x)$ funksiya grafigi 2-chizmada tasvirlangan.



2-chizma. $\eta(x)$ funksiya grafigi.

I s b o t. Faraz qilaylik, $\chi(x)-[a-h,b+h]$ kesmaning xarakteristik funksiyasi bo'lsin, ya'ni

$$\chi(x)=\begin{cases}1, & x\in[a-h,b+h],\\ 0, & x\overline{\in}[a-h,b+h].\end{cases}$$

U holda $\eta(x)=\int\limits_{\mathbb{R}}\chi(y)\omega_h(x-y)dy\in D(\mathbb{R})$, ya'ni bu funksiya talab etilayotgan xossalarga ega. Haqiqatan ham,

$$\omega_h\in D,\ 0\leq\omega_h(x),\ \mathrm{supp}\omega_h(x)=[-h,h]\ \int\limits_{\mathbb{R}}\omega_h(x)dx=1$$

ekanligidan

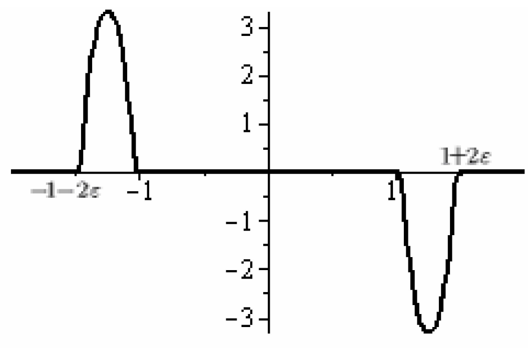
$$\eta(x)=\int\limits_{a-h}^{b+h}\omega_h(x-y)dy\in C^{\infty}(\mathbb{R});$$

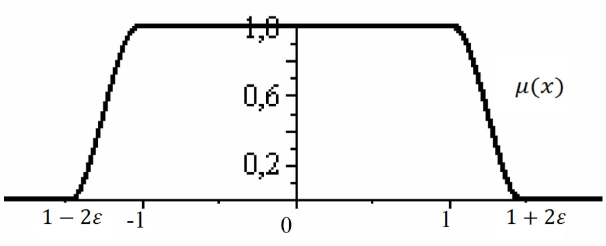
$$0\leq\eta(x)\leq\int\limits_{-\infty}^{\infty}\omega_h(x-y)dy=\int\limits_{-\infty}^{\infty}\omega_h(\xi)d\xi=1;$$

Tayin x lar uchun $\mathrm{supp}\omega_h(x-y)=[x-h,x+h]$ ekanligidan

$$\eta(x)=\int\limits_{x-h}^{x+h}\chi(y)\omega_h(x-y)dy=$$

$$=\begin{cases}\int\limits_{x-h}^{x+h}\omega_h(x-y)dy=\int_{-h}^{h}\omega_h(\xi)d\xi=1, & x\in[a,b],\\ 0, & x\overline{\in}[a-2h,b+2h].\end{cases}$$





3a-chizma. $g(x)$ funksiya grafigi. 3b-chizma. $\mu(x)$ funksiya grafigi.

Lemma isbotlandi.

Bunday funksiyalarni qurishning boshqa usuli ham mavjud. Masalan, ushbu

$$g(x) = \omega_h(x + 1 + h) - \omega_h(-x + 1 + h)$$

funksiyani qaraylik. Uning grafigi 3a-chizmada keltirilgan. Endi $\mu(x) = \int\limits_{-\infty}^{x} g(\tau)d\tau$ funksiyani quramiz. Uning grafigi 3b-chizmada tasvirlangan.

Ko‘rinib turibdiki, $\mu(x)$ funksiya cheksiz differensiallanuvchi va $[-1, 1]$ kesmada birga teng bo‘lgan qiymatni qabul qiladi. Agar

$$g_{a,b}(x) = \omega_h(x + a + h) - \omega_h(-x + a + h)$$

funksiya uchun

$$\mu_{a,b}(x) = \int\limits_{-\infty}^{x} g_{a,b}(\tau)d\tau$$

funksiyani qarasak, u holda u $[a, b]$ kesma uchun $\eta(x)$ funksiyaning yuqoridagi lemmada keltirilgan barcha xossalariga ega bo‘ladi.

Osongina payqash mumkinki, $\eta(x)$ funksiyaning finit bo‘lmagan cheksiz differensiallanuvchi funksiyaga ko‘paytmasi ham $D(\mathbb{R})$ ga tegishli bo‘ladi. Ko‘rilgan misollar $D(\mathbb{R})$ to‘plam yetarlicha ko‘p funksiyalarga ega ekanligini ko‘rsatadi.

Turli xil masalalar uchun funksiyalar ketma-ketligi qaralganda yaqinlashishning har xil ta’riflari bulan ish ko‘rishga to‘g‘ri keladi. Shuning uchun ko‘p ishlatiladigan yaqinlashishlarning ta’riflarini eslatamiz.

$$\{u_n(x)\} = \{u_1(x), u_2(x), ..., u_n(x)\} \qquad (2)$$

ketma - ketligi (a, b) oraliqda tekis yaqinlashadi deyiladi, agarda ixtiyoriy $\varepsilon > 0$ soni uchun shunday N sonini ko‘rsatish mumkin bo‘lsa, bunda $n, m > N$ natural son va ixtiyoriy $x \in (a, b)$ lar uchun

$$|u_n(x) - u_m(x)| < \varepsilon$$

shart bajarilsa.

(2) funksiyalar ketma - ketligi (a, b) oraliqda o‘rta kvadratik ma’noda yaqinlashadi deyiladi, agarda ixtiyoriy $\varepsilon > 0$ soni uchun shunday N sonini ko‘rsatish mumkin bo‘lsa, bunda $n, m > N$ natural son va ixtiyoriy $x \in (a, b)$ lar uchun

$$\int_a^b [u_n(x) - u_m(x)]^2 \, dx < \varepsilon$$

tengsizlik bajarilsa.

(2) funksiyalar ketma-ketligi (a, b) oraliqda kuchsiz yaqinlashadi deyiladi, agarda ixtiyoriy uzluksiz $f(x)$ funksiya uchun

$$\lim_{n \to \infty} \int_a^b f(x) u_n(x) dx$$

limit mavjud bo‘lsa.

(a, b) oraliqda aniqlangan uzluksiz funksiyalar sinfini qaraymiz. Yaqinlashuvchi ketma-ketliklar o‘rganilganda, odatda, limit elementlar kiritiladi. Tekis yaqinlashishda limit element ham aynan shu funksiyalar sinfiga tegishli bo‘ladi. Biroq bunday holat o‘rta kvadratik ma’noda va kuchsiz yaqinlashishlarda hamma vaqt ham o‘rinli bo‘lavermaydi. Agar limit element qaralayotgan funksiyalar sinfiga tegishli bo‘lmasa, u holda dastlabki sinf kengaytirilib, limit elementlar bu sinfga kiritiladi. Bunda kengaytma sinf sifatida dastlabki va limit elementlar to‘plami tushuniladi. Kengaytma tushunchasi bilan, masalan, haqiqiy sonlar nazariyasida duch kelish mumkin: irratsional sonlar ekvivalent ketma-ketligi sinfi orqali aniqlanuvchi limit elementlar tarzida kiritiladi. Kuchsiz ma’noda yaqinlashishdagi limit elementlar haqida fikr yuritganda, ikkita $\{u_n(x)\}$, $\{v_n(x)\}$ ketma-ketliklar bitta limit elementga yaqinlashadi deyiladi, agarda bu ketma-ketliklar ekvivalent bo'lsa, ya'ni quyidagi $\{u_n(x) - v_n(x)\}$ ketma-ketlik nolga kuchsiz yaqinlashsa:

$$\lim_{n\to\infty} \int\limits_a^b [u_n(x) - v_n(x)]^2 \, dx = 0.$$

1.3 Umumlashgan funksiya tushunchasi. Regulyar va singulyar funksiyalar

Quyidagi ta'riflarni kiritamiz:

T a ' r i f. $f : D(\mathbb{R}) \to \mathbb{C}-$ akslantirishga $D(\mathbb{R})$ da funksional deyiladi, bu yerda $\mathbb{C}$ - kompleks sonlar maydoni.

f funksionalning $\varphi \in D(\mathbb{R})$ funksiyadagi qiymati (f, φ) bilan belgilanadi. Ta'rifga asosan $(f, \varphi)-$ kompleks son.

T a ' r i f. Agar ixtiyoriy $\varphi, \psi \in D(\mathbb{R})$ va $\alpha, \beta \in \mathbb{C}$ lar uchun

$$(f, \alpha\varphi + \beta\psi) = \alpha(f, \varphi) + \beta(f, \psi)$$

tenglik o'rinli bo'lsa, $f : D(\mathbb{R}) \to \mathbb{C}$ funksional chiziqli deyiladi.

T a ' r i f. Agar $D(\mathbb{R})$ da nolga intiluvchi ixtiyoriy funksiyalar ketma-ketligi $\{\varphi_k(x)\} \in D(\mathbb{R})$ uchun $k \to \infty$ da $(f, \varphi_k) \to 0$ bo'lsa, $f : D(\mathbb{R}^n) \to \mathbb{C}$ funksional uzluksiz deyiladi.

T a ' r i f. Chiziqli, uzluksiz $f : D(\mathbb{R}) \to \mathbb{C}$ funksionalga umumlashgan funksiya deyiladi.

$D' = D'(\mathbb{R})$ orqali barcha umumlashgan funksiyalardan tuzilgan to'plamni belgilaymiz.

Agar ixtiyoriy $\mu, \chi \in \mathbb{C}$ sonlar va umumlashgan f va g funksiyalarning $\mu f + \chi g$ chiziqli kombinatsiyasini ixtiyoriy $\varphi \in D$ lar uchun

$$(\mu f + \chi g, \varphi) = \mu(f, \varphi) + \chi(g, \varphi)$$

tenglik bilan aniqlansa, $D'(\mathbb{R})$ to'plam chiziqli bo'ladi.

$\mu f + \chi g$ funksionalning D da chiziqli va uzluksiz ekanligini ko‘rsatamiz. Haqiqatan ham, agar $\varphi \in D$, $\psi \in D$ va α, β - kompleks sonlar bo‘lsa, u holda ta’rifga ko‘ra

$$(\mu f + \chi g, \alpha\varphi + \beta\psi) = \mu\,(f, \alpha\varphi + \beta\psi) + \chi\,(g, \alpha\varphi + \beta\psi) =$$

$$= \alpha\,[\mu(f,\varphi) + \chi(g,\varphi)] + \beta\,[\mu(f,\psi) + \chi(g,\psi)] =$$

$$= \alpha(\mu f + \chi g, \varphi) + \beta(\mu f + \chi g, \psi)$$

tengliklarga ega bo‘lamiz. Bu esa, $\mu f + \chi g$ funksionalning chiziqli ekanligini anglatadi. Uzluksizligi esa, f, g funksionallarning uzluksizligidan kelib chiqadi: agar $\varphi_k \in D$ ketma-ketlik $k \to \infty$ da 0 ga intilsa, u holda $k \to \infty$ da

$$(\mu f + \chi g, \varphi_k) = \mu\,(f, \varphi_k) + \chi\,(g, \varphi_k) \to 0$$

bo‘ladi.

D' da yaqinlashishni *kuchsiz yaqinlashish* kabi kiritamiz: Agar ixtiyoriy $\varphi \in D$ uchun $k \to \infty$ da $(f_k, \varphi) \to (f, \varphi)$ bo‘lsa, umumlashgan funksiyalar $f_k \in D'$ ketma - ketligi $f \in D'$ funksiyaga yaqinlashadi deyiladi. Bunday holda, D' da $f_k \to f$, $k \to \infty$ kabi yoziladi. D' chiziqli to‘plam unda aniqlangan yaqinlashish bilan D' *umumlashgan funksiyalar fazosi* deyiladi.

Lokal integrallanuvchi funksiyaning ta’rifini $n-$ o‘lchovli funksiya uchun keltiramiz.

T a ’ r i f. $f : \mathbb{R}^n \to \mathbb{R}$ funksiya ixtiyoriy chegaralangan $Q \subset \mathbb{R}^n$ to‘plam uchun u Q da absolyut integrallanuvchi bo‘lsa, u $\mathbb{R}$ da lokal integrallanuvchi deyiladi.

Bu ta’rif yanada tushunarli bo‘lishi uchun uni bir o‘lchovli hol uchun quyidagicha ta’riflaymiz: $\mathbb{R}$ da barcha lokal integrallanuvchi funksiyalar to‘plamini $\mathrm{L}_1^{loc}(\mathbb{R})$ ko‘rinishda belgilaymiz, u holda

$$\mathrm{L}_1^{loc}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : \int_a^b |f(x)|dx < \infty\},$$

bu yerda $a, b-$ chekli haqiqiy sonlar.

Keyinchalik biz qiymatlari haqiqiy sonlar to‘plami $\mathbb{R}$ ga tegishli bo‘lgan funksionallarni qaraymiz.

M i s o l. Faraz qilaylik, $f(x)$ $\mathbb{R}$ da lokal integrallanuvchi funksiya bo‘lsin. U holda ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$(f, \varphi) = \int\limits_{\mathbb{R}} f(x)\varphi(x)dx \tag{3}$$

funksional chiziqli va uzluksizdir. Shuning uchun bu funksional umumlashgan funksiyadir.

T a ’ r i f. Lokal integrallanuvchi funksiya yordamida quriladigan umumlashgan funksiya *regulyar umumlashgan funksiya* (yoki *regulyar funksiya*) deyi- ladi.

T e o r e m a (Umumlashgan regulyar funksiyaning umumiy ko‘rinishi). (3) ko‘rinishdagi funksional (regulyar) umumlashgan funksiyadir.

I s b o t. f akslantirishni chiziqlilikka tekshiramiz. Ixtiyoriy $\varphi_1, \varphi_2 \in D(\mathbb{R})$ va $\alpha_1, \alpha_2 \in \mathbb{C}$ lar uchun

$$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) = \int\limits_{\mathbb{R}} f(x)(\alpha_1(x)\varphi_1(x) + \alpha_2(x)\varphi_2(x))dx =$$

$$= \alpha_1 \int\limits_{\mathbb{R}} f(x)\varphi_1(x)dx + \alpha_2 \int\limits_{\mathbb{R}} f(x)\varphi_2(x)dx = \alpha_1(f, \varphi_1) + \alpha_2(f, \varphi_2).$$

Endi f akslantirishning uzluksiz ekanligini ko‘rsatamiz. Faraz qilaylik, 0 ga intiluvchi $\{\varphi_n\}_{n=1}^{\infty}$ asosiy funksiyalar ketma-ketligi berilgan bo‘lsin. D dagi yaqinlashish ta’rifiga ko‘ra, shunday $M > 0$ soni mavjudki, $\{\varphi_n\}_{n=1}^{\infty}$ ketma-ketlikdan barcha funksiyalarining tashuvchisi $[-M, M]$ kesmada yotadi. $f \in \mathrm{L}_1^{loc}(\mathbb{R})$ shartdan quyidagi o‘rinli,

$$0 < P \stackrel{\text{def}}{:=} \int\limits_{-M}^{M} |f(x)|dx < \infty.$$

Endi, istalgancha kichik $\epsilon > 0$ soni berilgan bo‘lsin. $\{\varphi_n\}_{n=1}^{\infty}$ ning tekis yaqinlashuvchi ekanligidan shunday $N \in \mathbb{N}$ son topiladiki, $n > N$ lar uchun

$$\max_{x \in [-M,M]} |\varphi_n(x)| < \frac{\epsilon}{P+1}$$

baho bajariladi. Bundan esa, ixtiyoriy $n > N$ larda

$$|(f,\varphi_n)| = \left|\int\limits_{\mathbb{R}} f(x)\varphi_n(x)dx\right| = \left|\int\limits_{-M}^{M} f(x)\varphi_n(x)dx\right| \leq \int\limits_{-M}^{M} |f(x)||\varphi_n(x)|dx \leq$$

$$\leq \max_{x\in[-M,M]} |\varphi_n(x)| \int\limits_{-M}^{M} |f(x)|dx < \frac{\epsilon}{P+1}\cdot P < \epsilon$$

tengsizlik o‘rinli bo‘ladi.

T a ’ r i f. Regulyar bo‘lmagan umumlashgan funksiyaga *singulyar umumlashgan funksiya* deyiladi.

M i s o l. Ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ funksiyalarda $(\delta(x), \varphi(x)) = \varphi(0)$ tenglik bilan aniqlangan $\delta(x)$ funksionalni qaraymiz. Bu funksional chiziqli va uzluksizdir. Demak, u umumlashgan funksiyani aniqlaydi. $\delta(x)$ umumlashgan funksiya singulyardir. Uni (3) ko‘rinishda hech bir lokal integrallanuvchi f funksiya yordamida ifodalab bo‘lmasligini ko‘rsatamiz. Faraz qilaylik, bunday funksiya mavjud bo‘lsin. U holda, ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$(\delta,\varphi) = \int\limits_{\mathbb{R}} f(x)\varphi(x)dx$$

tenglik o‘rinli, ya’ni

$$\int\limits_{\mathbb{R}} f(x)\varphi(x)dx = \varphi(0).$$

$\varphi(x) = \omega_h(x)$ deb, ushbu

$$\int\limits_{\mathbb{R}} f(x)\omega_h(x)dx = \omega_h(0) = \frac{1}{e}$$

ifodaga ega bo‘lamiz. $f(x)$ lokal integrallanuvchi ekanligi uchun

$$\lim_{h\to 0}\int\limits_{-h}^{h} |f(x)|dx = 0.$$

Bu tenglikdan $\omega_h(x) \leq \frac{1}{e}$, $x \in (-h, h)$ ni hisobga olib, quyidagilarga ega bo‘lamiz:

$$\frac{1}{e} = \left| \int\limits_{\mathbb{R}} f(x)\varphi(x)dx \right| \leq \int\limits_{-h}^{h} |f(x)||\varphi(x)|dx \leq \frac{1}{e} \int\limits_{-h}^{h} |f(x)|dx.$$

Bu yerda $h \to 0$ da limitga o‘tsak, $\frac{1}{e} \leq 0$ ko‘rinishdagi ziddiyatga ega bo‘lamiz. Demak, δ funksiyani (3) ko‘rinishida ifodalab bo‘lmas ekan. $\delta(x)$ funksiya "shapkacha" $\omega_h(x)$ funksiyasining $h \to 0$ da umumlashgan funksiyalar fazosidagi limiti ekanligini isbotlash qiyin emas. Buning uchun, ixtiyoriy $\varphi \in D$ uchun

$$\lim_{h\to 0} \int\limits_{\mathbb{R}} \omega_h(x)\varphi(x)dx = \varphi(0)$$

tenglikning bajarilishini ko‘rsatish kifoya. Haqiqatan ham, $\varphi(x)$ funksiyaning uzluksizligidan, ixtiyoriy $\eta > 0$ soni uchun shunday $\epsilon_0 > 0$ soni mavjudki, barcha $\{x : |x| < \epsilon\}$ lar uchun $|\varphi(x) - \varphi(0)| < \eta$ tengsizlik o‘rinli. $\omega_h(x)$ funksiyaning xossasidan foydalanib,

$$\left| \int\limits_{\mathbb{R}} \omega_h(x)\varphi(x)dx - \varphi(0) \right| \leq \int\limits_{\mathbb{R}} \omega_h(x)|\varphi(x) - \varphi(0)|dx \leq \eta \int\limits_{\mathbb{R}} \omega_h(x)dx = \eta$$

ifodaga ega bo‘lamiz. Bu esa yuqoridagi tenglikni isbotlaydi. Formal kelishuvga asosan $\delta(x)$ funksiya uchun

$$(\delta(x), \varphi(x)) = \int\limits_{\mathbb{R}} \delta(x)\varphi(x)dx$$

yozuv ishlatiladi. Bunga asosan, ixtiyoriy $\varphi \in D$ uchun:

$$\int\limits_{\mathbb{R}^n} \delta(x)\varphi(x)dx = \varphi(0). \tag{4}$$

Eslatma. (4) tenglik $x = 0$ nuqtada uzluksiz ixtiyoriy $\varphi(x)$ funksiya uchun ham o‘rinli.

M i s o l. Ushbu ixtiyoriy $\varphi \in D$ uchun

$$\left(\rho\frac{1}{x}, \varphi\right) = v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x}dx \stackrel{\text{def}}{=} \lim_{\epsilon\to +0} \int\limits_{|x|>\epsilon} \frac{\varphi(x)}{x}dx$$

tenglik bilan aniqlangan chiziqli $\rho\frac{1}{x}$ funksionalni qaraymiz. Bu funksionalning uzluksizligini isbot qilish uchun $D(\mathbb{R})$ da $\varphi_k \to 0$, $k \to \infty$ ketma-ketlikni olamiz. Bu ketma-ketlik uchun $\varphi_k(x) = 0$, $|x| > M$ va $D^\alpha\varphi_k(x) \rightrightarrows 0$, $k \to \infty$ munosabatlar o‘rinli. Shu bilan birga

$$\left|\left(\rho\frac{1}{x}, \varphi_k\right)\right| = \left|v.p. \int\limits_{\mathbb{R}} \frac{\varphi_k(x)}{x} dx\right| =$$

$$= \Big|[-M, M]\,\text{kesmadan tashqarida}\,\varphi(x) \equiv 0\Big| =$$

$$= \left|v.p. \int\limits_{\mathbb{R}} \frac{\varphi_k(x) - \varphi_k(0) + \varphi_k(0)}{x} dx\right| \overset{\text{Lagranj formulasiga ko‘ra}}{=}$$

$$= \left|v.p. \int\limits_{\mathbb{R}} \frac{x\varphi_k'(\xi)}{x} dx + v.p. \int\limits_{\mathbb{R}} \frac{\varphi_k(0)}{x} dx\right| \le$$

$$\le \int\limits_{-M}^{M} |\varphi_k'(\xi)| dx \le 2M \max_{|x| \le M} |\varphi_k'(x)| \to 0,\ k \to \infty.$$

Demak, $\rho\frac{1}{x}-$ umumlashgan funksiya. Uni (3) ko‘rinishda lokal integrallanuvchi funksiya yordamida ifodalab bo‘lmasligini oldingi misoldagi kabi osongina ko‘rsatish mumkin. $\rho\frac{1}{x}-$funksiya ham $\delta(x)$ kabi singulyar umumlashgan funksiyadir. Unga $\frac{1}{x}$ integralining bosh qiymati deyiladi.

M i s o l.

$$v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x} dx = \int_{-M}^{M} \frac{\varphi(x) - \varphi(0)}{x} dx$$

tenglikni isbotlaymiz. $\varphi(x)$ funksiyaning finitligidan

$$v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x} dx = v.p. \int\limits_{-M}^{M} \frac{\varphi(x)}{x} dx =$$

$$= v.p. \int\limits_{-M}^{M} \frac{\varphi(x) - \varphi(0)}{x} dx + \varphi(0) v.p. \int\limits_{-M}^{M} \frac{1}{x}.$$

Oxirgi tenglikning o‘ng tomonidagi birinchi integralda bosh qiymat belgisini tashlab yuborish mumkin, ikkinchi integral esa nolga teng.

Yuqoridagiga o‘xshash $\rho\frac{1}{x^n}$ funksionalning ham umumlashgan funksiya ekanligini ko‘rsatish mumkin. Buning uchun, quyidagi munosabatdan foydalanish yetarli: ixtiyoriy $n \in \mathbb{N}$ va $\varphi \in D$ lar uchun

$$\left(\rho\frac{1}{x^n}, \varphi\right) =$$

$$= v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x) - \sum\limits_{k=0}^{n-2} \frac{\varphi^{(k)}(0)}{k!}x^k}{x^n} dx \stackrel{\text{def}}{=} \lim_{\epsilon\to+0} \int\limits_{|x|>\epsilon} \frac{\varphi(x) - \sum\limits_{k=0}^{n-2} \frac{\varphi^{(k)}(0)}{k!}x^k}{x^n} dx.$$

Quyidagi, ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun bajariladigan muhim tenglikni isbot qilamiz:

$$\lim_{\epsilon\to+0} \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x+i\epsilon} dx = -i\pi\varphi(0) + v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x}.$$

Haqiqatan ham, agar $|x| > M$ lar uchun $\varphi(x) \equiv 0$ bo‘lsa, u holda

$$\lim_{\epsilon\to+0} \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x+i\epsilon} dx = \lim_{\epsilon\to+0} \int\limits_{-R}^{R} \frac{x-i\epsilon}{x^2+\epsilon^2}\varphi(x)dx =$$

$$= \varphi(0) \lim_{\epsilon\to+0} \int\limits_{-R}^{R} \frac{x-i\epsilon}{x^2+\epsilon^2}\varphi(x)dx + \lim_{\epsilon\to+0} \int\limits_{-R}^{R} \frac{x-i\epsilon}{x^2+\epsilon^2}[\varphi(x)-\varphi(0)]dx =$$

$$= -2i\varphi(0) \lim_{\epsilon\to+0} \operatorname{arctg}\frac{R}{\epsilon} + \int\limits_{-R}^{R} \frac{\varphi(x)-\varphi(0)}{x} dx =$$

$$= -i\pi\varphi(0) + v.p. \int\limits_{\mathbb{R}} \frac{\varphi(x)}{x} dx.$$

Demak, $\frac{1}{x+i\epsilon}$ ketma-ketlikning $\epsilon \to +0$ dagi limiti $D(\mathbb{R})$ da mavjud ekan. Uni $\frac{1}{x+i0}$ bilan belgilasak, bu limit $-i\pi\delta(x) + \rho\frac{1}{x}$ ga teng bo‘lar ekan. Shunday qilib,

$$\frac{1}{x+i\epsilon} = -i\pi\delta(x) + \rho\frac{1}{x}.$$

Xuddi shunga o‘xshash

$$\frac{1}{x - i\epsilon} = i\pi\delta(x) + \rho\frac{1}{x}.$$

Oxirgi ikkita tenglik kvant fizikasida keng ishlatiladi va ular *Soxotskiy formulalari* deyiladi.

T a ’ r i f. Ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$, $\mathrm{supp}\varphi(x) \subset Q$ uchun $(f, \varphi) = 0$ bo‘lsa, umumlashgan $f(x)$ funksiya Q ochiq to‘plamda nolga teng deyiladi.

Bu ta’rif umumlashgan funksiyaning tashuvchisi tushunchasini kiritishga imkon beradi.

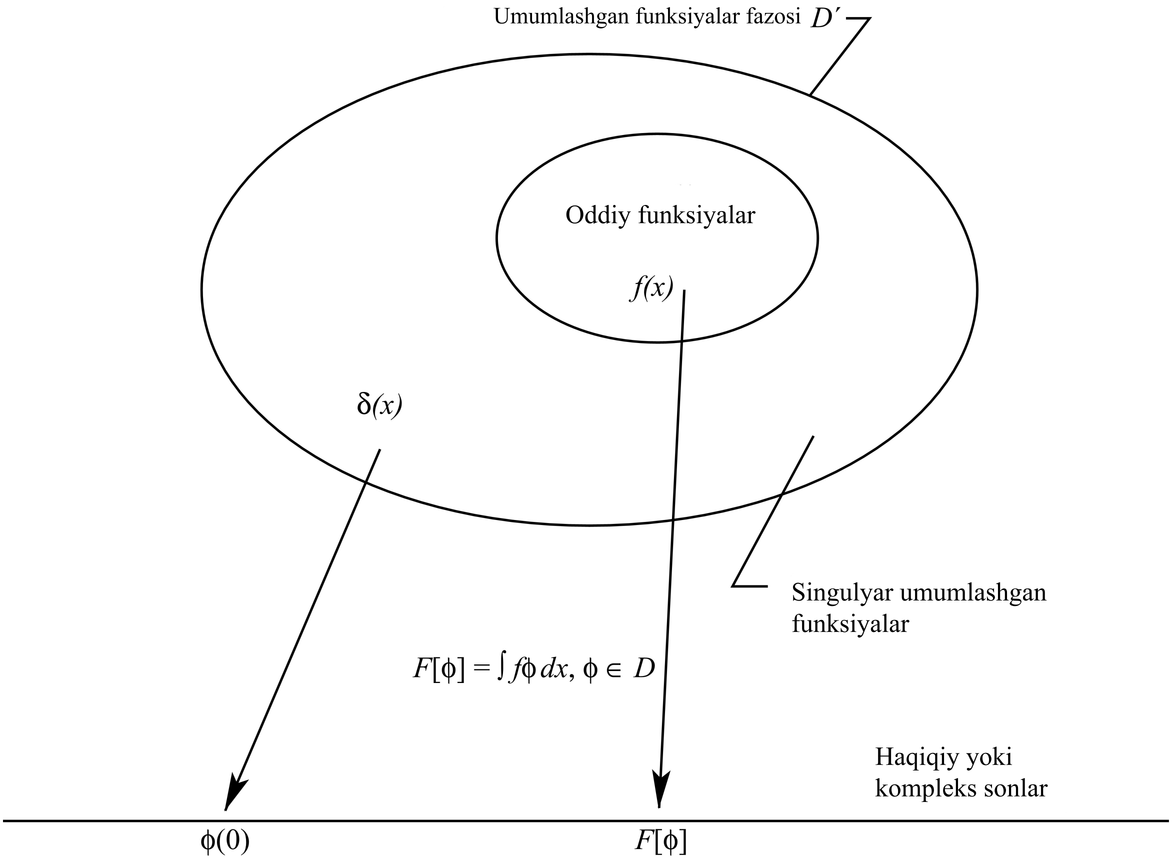
T a ’ r i f. $f(x) = 0$ tenglik bajariladigan barcha ochiq $Q \subset \mathbb{R}^n$ to‘plamlar birlashmasiga $f(x)$ funksiyaning nol to‘plami deyiladi, uning $\mathbb{R}^n$ gacha to‘ldirmasi umumlashgan funksiyaning tashuvchisi deyiladi va $\mathrm{supp}f(x)$ kabi belgilanadi.

Ta’rifdagi $\mathrm{supp}f(x)$ to‘plamni quyidagicha yozish mumkin.

$$\mathrm{supp}\varphi = \mathbb{R}^n / \{Q : \varphi|_Q = 0\} \tag{5}$$

M i s o l. $\delta(x) = \begin{cases} +\infty, \ x = 0 \\ 0, \quad x \neq 0, \end{cases}$ funksiyani qaraymiz. Anglash qiyin emaski, bu funksiyaning nolga aylanadigan nuqtalar to‘plami $(-\infty, 0) \cup (0, \infty)$. U holda (5) ga ko‘ra $\mathrm{supp}\delta(x) = \{0\}$.

T a ’ r i f. Agar umumlashgan n - o‘zgaruvchili $f(x)$ funksiyaning tashuvchisi chegaralangan bo‘lsa, u $\mathbb{R}^n$ fazoda finit deyiladi.



4-chizma. Klassik va umumlashgan funksiyalar to‘plamlari.

1.4 "$\delta-$ shaklli" ketma - ketliklar. Misollar

T a ’ r i f. Agar

1) har bir $k \in \mathbb{N}$ uchun $h_k : \mathbb{R}^n \to \mathbb{R}$ funksiya $\mathbb{R}^n$ da integrallanuvchi;
2) ixtiyoriy $x \in \mathbb{R}^n$ va $k \in \mathbb{N}$ lar uchun $h_k \geq 0$ tengsizlik o‘rinli;
3) har bir $k \in \mathbb{N}$ uchun shunday $\varepsilon_k > 0$ sonlar topilib, bunda $\varepsilon_k \to 0, k \to \infty$ da, $|x| > \varepsilon_k$ tengsizliknini qanoatlantiruvchi barcha $x \in \mathbb{R}^n$ lar uchun $h_k = 0$ o‘rinli bo‘ladi, bu yerda $k \to \infty$ da $\varepsilon \to 0$;
4) barcha $k \in \mathbb{N}$ lar uchun

$$\int\limits_{\mathbb{R}^n} h_k(x)dx = 1$$

bo‘lsa, $\mathbb{R}^n$ fazoda aniqlangan haqiqiy o‘zgaruvchili funksiyalarning

$$h_1, h_2, \ldots, h_k, \ldots$$

ketma-ketligi "$\delta-$ shaklli" deyiladi.

T e o r e m a. Aytaylik,

$$h_1, h_2, \ldots, h_k, \ldots$$

funksiyalar ketma ketligi "$\delta-$ shaklli" bo‘lsin. U holda,

$$\lim_{k\to\infty} h_k(x) = \delta(x)$$

bo‘ladi, bu yerda limit umumlashgan funksiya ma’nosida tushuniladi, ya’ni $D'(\mathbb{R}^n)$ da.

I s b o t. Faraz qilaylik, $\varphi-$ ixtiyoriy asosiy funksiya bo‘lsin. Umumlashgan funksiyalar ketma-ketligi limiti ta’rifidan va h_k asosiy funsiyaga integral yordamida ta’sir qiluvchi regulyar umumlashgan funksiya ekanidan foydalanib, quyidagi tenglikni yozishimiz mumkin:

$$\left(\lim_{k\to\infty} h_k(x), \varphi(x)\right) = \lim_{k\to\infty} (h_k(x), \varphi(x)) = \lim_{k\to\infty} \int\limits_{\mathbb{R}^n} h_k(x)\varphi(x)dx =$$

$$= \lim_{k\to\infty} \int\limits_{|x|<\varepsilon_k} h_k(x)\varphi(x)dx. \tag{6}$$

Bizga, matematik tahlil kursidan ma’lum bo‘lgan o‘rta qiymat haqidagi teoremaga ko‘ra va "$\delta-$shaklli" ketma-ketlikning 3 va 4 - xossalaridan foydalanib, (6) formuladagi oxirgi ifodani o‘zgartiramiz:

$$\lim_{k\to\infty} \varphi(x_k) \int\limits_{|x|<\varepsilon_k} h_k(x)dx = \lim_{k\to\infty} \varphi(x_k) \int\limits_{\mathbb{R}^n} h_k(x)dx =$$

$$= \lim_{k\to\infty} \varphi(x_k) = \varphi(0) = (\delta, \varphi),$$

bu yerda $x_k \in \{|x| < \varepsilon_k\}$ sharning biror nuqtasi.

Shu tariqa $\lim\limits_{k\to\infty} h_k$ va δ umumlashgan funksiyalar istalgan asosiy $\varphi(x)$ funksiyaga bir xilda ta’sir qiladi. O‘z-o‘zidan bu umumlashgan funksiyalar ustma-ust tushadi. Teorema isbot bo‘ldi.

M i s o l l a r.

I. $n = 1$ bo‘lsin. Quyidagi funksiyalarning $\varepsilon \to +0$ da $\delta(x)$ ga intilishini isbotlang:

I.1) $h_\varepsilon(x) = \frac{1}{2\sqrt{\pi\varepsilon}} \cdot e^{-\frac{x^2}{4\varepsilon}}$;

I.2) $h_\varepsilon(x) = \frac{1}{\pi} \cdot \frac{\varepsilon}{x^2+\varepsilon^2}$.

II. $t \to +\infty$ quyigagilarning o‘rinli ekanligini ko‘rsating:
II.1) $\frac{e^{ixt}}{x-i0} \to 2\pi i\delta(x)$; II.2) $\frac{e^{-ixt}}{x-i0} \to 0$; II.3) $\frac{e^{ixt}}{x+i0} \to 0$;
II.4) $\frac{e^{-ixt}}{x+i0} \to -2\pi i\delta(x)$.

Yuqoridagi munosabatlarni isbotlashga o‘tamiz.

I.1) ni ko‘rsatishda (1.9) formuladan foydalanamiz. Bunda

$$h_k(x) = \frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot e^{-\frac{x^2}{4\varepsilon_k}}$$

ga teng. Haqiqatan ham,

$$\Big(h_k(x), \varphi\Big) = \Big(\lim_{k\to\infty} \frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot e^{-\frac{x^2}{4\varepsilon_k}}, \varphi(x)\Big) =$$

$$= \lim_{k\to\infty} \Big(\frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot e^{-\frac{x^2}{4\varepsilon_k}}, \varphi(x)\Big) = \lim_{k\to\infty} \int\limits_{\mathbb{R}} \frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot e^{-\frac{x^2}{4\varepsilon_k}} \cdot \varphi(x)dx =$$

$$= \lim_{k\to\infty} \varphi(\varepsilon_k) \int\limits_{\mathbb{R}} \frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot e^{-\frac{x^2}{(2\sqrt{\varepsilon_k})^2}} dx = \lim_{k\to\infty} \varphi(\varepsilon_k) \frac{1}{2\sqrt{\pi\varepsilon_k}} \cdot 2\sqrt{\pi\varepsilon_k} =$$

$$= \varphi(0) = (\delta(x), \varphi(x)).$$

Shu kabi I.2) ning o‘rinli ekanligini ham ko‘rsatish mumkin. Bunda

$$h_k(x) = \frac{1}{\pi} \cdot \frac{\varepsilon_k}{x^2 + \varepsilon_k^2}.$$

$$\Big(h_k(x), \varphi(x)\Big) = \Big(\lim_{k\to\infty} \frac{1}{\pi} \cdot \frac{\varepsilon_k}{x^2 + \varepsilon_k^2}, \varphi(x)\Big) =$$

$$= \lim_{k\to\infty} \Big(\frac{1}{\pi} \cdot \frac{\varepsilon_k}{x^2 + \varepsilon_k^2}, \varphi(x)\Big) = \lim_{k\to\infty} \int\limits_{\mathbb{R}} \frac{1}{\pi} \cdot \frac{\varepsilon_k}{x^2 + \varepsilon_k^2} \cdot \varphi(x)dx =$$

$$= \lim_{k\to\infty} \varphi(\varepsilon_k) \int\limits_{\mathbb{R}} \frac{1}{\pi} \cdot \frac{\varepsilon_k}{x^2 + \varepsilon_k^2} dx = \lim_{k\to\infty} \varphi(\varepsilon_k) \int\limits_{\mathbb{R}} \frac{\varepsilon_k}{\pi} \cdot \frac{1}{\varepsilon_k} \frac{1}{\left(\frac{x}{\varepsilon_k}\right)^2 + 1} dx =$$

$$= \lim_{k\to\infty} \varphi(\varepsilon_k)\frac{\varepsilon_k}{\pi}\cdot\frac{1}{\varepsilon_k}\cdot 2\arctan\frac{x}{\varepsilon_k} = \varphi(0) = (\delta(x),\varphi(x)).$$

Endi misollarning ikkinchi qismidagi munosabatlarning o‘rinli ekanligini ko‘rsatamiz:

II.1) ni isbotlash uchun

$$\frac{e^{ixt}}{x-i0} := \lim_{\varepsilon\to -0}\frac{e^{ixt}}{x+i\varepsilon} \to 2\pi i\delta(x)$$

ekanligini ko‘rsatamiz. Buning uchun, $f_\varepsilon(x) = \frac{e^{ixt}}{x+i\varepsilon}$ funksiyalar ketma-ketligining kuchsiz limitini hisoblaymiz. Kuchsiz limitning aniqlanishiga ko‘ra, istalgan uzluksiz $\varphi(x)$ funksiya uchun bajariladigan ushbu

$$\Big(\lim_{\varepsilon\to 0} f_\varepsilon(x),\varphi(x)\Big) = \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^1} f_\varepsilon(x)\varphi(x)dx$$

tenglikdan foydalanamiz. Kompleks o‘zgaruvchili funksiyalar nazariyasidan ma‘lum bo‘lgan Eyler formulasiga asosan

$$\frac{e^{ixt}}{x+i\varepsilon} = \Big(\frac{cosxt}{x+i\varepsilon} + i\frac{sinxt}{x+i\varepsilon}\Big),$$

bo‘ladi. Bundan foydalanib,

$$\Big(\lim_{\varepsilon\to -0} f_\varepsilon(x),\varphi\Big) = \Big(\lim_{\varepsilon\to -0}\Big(\frac{cosxt}{x+i\varepsilon} + i\frac{sinxt}{x+i\varepsilon}\Big),\varphi(x)\Big) =$$

$$= \Big(\frac{cosxt}{x-i0},\varphi(x)\Big) + i\Big(\frac{sinxt}{x-i0},\varphi(x)\Big) =: J_1 + J_2.$$

limitni hisoblaymiz, bu yerda J_1, J_2 lar orqali mos ravishda oxirgi tenglikdagi birinchi va ikkinchi qo‘shiluvchilar belgilangan. Soxotskiy

$$\frac{1}{x\pm i0} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x}$$

formulasidan foydalanib, yuqoridagi ifodalarni soddalashtiramiz:

$$J_1 = \Bigg(\frac{cosxt}{x-i0},\varphi(x)\Bigg) =$$

$$= \Big(cosxt\Big(i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big) = \Big(i\pi\delta(x) + cosxt\cdot\mathcal{P}\frac{1}{x},\varphi(x)\Big) =$$

$$= (i\pi\delta(x), \varphi(x)) + \Big(cosxt \cdot \mathcal{P}\frac{1}{x}, \varphi(x)\Big) \tag{7}$$

ekanligi uchun $\Big(cosxt \cdot \mathcal{P}\frac{1}{x}, \varphi(x)\Big)$ ni alohida hisoblaymiz.

Lagranjning chekli orttirmalar haqidagi teoremasiga ko'ra

$$\varphi(x) = \varphi(0) + x\varphi'(\xi),\ \xi \in (0, x)$$

ekanligidan foydalanib,

$$\Big(cosxt \cdot \mathcal{P}\frac{1}{x}, \varphi(x)\Big) = \lim_{\varepsilon\to 0}\left[\left(\int\limits_{-\infty}^{-\varepsilon} + \int\limits_{\varepsilon}^{\infty}\right)\frac{cosxt}{x}(\varphi(0) + x\varphi'(\xi))dx\right] =$$

$$= \varphi(0)\lim_{\varepsilon\to 0}\left(\int\limits_{-\infty}^{-\varepsilon}\frac{\cos xt}{x}dx + \int\limits_{\varepsilon}^{\infty}\frac{\cos xt}{x}dx\right) +$$

$$+\lim_{\varepsilon\to 0}\left(\int\limits_{-\infty}^{-\varepsilon}\cos xt\varphi(\xi)dx + \int\limits_{\varepsilon}^{\infty}\cos xt\varphi(\xi)dx\right).$$

Birinchi qavsdagi integrallar integral osti funksiyasi toq bo'lgani uchun nolga teng. Ikkinchi qavsdagi integrallar $t \to \infty$ da nolga intiladi. Shunday qilib, (7) ga asosan

$$J_1 = \left(\frac{cosxt}{x - i0}, \varphi(x)\right) = (i\pi\delta(x), \varphi(x)).$$

J_2 ni hisoblashda ham Soxotskiy formulasidan foydalanamiz:

$$J_2 = \Big(i\frac{sinxt}{x - i0}, \varphi(x)\Big) = \Big(isinxt\Big(i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big), \varphi(x)\Big) =$$

$$= \Big(isinxt \cdot \mathcal{P}\frac{1}{x}, \varphi(x)\Big) = i\lim_{\varepsilon\to 0}\int\limits_{|x|>\varepsilon}\frac{sinxt}{x}\varphi(x)dx =$$

$$= i\lim_{\varepsilon\to 0}\Big(\int\limits_{-\infty}^{-\varepsilon}\frac{sinxt}{x}\varphi(0)dx + \int\limits_{\varepsilon}^{\infty}\frac{sinxt}{x}\varphi(0)dx\Big) +$$

$$+ i\lim_{\varepsilon\to 0}\Big(\int\limits_{-\infty}^{-\varepsilon}sinxt\varphi'(\xi)dx + \int\limits_{\varepsilon}^{\infty}sinxt\varphi'(\xi)dx\Big) =$$

$$= i\varphi(0)\int\limits_{-\infty}^{\infty}\frac{sinxt}{x}dx = i\pi\varphi(0) = i\pi(\delta(x),\varphi(x))$$

Natijalarni umumlashtirib,

$$\lim_{t\to\infty}\frac{e^{ixt}}{x-i0} = J_1 + J_2 = 2i\pi\delta(x).$$

formula o'rinli ekaniga ishonch hosil qilamiz.

II.2) $\frac{e^{-ixt}}{x-i\varepsilon}$ funksiyaning $\varepsilon\to -0$ dagi kuchsiz limitini hisoblaymiz. Yuqoridagiga o'xshash mulohazalar yuritib quyidagilarga ega bo'lamiz:

$$\Big(\lim_{\varepsilon\to -0} f_\varepsilon(x),\varphi\Big) = \Big(\lim_{\varepsilon\to -0}\Big(\frac{cosxt}{x+i\varepsilon} - i\frac{sinxt}{x+i\varepsilon}\Big),\varphi(x)\Big) =$$

$$= \Big(\frac{cosxt}{x-i0},\varphi(x)\Big) - i\Big(\frac{sinxt}{x-i0},\varphi(x)\Big) =: J_1 + J_2.$$

Soxotskiy formulalaridan foydalanib, hisoblashlarni davom ettiramiz:

$$J_1 = \Big(\frac{cosxt}{x-i0},\varphi(x)\Big) = \Big(cosxt\Big(i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big) =$$

$$= \Big(\Big(i\pi\delta(x) + cosxt\cdot\mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big) =$$

$$= (i\pi\delta(x),\varphi(x)) + \Big(cosxt\cdot\mathcal{P}\frac{1}{x},\varphi(x)\Big) =: J_{11} + J_{12}.$$

I.1) misolda $J_{12} = \Big(cosxt\cdot\mathcal{P}\frac{1}{x},\varphi(x)\Big) = 0$ ekanligi ko'rsatildi.

J_2 ni hisoblaymiz:

$$J_2 = \Big(-i\frac{sinxt}{x-i0},\varphi(x)\Big) = \Big(-isinxt\Big(i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big) =$$

$$= -i\lim_{\varepsilon\to -0}\int\limits_{|x|>\varepsilon}\frac{sinxt}{x}\varphi(x)dx =$$

$$= -i\lim_{\varepsilon\to -0}\Big[\int\limits_{-\infty}^{-\varepsilon}\frac{sinxt}{x}\Big[\varphi(0) + x\cdot\varphi'(\xi)\Big]dx +$$

$$+\int\limits_{\varepsilon}^{+\infty}\frac{sinxt}{x}[\varphi(0) + x\cdot\varphi'(\xi)]dx\Big] =$$

$$= -i\Big[\varphi(0)\int\limits_{-\infty}^{+\infty}\frac{sinxt}{x}dx + \int\limits_{-\infty}^{+\infty} sinxt\varphi^{'}(\xi)dx\Big] =$$

$$= -i\varphi(0)\int\limits_{-\infty}^{+\infty}\frac{sinxt}{x}dx = -i\pi\varphi(0) = (-i\pi\delta(x), \varphi(x))\,;$$

Olingan natijalarni umumlashtiramiz:

$$J_1 + J_2 = J_{11} + J_2 = (i\pi\delta(x), \varphi(x)) + (-i\pi\delta(x), \varphi(x)) = 0.$$

Demak,

$$\lim_{t\to\infty}\frac{e^{-ixt}}{x - i0} = 0.$$

II.3) $\frac{e^{ixt}}{x+i0}$ funksiyalar ketma-ketligining $t \to \infty$ da kuchsiz limitini hisoblaymiz. Bunda ham yuqoridagiga o‘xshash mulohazalar yuritib, quyidagilarga ega bo‘lamiz:

$$(f_\varepsilon(x), \varphi(x)) = \Big(\lim_{\varepsilon\to -0}\Big(\frac{cosxt}{x+i\varepsilon} + i\frac{sinxt}{x+i\varepsilon}\Big), \varphi(x)\Big) =$$

$$= \Big(\frac{cosxt}{x+i0}, \varphi(x)\Big) + i\Big(\frac{sinxt}{x+i0}, \varphi(x)\Big) =: J_1 + J_2.$$

$$J_1 = \Big(\frac{cosxt}{x+i0}, \varphi(x)\Big) = \Big(cosxt\Big(-i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big), \varphi(x)\Big) =$$

$$= (-i\pi\delta(x), \varphi(x)) + \Big(cosxt\cdot\mathcal{P}\frac{1}{x}, \varphi(x)\Big) = I_{11} + I_{12};$$

bunda $J_{11} = (-i\pi\delta(x), \varphi(x))$ ga teng.

$$J_{12} = \Big(cosxt\cdot\mathcal{P}\frac{1}{x}, \varphi(x)\Big) = 0$$

ekanligi II.2) misoldagi kabi ko‘rsatiladi. J_2 ni hisoblaymiz.

$$J_2 = \Big(i\frac{sinxt}{x+i0}, \varphi(x)\Big) = \Big(isinxt\Big(-i\pi\delta(x) + \mathcal{P}\frac{1}{x}\Big), \varphi(x)\Big) =$$

$$= \Big(isinxt\cdot\mathcal{P}\frac{1}{x}, \varphi(x)\Big) = (i\pi\delta(x), \varphi(x))$$

ekanligini II.2) ga o‘xshash ko‘rsatiladi.

Shunday qilib,

$$J_1 + J_2 = J_{11} + J_2 = (-i\pi\delta(x), \varphi(x)) + (i\pi\delta(x), \varphi(x)) = 0.$$

Bundan esa

$$\lim_{t\to\infty}\frac{e^{ixt}}{x+i0}=0$$

tenglik o‘rinli ekanligi kelib chiqadi.

II.4) $\frac{e^{-ixt}}{x+i0}$ funksiyalar ketma-ketligining $t\to\infty$ da kuchsiz limitini hisoblaymiz. Bu misolda ham yuqoridagiga o‘xshash mulohazalar yuritib, quyidagini olamiz:

$$(f_\varepsilon(x),\varphi(x))=\Big(\lim_{\varepsilon\to-0}\Big(\frac{cosxt}{x-i\varepsilon}-i\frac{sinxt}{x-i\varepsilon}\Big),\varphi(x)\Big)=$$

$$=\Big(\frac{cosxt}{x+i0},\varphi(x)\Big)-i\Big(\frac{sinxt}{x+i0},\varphi(x)\Big)=:J_1+J_2.$$

Soxotskiy formulasidan foydalanib, hisoblashlarda ifodalarni soddalashtiramiz:

$$J_1=\Big(\frac{cosxt}{x+i0},\varphi(x)\Big)=\Big(cosxt\Big(-i\pi\delta(x)+\mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big)=$$

$$=\Big(\Big(-i\pi\delta(x)+cosxt\cdot\mathcal{P}\frac{1}{x}\Big),\varphi(x)\Big)=$$

$$=(-i\pi\delta(x),\varphi(x))+\Big(cosxt\cdot\mathcal{P}\frac{1}{x},\varphi(x)\Big)=(-i\pi\delta(x),\varphi(x));$$

bu yerda $\Big(cosxt\cdot\mathcal{P}\frac{1}{x},\varphi(x)\Big)=0$ ekanligi va $J_2=\Big(-i\frac{sinxt}{x+i0},\varphi(x)\Big)=(-i\pi\delta(x),\varphi(x))$ ekanligi oldingi hisoblashlarda ko‘rsatilgan edi. Shuning uchun,

$$J_1+J_2=J_{11}+J_2=(-i\pi\delta(x),\varphi(x))+(-i\pi\delta(x),\varphi(x))=(-2\pi i\delta(x),\varphi(x))$$

va

$$\lim_{t\to\infty}\frac{e^{-ixt}}{x+i0}=-2\pi i\delta(x).$$

δ - funksiyaning Furye qatoriga yoyilmasi.

$(-l,\ l)$ oraliqda aniqlangan funksiyalardan iborat ushbu

$$\tilde{\delta}_n(x,x_0)=\frac{1}{2l}+\frac{1}{l}\sum_{m=1}^{n}\Big(\cos\frac{m\pi x_0}{l}\cos\frac{m\pi x}{l}+\sin\frac{m\pi x_0}{l}\sin\frac{m\pi x}{l}\Big)=$$

$$=\frac{1}{2l}+\frac{1}{l}\sum_{m=1}^{n}\cos\frac{m\pi}{l}(x-x_0)=\tilde{\delta}(x-x_0)$$

yoki kompleks ko‘rinishdagi

$$\tilde{\delta}_n(x-x_0)=\frac{1}{2l}\sum_{|m|\le n}\exp\left\{im\frac{\pi}{l}(x-x_0)\right\}$$

funksiyalar ketma-ketligini qaraymiz.

Osongina ko‘rsatish mumkinki, Furye qatoriga yoyiluvchi ixtiyoriy $g(x)$ funksiya uchun

$$\lim_{n\to\infty}\int\limits_{-l}^{l}\tilde{\delta}_n(x-x_0)g(x)dx=g(x_0)$$

tenglik bajariladi. Bu tenglik Furye qatoriga yoyiluvchi $\{g(x)\}$ funksiyalar sinfida yuqorida aniqlangan $\tilde{\delta}_n(x-x_0)$ ketma-ketlik bo‘sh yaqinlashish ma’nosida "$\delta-$shaklli" ketma-ketlikka ekvivalentligini ko‘rsatadi, ya’ni

$$\tilde{\delta}(x-x_0)=\frac{1}{2l}+\frac{1}{l}\sum_{m=1}^{\infty}\cos\frac{m\pi}{l}(x-x_0).$$

Shu nuqtayi nazardan, ushbu

$$\delta(x-x_0)=\sum_{m=1}^{\infty}\varphi_n(x)\varphi_n(x_0), \tag{8}$$

$$\delta(x-x_0)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{i\xi(x-x_0)}d\xi=\frac{1}{\pi}\int\limits_{0}^{\infty}\cos\xi(x-x_0)d\xi \tag{9}$$

ham o‘rinli bo‘ladi. (8) formulada $\{\varphi_n(x)\}-$ biror (a,b) oraliqda aniqlangan ortonormal funksiyalar sistemasi

1.5 Umumlashgan va cheksiz differensiallanuvchi funksiyalarning superpozitsiyasi

$f(x)-\mathbb{R}^n$ da aniqlangan, lokal integrallanuvchi funksiya va $y=\omega(x)-$ cheksiz marta differensiallanuvchi maxsus bo‘lmagan $\left(\det\frac{\partial\omega(x)}{\partial x}\neq 0\right)\mathbb{R}^n$

ni o'ziga akslantiruvchi vektor funksiya bo'lsin. U holda ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ uchun quyidagi tengliklar o'rinli:

$$(f(\omega(x)), \varphi(x)) =$$

$$= \int\limits_{\mathbb{R}^n} f(\omega(x))\varphi(x)dx = \int\limits_{\mathbb{R}^n} f(y)\varphi(\omega^{-1}(y))\Big|\det\frac{\partial\omega^{-1}(y)}{\partial y}\Big|dy =$$

$$= \Big(f(y), \varphi(\omega^{-1}(y))\Big|\det\frac{\partial\omega^{-1}(y)}{\partial y}\Big|\Big),$$

ya'ni

$$(f(\omega(x)), \varphi(x)) = \Big(f(y), \varphi(\omega^{-1}(y))\Big|\det\frac{\partial\omega^{-1}(y)}{\partial y}\Big|\Big) \tag{10}$$

$$\varphi(\omega^{-1}(y))\Big|\det\frac{\partial\omega^{-1}(y)}{\partial y}\Big| \in D(\mathbb{R}^n)$$

ekanligi sababli bu tenglikni berilgan $f(x)$ funksiya bo'yicha umumlashgan $f(\omega(x))$ funksiyani aniqlash uchun ishlatish mumkin. Xususan, $x = Ay + b$, $\det A \neq 0$ maxsus bo'lmagan $\mathbb{R}^n$ ni o'ziga akslantiruvchi chiziqli almashtirish bo'lsa, u holda ixtiyoriy $\varphi \in D$ lar uchun

$$(f(Ay+b), \varphi) = \int\limits_{\mathbb{R}^n} f(Ay+b)\varphi(y)dy =$$

$$= \frac{1}{|\det A|}\int\limits_{\mathbb{R}^n} f(x)\varphi[A^{-1}(x-b)]dx = \frac{1}{|\det A|}(f, \varphi[A^{-1}(x-b)]).$$

Shunday qilib, yuqoridagi tenglikka asosan,

$$f((Ay+b), \varphi) = \Big(f, \frac{\varphi[A^{-1}(x-b)]}{|\det A|}\Big)$$

munosabat o'rinli.

T a ' r i f. Umumlashgan $f(\omega(x))$ funksiya deb, qiymatlari $\varphi \in D(\mathbb{R}^n)$ funksiyalarda (10) tenglik bilan aniqlanadigan funksiyaga aytiladi.

M i s o l. $n = 1$ bo'lsin. $\delta(x - x^0)$, $x^0 \in \mathbb{R}$ funksiyani aniqlaymiz. (10) formuladan ixtiyoriy $\varphi \in D(\mathbb{R})$ funksiya uchun

$$(\delta(x - x^0), \varphi(x)) = (\delta(y), \varphi(y + x^0)) = \varphi(x^0)$$

tenglik o‘rinli, ya’ni $\delta(x-x^0)$ quyidagi tenglik bilan aniqlanadi:

$$(\delta(x-x^0),\varphi(x))=\varphi(x^0).$$

M i s o l. $\delta(x)=\delta(-x)$ ekanligini ko‘rsatamiz.

Ixtiyoriy $\varphi\in D(\mathbb{R})$ uchun

$$(\delta(-x),\varphi(x))=(\delta(x),\varphi(-x))=\varphi(0)=(\delta(x),\varphi(x))$$

ekanligidan yuqoridagi tenglik kelib chiqadi.

M i s o l. $n\in\mathbb{N}$ va $\omega^{-1}(0)=x^0\in\mathbb{R}^n$ bo‘lsin. U holda,

$$\delta(\omega(x))=\frac{1}{\left|\det\frac{\partial\omega(x^0)}{\partial x}\right|}\cdot\delta(x-x^0).$$

Bu tenglikning chap va o‘ng tomonidagi funksiyalarning qiymatlari asosiy funksiyalarda ustma-ust tushishini ko‘rsatamiz. Haqiqatdan ham, ixtiyoriy $\varphi(x)\in D(\mathbb{R}^n)$ uchun

$$(\delta(\omega(x)),\varphi(x))=$$
$$=\Big(\delta(y),\varphi(\omega^{-1}(y))\Big|\det\frac{\partial\omega^{-1}(y)}{\partial y}\Big|\Big)=\varphi(\omega^{-1}(0))\Big|\det\frac{\partial\omega^{-1}(0)}{\partial y}\Big|=$$
$$=\varphi(x^0)\frac{1}{\left|\det\frac{\partial\omega(x^0)}{\partial x}\right|}=\left(\frac{1}{\left|\det\frac{\partial\omega(x^0)}{\partial x}\right|}\delta(x-x^0),\varphi(x)\right)$$

tengliklar o‘rinli. Bu esa talab etilgan tenglikni isbotlaydi.

1.6 Umumlashgan funksiyalarning dekart ko‘paytmasi va cheksiz differensiallanuvchi funksiyalarga ko‘paytmasi

$f(x)$ va $g(y)$ lar mos ravishda $\mathbb{R}^n$ va $\mathbb{R}^m$ fazolarda aniqlangan umumlashgan regulyar funksiyalar bo‘lsin.

T a ’ r i f (umumlashgan regulyar funksiyalarning dekart ko‘paytmasi). $\varphi(x,y)\in D(\mathbb{R}^{n+m})$ funksiyalarda aniqlangan

$$(f(x)\cdot g(y),\varphi(x,y))=\big(f(x),(g(y),\varphi(x,y))\big)\qquad(11)$$

formula bilan berilgan funksionalga $f(x)$ va $g(y)$ umumlashgan funksiyalarning Dekart ko‘paytmasi deyiladi va $f(x) \cdot g(y)$ kabi belgilanadi.

T e o r e m a (Umumlashgan funksiyalar Dekart ko‘paytmasining korrektligi haqida). Ixtiyoriy $f \in D'(\mathbb{R}^n)$ va $g \in D'(\mathbb{R}^m)$, bu yerda $n, m \in \mathbb{N}$, lar uchun $f \cdot g \in D'(\mathbb{R}^{n+m})$ o‘rinli.

Teoremani **isbot**lash uchun ta’rifning korrekt ekanligi, ya’ni (11) tenglikning o‘ng tomoni haqiqatdan ham $D(\mathbb{R}^{n+m})$ da chiziqli va uzluksiz funksional ekanligini ko‘rsatamiz.

Dastlab quyidagi lemmani keltiramiz.

L e m m a. Ixtiyoriy $g(y) \in D'\left(\mathbb{R}^m\right)$ va $\varphi(x, y) \in D\left(\mathbb{R}^{n+m}\right)$ lar uchun $\psi(x) = (g(y), \varphi(x, y))$ funksiya $D\left(\mathbb{R}^n\right)$ sinfga tegishli, bunda barcha α lar uchun

$$D^\alpha \psi(x) = \left(g(y), D_x^\alpha \varphi(x, y)\right). \tag{12}$$

Shuningdek, agar φ_k $D\left(\mathbb{R}^{n+m}\right)$ da $k \to \infty$ lar uchun nolga yaqinlashuvchi ketma-ketlik bo‘lsa, u holda $D\left(\mathbb{R}^{n+m}\right)$ da $k \to \infty$ lar uchun

$$\psi_k(x) = (g(y), \varphi_k(x, y)) \to 0$$

bo‘ladi.

Isbot. Har bir $x \in \mathbb{R}^n$ uchun $\varphi(x, y) \in D\left(\mathbb{R}^m\right)$ bo‘lgani sababli $\psi(x)$ funksiya $\mathbb{R}^n$ da aniqlangan. Uning uzluksiz ekanini ko‘rsatamiz. x ni tayin qilib, $k \to \infty$ da $x_k \to x$ ketma-ketlikni olamiz. U holda $\mathbb{R}^m$ da $k \to \infty$ lar uchun

$$\varphi(x_k, y) \to \varphi(x, y), \tag{13}$$

$\varphi(x, y) \in D\left(\mathbb{R}^{n+m}\right)$ bo‘lgani sababli $\operatorname{supp}\varphi(x_k, y)$ to‘plam $\mathbb{R}^m$ da chegaralangan va k ga bog‘liq emas hamda barcha β lar uchun

$$D_y^\beta \varphi(x_k, y) \rightrightarrows D_y^\beta \varphi_k(x, y),\ k \to \infty,\ y \in \mathbb{R}^m.$$

$g(y)$ funksional $\mathbb{R}^m$ da uzluksiz bo‘lgani uchun (13) dan $\psi(x)$ funksiyaning x nuqtada uzluksiz ekani kelib chiqadi:

$$\psi(x_k) = (g(y), \varphi(x_k, y)) \to (g(y), \varphi(x, y)) = \psi(x),\ k \to \infty.$$

Endi (12) formulani isbotlaymiz. Buning uchun x ni tayin qilib, $h_i = (0, \cdots, h, \cdots, 0)$ (h soni h_i vektorning $i-$komponentasi) belgilash kiritamiz. U holda $\mathbb{R}^m$ da $h \to \infty$ lar uchun

$$\chi_h^{(i)}(y) = \frac{1}{h}\left[\varphi(x+h_i, y) - \varphi(x,y)\right] \to \frac{\partial \varphi(x,y)}{\partial x_i},$$

$\varphi(x,y) \in D\left(\mathbb{R}^m\right)$ bo'lgani uchun $\chi_h^{(i)}(y)$ funksiyalarning tashuvchisi h ga bogliq bo'lmagan tarzda $\mathbb{R}^m$ da chegaralangan va barcha β lar uchun

$$D^\beta \chi_h^{(i)}(y) =$$

$$= \frac{1}{h}\left[D_y^\beta \varphi(x+h_i, y) - D_y^\beta \varphi(x,y)\right] \to D_y^\beta \frac{\partial \varphi(x,y)}{\partial x_i}, \ k \to \infty, \ y \in \mathbb{R}^m.$$

$g(y) \in D'\left(\mathbb{R}^m\right)$ ekanligidan $h \to 0$ lar uchun

$$\frac{\psi(x+h_i) - \psi(x)}{h} = \frac{1}{h}\left[(g(y), \varphi(x+h_i, y)) - (g(y), \varphi(x,y))\right] =$$

$$= \left(g(y), \frac{\varphi(x+h_i, y) - \varphi(x,y)}{h}\right) = \left(g(y), \chi_h^{(i)}(y)\right) \to \left(g(y), \frac{\partial \varphi(x,y)}{\partial x_i}\right).$$

Bu yerdan (12) formula x_i koordinata bo'yicha birinchi tartibli hosila uchun o'rinli ekani kelib chiqadi. Shu tariqa, bajarilgan tekshirishlarni hosil qilingan formulaga yana qo'llash yo'li bilan (12) formulaning barcha α uchun o'rinli bo'lishiga ishonch hosil qilamiz. Demak, barcha α lar uchun $D^\alpha \psi(x)-$ $\mathbb{R}^n$ da uzluksiz funksiya. Shunday qilib, $\psi(x) \in C^\infty\left(\mathbb{R}^n\right)$. $\varphi(x,y)$ funksiyaning $\mathbb{R}^{n+m}$ da finit ekanligidan $\psi(x)$ funksiyaning $\mathbb{R}^n$ da finitligi kelib chiqadi. Demak, $\psi(x) \in D(\mathbb{R}^n)$.

$\mathbb{R}^{n+m}$ da $k \to \infty$ lar uchun $\varphi_k(x,y) \to 0$ bo'lsin. U holda $\mathbb{R}^n$ da $k \to \infty$ lar uchun $\psi_k(x) \to 0$ bo'lishini ko'rsatamiz. Buning uchun $\varphi_k(x,y)$ larning tashuvchisi $\mathbb{R}^{n+m}$ da chegaralanganligidan $\psi_k(x)$ funksiyalarning tashuvchisi ham k ga bog'liq bo'lmagan holda $\mathbb{R}^n$ da chegaralangan bo'lishini hisobga olsak, barcha α lar uchun

$$D^\alpha \psi_k(x) \rightrightarrows 0, \ k \to \infty, \ x \in \mathbb{R}^n$$

limit munosabat o'rinli bo'lishini ko'rsatish kifoya.

Teskarisini faraz qilamiz: ya'ni bunday bo'lmasin. U holda shunday ε_0 soni, α_0 indeks va x_k nuqtalar ketma-ketligi topiladiki, bunda

$$|D^{\alpha_0}\psi(x_k)| \geq \varepsilon_0, \quad k = 1, 2, \cdots \tag{14}$$

tengsizlik bajariladi. $\psi_k(x)$ funksiyalarning tashuvchisi k ga bog'liq bo'lmagan holda $\mathbb{R}^n$ da chegaralanganligidan x_k ketma-ketlikning $\mathbb{R}^n$ da chegaralanganligi kelib chiqadi. Shuning uchun bu ketma-ketlikdan, matematik tahlil fanidan ma'lum bo'lgan Bolsano-Veyershtrass teoremasiga ko'ra, yaqinlashuvchi ketma-ketlik ajratish mumkin. $x_{k_i} \to x_0,\ i \to \infty$ bo'lsin. U holda

$$D_x^{\alpha_0}\varphi_{k_i}(x_{k_i}, y) \rightrightarrows 0,\ i \to \infty,\ y \in \mathbb{R}^m.$$

Bu yerdan g funksionalning $\mathbb{R}^m$ da uzluksizligiga ko'ra (12) formuladan quyida- giga ega bo'lamiz:

$$D^{\alpha_0}\psi_{k_i}(x_{k_i}) \rightrightarrows (g(y),\ D_x^{\alpha_0}\varphi_{k_i}(x_{k_i}, y)) \to 0,\ i \to \infty.$$

Bu esa yuqoridagi farazimizga, ya'ni (14) tengsizlikka zid. Lemma isbotlandi.

Dekart ko'paytmaning ta'rifiga qaytamiz. Isbotlangan lemmaga asosan barcha $\varphi(x, y) \in D\left(\mathbb{R}^{n+m}\right)$ lar uchun $\psi(x) = (g(y), \varphi(x, y))$ funksiya $D\left(\mathbb{R}^n\right)$ sinfga tegishli. Demak, (11) o'ng (f, ψ) qismi ixtiyoriy f va g umumlashgan funksiyalar uchun ma'noga ega, va demak, $D\left(\mathbb{R}^{n+m}\right)$ da funksionalni aniqlaydi. f va g funksionallarning uzluksizligidan bu funksionalning uzluksizligi kelib chiqadi.

(f, ψ) funksionalning $D\left(\mathbb{R}^{n+m}\right)$ da uzluksizligini ko'rsatamiz. $D\left(\mathbb{R}^{n+m}\right)$ da $k \to \infty$ lar uchun $\varphi_k \to 0$ bo'lsin. U holda yuqoridagi lemmaga asosan

$$(g(y), \varphi_k(x, y)) \rightrightarrows 0,\ k \to \infty,\ y \in \mathbb{R}^n$$

ekanligi va f funksionalning $D\left(\mathbb{R}^n\right)$ da uzluksizligidan

$$(f(x), (g(y), \varphi_k(x, y))) \rightrightarrows 0,\ k \to \infty$$

kelib chiqadi. Bu esa (11) tenglikning o'ng tomonidagi funksionalning uzluksizligini bildiradi.

Shunday qilib, $f(x)\cdot g(y)\in D'\left(\mathbb{R}^{n+m}\right)$, ya'ni $f(x)\cdot g(y)$ - umumlashgan funksiya. Teorema isbotlandi.

Umumlashgan funksiyalarning Dekart koʻpaytmasi quyidagi xossalarga ega:
1) kommutativlik $f(x)\cdot g(y)=g(y)\cdot f(x)$;
2) assotsiativlik $[f(x)\cdot g(y)]\cdot h(z)=f(x)\cdot[g(y)\cdot h(z)]$;
3) $\mathrm{supp}(f(x)\cdot g(y))=\mathrm{supp}f(x)\cdot \mathrm{supp}g(y)$;
4) $D_x^\alpha D_y^\beta[f(x)\cdot g(y)]=D_x^\alpha f(x)\cdot D_y^\beta g(y)$;
5) $a(x)b(y)[f(x)\cdot g(y)]=[a(x)f(x)]\times[b(y)g(y)]$;
6) $[f(x)\cdot g(y)](x+x^0,y+y^0)=f(x+x^0)\cdot g(y+y^0)$.

Bu xossalarning isboti (11) tenglikdan kelib chiqadi. Masalan, 2-assotsiativlik xossasining oʻrinli ekanligini, ya'ni agar $f(x)\in D'\left(\mathbb{R}^n\right)$, $g(y)\in D'\left(\mathbb{R}^m\right)$, $h(z)\in D'\left(\mathbb{R}^k\right)$ boʻlsa, $[f(x)\cdot g(y)]\cdot h(z)=f(x)\cdot[g(y)\cdot h(z)]$ tenglik bajarilishini koʻrsatamiz.

Haqiqatan ham, ixtiyoriy $\varphi(x,y,z)\in D\left(\mathbb{R}^{n+m+k}\right)$ uchun

$$(\varphi(x,y,z),[f(x)\cdot g(y)]\cdot h(z))=(\varphi(x,y,z),(f(x)\cdot g(y))\cdot h(z))=$$

$$=(((\varphi(x,y,z),f(x)),g(y)),h(z))=$$

$$=(((\varphi(x,y,z),f(x)),g(y))\cdot h(z))=(\varphi(x,y,z),f(x)\cdot[g(y)\cdot h(z)]).$$

Endi umumlashgan funksiyalarning cheksiz differensiallanuvchi funksiyalarga koʻpaytmasini qanday aniqlanishini koʻrib chiqamiz. $a(x)\in C^\infty(\mathbb{R}^n)$ va $f(x)$ - $\mathbb{R}^n$ da lokal integrallanuvchi funksiya boʻlsin. U holda $a(x)f(x)$ koʻpaytma ham lokal integrallanuvchi funksiya va ixtiyoriy $\varphi\in D(\mathbb{R}^n)$ uchun quyidagi tenglik oʻrinli:

$$(a(x)f(x),\varphi(x))=\int\limits_{\mathbb{R}^n}a(x)f(x)\varphi(x)dx=(f(x),a(x)\varphi(x)).$$

Bu tenglik yordamida ixtiyoriy $f(x)$ lokal integrallanuvchi funksiyaning cheksiz marta differensiallanuvchi funksiyaga koʻpaytmasini aniqlash mumkin.

T a ' r i f. Agar $a(x)f(x)$ koʻpaytma ixtiyoriy $\varphi\in D(\mathbb{R}^n)$ uchun

$$(a(x)f(x),\varphi(x))=(f(x),a(x)\varphi(x))$$

tenglik bilan aniqlansa, unga umumlashgan funksiya deyiladi.

M i s o l. $f(x)=\delta(x)$, $a(x)\in C^{\infty}(\mathbb{R})$, $n=1$ bo'lsin. U holda ixtiyoriy $\varphi\in D(\mathbb{R})$ uchun

$$(a(x)\delta(x),\varphi(x))=(\delta(x),a(x)\varphi(x))=a(0)\varphi(0)=(a(0)\delta(x),\varphi(x))$$

Binobarin $a(x)\delta(x)=a(0)\delta(x)$.

Xuddi shunga o'xshash

$$a(x)\delta(x-x^0)=a(x^0)\delta(x-x^0) \tag{15}$$

ekanligini ko'rsatish mumkin.

E s l a t ma. (15) formula $x=x^0$ nuqtada uzluksiz ixtiyoriy $a(x)$ funksiya uchun ham o'rinli.

M i s o l. $\delta(x,y)=\delta(x)\cdot\delta(y)$ tenglikni isbotlang.

I s b o t. Ixtiyoriy $\varphi\in D(\mathbb{R}^2)$ uchun

$$(\delta(x)\cdot\delta(y),\varphi(x,y))=(\delta(x),(\delta(y),\varphi(x,y))=(\delta(x),\varphi(x,0))=$$

$$=\varphi(0,0)=(\delta(x,y),\varphi(x,y)).$$

E s l a t m a. $x=(x_1,\ x_2,\ ...,\ x_n)\in\mathbb{R}^n$ uchun

$$\delta(x)=\delta\left(x_1,\ x_2,\ ...,\ x_n\right)=\delta(x_1)\cdot\delta(x_2)\cdot...\cdot\delta(x_n) \tag{16}$$

tenglik ham yuqoridagi kabi ko'rsatiladi.

M i s o l. $x\rho\frac{1}{x}=1$, $n=1$ tenglikning o'rinli ekanligini ko'rsating.

Haqiqatan ham, ixtiyoriy $\varphi\in D(\mathbb{R})$ lar uchun

$$\left(x\rho\frac{1}{x},\varphi\right)=\left(\rho\frac{1}{x},x\varphi\right)=$$

$$=\lim_{\epsilon\to+0}\int\limits_{|x|>\epsilon}\frac{x\varphi(x)}{x}dx=\lim_{\epsilon\to+0}\int\limits_{|x|>\epsilon}\varphi(x)dx=\int\limits_{\mathbb{R}}\varphi(x)dx=(1,\varphi)$$

tengliklar bajariladi.

M i s o l.

$$x\rho\frac{1}{x^2}=\rho\frac{1}{x}$$

tenglikni isbotlang.

Yuqoridagi kabi, ixtiyoriy $\varphi \in D(\mathbb{R})$ lar uchun quyidagi tengliklar o‘rinli:

$$\Big(x\rho\frac{1}{x^2}, \varphi(x)\Big) = \Big(\rho\frac{1}{x^2}, x\varphi(x)\Big) =$$

$$= \lim_{\epsilon\to+0} \int\limits_{|x|>\epsilon} \frac{x\varphi(x) - 0\varphi(0)}{x^2} dx = \Big(\rho\frac{1}{x}, \varphi(x)\Big).$$

1.7 Umumlashgan funksiyalarning hosilasi. Misollar

Faraz qilaylik, $f(x) \in \mathbb{C}^n(\mathbb{R})$, $\varphi \in D(\mathbb{R})$ bo‘lsin. Bo‘laklab integrallash yordamida quyidagi formulaga ega bo‘lamiz:

$$(f^{(n)}(x), \varphi(x)) = \int\limits_{\mathbb{R}} f^{(n)}(x)\varphi(x)dx =$$

$$= (-1)^n \int\limits_{\mathbb{R}} f(x)\varphi^{(n)}(x)dx = (-1)^n(f(x), \varphi^{(n)}(x)).$$

Shunga o‘xshash, n o‘zgaruvchili funksiya holida, ya’ni $x = (x_1, ..., x_n)$ va $f(x) \in \mathbb{C}^{|\alpha|}$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ - multiindeks, α_i - manfiy bo‘lmagan butun sonlar bo‘lganda, ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ lar uchun

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}$$

umumlashgan hosila quyidagi formula yordamida aniqlanadi:

$$(D^\alpha f(x), \varphi(x)) = \int\limits_{\mathbb{R}^n} D^\alpha f(x)\varphi(x)dx =$$

$$= (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} f(x)D^\alpha\varphi(x)dx = (-1)^{|\alpha|}(f(x), D^\alpha\varphi(x)).$$

Demak,

$$(D^\alpha f(x), \varphi(x)) = (-1)^{|\alpha|}(f(x), D^\alpha\varphi(x))$$

tenglik yordamida aniqlangan $D^\alpha f(x)$ umumlashgan funksiya $f(x)$ umumlashgan funksiyadan olingan D^α umumlashgan hosila deyiladi.

M i s o l. $n = 1$, $f(x)-$ bo'lakli uzluksiz funksiya. Bu funksiya $\mathbb{R}$ da lokal integrallanuvchi bo'lgani sababli unga ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$(f(x), \varphi(x)) = \int\limits_{\mathbb{R}} f(x)\varphi(x)dx$$

tenglik bilan aniqlanuvchi $f(x)-$regulyar umumlashgan funksiya to'g'ri keladi. Ma'lumki, bo'lakli uzluksiz funksiyaning klassik hosilalari mavjud emas. Ammo uning ixtiyoriy tartibli umumlashgan hosilalari mavjud va quyidagi formula bilan aniqlanadi:

$$\Big(\frac{d^k}{dx^k}f(x), \varphi(x)\Big) = (-1)^k \int\limits_{\mathbb{R}} f(x)\frac{d^k}{dx^k}\varphi(x)dx.$$

M i s o l. $f(x)$ funksiya sonlar to'g'ri chiziqining hamma nuqtalarida aniqlangan, x^0 nuqtada chekli uzilishga ega va $f(x) \in C^1(x \geq x^0)$, $f(x) \in C^1(x \leq x^0)$ bo'lsin. U holda $\frac{d}{dx}f(x)$ ni aniqlovchi formulaga asosan, ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ uchun

$$\Big(\frac{d}{dx}f(x), \varphi(x)\Big) = -\Big(\frac{d}{dx}f(x), \varphi(x)\Big) =$$

$$= -\int\limits_{-\infty}^{x^0} f(x)\frac{d}{dx}\varphi(x)dx - \int\limits_{x^0}^{\infty} f(x)\frac{d}{dx}\varphi(x)dx =$$

$$= -f(x^0-0)\varphi(x^0) + \int\limits_{-\infty}^{x^0} \Big\{\frac{d}{dx}f(x)\Big\}\varphi(x)dx + f(x^0+0)\varphi(x^0) +$$

$$+ \int\limits_{x^0}^{\infty} \Big\{\frac{d}{dx}f(x)\Big\}\varphi(x)dx = \int\limits_{-\infty}^{\infty} \{f'(x)\}\varphi(x)dx +$$

$$+(f(x^0+0) - f(x^0-0))\varphi(x^0) = \Big(f'_{cl} + [f]_{x=x^0}\delta(x-x^0), \varphi(x)\Big),$$

ga ega bo'lamiz, bu yerda $f'_{cl} - f(x)$ funksiyaning $x \neq x^0$ nuqtadagi klassik hosilasi,

$$[f]_{x=x^0} = f(x^0+0) - f(x^0-0)$$

esa $f(x)$ funksiyaning $x = x^0$ nuqtadagi sakrashi. Shunday qilib, $f(x)$ funksiyaning umumlashgan hosilasi uchun

$$\frac{d}{dx}f(x) = f'_{cl} + [f]_{x=x^0}\delta(x - x^0) \tag{17}$$

formula oʻrinli.

M i s o l. $f(x) = |x|$ funksiya umuman olganda butun sonlar oʻqida klassik ma'noda hosilaga ega emas. $f(x)$ funksiyaning umumlashgan hosilasini topish uchun ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun quyidagiga ega boʻlamiz:

$$\Big(\frac{d}{dx}|x|, \varphi(x)\Big) = -(|x|, \varphi'(x)) = -\int\limits_{-\infty}^{\infty} |x|\varphi'(x)dx =$$

$$= \int\limits_{-\infty}^{0} x\varphi'(x)dx - \int\limits_{0}^{\infty} x\varphi'(x)dx =$$

$$= -\int\limits_{-\infty}^{0} \varphi(x)dx + \int_{0}^{\infty} \varphi(x)dx = \int\limits_{-\infty}^{\infty} sgn(x)\varphi(x)dx = (sgnx, \varphi(x))$$

Demak,

$$\frac{d}{dx}|x| = sgnx, \quad sgn(\cdot) = \begin{cases} 1,\ x > 0, \\ 0, \ \ x = 0, \\ -1, x < 0. \end{cases}$$

M i s o l. Quyidagi *Xevisayd funksiyasi*

$$\theta(t) = \begin{cases} 1,\ t \geq 0, \\ 0, \ \ t < 0. \end{cases}$$

uchun $\frac{d}{dt}\theta(t) = \delta(t)$ ekanligini koʻrsatish mumkin. Haqiqatan, $\theta'(t)_{cl} = 0$, $[\theta]_{t=1} = 1$ boʻlgani uchun (17) ga asosan $\frac{d}{dt}\theta(t) = \delta(t)$. Bu yerdan

$$\frac{d^{k+1}}{dt^{k+1}}\theta(t) = \frac{d^k}{dt^k}\delta(t) = \delta^{(k)}(t),\ k = 0, 1, 2, \dots$$

ekanligi kelib chiqadi.

M i s o l. $x \in \mathbb{R}$ va $n-$ butun son. U holda

$$x^n\delta^{(n)}(x) = (-1)^n n!\delta(x)$$

o'rinli.

I s b o t. Ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$(x^n\delta^{(n)}(x), \varphi(x)) = (\delta^{(n)}(x), x^n\varphi(x)) =$$

$$= (-1)^n\big(\delta(x), \frac{d^n}{dx^n}(x^n\varphi(x))\big) == (-1)^n\frac{d^n}{dx^n}(x^n\varphi(x))\Bigg|_{x=0} =$$

$$= (-1)^n n!\varphi(0) = \big((-1)^n n!\delta(x), \varphi(x)\big).$$

M i s o l. $(\ln|x|)' = \rho\frac{1}{x}$ tenglikni isbotlang.

I s b o t. Eslatib o'tamiz, $\ln|x|-$ regulyar umumlashgan funksiya. Bundan esa ixtiyoriy $x \in D(\mathbb{R})$ uchun

$$\big((\ln|x|)', \varphi(x)\big) = -(\ln|x|, \varphi'(x)) =$$

$$= -\int\limits_{\mathbb{R}} \ln|x|\varphi'(x)dx = -\lim_{\epsilon\to+0}\int\limits_{|x|>\epsilon} \ln|x|\varphi'(x)dx =$$

$$= \lim_{\epsilon\to+0}\left(-\int\limits_{-\infty}^{-\epsilon} \ln|x|\varphi'(x)dx - \int\limits_{\epsilon}^{\infty} \ln|x|\varphi'(x)dx\right) =$$

$$= \lim_{\epsilon\to+0}\left(-\ln|x|\varphi(x)\Bigg|_{-\infty}^{-\epsilon} + \int\limits_{-\infty}^{-\epsilon}(\ln|x|)'\varphi(x)dx - \ln|x|\varphi(x)\Bigg|_{\epsilon}^{\infty} + \right.$$

$$\left. + \int\limits_{\epsilon}^{\infty}(\ln|x|)'\varphi(x)dx\right) = \lim_{\epsilon\to+0}\left(\ln|\epsilon|\left(\varphi(\epsilon) - \varphi(-\epsilon)\right) + \int\limits_{|x|>\epsilon}\frac{\varphi(x)}{x}dx\right) =$$

$$= \lim_{\epsilon\to+0}\left(O(\epsilon\ln|\epsilon| + \int\limits_{|x|>\epsilon}\frac{\varphi(x)}{x}dx)\right) =$$

$$= \lim_{\epsilon\to+0}\int\limits_{|x|>\epsilon}\frac{\varphi(x)}{x}dx = \Big(\rho\frac{1}{x}, \varphi(x)\Big).$$

M i s o l.

$$\left(\rho\frac{1}{x}\right)' = -\rho\frac{1}{x^2}$$

ekanligini ko‘rsating. Haqiqatan ham, ixtiyoriy $\varphi \in D(\mathbb{R})$ lar uchun

$$\left(\left(\rho\frac{1}{x}\right)', \varphi(x)\right) = -v.p.\int\limits_{\mathbb{R}} \frac{\varphi'(x)}{x}dx = -\lim_{\epsilon\to 0}\left(\int\limits_{-\infty}^{-\epsilon} + \int\limits_{\epsilon}^{\infty}\right)\frac{\varphi'(x)}{x}dx =$$

$$= -\lim_{\epsilon\to 0}\left[\left(\int\limits_{-\infty}^{-\epsilon} + \int\limits_{\epsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}dx - \left(\frac{\varphi(x)-\varphi(0)}{x} + \frac{\varphi(0)}{x}\right)\Big|_{-\epsilon}^{\epsilon}\right] =$$

$$= -\lim_{\epsilon\to 0}\left(\int\limits_{-\infty}^{-\epsilon} + \int\limits_{\epsilon}^{\infty}\right)\frac{\varphi(x)-\varphi(0)}{x^2}dx =$$

$$= v.p.\int\limits_{\mathbb{R}}\frac{\varphi(x)-\varphi(0)}{x^2}dx = \left(\rho\frac{1}{x^2}, \varphi(x)\right).$$

Bu yerda, ushbu

$$\lim_{\epsilon\to 0}\left(\frac{\varphi(x)-\varphi(0)}{x}\right)\Big|_{-\epsilon}^{\epsilon} = 0$$

(Lopital qoidasiga ko‘ra),

$$-\frac{\varphi(0)}{x}\Big|_{-\epsilon}^{\epsilon} = \int\limits_{-\epsilon}^{\epsilon}\frac{\varphi(0)}{x^2}dx.$$

tengliklardan foydalanildi.

M i s o l. $\mathbb{R}^n$ da markazi x^0 nuqtada bo‘lgan sferik koordinatalar sistemasini qaraymiz:

$$x = x^0 + r\nu,\ r = |x - x^0|,\ \nu = (\nu_1, \nu_2, \ldots, \nu_n),\ |\nu| = 1,$$

$$\nu_1 = \cos\theta_1,$$

$$\nu_2 = \sin\theta_1\cos\theta_2,\ \ 0 \le \theta_k \le \pi,\ k = 1, 2, \ldots, n-2,$$

$$\nu_3 = \sin\theta_1\sin\theta_2\cos\theta_3,\ \ 0 < \theta_{n-1} < 2\pi,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\nu_{n-1} = \sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1},$$

$$\nu_n = \sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1}.$$

Delta funksiyasi uchun quyidagi tenglik o'rinli:

$$\delta(x - x^0) = \frac{(-1)^{n-1}2}{\omega_n(n-1)!}\delta^{n-1}(r),$$

bu yerda $\omega_n - \mathbb{R}^n$ da birlik sfera sirti yuzi.

I s b o t. $\delta(t)$ funksiyaning juftligidan ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$\int_0^\infty \delta(t)\varphi(t)dt = \frac{1}{2}\int_{\mathbb{R}} \delta(t)\varphi(t)dt = \frac{1}{2}\varphi(0)$$

ekanligi kelib chiqadi. Ma'lumki, hajm elementi sferik koordinatalar sistemasida quyidagicha aniqlanadi:

$$dx = r^{n-1}drd\omega_\nu$$

bu yerda $d\omega_\nu$−birlik sfera sirt elementi. Quyidagi tengliklar ketma-ketligi asosiy formulaning to'g'riligini ko'rsatadi:

$$\Big(\frac{(-1)^{n-1}2}{\omega_n(n-1)!}\delta^{n-1}(|x - x^0|), \varphi(x)\Big) =$$

$$= \frac{(-1)^{n-1}2}{\omega_n(n-1)!}\int_0^\infty r^{n-1}\delta^{n-1}(r)\int_{|\nu|=1} \varphi(x^0 + r\nu)drd\omega_\nu =$$

$$= \frac{2}{\omega_\nu}\int_0^\infty \delta(r)\int_{|\nu|=1} \varphi(x^0 + r\nu)drd\omega_\nu = \frac{1}{\omega_\nu}\int_{|\nu|=1} \varphi(x^0 + r\nu)d\omega_\nu\Big|_{r=0} =$$

$$= \varphi(x^0) = (\delta(x - x^0), \varphi(x)).$$

M i s o l. $x^m f(x) = 0$ tenglamaning umumlashgan yechimlari topilsin.

Dastlab, $xf(x) = 0$ tenglamaning umumlashgan yechimi $f(x) = C\delta(x)$ ekanligini isbotlaymiz.

Buning uchun $\varphi_0(0) = 1$ qo'shimcha shartni qanoatlantiruvchi ixtiyoriy $\varphi_0(x)$ asosiy funksiya uchun

$$\psi(x) = \frac{1}{x}\left[\varphi(x) - \varphi(0)\varphi_0(x)\right]$$

yordamchi funksiyani qaraymiz, bu yerda $\varphi(x)$ - ixtiyoriy asosiy funksiya. $\psi(x) \in D(\mathbb{R})$ ekanligini ko‘rsatamiz. Bu funksiyaning finitligi $\varphi_0(x)$ va $\varphi(x)$ funksiyalarning finitligidan kelib chiqadi.

$\psi(x)$ funksiyani ushbu

$$\psi(x) = \frac{1}{x}\int\limits_0^x [\varphi'(x) - \varphi(0)\varphi_0'(x)] = \int\limits_0^1 [\varphi'(xt) - \varphi(0)\varphi_0'(xt)]$$

ko‘rinishda yozish mumkin. Bundan va $\varphi_0(x)$, $\varphi(x)$ funksiyalarning cheksiz differensiallanuvchanlidan, $\psi(x)$ ning ham cheksiz differensialga ega bo‘lishi kelib chiqadi.

$xf(x) = 0$ tenglamani qanoatlantiruvchi $f(x)$ funksionalning qiymatini $\psi(x)$ funksiya yordamida ifodalangan ixtiyoriy $\varphi(x)$ asosiy funksiyada quyidagicha bo‘ladi:

$$(f, \varphi) = (f, x\psi + \varphi(0)\varphi_0) = (xf, \psi) + \varphi(0)\,(f, \varphi_0) = C\,(\delta, \varphi)\,,$$

bu yerda $C = (f, \varphi_0)$ (eslatib o‘tamiz φ_0(x) ixtiyoriy ravishda tanlangan) va $\psi(x) \in D(\mathbb{R})$ ekanligi uchun $(xf(x), \psi(x)) = 0$. Shunday qilib, $f(x) = C\delta(x)$.

$m = 2$ bo‘lganda, ya’ni $x^2 f(x) = 0$ tenglamaning umumlashgan yechimi $f(x) = C_1\delta(x) + C_2\delta'(x)$ bo‘lishi quyidagi tengliklardan kelib chiqadi:

$$\big(x^2 f(x), \varphi(x)\big) = \big(x^2\left[C_1\delta(x) + C_2\delta'(x)\right], \varphi(x)\big) =$$
$$= C_1\left(x^2\delta(x), \varphi(x)\right) + C_2\left(x^2\delta'(x), \varphi(x)\right) =$$
$$= C_1\left(\delta(x), x^2\varphi(x)\right) + C_2\left(\delta'(x), x^2\varphi(x)\right) =$$
$$= C_1\left[x^2\varphi(x)\right]_{x=0} + C_2\left[x^2\varphi(x)\right]'_{x=0} = 0 = (0, \varphi(x))\,,$$

bu yerda $\varphi(x) \in D(\mathbb{R})$.

Ixtiyoriy natural m soni uchun $x^m f(x) = 0$ tenglamaning yechimi

$$f(x) = \sum_{i=0}^{m-1} C_i \frac{d^i}{dx^i}\delta(x)$$

ekanligi yuqoridagi kabi ko‘rsatiladi. Haqiqatan ham, ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ uchun

$$(x^m f(x), \varphi(x)) =$$

$$= \left(x^m \sum_{i=0}^{m-1} C_i \frac{d^i}{dx^i}\delta(x), \varphi(x) \right) = \left(\sum_{i=0}^{m-1} C_i x^m \frac{d^i}{dx^i}\delta(x), \varphi(x) \right) =$$

$$= \left(\sum_{i=0}^{m-1} C_i \frac{d^i}{dx^i}\delta(x), x^m\varphi(x) \right) = \sum_{i=0}^{m-1} C_i \left\{ \frac{d^i}{dx^i}\left[x^m\varphi(x)\right] \right\}_{x=0} = 0 = (0, \varphi(x))$$

tengliklar yuqoridagi mulohazani isbotlaydi.

M i s o l.

$$H_n(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4a^2 t}}$$

funksiyani qaraymiz va

$$\int\limits_{\mathbb{R}^n} H_n(x,t)dx = 1,$$

$$H_n(x,t) = \frac{1}{(4\pi a^2 t)^{n/2}} e^{-\frac{|x|^2}{4a^2 t}} \longrightarrow \delta(x),\ t \to +0$$

munosabatlarning bajarilishini ko‘rsatamiz.

Haqiqatan ham, $t < 0$ da $H_n = 0$ va $t > 0$ da

$$\int\limits_{\mathbb{R}^n} H_n(x,t)dx = \frac{1}{(2a\sqrt{\pi t})^n} \int\limits_{\mathbb{R}} e^{-\frac{|x|^2}{4a^2 t}}dx = \prod_{i=1}^{n} \frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} e^{-\xi_i^2} d\xi_i = 1.$$

Endi $\varphi(x) \in D(\mathbb{R}^n)$ bo‘lsin. U holda $|\varphi(x) - \varphi(0)| \le K|x|,\ K = const$ dan

$$\left| \int\limits_{\mathbb{R}^n} H_n(x,t)[\varphi(x) - \varphi(0)]dx \right| \le \frac{K}{(4\pi a^2 t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\frac{|x|^2}{4a^2 t}} |x| dx =$$

$$= \frac{K\omega_n}{(4\pi a^2 t)} \int_0^\infty e^{-\frac{r^2}{4a^2 t}} r^n dr = \frac{2K\omega_n \sqrt{ta}}{n^{n/2}} \int\limits_0^\infty e^{-u^2} u^n du = C\sqrt{t},$$

$C = const$, $\omega - \mathbb{R}^n$ da birlik sfera yuzi. Bu yerda $t \to +0$ deb limitga o‘tib,

$$(H_n(x,t), \varphi) = \int\limits_{\mathbb{R}^n} H_n(x,t)\varphi(x)dx = \varphi(0) \int\limits_{\mathbb{R}^n} H_n(x,t)dx +$$

$$+ \int\limits_{\mathbb{R}^n} H_n(x,t)[\varphi(x) - \varphi(0)]dx \to \varphi(0) = (\delta, \varphi)$$

yuqoridagi limit munosabatning o‘rinli ekanligiga ishonch hosil qilamiz.

1.8 Umumlashgan funksiyalarning yig‘masi va uning xossalari

T a ’ r i f (Klassik funksiyalarning yig‘masi). $F:\mathbb{R}\to\mathbb{R}$ va $G:\mathbb{R}\to\mathbb{R}$ klassik funksiyalarning yig‘masi deb,

$$(f*g)(x)=\int_{\mathbb{R}} f(y)g(x-y)dy$$

ko‘rinishdagi funksiyaga aytiladi, bunda integral barcha $x\in\mathbb{R}$ da yaqinlashuvchi bo‘lishi talab etiladi.

Yigma mavjudligining (integralning yaqinlashishi) yetarli shartlari sifatida $F(y)G(x-y)$ funksiyaning ixtiyoriy tayin x uchun lokal integrallanuvchi bo‘lishi va quyidagi shartlardan birortasining bajarilishi talab etiladi: 1) $F(x)$ va $G(x)$ funksiyalarning birortasi finit, 2) ikkala funksiya ham yarimfinit, ya’ni $x\le a$ yoki $x\ge b$ lar uchun nolga teng. Ravshanki, funksiyalarning finitligi o‘rniga ularning cheksizlikda tez kamayuvchanlik shartini talab qilsa ham bo‘ladi.

Aytaylik, $f(x),g(y)$ lar $\mathbb{R}$ da lokal integrallanuvchi funksiyalar bo‘lsin.

T a ’ r i f (D' dagi umumlashgan funksiyalarning yig‘masi). $f\in D'(\mathbb{R})$ va $g\in D'(\mathbb{R})$ funksiyalarning yig‘masi deb, ixtiyoriy $\varphi\in D(\mathbb{R})$ uchun

$$(f*g(x),\varphi(x))=(f(x),(g(y),\varphi(x+y))) \tag{17}$$

formula yordamida aniqlangan akslantirishga aytiladi, bunda $D'(\mathbb{R})$ dagi umumlashgan funksiyalarda (17) formula korrekt aniqlanadi.

T e o r e m a (Umumlashgan funksiyalar yig‘masining mavjudligi haqida). Faraz qilaylik, $f\in D'(\mathbb{R})$, $g\in D'(\mathbb{R})$ va quyidagi shartlarning hech bo‘lmaganda bittasi o‘rinli bo‘lsin:
(1) shunday $M>0$ son mavjud bo‘lib, $\mathrm{supp} f\subset(-M,M)$ bo‘lsa;
(2) shunday $M>0$ son mavjud bo‘lib, $\mathrm{supp} g\subset(-M,M)$ bo‘lsa;
(3) shunday $M\ge 0$ son mavjud bo‘lib, $\mathrm{supp} f\subset(M,+\infty)$ va $\mathrm{supp} g\subset(M,+\infty)$ bo‘lsa;
(4) shunday $M\ge 0$ son mavjud bo‘lib, $\mathrm{supp}\ f\subset(-\infty,M)$ va $\mathrm{supp} g\subset(-\infty,M)$ bo‘lsa, u holda $f*g\in D'(\mathbb{R})$ yig‘ma korrekt aniqlangan.

Teoremaning isboti yig'maning ta'rifidan kelib chiqadi, masalan, 3-bandni isbotlaylik. Haqiqatan ham, ixtiyoriy katta R ($R > M$) soni uchun

$$\left|\int_{-R}^{R}\int_{\mathbb{R}} f(y)g(x-y)dydx\right| = \int_{M}^{R}\int_{M}^{x} |g(y)|\,|f(x-y)|\,dydx =$$

$$= \int_{M}^{R} |g(y)| \int_{y}^{R} |f(x-y)|dydx \leq \int_{M}^{R} |g(y)|dy \int_{M}^{R} |f(\xi)|d\xi < \infty.$$

T e o r e m a (Umumlashgan funksiyalar yig'masining xossalari).

Faraz qilaylik, $f \in D'(\mathbb{R})$, $g \in D'(\mathbb{R})$ va yuqoridagi teoremaning (1)-(4) shartlaridan kamida bittasi bajarilsin. U holda

(1) $f * g = g * f$;

(2) $(f * g)' = f' * g = f * g'$ munosabatlar o'rinli bo'ladi.

I s b o t. Oldingi teoremaning (1)-sharti o'rinli bo'lgan holda isbotlaymiz. Qolgan hollar shunga o'xshash isbotlanadi. Aytaylik, $\chi-$funksiya quyidagicha aniqlangan bo'lsin:

- $\chi \in C^{\infty}(\mathbb{R})$;
- $x \in [-R, R]$, $\chi(x) = 1$;
- $x\overline{\in}[-(R+1), R+1]$, $\chi(x) = 0$.

Funksiyalarning yig'masi ta'rifidan ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$(f * g(x), \varphi(x)) = \Big(f(x) \cdot g(y), \chi(x)\varphi(x+y)\Big) =$$

$$= \Big(g(y) \cdot f(x), \chi(x)\varphi(x+y)\Big) = (g * f(x), \varphi(x)).$$

Teoremaning (2)-shartining bajarilishini ko'rsatamiz. Ixtiyoriy $\varphi \in D(\mathbb{R})$ uchun

$$((f * g)'(x), \varphi(x)) = -(f * g(x), \varphi'(x)) =$$

$$= -\Big(f(x) \cdot g(y), \chi(x)\varphi'(x+y)\Big) \stackrel{\mathrm{d}}{=}$$

$$\stackrel{\mathrm{d}}{=} -\Big(f(x) \cdot g(y), \partial_x(\chi(x)\varphi'(x+y))\Big) + \Big(f(x) \cdot g(y), \chi'(x)\varphi(x+y)\Big) =$$

$$= \Big(\partial_x(f(x) \cdot g(y)), \chi(x)\varphi(x+y)\Big) + \Big((\chi'(x)f(x)) \cdot g(y), \varphi(x+y)\Big) =$$

$$= \Big(f'(x)\cdot g(y), \chi(x)\varphi(x+y)\Big) = \Big(f' * g(x), \varphi(x)\Big),$$

$$\overset{\mathrm{d}}{=} -\Big(f(x)\cdot g(y), \partial_y(\chi(x)\varphi'(x+y))\Big) = \Big(\partial_y(f(x)\cdot g(y)), \chi(x)\varphi(x+y)\Big) =$$

$$= \Big(f(x)\cdot g'(y), \chi(x)\varphi(x+y)\Big) = \Big(f * g'(x), \varphi(x)\Big).$$

Bu yerda $\chi'(x)f(x) = 0$ ekanligidan foydalanildi.

M i s o l. $\delta(x-a) * f(x)$ ifodani soddalashtiring, bu yerda $a \in \mathbb{R}$ va $f \in D'(\mathbb{R})$.

Ixtiyoriy $\varphi \in D(\mathbb{R})$ lar uchun

$$(\delta(x-a) * f(x), \varphi) = \Big(\delta(x-a), (f(y), \varphi(x+y))\Big) =$$

$$= (f(y), \varphi(y+a)) = (f(x-a), \varphi(x)).$$

Demak, $\delta(x-a) * f(x) = f(x-a)$.

M i s o l. $\theta * \theta(x)$ ifodani soddalashtiring.

Ixtiyoriy $\varphi \in D(\mathbb{R})$ lar uchun

$$(\theta * \theta, \varphi) = \Big(\theta(x), (\theta(y), \varphi(x+y))\Big) = \int_0^\infty dx \int_0^\infty \varphi(x+y)dy =$$

$$= \int_0^\infty dx \int_x^\infty \varphi(z)dz = \int_0^\infty dz \int_0^z \varphi(z)dy =$$

$$= \int_0^\infty z\varphi(z)dz = (z\theta(z), \varphi(z))$$

Demak, $\theta * \theta(x) = x\theta(x)$.

Endi $x \in \mathbb{R}^n$ bo‘lsin. Umumlashgan funksiyalar yig‘masining ba’zi xossalarini keltirib o‘tamiz.

a) (Ko‘chish). Agar ushbu, $f * g$ yig‘ma mavjud bo‘lsa, u holda barcha $h \in \mathbb{R}^n$ larda $f(x+h) * g(x)$ mavjud bo‘lib,

$$f(x+h) * g(x) = (f * g)(x+h)$$

munosabat o‘rinli.

b) (Qaytarish). Agar ushbu $f * g$ yig‘ma mavjud bo‘lsa, u holda

$$f(-x) * g(-x) = (f * g)(-x)$$

munosabat o‘rinli.

c) (Differensiallash). Agar ushbu, $f * g$ yig‘ma mavjud bo‘lsa, u holda $D^\alpha f * g$, $f * D^\alpha g$ lar ham mavjud bo‘lib,

$$D^\alpha f * g = D^\alpha (f * g) = f * D^\alpha g.$$

Bu xossalarning o‘rinli ekanligi bevosita yig‘maning ta’rifidan kelib chiqadi.

1.9 Dirakning delta funksiyasi xossalari

Quyida keltirilgan dirakning delta funksiyasi xossalari xususiy hosilali differensial tenglamalar uchun qo‘yilgan boshlang‘ich, boshlang‘ich-chegaraviy va chegaraviy masalalarni yechishda keng qo‘llaniladi.

1) $y = \omega(x) : \mathbb{R}^n \to \mathbb{R}^n-$ cheksiz differensiallanuvchi akslantirish uchun $x = \omega^{-1}(y)-$ teskari akslantirish mavjud va $\det \frac{\partial \omega}{\partial x} \neq 0$ bo‘lsin. U holda

$$\delta(\omega(x)) = \frac{\delta(x - x^0)}{\left| \det \frac{\partial \omega}{\partial x} \right|_{x=x^0}};$$

2) agar $x \in \mathbb{R}^n$ bo‘lsa, u holda

$$\delta(x - x^0) = \frac{2(-1)^{n-1}}{\omega_n (n-1)!} \delta^{(n-1)}(r),$$

bu yerda $r = |x - x^0|$, $\omega_n - \mathbb{R}^n$ da birlik sferaning yuzi.

3) agar $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$ bo‘lsa, u holda quyidagi tengliklar o‘rinli bo‘ladi:

a) $\delta(x,t) = -\frac{1}{2a}\delta'(t) \begin{cases} \frac{1}{\pi}\delta(\mathit{\Gamma}), \ n = 3, \\ \frac{\theta(\mathit{\Gamma})}{\pi\sqrt{\mathit{\Gamma}}}, \ n = 2, \\ \theta(\mathit{\Gamma}), \ n = 1. \end{cases}$

b) $0 = -\frac{1}{2a}\delta(t)\begin{cases} \frac{1}{\pi}\delta(\mathit{\Gamma}),\ n=3, \\ \frac{\theta(\mathit{\Gamma})}{\pi\sqrt{\mathit{\Gamma}}},\ n=2, \\ \theta(\mathit{\Gamma}),\ n=1, \end{cases}$

bu yerda $\mathit{\Gamma} = a^2t^2 - |x|^2$, $a = const > 0$.

1) - 2) xossalarning o‘rinli ekanligi oldingi paragraflarda ko‘rsatilgan edi. 3) xossaning birinchi tengligini isbot qilamiz. Ixtiyoriy $\varphi(x,t) \in D(\mathbb{R}^4)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}^1$ lar uchun

$$\Big(-\delta'(t)\delta(a^2t^2-|x|^2), \varphi(x,t)\Big) = -\int\limits_{\mathbb{R}^3} \delta'(t)\delta(a^2t^2-|x|^2)\varphi(x,t)dxdt =$$

$$= \Big(\delta'(t), \int\limits_{\mathbb{R}^3} \delta(a^2t^2-|x|^2)\varphi(x,t)dxdt\Big) =$$

$$= \frac{\partial}{\partial t}\Big[\int\limits_0^\infty\int\limits_0^{2\pi}\int\limits_0^\pi \delta(a^2t^2-r^2)\varphi(r\nu,t)r^2\sin\theta d\theta d\varphi dr\Big]_{t=0}$$

tengliklar o‘rinli, bu yerda

$$\nu = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi).$$

Oxirgi tenglik bo‘laklab integrallash yordamida hosil bo‘ladi. $w(r) = r^2 - a^2t^2$ deb, 1) xossadan foydalanamiz:

$$\delta(a^2t^2-r^2) = \frac{\delta(r-at)}{2r},\ \ r > 0.$$

Bu formulani e’tiborga olib quyidagiga ega bo‘lamiz:

$$\frac{\partial}{\partial t}\Big[\int\limits_0^\infty\int\limits_0^{2\pi}\int\limits_0^\pi \delta(a^2t^2-r^2)\varphi(r\nu,t)r^2\sin\theta d\theta d\varphi dr\Big]_{t=0} =$$

$$= \frac{\partial}{\partial t}\Big[\int\limits_0^\infty\int\limits_0^{2\pi}\int\limits_0^\pi \frac{\delta(r-at)}{2r}\varphi(r\nu,t)r^2\sin\theta d\theta d\varphi dr\Big]_{t=0} =$$

$$= \frac{\partial}{\partial t}\Big[\frac{at}{2}\int\limits_0^{2\pi}\int\limits_0^\pi \varphi(at\nu,t)\sin\theta d\theta d\varphi\Big]_{t=0} =$$

$$= \frac{a}{2}\Big[\int_0^{2\pi}\int_0^{\pi} \varphi(at\nu, t)\sin\theta d\theta d\varphi\Big]_{t=0} +$$

$$+\Big[\frac{at}{2}\cdot\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} \varphi(at\nu, t)\sin\theta d\theta d\varphi\Big]_{t=0} = 2\pi a\varphi(0,0) =$$

$$= (2\pi a\delta(x,t), \varphi(x,t)).$$

Bu tengliklar 3.a) ning birinchi formulasini isbot qiladi. 3.a) ning ikkinchi formulasini asoslaymiz. Ixtiyoriy $\varphi(x,t) \in D(\mathbb{R}^3)$, $x \in \mathbb{R}^2$, $t \in \mathbb{R}^1$ uchun quyidagi munosabatlar oʻrinli:

$$\Big(-\delta'(t)\frac{\theta(a^2t^2-|x|^2)}{\sqrt{a^2t^2-|x|^2}}, \varphi(x,t)\Big) = -\Big(\delta'(t)\int_{\mathbb{R}^2}\frac{\theta(a^2t^2-|x|^2)}{\sqrt{a^2t^2-|x|^2}}\varphi(x,t)dx\Big) =$$

$$= \frac{\partial}{\partial t}\Big[\int_{\mathbb{R}^2}\frac{\theta(a^2t^2-|x|^2)}{\sqrt{a^2t^2-|x|^2}}\varphi(x,t)dx\Big]_{t=0} =$$

$$= \frac{\partial}{\partial t}\Big[\int_0^{2\pi}\int_0^{\infty}\frac{\theta(a^2t^2-|x|^2)}{\sqrt{a^2t^2-|x|^2}}\varphi(\nu r,t)rdrd\varphi\Big]_{t=0},$$

bu yerda $\nu = (\cos\varphi, \sin\varphi)$. Oxirgi tenglikda oʻzgaruvchini almashtiramiz: $r = at\cdot z$:

$$\frac{\partial}{\partial t}\Big[\int_0^{2\pi}\int_0^{\infty}\frac{\theta(a^2t^2-|x|^2)}{\sqrt{a^2t^2-|x|^2}}\varphi(\nu r,t)rdrd\varphi\Big]_{t=0} =$$

$$= \frac{\partial}{\partial t}\Big[at\int_0^{2\pi}\int_0^{\infty}\frac{\theta(1-z^2)}{\sqrt{1-z^2}}\varphi(atz\nu,t)zdzd\varphi\Big]_{t=0} =$$

$$= a\varphi(0,0)\int_0^{2\pi}\int_0^{\infty}\frac{\theta(1-z^2)}{\sqrt{1-z^2}}\varphi(atz\nu,t)zdzd\varphi =$$

$$2\pi a\int_0^1 \frac{zdz}{\sqrt{1-z^2}} = 2\pi a\varphi(0,0) = (2\pi a\delta(x,t), \varphi(x,t)).$$

3.a) ning oxirgi tengligini asoslashga o‘tamiz. Ixtiyoriy $\varphi(x,t) \in D(\mathbb{R}^2)$, $x \in \mathbb{R}^1$, $t \in \mathbb{R}^1$ uchun:

$$(-\delta'(t)\theta(a^2t^2 - x^2), \varphi(x,t)) =$$

$$= -\Big(\delta'(t), \int_{\mathbb{R}^1} \theta(a^2t^2 - x^2)\varphi(x,t)dx\Big) = \frac{\partial}{\partial t}\Big[\int_{-at}^{at} \frac{\partial}{\partial t}\varphi(x,t)dx\Big]_{t=0} =$$

$$= 2a\varphi(0,0) = (2a\delta(x,t), \varphi(x,t)).$$

3.b) tengliklar quyidagicha isbot qilinadi. 3. a) tengliklari t ga ko‘paytiriladi. $t\delta(x,t) = t\delta(t)\delta(x) = 0\delta(x) = 0$, $-t\delta'(t) = \delta(t)$ ekanligini hisobga olsak, 3. b) tengliklar kelib chiqadi.

1.10 Sekin o‘suvchi umumlashgan funksiyalar va ularning Furye almashtirishi

1.10.1 Klassik Furye almashtirishi

Bu paragrafda klassik Furye almashtirishi haqida ma’lumot keltirib o‘tamiz. Bu yerda bajariladigan amallar ma’noga ega bo‘lishi uchun Furye almashtirishini aniqlanadigan funksiyani yetarlicha silliq va cheksizlikda tez kamayuvchi (Shvars ma’nosida) deb faraz qilamiz.

Ta’rif. $f(x)$, $x \in \mathbb{R}$ funksiyaning Furye almashtirishi deb,

$$\widetilde{f}(\xi) := \int_{\mathbb{R}} e^{i\xi x} f(x)dx$$

tenglik yordamida aniqlangan ξ o‘zgaruvchili funksiyaga aytiladi.

Zarurat bo'lganda $\widetilde{f}(\xi)$ o'rniga umumiyroq bo'lgan $F[f(x)](\xi)$ belgilashdan ham foydalanamiz.

Tabiiy ravishda Furye almashtirishi kompleks qiymatli funksiyalar sinfida qaraladi, ya'ni $\widetilde{f}(\xi)$ funksiya kompleks qiymat qabul qiladi va hattoki f haqiqiy qiymatli funksiya bo'lganida ham. Juft funksiyalar bundan istisno.

Furye almashtirishining ba'zi xossalarini keltiramiz:

$$1)\ F[x^m f(x)](\xi) = -i\frac{\partial}{\partial\xi}\int\limits_{\mathbb{R}} x^{m-1}f(x)e^{i\xi x}dx = \cdots = (-i)^m \widetilde{f}^{(m)}(\xi);$$

$$2)\ F[f^{(m)}(x)](\xi) = -\int\limits_{\mathbb{R}} f^{(m-1)}(x)\frac{\partial}{\partial x}e^{i\xi x}dx = \cdots = (-i\xi)^m \widetilde{f}(\xi);$$

$$3)\ F[f(x-x_0)](\xi) = \int\limits_{\mathbb{R}} f(y)e^{i\xi(y+x_0)}dy = e^{i\xi x_0}\widetilde{f}(\xi);$$

$$4)\ F[f(x)e^{i\xi_0 x}](\xi) = \int\limits_{\mathbb{R}} f(x)e^{i(\xi+\xi_0)x}dx = \widetilde{f}(\xi+\xi_0).$$

1 va 2-xossalarning isboti bo'laklab integrallash formulasidan $f(x)$ funksiya va uning hosilalarining cheksizlikda tez kamayuvchi ekanligini hisobga olgan holda kelib chiqadi. 3 va 4-xossalar esa sodda almashtirishlar natijasida hosil bo'ladi.

E s l a t m a. Tasdiqning 1-xossasidan ko'rinib turibdiki, agar $f(x)$ funksiya cheksizlikda $o\left(\frac{1}{|x|^m}\right)$ tartibda kamayuvchi bo'lsa (bunda 1-xossadagi $F[x^m f(x)](\xi)$ mavjud bo'ladi), u holda uning Furye almashtirishi m tartibli hosilaga ega bo'ladi; shunga o'xshash 2) dan ko'rish mumkinki, agar $f(x)$ original funksiya qanchalik ko'p hosilalarga ega bo'lsa, u holda uning $\widetilde{f}(\xi)$ Furye almashtirishi cheksizlikda shuncha tez kamayadi.

Faraz qilamiz, o'z navbatida $\widetilde{f}(\xi)$ funksiyaning ham Furye almashtirishi mavjud bo'lsin.

T a ' r i f. $\widetilde{f}(\xi)$ funksiyaning teskari Furye almashtirishi deb,

$$F^{-1}[\widetilde{f}(\xi)](x) := \frac{1}{2\pi}F[\widetilde{f}(\xi)](-x) \equiv \frac{1}{2\pi}\int \widetilde{f}(\xi)e^{-i\xi x}d\xi \equiv \frac{1}{2\pi}F[\widetilde{f}(-\xi)](x)$$

tenglik bilan aniqlangan $F^{-1}[\widetilde{f}(\xi)](x)$ funksiyaga aytiladi.

T e o r e m a. $F^{-1}[\widetilde{f}(\xi)](x) = f(x)$ munosabat o‘rinli.

Teoremaning isboti Furye qatorining qaytish formulasidan kelib chiqadi. Buni matematik tahlil fani biror darsligidan ko‘rib olish mumkin. Yuqoridagi ta’rifga asosan $F[\widetilde{f}(\xi)] = 2\pi f(-x)$ tenglik o‘rinli.

T a s d i q (Furye almashtirishining klassik yig‘masi). Funksiyalar yig‘masining Furye almashtirishi, bu funksiyalar Furye almashtirishlari ko‘paytmasiga teng:

$$F[(f * g)(x)](\xi) = \widetilde{f}(\xi) \cdot \widetilde{g}(\xi).$$

Isbot. Quyidagi tengliklar tasdiqning o‘rinli ekanligini ko‘rsatadi:

$$F[(f * g)(x)](\xi) = \int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}} f(t)g(x-t)dt \Big) e^{i\xi x} dx =$$

$$= \int\limits_{\mathbb{R}} f(t)dt \Big(\int\limits_{\mathbb{R}} g(x-t)e^{i\xi x} dx \Big) =$$

$$= \int\limits_{\mathbb{R}} f(t)e^{i\xi t} dx \int\limits_{\mathbb{R}} g(y)e^{i\xi y} dx = \widetilde{f}(\xi)\widetilde{g}(\xi).$$

T a s d i q (Parseval tengligi). Parseval nomi bilan yuritiluvchi ushbu

$$\int\limits_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \left| \widetilde{f}(\xi) \right|^2 d\xi$$

tenglik o‘rinli.

Isbot. Ushbu

$$\int\limits_{\mathbb{R}} |f(x)|^2 \, dx = \int\limits_{\mathbb{R}} f(x) \cdot \overline{f}(x) dx = \int\limits_{\mathbb{R}} \Big(\frac{1}{2\pi} \int\limits_{\mathbb{R}} \widetilde{f}(\xi) e^{-i\xi x} \Big) \overline{f}(x) dx =$$

$$= \frac{1}{2\pi} \int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}} \overline{f}(x) e^{-i\xi x} dx \Big) \widetilde{f}(\xi) d\xi = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \left| \widetilde{f}(\xi) \right|^2 d\xi$$

tengliklar tasdiqning o‘rinli ekanligini isbotlaydi.

E s l a t m a. Agar funksiyaning normasini

$$||f||^2 := \int\limits_{\mathbb{R}} |f(x)|^2 dx$$

ko'rinishda kiritsak, u holda Parseval tengligini

$$\frac{1}{2\pi}||Ff|| = ||f||$$

shaklda yozish mumkin, ya'ni Furye almashtirishi ($\frac{1}{2\pi}$ ko'paytuvchi aniqligigacha) unitarlik xossasiga ega.

Kvant mexanikasida ko'p ishlatiladigan quyidagi tengsizlik Furye almashtirishi xossalaridan kelib chiqadi:

$$\int\limits_{\mathbb{R}} |xf(x)|^2\, dx \int\limits_{\mathbb{R}} \left|\xi \tilde{f(\xi)}\right|^2 dx \geq \frac{\pi}{2}\left(\int\limits_{\mathbb{R}} |f(x)|^2\, dx\right)^2. \tag{18}$$

Bu tenglikni isbotlash uchun t haqiqiy parametrga bog'liq bo'lgan

$$J(t) := \int\limits_{\mathbb{R}} \left|txf(x) + f^{'}(x)\right|^2 dx \tag{19}$$

integralni qaraymiz. Osongina payqash mumkinki, $J(t)$ funksiya t bo'yicha nomanfiy kvadratik uchhad. Undan tashqari, agarda ushbu

$$\int\limits_{\mathbb{R}} \left|f^{'}(x)\right|^2 dx = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left|\widetilde{f'}(\xi)\right|^2 d\xi = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left|\xi\widetilde{f}(\xi)\right|^2 d\xi$$

va

$$\int\limits_{\mathbb{R}} \left(xf(x)\overline{f'} + x\overline{f}(x)f^{'}(x)\right)dx = -\int\limits_{\mathbb{R}} |f(x)|^2\, dx$$

munosabatlarning o'rinli ekanligini hisobga olsak, (18) tengsizlik (19) kvadrat uchhadning diskriminanti noldan katta bo'lmasligini anglatadi.

1.10.2 Asosiy va umumlashgan funksiyalarning Furye almashtirishi

$D(\mathbb{R}^n)$ dan olingan $\varphi(x)$ funksiya $\mathbb{R}^n$ da lokal integrallanuvchi bo'lganligi sababli, bunday funksiyalar uchun Furye almashtirishi aniqlangan:

$$F[\varphi](\xi) = \int\limits_{\mathbb{R}^n} \varphi(x)e^{i(\xi,x)}dx, \tag{20}$$

bu yerda

$$(\xi, x) = \sum_{i=1}^{n} \xi_i x_i,$$

$F[\varphi](\xi)-$ $\varphi(x)$ funksiyaning Furye almashtirishi.

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ manfiy bo'lmagan, komponentalari butun α_j sonlardan iborat vector bo'lsin (multiindex). $D^\alpha f(x)$ orqali $f(x)$ funksiyaning $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ tartibli hosilasini belgilaymiz:

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, x_2, ..., x_n)}{\partial x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}}, \; D^0 f(x) = f(x);$$

$$D = (D_1, \; D_2, \; ..., \; D_n), \; D_j = \frac{\partial}{\partial x_j}, \; j = 1, 2, ..., n.$$

Shuningdek, keyinchalik yozuvlarni qisqartirish maqsadida

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}, \;\; \alpha! = \alpha_1! \alpha_2! ... \alpha_n!$$

kabi belgilashlar ham ishlatiladi.

$\varphi(x)$ asosiy funksiyalar uchun (20) integral aslida chekli soha bo'yicha integraldan iborat. Shuning uchun Furye almashtirishini ξ o'zgaruvchi bo'yicha integral ostida istalgancha differensiallash mumkin:

$$D^\alpha F[\varphi](\xi) = \int (ix)^\alpha \varphi(x) e^{i(\xi,x)} dx = F[(ix)^\alpha \varphi](\xi).$$

$D(\mathbb{R}^n)$ dan olingan $\varphi(x)$ funksiyalarning Furye almashtirishi $\mathbb{R}^n$ da absolyut integrallanuvchi va uzluksiz differensiallanuvchi bo'lgani uchun, Furye almashtirishlarining umumiy nazariyasidan unga teskari F^{-1} almashtirishning mavjudligi kelib chiqadi:

$$\varphi(x) = F^{-1}[F[\varphi]] = F[F^{-1}[\varphi]],$$

bunda

$$F^{-1}[\varphi](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi) e^{-i(\xi,x)} d\xi =$$

$$= \frac{1}{(2\pi)^n} F[\varphi](-x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(-\xi) e^{-i(\xi,x)} d\xi =$$

$$= \frac{1}{(2\pi)^n} F[\varphi(-\xi)].$$

Endi $f(x)$ funksiya $\mathbb{R}^n$ da absolyut integrallanuvchi bo'lsin. U holda uning Furye almashtirishi

$$F[f](\xi) = \int\limits_{\mathbb{R}^n} f(x) e^{i(\xi,x)} dx, \quad \Big| F[f](\xi) \Big| \leq \int\limits_{\mathbb{R}^n} |f(x)| dx < \infty$$

$\mathbb{R}^n$ da uzduksiz va chegaralangan bo'lib, ixtiyoriy $\varphi \in D(\mathbb{R}^n)$ funksiyalar uchun umumlashgan funksiyani aniqlaydi:

$$(F[f], \varphi) = \int\limits_{\mathbb{R}^n} F[f](\xi) \varphi(\xi) d\xi.$$

Integrallash tartibini o'zgartirish haqidagi Fubini teoremasidan foydalanib, oxirgi integralni o'zgartiramiz:

$$\int\limits_{\mathbb{R}^n} F[f](\xi) \varphi(\xi) d\xi = \int\limits_{\mathbb{R}^n} \Big[\int\limits_{\mathbb{R}^n} f(x) e^{i(\xi,x)} dx \Big] \varphi(\xi) d\xi =$$

$$= \int\limits_{\mathbb{R}^n} f(x) \int\limits_{\mathbb{R}^n} \varphi(\xi) e^{i(\xi,x)} d\xi dx = \int\limits_{\mathbb{R}^n} f(x) F[\varphi](x) dx.$$

Demak,

$$(F[f], \varphi) = (f, F[\varphi]), \ f \in D'(\mathbb{R}^n), \ \varphi \in D(\mathbb{R}^n). \tag{21}$$

Bu tenglikni umumlashgan funksiyalarning Furye almashtirishi sifatida qabul qilamiz.

M i s o l.

$$F[\delta(x - x_0)] = e^{i(\xi, x_0)} \tag{22}$$

tenglikning o'rinli ekanligini ko'rsatamiz.

Haqiqatan ham, (21) ga asosan, ixtiyoriy $\varphi \in D(\mathbb{R}^n)$ uchun

$$(F[\delta(x - x_0)], \varphi) = (\delta(x - x_0), F[\varphi]) =$$

$$= F[\varphi](x_0) = \int\limits_{\mathbb{R}^n} \varphi(\xi) e^{i(\xi, x_0)} d\xi = (e^{i(\xi, x_0)}, \varphi).$$

Agar (22) da $x_0 = 0$ bo'lsa,

$$F[\delta] = 1$$

bo'ladi. Bu yerdan

$$\delta(x) = F^{-1}[1] = \frac{F[1]}{(2\pi)^n}.$$

Shuning uchun

$$F[1] = (2\pi)^n \delta(\xi).$$

M i s o l. $n = 1$ bo'lsin. Quyidagi tengliklarni isbotlang:

$$F[\theta(r - |x|)] = \int_{-r}^{r} e^{ix\xi} dx = \frac{2 \sin r\xi}{\xi}, \tag{23}$$

$$F[e^{-\alpha^2 x^2}] = \frac{\sqrt{\pi}}{\alpha} e^{-\frac{\xi^2}{4\alpha^2}}. \tag{24}$$

(23) tenglikning o'rinli ekanligiga integralni hisoblab ishonch hosil qilish mumkin. (24) ni isbotlaymiz. Haqiqatan ham,

$$F[e^{-\alpha^2 x^2}] = \int_{\mathbb{R}} e^{-\alpha^2 x^2 + ix\xi} dx =$$

$$= \frac{1}{\alpha} \int_{\mathbb{R}} e^{-y^2 + i\frac{\xi}{\alpha} y} dy = \frac{1}{\alpha} e^{-\frac{\xi^2}{4\alpha^2}} \int_{\mathbb{R}} e^{-\left(y - \frac{i\xi}{2\alpha}\right)^2} dy =$$

$$= \frac{1}{\alpha} e^{-\frac{\xi^2}{4\alpha^2}} \int_{-\infty - \frac{i\xi}{2\alpha}}^{\infty - \frac{i\xi}{2\alpha}} e^{-\eta^2} d\eta.$$

Bu integralda integrallash $\mathrm{Im}\eta = \frac{\xi}{2\alpha}$ chiziqi bo'ylab amalga oshirilyapdi. Bu chiziqni haqiqiy o'qqa siljitish mumkinligi va ixtiyoriy $a \in \mathbb{R}$ uchun

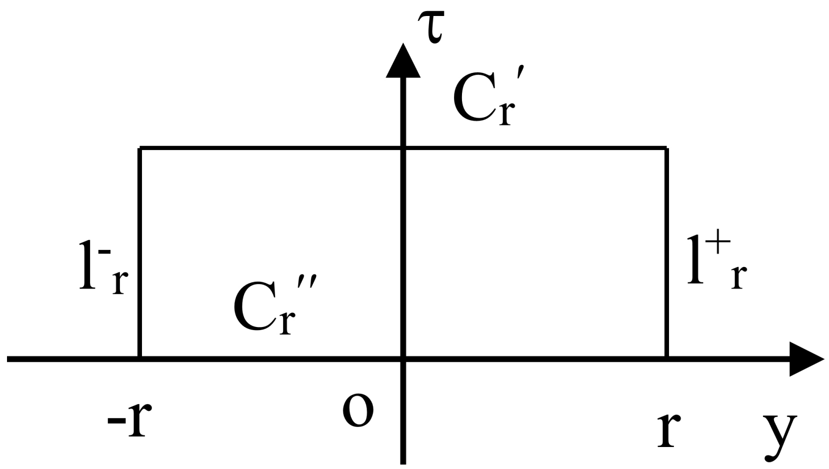
$$\int_{Im\eta = a} e^{-\eta^2} d\eta = \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi} \tag{25}$$

bo'lishini ko'rsatamiz.

Buning uchun Koshi teoremasiga asosan

$$\int_{C_r} e^{-\eta^2} d\eta = 0, \ \ \eta = y + i\tau \qquad (26)$$

tenglikning o'rinli ekanligidan foydalanamiz. Bu yerda $C_r = C_r' \cup C_r'' \cup e_r^+ \cup e_r^-$ kontur chizmada tasvirlangan.



5-chizma. C_r konturning tasviri.

$e_r^{\pm} := [0 \leq \tau \leq a, \, \tau = \pm r]$ kesmalarda

$$|e^{-\eta^2}| = |e^{-y^2+\tau^2-2iy\tau}| = e^{-r^2+\tau^2}$$

formula o'rinli bo'lib, $r \to \infty$ da $e^{-r^2+\tau^2} \rightrightarrows 0$ ekanligi uchun

$$\lim_{r\to\infty} \Big(\int_{e_r+} + \int_{e_r-} \Big) e^{-\eta^2} d\tau = 0$$

tenglik kelib chiqadi. Bunga va (26) formulaga asosan, (25) ning birinchi tengligiga ega bo'lamiz:

$$\lim_{r\to\infty} \int_{C_r} e^{-\eta^2} d\eta = \lim_{r\to\infty} \Big(\int_{C_r'} + \int_{C_r''} \Big) e^{-\tau^2} d\tau = \int_{-\infty}^{\infty} e^{-y^2} dy - \int_{\tau = ia} e^{-\tau^2} d\tau = 0.$$

Ikkinchi tenglikni isbot qilish uchun $f(x,y) = e^{-x^2-y^2}$ musbat, juft funksiyani va

$$\gamma_r = \Big\{ x, y : \, \sqrt{x^2+y^2} \leq r, \, \varphi = \arctan \frac{y}{x} \in \Big[0, \frac{\pi}{2}\Big] \Big\},$$

$$\gamma_0 = \Big\{x, y:\ 0 \leq x \leq r,\ 0 \leq x \leq r\Big\},$$

$$\gamma_{\sqrt{2}r} = \Big\{x, y:\ \sqrt{x^2+y^2} \leq \sqrt{2}r,\ \varphi = \arctan\frac{y}{x} \in \Big[0, \frac{\pi}{2}\Big]\Big\}$$

sohalarni olib va $\gamma_r \subset \gamma_0 \subset \gamma_{\sqrt{2}r}$ ekanligidan foydalanib,

$$\int\limits_{\gamma_r} e^{-x^2-y^2}dxdy \leq \int\limits_{\gamma_0} e^{-x^2-y^2}dxdy \leq \int\limits_{\gamma_{\sqrt{2}r}} e^{-x^2-y^2}dxdy$$

tengsizlikni hosil qilamiz. Soddalashtirishlarni bajarib,

$$I_1 = \int\limits_{\gamma_{\sqrt{2}r}} e^{-x^2-y^2}dxdy = \int\limits_0^r \int\limits_0^{\frac{\pi}{2}} e^{-\rho^2}\rho d\varphi d\rho =$$

$$= \frac{\pi}{2}\int\limits_0^r e^{-\rho^2}\rho d\rho = \frac{\pi}{4}\Big(1 - e^{-r^2}\Big);$$

$$I_2 = \int\limits_{\gamma_0} e^{-x^2-y^2}dxdy = \int\limits_0^r e^{-x^2}dx \cdot \int\limits_0^r e^{-y^2}dy = \Big(\int\limits_0^r e^{-x^2}dx\Big)^2;$$

$$I_3 = \int\limits_{\gamma_{\sqrt{2}r}} e^{-x^2-y^2}dxdy = \int\limits_0^{\sqrt{2}r} \int\limits_0^{\frac{\pi}{2}} e^{-\rho^2}\rho d\varphi d\rho =$$

$$= \frac{\pi}{2}\int\limits_0^{\sqrt{2}r} e^{-\rho^2}\rho d\rho = \frac{\pi}{4}\Big(1 - e^{-2r^2}\Big)$$

formulalarni yozamiz. Bularni yuqoridagi tengsizlikka qo'yib,

$$\frac{\pi}{4}\Big(1 - e^{-r^2}\Big) \leq \Big(\int\limits_0^r e^{-x^2}dx\Big)^2 \leq \frac{\pi}{4}\Big(1 - e^{-2r^2}\Big)$$

ga ega bo'lamiz. Bu yerda $r \to \infty$ da limitga o'tib,

$$\left(\int\limits_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}$$

yoki

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \tag{27}$$

tenglikni hosil qilamiz. Integral ostidagi funksiyaning juftligidan (25) tenglikning ikkinchi qismi kelib chiqadi.

1.10.3 Sekin o'suvchi S asosiy funksiyalar fazosi

Regulyar umumlashgan $f(x),\ x \in \mathbb{R}$ funksiyaning Furye almashtirishi, tabiiyki, ushbu

$$(\widetilde{f}, \phi) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(x)e^{i\xi x}dx\Big)\varphi(\xi)d\xi = \int_{\mathbb{R}} f(x)\Big(\varphi(\xi)e^{i\xi x}d\xi\Big)dx = (f, \widetilde{\varphi}).$$

integral tengliklar ko'rinishida aniqlanadi. Biroq, hamma vaqt ham $\varphi(x) \in D(\mathbb{R})$ dan $\widetilde{\varphi}(\xi) \in D(\mathbb{R})$ kelib chiqavermaydi. Masalan,

$$f(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| \geq a \end{cases}$$

funksiya finit (ammo silliq emas) va uning Furye almashtirishi

$$\widetilde{f}(\xi) = \frac{2sina\xi}{\xi}$$

dan iborat bo'lib, u finit funksiya emas.

Shunday qilib, $D^{'}(\mathbb{R})$ dagi barcha funksionallar uchun Furye almashtirishini aniqlab bo'lmas ekan. Ammo funksionallar sinfini toraytirib uni amalga oshirish mumkin. Buning uchun ularni kengroq asosiy funksiyalar sinfida (xususiy hol sifatida finit funksiyalarni o'z ichiga oluvchi) qarash zarur.

T a ' r i f. Aytaylik, $\phi(x)$ funksiyalar uchun quyidagi shartlar o'rinli bo'lsin:
1) $\phi \in C^\infty(\mathbb{R})$; 2) ixtiyoiy $k, l \in \mathbb{N} \cup 0$ sonlar uchun $|x| \to \infty$ da $x^k \phi^{(l)}(x) \to 0$.

U holda bu funksiyalar to'plamiga $S(\mathbb{R})$ asosiy funksiyalar fazosi deb aytiladi. Demak, $S(\mathbb{R})$ to'plam cheksiz differensiallanuvchi va $|x| \to \infty$ da hosilalari bilan birga $|x|^{-1}$ ning ixtiyoriy darajasiga nisbatan tezroq kamayuvchi funksiyalardan iborat.

S ga tegishli oddiy funksiya sifatida $\phi(x) = e^{-x^2}$ ni misol keltirish mumkin.

Ravshanki, $D \subset S$. Demak, ixtiyoriy finit va cheksiz differensiallanuvchi funksiya S ga tegishli bo'ladi. Bundan tashqari, 1) S chiziqli fazo hisoblanadi; 2) S dan olingan funksiyalar ko'paytmasi yana S ga tegishli; 3) ixtiyoriy $k, l \in \mathbb{N} \cup 0$ sonlar uchun $|x| \to \infty$ da

$$\left|h^l(x)\right| \leq C_{lk}|x|^k, \ C_{lk} = const$$

tengsizlikni qanoatlantiruvchi silliq $h(x)$ funksiyaning ixtiyoriy $\phi \in \mathcal{S}$ funksiyaga ko'paytmasi yana $\mathcal{S}$ ga tegishli bo'ladi.

T a ' r i f. Agar ixtiyoriy k, l lar uchun $x^k\phi^{(l)}(x) \rightrightarrows 0$ bo'lsa, $\phi_n(x)$ funksiyalar ketma-ketligi S da nol funksiyaga yaqinlashuvchi deyiladi, ya'ni $n \to \infty$ da $\phi_n(x) \xrightarrow{S} 0$. Agar $n \to \infty$ da $\phi_n(x) - \phi(x) \xrightarrow{S} 0$ bo'lsa, $n \to \infty$ da $\phi_n(x) \xrightarrow{S} \phi(x)$ bo'ladi.

Masalan, $\frac{1}{n}e^{-x^2}$ funksiyalar ketma-ketligining $n \to \infty$ da S asosiy funksiyalar fazosida 0 ga yaqinlashishini ko'rish mumkin.

T a s d i q. D asosiy funksiyalar fazosi S da to'la ($\overline{D} = S$), ya'ni ixtiyoriy $\phi \in S$ uchun shunday $\phi_n \in D$ ketma-ketlik mavjudki, u S da ϕ ga yaqinlashadi.

Isbot. Haqiqatdan ham, yuqoridagilarni hisobga olib, ko'rsatilgan ketma-ketlikni har doim

$$\phi_n(x) = \phi(x)\eta\left(\frac{x}{n}\right)$$

ko'rinishda olish mumkin, bu yerda $\eta(x) \in D$ va $|x| < 1$ da aynan birga teng. Bunday funksiyalarga "qirqish funksiyalari" deb yuritiladi. Misol uchun, $n \to \infty$ da

$$\phi_n^{'}(x) = \phi^{'}(x)\eta\left(\frac{x}{n}\right) + \frac{1}{n}\phi(x)\eta^{'}\left(\frac{x}{n}\right) \rightrightarrows \phi^{'}(x)$$

va boshqa hosilalari uchun ham yaqinlashish shu kabi ko'rsatiladi.

Endi S dagi asosiy funksiyalarning zarur xossalarini keltirib o'tamiz.

T a s d i q. Ixtiyoriy $\phi \in S$ uchun $\tilde{\phi} \in S$ munosabat o'rinli.

Isbot. 1) $|x| \to \infty$ da $|\phi(x)| \leq \frac{C}{x^2}$ baho bajarilishidan $\int\limits_{\mathbb{R}} \phi(x)e^{i\xi x}dx$ integralning yaqinlashuvchi ekanligi kelib chiqadi, ya'ni ixtiyoriy $\phi \in \mathcal{S}$ uchun $\tilde{\phi}$ mavjud, bu yerda $C-$ biror o'zgarmas son;

2) hosilani

$$\tilde{\phi}^{(l)}(\xi) = \int\limits_{\mathbb{R}} (ix)^l \phi(x) e^{i\xi x} dx$$

integral ko'rinishda tasvirlash mumkinligi va $|\phi(x)| \leq \frac{C}{x^{l+2}}$ bahodan ixtiyoriy tartibli $\tilde{\phi}^{(l)}(\xi)$ hosilaning mavjudligi kelib chiqadi;

3) 1-xossadan ixtiyoriy $k \in \mathbb{N}$ uchun $|\xi| \to \infty$ da $\xi^{k-1}\tilde{\phi}(\xi) \to 0$ ekanligi kelib chiqadi. 2-xossadan esa bu mulohazalar $\tilde{\phi}^{(l)}(\xi)$ uchun ham o'rinli ekanligini olamiz. Shunday qilib, $\tilde{\phi}(\xi)$ funksiya $\mathcal{S}$ dagi asosiy funksiyalarning barcha shartlarni qanoatlantiradi.

T a s d i q. Ushbu $F[\mathcal{S}] \subseteq \mathcal{S}$ munosabat o'rinli.

Isbot. Teskarisini faraz qilaylik, ya'ni shunday $\phi_0 \in \mathcal{S}$ funksiya topilib, S dagi qandaydir funksiyani Furye almashtirishi ko'rinishda tasvirlab bo'lmasin. Ammo, $\phi_0 \equiv F^{-1}[\tilde{\phi}_0(\xi)] = \frac{1}{2\pi}F[\tilde{\phi}_0(-\xi)]$ tengliklar bu farazning noto'g'ri ekanligini ko'rsatadi.

1.10.4 Sekin o'suvchi umumlashgan funksiyalar va umumlashgan Furye almashtirishi

T a ' r i f. S da aniqlangan chiziqli uzluksiz funksionallar sekin o'suvchi umumlashgan funksiya deb aytiladi va bu funksiyalar to'plamini $S^{'}$ kabi belgilanadi.

Ko'rish mumkinki, $S^{'} \subset D^{'}$, ya'ni finit funksiyalarda aniqlangan funksionallar kamayuvchi funksiyalarda aniqlangan kengroq sinfda, ma'noga ega bo'lmasligi mumkin.

M i s o l. Klassik e^{x^2} funksiya $D^{'}$ da regulyar umumlashgan funksiyani hosil qiladi, ammo $S^{'}$ da u regulyar umumlashgan funksiya emas (ixtiyoriy $\phi(x) \in S$ uchun $\int_{\mathbb{R}} e^{x^2}\phi(x)dx$ integral uzoqlashuvchi).

Biroq x^m klassik funksiya yordamida qurilgan regulyar umumlashgan funksiya ixtiyoriy m uchun $\mathcal{S}'$ ga tegishli.

T a s d i q. Agar $f(x)-$ lokal integrallanuvchi funksiya uchun $|x| \to \infty$ da $|f(x)| \leq c_0|x|^m,\ c_0 = const > 0$ tengsizlik o‘rinli bo‘lsa, u holda

$$(f(x), \phi(x)) = \int\limits_{\mathbb{R}} f(x)\phi(x)dx \tag{28}$$

funksional chiziqli va uzluksizdir.

Isbot. S asosiy funksiyalarning xossasiga ko‘ra $|\phi(x)| \leq \frac{C}{x^k}$ tengsizlik o‘rinli, bu yerda k ni ixtiyoriy ravishda tanlash mumkin. Uni $k = m+2$ ko‘rinishda olsak, u holda $|f(x)\phi(x)| \leq \frac{C}{x^2}$. Bu yerdan, (28) integralning yaqinlashuvchi ekanligi kelib chiqadi. Xuddi shunday mulohaza yuritib va $\phi(x)$ ni ixtiyoriy k, l larda $x^k\phi^{(l)}(x)$ ga almahstirilib, (28) integralning mavjudligi isbotlanadi, va agar

$$x^k\phi^{(l)}(x) \rightrightarrows 0$$

bo‘lsa, $|x| \to \infty$ da integral nolga yaqinlashuvchi bo‘ladi.

Oson ko‘rish mumkinki, $\delta,\ \mathcal{P}\frac{1}{x}, \theta, \dots \in \mathcal{S}'$. Shuning uchun quyidagi tasdiq o‘rinli.

T a s d i q. Agar $f \in D'-$finit funksiya bo‘lsa, u holda $f \in \mathcal{S}'$ bo‘ladi, undan tashqari ixtiyoriy $\phi \in \mathcal{S}$ uchun $(f, \phi) := (f, \eta\phi)$, bu yerda $\eta(x) \in D-$ "qirqish funksiyasi" bo‘lib, supp$f$ atrofida aynan birga teng.

Isbot. Modomiki, $\eta(x)\phi(x) \in D$ ekan, u holda $(f, \eta\phi)$ ni $\eta(x)$ "qirqish" funksiyasiga bog‘liq emasligi ko‘rsatish yetarli. Haqiqatan ham, ixtiyoriy $\phi(x) \in D$ va supp$f$ atrofida $\eta_1(x) - \eta_2(x) = 0$ bo‘lgani uchun

$$(f, \eta_1\phi) - (f, \eta_2\phi) = (f, (\eta_1 - \eta_2)\phi) = 0.$$

Endi yuqoridagi mulohazalarga asoslanib, S' fazoda Furye almashtirishini kiritish mumkin.

T a ’ r i f. $f(x) \in S'$ funksionalning Furye $\tilde{f}$ almashtirishi S' dagi funksional bo‘lib, $(\tilde{f}, \phi) = (f, \tilde{\phi})$ qoida orqali aniqlanadi.

M i s o l l a r. 1) $\tilde{\delta} = 1$. Haqiqatan ham,

$$(\tilde{\delta}, \phi) = (\delta, \tilde{\phi}) = \tilde{\phi}(0) = \int_{\mathbb{R}} \phi(x)dx = (1, \phi).$$

2) $F[\delta(x+a) - \delta(x-a)] = -2i\sin(a\xi),\ a = const.$

Bu tenglikning isboti (22) dan kelib chiqadi.

3)

$$(F[\theta(x)], \phi(x)) = \left(\theta(x), \tilde{\phi}\right) = \int_0^\infty \left(\int_{\mathbb{R}} \phi(\xi)e^{i\xi x}d\xi\right) dx. \qquad (29)$$

Modomiki, (29) integral x bo'yicha uzoqlashuchi bo'lgani uchun, integrallash tartibini almashtirish mumkin emas. Biroq integral ostida limitga o'tib, soddalashtirishlarni davom ettirish mumkin:

$$\int_0^\infty \Big(\int_{\mathbb{R}} \phi(\xi)e^{i\xi x}d\xi\Big)dx = \lim_{\varepsilon\to 0}\int_0^\infty \Big(\int_{\mathbb{R}} \phi(\xi)e^{i(\xi+i\varepsilon)x}d\xi\Big)dx =$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \phi(x)\Big(\int_{\mathbb{R}} \phi(\xi)e^{i(\xi+i\varepsilon)x}dx\Big)d\xi = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{\phi(\xi)}{-i(\xi+i\varepsilon)}d\xi.$$

Shunday qilib, $F[\theta] = \frac{i}{\xi+i0}$.

Ushbu paragrafning 1-bandidagi xossalar umumlashgan Furye almashtirishlari uchun ham o'rinli bo'ladii. Ulardan birinchisini tekshirib ko'raylik:

$$\left(\frac{d}{d\xi}\tilde{f}, \phi\right) = \left(\tilde{f}, -\phi'\right) = \left(f, -F[\phi']\right) = (f, (ix)\tilde{\phi}) =$$

$$= \left((ix)f, \tilde{\phi}\right) = (F[(ix)f], \phi),$$

ya'ni

$$\frac{d}{d\xi}\tilde{f} = F[(ix)f].$$

Qolganlari ham shunga o'xshash ko'rsatiladi.

Teskari umumlashgan Furye almashtirishini

$$F^{-1}[\tilde{f}(\xi)](x) := \frac{1}{2\pi}F[\tilde{f}(\xi)](-x)$$

yoki

$$F^{-1}[\tilde{f}(\xi)](x) := \frac{1}{2\pi} F[\tilde{f}(-\xi)](x)$$

ko'rinishda aniqlash mumkin. Bu formulalar ekvivalentdir.

T a s d i q. Umumlashgan Furye almashtirishi uchun

$$FF^{-1}[f] = F^{-1}Ff = f$$

munosabat o'rinli. Bundan $F^2[f] = 2\pi f(-x)$ ekanligi kelib chiqadi.

Isbot. Ushbu

$$(FF^{-1}[f], \phi) = (F^{-1}[f], \tilde{\phi}) =$$
$$= (\frac{1}{2\pi} F[f](-\xi), \tilde{\phi}) = (f, \frac{1}{2\pi} F[\tilde{\phi}(-\xi)]) = (f, \phi)$$

tengliklar tasdiqning o'rinli ekanligini ko'rsatadi.

Masalan, $F^{-1}[1] = \delta$ va $F^{-1}[1] = \frac{1}{2\pi} F[1]$ ekanligidan $F[1] = 2\pi\delta$ tenglik kelib chiqadi.

$F[\mathcal{P}\frac{1}{x}]$ ni topamiz. Soxotskiy formulasi va $F[\theta] = \frac{i}{\xi + i0}$, $\mathcal{P}\frac{1}{x} = -i\tilde{\theta} + i\pi\delta$ tengliklardan

$$F[\mathcal{P}\frac{1}{x}] = -2\pi i\theta(-\xi) + i\pi = i\pi sign(\xi)$$

ekanligiga ishonch hosil qilamiz.

Yuqorida ta'kidlab o'tilganidan, klassik funksiya cheksizlikda qanchalar tez kamayuvchi bo'lsa, uning Furye almashtirishi shuncha yuqori tartibli silliq bo'ladi.

Shunday qilib, quyidagi mulohaza o'rinli:

Agar $f \in D'-$ umumlashgan funksiya finit bo'lsa, u holda $F[f] \in C^{\infty}$, undan tashqari

$$F[f](\xi) = (f, \eta(x)e^{i\xi x})$$

o'rinli, bu yerda $\eta(x) \in D$ funksiya supp$f$ atrofida birga teng.

T a s d i q (Funksiyalar yig'masining umumlashgan Furye almashtirishi). Faraz qilaylik, $f \in S'$, $g \in S$. U holda $F[f * g] = \tilde{f}\tilde{g}$.

Isbot. $(F[f * g], \phi) = (f * g, \phi) = (f, (g, \tilde{\phi}(x + y)))$. Oxirgi ifodaning ichki qavsida g regulyar funksiyaning klassik Furye almashtirishidagi qiymati keltirilgan. Shuning uchun

$$(g, \tilde{\phi}(x + y)) =$$

$$= \int_{\mathbb{R}} g(y)dy \int_{\mathbb{R}} e^{i\xi(x+y)}\phi(\xi)d\xi = \int_{\mathbb{R}} t\phi(\xi)e^{i\xi x}\left(\int_{\mathbb{R}} e^{i\xi y}g(y)dy\right)d\xi = F[\phi\tilde{g}].$$

Bu tengliklarni davom ettirib,

$$(f, F[\phi\tilde{g}]) = (\tilde{f}, \phi\tilde{g}) = (\tilde{f}\tilde{g}, \phi)$$

ga ega bo'lamiz.

M i s o l. $n = 2$ da $\rho\frac{1}{|x|^2} \in D'(\mathbb{R}^2)$ funksiyani $\varphi \in D$ lar uchun quyidagi tenglik bilan kiritamiz:

$$\left(\rho\frac{1}{|x|^2}, \varphi\right) = \int_{|x|\leq 1} \frac{\varphi(x) - \varphi(0)}{|x|^2}dx + \int_{|x|>1} \frac{\varphi(x)}{|x|^2}dx.$$

Bunday aniqlangan funksiyaning Furye almashtirishi

$$F\left(\rho\frac{1}{|x|^2}\right) = -2\pi \ln|\xi| - 2\pi c_0 \tag{13}$$

ga teng, bu yerda

$$c_0 = \int_0^1 \frac{1 - J_0(u)}{u}du - \int_1^\infty \frac{J_0(u)}{u}du$$

va

$$J_0(x) = \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} -$$

0 indexli Bessel funksiyasi (7-bobning 2-paragrafiga qarang). (13) tenglikning o'rinli ekanligini ko'rsatamiz. Haqiqatdan ham, ixtiyoriy $\varphi \in D$ lar uchun

$$\left(F\left[\rho\frac{1}{|x|^2}\right], \varphi\right) = \left(\rho\frac{1}{|x|^2}, F[\varphi]\right) =$$

$$= \int_{|x|<1} \frac{F[\varphi](x) - F[\varphi](0)}{|x|^2}dx + \int_{|x|>1} \frac{F[\varphi](x)}{|x|^2}dx =$$

$$= \int_{|x|<1} \frac{1}{|x|^2}\int_{\mathbb{R}^2} \varphi(\xi)[e^{i(x,\xi)} - 1]d\xi dx + \int_{|x|>1} \frac{1}{|x|^2}\int_{\mathbb{R}^2} \varphi(\xi)e^{i(x,\xi)}d\xi dx =$$

$$= \int\limits_0^1 \frac{1}{r} \int\limits_{\mathbb{R}^2} \varphi(\xi) \int\limits_0^{2\pi} \left(e^{ir|\xi|\cos\theta} - 1 \right) d\theta d\xi dr +$$

$$+ \int\limits_1^{\infty} \frac{1}{r} \int\limits_{\mathbb{R}^2} \varphi(\xi) \int\limits_0^{2\pi} e^{ir|\xi|\cos\theta} d\theta d\xi dr =$$

$$= 2\pi \int\limits_0^1 \frac{1}{r} \int\limits_{\mathbb{R}^2} \varphi(\xi) \left[J_0(r|\xi| - 1) \right] d\xi dr + 2\pi \int\limits_1^{\infty} \frac{1}{r} \int\limits_{\mathbb{R}^2} \varphi(\xi) J_0(r|\xi|) d\xi dr =$$

$$= 2\pi \int\limits_{\mathbb{R}^2} \varphi(\xi) \left[\int\limits_0^1 \frac{J_0(r|\xi|) - 1}{r} dr + \int\limits_1^{\infty} \frac{J_0(r|\xi|)}{r} dr \right] d\xi =$$

$$= 2\pi \int\limits_{\mathbb{R}^2} \varphi(\xi) \left[\int_0^1 \frac{J_0(u) - 1}{u} du + \int\limits_1^{\infty} \frac{J_0(u)}{u} du \right] d\xi =$$

$$= 2\pi \int\limits_{\mathbb{R}^2} \varphi(\xi) \left[\left(\int\limits_0^1 + \int\limits_0^{|\xi|} \right) \frac{J_0(u) - 1}{u} du + \left(\int\limits_{|\xi|}^1 + \int\limits_1^{\infty} \right) \frac{J_0(u)}{u} du \right] d\xi =$$

$$= 2\pi \int\limits_{\mathbb{R}^2} \varphi(\xi) \left[\int\limits_0^1 \frac{J_0(u) - 1}{u} du + \int\limits_1^{\infty} \frac{J_0(u)}{u} du - \int\limits_1^{|\xi|} \frac{du}{u} \right] d\xi =$$

$$= -2\pi \int\limits_{\mathbb{R}^2} \varphi(\xi) \left(\ln|\xi| + c_0 \right) d\xi = \left(-2\pi \ln|\xi| - 2\pi c_0, \varphi(x) \right)$$

tengliklar (13) formulaning oʻrinli ekanligini koʻrsatadi.

1.11 Oddiy differensial tenglamalarni yechishning Furye almashtirishi usuli

Bu paragrafda biz

$$\mathcal{L}\mathcal{E}(x) = \delta(x), \; x \in \mathbb{R}$$

koʻrinishdagi tenglamaning yechimini Furye almashtirishi usuli bilan topishni organamiz. (1.13) da

$\mathcal{L} = \sum\limits_{k=0}^{n} p_k \frac{d^k}{dx^k}$, p_k- o‘zgarmas sonlar .

(1.13) ning har ikkala tomoniga Furye almashtirishini qo‘llab,

$$L(-i\xi)\mathcal{E} = 1$$

ga ega bo‘lamiz, bunda $L(\lambda)$ – ko‘phad, $L(\lambda) := \sum\limits_{k=0}^{n} p_k\lambda^k$.

$L(-i\xi)\mathcal{E} = 1$ tenglamaning yechimini hozircha

$$\tilde{\mathcal{E}} = \frac{1}{L(-i\xi)} + \text{"}\textit{bir jinsli tenglamaning umumiy yechimi}\text{"}$$

ko‘rinishda yozamiz, bunda birinchi qo‘shiluvchi qandaydir xususiy yechim. Ma’lumki, $\tilde{\mathcal{E}}$ ga teskari Furye almashtirishini qo‘llab, $\mathcal{E}$ ni topishimiz mumkin.

Ba’zi hollarni muhokama etamiz. Birinchidan, oldin aytilgani kabi, bir jinsli tenglamaning umumiy yechimi $c_i\delta(x - \xi_m)$ ko‘rinishdagi ifodalarni yig‘indisidan iborat, bu yerda c_i- ixtiyoriy koeffitsientlar, ξ_m- $\mathcal{L}(-\xi)$ ko‘phadning ildizlari (agar $\mathcal{L}(-\xi)$ ko‘phad ildizlari karrali bo‘lsa, bu yig‘indiga delta funksiyaning hosilalari ham kiradi). Shuning uchun bir jinsli tenglama umumiy yechimining teskari Furye almashtirishi ixtiyoriy koeffitsientli $\exp(i\xi_m x)$ ko‘rinishdagi ifodalardan tuzilgan yig‘indiga keladi (agar $L(-\xi)$ ko‘phad ildizlari karrali bo‘lsa, bu yig‘indida $x^n \exp(i\xi_m x)$ kabi qo‘shiluvchilar ham ishtirok etadi). Shunday qilib, bir jinsli $L(-i\xi)\tilde{y} = 0$ algebraik tenglamaning umumiy yechimi va $\mathcal{L}y = 0$ differensial tenglamaning umumiy yechimi Furye almashtirishi orqali bog‘langan.

Ikkinchidan, $L(-i\xi)\tilde{y} = 1$ tenglamaning xusuiy yechimini ifodalovchi $\frac{1}{L(-i\xi)}$ funksionalga aniq ma’no berish zarur. Agarda $L(-i\xi)$ ko‘phadning haqiqiy ildizi mavjud bo‘lmasa, $\frac{1}{L(-i\xi)}$ ifoda haqiqiy sonlar o‘qida silliq funksiya bo‘ladi va bu qo‘shiluvchining teskari Furye almashtirishi klassik ma’noda tushuniladi (bu holda, odatda, integral qoldiqlar yordamida hisoblanadi). $L(-i\xi)$ ko‘phadning haqiqiy ildizlari mavjud bo‘lgan holda (aytaylik, $\xi_j - n_j$ karrali bitta ildiz) $\frac{1}{L(-i\xi)}$ ifodani oddiy kasrlarning yig‘indisi shaklida yozish kerak. Haqiqiy sonlar o‘qida maxraj nolga aylanmaydigan kasrlar bilan yuqorida bayon etilgani kabi ish tutiladi. $\frac{1}{(\xi-\xi_j)^{n_i}}$, $n_i \leq n_j$ ko‘rinishdagi qo‘shiluvchilarni $\rho\frac{1}{(\xi-\xi_j)^{n_i}}$ funksional ma’nosida tushunish mumkin (3-paragrafga qarang).

M i s o l. Ikkita $\mathcal{L}_{\pm} = -\frac{d^2}{dx^2} \pm a^2$ operatorni qaraymiz. Furye almashtirishini qo‘llab, $(\xi^2 \pm a^2)\tilde{\mathcal{E}} = 1$ ni olamiz. "+" ishora bo‘lgan holda, quyidagiga ega bo‘lamiz

$$\tilde{\mathcal{E}}(\xi) = \frac{1}{\xi^2 + a^2} + C_1\delta(\xi - ia) + C_2\delta(\xi + ia)$$

yoki

$$\mathcal{E}(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{e^{-\xi x}}{\xi^2 + a^2}d\xi + \frac{1}{2\pi}C_1 e^{ax} + \frac{1}{2\pi}C_2 e^{-ax}$$

(keyinchalik tenglamaning umumiy yechimini keltiramiz). Integralni hisoblab.

$$\mathcal{E}(x) = \frac{e^{-a|x|}}{2a}$$

ni hosil qilamiz. "$-$" holda esa,

$$\tilde{\mathcal{E}}(\xi) = \frac{1}{2a}\Big(\rho\frac{1}{\xi - a} + \rho\frac{1}{\xi + a}\Big)$$

yoki (4-paragrafga qarang)

$$\mathcal{E}(x) = \frac{i}{4}\left(-sign(x)e^{-iax} + sign(x)e^{iax}\right) = \frac{1}{2a}\sin(ax)sign(x).$$

M i s o l. Oddiy $y' = 1$ ko‘rinishdagi tenglamani Furye almashtirishi usuli yordamida yechamiz. Furye almashtirishini qo‘llagandan so‘ng $-i\xi\tilde{y} = 2\pi\delta(\xi)$ ni olamiz. Modomiki, $\delta(\xi)$ funksiyaga faqat uzluksiz yoki silliq funksiyani ko‘paytirish ma’noga ega ekan, oxirgi tenglamani ξ ga bo‘lib xususiy yechimini topish mumkin emas. Shunday qilib, bu yerda Furye almashtirishi usulini qo‘llash qiyinchilik tug‘diradi (chunki, keltirilgan tenglamaning o‘ng tomoni ∞ da kamayuvchi emas). Biroq misol oddiy bo‘lgani uchun bu qiyinchilikni chetlab o‘tish mumkin. Haqiqatan ham, $-i\xi\tilde{y} = 2\pi\delta(\xi)$ tenglamaning umumiy yechimi

$$\tilde{y}(\xi) = C\delta(\xi) + \frac{2\pi}{i}\delta'(\xi) \Rightarrow y(x) = \frac{C}{2\pi} + x$$

dan iborat bo‘lishini oson tekshirish mumkin.

M i s o l.

$$xy' + (1 - \lambda)y = \delta(x)$$

tenglamaning yechimini quramiz. Furye almashtirishini qo‘llagandan so‘ng

$$\xi\tilde{y}^{'} + \lambda\tilde{y} = -1$$

tenglikni olamiz. Oson tekshirish mumkinki, bu tenglamaning xususiy yechimi $\tilde{y_*} = -\frac{1}{\lambda}$ agar $\lambda \neq 0$ bo‘lsa, va aksincha holda, $\tilde{y_*}' = -\rho\frac{1}{\xi}$ ga teng.

Bu misolda oddiylik uchun manfiy bo‘lmagan butun λ larni garaymiz, ya’ni $\lambda = m \in \mathbb{Z}$, $m \geq 0$.

$m = 0$ da

$$\tilde{y}^{'}(\xi) = -\rho\frac{1}{\xi} + \tilde{C}_1\delta(\xi)$$

ga ega bo‘lamiz. $y(x)$ yechimni topish uchun oxirgi tenglamaga Furye almashtirishini qo‘llaymiz va

$$xy(x) = \frac{sign(x)}{2} + C_1$$

yoki

$$y(x) = C_1\rho\frac{1}{x} + C_2\delta(x) + y_*(x)$$

ga ega bo‘lamiz, bunda y_*- funksional $xy(x) = sign(x)/2$ tenglamaning xususiy yechimi hisoblanadi. Modomiki,

$$(sign(x), \phi) = \int\limits_0^\infty [\phi(x) - \phi(-x)]dx$$

ekan, y_* sifatida

$$(y_*, \phi) = \frac{1}{2}\int\limits_0^R \frac{\phi(x) + \phi(-x) - 2\phi(0)}{x}dx$$

formula yordamida quruladigan funksiyani olish mumkin.

Endi faraz qilaylik, $m \neq 0$ bo‘lsin. U holda

$$\tilde{y}(\xi) = -\frac{1}{m} + \tilde{C}_1\mathcal{P}\frac{1}{\xi^m} + \tilde{C}_2\delta^{(m)}(\xi)$$

bo‘lib, Furyening teskari almashtirishini qo‘llash natijasida berilgan tenglamaning umumiy yechimini hosil qilamiz:

$$y(x) = -\frac{1}{m}\delta(x) + C_1x^{m-1}sign(x) + C_2x^m.$$

2-Bob. Xususiy hosilali differensial tenglamalarning klassifikatsiyasi. Asosiy masalalarning qo‘yilishi

2.1 Sterjenda issiqlik tarqalishi. Issiqlik o‘tkazuvchanlik tenglamasi

Sterjenda issiqlik tarqalishi jarayonini qaraymiz. Ox o‘qni sterjen bo‘ylab yo‘naltiramiz va $u(x,t)$ orqali x nuqtaning t vaqtdagi haroratini belgilaymiz. Faraz qilaylik, sterjenning harorati y va z koordinatalarga bog‘liq bo‘lmasin. Bu esa har bir tayin vaqtda sterjenning izotermik kesimi uning ko‘ndalang $x = const$ kesimi bilan ustma-ust tushishini va barcha ko‘ndalang kesimlari bir xil S yuzaga ega bo‘lishini bildiradi.

Haroratning o‘zgarishi faqat issiqlik tarqalishi natijasida ro‘y bersa, energiyaning saqlanish qonuniga asosan sterjenning $[x_1, x_2]$ oraliqdagi qismining $[t_1, t_2]$ vaqt mobaynida haroratini o‘zgartirishga sarflangan issiqlik energiyasi sterjenning $[x_1, x_2]$ qismi uchlari orqali olgan issiqlik miqdoriga teng bo‘ladi. Bu faqat ajratilgan $[x_1, x_2]$ qismning $x = x_1$ va $x = x_2$ uchlari orqali o‘tgan issiqlik oqimi $u(x,t)$ issiqlik o‘zgarishini aniqlashni bildiradi. Sterjenning x koordinatali kesimi orqali issiqlik oqimi birlik vaqt mobaynida

Ox bo'ylab bu kesim orqali o'tuvchi issiqlik miqdoriga teng.

Issiqlik oqimining zichligi ω (x nuqtada) deb, ko'ndalang kesimning birlik yuzasi orqali o'tuvchi oqimga aytiladi. Ravshanki, issiqlik sterjenning yuqori haroratli qismidan past haroratli qismiga uzatiladi. Bu fakt ω va va u miqdorlarni bog'lovchi quyidagi Furye qonuni bilan ifodalanadi:

$$\omega = -k\frac{\partial u}{\partial x},$$

bu yerda $k > 0$ issiqlik o'tkazuvchanlik koeffitsienti bo'lib, u sterjen materiali xossalariga bog'liq. $[x_1, x_2]$ kesma orqali τ vaqtda o'tuvchi issiqlik oqimi

$$S\left[k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_2} - k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_1}\right]$$

ga teng. Faraz qilaylik, sterjenning solishtirma issiqlik sig'imi (birlik massaga ega bo'lgan jismning issiqlik sig'imi) $c = const > 0$, chiziqli zichligi $\rho = const > 0$ va issiqlik o'tkazuvchanlik koeffitsienti $k = const > 0$ bo'lsin,

$$[c] = \frac{J}{grad \cdot gr}, \quad [\rho] = \frac{gr}{sm^3}, \quad [k] = \frac{J}{sm \cdot s \cdot grad}.$$

U holda $[x_1, x_2]$ kesma va $[t_1, t_2]$ vaqt oralig'i uchun issiqlik muvozanati tenglamasi

$$S\int_{x_1}^{x_2} c\rho\left[u(x,t_2) - u(x,t_1)\right]dx = S\int_{t_1}^{t_2}\left[\omega(x_1,\tau) - \omega(x_2,\tau)\right]d\tau =$$

$$= S\int_{t_1}^{t_2}\left[k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_2} - k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_1}\right]d\tau \qquad (1)$$

ko'rinishida bo'ladi.

$u(x,t)$ funksiya uzluksiz u_t va u_{xx} hosilalarga ega bo'lsin. (1) tenglamaning chap va o'ng tomonlarini $(t_2 - t_1)(x_2 - x_1)$ ga bo'lamiz. Hosil bo'lgan ifodalarga integral ko'rinishdagi o'rta qiymat haqidagi teoremani qo'llab, $t_1,\ t_2$ larni t ga va $x_1,\ x_2$ larni esa x ga intiltiramiz. Natijada issiqlik muvozanati tenglamasi

$$u_t = a^2 u_{xx}, \qquad a^2 = \frac{k}{c\rho} \ \ (a = const > 0)$$

ko‘rinishda yoziladi.

Bu differensial tenglamaga issiqlik o‘tkazuvchanlik tenglamasi deyiladi.

Sterjenning harorati nafaqat unda issiqlik tarqalishi natijasida, balki tashqi kuchlarning ta’siri ostida ham o‘zgarishi mumkin. Bu ta’sirlar sterjenning $u(x,t)$ haroratiga bog‘liq bo‘lmasin. Faraz qilaylik, oniy issiqlik manbalarining hajm zichligi $F(x,t)$ ma’lum bo‘lsin. Bu esa sterjenning $[x, x+dx]$ kichik qismidan $[t, t+dt]$ vaqt oralig‘ida

$$F(x,t)Sdxdt, \quad [F] = \frac{J}{s \cdot sm^2}$$

issiqlik ajralishini bildiradi. U holda (1) issiqlik muvozanati tenglamasining o‘ng tomoniga

$$S\int\limits_{t_1}^{t^2}\int\limits_{x_1}^{x^2} F(x,\tau)dxd\tau$$

ifoda qo‘shiladi. $F(x,t)$ funksiyani o‘zining argumentlari bo‘yicha uzluksiz deb faraz qilib, integral ko‘rinishidagi o‘rta qiymat haqidagi teoremadan foydalansak,

$$u_t = a^2 u_{xx} + f(x,t), \qquad f = \frac{F}{c\rho} \tag{2}$$

bir jinsli bo‘lmagan issiqlik o‘tkazuvchanlik tenglamasi hosil bo‘ladi.

Agar sterjen bir jinsli bo‘lmasa, uning materialini xarakterlovchi funksiyalar x ga bog‘liq bo‘ladi, ya‘ni $c = c(x), \quad \rho = \rho(x), \quad k = k(x)$. (1) issiqlik muvozanati tenglamasida bular hisobga olinsa, (2) chiziqli o‘zgarmas koeffitsientli tenglama o‘rniga quyidagi o‘zgaruvchan koeffitsientli issiqlik o‘tkazuvchanlik tenglamasi hosil bo‘ladi:

$$c(x)\rho(x)u_t(x,t) = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right) + F(x,t). \tag{3}$$

Sterjen materialining xossalari uning haroratiga ham bog‘liq bo‘lishi mumkin, masalan, issiqlik o‘tkazuvchanlik koeffitsienti haroratga bog‘liq, ya‘ni $k = k(x,u)$ bo‘lsa, (3) tenglama o‘rniga chiziqli bo‘lmagan

$$c(x)\rho(x)u_t(x,t) = \frac{\partial}{\partial x}\left(k(x,u(x,t))\frac{\partial u(x,t)}{\partial x}\right) + F(x,t) \tag{4}$$

tenglamani qarashga to'g'ri keladi. $c = c(x,u)$, $\rho = \rho(x,u)$ bo'lgan hollar ham uchrab turadi. (3) va (4) tenglamalarni keltirib chiqarishda c va ρ funksiyalarni uzluksiz, k funksiyani esa uzluksiz differensiallanuvchi deb hisobladik.

Sterjen haroratining o'zgarishi uning yon sirti orqali tashqi muhit bilan issiqlik almashinuvi natijasida ham ro'y berishi mumkin. Faraz qilaylik, tashqi muhitning harorati x va t larga bog'liq bo'lmasin va sterjen yon sirti orqali tark etayotgan oqimining zichligi Nyuton qonuniga bo'ysunsin, ya'ni $u(x,t) - u_m$ ayirmaga proporsional bo'lsin, bu yerda u_m- tashqi muhit harorati. Bu holda $[x_1, x_2]$ kesma va $[t_1, t_2]$ vaqt uchun issiqlik muvozanati tenglamasini yozamiz:

$$S\int_{x_1}^{x_2} c\rho\left[u(x,t_2) - u(x,t_1)\right]dx =$$

$$= S\int_{t_1}^{t_2}\left[k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_2} - k\frac{\partial u(x,\tau)}{\partial x}\bigg|_{x=x_1}\right]d\tau +$$

$$+S\int_{t_1}^{t_2}\int_{x_1}^{x_2} F(x,\tau)dxd\tau = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \alpha\left(u(x,\tau) - u_m\right)dxd\tau,$$

bu yerda p - sterjen izotermik ko'ndalang kesimining perimetri (u holda pdx - yon sirt yuzasi elementi), $\alpha > 0$ - sterjen sirti va muhit o'rtasidagi issiqlik almashinuv koeffitsienti.

Bir jinsli sterjen (k, c, ρ, α, S, p - o'zgarmaslar) uchun (1) issiqlik muvozanati tenglamasida bo'lgani kabi, issiqlik yo'qotish effekti inobatga olingan quyidagi issiqlik o'tkazuvchanlik tenglamasini olamiz:

$$u_t = a^2 u_{xx} + f(x,t) - b(u(x,t) - u_m), \tag{5}$$

bu yerda $b = \frac{\alpha\rho}{Sc\rho} = const > 0$.

Agarda $u(x,t)$ funksiya o'rniga $v(x,t) = u(x,t) - u_m$ funksiya kiritilsa, bu funksiya

$$v_t = a^2 v_{xx} + f(x,t) - bv$$

tenglamani qanoatlantiradi. Bu tenglamaning o‘ng tomonidagi oxirgi hadini yo‘qotish uchun yangi $\omega(x,t)$ funksiyani $v(x,t) = e^{-bt}\omega(x,t)$ formula bilan kiritish kifoya. U holda $\omega(x,t)$ funksiya uchun

$$\omega_t = a^2\omega_{xx} + e^{bt}f$$

tenglama hosil bo‘ladi.

Ba‘zi hollarda issiqlik tarqalish jarayonlari vaqtga bog‘liq bo‘lmaydi, ya‘ni $u = u(x)$. Masalan, (2) tenglama bu hollarda

$$u_{xx} = -\frac{F(x)}{k}$$

stasionar issiqlik o‘tkazuvchanlik tenglamasiga o‘tadi.

x va t erkli o‘zgaruvchilarning o‘rniga yangi x' va t' o‘zgaruvchilarni $x' = x,\ \ t' = a^2t$ formulalar bilan kiritsak, $u_t = a^2u_{xx}$ tenglama $\widetilde{u}(x',t')$ funksiya uchun $\widetilde{u}_t = \widetilde{u}_{x'x'}$ ko‘rinishni qabul qiladi. Unda $u = \widetilde{u}\left(x'(x), t'(t)\right)$ bo‘ladi.

Tenglamani noma‘lum funksiya o‘rniga boshqa yangi funksiya kiritish natijasida ham soddalashtirish mumkin. Quyidagi misolni qaraymiz:

M i s o l (Byurgers tenglamasi).

Issiqlik o‘tkazuvchanlik tenglamasi bilan bog‘langan Byugers tenglamasi deb ataluvchi ushbu

$$u_t = u_{xx} + (u_x)^2 \tag{6}$$

chiziqli bo‘lmagan tenglamani qaraymiz.

Agar (6) tenglamani x bo‘yicha diffirensiallab, $v(x,t) = u_x(x,t)$ deb belgilash kiritsak,

$$v_t = v_{xx} + 2vv_x$$

tenglama hosil bo‘ladi. Yana bitta yangi funksiyani $\omega(x,t) = e^{u(x,t)}$ tenglik bilan kiritib, quyidagi hosilalarni hisoblaymiz:

$$\omega_t = u_te^u, \ \ \omega_x = u_xe^u, \ \ \omega_{xx} = (u_{xx} + (u_x)^2)e^u.$$

U holda (6) dan $\omega(x,t)$ funksiyaning $\omega_t = \omega_{xx}$ issiqlik o‘tkazuvchanlik tenglamasini qanoatlantirishi kelib chiqadi. Bunda faqat musbat $\omega = e^u > 0$ yechimlar qaraladi.

2.2 Fazoda issiqlik o'tkazuvchanlik tenglamasi. Chegaraviy shartlarning qo'yilishi

Bu paragrafda issiqlik tarqalish tenglamasini uch o'lchovli (fazoviy o'zgaruvchilar bo'yicha) hol uchun keltirib chiqaramiz. Muhit $x = (x_1, x_2, x_3)$ nuqtasining t vaqtdagi haroratini $u(x,t)$ orqali, shu nuqtani o'z ichiga olgan biror hajm (soha) ni D orqali belgilab olamiz. D ning chegarasi S bo'lsin. Ravshanki, muhit turli qismlarining harorati turlicha bo'lsa, u holda ko'proq qizigan qismidan ozroq qizigan qismga qarab issiqlik harakatlanadi. D hajmda $[t_1, t_2]$ vaqt oralig'i issiqlik o'zgarishini tekshiramiz.

ds sirt elementi orqali issiqlik oqimi deb ds dan birlik vaqt ichida o'tuvchi issiqlik miqdoriga aytiladi. Uni issiqlik oqimi zichligining $\omega(x,t)$ vektori orqali aniqlash mumkin. Agar $n - ds$ sirtga o'tkazilgan tashqi birlik normal bo'lsa, u holda ds orqali o'tuvchi issiqlik oqimi $(\omega, n)ds$ ga teng bo'ladi.

Izotron muhitlarda (issiqlik o'tkazuvchanlik koeffitsienti k issiqlik tarqalish yo'nalishiga bog'liq emas) Furye qonuniga asosan $\omega = -k\text{grad}_x u$, bu yerda $k > 0$ issiqlik o'tkazuvchanlik koeffitsienti bo'lib, bir jinsli muhitlar uchun $k = const$ va bir jinsli bo'lmagan muhitlar uchun $k = k(x)$. Yopiq bo'lakli silliq S sirt bilan chegaralangan $D' \subset D$ sohani ajratamiz. Faraz qilaylik, $\rho(x)$ muhitning zichligi, $c(x)$ uning solishtirma issiqlik sig'imi bo'lsin. $F(x,t)$ orqali issiqlik manbalarining zichligi (birlik vaqt ichida birlik hajmdan ajralgan, unga o'tgan issiqlik miqdori) ni belgilaymiz. D' soha va $[t_1, t_2]$ vaqt oralig'i uchun issiqlik muvozanati tenglamasini yozamiz:

$$\iiint\limits_{D'} c(x)\rho(x)\left[u(x,t_2) - u(x,t_1)\right] dx_1 dx_2 dx_3 =$$

$$= -\int\limits_{t_1}^{t_2} dt \iint\limits_{S'} (\omega, n) ds + \int_{t_1}^{t_2} \iiint\limits_{D'} F(x,t) dx_1 dx_2 dx_3.$$

Ravshanki, Gauss-Ostragradskiy formulasiga ko'ra

$$\iint\limits_{S'} (\omega, n) ds = \iiint\limits_{D'} div_x \omega dx_1 dx_2 dx_3,$$

bu yerda $div_x\omega = \omega_{x_1} + \omega_{x_2} + \omega_{x_3}$. Faraz qilaylik, $u(x,t)$ funksiya bir marta t bo‘yicha va ikki marta x_1, x_2, x_3 o‘zgaruvchilari bo‘yicha uzluksiz differensiallanuvchi bo‘lsin. U holda Furye qonuniga ko‘ra

$$div_x\omega = -div_x(k\mathrm{grad}_x u),$$

va issiqlik muvozanati integral tenglamasi ushbu

$$\int\int\limits_{D'}\int c(x)\rho(x)\left[u(x,t_2) - u(x,t_1)\right] dx_1dx_2dx_3 =$$

$$= \int\limits_{t_1}^{t_2} dt \int\int\limits_{D'}\int div_x(k\mathrm{grad}_x u)dx_1dx_2dx_3 +$$

$$+ \int\limits_{t_1}^{t_2} dt \int\int\limits_{D'}\int F(x,t)dx_1dx_2dx_3 \tag{7}$$

ko‘rinishni oladi.

$F(x,t)$ funksiya barcha argumentlari bo‘yicha uzluksiz bo‘lsin. (7) formuladagi har bir $\int_{t_1}^{t_2}$ va $\int\int\limits_{D'}\int$ integrallarga o‘rta qiymat haqidagi teoremani qo‘llab, natijani D' sohaning hajmiga va $t_2 - t_1$ ga bo‘lamiz. D' soha hajmini kichraytirib, x nuqtaga, t_1, t_2 larni t ga intiltirib limitga o‘tamiz. $u(x,t)$ funksiyaning yuqorida qayd etilgan hosilalari mavjudligi to‘g‘risidagi farazimizga asosan quyidagi tenglamani olamiz:

$$c(x)\rho(x)u_t(x,t) = div_x(k(x)\ \mathrm{grad}_x u(x,t)) + F(x,t). \tag{8}$$

Bu tenglamaga uch o‘lchovli o‘tkazuvchanlik tenglamasi deyiladi. Agar $c = const$, $\rho = const$, $k = const$ (bir jinsli muhit) bo‘lsa, (8) tenglama

$$u_t(x,t) = a^2\Delta_x u(x,t) + f(x,t) \tag{9}$$

soddaroq ko‘rinishni oladi. Bu yerda $\Delta_x u = div_x\ \mathrm{grad}_x u(x,t)$ - x o‘zgaruvchilar bo‘yicha Laplas operatori,

$$a^2 = \frac{k}{c\rho}, \quad f(x,t) = \frac{F(x,t)}{c\rho}.$$

Issiqlik tarqalish jarayonini to'la ifodalash uchun muhitda haroratning boshlang'ich tarqalishi (boshlang'ich shart) hamda muhitning chegarasidagi harorat berilishi zarur. Boshlang'ich shart $u(x,t)$ funksiyaning boshlang'ich t vaqtdagi qiymatini berishdan iborat, ya'ni

$$u|_{t=t_0} = \varphi(x). \tag{10}$$

Chegaraviy shartlar haroratining chegaradagi rejimiga qarab turlicha bo'lishi mumkin. Ularning ba'zilarini ko'rib chiqamiz.

Agar S chegarada berilgan bir xil u_0 harorat saqlanayotgan bo'lsa, u holda

$$u|_S = u_0, \quad u_0 = const. \tag{11}$$

Agar S ga issiqlik oqimi bir xil bo'lsa, u holda

$$k\frac{\partial u}{\partial n}\bigg|_S = u_1, \quad u_1 = const. \tag{12}$$

Agar S ga issiqlik oqimi bir xil bo'lsa, u holda

$$\left[k\frac{\partial u}{\partial n} + h(u - u_m)\right]\bigg|_S = 0 \tag{13}$$

bo'ladi, bunda h - issiqlik almashinish koeffitsienti, u_m- atrofdagi muhitning harorati.

Xuddi issiqlik o'tkazuvchanlik tenglamasiga o'xshash zarrachalar diffuziyasi tenglamasi keltirib chiqariladi. Faqat bunda Furye qonuni o'rniga birlik vaqtda ds elementar qismdan o'tuvchi zarrachalar oqimi uchun Nerist qonunidan foydalanish kerak. Bunga asosan

$$dQ = -E\frac{\partial u}{\partial n}ds,$$

bu yerda $E(x)-$ diffuziya koeffisenti, $u(x,t)$ - t vaqtda x nuqtadagi zarralar zichligi. $u(x,t)$ zichlik uchun (8) ko'rinishidagi tenglamaga ega bo'lamiz, unda ρ g'ovaklik koeffisentini belgilaydi, $\rho = E(x)$, q esa muhitning singdirishlik xossasini ifodalaydi.

2.3 Statsionar issiqlik o‘tkazuvchanlik tenglamasi: Laplas va Puassan tenglamalari

Issiqlik o‘tkazuvchanlik tenglamasi uchun shunday qo‘shimcha shartlarni tanlash mumkinki, bunda harorat t vaqtga bog‘liq bo‘lmay qoladi: $u_t \equiv 0$. Bunday issiqlik tarqalish jarayoni statsionar deyiladi. Agar issiqlik o‘tkazuvchanlik koeffitsienti $k = const$ bo‘lsa, u holda statsionar $u(x)$, $x \in D$ harorat

$$\Delta u(x) = -\frac{F(x)}{k} \tag{14}$$

tenglamani qanoatlantiradi. (14) tenglamaga Puassan tenglamasi deyiladi.

Agarda D sohada issiqlik manbalari bo‘lmasa, statsionar $u(x)$ harorat D da

$$\Delta u(x) = 0 \tag{15}$$

tenglamani qanoatlantiradi. (15) tenglamaga Laplas tenglamasi deyiladi.

Nostatsionar issiqlik o‘tkazuvchanlik va diffuziya tenglamalari, Laplas va Puassan tenglamalarini D sohaning aniq shakliga bog‘liq ravishda maxsus tanlangan koordinatalar sistemasida yozish qulaydir. Masalan, agar D to‘g‘ri burchakli parallelepiped ko‘rinishida bo‘lsa, u holda tabiiyki, bu tenglamalar uchun qo‘yilgan masalalarni dekart koordinatalar sistemasida tanlagan ma‘qul. Shardan iborat D soha uchun sferik koordinatalar sistemasi qulay hisoblanadi.

Ba‘zi hollarda harorat fazoviy koordinatalardan biriga bog‘liq bo‘lmay qolishi mumkin. Masalan, $Ox_1x_2x_3$ dekart koordinatalar sistemasida bunday koordinata sifatida x_3 ni olaylik, ya’ni $u = u(x_1, x_2, t)$ bo‘lsin. U holda D sifatida Ox_1x_2 tekislikka tegishli soha qaraladi. Bunda ham (8), (9), (14), (15) tenglamalar o‘z ko‘rinishini saqlaydi (faqat u funksiyasining x_3 bo‘yicha hosilasi nolga tengligini hisobga olish zarur). Bu holda D sohaning shakliga mos ravishda Ox_1x_2 to‘g‘ri burchakli koordinatalar sistemasi o‘rniga boshqa, masalan, qutb koordinatalaridan foydalanish qulaydir.

Laplas operatorini yuqorida aytilgan koordinatalar sistemasida yozamiz:
$Ox_1x_2x_3$ *dekart koordinatalar sistemasida:*

$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2};$

Sferik koordinatalar $x_1 = r\sin\psi\cos\varphi, \quad x_2 = r\sin\psi\sin\varphi \quad x_3 = r\cos\psi$ *sistemasida:*

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) +$$

$$+\frac{1}{r^2\sin\psi}\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial u}{\partial\psi}\right) + \frac{1}{r^2\sin\psi^2}\frac{\partial^2 u}{\partial\varphi^2};$$

$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi \quad x_3 = z$ *sistemasida:*

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z}.$$

Tekislikda bu operator quyidagi koordinatalar sistemasida ko‘p uchraydi: Ox_1x_2 *dekart koordinatalar sistemasida:*

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2};$$

qutb koordinatalar sistemasida:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}.$$

Eslab o‘tamiz, statsionar jarayonlarni to‘la ifodalash uchun chegaradan holatni, ya‘ni (11), (12) va (13) chegaraviy shartlardan birini berish zarurdir.

Endi haroratning statsionar taqsimotiga doir masalalarni ko‘ramiz.

M i s o l. Issiqlik manbalariga ega bo‘lmagan bir jinsli sterjen ($D = (0,l)$) boshlang‘ich $t = 0$ vaqtda $u(x,0) = u_0x, \ \ u_0 = const$ haroratga ega hamda $t \geq 0$ vaqtlarda sterjenning uchlarida $u(0,t) = 0$ va $u(l,t) = u_0e$, sterjenning yon sirti muhitdan ajralgan issiqlik almashinuvi yo‘q bo‘lsin. U holda $u(x,t) = u_0x$ funksiya $u_t = a^2u_{xx}$ tenglama va barcha shartlarni qanoatlantiradi, ya‘ni u sterjenda haroratining statsionar taqsimoti bo‘ladi.

M i s o l. Agar $t = 0$ boshlang‘ich vaqtda bir jinsli tekis doiraviy plastinka ($D = \{(r,\varphi) : 0 \leq r < r_0, \ \ 0 \leq \varphi < 2\pi\}$, $r, \varphi -$ qutb koordinatalari)

$$u(r,\varphi,0) = u_0 r\sin\varphi, \quad u_0 = const$$

haroratga ega bo‘lib, barcha $t \geq 0$ vaqtlarda u chegarasi orqali

$$u(r, \varphi, t) = u_0 r_0 \sin\varphi$$

harorat bilan taminlanib turilsin. U holda bevosita hisoblashlar yordamida $u = u_0 r \sin\varphi$ funksiyaning $u_t = a^2 \Delta u$ tenglamani qanoatlantirishini, yani haroratning plastinkadagi statsionar taqsimoti bo‘lishini ko‘rish mumkin.

M i s o l. Bir jinsli $D = \{(r, \varphi, \psi) : 0 \leq r < r_0, \quad 0 \leq \varphi < 2\pi\, 0 \leq \psi \leq \pi\}$, shar uchun uning chegarasida barcha $t \leq 0$ vaqtlar uchun doimiy $u(r_0, \varphi, \psi, t) = u_0 r_0 (u_0 = const)$ harorat va sharning ichki nuqtalarida vaqtga nisbatan doimiy bo‘lgan haroratning markaziy simmetrik

$$F(r, \varphi, \theta, t) = F(r) = -\frac{2u_0}{r}$$

taqsimoti berilgan bo‘lsin. Bu holda shardagi harorat φ, ψ burchaklarga bog‘liq bo‘lmaydi va

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial^2 (ru)}{\partial r^2}.$$

Bevosita hisoblashlar yordamida $u(r, \varphi, \psi, t) = u_0 r$ funksiyaning

$$u_t = a^2 \Delta u - \frac{2a^2 u_0}{r} \quad \left(\text{yoki } \Delta u = \frac{2u_0}{r}\right)$$

tenglamani qanoatlantirishiga ishonch hosil qilish mumkin.

2.4 Elastik sterjenning bo‘ylama tebranishi

Tashqi kuch ostida elastik sterjen cho‘ziladi yoki qisqaradi, ya‘ni sterjen zarralarida ko‘chish ro‘y beradi. Bunda sterjenda bu kuchlarga qarshilik ko‘rsatuvchi o‘zaro qo‘shni zarralar o‘rtasida ichki elastik kuchlar paydo bo‘ladi, ya’ni sterjen deformatsiyalanadi. Agarda sterjendan tashqi kuchlar olinganda deformatsiya yo‘qolsa, bu deformatsiya elastik deyiladi. Tinch holatda sterjenning barcha ko‘ndalang qismlari bir xil S yuzaga ega deb hisoblaymiz. Faraz qilaylik, Ox o‘qi bo‘ylab yo‘nalgan tashqi kuchlar ta‘sirida sterjenning istalgan ko'ndalang kesimi o'zining shaklini o'zgartirmaydi. Bunday sterjenni bir o'lchovli deb hisoblash mumkin. Elastik tutash muhitlarda paydo bo'ladigan kuchlanishlar va kichik deformatsiyalar orasidagi bog'lanishi Guk qonuni bilan ifodalanadi. Kuchlanish bu jism kesimining birlik yuzasiga to'g'ri keluvchi elastik kuchdir. Agar elastik F_{el} kuch sirt kesimiga perpendikulyar bo'lsa, u normal elastik kuch deyiladi. Faqat sterjen bo'ylab yo'nalgan tashqi kuchlar ta'sirida unda hosil bo'ladigan normal kuchlanishlarni qaraymiz, ya'ni kuchlanishni σ orqali belgilasak, u holda $\sigma = F_{el}/S$. Guk qonuniga binoan normal kuchlanishli nisbiy uzayishga proporsional: agar sterjen tinch holatdagi l uzunligi tashqi kuchlar ta'siri ostida $l + \Delta l$ ga teng bo'lib qolsa, u holda $\sigma E\frac{\Delta l}{l}$ bo'ladi. Bunda $E > 0$ sterjenning materialiga bog'liq bo'lib, unga Yung moduli deyiladi, $[E] = gr/(sm \cdot s^2)$.

Dastlab sterjen bo'ylab taqsimlangan kuch ostida hosil bo'ladigan statsionar deformatsiyani qaraymiz. Tinch holda $[x, x+\Delta x]$ holatga ega bo'lgan sterjenning qismi uchun Guk qonunini yozamiz. Tinch holda x koordinataga sterjenning biror zarrasi to'g'ri kelsin. $F(x)$ - Ox bo'ylab taqsimlangan va $f(x)$ chiziqli zichlikka ega bo'lgan sterjenga ta'sir etuvchi tashqi kuch bo'lsin. Ma'lumki, $f(x)$ - birlik uzunlikka to'g'ri keluvchi kuch: $f = \frac{\partial F}{\partial x}$. Tashqi kuch ostida x zarra (nuqta) biror $u(x)$ miqdorga ko'chadi. Tinch holda $x + \Delta x$ koordinatali zarra esa $u(x + \Delta x)$ ga o'tadi. Demak, sterjenning ajratilgan qismi

$$[x + u(x), x + \Delta x + u(x + \Delta x)]$$

holatga siljiydi. Bu qismning nisbiy uzayishi

$$\frac{u(x + \Delta x) - u(x)}{\Delta x}$$

ga teng. $u(x)$ funksiya uzluksiz va differensiallangan deb, $u(x + \Delta x) - u(x)$ ayirmani Lagranj formulasiga asosan yozib olamiz. U holda Guk qonuni

$$\sigma(x) = E(x)u_x(x + \theta\Delta x),\ 0 < \theta < 0$$

bo'lishini ko'rsatadi. Agar $\Delta x \to 0$ desak, u holda $\sigma(x) = E(x)u_x(x)$, ya'ni elastik kuch x nuqtada

$$F_{el}(x) = \sigma(x)S = SE(x)u_x(x)$$

ga teng. $E = E(x)$ sterjen bir jinsli emasligini bildiradi.

Endi $[x_1, x_2]$ kesmani ajratamiz. Agar sterjen holati statsionar, ya‘ni cho‘zilish jarayoni tugagan bo‘lsa, u holda sterjenning ixtiyoriy $[x_1, x_2]$ kesmasi uchun

$$\int_{x_1}^{x_2} f(x)dx + SE(x_2)u_x(x_2) - SE(x_1)u_x(x_1) = 0$$

muvozanat tenglamasi o‘rinli. Agar sterjen nuqtalari uchun $f(x) > 0$ bo‘lsa, x_2 nuqtada $[x_1, x_2]$ sterjenning ajratilgan qismiga Ox o‘qining musbat tomoniga yo‘naltirilgan $x > x_2$ tomondan, x_1 nuqtada esa Ox o‘qining manfiy tomoniga yo‘naltirilgan $x < x_1$ tomondan elastik kuch ta’sir etadi. Faraz qilaylik, $f(x)$ uzluksiz, $E(x)$ uzluksiz differensiallanuvchi, $u(x)$ esa ikki marta uzluksiz differensiallanuvchi funksiyalar bo‘lsin. $\int_{x_1}^{x_2} f(x)dx$ integralni o‘rta qiymat haqidagi teorema yordamida yozib olamiz va $SE(x) = k(x)$ belgilash kiritamiz. Kuchlar muvozanati tenglamasini $x_2 - x_1$ ga bo‘lib olib, x_1 va x_2 larni x ga intiltiramiz. U holda $u(x)$ funksiya sterjenning statsionar cho‘zilishini ifodalovchi

$$\frac{d}{dx}\left(k(x)\frac{du(x)}{dx}\right) = -f(x)$$

oddiy differensial tenglamani qanoatlantiradi. Bir jinsli sterjen ($k = const$) uchun

$$u_{xx}(x) = -\frac{f(x)}{k}$$

tenglama hosil bo‘ladi. Bu tenglamaga, odatda, bir o‘lchovli Puassan tenglamasi deb ham aytiladi. Agar elastik sterjenning holati nostatsionar bo‘lsa, u holda u siljish vaqtga ham bog‘liq bo‘ladi. $u(x,t)$ funksiya tinch holda x koordinataga ega bo‘lgan nuqtaning t vaqtdagi sterjen bo‘ylab ko‘chishi bo‘lsin. U holda $u_t(x,t)$ funksiya t vaqtdagi x nuqtaning harakat tezligi. Umumiylik uchun sterjenni bir jinsli emas deb hisoblaymiz. Agar $\rho(x)$ x nuqtadagi massaning hajm zichligi bo‘lsa, $\rho(x)Sdx$ sterjenning dx uzunlikka ega bo‘lgan qismining massasi bo‘ladi. Sterjenning ixtiyoriy $[x_1, x_2]$ qismini tanlaymiz. Bu kesma uchun $[t_1, t_2]$ vaqt oralig‘ida harakat miqdorining o‘zgarishi (harakat miqdorini o‘zgarishi sterjen bo‘ylab $f(x,t)$ chiziqli zichlik bilan taqsimlangan $F(x,t)$ tashqi kuchlar va ko‘rsatilgan vaqt oralig‘ida ta'sir etuvchi elastik kuchlar hisobiga ro'y beradi) quyidagiga teng:

$$S\int_{x_1}^{x_2}\left[u_t(x,t_2)-u_t(x,t_1)\right]\rho(x)dx=$$

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} f(x,t)dxdt+\int_{t_1}^{t_2}\left[SE(x_2)u_x(x_2,t-SE(x_1)u_x(x_1,t)\right]dt.$$

$f(x,t)$ funksiyani har ikkiala argumenti bo'yicha uzluksiz deb

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} f(x,t)dxdt$$

integralni o'rta qiymat haqidagi teoremaga ko'ra yozib olamiz. Faraz qilaylik, u_{xx} va u_{tt} hosilalar mavjud va uzluksiz bo'lsin. Qolgan ikkita integralni ham o'rta qiymat haqidagi teoremaga ko'ra yozib olamiz. Hosil bo'lgan tenglikni x_2-x_1 va t_2-t_1 larga bo'lib, x_1,x_2 larni x ga, t_1,t_2 larni esa t ga intiltiramiz. U holda $u(x,t)$ funksiya

$$\rho(x)S\frac{\partial^2u(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)+f(x,t)$$

tenglamani qanoatlantiradi. Agar $k=const$, $\rho=const$ bo'lsa, tenglamani

$$u_{tt}=a^2u_{xx}+\frac{f}{\rho^s}$$

ko'rinishida yozish mumkin, bu yerda $a^2=\frac{k}{\rho S}>0$, $a=const>0$.. Bu tenglamalar to'lqin yoki tebranishlar tenglamasi deb yuritiladi.

Agar $F(x,t)\equiv 0$ bo'lib, tebranishi faqat elastik kuchlar (masalan, sterjenning biror boshlang'ich cho'zilishidan so'ng) ta'sirida sodir bo'lsa, u holda

$$u_{tt}(x,t)=a^2u_{xx}(x,t). \qquad (16)$$

(16) ga erkin tebranishlar tenglamasi deyiladi. Tebranish jarayonida sterjenga ishqalanish kuchlari ta'sir etishi mumkin. Agar ishqalanish kuchi $u_t(x,t)$ tezlikka proporsional bo'lsa, (16) tenglama o'rniga $u_{tt}=a^2u_{xx}-bu_t$, $b=const>0$ tenglama qaraladi.

2.5 Tor va membrananing ko‘ndalang tebranish tenglamasi. Fazoda tovush to‘lqinlari

Tor deganda erkin egiladigan, ingichka ip tushuniladi, boshqacha aytganda, tor shunday qattiq jismki, uning uzunligi boshqa o‘lchamlaridan anchagina ortiq bo‘ladi. Muvozanat holatida torni $0xyz$ fazoda butun Ox o‘q yoki uning biror qismi kabi tasavvur etish mumkin. Tor $(x, 0, 0)$ nuqtasining muvozanatdan chiqarilgandan keyingi holatini berish uchun

$$\{u_1(x,t), u_2(x,t), u_3(x,t)\}$$

vektorni kiritamiz. Biz torning tekis ko‘ndalang tebranishini tekshiramiz, ya‘ni bu shunday tebranishki, tor hamma vaqt Oxz tekislikda yotadi va uning har bir nuqtasi Ox o‘qqa perpendikulyar to‘g‘ri chiziq bo‘yicha siljiydi. Bu degani, muvozanat vaqtida x absitsaga ega bo‘lgan torning nuqtasi tebranishi jarayonida ham shu absissaga ega bo‘ladi. U holda x nuqtaning t vaqtda muvozanat holatga nisbatan Oz o‘q yo‘nalishida siljishini belgilovchi $u(x, t)$ skalyar funksiyaga qarash yetarli. Torni egish natijasida unda ichki elastik kuchlar paydo bo‘ladi. Torning elastikligi bu kuchlarning har bir vaqtda torga o‘tkazilgan urinma bo‘ylab yo‘nalishini bildiradi. Tor tebranishini kichik va Guk qonuniga bo‘ysinadi deb hisoblaymiz. U holda T elastiklik kuchi x, t larga bog‘liq bo‘lmaydi va uning Oz o‘qqa proyeksiyasini $Tu_x(x, t)$ ga teng deb olish mumkin. Bundan tashqari, ko‘ndalang tebranishlar qaralayotganligi uchun torga ta‘sir etuvchi barcha kuchlarning Oz o‘qqa proyeksiyalari yig‘indisi nolga teng bo‘ladi.

Faraz qilaylik, torga Oz o‘q yo‘nalishida $f(x, t)$ chiziqli zichlik bilan taqsimlangan tashqi kuch ta‘sir qilsin. Xuddi oldin bo‘lgani kabi torda $[x_1, x_2]$ kesmani ajratib olib, $[t_1, t_2]$ vaqt oralig‘ida uning harakat miqdorining o‘zgarishini topamiz. Agar $\rho(x)$ tor massasining chiziqli zichligi (bir jinsli tor uchun $\rho(x) = const$) bo‘lsa, $\rho u_t(x, t)dx$ - dx uzunlikka ega bo‘lgan tor qismining impulsi bo'ladi. Bu holda muvozanat tenglamasi

$$\int_{x_1}^{x_2} \rho \left(u_t(x,t_2) - u_t(x,t_1) \right) dx = \int_{t_1}^{t_2} T \left(u_t(x_2,t) - u_t(x_1,t) \right) dt +$$

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} f(x,t) dx dt$$

ko'rinishda bo'ladi. Bu tenglamadagi har bir integralda o'rta qiymat haqidagi formulani qo'llab, hosil bo'lgan tenglamalarni $x_2 - x_1$ va $t_2 - t_1$ ayirmalarga bo'lamiz. $u(x,t)$ funksiya uzluksiz ikkinchi tartibli $u_{xx}(x,t), u_{tt}(x,t)$ hosilalarga ega deb faraz qilib, xuddi oldin bo'lganidek, x_1, x_2 larni x ga va t_1, t_2 larni t ga intiltiramiz. U holda quyidagi tenglamaga ega bo'lamiz:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) + g(x,t), \tag{17}$$

bu yerda $a^2 = \frac{T}{\rho} = const$, $g(x,t) = \frac{1}{\rho} f(x,t)$. (17) ga torning ko'ndalang tebranishlar tenglamasi deyiladi. Agar $f(x,t) = 0$ (u holda $g(x,t) = 0$) bo'lsa, unga erkin tebranishlar tenglamasi deyiladi.

$a > 0$ tor materiali xossalari bilan aniqlanadi. Bu miqdor Ox bo'yicha torning tebranish tezligiga teng ekan. Endi Oxy tekislikda chegarasi S ga teng D sohani qaraymiz. Uni ikki o'lchovli bir jinsli muvozanat holdagi membrana deb hisoblash mumkin. Faraz qilaylik, membrananing $(x, y, 0)$ nuqtasi muvozanat holatdan chiqarilgandan so'ng t vaqtda Oz o'qi bo'yicha $u(x, y, t)$ miqdorga siljisin. Membranada to'lqin tarqalish tenglamasini olish uchun xuddi torning ko'ndalang yoki sterjenning bo'ylama tebranishlaridagi kabi farazlarni qabul qilamiz. U holda

$$u_{tt}(x,y,t) = a^2 (u_{xx}(x,y,t) + u_{yy}(x,y,t))$$

membrananing kichik erkin ko'ndalang tebranishlar tenglamasini hosil qilamiz. Agar membranaga ko'ndalang tashqi kuchlar ta'sir qilsa, unda membrananing tebranish tenglamasi

$$u_{tt}(x,y,t) = a^2 (u_{xx}(x,y,t) + u_{yy}(x,y,t)) + g(x,t,y)$$

ko‘rinishida bo‘ladi.

$$u_{tt}(M,t) = a^2\Delta u(M,t),\ M = M(x,y,z)$$

ko‘rinishida to‘lqin tenglamasi bilan fazoda tovush tarqalish jarayonlarini ifodalash mumkin. Tovush chiqaruvchi jismlar: tor, membrana, musiqali asboblar va har qanday boshqa tovush manbalari havoning bo‘ylama tebranishini hosil qiladi. Tovush to‘lqinlarining tarqalishini o‘rganish uchun havoni Guk qonuniga bo‘ysunuvchi elastik muhit deb hisoblab, Gyugensning to‘lqin harorati prinsipidan foydalanish mumkin.

2.6 Moddiy nuqtaning og‘irlik kuchi ta’siridagi harakat tenglamasi

Dekart ortogonal koordinatalar sistemasida Ox_1x_2 vertikal tekislikda moddiy $M(x_1,x_2)$ nuqta og‘irlik kuchi ta’sirida (ξ_1,ξ_2), $\xi_2 > 0$ holatdan $(\xi_0, 0)$, $\xi_0 > \xi_1$ holatni egallash uchun harakatlanayotgan bo‘lib, t vaqt ξ_2 balandlikning funksiyasi bo‘lsin, ya’ni $t = t(\xi_2)$. $M(x_1,x_2)$ nuqta harakat trayektoriyasini topish talab etilsin.

Ma’lumki, $M(x_1,x_2)$ nuqta tezlik vektori $\vec{v} = (\frac{dx_1}{dt}, \frac{dx_2}{dt})$ moduli uchun

$$|v|^2 = 2g(\xi_2 - x_2)$$

tenglik o‘rinli, bu yerda $g-$ og‘irlik kuchi tezlanishi.

Agar $\gamma(x_1,x_2)-$ soat mili harakati yo‘nalishiga teskari yo‘nalish hisoblanadigan tezlik vektori bilan x_1 o‘qning musbat yo‘nalishi orasidagi burchak bo‘lsa, yuqoridagi tenglikka asosan

$$\frac{dx_2}{dt} = |v|\sin\gamma = \sqrt{2g(\xi_2 - x_2)}\sin\gamma$$

formulani hosil qilamiz. $x_1 = x_1(x_2)$ traektoriya noma’lum bo‘lgani uchun, ushbu

$$\varphi(x_2) = \frac{1}{\sin\gamma\,(x_1(x_2), x_2)},\ |\varphi(x_2)| > 1$$

funksiya ham noma'lum bo'ladi. Bu va oldingi formuladan

$$dt = \frac{\varphi(x_2)dx_2}{\sqrt{2g(\xi_2 - x_2)}}$$

tenglikni olamiz. Bu tenglikni 0 dan ξ_2 gacha oraliqda integrallab,

$$t(\xi_2) = -\int_0^{\xi_2} \frac{\varphi(x_2)dx_2}{\sqrt{2g(\xi_2 - x_2)}}$$

yoki

$$\int_0^{\xi_2} \frac{\varphi(x_2)dx_2}{\sqrt{2g(\xi_2 - x_2)}} = f(\xi_2), \;\; f(\xi_2) = -\sqrt{2g}t(\xi_2)$$

tenglamaga ega bo'lamiz. Bu tenglamada noma'lum funksiya integral ostida bo'lgani uchun, u integral tenglamadir.

Bu tenglamaning $|\varphi(x_2)| > 1$ shartni qanoatlantiruvchi yechimini topsak, u holda

$$\frac{dx_1}{dx_2} = \operatorname{ctg}\gamma = \sqrt{\operatorname{cosec}^2 - 1}$$

tenglikdan izlanayotgan traektoriyaning

$$x_1(x_2) = \int_0^{\xi_2} \sqrt{\varphi^2(t) - 1}dt$$

tenglamasini topamiz. Bunga *tautoxron to'g'risidagi masala* deyiladi.

2.7 Xususiy hosilali differensial tenglamalar va ularning yechimi to'g'risida tushuncha

n o'lchovli $\mathbb{R}^n$ Evklid fazosida nuqtaning dekart koordinatalarini $x_1, x_2, \ldots, x_n$, $n \geq 2$ orqali belgilaymiz. Tartiblangan manfiy bo'lmagan n ta butun sonning $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ ketma-ketligi *n tartibli multiindeks* deyiladi, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ soniga *multiindeks uzunligi* deyiladi. Q - $\mathbb{R}^n$ fazodagi biror soha (ochiq bog'langan toplam) bo'lsin.

$u(x) = u(x_1, x_2, \ldots, x_n)$ funksiyaning $x \in Q$ nuqtadagi $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ tartibli hosilasini

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \; D^0 u = u(x)$$

ko'rinishda yozamiz. Masalan, $\alpha = \alpha_i$ xususiy hol uchun

$$D^\alpha u = \frac{\partial^{\alpha_i} u}{\partial x_i^{\alpha_i}} = D_i^{\alpha_i} u, \; D_i u = \frac{\partial u}{\partial x_i} = u_{x_i}, \; D_i^2 u = \frac{\partial^2 u}{\partial x_i^2} = u_{x_i x_i}.$$

$F = F(x, ..., q_\alpha, ...)$ funksiya Q soha x nuqtalarining va $q_\alpha = q_{\alpha_1 \alpha_2 \ldots \alpha_n} = D^\alpha u$, $\alpha_i = 0, 1, \ldots$ haqiqiy o'zgaruvchinig berilgan funksiyasi bo'lsin.

T a' r i f. Ushbu

$$F(x, \ldots, D^\alpha u, \ldots) = 0 \tag{18}$$

tenglik noma'lum $u(x) = u(x_1, x_2, \ldots, x_n)$ funksiyaga nisbatan *xususiy hosilali differensial tenglama* deyiladi.

(18) da qatnashayotgan hosilaning eng yuqori tartibiga tenglamaning tartibi deyiladi.

Agar F barcha q_α ($|\alpha| = 0, 1, \ldots, m$) o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (18) tenglama *chiziqli differensial tenglama* deyiladi.

Agar tenglamaning tartibi m bo'lib, F barcha q_α, , $|\alpha| = m$ o'zgaruvchilarga nisbatan chiziqli funksiya bo'lsa, (18) tenglama *kvazichiziqli differensial tenglama* deataladi.

T a' r i f. Q sohada aniqlangan $u(x)$ funksiya (18) tenglamada ishtirok etuvchi barcha hosilalari bilan uzluksiz bo'lib, uni ayniyatga aylantirsa, $u(x)$ ga (18) tenglamaning *klassik yechimi* deyiladi.

Xususiy hosilali $m-$tartibli chiziqli differensial tenglamani ushbu

$$Lu \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u = f(x) \tag{19}$$

ko'rinishda yozish mumkin, bu yerda a_α lar tenglama koeffitsientlari,

$$L = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

esa xususiy hosilali $m-$ tartibli differensial operator deyiladi.

Barcha $x \in Q$ lar uchun (19) tenglamaning o'ng tomoni $f(x)$ nolga teng bo'lsa, (19) tenglama *bir jisnli*, $f(x)$ nolga teng bo'lmasa, *bir jinsli bo'lmagan tenglama* deyiladi.

Agar $u(x)$ va $v(x)$ funksiyalar bir jinsli bo'lmagan (19) tenglamaning yechimlari bo'lsa, ravshanki (tenglama chiziqli bo'lgani sababli) $w(x) = u(x) - v(x)$ ayirma bir jinsli ($f = 0$) tenlamaning yechimi bo'ladi.

Agarda $u_i(x)$, $i = 1, \dots, k$ funksiyalar bir jinsli ($f = 0$) tenglamaning yechimlari bo'lsa, $u(x) = \sum_{i=1}^{k} c_i u_i(x)$ funksiya ham, bu yerda c_i- haqiqiy o'zgarmaslar, shu tenglamaning yechimi bo'ladi.

Xususiy hosilali ikkinchi tartibli chiziqli differensial tenglama

$$\sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial u}{\partial x_i} + C(x)u = f(x) \qquad (20)$$

ko'rinishda yoziladi, bu yerda A_{ij}, B_i, C, f - Q sohada berilgan haqiqiy funksiyalardir.

(20) tenglama berilgan sohada hech bo'lmaganda bitta $A_{ij}, i, j = 1, \dots, n$ koeffitsient noldan farqli deb hisoblaymiz. Tenglamada $i \neq j$ bo'lganda alohida-alohida $A_{ij} u_{x_i x_j}$, $A_{ji} u_{x_j x_i}$ qo'shiluvchilar ishtirok etmay balki ularning yig'indisi $(A_{ij} + A_{ji}) u_{x_i x_j}$ ishtirok etadi. Shu sababli umumiylikka ziyon yetkazmay hamma vaqt $A_{ij} = A_{ji}$ deb hisoblaymiz.

Eslatib o'tamiz, Q sohada aniqlangan va $k-$tartibgacha xususiy hosilalari bilan uzluksiz bo'lgan haqiqiy $u(x)$ funksiyalar sinfi $C^k(Q)$ orqali belgilanadi, $C(Q)-$ Q sohada uzluksiz funksiyalar sinfi. $g(x) \in C^k(Q)$ funksiyaning normasi

$$\|g\| = \sum_{i=0}^{k} \max_{x \in Q} |D^\alpha g(x)|$$

kabi aniqlanadi.

2.8 Xarakteristik forma tushunchasi, ikkinchi tartibli differensial tenglamalarning klassifikatsiyasi va kanonik koʻrinishi

Faraz qilaylik, (18) tenglamaning tartibi m boʻlib, $F(x,\dots,q_\alpha,\dots)$ funksiya $q_\alpha = q_{\alpha_1,\alpha_2,\dots,\alpha_n}$, $\mid \alpha \mid = m$ oʻzgaruvchilar boʻyicha uzluksiz birinchi tartibli hosilalarga ega boʻlsin. $\lambda_1,\dots,\lambda_n$ haqiqiy oʻzgaruvchilarga nisbatan ushbu

$$K(\lambda_1,\dots,\lambda_n) = \sum_{\alpha=m} \frac{\partial F}{\partial q_\alpha}\lambda^\alpha,\ \ \lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\dots\lambda_n^{\alpha_n} \tag{21}$$

m darajali bir jinsli koʻphadni yozib olamiz. Bu koʻphadga (18) tenglamaga mos boʻlgan *xarakteristik forma* deyiladi.

Ikkinchi tartibli kvazichiziqli

$$\sum_{i,j=1}^{n} A_{ij}(x)u_{x_ix_j} + G(x,u,u_{x_1},\dots,u_{x_n}) = 0 \tag{22}$$

differensial tenglama uchun, bu yerda $A_{ij}(x) \in C(D)$, (21) forma

$$\Omega(\lambda_1,\dots,\lambda_n) = \sum_{i,j=1}^{n} A_{ij}(x)\lambda_i\lambda_j \tag{23}$$

kvadratik formadan iborat boʻladi.

Xususiy hosilali differensial tenglamalar, shu jumladan (22) koʻrinishidagi ikkinchi tartibli tenglama oʻrganilganda, erkli oʻzgaruvchilarni almashtirib, tenglamalarni soddaroq koʻrinishga olib kelishga harakat qilinadi. Shu maqsadda dastlab (22) tenglamada erkli oʻzgaruvchilarni almashtirganda uning $A_{ij}(x)$ koeffisiyentlari qanday oʻzgarishini qarab chiqamiz. $x = (x_1,\dots,x_n)$ oʻzgaruvchilar oʻrniga $y = y(x)$, ya'ni

$$y_i = y_i(x_1,x_2,\dots,x_n),\ \ i = 1,\dots,n$$

oʻzgaruvchilarni kiritamiz. x nuqtaning Q sohaga tegishli biror atrofida $y_i \in C^2$ boʻlsin va ushbu yakobian

$$\frac{D(y_1,\dots,y_n)}{D(x_1,\dots,x_n)} \neq 0.$$

Bu shartga ko‘ra x o‘zgaruvchilarni y lar orqali bir qiymatli ifodalashimiz mumkin, ya‘ni $x = x(y)$. (22) tenglamadagi $u(x)$ funksiyaning hosilalarini yangi y o‘zgaruvchilarga nisbatan hisoblaymiz:

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i},$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial y_k \partial y_l}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial u}{\partial y_k}\frac{\partial^2 y_k}{\partial x_i \partial x_j}.$$

Hosil qilingan ifodalarni (22) tenglamaga qo‘yib, uni ushbu

$$\sum_{k,l=1}^{n} \tilde{A}_{kl} u_{y_k y_l} + \tilde{G}(y, u, u_{y_1}, \ldots, u_{y_n}) = 0, \tag{24}$$

ko‘rinishda yozib olamiz, bu yerda

$$\tilde{A}_{kl} = \sum_{i,j=1}^{n} A_{ij}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}, \tag{25}$$

$\tilde{G}$ orqali G dan va birinchi tartibli hosilalar ishtirok etgan hadlardan tashkil topgan ifoda belgilangan.

$x_0 \in Q$ nuqtani tayin qilib,

$$y_0 = y(x_0), \beta_{kl} = \frac{\partial y_k(x_0)}{\partial x_i}$$

belgilashlarni kiritamiz.

(25) formula x_0 nuqtada quyidagicha yoziladi:

$$\tilde{A}_{kl}(y_0) = \sum_{k,l=1}^{n} A_{i,j}(x_0)\beta_{ki}\beta_{lj}. \tag{26}$$

(23) kvadratik formani x_0 nuqtada yozib olamiz:

$$\Omega = \sum_{i,j=1}^{n} A_{i,j}(x_0)\lambda_i\lambda_j. \tag{27}$$

Maxsus bo‘lmagan ushbu

$$\lambda_i = \sum_{k=1}^{n} \beta_{ki}\xi_k, det(\beta_{ki}) \neq 0 \tag{28}$$

Affin almashtirish natijasida (27) kvadratik forma

$$\Omega = \sum_{k,l=1}^{n} \tilde{A}_{kl}(y_0)\xi_k\xi_l \tag{29}$$

ko‘rinishga keladi. Bu kvadratik formaning koeffisiyentlari (9) formula bilan aniqlaniladi.

Shunday qilib, (22) tenglamani x_0 nuqtada x o‘zgaruvchilar o‘rniga yangi $y = y(x)$ o‘zgaruvchilarni kiritib soddalashtirish uchun, shu nuqtada (27) kvadratik formani maxsus bo‘lmagan (28) chiziqli almashtirish yordamida soddalashtirirish yetarlidir.

Algebra kursidan ma’lumki, hamma vaqt shunday maxsus bo‘lmagan (28) almashtirish mavjud bo‘lib, u yordamida (27) kvadratik forma quyidagi ko‘rinishga keltiriladi:

$$Q = \sum_{k=1}^{n} \mu_k \xi_k^2, \tag{30}$$

bu yerda $\mu_k, k = 1, \ldots, n$ koeffitsientlar $1, -1, 0$ qiymatlarni qabul qiladi. Shu bilan birga musbat (manfiy) koeffitsientlar soni va nolga teng bo‘lgan koeffitsientlar soni (forma defekti) affin invariantdir, ya’ni bu sonlar faqat (27) forma bilan aniqlanib, (28) almashtirishning tanlanishiga bog‘liq bo‘lmaydi. Bu esa (22) differensial tenglama $A_{ij}(x)$ koeffitsientlarining x_0 nuqtada qabul qiladigan qiymatlariga qarab, klassifikatsiya qilish imkonini beradi.

Demak, yuqorida almashtirishlardan so‘ng (25) tenglama

$$\sum_{k=}^{n} \mu_k u_{y_k y_k} + \tilde{G}(y, u, u_1, \ldots, u_{y_n}) = 0 \tag{31}$$

ko‘rinishga olib kelinadi.

Ikkinchi tartibli differensial tenglamaning aralash hosilalar qatnashmagan bunday ko‘rinishi, odatda uning *kanonik (sodda) ko‘rinishi* deyiladi.

Agar barcha $\mu_k = 1$ yoki barcha $\mu_k = -1, k = 1, \ldots, n$ bo‘lsa, ya’ni Ω forma mos ravishda musbat yoki manfiy aniqlangan bo‘lsa, (22) tenglama $x \in Q$ nuqtada *elliptik tipdagi yoki elliptik tenglama* deyiladi.

Agar μ_k koeffitsientlardan bittasi manfiy, qolganlari musbat (yoki aksincha) bo‘lsa, (22) tenglama $x \in Q$ nuqtada giperbolik tenglama deyiladi.

μ_k koeffitsientlardan l tasi, $1 < l < n-1$, musbat, qolgan $n-l$ tasi manfiy bo'lsa, (22) tenglama *ultragiperbolik tipdagi tenglama* deyiladi.

Agar μ_k koeffitsientlardan bittasi nolga teng, qolganlari noldan farqli bo'lsa, (22) tenglama $x \in Q$ nuqtada *parabolik tenglama* deyiladi.

Agar koeffitsiyentlardan kamida bittasi nolga teng bo'lsa, (22) tenglama keng ma'noda $x \in Q$ nuqtada *parabolik tenglama* deb ataladi.

Agar (22) tenglamada Q sohaning har bir nuqtasida elliptik, giperbolik yoki parabolik bo'lsa D sohada mos ravishda *elliptik, giperbolik yoki parabolik tipdagi tenglama* deb ataladi.

Agar noldan farqli bo'lgan, bir xil ishorali α_0, α_1 haqiqiqy sonlar mavjud bo'lib, barcha $x \in Q$ nuqtalar uchun ushbu

$$\alpha_0 \sum_{i=1}^{n} \lambda_i^2 \leq \Omega(\lambda_1, \ldots, \lambda_n) \leq \alpha_1 \sum_{i=1}^{n} \lambda_i^2$$

tengsizlik bajarilsa, Ω sohada elliptik bo'lgan (22) tenglamaga *tekis elliptik tenglama* deyiladi.

Masalan, ushbu Trikomi

$$u_{x_1x_1} + x_1 u_{x_2x_2} = 0$$

tenglamasi $x_1 > 0$ yarim tekislikning har bir nuqtasida elliptik bo'lishi bilan birga, u bu sohada tekis elliptik emasdir.

Q sohaning turli qismlarida (22) tenglama har xil tipga tegishli bo'lsa, unga aralash tipdagi tenglama deyiladi.

Yuqorida keltirilgan Trikomi tenglamasi $x_2 = 0$ o'qning ixtiyoriy qismini o'z ichiga olgan ixtiyoriy Q sohada aralash tipdagi tenglamaga misol bo'ladi.

Yoqorida bayon qilingan (22) tenglamaning klassifikatsiyasini ekvivalent tarzda $A = \|A_{ij}\|$ matritsaning xarakteristik sonlariga asoslanib ham berish mumkin. Buning uchun algebradan ma'lum bo'lgan (27) kvadratik formaning (30) kanonik ko'rinishidagi μ_k, $k = 1, \ldots, n$ sonlar A matritsaning xarakteristik sonlaridan iborat ekanligini eslash yetarli. Ma'lumki, simmetrik ($A_{ij} = A_{ji}$) matrisaning barcha xarakteristik sonlari haqiqiydir.

Eslatib o'tamiz, A matrisaning xarakteristik sonlari ushbu

$$det(A - \lambda I) = 0$$

algebraik tenglamaning ildizlaridan iborat, bu yerda I - birlik matritsa.

Demak, (22) tenglama berilgan Q sohaning ixtiyoriy x nuqtasida A matritsa xarakteristik sonlarining ishorasini aniqlab, (22) tenglamaning qaysi tipga tegishli ekanini bilib olish mumkin.

Endi xarakteristik sirt tushunchasini kiritamiz. Ushbu

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial\omega}{\partial x_i}\frac{\partial\omega}{\partial x_j} = 0$$

tenglamaga (22) differensial tenglama xarakteristikalarining tenglamasi deyiladi.

Agar $\omega(x_1,\ldots,x_n)$ funksiya xarakteristikalar tenglamasini qanoatlantirsa

$$\omega(x_1,\ldots,x_n) = c, \;\; c = const$$

tenglik bilan aniqlanadigan sirt berilgan (22) differensial tenglamaning *xarakteristik sirti yoki xarakteristikasi* deyiladi. O‘zgaruvchilar soni ikkita bo‘lganda xarakteristik egri chiziq haqida so‘z boradi.

Xarakteristikalar tenglamasini qurish uchun (22) differensial tenglamaga mos bo‘lgan (23) kvadratik formani tuzib, unda $\lambda_i = \frac{\partial\omega}{\partial x_i}$, $\lambda_j = \frac{\partial\omega}{\partial x_j}$ deb, hosil bo‘lgan ifodani nolga tenglashtirish zarur.

Faraz qilaylik, $\omega \in C^2$ bo‘lsin. (22) tenglamani kanonik ko‘rinishga keltirish maqsadida x_i o‘zgaruvchilar o‘rniga kiritilgan y_i o‘zgaruvchilardan bittasini, masalan, y_1 ni $y_1 = \omega(x_1, x_2, \ldots, x_n)$ deb hisoblasak, u holda xarakteristikalar tenglamasiga ko‘ra $A_{11} = 0$ bo‘ladi. Shuning uchun ham diffe- rensial tenglamaning bitta yoki bir nechta xarakteristikalar oilasini bilish, bu tenglamani soddaroq ko‘rinishga keltirish imkonini beradi.

2.9 Yuqori tartibli differensial tenglamalarning va sistemalarning klassifikatsiyasi

m-tartibli xususiy hosilali kvazichiziqli differensial tenglama

$$\sum_{|\alpha|=m} a_\alpha(x)D^\alpha u + G(x, u, \ldots, D^\beta u, \ldots) = 0 \qquad (32)$$

ko‘rinishda yoziladi, bu yerda G orqali x o‘zgaruvchilar, noma’lum $u = u(x)$ funksiya va uning $1, \ldots,\ m-1$ tartibli hosilalarini o‘z ichiga olgan ifoda belgilangan.

(32) tenglamaga mos bo‘lgan xarakteristik forma (21) ga asosan

$$K(\lambda_1, \ldots, \lambda_n) = \sum_{|\alpha|=m} a_\alpha(x)\lambda^\alpha \tag{33}$$

ko‘rinishda yoziladi.

Agar $x \in Q$ nuqtada $\lambda_1, \ldots, \lambda_n$ o‘zgaruvchilarning shunday $\lambda_i = = \lambda_i(\mu_1, \ldots, \mu_n)$, $i = 1, \ldots, n$ affin almashtirishini topish mumkin bo‘lsaki, natijada (33) formadan hosil bo‘lgan forma μ_i o‘zgaruvchilarning faqat l tasini, $0 < l < n$, o‘z ichiga olsa, (32) tenglama x nuqtada *parabolik* yoki *parabolik buziladi* deb aytiladi.

Parabolik buzilish bo‘lmaganda, $K(\lambda_1, \ldots, \lambda_n) = 0$ tenglik faqat $\lambda_1 = 0, \ldots, \lambda_n = 0$ bo‘lganda bajarilsa, (32) tenglama $x \in Q$ nuqtada *elliptik* deyiladi.

Agarda $\lambda_1, \ldots, \lambda_n$ o‘zgaruvchilardan biror λ_i ni ajratib olish mumkin bo‘lsa (zarur bo‘lgan holda bu o‘zgaruvchilarning Affin almashtirishidan so‘ng), barcha $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) \in \mathbb{R}^{n-1}$ nuqtalar uchun λ ga nisbatan xarakteristik

$$K(\lambda', \lambda) = 0$$

tenglamaning barcha ildizlari haqiqiy bo‘lsa, (32) tenglama $x \in Q$ nuqtada *giperbolik* deyiladi.

Agarda λ ildizlarning bir qismi haqiqiy, qolganlari esa kompleks bo‘lsa, (32) tenglama $x \in Q$ nuqtada *qo‘shma tipdagi tenglama* deyiladi.

Bunga asosan $m \geq 3$ bo‘lgandagina (32) qo‘shma tipdagi tenglama bo‘lishi mumkin.

Qo‘shma tipdagi tenglamaga

$$\frac{\partial}{\partial x_1}(u_{x_1x_1} + u_{x_2x_2})$$

tenglama misol bo‘la oladi.

Xuddi shunga o‘xshash, (18) tenglama chiziqli bo‘lmagan holda (21) forma xususiyatiga asosan tiplarga ajratiladi. (21) forma koeffitsientlari x

nuqta bilan birga izlanayotgan $u(x)$ yechim va uning hosilalariga bog'liq bo'lgani sababli, tiplarga ajratish bu holda faqat shu yechim uchungina ma'noga ega bo'ladi.

Masalan,

$$u(x)u_{x_1x_1} + \sum_{i=2}^{n} u_{x_ix_i} = 0$$

tenglama $u(x) > 0$ bo'lgan $x \in Q$ nuqtalarda elliptik, $u(x) < 0$ bo'lganda giperbolik va $u(x) = 0$ bo'lgan $x \in Q$ nuqtalarda parabolik buziladi.

M i s o l. Chiziqli bo'lmagan

$$F\left(x_1, x_2, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}\right) = u_{x_1x_1} + u_{x_1x_2}u_{x_2x_2} + u_{x_2x_2}^2 - 4u_{x_2x_2} = 0$$

tenglamani qaraymiz. Bevosita tekshirish yordamida ishonch hosil qilish mumkinki, $u_1(x_1, x_2) = 2x_2^2$, $u_2(x_1, x_2) = 5x_1x_2$, $u_3(x_1, x_2) = x_1$ lar bu tenglamaning xususiy yechimlaridir. Tenglamani

$$u_{x_1x_1} + \frac{1}{2}u_{x_1x_2}u_{x_2x_2} + \frac{1}{2}u_{x_2x_1}u_{x_2x_2} + u_{x_2x_2}^2 - 4u_{x_2x_2} = 0$$

ko'rinishda yozib olib, (16) kvadratik formaning

$$a_{11} = \frac{\partial F}{\partial q_{11}} = \frac{\partial F}{\partial u_{x_1x_1}} = 1, \quad a_{12} = \frac{\partial F}{\partial q_{12}} = \frac{\partial F}{\partial u_{x_1x_2}} = \frac{1}{2}u_{x_2x_2},$$

$$a_{21} = \frac{\partial F}{\partial q_{21}} = \frac{\partial F}{\partial u_{x_2x_1}} = a_{12}, \quad a_{22} = \frac{\partial F}{\partial q_{22}} = \frac{\partial F}{\partial u_{x_2x_2}} = u_{x_1x_2} + 2u_{x_2x_2} - 4$$

koeffitsientlarini aniqlaymiz.

1. u_1 yechimni qaraymiz. Bu yechimda $a_{11} = 1, a_{12} = a_{21} = 2, a_{22} = 4$ bo'lib, (33) kvadratik forma

$$K(\lambda_1, \lambda_2) = \lambda_1^2 + 4\lambda_1\lambda_2 + 4\lambda_2^2$$

ko'rinishga ega bo'ladi. Osongina tekshirish mumkinki, bu kvadratik forma koeffitsientlaridan tuzilgan matritsaning xos sonlari $\gamma_1 = 0$, $\gamma_2 = 5$ lardan iborat. Shunung uchun ν_1, ν_2 o'zgaruvchilarni shunday tanlash mumkinki, kvadratik forma

$$\bar{K}(\nu_1, \nu_2) = \gamma_1\nu_1^2 + \gamma_2\nu_2^2 = 5\nu_2^2$$

ko‘rinishga o‘tadi. Demak, u_1 yechimda tenglama parabolik tipga mansub ekan.

2. u_2 yechimni qaraymiz. Bu holda $a_{11} = 1, a_{12} = a_{21} = 0, a_{22} = 1$ bo‘lib, (33) ga ko‘ra kvadratik

$$K(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2$$

formaning o‘zi kanonik ko‘rinishga ega. Bundan u_2 yechimda tenglama elliptik tipda ekan.

3. u_3 yechimni qaraymiz. Bu yechimda $a_{11} = 1, a_{12} = a_{21} = 0, a_{22} = -4$ bo‘lib,

$$K(\lambda_1, \lambda_2) = \lambda_1^2 - 4\lambda_2^2.$$

Demak, u_3 yechimda tenglama giperbolik tipda ekan.

Qaralgan har uchala holda ham $a_{ij}, i, j = 1, 2,$ koeffitsientlar x_1, x_2 o‘zgaruvchilarga bog‘liq emas. Shuning uchun har bir holda tenglama Q sohaning barcha nuqtalarida faqat bitta tipga ega bo‘ladi.

(32) tenglamaning quyidagi xususiy holini o‘rganamiz.

Biror erkli o‘zgaruvchiga (bu o‘zgaruvchini t orqali belgilaymiz) nisbatan m - tartibli hususiy hosilasi ajratib yozilgan

$$D_t^m u + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t, x) D_x^\alpha D_t^k u = f(t, x) \tag{32'}$$

tenglama *Kovalevskaya tipidagi* chiziqli m - tartibli hususiy hosilali differensial tenglama deyiladi. Bunda, avvalgi kabi, $\alpha = (\alpha_1, \ldots, \alpha_n)$-multiindeks. (32′) tenglamaning chap tomonini m-tartibli hosilalar qatnashgan

$$D_t^m u + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t, x) D_x^\alpha D_t^k u,$$

berilgan tenglamaning *bosh qismi* deb ataluvchi hamda $u(t, x)$ va uning $1, \ldots, m-1$ tartibli hosilalarini o‘z ichiga olgan

$$\sum_{k+|\alpha| < m} a_{k,\alpha}(t, x) D_x^\alpha D_t^k u$$

qismlarga ajratamiz.

$$D_x^\alpha D_t^k = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \dots D_{x_n}^{\alpha_n} D_t^k$$

differensial operatorni

$$\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_n^{\alpha_n} \tau^k = \lambda^\alpha \tau^k$$

bilan almashtirib, tenglamaning bosh qismini unga mos keluvchi

$$\tau^m + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t,x) \lambda^\alpha \tau^k$$

ko‘rinishdagi xaraktiristik formaga keltiramiz.

Faraz qilaylik, (32′) tenglamaning koeffisientlari $D \subset \mathbb{R}^{n+1}$ sohada haqiqiy va uzluksiz funksiyalar bo‘lsin.

T a’ r i f. Agar

$$\tau^m + \sum_{k+|\alpha| \le m, k \ne m} a_{k,\alpha}(t_0,x_0) \lambda^\alpha \tau^\alpha \tag{33′}$$

xarakteristik tenglama $\xi \neq 0$ lar uchun $(t_0, x_0) \in D$ nuqtada faqat haqiqiy $\tau_1 = \tau_1(t_0, x_0, \xi), \dots, \tau_m = \tau_m(t_0, x_0, \xi)$ ildizlarga ega bo‘lsa, u holda (32′) tenglama $(t_0, x_0) \in D$ nuqtada giperbolik deyiladi.

Agar (33′) xarakteristik tenglama $\xi \neq 0$ lar uchun faqat haqiqiy va har xil $\tau_1 = \tau_1(t_0, x_0, \xi), \dots, \tau_m = \tau_m(t_0, x_0, \xi)$ ildizlarga ega bo‘lsa, u holda (32′) tenglama

$(t_0, x_0) \in D$ nuqtada qat’iy giperbolik deyiladi.

Agar $\xi \neq 0$ lar uchun (32′) tenglama haqiqiy ildizlarga ega bo‘lmasa, u holda

(32′) tenglama $(t_0, x_0) \in D$ nuqtada elliptik deyiladi,

Agar (32′) tenglama D sohaning har bir nuqtasida giperbolik (qat’iy giperbolik, elliptik) bo‘lsa, u holda u D sohada giperbolik (qat’iy giperbolik, elliptik) deyiladi

M i s o l. $\mathbb{R}^2$ tekislikda

$$\frac{\partial^2 u}{\partial t^2} - t^2 \frac{\partial^2 u}{\partial x^2} = 0$$

tenglamani qaraymiz. bu tenglamaga mos keluvchi xarakteristik tenglamaning yechimlari $\tau_1 = -t\xi$ va $\tau_2 = t\xi$ lardan iborat. Demak, ta'rifga ko'ra tenglama giperbolik tipga ega. Ravshanki, $\{(x,t) \in \mathbb{R}^2 : t \neq 0\}$ sohada tenglama qat'iy giperbolik bo'ladi.

Endi xususiy hosilali differensial tenglamalar sistemasining klassifikatsiyasiga qisqacha to'xtalib o'tamiz.

(18) tenglamada F funksiya k o'lchovli $F = (F_1, \ldots, F_k)$ vektor funksiyadan iborat bo'lsin. Faraz qilaylik, bu vektorning $F_1, \ldots, F_k$ komponentlari Q soha x nuqtalari va $q_0^j = u_j, q_\alpha^j = D^\alpha u_j, j = 1, \ldots, l)$ haqiqiy o'zgaruvchilarning berilgan haqiqiy funksiyalari bo'lsin.

Ushbu

$$F_i(x, \ldots, D^\alpha u_j, \ldots) = 0, \ i = 1, \ldots, k, \ j = 1, \ldots, l \tag{34}$$

ko'rinishdagi tengliklar, noma'lum $u_1, \ldots, u_l$ funksiyalarga nisbatan xususiy hosilali differensial tenglamalar sistemasi deyiladi. (34) tenglamalar sistemasida qatnashayotgan noma'lum funksiyalar hosilalarining eng yuqori tartibiga shu sistemaning tartibi deyiladi.

(34) sistemasining tartibi m ga teng bo'lsin. Agar F_i, $i = 1, \ldots, k$ funksiyalar barcha q_α^j o'zgaruvchilarga nisbatan chiziqli bo'lsa, (34) sistema chiziqli, agarda F_i lar, $i = 1, \ldots, k$, barcha $q_\alpha^j, |\alpha| = m$ larga nisbatan chiziqli bo'lsa, (34) sistema kvazichiziqli deyiladi.

Agar (34) sistemada $k = l, k > l, k < l$ bo'lsa, u mos ravishda aniq, ortig'i bilan aniqlangan va yetarlicha aniqlanmagan deyiladi. (34) sistema aniq bo'lib, uning har bir tenglamasining tartibi m ga teng bo'lsin.

Ushbu

$$a_\alpha = \left\| \frac{\partial F_i}{\partial q_\alpha^j} \right\|, \ i, j = 1, \ldots, k, \quad \sum_{i=1}^{n} \alpha_i = m$$

kvadratik matritsani tuzamiz.

$$K(\lambda_1, \ldots, \lambda_n) = det \sum_{|\alpha|=m} a_\alpha \lambda^\alpha = det \sum_{|\alpha|=m} a_{\alpha_1,\ldots,\alpha_n} \lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n} \tag{35}$$

ifoda haqiqiy skalyar $\lambda_1, \ldots, \lambda_n$ parametrlarga nisbatan km tartibli formadan iboratdir. Bu forma (34) *sistemaning xarakteristik determinanti* deyiladi.

(35) formaning xarakteriga qarab, xuddi (32) tenglamaga o‘xshash, (34) sistema ham tiplarga ajratiladi. Ko‘p hollarda tadbiqiy masalalarda uchraydigan tenglamalar sistemasini ushbu matritsali

$$\sum_{|\alpha|\le m} a_\alpha D^\alpha u = f \tag{36}$$

tenglama ko‘rinishida yozish mumkin. (36) ifodada D^α differensial operator $u = (u_1(x), \dots, u_k(x))$ vektor-funksiya yoki $u = \|u_j\|, j = 1, \dots, k$ matritsa-ustunining har bir komponentiga ta’sir qiladi, a_α - koeffitsientlar k-tartibli matritsadan iborat bo‘lib, ular hamda (36) sistemaning o‘ng tomoni $f = (f_1, \dots, f_N)$ yoki $f = \|f_j\|$ $x = (x_1, \dots, x_n)$ o‘zgaruvchilarga, noma’lum $u_j(x)$ funksiyalarga va ularning tartibi $m-1$ dan katta bo‘lmagan hosilalariga bog‘liq bo‘lishi mumkin. $m = 1$ bo‘lganda (36) sistemadan birinchi tartibli xususiy hosilali differensial tenglamalar sistemasi

$$\sum_{j=1}^{n} a_j D_j u + bu = f \tag{37}$$

differensial tenglamalar sistemasi kelib chiqadi, bu yerda $b-k$-tartibli kvadratik matritsa.

(37) tenglamalar sistemasiga to‘g‘ri keluvchi k-tartibli xarakteristik forma ushbu

$$K(\lambda_1, \dots, \lambda_n) = det \sum_{j=1}^{n} a_j \lambda_j$$

ko‘rinishga ega. Soddalik uchun (37) tenglamaning xususiy holini qaraymiz, ya’ni (37) da $n = k = 2,\ b = 0,\ f = 0,$

$$a_1 = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad a_2 = \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\|$$

bo‘lsin. Bu holda quyidagi sistemaga ega bo‘lamiz:

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| \left\| \frac{\partial}{\partial x_1} \right\| \left\| \begin{matrix} u_1 \\ u_2 \end{matrix} \right\| + \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\| \left\| \frac{\partial}{\partial x_2} \right\| \left\| \begin{matrix} u_1 \\ u_2 \end{matrix} \right\| =$$

$$\left\| \begin{matrix} D_1u_1 + 0 \\ 0 + D_1u_2 \end{matrix} \right\| + \left\| \begin{matrix} 0 - D_2u_2 \\ D_2u_1 + 0 \end{matrix} \right\| = \left\| \begin{matrix} D_1u_1 - D_2u_2 \\ D_2u_1 + D_1u_2 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$$

yoki

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2}, \quad \frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1}. \tag{38}$$

Ma'lumki, (38) kompleks o'zgaruvchili funksiyalar nazariyasidan tanish bo'lgan *Koshi-Riman tenglamalari* sistemasidir. Bu sistemaga mos bo'lgan xarakteristik forma

$$K = det(a_1\lambda_1 + a_2\lambda_2) = det \begin{Vmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{Vmatrix} = \lambda_1^2 + \lambda_2^2$$

ko'rinishga ega bo'ladi.

Demak, Koshi-Riman tenglamalari sistemasi elliptik tipda ekan. $z = x_1 + ix_2$ kompleks o'zgaruvchini hamda

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

differensial operatorni kiritib, (38) sistemani bitta differensial tenglama

$$\frac{\partial w}{\partial z} = 0$$

ko'rinishida yozish mumkin, bu yerda

$$w(z) = u_(x_1, x_2) + iu_2(x_1, x_2).$$

2.10 Ikkinchi tartibli ikki o'zgaruvchili differensial tenglamalarni kanonik ko'rinishga keltirish

Ikkita x va y haqiqiy o'zgaruvchili ikkinchi tartibli kvazichiziqli (yuqori tartibli hosilalariga nisbatan chiziqli) differensial tenglama ushbu

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + G\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \tag{39}$$

ko'rinishda yoziladi.

(39) tenglama xarakteristikalarining tenglamasi chiziqli bo'lmagan

$$a\left(\frac{\partial \omega}{\partial x}\right)^2 + 2b\frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y} + c\left(\frac{\partial \omega}{\partial y}\right)^2 = 0 \tag{40}$$

tenglamadan iboratdir.

Bu tenglamani oddiy differensial tenglamaga keltiramz. Agar $\omega(x, y)$ funksiya (40) tenglamaning yechimi bo‘lsa, u holda $\omega(x, y) = const$ egri chiziq (39) tenglamaning xarakteristikasidir. Bu xarakteristika atrofida

$$\frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy = 0$$

yoki

$$\frac{\partial \omega}{\partial x} : \frac{\partial \omega}{\partial y} = -\frac{dy}{dx}$$

munosabat bajariladi.

(40) tenglamada

$$\frac{\partial \omega}{\partial x} : \frac{\partial \omega}{\partial y}$$

nisbatni $-dy : dx$ ga almashtirib,

$$ady^2 - 2bdydx + cdx^2 = 0 \tag{41}$$

oddiy differensial tenglamaga ega bo‘lamiz.

Aksincha, agar $\omega(x, y) = const$ (41) tenglamaning integrali bo‘lsa, u holda $\omega(x, y)$ funksiya (5) xarakteristikalar tenglamasini qanoatlantiradi. (41) tenglik xarakteristik egri chiziqlarning oddiy differensial tenglamasidan iboratdir.

(41) tenglik bilan aniqlangan(dx, dy) yo‘nalish odatda xarakteristik yo‘nalish deyiladi.

Agar (39) tenglamaning a, b, c koeffitsientlari tenglama berilgan biror $(x, y) \in Q$ sohada yetarli silliq funksiyalar bo‘lsa, u holda x, y o‘zgaruvchilarning shunday maxsus bo‘lmagan

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

almashtirish mavjud bo‘ladiki, (39) tenglama Q sohada bu almashtirish yordamida quyidagi kanonik ko‘rinishlardan biriga keladi:

Elliptik holda

$$u_{\xi\xi} + u_{\eta\eta} + \tilde{G}(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{42}$$

giperbolik holda

$$u_{\xi\xi} - u_{\eta\eta} + \tilde{G}(\xi, \eta, u, u_\xi, u_\eta) = 0 \tag{43}$$

yoki

$$u_{\xi\eta} + \tilde{G}(\xi, \eta, u, u_\xi, u_\eta) = 0 \tag{44}$$

va parabolik holda

$$u_{\eta\eta} + \tilde{G}(\xi, \eta, u, u_\xi, u_\eta) = 0. \tag{45}$$

Faraz qilamiz, (39) tenglamaning a, b, c koeffitsiyentlari (x, y) nuqtaning biror atrofida C^2 sinfga tegishli bo‘lib,

$$a^2(x, y) + b^2(x, y) + c^2(x, y) \neq 0$$

bo‘lsin.

Umumiylikka ziyon yetkazmay $a(x, y) \neq 0$ deb hisoblashimiz mumkin. Haqiqatan ham, aks holda $c(x, y) \neq 0$ bo‘lishi mumkin. Bu holda x va y o‘rnini almashtirib, shunday tenglama hosil qilamizki, unda $a(x, y) \neq 0$ bo‘ladi.

Agarda a va c lar bir vaqtda biror nuqtada nolga teng bo‘lsa, bu nuqta atrofida $b(x, y) \neq 0$ bo‘ladi. Bu holda $x' = x + y,\ y' = x - y$ almashtirishlar natijasida hosil bo‘lgan tenglamada $a(x, y) \neq 0$ bo‘ladi.

(39) tenglamada erkli o‘zgaruvchilarni ixtiyoriy (qaytariluvchi) almashtiramiz:

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

Birinchi va ikkinchi tartibli hosilalarni quyidagicha hisoblab:

$$u_x = u_\xi \xi_x + u_\eta \eta_x,\ u_y = u_\xi \xi_y + u_\eta \eta_y,$$

$$u_{xx} = \xi_x^2 u_{\xi\xi} + 2\xi_x \eta_x u_{\eta\xi} + \eta_x^2 u_{\eta\eta} + \xi_{xx} u_\xi + \eta_{xx} u_\eta,$$

$$u_{xy} = \xi_x \xi_y u_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x) u_{\eta\xi} + \eta_x \eta_y u_{\eta\eta} + \xi_{xy} u_\xi + \eta_{xy} u_\eta,$$

$$u_{yy} = \xi_y^2 u_{\xi\xi} + 2\xi_y \eta_y u_{\eta\xi} + \eta_y^2 u_{\eta\eta} + \xi_{yy} u_\xi + \eta_{yy} u_\eta,$$

(39) tenglamaga qo‘yamiz. Natijada (39) tenglama

$$a_1 \frac{\partial^2 u}{\partial \xi^2} + 2b_1 \frac{\partial^2 u}{\partial \xi \partial \eta} + c_1 \frac{\partial^2 u}{\partial \eta^2} + \bar{G}(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0 \tag{46}$$

ko'rinishda yoziladi. Bu yerda

$$a_1(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2,$$

$$b_1(\xi,\eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$$

$$c_1(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,$$

$$u(\xi,\eta) = u[x(\xi,\eta), y(\xi,\eta)],$$

$x = x(\xi,\eta)$, $y = y(\xi,\eta)$ almashtirish esa $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ ga teskari almashtirishdir.

(40) xarakteristikalar tenglamasini

$$a\left(\omega_x + \frac{b+\sqrt{-\Delta}}{a}\omega_y\right)\left(\omega_x + \frac{b-\sqrt{-\Delta}}{a}\omega_y\right) = 0 \qquad (47)$$

ko'rinishda yozib olamiz, bu yerda $\Delta = ac - b^2$. Dastlab (39) tenglama elliptik tipga tegishli bo'lsin, ya'ni (x,y) nuqta atrofida $\Delta(x,y) > 0$, shu bilan birga $\sqrt{-\Delta} = i\sqrt{\Delta}$.

Bu holda (47) tenglama haqiqiy yechimlarga ega emas. Shuning uchun

$$\omega(x,y) = \xi(x,y) + i\eta(x,y)$$

deb belgilab olamiz. Agar (40) tenglamaning chap tomonini q orqali belgilasak, bevosita hisoblashlarni amalga oshirib,

$$q = a_1 - c_1 + 2ib \qquad (48)$$

bo'lishiga ishonch hosil qilish mumkin. Shunga muvofiq $\omega(x,y)$ funksiyani ushbu ikkita

$$\omega_x + \frac{b+i\sqrt{\Delta}}{a}\omega_y = 0,$$

$$\omega_x + \frac{b-i\sqrt{\Delta}}{a}\omega_y = 0 \qquad (49)$$

tenglamadan birining yechimi sifatida izlash tabiiydir. Bu tenglamalarda ω_y oldidagi koeffitsiyentlar o'zaro qo'shma kompleks ifodalardan iborat. (49) tenglama ikkita birinchi tartibli

$$a\xi_x + b\xi_y - \sqrt{\Delta}\eta_y = 0$$

$$a\eta_x + b\eta_y - \sqrt{\Delta}\xi_y = 0 \tag{50}$$

tenglamalar sistemasiga ekvivalentdir. (50) tenglamalar *Beltrami tenglamalari* deb ataladi.

Endi $\xi = \xi(x, y)$va $\eta = \eta(x, y)$ sifatida Beltrami tenglamalarining yakobiani

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0 \tag{51}$$

bo'lgan yechimlarni olamiz. $\omega(x, y)$ funksiya (40) xarakteristikalar tenglamasining yechimi bo'lgani uchun (38) tenglik nolga teng bo'ladi. Bundan

$$a_1 = c_1, b_1 = 0$$

kelib chiqadi. (50) tenglamalardan ξ_x va η_x ni topib $a_1,\ b_1$ koeffitsiyentlarning ifodalariga qo'ysak,

$$a_1 = c_1 = \frac{\Delta}{a}(\xi_y^2 + \eta_y^2) \neq 0$$

bo'lishiga ishonch hosil qilamiz. (46) tenglamaning barcha hadlarini noldan farqli bo'lgan

$$\frac{\Delta}{a}(\xi_y^2 + \eta_y^2)$$

ifodaga bo'lib, (42) tenglamani hosil qilamiz.

Endi (x, y) nuqta atrofida (4) giperbolik tipga tegishli bo'lsin, ya'ni $\Delta < 0,\ \xi(x, y),\ \eta(x, y)$ funksiyalar esa mos ravishda

$$\xi_x + \frac{b + \sqrt{\Delta}}{a}\xi_y = 0,$$

$$\eta_x + \frac{b - \sqrt{\Delta}}{a}\eta_y = 0 \tag{52}$$

tenglamaning (51) shartlarini qanoatlantiruvchi yechimi bo'lsin.

Bu holda, (52) ga asosan

$$a_1 = c_1 = 0, \quad b_1 = 2\frac{\Delta}{a}\xi_y\eta_y \neq 0.$$

Agarda $\xi_y = 0$ yoki $\eta_y = 0$ bo'lsa, (52) asosan (51) yakobian nolga teng bo'ladi. $\xi(x, y),\ \eta(x, y)$ funksiyalarning tanlanilishiga muvofiq bunday bo'lishi mumkin emas. Shu sababli (x, y) nuqta atrofida $b_1 \neq 0$ bo‘ladi. (46) tenglama $2b_1$ ga bo‘lingandan so‘ng (44) ko‘rinishga keladi. $\alpha = \xi + \eta$, $\beta = \xi - \eta$ almashtirish natijasida (44) tenglamadan (43) tenglama kelib chiqadi.

Nihoyat $\Delta = 0$, ya’ni (x, y) nuqta atrofida (39) tenglama parabolik bo‘lsin. $\xi(x, y)$ sifatida

$$a\xi_x + b\xi_y = 0 \tag{53}$$

tenglamaning o‘zgarmas sondan farqli bo‘lgan yechimini olamiz.

Bu holda $a_1 = b_1 = 0$ bo‘ladi. $\eta(x, y)$ funksiyani shunday tanlab olamizki, natijada $c_1(\xi, \eta) \neq 0$ bo‘lsin. Natijada, (46) dan (45) tenglama hosil bo‘ladi.

Yuqorida hosil qilingan xususiy hosilali birinchi tartibli chiziqli differensial tenglamalar yechimlarining mavjudligi masalasi birinchi tartibli oddiy differensial tenglamalar nazariyasi bilan uzviy bog‘liqdir. Oddiy differensial tenglamalar nazariyasidan ma’lumki, a, b, c funksiyalar yetarlicha silliq bo‘lganda xususiy hosilali chiziqli tenglamalarning (50) sistemasi hamda (52) va (53) chiziqli tenglamalar (39) tenglama bilan berilgan $(x, y) \in D$ nuqtasining kichik atrofida yakobiani noldan farqli bo‘lgan yechimlarga egadir.

Shu bilan (39) tenglamani (x, y) nuqta atrofida, ya’ni lokal (42), (43) (yoki (44)) va (45) kanonik ko‘rinishlarga keltirish mumkin.

M i s o l. Ushbu

$$u_{xx} + xu_{yy} = 0$$

Trikomi tenglamasini Oxy tekislikning tenglama tipi saqlanadigan sohalarida kanonik ko‘rinishga keltiramiz.

$\triangle = ac - b^2 = x$ bo‘lgani uchun, $x < 0$ sohada tenglama giperbolik, $x > 0$ sohada esa elliptik tipga ega. Dastlab giperbolik holni qaraylik. Bu holda xarakteristikaning differensial tenglamasi $(dy)^2 + x\,(dx)^2 = 0$ yoki $dy = \pm\sqrt{-x}dx$ ko‘rinishga ega bo‘lib, u ikkita $y + \frac{2}{3}(-x)^{3/2} = const$ va $y - \frac{2}{3}(-x)^{3/2} = const$ haqiqiy integrallarga ega. $\xi = y + \frac{2}{3}(-x)^{3/2}$, $\eta = y - \frac{2}{3}(-x)^{3/2}$ almashtirishlarni bajaramiz. Buning uchun u_x, u_y, u_{xx}, u_{yy} hosilalarni ξ, η lar orqali ifodalab, tenglamada u_{xx}, u_{yy} larning o‘rniga qo'yamiz. Natijada

$$u_{\xi\eta} - \frac{u_\xi - u_\eta}{8(-x)^{\frac{3}{2}}} = 0$$

ni hosil qilamiz. $(-x)^{\frac{3}{2}} = \frac{3}{4}(\xi - \eta)$ ekanligini inobatga olsak,

$$u_{\xi\eta} - \frac{u_\xi - u_\eta}{6(\xi - \eta)} = 0, \ \xi > \eta.$$

Bu qaralayotgan tenglamaning birinchi kanonik formasi. ξ, η lar o'rniga boshqa $\tilde{\xi} = y$, $\eta = \frac{2}{3}(-x)^{3/2}$ o'zgaruvchlarni kiritamiz. Yana u_x, u_y, u_{xx}, u_{yy} larni $\tilde{\xi}, \tilde{\eta}$ lar orqali ifodalab, tenglamani ushbu

$$u_{\tilde{\xi}\tilde{\xi}} - u_{\tilde{\eta}\tilde{\eta}} - \frac{u_{\tilde{\eta}}}{3\tilde{\eta}} = 0$$

ikkinchi kanonik formaga keltiramiz. Ravshanki, $\tilde{\xi} = \frac{1}{2}(\xi + \eta)$, $\tilde{\eta} = \frac{1}{2}(\xi - \eta)$ almashtirishlar yordamida bu kanonik formalarni biridan ikkinchisiga o'tkazish mumkin.

Endi $x > 0$ sohani qaraymiz. Bu sohada tenglama elliptik bo'lib, xarakteristik tenglamaning ikkita kompleks-qo'shma $y + \frac{2}{3}ix^{3/2} = const$ va $y - \frac{2}{3}ix^{3/2} = const$ yechimlarini olamiz. Yangi o'zgaruvchilarni $\tilde{\xi} = y$, $\tilde{\eta} = \frac{2}{3}x^{3/2}$ kabi tanlab, u_x, u_y, u_{xx}, u_{yy} larni yangi o'zgaruvchilar orqali ifodalaymiz. Natijada tenglamaning

$$u_{\tilde{\xi}\tilde{\xi}} + u_{\tilde{\eta}\tilde{\eta}} + \frac{u_{\tilde{\eta}}}{3\tilde{\eta}} = 0, \ \tilde{\eta} > 0$$

kanonik formasini hosil qilamiz.

$x = 0$ hol parabolik tipga to'g'ri keladi. Bunda qalayotgan tenglama xususiy hosilali differensial tenglama sifatida qiziqish tug'dirmaydi.

E s l a t m a. Osongina tekshirib ko'rish mumkinki, birinchi paragrafda keltirib chiqarilgan bir, ikki va uch o'lchovli issiqlik o'tkazuvchanlik tenglamalari parabolik tipga, Laplas va Puasson tenglamalari elliptik tipga, to'lqin tarqalish tenglamalari giperbolik tipga tegishli bo'ladi.

2.11 Koshi masalasi va uning qo'yilishida xarakteristikalarning roli

Bir jinsli bo'lmagan tor, sterjen, membrana, uch o'lchovli hajmlarning tebranishlari hamda bir jinsli bo'lmagan muhitlarda elektromagnit, elastik va akustik to'lqinlarning tarqalishi kabi ko'p tadbiqiy masalalar 4-paragrafda keltirib chiqarilgan

$$\rho\frac{\partial^2 u}{\partial t^2} = div\,(p\nabla u) - qu + F(x,t) \tag{54}$$

ko'rinishdagi tenglamalar yordamida ifodalanadi. Bundagi $u(x,t)$ noma'lum funksiya umumiy holda n ta fazoviy $x = (x_1, ..., x_n)$ koordinatalarga va t vaqt o'zgaruvchisiga bog'liqdir. $\rho,\ p,\ q$ - koeffisientlar n ta fazoviy $x = (x_1, ..., x_n)$ koordinatalarning funksiyalari bo'lib, fizik jarayon sodir bo'layotgan muhitning xossalari bilan aniqlanadi, ozod had $F(x,t)$ esa ta'sir qilayotgan tashqi kuchlarning intensivligini ifodalaydi. (54) tenglamadagi div va ∇ operatorlari qatnashgan ifoda yig'indi ko'rinishida quyidagicha yoziladi :

$$div\,(p\nabla u) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i}\right).$$

(54) tenglama uchun (giperbolik tip) klassik Koshi masalasi quyidagicha qo'yilishi mumkin:

$C^2\left(\mathbb{R}^n \times \mathbb{R}_+\right) \cap C^1\left(\mathbb{R}^n \times \bar{\mathbb{R}}_+\right)$, *bu yerda* $\mathbb{R}_+ = \left\{t \in \mathbb{R} \middle| t > 0\right\}$, $\bar{\mathbb{R}}_+ = \left\{t \in \mathbb{R} \middle| t \geq 0\right\}$ *sinfga tegishli,* $(\mathbb{R}^n \times \mathbb{R}_+)$ *yarim fazoda (54) tenglamani va* $t = +0$ *da*

$$u\big|_{t=+0} = u_0(x), \quad \frac{\partial u}{\partial t}\big|_{t=+0} = u_1(x) \tag{55}$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,t)$ *funksiya topilsin.*

$$\rho\frac{\partial u}{\partial t} = div(p\nabla u) - qu + F(x,t) \tag{56}$$

diffuziya tenglamasi uchun (parabolik tip) klassik Koshi masalasi quyidagicha qo‘yiladi:

$C^2(\mathbb{R}^n \times \mathbb{R}_+) \cap C\left(\mathbb{R}^n \times \bar{\mathbb{R}}_+\right)$ *sinfga tegishli,* $(\mathbb{R}^n \times \mathbb{R}_+)\,0$ *yarim fazoda (55) tenglamani va* $t = +0$ *da*

$$u\Big|_{t=+0} = u_0(x) \tag{57}$$

boshlang‘ich shartni qanoatlantiruvchi $u(x,t)$ *funksiya topilsin.*

Koshi masalasini umumiy hol uchun ham qo‘yish mumkin. Shu maqsadda $x = (x_1, ..., x_n)$ o‘zgaruvchili ikkinchi tartibli ushbu

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \Phi\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0,\ x \in Q. \tag{58}$$

kvazichiziqli differensial tenglamani qaraymiz. (58) da barcha A_{ij} koeffitsientlar va Φ funksiya uzluksiz hamda $\sum\limits_{i,j=1}^{n} A_{ij}^2(x) \neq 0$ deb hisoblaymiz.

Quyidagi umumiyroq bo‘lgan Koshi masalasini o‘rganamiz: *yetarlicha silliq* $S := \{x \in \mathbb{R}^n \mid \omega(x_1, \dots, x_n) = 0\}$ *sirt va bu sirtga urinma bo‘lmagan, uning har bir nuqtasida biror* l *yo‘nalish berilgan bo‘lsin.* S *sirtning biror atrofida (58) tenglama va Koshi (boshlang‘ich)*

$$u\Big|_S = u_0(x), \quad \frac{\partial u}{\partial l}\Big|_S = u_1(x) \tag{59}$$

shartlarini qanoatlantiruvchi $u(x)$ *funksiya topilsin.*

M i s o l. Sterjenning ko‘ndalang tebranish tenglamasi uchun $u(x,t)$ funksiyaga nisbatan S vazifasini $t = t_0$ vaqtda Ox o‘qining $[0, l]$ kesmasi yoki butun Ox o‘q o‘ynashi mumkin. Bunda $u\big|_{(x,t)\in S} = u_0(x)$ shart $t = t_0$ vaqtda Ox o‘q ko‘rsatilgan qismining $u(x,t_0) = u_0(x)$ siljishini bildiradi. Agar x, t o‘zgaruvchilarning $\mathbb{R}^2$ fazosida tanlangan S da Ot o‘qning musbat tomoniga qaratilgan yo‘nalishni bersak, $\frac{\partial u}{\partial l}\Big|_{(x,t)\in S} = u_1(x)$ shart $t = t_0$ vaqtda Ox o‘qning ko‘rsatilgan qismida $\frac{\partial u}{\partial t}(x,t_0) = u_0(x)$ sterjenning ko‘ndalang siljish tezligi ma’lum ekanligini bildiradi. Bu holda sterjenning $t \geq t_0$ vaqtlarda S ning bir yoqlama atrofida tebranishlari o‘rganiladi.

$\mathbb{R}^3$ da siqilmaydigan suyuqlik oqimi potensialini ifodalovchi Laplas tenglamasi uchun S sirt sifatida, masalan, to‘g‘ri burchakli dekart koordinatalari $Ox_1x_2x_3$ sistemasida Ox_1x_2 tekislikni tanlash mumkin. $u(x_1,x_2,x_3)\big|_{(x_1,x_2,x_3)\in S} = u_0(x)$ shart $x_3=0$ da potensialning $u(x_1,x_2,0)=u_0(x_1,x_2)$ qiymatlari ma’lum ekanligini bildiradi. Agar x_1,x_2,x_3 o‘zgaruvchilarning $\mathbb{R}^3$ fazosida S $(x_3=0)$ ning barcha nuqtalarida Ox_3 o‘qqa parallel yo‘nalishni bersak, $\frac{\partial u}{\partial l}\Big|_{(x_1,x_2,x_3)\in S} = u_1(x_1,x_2)$ shart Ox_1x_2 da oqim tezligi normal tashkil etuvchisi $\frac{\partial u(x_1,x_2,x_3)}{\partial x_3}\Big|_{x_3=0} = u_1(x_1,x_2)$ oqim tezligi normal tashkil etuvchisining berilayotganini bildiradi. Bu holda $u(x)$ yechimni $x_3>0$ da yoki barcha $\mathbb{R}^3$ fazoda topish masalasi o‘rganilishi mumkin.

Boshlang‘ich shartlar berilayotgan S sirt etarlicha silliq ekanligi va l yo‘nalish bu sirtga o‘tkazilgan urinmada yotmasligi sababli, (59) shartlardan foydalanib, S sirtda izlanayotgan funksiyaning barcha birinchi tartibli hosilalarini topish mumkin. Endi (58) tenglama va (59) shartlardan foydalanib, S sirtda $u(x)$ funksiyaning ((58) tenglamaning $u(x)$ yechimi mavjud deb faraz qilamiz) ikkinchi tartibli hosilalarni topish mumkinmi savolini qo‘yamiz.

T a ’ r i f. $\mathbb{R}^n$ - n o‘lchovli Evklid fazosidagi $(n-1)$ o‘lchovli tekislikka gipertekislik deyiladi; $n=3$ da gipertekislik oddiy tekislikdan, $n=2$ esa to‘g‘ri chiziqdan iboratdir.

Avvalo, boshlang‘ich shartlar $x_1=x_1^0$ gipertekislikda ($\mathbb{R}^n$ - n o‘lchovli evklid fazosidagi $(n-1)$ o‘lchovli tekislik *gipertekislik* deyiladi; $n=3$ da gipertekislik oddiy tekislikdan, $n=2$ esa to‘g‘ri chiziqdan iboratdir) berilgan holni ko‘ramiz:

$$u\Big|_{x_1=x_1^0} = \varphi_0(x_2,\dots,x_n), \quad \frac{\partial u}{\partial x_1}\Big|_{x_1=x_1^0} = \varphi_1(x_2,\dots,x_n), \tag{60}$$

bu yerda l yo‘nalish sifatida normal olinyapti, (60) shartlar asosida $x_1=x_1^0$ gipertekislikda $\frac{\partial u}{\partial x_1}$ hosiladan tashqari $u(x)$ funksiyaning birinchi va ikkinchi tartibli hosilalarini aniqlash mumkin. $\frac{\partial^2 u}{\partial x_1^2}$ ni aniqlash uchun (58) tenglamadan foydalanishimiz kerak. Bunda ikki hol bo‘lishi mumkin:

$$1)\ A_{11}(x_1^0, x_2, \dots, x_n) \neq 0, \qquad 2)\ A_{11}(x_1^0, x_2, \dots, x_n) = 0.$$

1-holda $x_1 = x_1^0$ gipertekislikda $\frac{\partial^2 u}{\partial x_1^2}$ ni yagona ravishda aniqlash mumkin; 2-holda esa aniqlab bo‘lmaydi. Endi umumiy holni, ya’ni boshlang‘ich shartlar biror S:

$$\omega(x_1, x_2, \dots, x_n) = 0$$

sirtda berilgan holni ko‘ramiz. S sirt atrofida $x_1, \dots, x_n$ o‘zgaruvchilar o‘rniga yangi $y_1, \dots, y_n$ o‘zgaruvchilarni

$$y_1 = \omega_1(x), \quad y_i = \omega_i(x), \quad i = 2, \dots, n \tag{61}$$

formulalar bilan kiritamiz.

Shu bilan birga, $\omega_i(x)$ funksiyalar yetarli silliq va (61) almashtirishning yakobiani noldan farqli qilib tanlab olinadi. Yangi o‘zgaruvchilarga nisbatan (58) tenglamaning koeffisentlarini $\overline{A}_{ke}$ orqali belgilab olsak, $\overline{A}_{ke} = \overline{A}_{ek}$ tenglikni e’tiborga olib, (58) tenglamani quyidagi ko‘rinishda yozib olish mumkin:

$$\overline{A}_{11}\frac{\partial^2 u}{\partial y_1^2} + 2\sum_{e=2}^{n}\overline{A}_{1e}\frac{\partial^2 u}{\partial y_1 \partial y_e} + \sum_{k,e=2}^{n}\overline{A}_{ke}\frac{\partial^2 u}{\partial y_k \partial y_e} + \overline{\Phi}\left(y, u, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right) = 0. \tag{62}$$

S sirt tenglamasi esa $y_1 = 0$ dan iborat bo‘ladi, ya’ni bu holda masala avvalgi xususiy holga keladi:

$$u\Big|_{y_1=0} = \overline{\varphi}_0(y_2,\ \dots\ , y_n), \quad \frac{\partial u}{\partial y_1}\Big|_{y_1=0} = \overline{\varphi}_1(y_2, \dots, y_n).$$

Agar S sirt (58) tenglamaning xarakteristik sirti bo‘lmasa, $\overline{A}_{11} \neq 0$ bo‘ladi. Bu holda (62) tenglamaga kirgan barcha hosilalarni $y_1 = 0$ da hisoblash mumkin. Agarda S xarakteristik sirt bo‘lsa, $\overline{A}_{11} = 0$ bo‘ladi. Natijada (62) tenglamada $\frac{\partial^2 u}{\partial y_1^2}$ hosila ishtirok etmaydi. (62) dan boshlang‘ich shartlarga asosan $y = 0$ bo‘lganda ushbu

$$\sum_{k,e=2}^{n} \overline{A}_{ke} \frac{\partial^2 \overline{\varphi}_0}{\partial y_k \partial y_e} + 2\sum_{e=2}^{n} \overline{A}_{1e} \frac{\partial^2 \overline{\varphi}_1}{\partial y_e} + \overline{\Phi}\left(y_2, \dots, y_n, \overline{\varphi}_0, \overline{\varphi}_1, \frac{\partial \overline{\varphi}_0}{\partial y_2}, \dots, \frac{\partial \overline{\varphi}_0}{\partial y_n}\right) = 0.$$

tenglik hosil bo‘ladi.

Bu tenglikdan agar S xarakteristik sirt bo‘lsa, boshlang‘ich shartlarda berilgan $\overline{\varphi}_0$ va $\overline{\varphi}_1$ funksiyalar o‘zaro bog‘langan bo‘lib qolishi kelib chiqadi. Demak, xarakteristik sirtda boshlang‘ich shartlarni ixtiyoriy berilish mumkin emas. Bu holda Koshi masalasi umuman yechimga ega bo‘lmasligi mumkin yoki yechimga ega bo‘lsa ham u yagona bo‘lmaydi.

Yuqoridagi fikrlarning tasqig‘i sifatida quyidagi misolni keltiramiz:

M i s o l. Ushbu

$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

tenglamaning

$$u\Big|_{y=+0} = \varphi_0(x), \quad \frac{\partial u}{\partial y}\Big|_{y=+0} = \varphi_1(x)$$

boshlang‘ich shartni qanoatlantiruvchi yechimi topilsin.

Ravshanki, $x = x_0 = const$, $y = y_0 = const$ to‘g‘ri chiziqlar oilasi, jumladan $y = 0$ ham berilgan tenglamaning xarakteristikalaridan iborat. Demak, boshlang‘ich shartlar xarakteristikada berilyapti. O‘rganilayotgan tenglamaning umumiy yechimi

$$u(x, y) = f_1(x) + f_2(y)$$

dan iborat. Umumiylikka ziyon yetkazmay $f_2(0) = 0$ deb hisoblashimiz mumkin.

Boshlang‘ich shartlarga asosan

$$u\Big|_{y=+0} = f_1(x) = \varphi_0(x), \quad \frac{\partial u}{\partial y}\Big|_{y=+0} = f_2'(y)\Big|_{y=+0} = \varphi_1(x).$$

Agar $\varphi_1(x) \neq const$ bo‘lsa, oxirgi tenglikning bajarilishi mumkin emas, bu holda Koshi masalasi yechimga ega bo‘lmaydi.

Shunday qilib, $\varphi_1(x) = const = a$ bo‘lgandagina Koshi masalasi yechimga ega bo‘lishi mumkin. Bu holda $f_2(y)$ sifatida ushbu funksiyani olishimiz mumkin:

$$f_2(y) = ay + c(y).$$

bu yerda $c(y)$ funksiya $C^2(y \geq 0)$ sinfga tegishli va $c(0) = c'(0) = 0$ shartlarni qanoatlantiruvchi ixtiyoriy funksiya.

Agar $\varphi_0(x) \in C^2$ bo‘lsa, Koshi masalasining haqiqatdan yechimi mavjud bo‘lib, u yechim

$$u(x, y) = \varphi_0(x) + ay + c(y)$$

formula bilan aniqlanadi, lekin yechim yagona emas.

Matematik fizika tenglamalari uchun qo‘yilgan masalalar aniq fizik jarayonlarning matematik modeli bo‘lganligi sababli, bu masalalar quyidagi shartlarga bo‘ysunishi lozim:
a) biror M_1 funksiyalar sinfida yechim mavjud bo‘lishi,
b) biror M_2 funksiyalar sinfida yechim yagona bo‘lishi,
c) yechim berilganlardan (boshlang‘ich va chegaraviy shartlardagi funksiyalardan, ozod haddan, tenglama koeffitsientlaridan va shu kabilardan) uzluksiz bog‘liq bo‘lishi zarur.

u yechimning berilgan $\bar{u}$ funksiyadan uzluksiz bog‘liqligi quyidagini bildiradi: faraz qilaylik berilgan $\bar{u}_k$, $k = 1, 2, \cdots$, funksiyalar ketma-ketligi $\bar{u}$ ga biror ma’noda yaqinlashsin va u_k, $k = 1, 2, \cdots$, $u-$ masalaning mos yechimlari bo‘lsin. U holda mos ravishda tanlangan yaqinlashish ma’nosida $k \to \infty$ da $u_k \to u$ bo‘lishi kerak.

Masalalarda berilgan funksiyalar, odatda, tajriba asosida aniqlanadi, shuning uchun ham ularni absolyut aniq topish mumkin emas. Demak, bu funksiyalarda biror xatolikning bo‘lishi tabiiydir. Bu xatolik o‘z navbatida yechimga ham ta’sir ko‘rsatadi. Masalalarni tekshirishda, yechimning mavjudligi va yagonaligidan tashqari berilgan funksiyalardagi xatoliklarning yechimga sezilarli ta’sir ko‘rsatmasligi, ya’ni yechimning berilganlardan uzluksiz bog‘liqligi muhim ahamiyat kasb etadi.

Yuqoridagi shartlarni qanoatlantiruvchi masala $M_1 \cap M_2$ funksiyalar sinfida *korrekt qo'yilgan masala* yoki to'g'ridan-to'g'ri *korrekt masala* deyiladi. a)-c) shartlardan kamida bittasi bajarilmasa ham masala *korrekt qo'yilmagan masala* deyiladi.

2.12 Koshi - Kovalevskaya teoremasi va Adamar misoli

Oddiy differensial tenglamalar va tenglamalar sistemasi nazariyasida Koshi masalasi yechimining lokal mavjudlik va yagonalik teoremalari (Pikar teoremalari va uning analoglari) yaxshi ma'lum. Ushbu bandda biz o'xshash teoremani xususiy hosilali tenglamalar uchun keltiramiz. O'rganiladigan tenglamalardagi noma'lum funksiyalar $n+1$ o'zgaruvchiga bog'liq bo'lib, bulardan bittasini t orqali, qolganlarini esa $x = (x_1, \ldots, x_n)$ orqali belgilab olamiz. Avvalo ikkita ta'rif kiritamiz.

T a ' r i f. N ta noma'lum $u_1, \ldots, u_N$ funksiyali ushbu

$$\frac{\partial^{k_i} u_i}{\partial t^{k_i}} = \Phi_i \left(x, t, u_1, \ldots, u_N, \ldots, D_t^{\alpha_0}, D_x^{\alpha} u_i, \ldots\right) \qquad (63)$$
$$i = 1, 2, \ldots, N$$

differensial tenglamalar sistemasining o'ng tomonidagi φ_i funksiyalarda t o'zgaruvchi bo'yicha $k_i - 1$ dan yuqori tartibli boshqa o'zgaruvchilar bo'yicha k_i dan yuqori tartibli hosilalar ishtirok etmasa, ya'ni $\alpha_0 + \alpha_1 + \ldots + \alpha_n \leq k_i$, $\alpha_0 \leq k_i - 1$ bo'lsa, (63) sistema t o'zgaruvchiga nisbatan normal sistema deyiladi.

Masalan, to'lqin tenglamasi, Laplas tenglamasi, issiqlik tarqalish tenglamasi har bir x o'zgaruvchiga nisbatan normal tenglamadir, bundan tashqari to'lqin tenglamasi t ga nisbatan ham normal tenglamadir.

Ixtiyoriy xususiy hosilali differensial tenglamalar sistemasini umuman (63) ko'rinishga keltirish mumkin emasligini eslatib o'tamiz.

T a ' r i f. Agar $f(x)$, $x = (x_1, \ldots, x_n)$ funksiya, x_0 nuqtaning biror atrofida tekis yaqinlashuvchi

$$f(x) = \sum_{\alpha \geq 0} C_\alpha (x - x_0)^\alpha = \sum_{\alpha \geq 0} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha,$$

$$C_\alpha = C_{\alpha_1, \dots, \alpha_n}, \quad (x - x_0)^\alpha = (x_1 - x_{01})^{\alpha_1} \ \dots \ (x_1 - x_{0n})^{\alpha_n}$$

darajali qator bilan ifodalansa, u x_0 nuqtada analitik funksiya deyiladi. x_0 nuqta kompleks bo'lishi ham mumkin.

Agar $f(x)$ funksiya G sohaning har bir nuqtasida analitik bo'lsa, u G *sohada analitik* deyiladi.

t ga nisbatan normal sistema uchun Koshi masalasi quyidagicha qo'yiladi: *(63) sistemaning* $t = t_0$ *da ushbu*

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=t_0} = \varphi_{ik}(x), \quad k = 0, 1, \dots, k_i - 1; \quad i = 1, 2, \dots, N \tag{64}$$

boshlang'ich shartlarni qanoatlantiruvchi $u_1, \dots, u_N$ *yechimi topilsin.* Bu yerda $\varphi_{ik}(x)$ - biror $G \subset \mathbb{R}^n$ sohada berilgan funksiyadir.

Berilgan (64) boshlang'ich shartlarga asosan Φ_i funksiyalar ishtirok etayotgan barcha hosilalar $t = t_0$ va $x = x_0$ da ma'lum bo'ladi, masalan

$$D_x^\alpha u_j(x,t)\big|_{t=t_0, x=x_0} = D^\alpha \varphi_{j0}(x_0), \quad D_t^{\alpha_0} D_x^\alpha u_j\big|_{t=t_0, x=x_0} = D^\alpha \varphi_{j\alpha_0}(x_0).$$

Xususan, birinchi tartibli xususiy hosilali differensial tenglamalarning normal sistemasi

$$\frac{\partial u_i}{\partial t} = \Phi_i\left(x, t, u_1, \dots, u_N, \frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_N}{\partial x_n}\right), \quad i = 1, 2, \dots, N \tag{65}$$

ko'rinishda bo'ladi. Bunda tenglamalarning o'ng tomoni noma'lum funksiyalarning t bo'yicha hosilasiga, boshqa o'zgaruvchilar bo'yicha birinchi tartibdan yuqori bo'lgan hosilalarga bog'liq emas. Birinchi tartibli normal sistema uchun boshlang'ich shartlar

$$u_i\big|_{t=t_0} = \varphi_{i0}(x), \quad i = 1, 2, \dots, N$$

ko‘rinishga ega bo‘ladi. Bu shartlarga ko‘ra, (65) sistema o‘ng tomonidan Φ_i funksiyalarning argumentlari (x_0, t_0) nuqtada darhol aniqlanadi.

Koshi-Kovalevskaya teoremasi. Agar barcha $\varphi_{ik}(x)$ funksiyalar x_0 nuqtaning biror atrofida analitik, $\Phi_i(x, t, \ldots, u_{j\alpha_0\alpha_1\ldots\alpha_n}, \ldots)$ funksiya esa $(x_0, t_0, \ldots, D^\alpha\varphi_{i\alpha_0}(x_0), \ldots)$ nuqtaning biror atrofida analitik bo‘lsa, u holda (63), (64) Koshi masalasi (x_0, t_0) nuqtaning biror atrofida analitik yechimga ega bo‘ladi, shu bilan birga bu yechim analitik funksiyalar sinfida yagona bo‘ladi.

Bu teorema analitik funksiyalar sinfida Koshi masalasining yechimi yetarli kichik sohada mavjud va yagona ekanligini tasdiqlaydi.

Analitik bo‘lmagan, lekin yetarli silliq funksiyalar sinfida (63), (64) masala yechimining yagonaligi Holmgren tomonidan isbotlangan. Koshi - Kovalevskaya teoremasining to‘la isbotini R. Kurantning kitobidan o‘qib olish mumkin.

Shuni aytish lozimki, Koshi - Kovalevskaya teoremasi (63), (64) masalaning korrekt qo‘yilganmi yoki yo‘qligi haqidagi savolga to‘liq javob bera olmaydi: birinchidan, masalaning bir qiymatli yechilishini (x_0, t_0) nuqtaning biror atrofida (lokal ma’noda) ifodalaydi; ikkinchidan, teoremada yechimning berilgan funksiyalarga uzluksiz bog‘liqligi haqida hech narsa deyilmayapdi. Quyidagi misol, agar (63) tenglama o‘rniga elliptik tenglama qaralganda, yechimning boshlang‘ich shartlarga uzluksiz bog‘liq emasligini ko‘rsatadi:

Adamar misoli.

Ikki o‘lchovli Laplas tenglamasini

$$\frac{\partial^2 u}{\partial x_1^2} = -\frac{\partial^2 u}{\partial x_2^2}$$

ko‘rinishda yozib olamiz. Ravshanki, bu normal tenglama va u uchun ikkita Koshi masalasini qo‘yamiz:

1) $u\big|_{x_1=0} = \frac{\partial u}{\partial x_1}\big|_{x_1=0} = 0$

2) $u\big|_{x_1=0} = 0,\ \frac{\partial u}{\partial x_1}\big|_{x_1=0} = \frac{1}{k}\sin(kx_2).$

Yetarlicha katta k lar uchun bu boshlang‘ich shartlar, mos ravishda, bir - biriga istalgancha yaqin ekanligini payqash qiyin emas. Shu bilan birga, bu masalalar yechimlari bo'lgan

$$u^1(x_1, x_2) = 0$$

va

$$u^2(x_1, x_2) = \frac{1}{k^2}\sin(kx_2)sh(kx_1),$$

bu yerda $shx_1 = \frac{(e^{kx_1} - e^{-kx_1})}{2}$ - giperbolik sinus, (yechim ekanligi to'g'ridan - to'g'ri tekshiriladi) funksiyalar yetarlicha katta k larda $x_1 > 0,\ x_2 \in \mathbb{R}$ lar uchun bir - birigan istalgancha katta miqdorga farq qiladi.

Ushbu kitobda korrekt qo'yilgan masalalar o'rganiladi.

3-Bob. Fundamental yechim. Koshi masalasi

Oldingi bobdada ko‘rilgan xususiy hosilali differensial tenglamalar uchun boshlang‘ich va chegaraviy shartlarni ifodalovchi funksiyalar etarlicha silliq bo‘lganda mos masalalarning klassik yechimi tushinchasi kiritildi. Ammo fizik masalalarda hamma vaqt ham bu funksiyalar etarlicha silliq bo‘lavermaydi. Agar boshlang‘ich va chegaraviy shartlardagi funksiyalar uzluksiz va keraklicha diffrensiallanuvchi bo‘lmasa, masalaning differensiallanuvchi yechimi mavjud bo‘lmasligi ham mumkin. Bunday hollarda differensial tenglamalarning *umumlashgan yechimi* tushunchasini kiritish muhim ahamiyatga egadir. Bu tushunchani oddiy differensial tenglamalar uchun aniqlashdan boshlaymiz.

3.1 Oddiy differensial tenglamalarning umumlashgan yechimlari va fundamental yechim tushunchasi

Ushbu

$$Ly(x) = \sum_{k=0}^{n} p_k(x) \frac{d^k}{dx^k} y(x) = f(x) \tag{1}$$

ko‘rinishdagi silliq koeffitsientli ($p_n(x) \neq 0$) $n-$tartibli chiziqli oddiy differensial tenglamani qaraymiz.

T a ’ r i f. n marta uzluksiz differensiallanuvchш va (1) tenglamani qanoatlantiruvchi $y(x)$ funksiyaga bu tenlamaning klassik echimi deyiladi.

T a ' r i f. L operatoriga qo'shma operator deb quyidagi tenglik bilan aniqlanuvchi L^* operatoriga aytiladi:

$$L^*(x) = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} \left(p_k(x) y(x) \right).$$

T a ' r i f. (1) differensial tenglamaning umumlashgan yechimi deb, ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ uchun

$$(y(x), L^* \varphi(x)) = (f(x), \varphi(x))$$

tenglik (integral ayniyat) ni qanoatlantiruvchi $y(x)$ funksionalga aytiladi.

L e m m a. a) klassik yechim umumlashgan yechim hamdir; b) n marta uzluksiz differensialga ega bo'lgan umumlashgan yechim klassik yechim hamdir.

Isbot. Lemmaning isboti silliq yechimlar uchun bo'laklab integrallash yordamida olinadigan quyidagi ayniyatlardan kelib chiqadi:

$$(y(x), L^* \varphi(x)) = \int\limits_{\mathbb{R}} y(x) \left(L^* \varphi(x) \right) dx = \int\limits_{\mathbb{R}} \left(L y(x) \right) \varphi(x) dx =$$

$$= \int\limits_{\mathbb{R}} y(x) f(x) dx = (y(x), f(x)).$$

T a s d i q. $y'(x) = 0$ tenglama faqat klassik yechimga ega.

Isbot. $\int\limits_{\mathbb{R}} \varphi_0(x) dx = 1$ shartni qanoatlantiruvchi ixtiyoriy $\varphi_0(x) \in D(\mathbb{R})$ funksiyani tayin qilamiz. Ma'lumki, ixtiyoriy $\varphi(x) \in D(\mathbb{R})$ funksiyani $\varphi(x) = \psi'(x) + c\varphi_0(x)$, bu yerda $\psi(x) \in D(\mathbb{R})$ va $c = \int\limits_{\mathbb{R}} \varphi(x) dx$, ko'rinishda yozish mumkin. Haqiqatan ham, bu tenglikni $\psi'(x)$ ga nisbatan yechib va so'ng integrallab,

$$\psi(x) = \psi(-\infty) + \int\limits_{-\infty}^{x} \varphi(t) dt - c \int\limits_{-\infty}^{x} \varphi_0(t) dt$$

ga ega bo'lamiz. Bundan $\psi(x)$ funksiyaning cheksiz differensiallanuvchanligi kelib chiqadi. $\psi(-\infty) = 0$ deb olamiz. $\varphi(x)$ va $\varphi_0(x)$ larning finitligidan, yetarlicha katta M uchun $x < -M$ larda $\psi(x) = 0$ kelib chiqadi. Shuningdek, $x > M$ lar uchun

$$\psi(M) = \int\limits_{-\infty}^{M} \varphi(t)dt - c\int_{-\infty}^{M} \varphi_0(t)dt = \int\limits_{\mathbb{R}} \varphi(t)dt - c = 0$$

tenglik o‘rinli, ya’ni $\psi(x)-$ finit funksiya. Shunday qilib, $\psi(x) \in D(\mathbb{R})$.

$y'(x) = 0$ tenglamaning ixtiyoriy yechimi uchun

$$(y(x), \varphi(x)) = (y(x), \psi'(x) + c\varphi_0(x)) = (y'(x), -\psi(x)) + c\,(y(x), \varphi_0(x))$$

tengliklar bajariladi. Bu yerda $y'(x) = 0$ tenglamaga asosan, birinchi qo‘shiluvchi nolga, ikkinchi qo‘shiluvchi $(y(x), \varphi_0(x))$ esa $\varphi_0(x)$ ga bo‘g‘liq bo‘lgan o‘zgarmasga teng. Shunday qilib, bu o‘zgarmasni c_0 orqali belgilab, quyidagiga ega bo‘lamiz:

$$(y(x), \varphi(x)) = cc_0 = c_0 \int\limits_{\mathbb{R}} \varphi(x)dx = (c_0, \varphi(x))\,,$$

ya’ni $y(x) = c_0 = const$. Tasdiq isbot bo‘ldi.

M i s o l. $xy'(x) = 0$ tenglamani qaraymiz. Ma’lumki, bu yerdan $y'(x) = c_1\delta(x)$, c_1- ixtiyoiy o‘zgarmas, chunki 1-bobdagi (15) formulaga asosan umumlashgan funksiyalar ma’nosida $c_1x\delta(x) = 0$. $y'(x) = c_1\delta(x)$ tenglamaning xususiy yechimi $y(x) = c_1\theta(x)$ dan iborat. Unga mos keluvchi bir jinsli tenglamaning umumiy yechimi $c_2 = const$ ga teng. Shuning uchun $y(x) = c_1\theta(x) + c_2$.

M i s o l. $xy'(x) + (1-\lambda)y(x) = 0$ tenglamani qaraymiz. Bu Eyler tenglamasidir. Ma’lumki, uning klassik yechimi $y_{cl}(x) = x^{1-\lambda}$ dan iborat. Bu tenglama qanday umumlashgan yechimlarga ega degan savol tug‘iladi. $y(x)$ funksiyaning yuqori tartibli hosilasi oldidagi koeffitsient nolga faqat $x = 0$ da erishgani uchun, umumlashgan yechimni $\delta-$ funksiya va uning hosilalarining chiziqli kombinatsiyasi ko‘rinishida qidirish maqsadga muvofiqdir. $x\delta^{(p)} = -p\delta^{(p-1)}$ ekanligidan, $-p - \lambda = 0$ tenglikni qanoatlantiruvchi, biror manfiy bo‘lmagan butun p va musbat bo‘lmagan butun λ sonlari uchun bunday yechim mavjud. Bu yechim $y_{gen}(x) = \delta^{(-\lambda)}(x)$, $\lambda \leq 0$ ga teng.

Bu funksiyaning umumlashgan funksiyalar ma'nosida tenglamani qanoatlantirishi to'g'ridan-to'g'ri ko'rsatiladi.

T a ' r i f. *L operatorning fundamental (elementar) yechimi deb*

$$L\varepsilon(x) = \delta(x)$$

tenglamani qanoatlantiruvchi $\varepsilon(x)$ umumlashgan funksiyaga aytiladi.

Ravshanki, fundamental yechim unga bir jinsli $Ly(x) = 0$ tenglamaning ixtiyoriy yechimini qo'shgan bilan o'zgarmaydi.

M i s o l. $L = \frac{d^2}{dx^2} - 1$ operatorning fundamental yechimi $\varepsilon(x) = \theta(x)\sinh(x)$ bo'lishini ta'rifdan foydalanib, osongina ko'rsatish mumkin. Biroq bu yechimdan bir jinsli $Ly(x) = 0$ tenglamaning yechimi $y(x) = (1/2)\exp(x)$ ni ayirib, uni simmetrik

$$\varepsilon(x) = -\frac{1}{2}e^{-|x|}$$

ko'rinishga keltirish ham mumkin.

Keyinchalik biz $\varepsilon|_{x<0} = 0$ shartni qanoatlantiruvchi $\varepsilon_+(x)$ fundamental yechimlarni qaraymiz.

T a s d i q. $y(x) = (\varepsilon * f)(x)$ yig'ma (agar u mavjud bo'lsa) $Ly(x) = f(x)$ tenglamani qanoatlantiradi.

Haqiqatan ham,

$$L(\varepsilon * f)(x) = (L\varepsilon)(x) * f(x) = (\delta * f)(x) = f(x).$$

Masalan, $y''(x) - y(x) = f(x)$ tenglamaning xususiy yechimi bo'lib

$$y_{par}(x) = \int\limits_{\mathbb{R}} \varepsilon(t)f(x-t)dt = \int\limits_0^{\infty} \sinh(t)f(x-t)dt$$

yoki

$$y_{par}(x) = -\frac{1}{2}\int\limits_{\mathbb{R}} e^{-|t|}f(x-t)dt$$

funksiya xususiy yechimidir.

$p_n(0) \neq 0$ shart bajarilganda, (1) tenglamadagi L operatorning fundamental yechimi haqida quyidagi tasdiq o‘rinli:

T a s d i q. (1) L operatorning $n \geq 2$ uchun fundamental yechimi quyidagidan iborat: $\varepsilon_+(x) = u(x)$, agar $x > 0$ va $\varepsilon_+(x) = 0$, agar $x < 0$ bo‘lsa; bu yerda $u(x)$ funksiya

$$u(0) = 0,\ u'(0) = 0,\ ...,\ u^{(n-2)}(0) = 0,\ u^{(n-1)}(0) = 1/p_n(0)$$

boshlang‘ish shartlarni qanoatlantiruvchi bir jinsli $Lu(x) = 0$ tenglamaning xususiy yechimi.

Isbot. Ko‘rinib turibdiki, $\varepsilon_+(x)-$ bo‘laklari silliq regulyar umumlashgan funksiya va u uchun $\varepsilon_+(x)|_{x=0} = \varepsilon'_+(x)|_{x=0} = ... = \varepsilon_+^{n-2}(x)|_{x=0} = 0$, $\varepsilon_+^{n-1}(x)|_{x=0} = 1/p_n(0)$ shartlar bajariladi. Binobarin, $\varepsilon_+(x)$ ning klassik hosilalarini figurali qavslar bilan belgilab,

$$\frac{d}{dx}\varepsilon_+(x) = \{\varepsilon'_+(x)\} + \varepsilon_+(x)|_{x=0}\delta(x) = \{\varepsilon'_+(x)\},$$

$$\frac{d^2}{dx^2}\varepsilon_+(x) = \{\varepsilon''_+(x)\} + \varepsilon'_+(x)|_{x=0}\delta(x) = \{\varepsilon''_+(x)\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$\frac{d^{n-1}}{dx^{n-1}}\varepsilon_+(x) = \left\{\varepsilon_+^{(n-1)}(x)\right\} + \varepsilon_+^{(n-2)}(x)|_{x=0}\delta(x) = \left\{\varepsilon_+^{(n-1)}(x)\right\},$$

$$\frac{d^n}{dx^n}\varepsilon_+(x) = \left\{\varepsilon_+^{(n)}(x)\right\} + \varepsilon_+^{(n-1)}(x)|_{x=0}\delta(x) = \left\{\varepsilon_+^{(n)}(x)\right\} + \frac{1}{p_n(0)}\delta(x)$$

tengliklarga ega bo‘lamiz. Natijada,

$$L\varepsilon_+ = Lu(x) + \frac{p_n(x)}{p_n(0)}\delta(x) = \delta(x)$$

ifodalar tasdiqni isbotlaydi.

Ushbu paragrafning so‘nggi misolidagi aynan $\sinh(x)$ funksiya, yuqoridagi tasdiqning boshlang‘ich shartlarini qanoatlantiruvchi $u(x)$ funksiya bo‘lib xizmat qiladi.

Endi (1) tenglama uchun $p_i(x) \in C^\infty\{\mathbb{R}_+\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$, $i = 1, 2, ..., n$ va $p_n(x) \neq 0$ shartlar bajarilganda ushbu

$$Ly(x) = f(x), \ y^{(i)}|_{x=0} = y_i, \ i = 1, 2, ..., n-1 \tag{2}$$

Koshining klassik masalasini qaraymiz. Masalaning yechimini $x \in \mathbb{R}_+$ lar uchun quramiz.

Quyidagi masalani qo‘yamiz: $x > 0$ *larda (2) Koshi masalasining yechimi bo‘lgan va* $x < 0$ *sohaga nolga teng qilib (umuman olganda silliq emas) davom ettirilgan* $y_+(x)$ *funksiya topilsin.* Xuddi shu tarzda $f_+(x)$ funksiyani kiritamiz, biroq davom ettirilgan funksiyalar uchun oldingi belgilashlarni qoldiramiz: $y_+ \rightsquigarrow y$, $f_+ \rightsquigarrow f$. Bu umumlashgan funksiyalar qanday tenglamani qanoatlantirishini ko‘ramiz. Yuqoridagi kabi umumlashgan hosilarni hisoblaymiz:

$$y'(x) = \{y'(x)\} + y_0\delta(x), \ y''(x) = \{y''(x)\} + y_0\delta'(x) + y_1\delta(x),$$

$$..., \ y^{(i)}(x) = \left\{y^{(i)}(x)\right\} + \sum_{j=0}^{i-1} y_j\delta^{(i-j-1)}(x),$$

bu yerda figurali qavslar bilan klassik hosilalar belgilangan. Bu formulalarni (2) tenglamaga qo‘yib,

$$Ly(x) = \{Ly(x)\} + \sum_{i=0}^{n-1} c_i\delta^{(i)}(x) =$$

$$= f(x) + \sum_{i=0}^{n-1} c_i\delta^{(i)}(x), \ c_i = \sum_{j=0}^{n-i-1} p_{i+j+1}(x)y_j \tag{3}$$

tengliklarni hosil qilamiz.

Shunday qilib, quyidagi ta’rifga kelamiz:

T a ’ r i f. (3) tenglamadan $y(x)$ funksiyani topish masalasiga L operatori uchun *umumlashgan Koshi masalasi* deyiladi.

Umumlashgan Koshi masalasi fundamental yechim yordamida hal qilinishi mumkin:

$$y(x) = \varepsilon_+ * \left(f(x) + \sum_{i=0}^{n-1} c_i\delta^{(i)}(x)\right) = \varepsilon_+ * f(x) + \varepsilon_+ * \sum_{i=0}^{n-1} c_i\delta^{(i)}(x).$$

Yig‘maning

$$\varepsilon_+ * \sum_{i=0}^{n-1} c_i \delta^{(i)}(x) = \sum_{i=0}^{n-1} c_i \varepsilon_+ * \delta^{(i)}(x)$$

xossasiga ko‘ra

$$y(x) = \varepsilon_+ * \left(f(x) + \sum_{i=0}^{n-1} c_i u^{(i)}(x) \right),$$

bunda $u(x)-$ so‘nggi tasdiqda keltirilgan $Lu(x) = 0$ tenglamaning xususiy yechimi.

(2) masalaning xususiy, ya’ni $f(x) = 0$, $y_0 = y_1 = ... = y_{n-2} = 0$, $y_{n-1} = 1/p_n(0)$ tengliklar bajarilgan holini qaraymiz. Ma’lumki, umumlashgan yechim

$$Ly(x) = \delta(x)$$

tenglamani umumlashgan funksiyalar ma’nosida qanoatlantiradi. $y|_{x<0} = 0$ shartni hisobga olsak, bu yechim fundamental yechim ε_+ ekanligini ko‘rish qiyin emas. Bu ε_+ funksiyani $Ly(x) = 0$ tenglamaning $\varepsilon_+|_{x=0} = 0$, $\varepsilon_+'|_{x=0} = 0$, ..., $\varepsilon_+^{(n-2)}|_{x=0} = 0$, $\varepsilon_+^{(n-1)}|_{x=0} = 1/p_n(0)$ shartlarni qanoatlantiruvchi Koshi masalasining yechimi sifatida qarashimiz mumkin.

3.2 Xususiy hosilali differensial tenglamaning umumlashgan yechimi tushunchasi. Fundamental yechimlar

$\Omega \in \mathbb{R}^n$ sohada $m-$tartibli chiziqli

$$Lu \equiv \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f(x) \tag{4}$$

xususiy hosilali differensial tenglamani qaraymiz. $a_\alpha(x)$ koeffitsientlar yetarlicha silliq funksiyalar bo‘lsin.

Ma‘lumki, bu tenglamaning *klassik yechimi* deb Ω sohada (4) tenglikni qanoatlantiruvchi barcha $D^\alpha u$, $|\alpha \leq m|$ hosilalari bilan uzluksiz bo‘lgan $u(x)$ funksiyaga aytiladi.

Ta'rifdan ko'rinib turibdiki, (4) tenglamaning har qanday o'ng qismi uchun ham yechim mavjud bo'lavermaydi. Yechim mavjud bo'lishi uchun $f(x)$ funksiya hech bo'lmaganda uzluksiz bo'lishi zarur. Differensial tenglamalarning tadbiqi nuqtai nazaridan bu talab juda ko'p hollarda bajarilmaydi. $f(x)$ funksiya fizik jarayonni qo'zg'atuvchi tashqi manbalarni ifodalaydi. Shu sababli u, masalan, uzilishga ega bo'lishi mumkin. Ko'p hollarda tashqi manba $\mathbb{R}^n$ da x^0 nuqtaning biror atrofida jamlangan bo'ladi. Bunday manbalarni $f(x)$ funksiya sifatida $\delta(x-x^0)$ delta-funksiyani olib modellashtirish qulaydir. Shuning uchun (4) tenglamaning, umuman aytganda, umumlashgan $f(x)$ funksiyaga mos keluvchi yechimini qarash maqsadga muvofiqdir. Tabiiyki, bunda yechim ham umumlashgan funksiya bo'ladi.

Ushbu $\varphi(x) \in D(\mathbb{R}^n)$, $supp\varphi(x) \subset \Omega$ funksiyalarni kiritamiz. Bunday funksiyalar sinfini $D(\Omega)$ bilan belgilaymiz. (4) tenglamani $\varphi(x)$ funksiyalarda qarab, quyidagilarni hosil qilamiz:

$$(f,\varphi) = \Big(\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u, \varphi\Big) = \sum_{|\alpha|\le m} (a_\alpha D^\alpha u, \varphi) = \sum_{|\alpha|\le m} (D^\alpha u, a_\alpha \varphi) =$$

$$= \sum_{|\alpha|\le m} (-1)^{|\alpha|}(u, D^\alpha(a_\alpha\varphi)) = \Big(u, (-1)^{|\alpha|} D^\alpha(a_\alpha\varphi)\Big).$$

T a ' r i f. $u(x)$ umumlashgan funksiya (4) tenglamaning Ω sohada umumlashgan yechimi deyiladi, agarda har qanday $\varphi(x) \in D(\Omega)$ uchun

$$(f,\varphi) = \Big(u, \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha\varphi)\Big) \tag{5}$$

tenglik o'rinli bo'lsa. L orqali (4) tenglikning chap tomonidagi differensial operatorni belgilaymiz:

$$L = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha. \tag{6}$$

$$L^*\varphi = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha\varphi)$$

tenglik bilan aniqlangan L^* operatorni Lagranj ma'nosida L operatorga qo'shma operator deyiladi. Kiritilgan belgilashlar yordamida (5) tenglikni quyidagicha yozish mumkin:

$$(f,\varphi) = (u, L^*\varphi).$$

T a ’ r i f. Agar ixtiyoriy $\varphi \in D(\Omega)$ va $x^0 \in \Omega$ tayin nuqta uchun

$$\varphi(x^0) = (G, L^*\varphi)$$

tenglik bajarilsa, yoki $G(x, x^0)$ funksiya (4) tenglamaning

$$f(x) = \delta(x - x^0),\ x^0 \in \Omega$$

bo‘lgandagi yechimi bo‘lsa, $G(x, x^0)$, $x \in \Omega$, $x^0 \in \Omega$ umumlashgan funksiya L operatorining Ω sohadagi fundamental yechimi deyiladi.

M i s o l. $n = 1, \Omega = \mathbb{R}$ bo‘lsin. Bevosita umumlashgan hosilalarni hisoblab, ishonch hosil qilish mumkinki,

$$G(x, x^0) = (x - x^0)\theta(x - x^0)$$

funksiya

$$U''(x) = \delta(x - x^0),\ x \in \mathbb{R}$$

differensial tenglamaning, binobarin $L = \frac{d^2}{dx^2}$ differensial operatorning fundamental yechimidir.

(6) differensial operator fundamental yechimining mavjudligi o‘zgarmas a_α koeffitsientlar holida va ikkinchi tartibli giperbolik, elliptik, parabolik turdagi operatorlar uchun esa $a_\alpha(x)$ funksiyalarga qo‘yilgan ancha umumiy farazlar bilan isbot qilingan.

Qayd qilmoq lozimki, L differensial operatorning fundamental yechimi bir qiymatli aniqlanmaydi. Haqiqatan ham, agar $G(x, x^0)$ fundamental yechim bo‘lsa, u holda

$$\bar{G}(x, x^0) = G(x, x^0) + g(x)$$

funksiya ham L operatorining fundamental yechimi bo‘ladi, bu yerda $Lg = 0$. Xususan, yuqorida keltirilgan misolda

$$(x - x^0)\theta(x - x^0) + c_1 x + c_2,$$

bu yerda c_1, c_2- ixtiyoriy o‘zgarmaslar funksiya, ham fundamental yechim bo‘ladi.

L operator o‘zgarmas koeffitsientli bo‘lgan holda fundamental yechimni x, x^0 nuqtalarning $x - x^0$ kombinatsiyasiga bog‘liq holda tanlash mumkin, ya'ni $G(x, x^0) = G_1(x - x^0)$. Bu faqat L differensial operatorning $x - x^0 = y$ o'zgaruvchilarni almashtirishga nisbatan invariantligi natijasida namoyon bo'ladi. Bu holat L operatorning koeffitsientlari x ning biror koordinatasiga bog'liq bo'lmaganda ham o'rinli. Masalan, L operatorning koeffitsientlari x_1 ga bog'liq bo'lmasa, fundamental yechimni

$$G(x, x^0) = G_1(x_1 - x_1^0, \overline{x}, \overline{x}_0),\ \overline{x} = (x_2, \ldots, x_n),\ \overline{x}^0 = (x_2^0, \ldots, x_n^0)$$

ko'rinishda tanlash mumkin.

Ω soha $\mathbb{R}^n$ bilan ustma-ust tushganda (6) tenglik bilan aniqlangan L operatorning G fundamental yechimi yordamida (4) tenglamaning yechimini hosil qilish mumkin.

T e o r e m a. $f(x) \in D'(\mathbb{R}^n)$ funksiya uchun $G * f$ yig'ma mavjud va $D'(\mathbb{R}^n)$ sinfga tegishli bo'lsin. U holda (4) tenglamaning yechimi $D'(\mathbb{R}^n)$ sinfda mavjud va

$$u = G * f \tag{7}$$

formula bilan aniqlanadi.

Bu yechim G funksiya bilan yig'masi mavjud bo'luvchi $D'(\mathbb{R}^n)$ dan olingan funksiyalar sinfida yagonadir.

Isbot. 1-bobning 8-paragrafidagi yig'mani differensiallash formulasiga ko'ra va G fundamental yechimning umumlashgan funksiyalar ma'nosida $LG = \delta(x)$ tenglamani qanoatlantirishini inobatga olib

$$L(G * f) = \sum_{|\alpha| \le m} a_\alpha D^\alpha (G * f) =$$

$$= \left(\sum_{|\alpha| \le m} a_\alpha D^\alpha G \right) * f = LG * f = \delta * f = f$$

tengliklarni hosil qilamiz. Bu esa, haqiqatan ham, (4) tenglamaning yechimi $u = G * f$ bilan berilishini bildiradi.

Bu yechimning G funksiya bilan yig'masi mavjud bo'luvchi $D'(\mathbb{R}^n)$ ga tegishli funksiyalar sinfida yagona bo'lishini ko'rsatish uchun (4) tenglamaga mos keluvchi bir jinsli $Lu = 0$ tenglamaning faqat nol yechimga ega bo'lishini ko‘rsatish yetarli. Haqiqatan ham,

$$u = u * \delta = u * LG = Lu * G = 0.$$

Teorema isbotlandi.

Agar (4) tenglama $\Omega \in \mathbb{R}^n$ sohada qaralib, $G(x, x^0)$ funksiya L operatorning fundamental yechimi bo‘lsa, u holda tenglamaning yechimi

$$u(x) = \int_{\Omega} G(x, x^0) f(x^0) dx^0$$

formula bilan aniqlanadi.

3.3 Differensial operatorlarning fundamental yechimlari

Mazkur paragrafda umumlashgan funksiyalar nazariyasi oldingi bobda keltirib chiqarilgan tadbiqiy masalalarda ko‘p uchraydigan xususiy hosilali differensial tenglamalarning fundamental (elementar) yechimlarini topishda hamda to‘lqin va issiqlik o‘tkazuvchanlik tenglamalari uchun Koshi masalasining yechimini qurishda qo‘llaniladi.

3.3.1 Bir o‘zgaruvchili chiziqli differensial operatorning fundamental yechimi

Bir o‘zgaruvchili $n-$tartibli o‘zgarmas koeffitsientli

$$L = \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_n$$

operatorning fundamental yechimi

$$H(t) = \theta(t) Z(t)$$

formula bilan ifodalanishini ko‘rsatamiz. Bu yerda, $Z(t)$ bir jinsli $LZ = 0$ tenglamani va

$$Z(0) = Z'(0) = \cdots = Z^{(n-2)}(0), \; Z^{(n-1)}(0) = 1$$

boshlang‘ich shartlarni qanoatlantiruvchi funksiya. $H(t)$ funksiya fundamental yechim bo‘lishi uchun uning

$$LH = \delta(t)$$

tenglamani qanoatlantirishini ko‘rsatish zarur. Haqiqatan ham,

$$\frac{dH(t)}{dt} = \theta'(t)Z(t) + \theta Z'(t) = \theta(t)Z'(t), \ (\theta'(t) = \delta(t)),$$

$$\frac{dH(t)}{dt} = \theta(t)Z^{(n-1)}(t), \ \frac{d^nH(t)}{dt^n} = \delta(t) + \theta(t)Z^{(n)}(t)$$

tengliklardan

$$LH = \theta(t)LZ + \delta(t) = \delta(t)$$

ekanligi kelib chiqadi.

Xususan,

$$H(t) = \theta(t)e^{-at} \tag{8}$$

va

$$H(t) = \theta(t)\frac{\sin at}{a} \tag{9}$$

funksiyalar mos ravishda

$$\frac{d}{dt} + a, \ \frac{d^2}{dt^2} + a^2$$

differensial operatorlarning fundamental yechimi ekanligini ko‘rish qiyin emas. Masalan, agarda $a = 0$ bo‘lsa, u holda bu yechimlar mos ravishda $\theta(t)$ va $t\theta(t)$ lardan iborat bo‘ladi.

3.3.2 Issiqlik o‘tkazuvchanlik operatorining fundamental yechimi

$$\frac{\partial H}{\partial t} - a^2\Delta H = \delta(x,t), \ x \in \mathbb{R}^n, \ \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{10}$$

tenglamaning yechimini topish uchun Furye almashtirishidan foydalanamiz.

(10) tenglikka x bo‘yicha F_x Furye almashtirishini qo‘llab,

$$F_x\Big[\frac{\partial H}{\partial t}\Big] - a^2F_x[\Delta H] = F_x[\delta(x,t)]$$

quyidagi formulalardan foydalanamiz:

$$F_x[\delta(x,t)] = F_x[\delta(x)\delta(t)] = F[\delta](\xi)\cdot\delta(t) = \delta(t),$$

$$F_x\Big[\frac{\partial H}{\partial t}\Big] = \frac{\partial}{\partial t}F_x[H],\ \ F_x[\Delta H] = -|\xi|^2 F_x[H].$$

Natijada, $\overline{H}(\xi,t) = F_x[H](\xi,t)$ umumlashgan funksiya uchun

$$\frac{\partial\overline{H}(\xi,t)}{\partial t} + a^2|\xi|^2\overline{H}(\xi,t) = \delta(t)$$

tenglamaga ega bo'lamiz. (8) formulaga asosan bu tenglamaning yechimi

$$\overline{H}(\xi,t) = \theta(t)e^{-a^2|\xi|^2 t} \tag{11}$$

funksiyadan iborat bo'ladi. $H(x,t)$ funksiyani aniqlashda 2-bobning (7) formulasidan kelib chiquvchi

$$F^{-1}\left[e^{-a^2|\xi|^2 t}\right] = \frac{1}{(2\pi)^n}F\left[e^{-a^2|\xi|^2 t}\right] =$$

$$= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{-a^2|\xi|^2 t + i(\xi,t)}d\xi = \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\int\limits_{\mathbb{R}} e^{-(a\xi_j\sqrt{t})^2 + i\xi_j x_j}d\xi_j =$$

$$= \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\frac{\sqrt{\pi}}{a\sqrt{t}}e^{-\frac{x_j^2}{4a^2t}} = \frac{1}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}$$

tengliklarni e'tiborga olib, (11) formulaga Furyening teskari almashtirishini qo'llaymiz:

$$H(x,t) = F^{-1}[\overline{H}(\xi,t)] = \frac{\theta(t)}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-a^2|\xi|^2 t - i(\xi,t)}d\xi = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}.$$

Shunday qilib, issiqlik o'tkazuvchanlik operatorining fundamental yechimi

$$H(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}$$

formula bilan ifodalanadi.

3.3.3 To‘lqin operatorining fundamental yechimi

$$\Box_a H_n = \delta(x,t)$$

tenglamani qaraymiz, bu yerda $\Box_a = \frac{\partial^2}{\partial t^2} - a^2\Delta$ - to‘lqin operatori. Bu tenglikka Furye almashtirishini qo‘llab, quyidagi tenglamani hosil qilamiz:

$$\frac{\partial^2 \overline{H}_n(\xi,t)}{\partial t^2} + a^2|\xi|^2\overline{H}_n(\xi,t) = \delta(t), \ \overline{H}_n(\xi,t) = F_x[H].$$

(9) formulaga asosan

$$\overline{H}_n(\xi,t) = \theta(t)\frac{\sin a|\xi|t}{a|\xi|}.$$

Bu yerdan esa

$$H_n(x,t) = F_\xi^{-1}\Big[\overline{H}_n(\xi,t)\Big] = \theta(t)F_\xi^{-1}\Big[\frac{\sin a|\xi|t}{a|\xi|}\Big]. \qquad (12)$$

$n = 3$ bo‘lsin. $F_\xi^{-1}\Big[\frac{\sin a|\xi|t}{a|\xi|}\Big]$ ni hisoblashda

$$F[\delta(at - |x|)] = 4\pi at\frac{\sin a|\xi|t}{|\xi|} \qquad (13)$$

formuladan foydalanamiz. Buning uchun (13) tenglikning o‘rinli ekanligini ko‘rsatamiz. Haqiqatdan ham,

$$F[\delta(at - |x|)] =$$

$$= \int\limits_{\mathbb{R}^3} \delta(at - |\xi|)e^{i(\xi,y)}dy = \int\limits_{S_{at}} e^{iat(\xi,s)}ds = (at)^2\int\limits_{S_1} e^{iat(\xi,s)}ds =$$

$$= (at)^2\int\limits_0^\pi\int\limits_0^{2\pi} e^{iat|\xi|\cos\theta}\sin\theta d\theta d\varphi = 4\pi at\frac{\sin a|\xi|t}{|\xi|},$$

bu yerda $S_{at} = \{x : |x| = at\}$. (12) formulaga teskari Furye almashtirishini qo‘llab,

$$H_3(x,t) = \frac{\theta(t)}{4\pi a^2 t}\delta(at - |x|) \equiv \frac{\theta(t)}{2\pi a}\delta(a^2t^2 - |x|^2) \qquad (14)$$

uch o‘lchovli to‘lqin operatorining fundamental yechimini hosil qilamiz. Bu umumlashgan funksiyadan quyidagi qoida bo‘yicha foydalaniladi:

$$(H_3(x,t),\varphi(x,t)) = \frac{1}{4\pi a^2}\int\limits_0^\infty (\delta(at-|x|),\varphi(x,t))\,\frac{dt}{t} =$$

$$= \frac{1}{4\pi a^2}\int\limits_0^\infty \frac{1}{t}\int\limits_{|x|=at} \varphi(x,t)dS_x dt,\ \ \varphi(x,t)\in D(\mathbb{R}^4).$$

Shunga o‘xshash to‘lqin operatorining fundamental yechimlarini mos ravishda $n=1$ va $n=2$ lar uchun hosil qilish mumkin:

$$H_1(x,t) = \frac{\theta(t)}{2a}\theta(at-|x|), \tag{14'}$$

$$H_2(x,t) = \theta(t)\frac{\theta(at-|x|)}{2\pi a\sqrt{a^2t^2-|x|^2}}. \tag{14''}$$

3.3.4 Laplas operatorining fundamental yechimi

$$\Delta H_n = \delta(x) \tag{15}$$

tenglamaning yechimini n ning turli natural qiymatlari uchun topamiz.

$n=1$ da (15) tenglama

$$\frac{\partial^2}{\partial x^2}H_1 = \delta(x)$$

ko‘rinishni oladi va uning yechimi 1-paragrafda o‘rganilgan $\frac{\partial^2}{\partial t^2}+a^2$ operatorning fundamental yechimi bilan $a=0$ da ustma-ust tushadi.

(9) formulada $a\to 0$ da limitga o‘tib, (15) ning $n=1$ dagi fundamental yechimiga ega bo‘lamiz:

$$H_1(x) = x\theta(x)$$

$n\geq 2$ lar uchun (15) tenglikka Furye almashtirishini qo‘llab, quyidagini hosil qilamiz:

$$-|\xi|^2F(H_n) = 1.$$

$n = 2$ bo'lsin. $-\rho\frac{1}{|\xi|^2}$ umumlashgan funksiya oldingi tenglikni qanoatlantirishini tekshiramiz. Haqiqatdan ham, ixtiyoriy $\varphi \in D(\mathbb{R}^2)$ uchun

$$\left(|\xi|^2\rho\frac{1}{|\xi|^2}, \varphi\right) = \left(\rho\frac{1}{|\xi|^2}, |\xi|^2\varphi\right) = \int\limits_{|\xi|<1} \frac{|\xi|^2\varphi(\xi) - |\xi|^2\varphi(\xi)|_{\xi=0}}{|\xi|^2}d\xi +$$

$$+ \int\limits_{|\xi|>1} \frac{|\xi|^2\varphi(\xi)}{|\xi|^2}d\xi = \int\limits_{\mathbb{R}^2} \varphi(\xi)d\xi = (1, \varphi).$$

Shuning uchun,

$$F(H_2) = -\rho\frac{1}{|\xi|^2}.$$

Bu yerdan 2-bobdagi (13) formulani Furyening teskari almashtirishi uchun qo'llab,

$$H_2(x) = F^{-1}\left[-\rho\frac{1}{|\xi|^2}\right] =$$

$$= -\frac{1}{4\pi^2}F\left[\rho\frac{1}{|\xi|^2}\right] = \frac{1}{2\pi}\ln|x| + \frac{C_0}{2\pi}$$

formulani hosil qilamiz. Har qanday o'zgarmas, bir jinsli Laplas tenglamasini qanoatlantirgani sababli oxirgi tenglikdan $\frac{C_0}{2\pi}$ qo'shiluvchini tashlab yuborib, fundamental yechimni

$$H_2(x) = \frac{1}{2\pi}\ln|x|$$

ko'rinishda tanlash mumkin.

$n = 3$ da $H_n(x)$ fundamental yechimni issiqlik o'tkazuvchanlik operatorining fundamental yechimidan t o'zgaruvchisi bo'yicha tushish usuli bilan olish mumkin. Issiqlik o'tkazuvchanlik operatorining fundamental yechimida $a = 1$ deb, quyidagi formulani olamiz:

$$H_n(x) = -\int\limits_{-\infty}^{\infty} H(x,t)dt =$$

$$= -\int\limits_0^{\infty} \frac{1}{(2\pi\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4a^2t}}dt = -\frac{|x|^{-n+2}}{4\pi^{\frac{n}{2}}}\int\limits_0^{\infty} e^{-u}u^{\frac{n}{2}-2}du =$$

$$= -\Gamma\Big(\frac{n}{2}-1\Big)\frac{|x|^{-n+2}}{4\pi^{\frac{n}{2}}} = -\frac{1}{(n-2)\omega_n}|x|^{-n+2},$$

bu yerda $\Gamma(z) = \int\limits_0^\infty e^{-t}t^{z-1}dt$ - Gamma funksiya,

$$\Gamma\Big(\frac{n}{2}-1\Big) = \frac{4\pi^{\frac{n}{2}}}{(n-2)\omega_n},$$

$\omega_n - \mathbb{R}^n$ da birlik sfera yuzasi.

3.4 To'lqin potensiallari

3.4.1 To'lqin operatori fundamental yechimining xossalari

To'lqin operatorining fundamental yechimlari oldingi paragrafdagi (14), $(14')$ va (14") formulalarga ko'ra

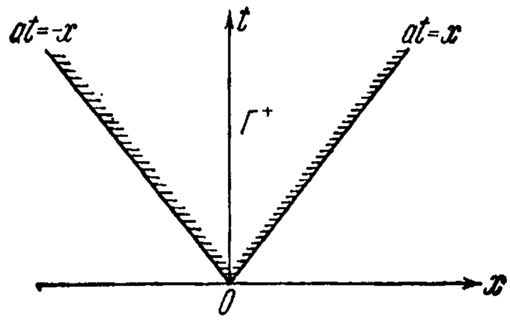
$$H_1(x,t) = \frac{\theta(t)}{2a}\theta(at-|x|),\ H_2(x,t) = \theta(t)\frac{\theta(at-|x|)}{2\pi a\sqrt{a^2t^2-|x|^2}},$$

$$H_3(x,t) = \frac{\theta(t)}{4\pi a^2 t}\delta(at-|x|) = \frac{\theta(t)}{2\pi a}\delta\left(a^2t^2-|x|^2\right)$$

umumlashgan funksiyalardan iborat. H_1 va H_2 lar lokal integrallanuvchi funksiyalar, H_3 umumlashgan funksiyaning asosiy $\varphi \in D(\mathbb{R}^n)$ funksiyalardagi qiymati esa

$$(H_3,\varphi) = \frac{1}{4\pi a^2}\int\limits_0^\infty \frac{1}{t}\int\limits_{\Sigma_x^{at}} \varphi(x,t)ds = \frac{1}{4\pi a^2}\int\limits_{\mathbb{R}^3} \frac{\varphi\left(x,\frac{|x|}{a}\right)}{|x|}dx \qquad (16)$$

qoida bilan anqlanadi, bu yerda Σ_x^{at} - markazi x va radiusi at bo'lgan sfera. H_1 va H_2 funksiyalarning tashuvchisi $\overline{\Gamma}_n^+ = \Big\{(x,t) \in \mathbb{R}^n \times (t \geq 0) : at \geq |x|\Big\}$, $n = 1,2$, yopiq konus bilan ustma-ust tushadi.



6-chizma. $\overline{\Gamma}_n^+$ soha.

H_3 umumlashgan funksiyaning tashuvchisi esa Γ_3^+ konusning $at = |x|$ chegarasida yotadi.

$f(x,t) \in D'(\mathbb{R}^{n+1})$ va $\varphi(x) \in D(\mathbb{R}^n)$ bo'lsin. $(f(x,t), \varphi(x)) \in D'(\mathbb{R})$ umumlashgan funksiyaning $\psi(t) \in D(\mathbb{R})$ lardagi qiymatini

$$((f(x,t), \varphi(x)), \psi(t)) = (f(x,t), \varphi(x)\psi(t)) \tag{17}$$

qoida bilan aniqlaymiz. Bundan quyidagi tenglik kelib chiqadi:

$$\left(\frac{\partial^k f(x,t)}{\partial t^k}, \varphi(x)\right) = \frac{d^k}{dt^k}(f(x,t), \varphi(x)), \ k = 1, 2, \ldots \tag{18}$$

Haqiqatan, har qanday $\psi \in D(\mathbb{R})$ uchun

$$\left(\left(\frac{\partial^k f(x,t)}{\partial t^k}, \varphi(x)\right), \psi(t)\right) =$$

$$= \left(\frac{\partial^k f(x,t)}{\partial t^k}, \varphi(x)\psi(t)\right) = (-1)^k \left(f(x,t), \varphi(x)\frac{d^k\psi(t)}{dt^k}\right) =$$

$$= (-1)^k \left((f(x,t), \varphi(x)), \frac{d^k\psi(t)}{dt^k}\right) = \frac{d^k}{dt^k}((f(x,t), \varphi(x)), \psi(t)).$$

Bu esa (18) tenglikning o'rinli ekanligini bildiradi.

T a' r i f. Agar ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ uchun $(f(x,t), \varphi(x))$ umumlashgan funksiya $C^p(a,b)$ sinfga (mos ravishda $C^p[a,b]$) tegishli bo'lsa, $f(x,t)$ umumlashgan funksiya t o'zgaruvchi bo'yicha (a,b) (mos ravishda $[a,b]$ da) C^p, $p \in [0,\infty]$ sinfga tegishli deyiladi.

L e m m a. $H_n,\ n=1,2,3,$ fundamental yechimlar t bo'yicha $C^\infty[0,\infty)$ sinfga tegishli va $D^{'}(\mathbb{R}^n)$ ga ushbu

$$H_3(x,t)\to 0,\ \frac{\partial H_3(x,t)}{\partial t}\to\delta(x),\ \frac{\partial^2 H_3(x,t)}{\partial t^2}\to 0 \qquad (19)$$

limit munosabatlar $t\to+0$ da bajariladi.

Isbot. $n=3$ va $\varphi(x)\in D(\mathbb{R}^3)$ bo'lsin. (16) ga ko'ra

$$(H_3(x,t),\varphi(x))=\frac{\theta(t)}{4\pi a^2 t}\int\limits_{\Sigma_x^{at}}\varphi(x)ds=\frac{\theta(t)t}{4\pi}\int\limits_{\Sigma_0^1}\varphi(ats)ds. \qquad (20)$$

(20) ning o'ng tomoni cheksiz differensiallanuvchi ekanligidan $H_3(x,t)$ ning t bo'yicha $C^\infty[0,\infty)$ sinfga tegishli bo'lishi kelib chiqadi. Bundan tashqari, (20) ga asosan

$$(H_3(x,t),\varphi(x))\to 0,\ t\to+0. \qquad (21)$$

(18) formulada $f=H_3$ va $k=1,2$ deb, (20) dan $t\to+0$ da quyidagilarni hosil qilamiz:

$$\left(\frac{\partial H_3(x,t)}{\partial t},\varphi(x)\right)=\frac{d}{dt}\left[\frac{t}{4\pi}\int\limits_{\Sigma_0^1}\varphi(ats)ds\right]=$$

$$=\frac{t}{4\pi}\int\limits_{\Sigma_0^1}\varphi(ats)ds+\frac{t}{4\pi}\frac{d}{dt}\int\limits_{\Sigma_0^1}\varphi(ats)ds\to\varphi(0)=(\delta,\varphi), \qquad (22)$$

$$\left(\frac{\partial^2 H_3(x,t)}{\partial t^2},\varphi(x)\right)=\frac{d^2}{dt^2}\left[\frac{t}{4\pi}\int\limits_{\Sigma_0^1}\varphi(ats)ds\right]=$$

$$=\frac{t}{2\pi}\frac{d}{dt}\int\limits_{\Sigma_0^1}\varphi(ats)ds+\frac{t}{4\pi}\frac{d^2}{dt^2}\int\limits_{\Sigma_0^1}\varphi(ats)ds\to 0, \qquad (23)$$

bu yerda $\int\limits_{\Sigma_0^1}\varphi(ats)ds=\int\limits_{\Sigma_0^1}\varphi(-ats)ds$ - t bo'yicha juft cheksiz differensiallanuvchi funksiya ekanligidan uning birinchi tartibli hosilasi $t=0$ da nolga teng bo'lishidan foydalanildi. $\varphi(x)\in D(\mathbb{R}^3)$ ning ixtiyoriy tanlanganligidan (21) – (23) limit munosabatlarning $n=3$ da (19) ga ekvivalentligi kelib chiqadi.

Endi $n = 2, 1$ va $\varphi(x) \in D(\mathbb{R}^n)$ bo‘lsin. U holda $t > 0$ uchun

$$(H_2(x,t), \varphi(x)) = \frac{1}{2\pi a} \int\limits_{K_x^{at}} \frac{\varphi(x)}{\sqrt{a^2t^2 - |x|^2}} dx = \frac{1}{2\pi} \int\limits_{K_0^1} \frac{\varphi(at\eta)}{\sqrt{1 - |\eta|^2}} d\eta, \qquad (24)$$

$$(H_1(x,t), \varphi(x)) = \frac{1}{2a} \int\limits_{-at}^{at} \varphi(x)dx = \frac{t}{2} \int\limits_{-1}^{1} \varphi(at\eta)d\eta. \qquad (25)$$

Bundan, $n = 3$ da bo‘lgani kabi, lemmaning barcha tasdiqlari kelib chiqadi.

3.4.2 Yig‘ma haqida qo‘shimcha ma’lumotlar

Keyingi o‘rinlarda U_x^r orqali markazi $x \in \mathbb{R}^n$ nuqtada va radiusi r bo‘lgan $\mathbb{R}^n$ fazodagi shar belgilanadi.

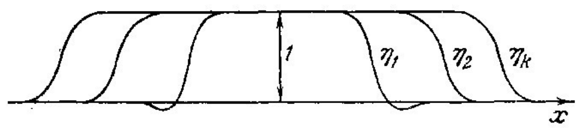
T a’ r i f. $\eta_k(x) \in D(\mathbb{R}^n)$ asosiy funksiyalar ketma-ketligi $D(\mathbb{R}^n)$ da 1 ga yaqinlashadi deyiladi, agarda quyidagi shartlar bajarilsa:

a) ixtiyoriy K kompakt to‘plam uchun shunday N raqam mavjud bo‘lib, barcha $x \in K$ va $k \geq N$ lar uchun $\eta_k(x) = 1$;

b) η_k, $k = 1, 2, \ldots$ funksiyalar ixtiyoriy tartibli hosilalari bilan chegaralangan .

Tasavvur etish uchun η_k, $k = 1, 2, \ldots$ ketma-ketlik grafigi $n = 1$ quyidagi chizmada keltirilgan.

Qayd qilish o‘rinliki, bunday ketma-ketlik har doim mavjud. Masalan, $\eta_k(x) = \eta(\frac{x}{k})$, bu yerda $\eta(x) \in D(\mathbb{R}^n)$, $\eta(x) = 1, x \in U_0^1 -$ ixtiyoriy funksiya, u yuqoridagi ketma-ketlik shartlarini qanoatlantiradi.



7-chizma. $\eta(x)$ funksiya grafigi.

Ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ uchun yig‘ma

$$(f * g, \varphi) = \int\limits_{\mathbb{R}^{2n}} f(x)g(y)\varphi(x+y)\, dxdy \qquad (26)$$

formula bilan aniqlanadi.

(26) tenglikni ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ uchun

$$(f * g, \varphi) = \lim_{k\to\infty} (f(x)g(x), \eta_k(x; y)\varphi(x + y)) \qquad (26')$$

ko‘rinishda yozish mumkinligini ko‘rsatamiz, bu yerda $\eta_k(x, y)$, $k = 1, 2, \ldots$ – $\mathbb{R}^{2n}$ da 1 ga yaqinlashuvchi ixtiyoriy ketma-ketlik.

Haqiqatan ham, $c_0 \left|f(x)g(x)\varphi(x + y)\right|$ funksiya $\mathbb{R}^{2n}$ da integrallanuvchi va

$$f(x)g(y)\eta_k(x; y)\varphi(x + y) \le c_0 \left|f(x)g(y)\varphi(x + y)\right|, \; k = 1, 2, \ldots$$

tengsizlik o‘rinli. Deyarli barcha $(x, y) \in \mathbb{R}^{2n}$ nuqtalar uchun $k \to \infty$ da

$$f(x)g(y)\eta_k(x; y)\varphi(x + y) \to f(x)g(y)\,\varphi(x + y)$$

bo‘ladi. Matematik tahlil fanidan ma’lum bo‘lgan Lebegning integral ostida limitga o‘tish haqidagi teoremasiga ko‘ra

$$\int\limits_{\mathbb{R}^{2n}} f(x)g(y)\varphi(x + y)\, dxdy = \lim_{k\to\infty} \int\limits_{\mathbb{R}^{2n}} f(x)g(y)\eta_k(x; y)\varphi(x + y)dxdy$$

tenglikni olamiz. Bu esa (26) ga asosan (26′) ga ekvivalent.

$f(x)$ va $g(y)$ funksiyalarning $f(x) \cdot g(y)$ dekart ko‘paytmasi va $\mathbb{R}^{2n}$ da 1 ga yaqinlashuvchi ixtiyoriy $\left\{\eta_k(x)\right\} \in D(\mathbb{R}^{2n})$ ketma-ketlik uchun quyidagi sonli ketma-ketlikning limiti mavjud va

$$\lim_{k\to\infty} (f(x) \cdot g(y), \eta_k(x; y) \cdot \varphi(x + y)) = (f(x) \cdot g(y), \varphi(x + y))$$

bo‘lsin, bu yerda limit $\left\{\eta_k(x)\right\}$ ga bog‘liq emas. Har bir k da $\eta_k(x; y)\cdot\varphi(x+y) \in D'(\mathbb{R}^{2n})$ bo‘lgani uchun yuqoridagi ketma-ketlik aniqlangan.

(26) va (26′) larga ko‘ra yig‘maning quyidagi ta’rifini qabul qilamiz.

T a’ r i f. $f * g$ yig‘ma deb ixtiyoriy $\varphi(x) \in D(\mathbb{R}^n)$ uchun

$$(f * g, \varphi) = (f(x) \cdot g(y), \varphi(x + y)) = \lim_{k\to\infty} (f(x) \cdot g(y), \eta_k(x, y)\varphi(x + y))$$

formula bilan aniqlangan funksionalga aytiladi. Bu funksional $D'(\mathbb{R}^n)$ ga tegishli. Ya’ni $f * g$ umumlashgan funksiyadir.

T e o r e m a. $f(x,t) \in D'(\mathbb{R}^{n+1})$ va $g(x,t) \in D'(\mathbb{R}^{n+1})$ umumlashgan funksiyalar $f(x,t)|_{t<0} = 0$ va $supp g \subset \bar{\Gamma}^+$, $\bar{\Gamma}^+ = \Big\{(x,t) \in \mathbb{R}^{n+1} : \ at \geq a|x|\Big\}$ shartlarni qanoatlantirsin. U holda $D'\left(\mathbb{R}^{n+1}\right)$ da $f * g$ yig'ma mavjud va u ixtiyoriy $\varphi(x,t) \in D\left(\mathbb{R}^{n+1}\right)$ uchun

$$(f * g, \varphi) = (f(\xi,t) \cdot g(y,\tau), \eta(t)\eta(\tau)\eta(a^2\tau^2 - |y|^2)\varphi(\xi + y, t + \tau)) \quad (27)$$

bo'ladi, bu yerda $\eta(\tau)$ uzluksiz funksiya bo'lib, $\eta(\tau) = 0, t < -\delta$ va $\eta(\tau) = 1, t > -\varepsilon$ (δ va ε ixtiyoriy sonlar, $\delta > \varepsilon > 0$). Shuningdek, $(f * g)\Big|_{t<0} = 0$ tenglik o'rinli va bu yig'ma f va g funksiyalarning har biri bo'yicha uzluksiz, ya'ni 1) agar f_k ketma-ketlik $D'\left(\mathbb{R}^{n+1}\right)$ da $k \to \infty$ uchun $f_k \to 0$, $f_k\big|_{t<0} = 0$ bo'lsa, $D'\left(\mathbb{R}^{n+1}\right)$ da $k \to \infty$ uchun $f_k * g \to 0$ bo'ladi; 1) agar g_k ketma-ketlik $D'\left(\mathbb{R}^{n+1}\right)$ da $k \to \infty$ uchun $g_k \to 0$, $g_k\big|_{t<0} = 0$ bo'lsa, $D'\left(\mathbb{R}^{n+1}\right)$ da $k \to \infty$ uchun $f * g_k \to 0$ bo'ladi.

Isbot. $\varphi(x,t) \in D\left(\mathbb{R}^{n+1}\right), supp\varphi(x,t) \in U_0^A$ - ixtiyoriy funksiya va $\eta_k(\xi,t;y,\tau) \in \mathbb{R}^{2n+2}$ funksiyalar $D\left(\mathbb{R}^{2n+2}\right)$ da $k \to \infty$ 1 ga yaqinlashuvchi ketma-ketlik bo'lsin. U holda yetarli katta k lar uchun

$$\psi_k = \eta(t)\eta(\tau)\eta\left(a^2\tau^2 - |y|^2\right)\eta_k(\xi,t;y,\tau)\varphi(\xi + y, t + \tau) =$$
$$= \eta(t)\eta(\tau)\eta\left(a^2\tau^2 - |y|^2\right)\varphi(\xi + y, t + \tau) = \psi \quad (28)$$

tengliklar o'rinli.

(28) formulani isbotlash uchun $\psi(x) \in D(\mathbb{R}^{2n+2})$ ekanini ko'rsatish yetarli. Bu esa uning cheksiz differensiallanuvchi ekanligi va tashuvchisi chegaralangan

$$A_0 = \Big\{(\xi,t,y,\tau) : t > -\delta, \tau > -\delta, a^2\tau^2 - |y|^2 > -\delta,$$
$$|y + \xi|^2 + |t + \tau|^2 \leq A^2\Big\} \subset A_1 == \Big\{(\xi,t,y,\tau) : -\delta \leq t \leq A + \delta,$$
$$-\delta \leq \tau \leq A + \delta, |y| \leq \sqrt{a^2(A+\delta)^2 + \delta}, |\xi| \leq \sqrt{a^2(A+\delta)^2 + \delta} + A\Big\}$$

to'plamda joylashganligidan kelib chiqadi.

Tanlanishiga ko'ra $f(\xi,t)$ funksiyaning tashuvchisi atrofida $\eta(t) = 1$ va $g(y,t)$ funksiyaning tashuvchisi atrofida $\eta(\tau)\eta(a^2\tau^2 - |y|^2) = 1$. Bundan $f(\xi,t) = \eta(t)f(\xi,t)$, $g(y,\tau) = \eta(\tau)\eta(a^2\tau^2 - |y|^2)g(y,\tau)$. Bu va (28) tenglikni inobatga olib, (27) tenglikning o'rinli ekanligiga ishonch hosil qilamiz.

$(f * g)\,|_{t<0} = 0$ tenglikni ko‘rsatamiz. $\varphi(x) \in D\left(\mathbb{R}^{n+1}\right)$ va $supp\varphi \subset \{\mathbb{R}^n \times (t<0)\}$ ixtiyoriy funksiya bo‘lsin. $supp\varphi$ ning $\mathbb{R}^{n+1}$ da kompakt to‘plamligi uchun shunday $\delta_1 > 0$ son mavjudki, bunda $supp\varphi \subset \mathbb{R}^n \times (t \leq -\delta)$ munosabat o‘rinli bo‘ladi. U holda δ sonini $\delta = \frac{\delta_1}{2}$ shartdan tanlab,

$$\eta(t)\eta(\tau)\eta(a^2\tau^2 - |y|^2)\varphi(\xi + y, t + \tau) = 0$$

tenglikka ega bo‘lamiz. Bundan esa (27) tenglikka ko‘ra $t < 0$ lar uchun $(f * g, \varphi) = 0$ o‘rinli.

Yig‘maning f va g funksiyalarga nisbatan uzluksizligi (27) tenglik va $f(\xi, t) \cdot g(y, \tau)$ dekart ko‘paytmaning f va g larning har biriga nisbatan uzluksizligidan kelib chiqadi. Bunda yordamchi η funksiyani k larga bog‘liq bo‘lmagan holda tanlash mumkin. Teorema isbotlandi.

N a t i j a. (27) formulada $f(x,t) = u(x) \cdot \delta(t)$, $u(x) \in D'(\mathbb{R}^n)$ deb olinsa, undan ixtiyoriy $g(x,t) \in D'\left(\mathbb{R}^{n+1}\right)$, $suppg \subset \bar{\Gamma}_+$ funksiya uchun

$$g * [u(x) \cdot \delta(t)] = g(x,t) * u(x) \tag{29}$$

formula kelib chiqadi. Shuningdek, bu formuladan hamda dekart ko‘paytma va yig‘mani differensiallash qoidalaridan foydalanib, quyidagi (29) munosabatni umumlashtiruvchi formulaga ega bo‘lamiz:

$$g * \left[u(x) \cdot \delta^(k)(t)\right] = \frac{\partial^k}{\partial t^k}\left[g(x,t) * u(x)\right] = \frac{\partial^k}{\partial t^k} g(x,t) * u(x). \tag{30}$$

Endi keyinchalik zarur bo‘lgan, $\mathbb{R}^n$ da umumlashgan funksiyalar yig‘masining ekvivalent ta’rifi va ba’zi xossalarini keltiramiz. $f(x)$ va $g(x)$ lar $\mathbb{R}^n$ da aniqlangan va lokal integrallanuvchi funksiyalar uchun

$$h(x) = \int\limits_{\mathbb{R}^n} |g(y) f(x-y)|\, dy$$

funksiya ham $\mathbb{R}^n$ da lokal integrallanuvchi funksiya bo‘lsin. $f * g$ *yig‘ma* deb

$$(f * g)(x) = \int\limits_{\mathbb{R}^n} |f(y) g(x-y)|\, dy = \int_{\mathbb{R}^n} |g(y) f(x-y)|\, dy = (g * f)(x) \tag{31}$$

tenglik bilan aniqlangan $(f * g)(x)$ funksiyaga aytiladi. $f * g$ va $|f| * |g| = h$ yig‘malar bir vaqtning o‘zida mavjud bo‘lib, $|(f * g)(x)| \leq h(x)$ tengsizlikni qanoatlantiradi. Shuning uchun yig'ma ham $\mathbb{R}^n$ da lokal integrallanuvchi funksiya bo'ladi. Binobarin, u $\varphi \in D(\mathbb{R}^n)$ asosiy funksiyalarga quyidagi qoida bilan ta'sir etuvchi umumlashgan funksiyani aniqlaydi:

$$(f * g, \varphi) = \int_{\mathbb{R}^n} (f * g)(\xi)\varphi(\xi)\, d\xi = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} g(y) f(\xi - y)\, dy \right] \varphi(\xi)\, d\xi =$$

$$= \int_{\mathbb{R}^n} g(y) \left[\int_{\mathbb{R}^n} f(\xi - y)\, \varphi(\xi)\, d\xi \right] dy = \int_{\mathbb{R}^n} g(y) \left[\int_{\mathbb{R}^n} f(x)\, \varphi(x + y)\, dx \right] dy.$$

Bundan, matematik tahlil fanidan ma'lum bo'lgan Fubini teoremasiga asosan

$$(f * g, \varphi) = \int_{\mathbb{R}^{2n}} f(x) g(y)\, \varphi(x + y)\, dx dy$$

tenglikka ega bo'lamiz.

$h(x)$ funksiya lokal integrallanuvchi va (31) formula bilan aniqlangan $(f * g)$ yig'ma mavjud bo'ladigan ikkita holni keltiramiz.

1) $f(x)$ va $g(x)$ funksiyalardan biri finit, masalan, $suppg(x) \subset U_0^{R_1}$ bo'lsin. Bu holda

$$\int_{U_0^R} h(x) dx =$$

$$= \int_{U_0^{R_1}} |g(y)| \int_{U_0^R} |f(x - y)| dx dy \leq \int_{U_0^{R_1}} |g(y)| dy \int_{U_0^{R+R_1}} |f(\xi)| d\xi < \infty,$$

$R > 0$ - ixtiyoriy son.

2) $f(x)$ va $g(x)$ funksiyalar $\mathbb{R}^n$ da integrallanuvchi. Bu holda

$$\int_{\mathbb{R}^n} h(x) dx =$$

$$= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x - y)| dx dy = \int_{\mathbb{R}^n} |g(y)| dy \int_{\mathbb{R}^n} |f(\xi)| d\xi < \infty$$

munosabatlardan $f * g$ yig'maning $\mathbb{R}^n$ da integrallanuvchanligi kelib chiqadi.

3.4.3 To‘lqin potensiali. Kechikuvchan potensial

$f(x,t) \in D'(\mathbb{R}^{n+1})$ funksiya $f(x,t)|_{t<0} \equiv 0$ shartni qanoatlantirsin.

$$V_n(x,t) = (H_n * f)(x,t)$$

umumlashgan funksiyaga *to‘lqin potensiali,* $f(x,t)$ funksiyaga esa uning *zichligi* deyiladi, bu yerda H_n- to‘lqin operatorining fundamental yechimi.

$suppH_n \subset \overline{\Gamma^+}$ bo‘lgani uchun oldingi paragrafdagi teoremaga ko‘ra $V_n(x,t)$ to‘lqin potensiali mavjud va $V_n \in D'(\mathbb{R}^{n+1})$ bo‘ladi. Bu funksiya ixtiyoriy asosiy $\varphi(x,t) \in D'(\mathbb{R}^{n+1})$ funksiyalarda

$$(V_n, \varphi) =$$

$$= \left(H_n(y,\tau) * f(\xi,\tau'),\ \eta(\tau)\eta(\tau')\eta\left(a^2\tau^2 - |y|^2\right)\varphi(y+\xi,\tau+\tau')\right) \qquad (32)$$

tenglik bilan aniqlanadi. Bu yerda $\eta(\tau) = 0,\ \tau < -\delta;\ \eta(\tau) = 1,\ \tau > -\varepsilon;\ \delta$, ε sonlar ixtiyoriy bo‘lib, $\delta > \varepsilon > 0$ shartni qanoatlantiradi va $\eta(\tau) \in C^\infty(\mathbb{R})$. Yana shu teoremaga ko‘ra $V_n|_{t<0} \equiv 0$ va V_n potensial $f(x,t)$ funksiyadan $D'(\mathbb{R}^{n+1})$ da uzluksiz bog‘liq.

Ushbu bobning 2-paragrafidagi teoremaga ko‘ra, to‘lqin potensiali

$$\Box_a V_n = f$$

tenglamani qanoatlantiradi. To‘lqin V_n potensialining keyingi xossalari uning f zichligining xossalariga bog‘liq.

L e m m a. Agar $f(x,t)$ funksiya $\mathbb{R}^{n+1}$ da lokal integrallanuvchi bo‘lsa, $V_n(x,t)$ ham $\mathbb{R}^{n+1}$ da lokal integrallanuvchi va u

$$V_3(x,t) = \frac{1}{4\pi a^2} \int\limits_{U_x^{at}} \frac{f(\xi, t - |\frac{x-\xi}{a}|)}{x-\xi} d\xi, \qquad (33)$$

$$V_2(x,t) = \frac{1}{2\pi a} \int\limits_0^t \int\limits_{K_x^{a(t-\tau)}} \frac{f(\xi,\tau)d\xi d\tau}{\sqrt{a^2(t-\tau)^2 - |x-\xi|^2}}, \qquad (33')$$

$$V_1(x,t) = \frac{1}{2a} \int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau \qquad (33'')$$

formulalar bilan aniqlanadi. $(33')$ formulada $K_x^{a(t-\tau)}$ - markazi x nuqtada va radiusi $a(t-\tau)$ bo'lgan doira.

(33) formulaning o'rinli ekanini isbotlaymiz. $\varphi(x,t) \in D'(\mathbb{R}^4)$ bo'lsin. $f(x,t) \in L_1^{loc}(\mathbb{R}^4)$ va $f(x,t)|_{t<0} \equiv 0$ bo'lgani uchun integrallash tartibini o'zgartirish haqidagi Fubini teoremasiga ko'ra (32) dan quyidagi tengliklarni olamiz:

$$(V_3, \varphi) =$$

$$= \left(H_3(y,\tau)\eta(\tau)\eta\left(a^2\tau^2 - |y|^2\right) \int\limits_{\mathbb{R}^4} f\left(\xi, \tau'\eta(\tau')\right)\varphi(y+\xi, \tau+\tau')d\xi d\tau \right) =$$

$$= \left(H_3(y,t), \eta(t)\eta(a^2t^2 - |y|^2) \int\limits_{\mathbb{R}^4} f(x-y, t-\tau)\varphi(x,t)dxdt \right) =$$

$$= \frac{1}{4\pi a^2} \int\limits_{\mathbb{R}^3} \frac{\eta(\frac{|y|}{a})\eta(0)}{|y|} \left[\int\limits_{\mathbb{R}^4} f\left(x-y, t-\frac{|y|}{a}\right)\varphi(x,t)dxdt \right] dy =$$

$$= \frac{1}{4\pi a^2} \int\limits_{\mathbb{R}^n} \varphi(x,t) \int\limits_{U_0^{at}} \frac{f\left(x-y, t-|\frac{|y|}{a}|\right)}{|y|} dydxdt.$$

Bundan ko'rinib turibdiki, V_3 - lokal integrallanuvchi va

$$V_3(x,t) = \frac{1}{4\pi a^2} \int\limits_{U_0^{at}} \frac{f\left(x-y, t-\frac{|y|}{a}\right)}{|y|} dy. \qquad (34)$$

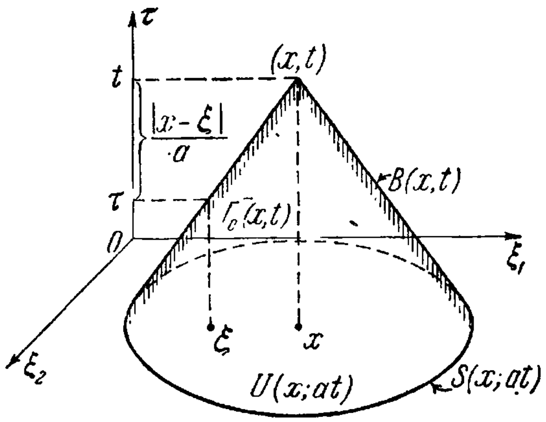
(34) da $x - y = \xi$ almashtirish bajarib, (33) formulaga ega bo'lamiz.

O'xshash tarzda tegishli soddalashtirishlar bilan, V_2 va V_1 potensiallar uchun $(33')$ va $(33'')$ formulalar asoslanadi.

Har bir $(x,t), t > 0$ nuqtaga uchi (x,t) da, asosi U_0^{at} va yon sirti $B(x,t)$ bo'lgan ochiq

$$\Gamma_0(x,t) = \Gamma^-(x,t) \cap (0 < \tau < t)$$

konusni mos qo'yamiz.



8-chizma. $\Gamma_0(x,t)$ soha.

T e o r e m a. Agar $f(x,t) \in C^2(\mathbb{R}^n \times (t \geq 0))$, $n = 3, n = 2$ va $f(x,t) \in C^1(\mathbb{R} \times (t \geq 0))$, $n = 1$ uchun bo‘lsa, u holda V_n potensial $V_n(x,t) \in C^2(\mathbb{R}^n \times (t \geq 0))$ va u

$$\left|V_3(x,t)\right| \leq \frac{t^2}{2} \max_{(x,t)\in B(x,t)} \left|f(x,t)\right|,$$

$$\left|V_n(x,t)\right| \leq \frac{t^2}{2} \max_{(x,t)\in\overline{\Gamma}_0} \left|f(x,t)\right|, \; n = 1, 2 \tag{35}$$

baholarni va

$$V_n|_{t=0} = 0, \; \frac{\partial V_n}{\partial t}\bigg|_{t=0} = 0, \; n = 3, 2, 1 \tag{36}$$

boshlang‘ich shartlarni qanoatlantiradi.

Isbot. Teoremani $n = 3$ uchun isbotlaymiz. (21) formulada $y = at\eta$, $t > 0$ tenglik bilan o‘zgaruvchilarni almashtirib, uni

$$V_3(x,t) = \frac{t^2}{4\pi} \int\limits_{U_0^1} \frac{f(x + at\eta, t(1 - |\eta|))}{|\eta|} d\eta \tag{37}$$

ko‘rinishida yozamiz. $f(x,t) \in C^2(\mathbb{R}^3 \times (t \geq 0))$ ekanligi va (37) ning integral osti ifodasi integrallanuvchi maxsuslikka egaligi uchun $V_3(x,t) \in C^2$ $(\mathbb{R}^3 \times (t \geq 0))$ bo‘ladi. (37) formuladan V_3 uchun (35) baho kelib chiqadi:

$$\left|V_3(x,t)\right| \leq \frac{t^2}{4\pi} \max_{(x,t)\in B(x,t)} \left|f(x,t)\right| \int\limits_{U_0^1} \frac{d\eta}{|\eta|} = \frac{t^2}{2} \max_{(x,t)\in B(x,t)} \left|f(x,t)\right|.$$

Bu yerda $\int\limits_{U_0^1} \frac{d\eta}{|\eta|} = 2\pi$ tenglikdan foydalanildi. $V_3(x,t) \in C^2(\mathbb{R}^3 \times (t \geq 0))$ ekanligidan boshlang‘ich shartlar o‘rinli bo‘lishiga ishonch hosil qilamiz.

Endi $n = 2$ bo'lsin. $\xi = x + at\eta$, $t = t - \alpha t$ almashtirishlar V_2 potensial uchun $(33^{'})$ formulani

$$V_2(x,t) = \frac{t^2}{2}\int\limits_0^1\int\limits_{K_0^\alpha}\frac{f(x+at\eta, t-\alpha t)}{\sqrt{\alpha^2 - |\eta|^2}}d\eta d\alpha$$

ko'rinishga o'zgartiradi. Ravshanki, bundan bu potensialning talab etilayotgan xossalari kelib chiqadi. V_1 potensialning xossalari esa $(33^{''})$ formuladan bevosita kelib chiqadi.

$V_3(x,t)$ potensialga *kechikuvchan potensial* ham deyiladi. Uning bu nomi potensialning kuzatilayotgan t vaqtda x nuqtadagi qiymati $f(\xi,\tau)$, $\xi \in \overline{U_x^{at}}$ manbaning orqada qoluvchi vaqtning $\tau = t - \frac{|x-\xi|}{a}$ qiymatiga bog'liqligidir. Bunda kechikish $\frac{|x-\xi|}{a}$ vaqti kattaligi to'lqinning ξ nuqtadan x gacha kelishi uchun sarflangan vaqtiga teng. Boshqacha aytganda, $V_3(x,t)$ potensial $f(\xi,\tau)$ manbaning $\Gamma_0^-(x,t)$ konus yon $B(x,t)$ sirtidagi qiymatlariga bog'liqdir.

3.4.4 Sirt to'lqin potensiallari

Ixtiyoriy $u_0(x)$, $u_1(x) \in D^{'}(\mathbb{R}^n)$ funksiyalar uchun $f(x,t) = u_1(x)\delta(t)$ va $f(x,t) = u_0(x)\delta^{'}(t)$ funksiyalarga mos keluvchi to'lqin

$$V_n^0 = H_n * \Big[u_1(x)\delta(t)\Big],\ V_n^1 = H_n * \Big[u_0(x)\delta^{'}(t)\Big],\ n = 1,2,3$$

potensiallariga *sirt potensiallari (mos ravishda u_1 va u_0 zichlikka ega bo'lgan oddiy va ikkilangan qatlam)* deyiladi.

(29) va (30) formulalarga asosan V_n^0 va V_n^1 to'lqin potensiallari

$$V_n^0 = H_3(x,t) * u_1(x), \tag{38}$$

$$V_n^1 = \frac{\partial H_3(x,t)}{\partial t} * u_0(x) = \frac{\partial}{\partial t}\Big[H_3(x,t) * u_0(x)\Big] \tag{39}$$

ko'rinishda ifodalanadi.

L e m m a. V_n^0 va V_n^1 sirt to'lqin potensiallari t bo'yicha $C^\infty[0,\infty)$ sinfga tegishli va $t \to 0$ da $D^{'}(\mathbb{R}^n)$ funksiyalar sinfida

$$V_n^0(x,t) \to 0,\ \ \frac{\partial V_n^0(x,t)}{\partial t} \to u_1(x), \tag{40}$$

$$V_n^1(x,t) \to u_0(x), \quad \frac{\partial V_n^1(x,t)}{\partial t} \to 0 \tag{41}$$

boshlang‘ich shartlarni qanoatlantiradi.

Isbot. To‘lqin operatorining fundamental yechimi xossalari haqidagi lemmaga asosan $H_n(x,t)$ umumlashgan funksiya t bo‘yicha C^∞ sinfga tegishli. Har bir tayin $t>0$ da $H_n(x,t)$ ning tashuvchisi U_x^{at} sharda yotadi, demak, u $t \to t_0 \geq 0$ uchun $\mathbb{R}^n$ da tekis chegaralangan. Yig‘maning $D^{'}$ da uzluk- sizligidan ixtiyoriy $\varphi \in D(\mathbb{R}^n)$ uchun

$$\left(\frac{\partial^k H_n(x,t)}{\partial t^k} * u_1(x), \varphi(x)\right) \in C[0,\infty), \; k = 0,1,2,\dots.$$

Bu yerdan

$$\frac{\partial^k}{\partial t^k}\left(H_n(x,t) * u_1(x), \varphi(x)\right) = \left(\frac{\partial^k}{\partial t^k}\Big[H_n(x,t) * u_1(x)\Big], \varphi(x)\right)$$

ekanligi uchun

$$(H_3(x,t) * u_1(x), \varphi) \in C^\infty[0,\infty)$$

kelib chiqadi. Bu esa $D^{'}(\mathbb{R}^n)$ da $V_n^0(x,t)$ ning t bo‘yicha $C^\infty[0,\infty)$ sinfga tegishli bo‘lishini bildiradi.

u_1 ni u_0 bilan almashtirib, V_n^1 potensial ham bu xossalarga ega bo‘lishiga ishonch hosil qilish mumkin.

(40) limit munosabatni isbotlaymiz. (31) limit munosabatlar va $H_3(x,t) * u_1(x)$ yig‘maning $D^{'}(\mathbb{R}^n)$ da uzluksizligini inobatga olib, $t \to +0$ da

$$V_n^0(x,t) = H_3(x,t) * u_1(x) \to 0 * u_1(x) = 0,$$

$$\frac{\partial V_n^0(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[H_3(x,t) * u_1(x)\right] = \frac{\partial H_3(x,t)}{\partial t} * u_1(x) \to \delta * u_1 = u_1(x)$$

munosabatlarning $D^{'}(\mathbb{R}^n)$ da bajarilishini ko‘ramiz. O‘xshash tarzda (41) ham ko‘rsatiladi. To‘lqin V_n potensiali uchun bo‘lgani kabi sirt to‘lqin potensiallarining keyingi xossalari u_1 ni u_0 zichliklarning xossalariga bog‘liq. Agar $u_1 \in \mathbb{L}_1^{loc}(\mathbb{R}^n)$ ($\mathbb{L}_1^{loc}\mathbb{R}^n$ - lokal integrallanuvchi funksiyalar sinfi) bo‘lsa, $V_n^0 \in \mathbb{L}_1^{loc}(\mathbb{R}^{n+1})$ bo‘ladi va V_n^0 uchun

$$V_3^0(x,t) = \frac{\theta(t)}{4\pi a^2 t}\int\limits_{\Sigma_x^{at}} u_1(\xi)ds, \tag{42}$$

$$V_2^0(x,t) = \frac{\theta(t)}{2\pi a}\int\limits_{K_x^{at}} \frac{u_1(\xi)d\xi}{\sqrt{a^2t^2 - |x-\xi|^2}}, \tag{42'}$$

$$V_1^0(x,t) = \frac{\theta(t)}{2\pi}\int\limits_{x-at}^{x+at} u_1(\xi)d\xi \tag{42''}$$

formulalar o'rinli. (42) formulani isbotlaymiz. (38), (41) formulalarga asosan ixtiyoriy $\varphi \in D(\mathbb{R}^n)$ lar uchun

$$\left(V_3^0, \varphi\right) = (H_3(x,t) * u_1(x), \varphi) =$$

$$= \left(H_3(y,t), \eta(a^2t^2 - |y|^2)\int\limits_{\mathbb{R}^n} u_1(\xi)\varphi(\xi)d\xi\right) =$$

$$= \frac{\theta(t)}{4\pi a^2}\int\limits_0^\infty \frac{\eta(0)}{t}\int\limits_{\sum_0^{at}}\int\limits_{\mathbb{R}^3} u_1(x-y)\varphi(x,t)dxds_ydt =$$

$$= \frac{\theta(t)}{4\pi a^2}\int\limits_0^\infty\int\limits_{\mathbb{R}^3} \frac{\varphi(x,t)}{t}\int\limits_{\sum_0^{at}} u_1(x-y)ds_ydx.dt$$

Bu yerdan $V_3^{(0)} \in \mathbb{L}_1^{loc}(\mathbb{R}^4)$ va

$$V_3^{(0)} = \frac{\theta(t)}{4\pi a^2 t}\int\limits_{\sum_0^{at}} u_1(x-y)ds_y$$

bo'lishi kelib chiqadi. Bu integralda $x - y = \xi$ almashtirishlarni bajarib, (42) tenglikni olamiz. O'xshash tarzda tegishli soddalashtirishlardan so'ng (42′) va (42″) formulalar $V_2^{(0)}$ va $V_1^{(0)}$ potensiallar uchun keltirib chiqariladi.

T e o r e m a. Agar $u_0 \in C^3(\mathbb{R}^n)$, $u_1 \in C^2(\mathbb{R}^n)$, $n = 3, 2$ va $u_0 \in C^2(\mathbb{R})$, $u_1 \in C^1(\mathbb{R})$, $n = 1$ bo'lsa, u holda $V_n^{(0)},\ V_n^{(1)} \in C^2\left(\mathbb{R}^n \times (t \geq 0)\right)$ va

$$\left|V_3^{(0)}(x,t)\right| \leq t\max_{\xi\in\sum_x^{at}} |u_1(\xi)|,$$

$$|V_n^{(0)}(x,t)| \leq t\max_{\xi\in K_x^{at}} |\ u_1(\xi)|,\ \ n = 1, 2, \tag{43}$$

$$|V_3^{(1)}(x,t)| \leq \max_{\xi\in\sum_x^{at}} |u_0(\xi)| + at\max_{\xi\in\sum_x^{at}} |grad\ u_0(\xi)|, \tag{44}$$

$$|V_2^{(1)}(x,t)| \le \max_{\xi\in K_x^{at}} |u_0(\xi)| + at \max_{\xi\in K_x^{at}} |grad\ u_0(\xi)|, \tag{44'}$$

$$|V_2^{(1)}(x,t)| \le \max_{\xi\in[x-at,x+at]} |grad\ u_0(\xi)| \tag{44''}$$

baholarni,

$$V_n^{(0)}\Big|_{t=0} = 0,\ \frac{\partial V_n^{(0)}}{\partial t}\Big|_{t=0} = u_1(x), \tag{45}$$

$$V_n^{(1)}\Big|_{t=0} = u_0,\ \frac{\partial V_n^{(1)}}{\partial t}\Big|_{t=0} = 0 \tag{46}$$

boshlang‘ich shartlarni qanoatlantiradi.

Isbot. $n = 3$ bo‘lsin. (47) formulada $x - \xi = ats,\ t > 0$ almashtirishlarni bajarib,

$$V_3^{(1)}(x,t) = \frac{\theta(t)t}{4\pi} \int\limits_{\sum_0^1} u_1(\alpha - ats)ds \tag{47}$$

formulani hosil qilamiz. Ravshanki, bundan $V_3^{(1)}(x,t) \in C^2(\mathbb{R}\times(t\ge 0))$ ekanligi kelib chiqadi. (47) tenglikni t bo‘yicha differensiallaymiz va (39) formuladan foydalanib, $V_3^{(1)}$ potensial uchun

$$V_3^{(1)}(x,t) = \frac{\theta(t)t}{4\pi} \int\limits_{\sum_0^1} u_0(\alpha - ats)ds - \frac{at\theta(t)}{4\pi} \int\limits_{\sum_0^1} (grad_x u_0(x - ats), s)ds$$

formulani olamiz. Bundan esa (44) kelib chiqadi:

$$|V_3^{(1)}(x,t)| \le \max_{|s|=1} |u_0(x - ats)| + at \max_{|s|=1} (grad_x u_0(x - ats), s)\,| \le$$

$$\le \max_{\xi\in\sum_x^{at}} |u_\xi| + at \max_{\sum_x^{at}} |grad_x u_0(x)|.$$

$u_0 \in C^3(\mathbb{R}^3)$ ekanligidan $V_3^{(1)} \in C^2\left(\mathbb{R}^3\times(t\ge 0)\right)$ bo‘ladi. $n = 2$ uchun $\xi = x - at\eta,\ t > 0$ almashtirish (42′) va (39) ko‘rinishdagi $V_2^{(0)}$ va $V_2^{(1)}$ potensiallarni

$$V_2^{(0)}(x,t) = \frac{\theta(t)t}{2\pi} \int\limits_{U_0^1} \frac{u_0(x - at\eta)d\eta}{\sqrt{1-|\eta|^2}},$$

$$V_2^{(1)}(x,t) = \frac{\theta(t)}{2\pi} \int\limits_{U_0^1} \frac{u_0(x - at\eta)d\eta}{\sqrt{1-|\eta|^2}} - \frac{a\theta(t)t}{2\pi} \int\limits_{U_0^1} \frac{(grad_x u_0(x - at\eta), \eta)\,d\eta}{\sqrt{1-|\eta|^2}}$$

ko'rinishiga o'tkazadi. Bu yerdan talab etilayotgan silliqlik va $V_2^{(0)}$, $V_2^{(1)}$ potensiallar uchun (43), $(44')$ baholar kelib chiqadi. Masalan,

$$V_2^{(0)}(x,t) \leq \frac{t}{2\pi} \max_{\eta \in U_0^1} |u_1(x - at\eta)| \int\limits_{U_0^1} \frac{d\eta}{\sqrt{1 - |\eta|^2}} = t \max_{\xi \in U_0^1} |u_1(\xi)| \int\limits_0^t \frac{\rho d\rho}{\sqrt{1 - \rho^2}}.$$

$V_1^{(0)}$, $V_1^{(1)}$ potensiallarning mos xossalari $(44'')$ va (38) formulalardan kelib chiqadi, masalan,

$$V_1^{(1)}(x,t) = \frac{\theta(t)}{2} \left[u_0(x + at) + u_0(x - at) \right].$$

Endi (45) va (46) boshlang'ich shartlarning bajarilishini ko'rsatamiz. (40) va (46) limit munosabatlarga ko'ra bu shartlar $D'(\mathbb{R}^n)$ fazoda yaqinlashish ma'nosida bajariladi. Biroq isbotlanganga asosan $V_n^{(0)}$, $V_n^{(1)}$ funksiyalar $C^2(\mathbb{R}^n \times (t \geq 0))$ sinfga tegishli. Demak, bu funksiyalar (45) va (46) shartlarni klassik ma'noda qanoatlantiradi. Teorema isbotlandi.

3.5 To'lqin tenglamasi uchun umumlashgan Koshi masalasining qo'yilishi

Ma'lumki, to'lqin tenglamasi uchun klassik Koshi masalasi

$$\Box_a u = f(x,t), \ x \in \mathbb{R}^n, \ t > 0 \tag{48}$$

tenglamaning

$$u|_{t=0} = u_0(x), \ \left. \frac{du}{dt} \right|_{t=0} = u_1(x), \ x \in \mathbb{R}^n \tag{49}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topishdan iborat. Bunda $f \in C(\mathbb{R}^n \times (t \geq 0))$, $u_0 \in C^1(\mathbb{R}^n)$, $u_1 \in C(\mathbb{R}^n)$ shartlar bajarilgan deb hisoblanadi.

Faraz qilaylik, (48), (49) masalaning klassik yechimi mavjud bo'lsin, ya'ni (48) tenglamani va (49) shartlarni qanoatlantiruvchi $C^2(\mathbb{R}^n \times (t \geq 0))$ sinfga tegishli $u(x,t)$ funksiya mavjud bo'lsin. t ning manfiy qiymatlari uchun u va f funksiyalarni nolga teng qilib, davom ettiramiz, ya'ni

$$\overline{u} = \begin{cases} u, \ t \geq 0, \\ 0, \ t < 0, \end{cases} \qquad \overline{f} = \begin{cases} f, \ t \geq 0, \\ 0, \ t < 0. \end{cases}$$

U holda $\overline{u}(x,t)$ funksiyaning $\mathbb{R}^{n+1}$ da ushbu

$$\Box_a u = \overline{f}(x,t) + u_0(x) \cdot \delta'(t) + u_1(x) \cdot \delta(t) \tag{50}$$

to‘lqin tenglamasini umumlashgan funksiyalar ma’nosida qanoatlantirishini ko‘rsatamiz. Haqiqatdan ham, ixtiyoriy $\varphi \in D(\mathbb{R}^{n+1})$ lar uchun quyidagi tengliklarga ega bo‘lamiz:

$$(\Box_a \overline{u}, \ \varphi) = (\overline{u}, \ \Box_a \varphi) =$$

$$= \int\limits_0^\infty \int\limits_{\mathbb{R}^n} u \Box_a \varphi dx dt = \lim_{\epsilon \to 0} \int\limits_\epsilon^\infty \int\limits_{\mathbb{R}^n} u \Big(\frac{d^2\varphi}{dt^2} - a^2 \Delta \varphi \Big) dx dt =$$

$$= \lim_{\epsilon \to 0} \Big[\int\limits_\epsilon^\infty \int\limits_{\mathbb{R}^n} \Big(\frac{d^2 u}{dt^2} - a^2 \Delta u \Big) dx dt -$$

$$- \int\limits_{\mathbb{R}^n} \frac{d\varphi(x,\epsilon)}{dt} u(x,\epsilon) dx + \int\limits_{\mathbb{R}^n} \varphi(x,\epsilon) \frac{du(x,\epsilon)}{dt} dx \Big] =$$

$$= \int\limits_\epsilon^\infty \int\limits_{\mathbb{R}^n} f \varphi dx dt - \int\limits_{\mathbb{R}^n} \frac{d\varphi(x,0)}{dt} u(x,0) dx + \int\limits_{\mathbb{R}^n} \varphi(x,0) \frac{du(x,0)}{dt} dx =$$

$$= \int\limits_{\mathbb{R}^{n+1}} \varphi \overline{f} dx dt - \int\limits_{\mathbb{R}^n} u_0(x) \frac{d\varphi(x,0)}{dt} dx + \int\limits_{\mathbb{R}^n} u_1(x) \varphi(x,0) dx =$$

$$= (\overline{f}(x,t) + u_0(x) \cdot \delta'(t) + u_1(x) \cdot \delta(t), \varphi).$$

Bu formulalarda oxirgi tenglikning o‘ng tomonidagi yig‘indini $F(x,t)$ orqali belgilaymiz, ya‘ni

$$F(x,t) = \overline{f}(x,t) + u_0(x) \cdot \delta'(t) + u_1(x) \cdot \delta(t).$$

(50) tenglikdan ko‘rinib turibdiki, (49) boshlang‘ich shartlardagi u_0 va u_1 funksiyalar $\overline{u}(x,t)$ funksiya uchun $t = 0$ da oniy ta’sir etuvchi $u_0(x) \cdot \delta'(t) + u_1(x) \cdot \delta(t)$ manba rolini bajaryapti. (48) va (49) Koshi masalasining

klassik yechimi (50) tenglamaning $t < 0$ da nolga aylanuvchi yechimlari orasida qamrab olingan. Bu esa (50) tenglamaning $t < 0$ da nolga aylanuvchi yechimini izlash masalasiga to'lqin tenglamasi uchun umumlashgan Koshi masalasi deyishga imkon beradi. U holda (50) tenglamaning o'ng qismini umumlashgan funksiya deb hisoblash mumkin.

T a ' r i f. $F \in D'(\mathbb{R}^{n+1})$ manbali to'lqin tenglamasi uchun umumlashgan Koshi masalasi deb,

$$\Box_a u = F(x,t) \tag{51}$$

tenglama va

$$u\Big|_{t<0} = 0 \tag{52}$$

shartni qanoatlantiruvchi $u \in D'(\mathbb{R}^{n+1})$ funksiyani topish masalasiga aytiladi. (51) tenglama ixtiyoriy $\varphi \in D'(\mathbb{R}^{n+1})$ uchun

$$(u,\ \Box_a \varphi) = (F,\ \varphi)$$

tenglamaga teng kuchli. To'lqin tenglamasi uchun umumlashgan Koshi masalasi bir qiymatli yechilishining zaruriy va yetarli sharti $t < 0$ da $F = 0$ bo'lishidan iboratdir.

$F(x,t) \in D'\left(\mathbb{R}^{n+1}\right)$, $F(x,t)\Big|_{t<0} = 0$ bo'lsin. U holda 4-paragrafning 2-bandidagi teoremaga ko'ra (51), (52) masalaning yechimi $D'\left(\mathbb{R}^{n+1}\right)$ sinfda mavjud, yagona va $D'\left(\mathbb{R}^{n+1}\right)$ da $F(x,t)$ ga uzluksiz bog'liq. Bu yechim 2-paragrafdagi teoremaga ko'ra

$$u(x,t) = (H_n * F)(x,t)$$

formula bilan beriladi, bu yerda $H_n(x,t)-$ to'lqin operatorining fundamental yechimi, $n = 1, 2, 3$.

(51) va (52) masala yechimini batafsilroq qaraymiz:

$$u(x,t) = (H_n * F)(x,t) =$$

$$= H_n(x,t) * \left(f(x,t) + u_0(x) \cdot \delta'(x) + u_1(x) \cdot \delta(x)\right). \tag{53}$$

Fundamental yechimning tashuvchisi $\overline{\Gamma}_n^+ = \Big\{(x,t) \in \mathbb{R}^n \times (t \geq 0) : at \geq |x|\Big\}$, $n = 1, 2, 3$ yopiq konusda joylashganligi va $F(x,t)$ funksiya t bo‘yicha yarim finit ekanligi sababli yig‘maning mavjudligi haqidagi xossaga ko‘ra (53) formuladagi yig‘ma mavjud. Yig‘maning ushbu

$$g(x,t) *_t \delta(t) = g(x,t),\ g(x,t) *_t \delta'(t) = \frac{\partial g(x,t)}{\partial t} *_t \delta(t) = \frac{\partial g(x,t)}{\partial t},$$

bu yerda $*_t-$ t bo‘yicha yig‘ma, xossalariga asosan (53) dagi t bo‘yicha yig‘malarni hisoblab, uni

$$u(x,t) = (H_n * f)(x,t) + \frac{\partial H_n(x,t)}{\partial t} *_x u_0(x) + H_n(x,t) *_x u_1(x) \qquad (54)$$

ko‘rinishga keltiramiz. (54) umumiy holda to‘lqin tarqalish tenglamasi uchun umumlashgan Koshi masalasi yechimini ifodalaydi. Shunday qilib, bu yechim zichliklari mos ravishda $f(x,t)$, $u_0(x)$ va $u_1(x)$ funksiyalardan iborat bo‘lgan to‘lqin va sirt to‘lqin potensiallari yig‘indisiga teng ekan. Keyinchalik biz bu formulani turli xil o‘chovlarda batafsilroq muhokama etamiz.

3.6 Koshining klassik masalasi yechimini beruvchi formulalar va ularni tekshirish. To‘lqinlarning diffuziyasi

4-paragrafning 2-bandidagi, 5- va 6-paragraflardagi teoremalardan quyidagi teorema kelib chiqadi.

T e o r e m a. $n = 3,\ 2$ larda $f \in C^2(\mathbb{R}^n \times [0,\infty))$, $u_0 \in C^3(\mathbb{R}^n)$ va $u_1 \in C^2(\mathbb{R}^n)$; $n = 1$ da $f_0 \in C^1(\mathbb{R} \times [0,\infty))$, $u_0 \in C^2(\mathbb{R})$ va $u_1 \in C^1(\mathbb{R})$ bo‘lsin. U holda Koshining (48), (49) klassik masalasi yechimi mavjud, yagona va quyidagi

$n = 3$ da *Kirxgof formulasi*

$$u(x,t) = \frac{1}{4\pi a^2} \int\limits_{U_x^{at}} \frac{f\left(\xi, t - \left|\frac{x-\xi}{a}\right|\right)}{|x-\xi|} d\xi +$$

$$+\frac{1}{4\pi a^2 t}\int\limits_{\Sigma_x^{at}} u_1(\xi)ds+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\Big[\frac{1}{t}\int\limits_{\Sigma_x^{at}} u_0(\xi)ds\Big]; \tag{55}$$

$n = 2$ da *Puasson formulasi*

$$u(x,t)=\frac{1}{2\pi a}\int\limits_0^t\int\limits_{K_x^{a(t-\tau)}}\frac{f(\xi,\tau)d\xi d\tau}{\sqrt{a^2(t-\tau)^2-|x-\xi|^2}}+$$

$$+\frac{1}{2\pi a}\int\limits_{K_x^{a(t-\tau)}}\frac{u_1(\xi)d\xi}{\sqrt{a^2t^2-|x-\xi|^2}}++\frac{1}{2\pi a}\frac{\partial}{\partial t}\int\limits_{K_x^{a(t-\tau)}}\frac{u_0(\xi)d\xi}{\sqrt{a^2t^2-|x-\xi|^2}}; \tag{56}$$

$n = 1$ da *Dalamber formulasi*

$$u(x,t)=\frac{1}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau+$$

$$+\frac{1}{2a}\int\limits_{x-at}^{x+at} u_1(\xi)d\xi+\frac{1}{2}\left[u_0(x+at)+u_0(x-at)\right] \tag{57}$$

bilan ifodalanadi. Shuningdek, bu yechimlar $f,\ u_0$ va u_1 berilganlarga uzluksiz bog‘liqdir.

Endi (53) yoki (54) formula orqali ifodalanuvchi yechimning fizik ma’nosini o‘rganamiz.

Avvalo, yig‘ma $F*H_n$ ko‘rinishida ifodalanuvchi $u(x,t)$ yechim x nuqtada va $t>0$ vaqtda

$$F(\xi,\tau)*H_n(x-\xi,t-\tau)$$

"elementar" ta’sirlarning superpozitsiyasidan iborat bo‘ladi.

Faraz qilamiz, $F(x,t)$ finit funksiya bo‘lsin. U holda $u(x,t)-$ ham finit funksiya va uning tashuvchisi (ξ,τ) nuqta $F(\xi,\tau)$ funksiyaning tashivchi to‘plamida o‘zgarganda $H_n(x-\xi,t-\tau)$ funksiyaning tashivchi to‘plamiga tegishli bo‘ladigan (x,t) nuqtalar to‘plamlarining birlashmasidan iboratdir, ya’ni

$$suppu=\bigcup_{(\xi,\tau)\in suppF} suppH_n(x-\xi,t-\tau).$$

Boshqacha aytganda, finit bo‘lgan $F(x,t)$ to‘lqin tarqatish "manba"lari har bir tayin t lar uchun fazoning chegaralangan sohasida noldan farqli yechimning hosil bo‘lishiga sabab bo‘ladi.

Bu jarayonning qanday kechishi $H_n(x,t)$ fundamental yechim tashuvchisining tuzilishiga bog‘liq va turli fazolarda turlichadir.

$n = 3$ bo‘lsin. Bu holda

$$H_3(x,t) = \frac{\theta(t)}{4\pi a^2 t}\delta(at - |x|)$$

bo‘lib, bu funksiyaning fazoviy tashuvchisi radiusi at bo‘lgan sferadan iborat. Shu sababli, $H_3(x-\xi, t-\tau)$ ning tashuvchisi markazi x nuqtada va radiusi $a|t-\tau|$ bo‘lgan sferadir. Bundan va yuqoridagi mulohazalardan, $u(x,t)$ funksiyaning tashuvchisi $(\xi,\tau) \in supp\ F$ bo‘lganda bunday sferalarning birlashmasidan iborat bo‘lishi kelib chiqadi. Faraz qilaylik, x nuqta $supp\ F$ dan tashqarida, ya’ni

$$x \bar{\in} supp\ f \cup supp\ u_0 \cup supp\ u_1$$

(boshqacha aytganda, bu nuqtada tashqi kuchlar va boshlang‘ich vaqtda to‘lqin yo‘q). U holda yetarlicha kichik t vaqtlarda $u(x,t) = 0$; to‘lqin x nuqtaga t vaqtning

$$a|t-\tau| \geq \inf_{(\xi,\tau)\in suppF} |x-\xi|$$

shartni qanoatlantiruvchi qiymatlarida yetib keladi. Bu t qiymatlarning eng kichigiga to‘lqin *old frontining* x nuqtadan o‘tish vaqti deyiladi. Ravshanki,

$$a|t-\tau| \geq \sup_{(\xi,\tau)\in suppF} |x-\xi|$$

shart bajarilganda x nuqta tinch holatda, ya’ni bu nuqtadan to‘lqin o‘tib ketgan bo‘ladi. Bu t qiymatlarning eng kichigiga to‘lqinning *orqa frontining* x nuqtadan o‘tish vaqti deyiladi. Bundan esa yechimning old va orqa frontlarining o‘tish vaqti oralig‘ida noldan farqli ekanligi kelib chiqadi. Shunday qilib, $n = 3$ bo‘lganda aniq old va orqa frontlarga ega bo‘lgan to‘lqinlarning tarqalishiga ega bo‘lamiz. Bu holat *Gyuygens prinsipi* deyiladi.

$n = 2$ bo'lgan holda

$$H_2(x,t) = \frac{\theta(at - |x|)}{2\pi a\sqrt{a^2t^2 - |x|^2}},$$

$x = (x_1, x_2) \in \mathbb{R}^2,\ t > 0$ fundamental yechimning fazoviy tashuvchisi $t > 0$ vaqtda markazi $x = 0$ nuqtada va radiusi at ga teng bo'lgan doiradan iborat bo'ladi. Demak, $suppH_2(x - \xi, t - \tau)-$ markazi x nuqtada va radiusi $a|t-\tau|$ bo'lgan doira. Bu holda $u(x,t)$ funksiyaning tashuvchisi (ξ, τ) nuqtalar $suppF$ to'plamga tegishli bo'lganda bunday doiralarning birlashmasidan iboratdir. Yuqoridagi kabi fikr yuritib, $suppF$ to'plamga tegishli bo'lmagan x nuqtalarda to'lqinning old frontining mavjudligi va uning a tezlik bilan tarqalishiga ishonch hosil qilamiz. Uch o'lchovli holdan farqli ravishda, old frontning orqa qismida hamma vaqt ham to'lqin mavjud bo'ladi. Shunday qilib, aniq old frontga ega bo'lgan orqa frontga esa ega bo'lmagan to'lqin tarqalishi ro'y beradi. Bunday holga *to'lqinlarning diffuziyasi* sodir bo'lishi deyiladi.

Endi $n = 1$ bo'lsin. Bu holda

$$H_1(x,t) = \frac{1}{2a}\theta(at - |x|),$$

$x \in \mathbb{R},\ t > 0$ fundamental yechimning fazoviy tashuvchisi $[-at, at]$ kesmadan iborat bo'ladi. Biroq ikki o'lchovli holdan farqli ravishda

$$\frac{\partial H_1(x,t)}{\partial t} = \frac{1}{2}\delta\left(at - |x|\right)$$

funksiyaning tashuvchisi ikkita $x = \pm at$ nuqtalardan iborat bo'ladi. Shuning uchun, bir o'lchovli holda to'lqinning old fronti hamma vaqt ham mavjud bo'lib, orqa frontning mavjud bo'lishi yoki yo'qligi boshlang'ich shartlarda berilgan funksiyalarga bog'liq. Haqiqatan ham, (54) formulada $f(x,t) = 0$, $u_1(x) = 0$ bo'lsa, yechim

$$u(x,t) = \frac{\partial H_1(x,t)}{\partial t} *_x u_0(x)$$

formula bilan ifodalanadi va bunday to'lqin har ikkala frontga ham ega bo'ladi. $f(x,t) \neq 0$ yoki $u_1(x) \neq 0$ bo'lsa, yechimda $H_1(x,t)$ funksiyaning o'zi qatnashib, to'lqin orqa frontga ega bo'lmaydi, ya'ni to'lqinlarning diffuziyasi sodir bo'ladi.

3.7 Issiqlik o‘tkazuvchanlik tenglamasi uchun Koshi masalasi

Issiqlik o‘tkazuvchanlik tenglamasi uchun Koshi masalasi yechimi ham oldingi paragrafdagi usul yordamida topilishi mumkin.

3.7.1 Issiqlik potensiali

1.3.2 paragrafda

$$H_n(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4a^2 t}}$$

funksiya issiqlik o‘tkazuvchanlik operatorining fundamental yechimi ekanligi ko‘rsatilgan edi. Bu funksiya uchun quyidagi xossalarning o‘rinli bo‘lishini osongina payqash mumkin: $H_n(x,t) \geq 0$, $H_n(x,t)|_{t<0} = 0$, $H_n(x,t) \in L_1^{loc}(\mathbb{R}^n)$ va $(x,t) \neq (0,0)$ nuqtalarda cheksiz differensiallanuvchi. Bundan tashqari, 1-bobning 7-paragrafida uchbu

$$\int\limits_{\mathbb{R}^n} H_n(x,t)dx = 1, \quad t > 0; \tag{58}$$

$$H_n(x,t) \to \delta(x), \quad t \to +0.$$

munosabatlarning bajarilishi ko‘rsatilgan edi.

Faraz qilaylik, $f(x,t) \in D'(\mathbb{R}^{n+1})$ funksiya $t < 0$ da nolga teng bo‘lsin.

T a’ r i f. $V = H_n * f$ umumlashgan funksiya issiqlik potensiali, f funksiya esa uning zichligi deyiladi.

Ma’lumki, V issiqlik potensiali $D'(\mathbb{R}^{n+1})$ da mavjud bo‘lib,

$$\frac{\partial V}{\partial t} = a^2 \Delta V + f(x,t)$$

tenglamani qanoatlantiradi.

$\mathcal{M}$ orqali $t < 0$ da nolga teng bo‘lgan va $\mathbb{R}^n \times [0,T]$ ($T > 0-$ixtiyoriy son) sohada chegaralangan funksiyalar sinfini belgilaymiz.

T e o r e m a. Agar $f \in \mathcal{M}$ bo'lsa, V issiqlik potensiali $\mathcal{M}$ funksiyalar sinfida mavjud va

$$V(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{f(\xi,\tau)}{\left[2a\sqrt{\pi(t-\tau)}\right]^n} e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}} d\xi d\tau \tag{59}$$

formula bilan ifodalanadi, V potensial

$$|V(x,t)| \leq t \sup_{\xi \in \mathbb{R}^n,\ \ 0 \leq \tau \leq t} |f(\xi,\tau)|, \qquad t > 0 \tag{60}$$

bahoni va $x \in \mathbb{R}^n$ uchun

$$|V(x,t)| \rightrightarrows 0,\ t \to +0 \tag{61}$$

boshlang'ich shartni qanoatlantiradi. Bundan tashqari, agar f funksiya $f(x,t) \in C^2(\mathbb{R}^n \times [0,+\infty))$ va uning ikkinchi tartibgacha bo'lgan hosilalari ixtiyoriy chekli $T > 0$ soni uchun $\mathbb{R}^n \times [0,T]$ sohada chegaralangan bo'lsa, u holda

$$V(x,t) \in C^2\left(\mathbb{R}^n \times [0,+\infty)\right) \cap C^1\left(\mathbb{R}^n \times [0,+\infty)\right)$$

bo'ladi.

Isbot. Agar

$$h(x,t) = \int_0^t \int_{\mathbb{R}^n} |f(\xi,\tau)| H_n(x-\xi, t-\tau) d\xi d\tau$$

funksiya $\mathbb{R}^{n+1}$ da lokal integrallanuvchi bo'lsa, u holda H_n va f funksiyalarning lokal integrallanuvchi ekanligidan ularning

$$H_n * f = \int_0^t \int_{\mathbb{R}^n} |f(\xi,\tau)| H_n(x-\xi, t-\tau) d\xi d\tau$$

yig'masi mavjud va $\mathbb{R}^{n+1}$da lokal integrallanuvchi bo'lishi kelib chiqadi.

Bu shartning bajarilishini tekshiramiz. $t < 0$ da $h = 0$ bo'lgani uchun h ning $t > 0$ da (60) bahoni qanoatlantirishini ko'rsatish yetarli. Bu esa (58) tenglik va integrallash tartibini o‘zgartirish haqidagi Fubini teoremasidan kelib chiqadi. Bunga asosan

$$h(x,y) \leq \sup_{\xi\in\mathbb{R}^n,\ 0\leq\tau\leq t} |f(\xi,\tau)| \int\limits_0^t \int\limits_{\mathbb{R}^n} H_h(x-\xi, t-\tau) d\xi d\tau =$$

$$= t \sup_{\xi\in\mathbb{R}^n,\ 0\leq\tau\leq t} |f(\xi,\tau)|,\ t > 0. \tag{62}$$

Shunday qilib, $V = H_n * f$ issiqlik potensial (59) formula bilan ifodalanadi. $|V| \leq h$ bo‘lgani uchun $t < 0$ da $V = 0$ va (62) tengsizlikka asosan (60) bahoni qanoatlantiradi. Bu esa $V \in \mathcal{M}$ bo‘lishini bildiradi. (60) bahodan V ning (61) boshlang‘ich shartni qanoatlantirishi kelib chiqadi.

(59) formulada integrallash o‘zgaruvchilarini

$$\xi = x - 2a\sqrt{s}y,\ \tau = t - s$$

kabi almashtirib, uni

$$V(x,t) = \frac{1}{\pi^{n/2}} \int\limits_0^t \int\limits_{\mathbb{R}^n} f(x - 2a\sqrt{s}y, t-s) e^{-|y|^2} dyds \tag{59'}$$

ko‘rinishda yozib olamiz.

Faraz qilaylik, $f(x,t) \in C^2(\mathbb{R}^n \times [0,+\infty))$ va bu funksiya ikkinchi tartibgacha barcha xosilalar bilan $\mathcal{M}$ sirtga tegishli bo‘lsin. U holda matematik tahlil kursidan ma’lum bo‘lgan parametrga bog‘liq integralning uzluksizligi va differensiallanuvchanligi to‘g‘risidagi teoremaga ko‘ra, (59′) formula va

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{\pi^{n/2}} \int\limits_{\mathbb{R}^n} \frac{\partial f}{\partial t}(x - 2a\sqrt{s}y, t-s) e^{-|y|^2} dyds +$$

$$+ \frac{1}{\pi^{n/2}} \int\limits_{\mathbb{R}^n} f(x - 2a\sqrt{t}y, +0) e^{-|y|^2} dy$$

tengsizlikdan $V,\ V_{x_i},\ V_t,\ V_{x_ix_j},\ V_{x_it}$ funksiyalarning $t \geq 0$ da, V_{tt} funksiyaning $t > 0$ da uzluksizligi kelib chiqadi. Teorema isbotlandi.

3.7.2 Sirt issiqlik potensiali

$f(x,t) = u_0(x)\cdot\delta(t)$ zichlikka ega $V^{(0)}$ issiqlik potensiali *sirt issiqlik potensiali* (u_0 zichlikka ega oddiy qatlam) deyiladi va

$$V^{(0)} = H_n * [u_0(x)\cdot\delta(t)] = H_n(x,t) * u_0(x)$$

kabi aniqlanadi.

T e o r e m a. Agar $u_0(x)$ funksiya $\mathbb{R}^n$ da chegaralangan bo'lsa, $V_{(0)}$ sirt issiqlik potensiali $\mathcal{M}$ da mavjud, ushbu

$$V^{(0)}(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}\int\limits_{\mathbb{R}^n} u_0(\xi)e^{-\frac{|x-\xi|^2}{4a^2t}}d\xi \tag{63}$$

formula bilan ifodalanadi va

$$|V^{(0)}(x,t)| \leq \sup_{\xi\in\mathbb{R}^n}|u_0(\xi)|,\ t>0 \tag{64}$$

tengsilikni qanoatlantiradi. Shuningdek, $u_0(x)\in C(\mathbb{R}^n)$ bo'lsa, $V^{(0)}\in C\left(\mathbb{R}^n\times[0,+\infty)\right)$ bo'ladi va

$$V^{(0)}|_{t=0} = u_0(x) \tag{65}$$

boshlang'ich shart bajariladi.

Isbot. Ushbu

$$h(x,t) = \int\limits_{\mathbb{R}^n}|U_0(\xi)|H_n(x-\xi,t)d\xi$$

funksiya $t<0$ da nolga teng va $t>0$ da (58) tenglikka asosan

$$h(x,t) \leq \sup_{\xi\in\mathbb{R}^n}|U_0(\xi)|\int\limits_{\mathbb{R}^n}H_n(x-\xi,t)d\xi = \sup_{\xi\in\mathbb{R}^n}|U_0(\xi)|$$

bo'lgani uchun uning $\mathbb{R}^{n+1}$da lokal integrallanuvchi ekanligi kelib chiqadi. Bundan esa

$$V^{(0)} = H_n(x,t) * u_0(x)$$

potensialning

$$V^{(0)}(x,t) = \int\limits_{\mathbb{R}^n}u_0(\xi)H_n(x-\xi,t)d\xi \tag{63'}$$

tenglik yoki (63) formula bilan ifodalanishi, $t < 0$ da $V^{(0)} = 0$ bo‘lishi va $|V^0| \leq h$ ga asosan (64) bahoni qanoatlantirishiga ishonch hosil qilamiz. Bular $V^{(0)} \in \mathcal{M}$ ekanini bildiradi.

Endi faraz qilaylik, $u_0 - \mathbb{R}^n$ da uzluksiz va chegaralangan funksiya bo‘lsin. U holda $V^{(0)} \in C(\mathbb{R}^n \times [0, +\infty))$ va (65) tenglikning bajarilishini ko‘rsatamiz. $\varepsilon > 0$ ixtiyoriy tayin son, $(x,t) \to (x_0, t_0)$, $t > 0$ bo‘lsin. $u_0(x)$ funksiya uzluksiz bo‘lgani uchun shunday $\delta > 0$ son mavjudki, $|\xi - x_0| < 2\delta$ tengsizlikni qanoatlantiruvchi ξ lar uchun $|u_0(\xi) - u_0(x_0)| < \varepsilon$ bo‘ladi. Shu sababli agar $|x - x_0| < \delta$ va $|y| < \delta$ bo‘lsa, $|x - y - x_0| < 2\delta$ bo‘ladi va (58), (63′) larga asosan $t > 0$ da

$$|V^0(x,t) - u_0(x_0)| \leq \int\limits_{\mathbb{R}^n} |u_0(\xi) - u_0(x_0)| H_n(x - \xi, t) d\xi =$$

$$\int\limits_{|y| \leq \delta} |u_0(x-y) - u_0(x_0)| H_n(y,t) dy + \int\limits_{|y| > \delta} |u_0(x-y) - u_0(x_0)| H_n(y,t) dy \leq$$

$$\leq \varepsilon + \frac{2}{\pi^{n/2}} \sup_{\xi \in \mathbb{R}^n} |u_0(\xi)| \int\limits_{|\xi| > \frac{\delta}{2a\sqrt{t}}} e^{-|\xi|^2} d\xi \tag{66}$$

munosabatlar hosil bo‘ladi. (66) ning o‘ng tomonidagi ikkinchi qo‘shiluvchini t ni nolga intiltirib, ε dan kichik qilish mumkin. Shu sababli biror $\delta_1 \leq \delta$ uchun

$$|V^0(x,t) - u_0(x_0)| < 2\varepsilon, \quad |x - x_0| < \delta_1, \quad |t| < \delta_1$$

tengsizliklar o‘rinli. Teorema isbotlandi.

3.7.3 Issiqlik o‘tkazuvchanlik tenglamasi uchun umumlashgan Koshi masalasi

To’lqin tarqalish tenglamasi uchun Koshi masalasini yechimda foydalanilgan usul bu yerda issiqlik o‘tkazuvchanlik tenglamasi uchun

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(x,t), \tag{67}$$

$$u|_{t=0} = u_0(x) \tag{68}$$

Koshi masalasining yechimini topishga qo‘llaniladi. Bunda $f(x,t) \in C$ $(\mathbb{R} \times [0,+\infty))$ va $u_0(x) \in C(\mathbb{R}^n)$ deb hisoblaymiz.

Faraz qilaylik (67), (68) masalaning klassik yechimi mavjud bo‘lsin. Bu $u(x,t) \in C^2\left(\mathbb{R}^n \times (0,+\infty)\right) \cap C\left(\mathbb{R}^n \times [0,+\infty)\right)$ bo‘lib, $u(x,t)$ funksiya (67) tenglamani $t > 0$ da qanoatlantirishini va $t \to 0$ da (68) boshlang‘ich shart bajarilishini bildiradi.

u va f funksiyalarni $t < 0$, $x \in \mathbb{R}^n$ sohada nolga teng deb davom ettiramiz. Ravshanki, davom ettirilgan $\widetilde{u}$ va $\widetilde{f}$ funksiyalar $\mathbb{R}^{n+1}$ da

$$\frac{\partial \widetilde{u}}{\partial t} = a^2 \Delta \widetilde{u} + \widetilde{f}(x,t) + u_0(x)\delta(t) \tag{69}$$

issiqlik o‘tkazuvchanlik tenglamasini qanoatlantiradi. (67), (68) Koshi masalasining klassik yechimlari (69) tenglamaning $t < 0$ da nolga aylanuvchi yechimlari ichida bo‘ladi.

Xuddi oldingi paragrafdagi kabi, quyidagi issiqlik o‘tkazuvchanlik tenglamasi uchun umumlashgan Koshi masalasining ta‘rifini keltiramiz.

T a ’ r i f. Issiqlik tarqalish tenglamasi uchun umumlashgan Koshi masalasi deb, ushbu

$$\left(\frac{\partial}{\partial t} - a^2\Delta\right) u(x,t) = f(x,t) + u_0(x) \cdot \delta(t)$$

tenglikni umumlashgan funksiyalar ma’nosida qanoatlantiruvchi $u(x,t)$ funksiyani topish masalasiga aytiladi.

Ushbu paragrafdagi teoremalardan quyidagi natija kelib chiqadi:

N a t i j a.

$$F(x,t) = f(x,t) + u_0(x) \cdot \delta(t), \ \ f \in \mathcal{M}$$

va $u_0 - \mathbb{R}^n$ da chegaralangan funksiya bo‘lsin. U holda issiqlik o‘tkauvchanlk tenglamasi uchun umumlashgan Koshi masalasining yagona yechimi mavjud, $\mathcal{M}$ sinfga tegishli va

$$u(x,t) =$$

$$= \frac{\theta(t)}{(2a\sqrt{\pi t})^n} \int\limits_{\mathbb{R}^n} u_0(\xi) e^{-\frac{|x-\xi|^2}{4a^2 t}} d\xi + \int\limits_0^t \int\limits_{\mathbb{R}} \frac{f(\xi,\tau)}{[2a\sqrt{\pi(t-\tau)}]^n} e^{-\frac{|x-\xi|^2}{4a^2(t-\tau)}} d\xi d\tau$$

formula bilan ifodalanadi.

Bu yechim f va u_0 funksiyalardan uzluksiz bog‘liq. Agar $f(x,t)$ funksiya qo‘shimcha ravishda $f \in C^2(\mathbb{R}^n \times [0,+\infty))$ bo‘lib, ikkinchi tartibli barcha hosilalari bilan $\mathcal{M}$ sinfga tegishli va $u_0 \in C(\mathbb{R}^n)$ bo‘lsa, u holda $u(x,t)$ klassik yechimdir.

Quyida muhim hisoblangan yana bir ta’rifni keltirib o‘tamiz.

T a ’ r i f. Issiqlik tarqalish tenglamasi uchun Koshi masalasining *Grin funksiyasi* (x ga nisbatan umumlashgan funksiya, t esa parametr) deb ushbu

$$\left(\frac{\partial}{\partial t} - a^2\Delta\right)\mathcal{G}(x,t) = 0, \quad \mathcal{G}|_{t=0} = \delta(x)$$

tengliklarni qanoatlantiruvchi $\mathcal{G}(x,t)$ funksiyaga aytiladi.

E s l a t m a. Yuqorida olingan fundamental $H_n(x,t)$ yechimdan

$$\mathcal{G}(x,t) = \frac{1}{\left(2a\sqrt{\pi t}\right)^n} e^{-\frac{|x|^2}{4a^2 t}}$$

ekanligi kelib chiqadi.

4-Bob. Giperbolik tenglamalar

4.1 Tor tebranish tenglamasi uchun Koshi masalasi. Dalamber yechimi

Bir jinsli cheksiz tor tebranish

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0 \tag{1}$$

tenglamasini qaraymiz, bu yerda $u = u(x,t)$ - ikki oʻzgaruvchili funksiya va c - musbat oʻzgarmas son. (1) ga bir oʻlchovli toʻlqin tenglamasi ham deb aytiladi. Bu tenglamada

$$\xi = x - ct, \quad \eta = x + ct$$

xarakteristik oʻzgaruvchilarga oʻtsak, u ushbu

$$\frac{\partial^2 \tilde{u}(\xi,\eta)}{\partial\xi\partial\eta} = 0$$

koʻrinishni oladi. Bundan esa $u_\eta(\xi,\eta) = g_0(\eta)$. Bu yerda g_0- η oʻzgaruvchining ixtiyoriy funksiyasi. Bu tenglikni η boʻyicha integrallab

$$\tilde{u}(\xi,\eta) = f(\xi) + g(\eta)$$

yechimga ega boʻlamiz, bunda $f(\xi)-$ ixtiyoriy funksiya va

$$g(x) = \int g_0(\eta)d\eta.$$

Endi (x,t) oʻzgaruvchilarga qaytib, (1) tenglamaning

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2}$$

umumiy yechimini olamiz.

Tabiiyki, (2) tenglik bilan aniqlangan $u(x,t)$ funksiya (1) tenglamaning yechimi bo'lishi uchun f va g funksiyalar $C^2(\mathbb{R})$ sinfga tegishli bo'lishi kerak.

Koshi masalasi: (1) tenglamaning ushbu

$$u(x,0) = \varphi(x),$$

$$u_t(x,0) = \psi(x)$$

shartlarni qanoatlantiruvchi $u(x,t) \in C^2(\mathbb{R} \times \mathbb{R}_+)$, $\mathbb{R}_+ = \{t \in \mathbb{R};\ t > 0\}$ yechimi topilsin. Bunda $\varphi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$.

T e o r e m a. Koshi masalasining $C^2(\mathbb{R} \times \mathbb{R}_+)$ funksiyalar sinfiga tegishli bo'lgan yechimi mavjud va u

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds \tag{3}$$

formula bilan beriladi.

Isbot. Faraz qilaylik, Koshi masalasining yechimi mavjud bo'lsin, u holda (2) ga asosan

$$u(x,0) = f(x) + g(x) = \varphi(x), \tag{4}$$

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x). \tag{5}$$

(4) tenglikdan

$$f'(x) + g'(x) = \varphi'(x)$$

ga ega bo'lamiz. Bu va (5) formuladan

$$f'(x) = \frac{\varphi'(x) + \frac{\psi(x)}{c}}{2},$$

$$g'(x) = \frac{\varphi'(x) - \frac{\psi(x)}{c}}{2}$$

tengliklar kelib chiqadi.

Bulardan

$$f(x) = \frac{\varphi(x)}{2} + \frac{1}{2c}\int\limits_0^x \psi(s)ds + C_1,$$

$$g(x) = \frac{\varphi(x)}{2} - \frac{1}{2c}\int_0^x \psi(s)ds + C_2$$

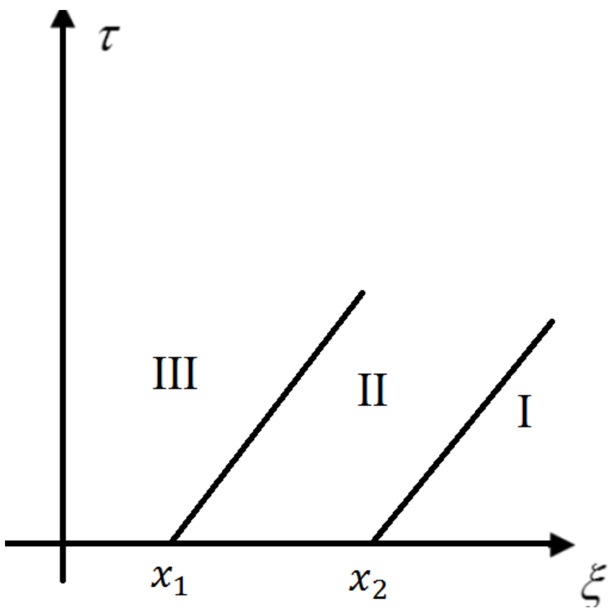
ekanligini topamiz. $C_1,\ C_2$ o'zgarmaslar, (4) ga ko'ra

$$C_1 + C_2 = f(x) + g(x) - \varphi(x) = 0$$

tenglikni qanoatlantiradi. $f(x)$ va $g(x)$ funksiyalar uchun hosil qilingan formulalarni (2) tenglikka qo'yib, (3) ni olamiz. (3) formulaning o'ng tomoni bilan aniqlanuvchi $u(x,t)$ funksiya Koshi masalasining yechimi ekanligiga to'g'ridan-to'g'ri hisoblashlarni bajarib ishonch hosil qilish mumkin.

4.2 Koshi masalasi yechimining fizik ma'nosi.

Tor tebranish tenglamasi uchun Koshi masalasining (3) formula bilan aniqlangan yechimi $t = 0$ vaqtda torning boshlang'ich tezligi va boshlang'ich siljishi ma'lum bo'lganda $t > 0$ vaqtlarda uning tebranish jarayonini ifodalaydi. Tor tebranish tenglamasining (2) umumiy yechimning fizikaviy xususiyatiga asosan (3) formula ikkita to'g'ri to'lqinning yig'indisidan iborat, ya'ni $f(x + ct) + g(x - ct)$, bulardan biri c tezlik bilan o'ng tomonga ikkinchisi esa shu tezlik bilan chap tomonga tarqaladi.



9-chizma. D soha

Bu holda

$$f(x+ct) = \frac{1}{2}\varphi(x+ct) + \Psi(x+ct),$$

$$g(x-ct) = \frac{1}{2}\varphi(x-ct) - \Psi(x-ct)$$

tengliklar o‘rinli bo‘ladi. Bu yerda

$$\Psi(\xi) = \frac{1}{2c}\int\limits_0^{\xi} \psi(s)ds.$$

(x,t) o‘zgaruvchilar tekisligida $x+ct = c_1 = const$ va $x-ct = c_2 = const$ to‘g‘ri chiziqlar (1) tenglamaning xarakteristikalari bo‘lgani uchun $u(x,t) = \varphi(x+at)$ funksiya $x+at = c_1$ xarakteristika bo‘ylab o‘zgarmas va bu qiymat $\varphi(c_1)$ ga teng. Xuddi shunday $u(x,t) = \varphi(x-at)$ funksiya $x-ct = c_2 = const$ xarakteristika bo‘ylab o‘zgarmasdir.

Faraz qilaylik, $\varphi(x)$ funksiya biror (x_1, x_2) intervalda noldan farqli va intervaldan tashqarida nolga teng, ya’ni finit funksiya bo‘lsin. $(x_1, 0)$ va $(x_2, 0)$ nuqtalardan (1) tenglamaning $x-ct = c_1$ va $x-ct = c_2$ xarakteristikalarini o‘tkazamiz. Bu xarakteristikalar $t > 0$ yarim tekislikni uchta I, II va III bo‘lakka bo‘ladi (5-chizma).

$u(x,t) = \varphi_0(x-ct)$ funksiya $II : x_1 < x-ct < x_2$ sohada noldan farqli, bunda $x-ct = x_1$ va $x+ct = x_2$ xarakteristikalar o‘ng tomonga c tezlik bilan tarqalayotgan to‘g‘ri to‘lqinning oldingi va orqa fronti deb yuritiladi.

Faraz qilaylik, $M = (x_0, t_0)$ nuqta $t > 0$ yarim tekislikda tayin nuqta bo‘lsin. Bu nuqtadan (1) tenglamaning $x-ct = x_0 - ct_0$ va $x+ct = x_0 + ct_0$ xarakteristikalarini o‘tkazamiz. Bu xarakteristikalar Ox o‘qi bilan $P = (x_1, 0) = (x_0 - ct_0, 0)$ va $Q = (x_2, 0) = (x_0 + ct_0, 0)$ nuqtalar kesishadi. Tor tebranish tenglamasi (2) umumiy yechimining M nuqtadagi qiymati $u(x_0, y_0) = g(x_1) + f(x_2)$ ga teng, ya‘ni $f(x)$ va $g(x)$ funksiyalarning qiymati mos ravishda MPQ uchburchak asosining $(x_1, 0)$ va $(x_2, 0)$ uchlaridagi qiymatlari orqali ifodalanadi (6-chizma).

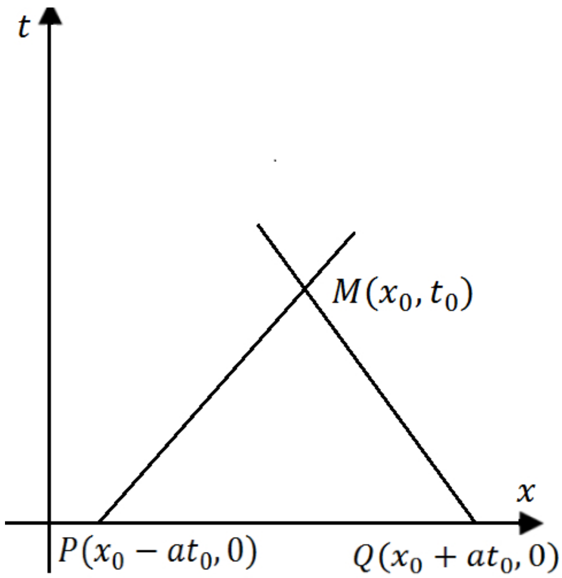
MP, MQ xarakteristikalar va Ox o‘qida PQ kesmadan tashkil topgan MPQ uchburchak M nuqtaning *xarakteristik uchburchagi* deyiladi.

Koshi masalasining yechimini ifidalovchi (3) formuladan torning x_0 nuqtasining t_0 vaqtdagi $u(x_0, t_0)$ siljishi mos ravishda boshlang‘ich holat va boshlang‘ich tezlikning P, Q nuqtalar va PQ kesmadagi qiymatlariga bog‘liq ekanligi ko‘rinadi. (3) formulani quyidagi

$$u(M) = \frac{\varphi(P) + \varphi(Q)}{2} + \frac{1}{2c}\int\limits_P^Q \varphi_1(s)ds$$

ko‘rinishda yozish mumkin. PQ kesmadan tashqarida berilgan boshlang‘ich shartlar $u(x, t)$ yechimning M nuqtadagi qiymatiga hech qanday ta‘sir ko‘rsatmaydi.

Demak, tor tebranish tenglamasi uchun boshlang‘ich shartlar butun o‘qda emas, balki PQ kesmada berilgan bo‘lsa, Koshi masalasining yechimi MPQ xarakteristik uchburchakning ichida aniqlanadi.



10-chizma. D soha tasviri.

Yuqoridagi mulohazalardan Koshi masalasining $u(x, t)$ yechimi (x_0, t_0) nuqtada $\varphi(x)$ funksiyaning $[x_0 - ct_0, x_0 + ct_0]$ kesma chetki nuqtalaridagi qiymatiga, $\psi(x)$ funksiyaning esa $[x_0 - ct_0, x_0 + ct_0]$ kesmaning barcha nuqtalardagi qiymatiga bog‘liq bo‘lishi kelib chiqadi. $[x_0 - ct_0, x_0 + ct_0]$ kesma yechimning boshlang‘ich funksiyalarga *bog‘liqlik intervali* deyiladi.

4.3 Bir jinsli bo‘lmagan tenglama. Dyuamel prinsipi. Dalamber formulasi. Yechimning berilganlarga uzluksiz bog‘liqligi

Bir jinsli bo‘lmagan cheksiz tor tebranish

$$\frac{1}{c^2}u_{tt} - u_{xx} = f(x,t) \tag{6}$$

tenglamasining

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topamiz. Buning uchun masalani ikkiga ajratamiz:

1). Bir jinsli tenglama uchun bir jinsli bo‘lmagan boshlang‘ich shartli masala

$$\frac{1}{c^2}v_{tt} - v_{xx} = 0,$$
$$v(x,0) = \varphi(x),\ v_t(x,0) = \psi(x).$$

2). Bir jinsli bo‘lmagan tenglama uchun bir jinsli boshlang‘ich shartli masala

$$\frac{1}{c^2}w_{tt} - w_{xx} = f(x,t),$$
$$w(x,0) = 0,\ w_t(x,0) = 0.$$

U holda, ravshanki, $u(x,t) = v(x,t) + w(x,t)$.

Birinchi masalaning yechimi (3) formula bilan ifodalanadi:

$$v(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(s)ds.$$

Ikkinchi masalaning yechimini topish uchun biror $p(x,t,\tau)$ ($\tau-$ parametr) funksiyaga nisbatan ushbu

$$\frac{1}{c^2}p_{tt} - p_{xx} = 0,$$
$$p(x,t,\tau)\Big|_{t=\tau} = 0,\ p_t(x,t,\tau)\Big|_{t=\tau} = f(x,\tau)$$

yordamchi masalani qaraymiz (*Dyuamel prinsipi*). Bu masalaning yechimi (3) formulaga ko'ra

$$p(x,t,\tau) = \frac{1}{2c} \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi.$$

Shuning uchun

$$w(x,t) = c^2 \int\limits_0^t p(x,t,\tau)d\tau = \frac{c}{2} \int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi d\tau$$

formula bilan ifodalangan $w(x,t)$ funksiya ikkinchi masalaning yechimi bo'lishiga bevosita hisoblashlar yordamida ishonch hosil qilish mumkin.

Shunday qilib, bir jinsli bo'lmagan cheksiz tor tebranish tenglamasi uchun Koshi masalasining yechimi

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c} \int\limits_{x-ct}^{x+ct} \psi(s)ds + \frac{c}{2} \int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi d\tau \tag{7}$$

formula bilan berilar ekan.

Eslatib o'tamiz, (7) formulaga Dalamber formulasi deyilib, u 3-bobning 6-paragrafida boshqacha usul bilan keltirib chiqarilgan edi.

Koshi masalasi yechimining yagonaligi (7) formuladan kelib chiqadi. Haqiqatan, aynan bir xil boshlang'ich shartlarni qanoatlantiruvchi (6) tenglamaning ikkita yechimi mavjud deb faraz qilib va (7) formula yordamida bu yechimlarning ayirmasini tuzib olib, uning aynan nol ekanligiga, ya'ni bu yechimlarning ustma-ust tushishiga ishonch hosil qilamiz.

(7) formula bilan aniqlangan $u(x,t)$ yechim (6) tenglama o'ng tomoni va boshlang'ich shartlarga uzluksiz bog'liq, ya'ni turg'un bo'ladi.

T e o r e m a. Faraz qilaylik, $u(x,t)$ va $v(x,t)$ funksiyalar mos ravishda

$$\frac{1}{c^2}u_{tt} - u_{xx} = f_1(x,t)$$

$$u(x,0) = \varphi_1(x), \quad u_t(x,0) = \psi_1(x)$$

va

$$\frac{1}{c^2}v_{tt} - v_{xx} = f_2(x,t)$$

$$v(x,0) = \varphi_2(x), \quad v_t(x,0) = \psi_2(x)$$

masalalarning yechimlari bo‘lsin. U holda ixtiyoriy $\varepsilon > 0$ va $T > 0$ sonlar uchun shunday $\delta > 0$ $(\delta = \delta(\varepsilon, T))$ son topiladiki,

$$|\varphi_1(x) - \varphi_2(x)| < \delta, \quad |\psi_1(x) - \psi_2(x)| < \delta, \quad x \in \mathbb{R},$$

$$|f_1(x,t) - f_2(x,t)| < \delta, \quad x \in \mathbb{R}, \quad t \le T \tag{8}$$

tengsizliklardan barcha $(x,t) \in (\mathbb{R} \times [0,T])$ lar uchun

$$|u(x,t) - v(x,t)| < \varepsilon \tag{9}$$

tengsizlik kelib chiqadi.

Isbot. $u(x,t)$ va $v(x,t)$ yechimlar uchun (7) formula va (8) tengsizlikdan foydalanib, quyidagilarni yozamiz:

$$|u(x,t) - v(x,t)| \le \frac{1}{2}|\varphi_1(x-ct) - \varphi_2(x-ct)| + \frac{1}{2}|\varphi_1(x+ct) - \varphi_2(x+ct)| +$$

$$+\frac{1}{2c}\int\limits_{x-ct}^{x+ct} |\psi_1(\xi) - \psi_2(\xi)|d\xi + \frac{c}{2}\int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} |f_1(\xi,\tau) - f_2(\xi,\tau)|d\xi d\tau \le$$

$$\le \delta + \frac{\delta}{2c}\int\limits_{x-ct}^{x+ct} d\xi + \frac{\delta c}{2}\int\limits_0^t \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi d\tau \le$$

$$\le \delta + \delta t + \frac{\delta c}{2}\int\limits_0^t 2c(t-\tau)d\tau = \delta(1+t) + \frac{\delta}{2}c^2t^2 \le \delta(1+T) + \frac{\delta}{2}c^2T^2.$$

Agar $\delta = 2\varepsilon\left[2 + 2T + c^2T^2\right]^{-1}$ deb olsak, (9) tengsizlik o‘rinli bo‘ladi. Bu esa Koshi masalasi yechimining berilganlarga uzluksiz bog‘liqligini ko‘rsatadi. Demak, tor tebranish tenglamasi uchun Koshi masalasi korrekt qo‘yilgan.

4.4 Yarim chegaralangan soha va davom ettirish usuli

Yarim chegaralangan $x \geq 0$ to‘g‘ri chiziqda bir jinsli torning tebranishi haqidagi masalani, ya’ni $x > 0$, $t > 0$ sohada (6) tenglamaning $u(0,t) = \mu(t)$ yoki $u_x(0,t) = \nu(t)$ yoki

$$\alpha u(0,t) + \beta u_x(0,t) = \nu(t), \quad t \geq 0 \tag{10}$$

chegaraviy va

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \geq 0 \tag{11}$$

boshlang‘ich shartlarni qanoatlantiruvchi $u(x,t)$ yechimini topish masalasini o‘rganamiz. Soddalik uchun dastlab $\mu(t) = 0$ deb olamiz, ya‘ni cheksiz uzunlikdagi torning $x = 0$ uchi maxkamlangan bo‘lsin.

Ma’lumki, bir jinsli tor tebranish tenglamasi uchun Koshi masalasining $u(x,t)$ yechimi (3) formula blan ifodalanadi. Bu yechim uchun quyidagi lemma o‘rinli.

L e m m a. Agar $\varphi(x)$ va $\psi(x)$ funksiyalar $x = 0$ nuqtaga nisbatan

1) toq funksiyalar bo‘lsa, $u(0,t) = 0$,

2) juft funksiyalar bo‘lsa, $u_x(0,t) = 0$ bo‘ladi.

Isbot. 1) $\varphi(x)$ va $\psi(x)$ funksiyalar $x = 0$ nuqtaga nisbatan toq bo‘lsin, ya’ni $\varphi(-x) = -\varphi(x)$, $\psi(-x) = -\psi(x)$. U holda (3) formulaga ko‘ra

$$u(0,t) = \frac{1}{2}\left[\varphi(-ct) + \varphi(ct)\right] + \frac{1}{2}\int_{-ct}^{ct} \psi(s)ds = 0$$

bo‘ladi, chunki juft funksiyaning hosilasi toq funksiyadir.

2) $\varphi(x)$ va $\psi(x)$ funksiyalar $x = 0$ nuqtaga nisbatan juft bo‘lsin, ya’ni $\varphi(-x) = \varphi(x)$, $\psi(-x) = \psi(x)$. U holda yuqoridagiga o‘xshash ravishda

$$u_x(0,t) = \frac{1}{2}\left[\varphi'(-ct) + \varphi'(ct)\right] + \frac{1}{2c}\left[\psi(ct) - \psi(-ct)\right] = 0$$

ekanligiga ishonch hosil qilamiz.

Bu lemmaga tayanib, yarim chekli sohada (6) tenglama uchun qoʻyilgan $u(0,t) = 0$ chegaraviy va (11) boshlangʻich shartli masalaning yechimini topamiz.

$\varphi(x)$ va $\psi(x)$ funksiyalarni $x = 0$ nuqtaga nisbatan $x < 0$ sohaga toq davom ettiramiz va hosil boʻlgan funksiyalarni mos ravishda $\Phi(x)$ va $\Psi(x)$ orqali belgilaymiz. U holda

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0, \\ -\varphi(-x), & x < 0, \end{cases} \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0, \\ -\psi(-x), & x < 0. \end{cases}$$

Ravshanki, bu funksiyalar butun $\mathbb{R}$ sohada aniqlangan. Dalamber formulasiga koʻra

$$u(x,t) = \frac{1}{2}\left[\Phi(x-ct) + \Phi(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \Psi(s)ds, \ x \in \mathbb{R}, \ t \geq 0. \quad (12)$$

Lemmaning 1-qismiga koʻra $u(0,t) = 0$. Bundan tashqari bu funksiya $t = 0$ va $x > 0$ larda (11) shartlarni qanoatlantiradi. Demak, (12) dan

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds, & t \leq \frac{x}{c}, \quad x \geq 0, \\ \frac{1}{2}\left[\varphi(x+ct) - \varphi(x-ct)\right] + \frac{1}{2c}\int\limits_{ct-x}^{x+ct} \psi(s)ds, & t > \frac{x}{c}, \quad x > 0 \end{cases} \quad (13)$$

yechimga ega boʻlamiz.

Shunga oʻxshash $x = 0$ da $u_x(0,t) = 0$ chegaraviy shart berilsa, $\varphi(x)$ va $\psi(x)$ boshlangʻich funksiyalar juft davom ettiriladi:

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0, \\ \varphi(-x), & x < 0, \end{cases} \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0, \\ \psi(-x), & x < 0. \end{cases}$$

U holda (6) tenglamaning $\{x \geq 0, \ t \geq 0\}$ sohada (11) boshlangʻich va $u_x(0,t) = 0$ chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u(x,t)=\begin{cases}\frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right]+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(s)ds, & t\le\frac{x}{c}, x\ge 0,\\ \frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right]+\frac{1}{2c}\left(\int\limits_{0}^{x+ct}+\int\limits_{0}^{ct-x}\right)\psi(s)ds, \\ t>\frac{x}{c}, x>0,\end{cases}$$

ko‘rinishda bo‘ladi.

Endi faraz qilaylik, $\{x\ge 0,\ t\ge 0\}$ sohada (6) tenglamaning $u(x,t)$ yechimi quyidagi

$$\overline{u}(x,0)=0,\quad \overline{u}_t(x,0)=0,\quad x\ge 0,$$

$$\overline{u}(0,t)=\mu(t),\quad t\ge 0$$

shartlarni qanoatlantirsin.

U holda (6) tenglamaning umumiy yechimi

$$\overline{u}(x,t)=f(x-ct)$$

ko‘rinishga ega bo‘lib, $\overline{u}(x,t)=\mu(t)$ shartdan

$$f(z)=\mu\left(-\frac{z}{c}\right)$$

tenglamaga ega bo‘lamiz. Bundan esa $\overline{u}(x,t)=\mu\left(t-\frac{x}{c}\right)$ kelib chiqadi. Ammo $\mu(t)$ funksiya $t\ge 0$ da aniqlangani uchun

$$\mu\left(t-\frac{x}{c}\right)=\mu\left(-\frac{x-ct}{c}\right)$$

funksiya $x-ct\le 0$ da aniqlangan. $\overline{u}(x,t)$ funksiyani barcha $x\in\mathbb{R},\ t\ge 0$ qiymatlarida aniqlash uchun $\mu(t)$ funksiyani $t<0$ qiymatlarda nol bilan davom ettirish lozim (bu $\overline{u}(x,t)$ funksiya uchun $x-ct>0$ bo‘ladi). Natijada $\overline{u}(x,t)$ butun $x\in\mathbb{R},\ t\ge 0$ sohada aniqlangan bo‘ladi va

$$\overline{u}(x,t)=\begin{cases}0, & agar\ \ t<\frac{x}{c}\ \ bo‘lsa,\\ \mu\left(t-\frac{x}{c}\right), & agar\ \ t\ge\frac{x}{c}\ \ bo‘lsa.\end{cases}\ x>0.$$

Nihoyat, (6) tenglamaning (11) boshlang'ich va $u(0,t) = \mu(t)$ chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u(x,t) = \overline{u}(x,t) + u(x,t)$$

bo'lib, bu yerda $u(x,t)$ funksiya (13) formula bilan aniqlangan.

Endi $x > 0,\ t > 0$ sohada bir jinsli bo'lmagan

$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{14}$$

tenglamaning

$$u(0,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 0, \quad t \geq 0, \quad x \geq t \tag{15}$$

bir jinsli shartlarni qanoatlantiruvchi yechimni topish masalasini qaraylik.

Agar $f(x,t)$ va $u(x,t)$ funksiyalar $x \in \mathbb{R},\ t > 0$ sohada aniqlanganda edi bu masalaning yechimi

$$u(x,t) = \frac{1}{2c}\int\limits_0^t d\tau \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi \tag{16}$$

bo'lar edi. Demak, yechimni $x < 0$ qiymatlarda aniqlash uchun tabiiyki, $f(x,t)$ funksiyani $x < 0$ da ham aniqlash zarur. Shu maqsadda quyidagi lemmani keltiramiz.

L e m m a. Agar (16) integralda $f(x,t)$ funksiya x o'zgaruvchiga nisbatan toq bo'lsa, $u(0,t) = 0$; juft bo'lsa, $u_x(0,t) = 0$ bo'ladi.

Haqiqatan, agar $f(x,t)$ funksiya $x = 0$ nuqtaga nisbatan
a) toq bo'lsa, (18) dan $u(0,t) = 0$,
b) juft bo'lsa, $u_x(0,t) = 0$ bo'lishini tekshirish qiyin emas. Demak, (14), (15) masala yechimini ifodalovchi formulani yozish uchun $f(x,t)$ funksiyani x o'zgaruvchi bo'yicha $x = 0$ nuqtaga nisbatan toq ravishda davom ettiramiz:

$$F(x,t) = \begin{cases} f(x,t), & x \geq 0, \\ -f(-x,t), & x < 0 \end{cases}$$

va

$$U_{tt} - a^2 U_{xx} = F(x,t), \quad x \in \mathbb{R},\ t > 0$$

$$U(x,0)=0,\quad U_t(x,0)=0,\ x\in\mathbb{R}$$

Koshi masalasini qaraymiz. Uning yechimi esa

$$U(x,t)=\frac{1}{2c}\int\limits_0^t d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}F(\xi,\tau)d\xi.$$

Quyidagi hollarni qaraylik:

1) $x>0,\quad x-ct>0\ \left(t<\frac{x}{c}\right)$.

Unda $x-c(t-\tau)=x-ct+c\tau>0$ bo'lib,

$$u(x,t)=U(x,t)=\frac{1}{2c}\int\limits_0^t d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(\xi,\tau)d\xi.$$

2) $x>0,\quad x-ct<0\left(t>\frac{x}{c}\right)$. Bu holda

$$x-c(t-\tau)=x-ct+c\tau\begin{cases}<0, & 0<\tau<t-\frac{x}{c},\\ >0, & \tau>t-\frac{x}{c}.\end{cases}$$

Shu sababli

$$u(x,t)=U(x,t)=\frac{1}{2c}\int\limits_0^{t-\frac{x}{c}}d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(\xi,\tau)d\xi+$$

$$+\frac{1}{2c}\int\limits_{t-\frac{x}{c}}^{t}d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(\xi,\tau)d\xi=$$

$$=\frac{1}{2c}\int\limits_0^{t-\frac{x}{c}}\left\{-\int\limits_{x-c(t-\tau)}^{0}f(-\xi,\tau)d\xi+\int\limits_0^{x+c(t-\tau)}f(\xi,\tau)d\xi\right\}d\tau+$$

$$+\frac{1}{2c}\int\limits_{t-\frac{x}{c}}^{t}d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(\xi\tau)d\xi=\frac{1}{2c}\int\limits_0^{t-\frac{x}{c}}d\tau\int\limits_{c(t-\tau)-x}^{x+c(t-\tau)}f(\xi,\tau)d\xi+$$

$$+\frac{1}{26}\int\limits_{t-\frac{x}{c}}^{t}d\tau\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(\xi,\tau)d\xi.$$

Nihoyat, 1 va 2-hollarni birlashtirib, yechimni

$$u(x,t)=\frac{1}{2c}\begin{cases}\int\limits_0^t d\tau\int\limits_{x_1}^{x_2} f(\xi,\tau)d\xi, \quad x>0, & t<\frac{x}{c},\\ \int\limits_0^{t_1} d\tau\int\limits_{-x_1}^{x_2} f(\xi,\tau)d\xi+\int\limits_{t_1}^{t} d\tau\int\limits_{x_1}^{x_2} f(\xi,\tau)d\xi, \quad x>0, & t>\frac{x}{c}\end{cases}$$

ko'rinishda hosil qilamiz. Bu yerda $x_1=x-c(t-\tau)$, $x_2=x+c(t-\tau)$, $t_1=t-\frac{x}{c}$.

4.5 Ko'p o'lchovli to'lqin tenglamasi uchun Koshi masalasi. To'lqin tenglamasi uchun Koshi masalasi yechimining yagonaligi

Endi to'lqin tenglamasini $n>1$ o'lchovli hollarda qaraymiz.

$$\Box u=u_{tt}-c^2\triangle u, \quad \triangle\equiv\triangle_x=\frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_1^n}$$

kabi belgilashlar kiritib, ushbu

$$\Box u=0, \quad (x,t)\in\mathbb{R}^n\times(t>0), \tag{17}$$

$$u(x,0)=f(x), \quad x\in\mathbb{R}^n, \tag{18}$$

$$u_t(x,0)=g(x), \quad x\in\mathbb{R}^n \tag{19}$$

Koshi masalasining yechimini qidiramiz. Bu yerda $f(x)$, $g(x)$ - berilgan va $C^2(\mathbb{R}^n)$ sinfga tegishli bolgan funksiyalar.

Sferik o'rta qiymatlar usuli va (3) formuladan foydalanib, to'lqin tenglamasi uchun Koshi masalasi yechimini topamiz.

Sferik o'rta qiymatlar usuli.

$u(x,t)\in C^2(\mathbb{R}^n\times(t>0))$ yechimning sferik o'rta qiymatini

$$M(r,t)=\frac{1}{\omega_n r^{n-1}}\int\limits_{\Sigma_x^r} u(y,t)dS_y \tag{20}$$

bilan aniqlaymiz, bu yerda Σ_x^r - markazi x nuqtada va radiusi r bo'lgan sfera, $\omega_n = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ - n o'lchovli birlik sfera yuzasi, ωr^{n-1} esa radiusi r ga teng bo'lgan sfera yuzasidir.

Integral hisobning o'rta qiymat haqidagi teoremasiga ko'ra biz qidirayotgan funksiya

$$u(x,t) = \lim_{r\to 0} M(r,t) \tag{21}$$

tenglikdan topiladi.

Boshlang'ich shartlardan foydalanib,

$$M(r,0) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\Sigma_x^r} f(y) dS_y = F(r), \tag{22}$$

$$M_t(r,0) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\Sigma_x^r} g(y) dS_y = G(r) \tag{23}$$

formulalarga ega bo'lamiz. Bular esa mos ravishda $f(x)$ va $g(x)$ funksiyalarning sferik o'rta qiymatlaridir.

Endiki qadam $M(r,t)$ funksiya uchun hususiy hosilali differensial tenglamani hosil qilishdan iborat. (20) formulada tayin x va r lar uchun $\xi = (y-x)/r$ almashtirish bajarib,

$$M(r,t) = \frac{1}{\omega_n} \int\limits_{\Sigma_0^1} u(x + r\xi, t) dS_\xi$$

ni olamiz.

Bu yerdan

$$M_r(r,t) = \frac{1}{\omega_n} \int\limits_{\Sigma_0^1} \sum_{i=1}^{n} u_{y_i}(x + r\xi, t)\xi_i dS_\xi = \frac{1}{\omega_n r^{n-1}} \int\limits_{\Sigma_x^r} \sum_{i=1}^{n} u_{y_i}(y,t)\xi_i dS_\xi.$$

Bo'laklab integrallab, $\xi = (y-x)/r$ ning Σ_x^r sirtga tashqi normal ekanligidan foydalanib,

$$M_r(r,t) = \frac{1}{\omega_n r^{n-1}} \int\limits_{U_x^r} \sum_{i=1}^{n} u_{y_i y_i}(y,t) dy$$

formulani hosil qilamiz, bu yerda U_x^r- markazi x nuqtada va radiusi r bo‘lgan shar.

Faraz qilaylik, $u(x,t)$ funksiya to‘lqin tenglamasining yechimi bo‘lsin. U holda

$$r^{n-1}M_r = \frac{1}{c^2\omega_n}\int\limits_{U_x^r} u_{tt}(y,t)dy = \frac{1}{c^2\omega_n}\int\limits_0^r \int\limits_{\partial B_\rho(x)} u_{tt}(y,t)dS_y d\rho.$$

Bu yerdan

$$(r^{n-1}M_r)_r = \frac{1}{c^2\omega_n}\int\limits_{\Sigma_x^r} u_{tt}(y,t)dS_y =$$

$$= \frac{r^{n-1}}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n r^{n-1}}\int\limits_{\Sigma_x^r} u(y,t)dS_y\right) = \frac{r^{n-1}}{c^2}M_{tt}.$$

Shunday qilib, $(r^{n-1}M_r)_r = c^{-2}r^{n-1}M_{tt}$ tenglamani hosil qilamiz. Uni

$$M_{rr} + \frac{n-1}{r}M_r = c^{-2}M_{tt} \tag{24}$$

ko‘rinishda yozib olamiz.

(24) tenglama *Eyler-Puasson-Darbu* tenglamasi deyiladi.

$n=3$ **bo‘lgan hol.**

Eyler-Puasson-Darbu tenglamasi bu holda $(rM)_{rr} = c^{-2}(rM)_{tt}$ ko‘rinishda bo‘ladi. Demak, rM funksiya bir o‘lchovli to‘lqin tenglamasi va

$$(rM)(r,0) = rF(r), \qquad (rM)_t(r,0) = rG(r) \tag{25}$$

boshlang‘ich shartlarni qanoatlantiradi. (3) formulaga asosan

$$M(r,t) = \frac{(r+ct)F(r+ct) + (r-ct)F(r-ct)}{2r} + \frac{1}{2cr}\int\limits_{r-ct}^{r+ct} \xi G(\xi)d\xi. \tag{26}$$

Bu formulaning o‘ng tomoni aniqlangan bo‘lishi uchun $[r-ct, r+ct] \subset (0,\infty)$ bo‘lishi kerak. $F(r)$ va $G(r)$ funksiyalarni $(-\infty,\infty)$ sohaga quyidagi tarzda davom ettiramiz:

$$F_0(r) = \begin{cases} F(r), & r > 0, \\ f(r), & r = 0, \\ F(-r), & r < 0, \end{cases} \qquad G_0(r) = \begin{cases} G(r), & r > 0, \\ g(r), & r = 0, \\ G(-r), & r < 0. \end{cases}$$

T a s d i q. $rF_0(r)$ va $rG(r)$ funksiyalar ikki marta uzluksiz differensiallanuvchi ya'ni, $rF_0(r),\ rG_0(r) \in C^2(\mathbb{R}^2)$.

Isbot. $F(r),\ G(r),\ r > 0$ funksiyalarning aniqlanishi va o'rta qiymat haqidagi teoremadan $\lim\limits_{r\to+0} F(r) = f(x), \quad \lim\limits_{r\to+0} G(r) = g(x)$ kelib chiqadi. Demak, $rF_0(r),\ rG_0(r) \in C(\mathbb{R})$. Bu funksiyalar $C^1(\mathbb{R})$ ga ham tegishli. Buni faqat $F_0(r)$ funksiya uchun ko'rsatamiz. $G_0(r)$ uchun shunga o'xshash ko'rsatiladi. Haqiqatan ham,

$$F'(r) = \frac{1}{\omega_n} \int\limits_{\Sigma_0^1} \sum_{j=1}^{n} f_{y_j}(x + r\xi)\xi_j dS_\xi,$$

$$F'(+0) = \frac{1}{\omega_n} \int\limits_{\Sigma_0^1} \sum_{j=1}^{n} f_{y_j}(x)\xi_j dS_\xi = \frac{1}{\omega_n} \sum_{j=1}^{n} f_{y_j}(x) \int\limits_{\Sigma_0^1} n_j dS_\xi = 0.$$

Bu yerdan esa F'' va G'' lar $r \to +0$ da chegaralangan bo'lsa, $rF_0(r)$, $rG_0(r) \in C^2(\mathbb{R})$ o'rinli bo'ladi. Bu funksiyalar ikkinchi tartibli hosilalarining $r \to +0$ da chegaralanganligi

$$F''(r) = \frac{1}{\omega_n} \int\limits_{\Sigma_0^1} \sum_{i,j=1}^{n} f_{y_i y_j}(x + r\xi)\xi_i \xi_j dS_\xi,$$

$$F''(+0) = \frac{1}{\omega_n} \sum_{i,j=1}^{n} f_{y_i y_j}(x) \int\limits_{\Sigma_0^1} n_i n_j dS_\xi$$

formulalar va $(f, g) \in C^2(\mathbb{R})$ ekanligidan kelib chiqadi. Tasdiq isbotlandi.

(26) formula F_0 va G_0 lar uchun

$$M_0(r,t) = \frac{(r+ct)F_0(r+ct) + (r-ct)F_0(r-ct)}{2r} + \frac{1}{2r} \int\limits_{r-ct}^{r+ct} \xi G_0(\xi) d\xi$$

ko'rinishni oladi.

F_0 va G_0 funksiyalar toq bo‘lgani uchun

$$\int\limits_{r-ct}^{ct-r} \xi G_0(\xi)d\xi = 0.$$

Shunday qilib,

$$M_0(r,t) = \frac{(r+ct)F_0(r+ct) + (r-ct)F_0(r-ct)}{2r} + \frac{1}{2r}\int\limits_{ct-r}^{ct+r} \xi G_0(\xi)d\xi,$$

$s > 0$ lar uchun.

$F_0(s) = F(s),\ G_0(s) = G(s)$ bo‘lgani sababli tayin $t > 0$ va $0 < r < ct$ larda $M_0(r,t)$ bir o‘lchovli to‘lqin tenglamasining (25) boshlang‘ich shartlarni qanoatlantiruvchi yechimi bo‘ladi. $u(x,t) = \lim\limits_{r\to 0} M_0(r,t),\ t > 0$ ekanligidan Lopital qoidasiga asosan

$$u(x,t) = ctF'(ct) + F(ct) + tG(t) = \frac{d}{dt}\left(tF(ct)\right) + tG(ct).$$

Bu formulaga asosan (22) va (23) larni $n = 3$ da inobatga olib,

$$u(x,t) = \frac{d}{dt}\left(\frac{1}{4\pi c^2 t}\int\limits_{\Sigma_x^{ct}} f(y)dS_y\right) + \frac{1}{4\pi c^2 t}\int\limits_{\Sigma_x^{ct}} g(y)dS_y \tag{27}$$

formulani hosil qilamiz.

E s l a t m a. (27) formula 4-bobdagi (48), (49) Koshi masalasining yechimini beruvchi (55) Kirxgof formulasining bir jinsli tenglama bo‘lgan holi ($f(x,t) = 0$) uchun ustma-ust tushadi. Shuningdek, $n = 2$ bo‘lgan hol uchun yuqoridagi mulohazalarni takrorlab, 3-bobdagi (56) Puasson formulasining bir jinsli tenglama bo‘lgan holidagi yechimini hosil qilish mumkin.

Koshi masalasi yechimining yagonaligi $n = 2, 3$ hollar uchun mos ravishda 3-bobdagi (55) va (56) formulalardan kelib chiqadi. Haqiqatan ham, aynan bir xil boshlang‘ich shartlarni qanoatlantiruvchi va bir xil o‘ng tomonga ega bo‘lgan tenglamaning ikkita yechimi mavjud deb faraz qilib hamda yechimlarni beruvchi formulalar yordamida ularning ayirmasini tuzib olib, uning aynan nol ekanligiga, ya’ni bu yechimlarning ustma-ust tushishiga ishonch hosil qilamiz.

4.6 Koshi, Gursa masalalari. Tor tebranish tenglamasi uchun Asgeyrsson prinsipi

Ushbu

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{28}$$

tenglama uchun Koshi masalasi umumiy qo'yilganda, ya'ni boshlang'ich shartlarni $t = 0$ dan farqli bo'lgan, berilish chizig'i L qanday bo'lishini va berilgan funksiyalar masala korrekt qo'yilgan bo'lishi uchun qanday shartlarni qanoatlantirishini ko'rsatamiz. D orqali x, t o'zgaruvchilar tekisligida chegarasi bo'laklari silliq S chiziqdan iborat bo'lgan sohani belgilaymiz. Faraz qilaylik, $u(x,t)$ funksiya D sohada (28) tenglamaning klasssi yechimi bo'lib, $D \cup S$ da birinchi tartibli uzluksiz hosilalarga ega bo'lsin. (28) tenglamada t va x larni mos ravishda τ va ξ o'zgaruvchilar bilan almashtirib, uni

$$\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial \tau}\right) + \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) = 0$$

ko'rinishda yozib olamiz. Bu tenglamani D soha bo'yicha integrallab, Gauss-Ostrogradskiy formulasini qo'llaymiz:

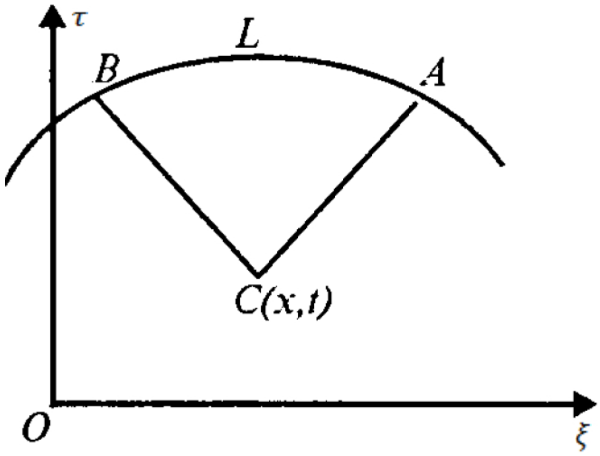
$$\int\limits_D \left[\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial \tau}\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)\right] d\xi d\tau = \int\limits_S \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi. \tag{29}$$

Faraz qilaylik, L - yopiq bo'lmagan silliq Jordan chizig'i bo'lib, quyidagi ikkita shartni qanoatlantirsin:

a) (28) tenglamaning $x - t = c_1$, $x + t = c_2$ xarakteristikalar oilasiga tegishli bo'lgan har bir to'g'ri chiziq L bilan bittadan ortiq nuqtada kesishmasin;

b) L egri chiziqqa uning ixtiyoriy nuqtasida o'tkazilgan urinmaning yo'nalishi hech bir nuqtada (28) tenglama xarakteristikalarining yo'nalishi bilan ustma-ust tushmasin.

$O\xi\tau$ tekislikda ixtiyoriy $C(x,t)$ nuqtadan o'tkazilgan $\xi - x = \tau - t$, $\xi - x = -(\tau - t)$ - (28) tenglamaning xarakteristikalari L egri chiziq bilan mos ravishda A va B nuqtalarda kesishsin.



11-chizma. D soha tasviri.

(29) formulani AB egri chiziq, CA va CB xarakteristikalar bilan chegaralangan sohada qo‘llab, ushbu

$$\int\limits_{AB+BC+CA} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi = 0$$

tenglikni hosil qilamiz. CA va BC da, mos ravishda, $d\xi = d\tau$ va $d\xi = -d\tau$ bo‘lgani uchun oldingi tenglik quyidagi ko‘rinishda yoziladi:

$$\int\limits_{AB} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi - \int\limits_{BC} du + \int\limits_{CA} du = 0.$$

Bundan

$$u(C) = \frac{1}{2} u(A) + \frac{1}{2} u(B) + \int\limits_{AB} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi \tag{30}$$

tenglik kelib chiqadi. Agar (28) tenglamaning $u(x,t)$ yechimi

$$u\Big|_L = \varphi, \ \frac{\partial u}{\partial l}\Big|_L = \psi \tag{31}$$

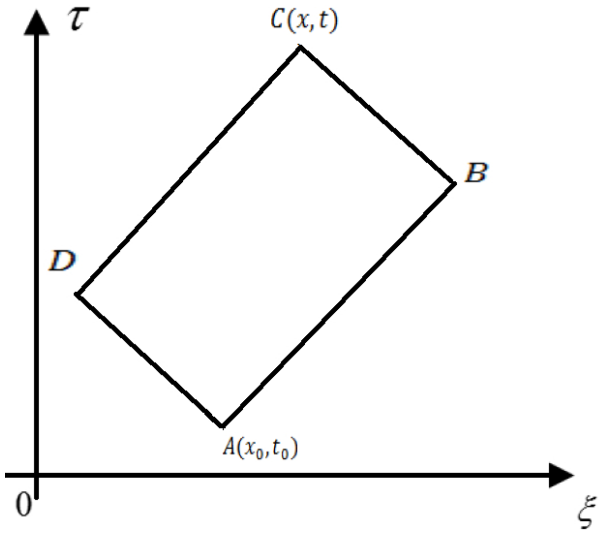
shartlarni qanoatlantirsa, bunda φ va ψ berilgan, mos ravishda ikki va bir marta uzluksiz differensiallanuvchi funksiyalar, l esa L da berilgan yo‘nalish bo‘lib, L ning urinmasi bilan ustma-ust tushmaydi, u holda $\frac{\partial u}{\partial \tau}$, $\frac{\partial u}{\partial \xi}$ noma’lum funksiyalarni

$$\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial s} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial s} = \frac{\partial \varphi}{\partial s},$$

$$\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial l} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial l} = \psi$$

tengliklardan aniqlab olamiz, bu yerda s - L ning yoy uzunligi. Aniqlangan u, $\frac{\partial u}{\partial \tau}$, $\frac{\partial u}{\partial \xi}$ miqdorlarni (30) tenglikning o‘ng tomoniga qo‘yib, (28) tenglamaning (31) shartlarni qanoatlantiruvchi yechimini hosil qilamiz. Shunday qilib, faqat shu tarzda Koshi masalasi qo‘yilsa, masala yagona va turg‘un yechimga ega bo‘ladi.

Endi *Gursa masalasini (yoki xarakteristik masala)* ko‘rishga o‘tamiz. Tayin $A(x_0, t_0)$ nuqtadan chiqarilgan (28) tenglamaning AB: $\xi - x_0 = \tau - t_0$, AD: $\xi - x_0 = -(\tau - t_0)$ xarakteristikalari va $C(x, t)$ nuqtadan o‘tuvchi CB: $\xi - x = \tau - t$ va CD: $\xi - x = -(\tau - t)$ xarakteristikalaridan tashkil topgan xarakteristik to‘rtburchak sohani G orqali belgilab olamiz.



12 - chizma. D soha

(29) formulani G soha uchun qo‘llab,

$$\int\limits_{AB+BC+CD+DA} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi = 0$$

tenglikka ega bo‘lamiz. AB va CD da $d\xi = d\tau$, BC va DA da $d\xi = -d\tau$ bo‘lgani uchun avvalgi tenglikni quyidagi ko‘rinishda yozib olamiz:

$$\int\limits_{AB} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi - \int\limits_{BC} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi + \int\limits_{CD} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi -$$

$$- \int\limits_{DA} \frac{\partial u}{\partial \xi} d\tau + \frac{\partial u}{\partial \tau} d\xi = \int\limits_{AB} du - \int\limits_{BC} du + \int\limits_{CD} du - \int\limits_{DA} du =$$

$$= 2u(B) - 2u(A) - 2u(C) + 2u(D) = 0.$$

Bundan

$$u(A) + u(C) = u(B) + u(D) \tag{32}$$

Asgeyrsson prinsipi yoki o'rta qiymat to'g'risidagi teoremani ifodalovchi tengliq kelib chiqadi. Bunga asosan (28) tenglama $u(x,t)$ yechimining xarakteristik to'rtburchak qarama-qarshi uchlaridagi qiymatlarining yig'indisi birbiriga teng. B va D nuqtalarning koordinatalari mos ravishda $\left(\frac{x+t+x_0-t_0}{2}, \ \frac{x+t-x_0+t_0}{2}\right)$ va $\left(\frac{x-t+x_0+t_0}{2}, \ \frac{-x+t+x_0+t_0}{2}\right)$ lardan iborat. Agar Gursa masalasining shartlari ma'lum bo'lsa, ya'ni $u|_{AB} = \varphi(\xi)$, $u|_{AD} = \psi(\xi)$, $\varphi(A) = \psi(B)$, u holda (32) ga asosan

$$u(x,t) = \varphi\left(\frac{x+t+x_0-t_0}{2}\right) + \psi\left(\frac{x-t+x_0+t_0}{2}\right) - \varphi(x_0) \tag{33}$$

va $\varphi(x)$, $\psi(x)$ funksiyalar ikki marta uzluksiz differensiallanuvchi funksiyalar bo'lsa, (33) formula bilan aniqlangan $u(x,t)$ funksiya Gursa masalasining yechimidan iboratdir.

4.7 Xarakteristikalarda berilgan masala. Integral tenglamalarning ekvivalent sistemasi

Ushbu

$$u_{xy}(x,y) = a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + f(x,y,u(x,y)), \tag{34}$$

$$0 < x < l_1, \ 0 < x < l_2$$

tenglamani va

$$u(x,0) = \varphi(x), \quad 0 \le x \le l_1, \tag{35}$$

$$u(0,y) = \phi(y), \quad 0 \le y \le l_2 \tag{36}$$

shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani topish masalasini qaraymiz. Bu yerda l_1, l_2- berilgan musbat haqiqiy sonlar va φ, $\phi-$ C^1 funksiyalar sinfidan bo'lgan $\varphi(0) = \phi(0)$ shartni qanoatlantiruvchi berilgan funksiyalar.

Bu masalaga (34) chiziqli bo'lmagan giperbolik tipdagi tenglama uchun qo'yilgan Gursa masalasi deb ataladi. Ma'lumki, (34) tenglamaning xarakteristikalari $dxdy = 0$ tenglamaning yechimidan iborat. Bu yechimlar $x =$

const, $y = const$ ko‘rinishdagi to‘g‘ri chiziqlar oilasidir. Shunday qilib, (34) tenglamani va (35), (36) xarakteristik $x = 0$, $y = 0$ chiziqlarda berilgan shartlarni qanoatlantiruvchi $u(x, t)$ funksiyani topish talab etiladi.

T a ’ r i f. (34)-(36) tengliklarni qanoatlantiruvchi $u\,(x, y) \in C^2$ $([0, l_1] \times [0, l_2])$ funksiyaga (34)-(36) Gursa masalasining yechimi deyiladi.

Qo‘yilgan masala yechimining mavjudligi va yagonligini bir necha etaplarda isbotlaymiz. Dastlab (34) – (36) masalaning qandaydir chiziqli bo‘lmagan integral tenglamalar sistemasiga ekvivalent ekanligini ko‘rsatamiz. Faraz qilaylik, $u(x, y)$ funksiya (34) – (36) masalaning yechimi bo‘lsin. U holda (34) tenglamani dastlab y bo‘yicha keyin x bo‘yicha integrallab, quyidagilarni hosil qilamiz:

$$u_x(x, y) = u_x(x, 0) + \int_0^y a(x, \eta) u_x(x, \eta) d\eta +$$

$$+ \int_0^y b(x, \eta) u_y(x, \eta) d\eta + \int_0^y f(x, \eta, u(x, n)) d\eta;$$

$$u(x, y) = u(0, y) + u(x, 0) - u(0, 0) +$$

$$+ \int_0^x \int_0^y a(\xi, \eta) u_x(\xi, \eta) d\eta d\xi + \int_0^x \int_0^y b(\xi, \eta) u_y(\xi, \eta) d\eta d\xi +$$

$$+ \int_0^x \int_0^y f(\xi, \eta, u(\xi, \eta)) d\eta d\xi. \tag{37}$$

Ikkita yangi funkiyalarni $v(x, y) = u_x(x, y)$, $w(x, y) = u_y(x, y)$ kabi kiritamiz. U holda, (34) – (36) boshlang‘ich shartlarni qo‘llab, (34) tenglamani quyidagi ko‘rinishda yozish mumkin:

$$u(x, y) = \varphi(y) + \phi(x) - \phi(0) +$$

$$+ \int_0^x \int_0^y \left[a(\xi, \eta) v(\xi, \eta) + b(\xi, \eta) w(\xi, \eta) \right] d\xi d\eta + \int_0^x \int_0^y f(\xi, \eta, u(\xi, \eta)) d\xi d\eta. \tag{38}$$

Bu tenglamani x bo‘yicha differensiallab

$$v(x,y)=\phi^{'}(x)+$$

$$+\int_0^y [a(x,\eta)v(x,\eta)+b(x,\eta)w(x,\eta)]\,d\eta+\int_0^y f(x,\eta,u(x,\eta))d\eta \qquad (39)$$

tenglikni va y bo‘yicha differensiallab esa

$$w(x,y)=\varphi^{'}(y)+$$

$$+\int_0^x [a(\xi,y)v(\xi,y)+b(\xi,y)w(\xi,y)]\,d\xi+\int_0^x f(\xi,y,u(\xi,y))d\xi \qquad (40)$$

tenglikni hosil qilamiz.

Demak, agar $u(x,t)$ (34)$-$(36) masalaning yechimi bo‘lsa, u holda (38)$-$(40) tenglamalarni qanoatlantiruvchi $v(x,t),\ w(x,t)$ funksiyalar mavjud bo‘ladi. Teskarisi: (38) – (40) tenglamalarning yechimlari bo‘lgan $u,\ v,\ w$- uzluksiz funksiyalarning mavjudligidan $v=u_x, \quad w=u_y$ ekanligi kelib chiqadi. (38) tenglamada navbat bilan $x=0$ va $y=0$ deb (35) va (36) shartlar hosil qilinadi. Shuningdek, (39) va (40) tenglamalarni bevosita differensiallash orqali $u(x,t)$ funksiyaning (34) tenglamani qanoatlantirishini ko‘rish mumkin.

Xarakteristikalarda berilgan masala yechimning mavjudligi.

T e o r e m a (Mavjudlik teoremasi). Quyidagi shartlar bajarilgan bo‘lsin:

1) $a(x,y),\quad b(x,y)\in C\left([0,l_1]\times[0,l_2]\right);$

2) $f(x,y,p)\in C\left([0,l_1]\times[0,l_2]\times\mathbb{R}\right),$ bu yerda (34) tenglamadagi f funksiyaning $u(x,y)$ argumenti p ixtiyoriy qiymat qabul qiluvchi o‘zgaruvchi bilan almashtirildi;

3) ixtiyoriy $x\in[0,l_1]\,,\ y\in[0,l_2]$ va $p_1,p_2\in\mathbb{R}$ lar uchun

$$|f(x,y,p_2)-f(x,y,p_2)|\le L\,|p_1-p_2| \qquad (41)$$

tengsizlik o‘rinli, bunda $L>0$;

4) $\phi(x)\in C^1[0,l_1],\quad \varphi(y)\in C^1[0,l_2],\quad \phi(0)=\phi(0).$

U holda (34) – (36) masalaning yechimi mavjud. (41) tengsizlikka *Lipshits sharti*, L soniga esa *Lipshits o‘zgarmasi* deyiladi.

Isbot. (34) – (36) masala (38) – (40) integral tenglamalargaga ekvivalentligini hisobga olib, (38)–(40) ni qanoatlantiruvchi $u(x,y),\ v(x,y),\ w(x,y)$ uzluksiz funksiyalar mavjudligini isbotlaymiz. Bu funksiyalarni ketma-ketligi yaqinlashish usuli yordamida topamiz. Ketma-ket yaqinlashish jarayonini quyidagicha quramiz:

$$u_0(x,y) = v_0(x,y) = w_0(x,y) = 0,$$

$$u_{n+1}(x,y) = \varphi(y) + \phi(x) - \phi(0) +$$

$$+\int_0^x\int_0^y [a(\xi,\eta)v_n(\xi,\eta) + b(\xi,\eta)w_n(\xi,\eta)]\,d\eta d\xi + \int_0^x\int_0^y f\left(\xi,\eta,u_n(\xi,\eta)\right)d\eta d\xi$$

$$v_{n+1}(x,y) = \phi'(y) +$$

$$+\int_0^y [a(x,\eta)v_n(x,\eta) + b(x,\eta)w_n(x,\eta)]\,d\eta + \int_0^y f(x,\eta,u_n(x,\eta))d\eta,$$

$$w_{n+1}(x,y) = \varphi'(y) +$$

$$\int_0^x [a(\xi,y)v_n(\xi,y) + b(\xi,\eta)w_n(\xi,y)]\,d\xi + \int_0^x f\left(\xi,y,u_n(\xi,y)\right)d\xi,\ n = 1,2,\ldots.$$

Bu jarayonning yaqinlashuvchi ekanligini isbotlaymiz. Buning uchun $u_n,\ v_n,\ w_n$ ketma-ketliklarning hadlari orasidagi farqlarni baholaymiz. u_n hadlarning aniqlanishi va teoremaning 3-shartidan quyidagi tengsizlik kelib chiqadi:

$$|u_{n+1} - u_n| \le \int_0^x\int_0^y \Big[\, |a(\xi,\eta)|\,|v_n(\xi,\eta) - v_{n-1}(\xi,\eta)| +$$

$$+|b(\xi,\eta)|\,|w_n(\xi,\eta) - w_{n-1}(\xi,\eta)|\,\Big]d\xi d\eta + \int_0^x\int_0^y L\,|u_n(\xi,\eta) - u_{n-1}(\xi,\eta)|\,d\xi d\eta.$$

Faraz qilaylik, $(x,y) \in ([0,l_1] \times [0,l_2])$ da

$$M = \max\left\{\max|a(x,y)|,\ \max|b(x,y)|,\ L\right\}$$

bo‘lsin. U holda

$$|u_{n+1} - u_n| \leq M \int_0^x \int_0^y \Big[|v_n(\xi,\eta) - v_{n-1}(\xi,\eta)| +$$

$$+ |w_n(\xi,\eta) - w_{n-1}(\xi,\eta)| + |u_n(\xi,\eta) - u_{n-1}(\xi,\eta)| \Big] d\xi d\eta$$

$v_n, \ w_n$ funksiyalar uchun ham shunga o‘xshash quyidagi baholarni olamiz:

$$|u_{n+1} - u_n| \leq M \int_0^y \Big[|u_n(x,\eta) - u_{n-1}(x,\eta)| +$$

$$+ |w_n(x,\eta) - w_{n-1}(x,\eta)| + |u_n(x,\eta) - u_{n-1}(x,\eta)| \Big] d\eta,$$

$$|w_{n+1} - w_n| \leq M \int_0^x \Big[|v_n(\xi,y) - v_{n-1}(\xi,y)| +$$

$$+ |w_n(\xi,y) - w_{n-1}(\xi,y)| + |u_n(\xi,y) - u_{n-1}(\xi,y)| \Big] d\xi.$$

Ketma-ketlikning barcha elementlari uzluksiz funksiyalar bo‘lganligi sababli, $(x, y) \in ([0, l_1] \times [0, l_2])$ lar uchun

$$|u_n|, \ |v_n|, \ |w_n|$$

larning biror $H > 0$ o‘zgarmas bilan chegaralanganligi kelib chiqadi. U holda ketma-ketlikning no‘linchi hadlarining aniqlanishiga ko‘ra

$$|u_1 - u_0| \leq M, \ |v_1 - v_0| \leq M, \ |w_1 - w_0| \leq M$$

kelib chiqadi. Bularni inobatga olib, quyidagi baholarni olamiz:

$$|u_2 - u_1| \leq M \int_0^x \int_0^y 3H d\xi d\eta = 3HMxy \leq 3HM \frac{(x+y)^2}{2},$$

$$|v_2 - v_1| \leq M \int_0^y 3H d\eta = 3HMy \leq 3HM(x+y),$$

$$|w_2 - w_1| \leq M \int_0^x 3H d\xi = 3HMx \leq 3HM(x+y).$$

Ketma-ketlikning tekis yaqinlashuvchi ekanligini isbotlash uchun majorant qator qurishga to‘g‘ri keladi. Dastlab uchbu

$$|u_n(x,y) - u_{n-1}(x,y)| \leq 3HM^{n-1}K^{n-2}\frac{(x+y)^n}{n!},$$

$$|v_n(x,y) - v_{n-1}(x,y)| \leq 3HM^{n-1}K^{n-2}\frac{(x+y)^{n-1}}{(n-1)!},$$

$$|w_n(x,y) - w_{n-1}(x,y)| \leq 3HM^{n-1}K^{n-2}\frac{(x+y)^{n-1}}{(n-1)!}, \; n = 1, 2, \ldots$$

baholarni isbotlaymiz. Bu yerda $K = 2 + l_1 + l_2$. Isbotlash uchun induksiya usulidan foydalanamiz. Faraz qilaylik, yuqoridagi baholar $n = j$ uchun o‘rinli. $n = j+1$ uchun isbotlaymiz. Induksiya farazidan foydalanib, $|u_{j+1} - u_j|$ ayirmani baholaymiz:

$$|u_{j+1} - u_j| \leq$$

$$\leq M\int_0^x\int_0^y \left[3HM^{j-1}K^{j-2}\frac{(\xi+\eta)^j}{j!} + 2\cdot 3HM^{j-1}K^{j-2}\frac{(\xi+\eta)^{j-1}}{(j-1)!}\right] d\xi d\eta \leq$$

$$\leq 3HM^jK^{j-2}\left[\int_0^x \frac{(\xi+\eta)^{j+1}}{(j+1)!}\Big|_{\eta=0}^{\eta=y} d\xi + \frac{(\xi+\eta)^j}{j!}\Big|_{\eta=0}^{\eta=y} d\xi\right].$$

Ravshanki, bu tengsizlikning o‘ng tomonida integral osti ifodalarning $\eta = 0$ dagi qiymatlarini tashlab yuborishdan tengsizlik faqat kuchayadi. Natijada, qolgan integrallarni hisoblab va

$$\frac{x+y}{j+2} + 2 \leq l_1 + l_2 + 2 = K$$

ekanligidan foydalanib,

$$|u_{j+1} - u_j| \leq 3HM^jK^{j-2}\left[\frac{(x+y)^{j+2}}{(j+2)!} + 2\frac{(x+y)^{j+1}}{(j+1)!}\right] =$$

$$= 3HM^{j-1}K^{j-2}\frac{(x+y)^{j+1}}{(j+1)!}\left[\frac{x+y}{j+2} + 2\right] \leq 3HM^jK^{j-1}\frac{(x+y)^{j+1}}{(j+1)!}$$

tengsizliklarni hosil qilamiz. Shunday qilib, u_n ketma-ketlik uchun induksiya farazi isbotlandi. Qolgan ikkita ketma-ketlik uchun baholarning o'rinli bo'lishi shunga o'xshash isbotlanadi. Bu baholarning bir xilligi uchun faqat

$$|v_n(x,y) - v_{n-1}(x,y)|$$

uchun bahoni isbotlaymiz. Yuqoridagi kabi faraz qilamiz, bu baho $n = j$ uchun o'rinli bo'lsin va $n = j+1$ uchun bajarilishini ko'rsatamiz. Induksiya farazidan foydalanib, $|v_{j+1} - v_j|$ ayirmani quyidagicha baholaymiz:

$$|v_{j+1}(x,t) - u_j(x,t)| \leq$$

$$\leq M\int\limits_0^y \left[3HM^{j-1}K^{j-2}\frac{(\xi+\eta)^j}{j!} + 2\cdot 3HM^{j-1}K^{j-2}\frac{(\xi+\eta)^{j-1}}{(j-1)!}\right]d\eta \leq$$

$$\leq 3HM^jK^{j-2}\left[\frac{(x+y)^{j+1}}{(j+1)!} + 2\frac{(x+y)^j}{j!}\right] =$$

$$= 3HM^jK^{j-2}\frac{(x+y)^j}{j!}\left[\frac{x+y}{j+1}+2\right] \leq\leq 3HM^jK^{j-1}\frac{(x+y)^j}{j!}.$$

Endi $u_n,\ v_n,\ w_n$, ketma -ketliklarning tekis yaqinlashuvchi ekanligini isbotlaymiz. Ko'rinib turibdiki, bunday ketma-ketliklarning har bir hadini tegishli qatorning xususiy yig'indisi shaklida ifodalash mumkin:

$$u_n(x,y) = \sum_{m=1}^{n}\left(u_m(x,y) - u_{m-1}(x,y)\right),$$

$$v_n(x,y) = \sum_{m=1}^{n}\left(v_m(x,y) - v_{m-1}(x,y)\right),$$

$$w_n(x,y) = \sum_{m-1}^{n}\left(w_m(x,y) - w_{m-1}(x,y)\right).$$

Birinchi qatorning hadlari uchun baholar olgan edik. Ulardan foydalanib,

$$|u_n(x,y) - u_{n-1}(x,y)| \leq 3HM^{n-1}K^{n-2}\frac{(x+y)^n}{n!} \leq$$

$$\leq 3HM^{n-1}K^{n-2}\frac{(l_1+l_2)^n}{n1} = C\frac{a^n}{n!},\ C,\ a = const$$

munosabatlarni hosil qilamiz. Bu yerda $C = 3HM^{n-1}K^{n-2}$, $a = l_1 + l_2$. Malumki, $\sum\limits_{n=1}^{\infty} c\frac{a^n}{n!}$ qator yaqinlashuvchi. Bundan Veyershtrass alomatiga ko‘ra, u_n ketma-ketlikning tekis yaqinlashuvchi bo‘ladi. Hadlarning uzluksizligidan limit funksiyaning uzluksizligi kelib chiqadi.

Shunga o‘xshash qolgan ikki ketma-ketlik uchun ham ushbu

$$v_n(x,y) \rightrightarrows v(x,y) \in C([0,l_1] \times [0,l_2])$$

$$w_n(x,y) \rightrightarrows w(x,y) \in C([0,l_1] \times [0,l_2])$$

munosabatlarni ko‘rsatish mumkin.

Shuday qilib, $n \to \infty$ da yuqorida yozilgan ketma-ketliklar (38)-(40) integral tenglamalar sistemasining yechimi bo‘lgan $u,\ v,\ w$ funksiyalarga tekis yaqinlashishini ko‘rsatdik. Bundan bu integral tenglamalarning yechimlari mavjudligi kelib chiqadi. (38)-(40) integral tenglamalar sistemasining (34)-(36) masalaga ekvivalent ekanligini esga olsak, teorema isbot bo‘ldi.

Xarakteristikalarda berilgan masala yechimining yagonaligi.

Endi (34)-(36) masala yechimining yagonaligini isbotlaymiz. Ravshanki, bu (38)-(40) integral tenglamalar sistemasi yechimining yagonaligiga teng kuchli.

T e o r e m a (Yagonalik). Faraz qilaylik,

$$\{u_1(x,y),\ v_1(x,y),\ w_1(x,y)\},$$
$$\{u_2(x,y),\ v_2(x,y),\ w_2(x,y)\}$$

ikkita funksiyalar sistemasi mavjud bo‘lib, ular (38)-(40) integral tenglamalar sistemasining yechimlari va bunda $(34)-(36)$ masala yechimining mavjudligi haqidagi teorema shartlari bajarilgan bo‘lsin. U holda

$$u(x.y) = u_1(x,y) - u_2(x,y), v(x,y) = v_1(x,y) - v_2(x,y),$$

$$w(x,y) = w_1(x,y) - w_2(x,y)$$

funksiyalar $[0,l_1] \times [0,l_2]$ to‘g‘ri to‘rtburchakda aynan 0 ga teng bo‘ladi.

Isbot. Teoremaning shartiga ko‘ra quyidagi tengliklar o‘rinli:

$$u_1(x,y) = \phi(y) + \varphi(x) - \varphi(0) +$$

$$+\int\limits_0^x\int\limits_0^y\left[a\left(\xi,\eta\right)v_1\left(\xi,\eta\right)+b\left(\xi,\eta\right)w_1\left(\xi,\eta\right)\right]d\eta d\xi+$$

$$+\int\limits_0^x\int\limits_0^y f\left(\xi,\eta,u_1\left(\xi,\eta\right)\right)d\eta d\xi,$$

$$u_2\left(x,y\right)=\phi\left(y\right)+\varphi\left(x\right)-\varphi\left(0\right)+$$

$$+\int\limits_0^x\int\limits_0^y\left[a\left(\xi,\eta\right)v_2\left(\xi,\eta\right)+b\left(\xi,\eta\right)w_2\left(\xi,\eta\right)\right]d\eta d\xi+$$

$$+\int\limits_0^x\int\limits_o^y f\left(\xi,\eta,u_2\left(\xi,\eta\right)\right)d\eta d\xi.$$

Bu tenglamalarning biridan ikkinchisini ayirib va $f(x,y,p)$ uchun Lipshits shartini qo‘llab,

$$|u_2-u_1|\leq\int\limits_0^x\int\limits_0^y\Big[M\left|v_2\left(\xi,\eta\right)-v_1\left(\xi,\eta\right)\right|+$$

$$+M\left|w_2\left(\xi,\eta\right)-w_1\left(\xi,\eta\right)\right|+M\left|u_2\left(\xi,\eta\right)-u_1\left(\xi,\eta\right)\right|\Big]d\eta d\xi$$

$$\leq\int\limits_0^x\int\limits_0^y\Big[M\left|v\left(\xi,\eta\right)\right|+M\left|w\left(\xi,\eta\right)\right|+M\left|u\left(\xi,\eta\right)\right|\Big]d\eta d\xi \qquad (42)$$

tengsizliklarni hosil qilamiz. Shunga o‘xshash munosabatlar $v\left(x,y\right),\ w\left(x,y\right)$ funksiyalar uchun ham o‘rinli:

$$\left|v\left(x,y\right)\right|\leq\int\limits_0^y\Big[M\left|v\left(x,\eta\right)\right|+M\left|w\left(x,\eta\right)\right|+M\left|u\left(x,\eta\right)\right|\Big]d\eta,$$

$$\left|w\left(x,y\right)\right|\leq\int\limits_0^x\left[M\left|v\left(\xi,y\right)\right|+M\left|w\left(\xi,y\right)\right|+M\left|u\left(\xi,y\right)\right|\right]d\xi.$$

Bu tengsizliklardan $u\left(x,y\right),\ v\left(x,y\right),\ w\left(x,y\right)$ funksiyalarning $[0,l_1]\times[0,l_2]$ to‘g‘ri to‘rtburchakda 0 ga tengligi kelib chiqishini isbotlaymiz. Dastlab ular

$[0, x_0] \times [0, y_0]$ to‘g‘ri to‘rtburchakda 0 ga tengligini ko‘rsatamiz. Bu yerda $x_0 \in (0, l_1)$, $y_0 \in (0, l_1)$ lar ushbu

$$\begin{cases} 3x_0 y_0 M < 1, \\ 3x_0 M < 1, \\ 3y_0 M < 1 \end{cases}$$

shartlarni qanoatlantiradi.

Faraz qilaylik,

$$\bar{u} = \max_{(x,y)\in[0,x_0]\times[0,y_0]} |u(x, y)|,$$

$$\bar{v} = \max_{(x,y)\in[0,x_0]\times[0,y_0]} |v(x, y)|,$$

$$\bar{w} = \max_{(x,y)\in[0,x_0]\times[0,y_0]} |w(x, y)|.$$

Ravshanki, umumiylikka ziyon yetkazmasdan $\bar{u} \geq \max\{\bar{v}, \bar{w}\}$ deb olish mumkin. U holda (42) tengsizliklardan quyidagilar kelib chiqadi:

$$|\bar{u}(x, y)| \leq M \int\limits_0^x \int\limits_0^y [\bar{u} + \bar{u} + \bar{u}]\, d\xi d\eta \leq 3M x_0 y_0 \bar{u}$$

yoki

$$\bar{u} \leq 3M x_0 y_0 \bar{u}, \ (x, t) \in [0, x_0] \times [0, y_0].$$

$3x_0 y_0 M < 1$ bo‘lganligi sababli, oxirgi tengsizlik faqat $\bar{u} = 0$ bo‘lganda bajariladi. Bundan ko‘rinib turibdiki $u(x, y), v(x, y), w(x, y)$ funksiyalar $[0, x_0] \times [0, y_0]$ da aynan 0 ga teng. Keyingi qadamda biz shunday x_1 $(x_1 > x_0)$ ni olamizki,

$$\begin{cases} 3(x_1 - x_0) y_0 M < 1, \\ 3(x_1 - x_0) M < 1, \\ 3y_0 M < 1 \end{cases}$$

shartlar bajarilsin. $[0, x_1] \times [0, y_0]$ to‘g‘ri to‘rtburchakni qaraymiz.U holda (42) tengsizliklar quyidagi ko‘rinishga ega bo‘ladi:

$$|\bar{u}(x, y)| \leq M \int\limits_{x_0}^x \int\limits_0^y [\bar{u} + \bar{u} + \bar{u}] d\xi d\eta, \ (x, y) \in [0, x_1] \times [0, y_0].$$

Oldingi isbotdagi kabi, $u(x,y)$, $v(x,y)$, $w(x,y)$ funksiyalar $[0,x_1]\times[0,y_0]$ to'g'ri to'rtburchakda aynan 0 ga tengligini hosil qilamiz. Bu kabi mulohazalarni davom etib, chekli sonli qadamlardan keyin bu funksiyalarning $[0,l_1]\times[0,y_0]$ da 0 ga teng ekanligini ko'rsatish mumkin. So'ngra, $[0,y_0]$ kesma uchun ham yuqoridagi kabi ish tutib, ularning $[0,l_1]\times[0,l_2]$ da aynan 0 ga teng ekanligi ko'rsatiladi. Teorema isbotlandi.

4.8 Qo'shma differensial operatorlar. Riman usuli

Ushbu ikkinchi tartibli

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}a_i(x)\frac{\partial u}{\partial x_i}+a_0(x)u$$

chiziqli differensial operatorni qaraymiz. D - $\mathbb{R}^n$ fazoda bo'laklari silliq S sirt bilan chegaralangan soha bo'lsin. Faraz qilamizki, $a_{ij}(x)$ koeffisiyentlar ikkinchi tartibli, $a_i(x)$ - birinchi tartibli uzluksiz hosilalarga ega, $a_0(x)$ koeffisiyent esa uzluksiz bo'lsin.

$$Mv=\sum_{i,j=1}^{n}\frac{\partial^2\left(a_{ij}(x)v\right)}{\partial x_i\partial x_j}+\sum_{i=1}^{n}\frac{\partial\left(a_i(x)v\right)}{\partial x_i}+a_0(x)v \tag{43}$$

ifoda Lu ga qo'shma differensial operator deyiladi. Bevosita hisoblashlarni bajarib,

$$vLu-uMv=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{\sum_{j=1}^{n}\left[va_{ij}(x)\frac{\partial u}{\partial x_j}-u\frac{\partial(a_{ij}(x)v)}{\partial x_j}\right]+a_i(x)uv\right\}$$

tenglikning to'g'riligiga ishonch hosil qilamiz. Agar

$$P_i(x):=\left\{\sum_{j=1}^{n}\left[va_{ij}(x)\frac{\partial u}{\partial x_j}-u\frac{\partial(a_{ij}(x)v)}{\partial x_j}\right]+a_i(x)uv\right\} \tag{44}$$

belgilash kiritsak, avvalgi tenglik quyidagi ko'rinishda yoziladi:

$$vLu-uMv=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}P_i(x)$$

Agar $L \equiv M$ bo'lsa, L operator o'zi-o'ziga qo'shma operator deyiladi. L operatorni quyidagi ko'rinishda yozib olamiz:

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \sum_{i,j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u.$$

Agar

$$b_i(x) := a_i(x)\sum_{j=1}^{n} \frac{a_{ij}(x)}{\partial x_j},\ a_0(x) := c(x)$$

belgilashlarni kiritsak, L ushbu

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + c(x)u \quad (45)$$

ko'rinishga keladi. U holda M operatorni quyidagicha yozish mumkin:

$$Mv = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}(x)}{\partial x_j}v + a_{ij}(x)\frac{\partial v}{\partial x_j}\right) - \sum_{i=1}^{n} \frac{\partial\left(b_i(x)v\right)}{\partial x_i} + a_0(x)v =$$

$$= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}(x)}{\partial x_j}v\right) + \sum_{i=1}^{n} \frac{\partial\left(b_i(x)v\right)}{\partial x_i} + a_0(x)v. \quad (46)$$

Agar M operator berilgan bo'lsa, unga qo'shma operator L bo'lishini tekshirib ko'rish qiyin emas. (43) va (45) formulalardan ko'riniyaptiki, qo'shma operatorlar faqat o'rta hadlari bilan bir-biridan farq qiladi. $b_i(x) = 0$, $i = 1, 2, ..., n$ shartlar bajarilganda $L \equiv M$ bo'ladi. Demak, o'z-o'ziga qo'shma operatorni

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u,\ a_{ij}(x) = a_{ji}(x)$$

ko'rinishga keltirish mumkin. Laplas va to'lqin operatorlari o'z-o'ziga qo'shma operatorlardir, lekin issiqlik tarqalish operatori o'z-o'ziga qo'shma operator bo'la olmaydi.

$\mathbb{R}^2$ fazoda $u(x, y)$ funksiya uchun quyidagi differensial ifodanini qaraymiz:

$$Lu = u_{xy} + a\left(x, y\right)u_x\left(x, y\right) + b\left(x, y\right)u_y\left(x, y\right) + c\left(x, y\right)u\left(x, y\right) \quad (47)$$

Ta'rifga ko'ra, unga qo'shma operator quyidagi ko'rinishga ega:

$$M[v] = u_{xy} - (a\,(x,y)\,v)_x - (b\,(x,y)\,v)_x + c\,(x,y)\,v$$

Ravshanki, (46) formulada

$$n = 2,\ a_{11} = a_{22} = 0,\ \ a_{12} = a_{21} = \frac{1}{2},\ \ b_1 = a,\ b_2 = b,\ c = c.$$

Ko'rinib turibdiki, (44) formuladagi P_1, P_2 lar

$$P_1 = \frac{1}{2}(vu_y - uv_y) + auv,$$

$$P_2 = \frac{1}{2}(vu_x - uv_x) + buv$$

kabi hisoblanadi.

Endi Oxy tekisligida biror $y = f(x)$ egri chiziq berilgan va ixtiyoriy x lar uchun $f'(x) < 0$ shart bajarilgan bo'lsin. $y = f(x)$ funksiyaning grafigini L_f orqali belgilaymiz. $\mathbb{R}^2$ tekislikning $f\,(x)$ funksiya grafigidan yuqorida joylashgan qismini $\mathbb{R}_f^+$ bilan belgilaymiz, ya'ni

$$\mathbb{R}_f^+ = \{(x,y):\ y > f(x)\}.$$

Quyidagi chegaraviy masalani qo'yamiz:

$$Lu = F(x,y) \qquad (x,y) \in \mathbb{R}_f^+ \tag{48}$$

tenglamaning

$$u(x,y) = \phi(x,y),\ \ (x,y) \in L_f,\ \ \frac{\partial u}{\partial n}(x,y) = \psi(x,y),\ \ (x,y) \in L_f \tag{49}$$

shartlarni qanoatlantiruvchi yechimi topilsin, bu yerda Lu (47) formula bilan aniqlanadi, n - L_f chiziqqa o'tkazilgan $\mathbb{R}_f^+$ sohaga nisbatan tashqi normal.

Ixtiyoriy $A(x_0,y_0) \in \mathbb{R}_f^+$ nuqtada (48), (49) masalaning yechimini ifodalovchi formulani topamiz. Buning uchun A nuqtani L_f egri chiziq bilan koordinata o'qlariga parallel bo'lgan kesmalar yordamida birlashtiramiz va mos ravishda $B(x,y_0)$, $C(x_0,y)$ kesishish nuqtalarini hosil qilamiz. $AB,\ AC$ kesmalar hamda BC yoydan hosil bo'lgan konturni S deb, uning ichki qismini esa D bilan belgilaymiz.

L va unga qo'shma differensial M operator uchun o'rinli bo'lgan

$$vLu - uMv = \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y}$$

tenglikni D soha bo'yicha integrallaymiz:

$$\iint\limits_D (vLu - uMv)\, ds = \iint\limits_D \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right) ds$$

Bu tenglikning o'ng qismini matematik tahlil fanidan ma'lum bo'lgan

$$\int\limits_L Rdx + Qdy = \iint\limits_D (Q_x - R_y)\, ds$$

Grin formulasi yordamida o'zgartiramiz. U holda quyidagiga ega bo'lamiz:

$$\iint\limits_D (vLu - uMv)\, ds = \int\limits_L -P_2 dx - P_1 dy.$$

L kontur qismlarining koordinata o'qlariga parallelligi va P_1, P_2 larning yuqoridagi ifodalarini inobatga olib, avvalgi tenglikni davom ettiramiz

$$\int\limits_L -P_2 dx - P_1 dy =$$

$$= \int\limits_B^C \left(\left[\frac{1}{2}(vu_y - uv_y) + auv \right] dy - \left[\frac{1}{2}(vu_x - uv_x) + buv \right] dx \right) +$$
$$+ \int\limits_C^A \left[\frac{1}{2}(vu_y - uv_y) + auv \right] dy + \int\limits_B^A \left[\frac{1}{2}(vu_x - uv_x) + buv \right] dx. \qquad (50)$$

Ravshanki,

$$\int\limits_C^A \left[\frac{1}{2}(vu_y - uv_y) + auv \right] dy + \int\limits_B^A \left[\frac{1}{2}(vu_x - uv_x) + buv \right] dx =$$

$$= \int\limits_C^A \left[\frac{1}{2}(vu_y - uv_y) + auv \right] dy + \int\limits_B^A \left[\frac{1}{2}(vu_x - uv_x) + buv \right] dx$$

tenglik bajariladi. Bu formulaning o‘ng tomonidagi integrallarni mos ravishda I_{CA} va I_{BA} orqali belgilaymiz.

Hozirgach v funksiyaga hech qanday shart qo‘yilmagan edi. Endi faraz qilaylik, v funksiya ikki marta uzluksiz differensiallanuvchi bo‘lib, ushbu

$$Mv = v_{xy} - (a(x,y)v)_x - (b(x,y)v)_y + c(x,y)v = 0,\ x \le x_0,\ y \le y_0$$

tenglamani va

$$v(x_0,y) = \exp\left[\int_{y_0}^{y} a(x_0,s)ds\right],\ y \le y_0, \tag{51}$$

$$v(x,y_0) = \exp\left[\int_{x_0}^{x} b(s,y_0)ds\right],\ x \le x_0 \tag{52}$$

shartlarni qanoatlantirsin. Bu masalada chegaraviy shartlar xarakteristikalarda berilgan. Oldingi paragrafda ko‘rsatilgani kabi bunday masalalar yagona $v(x,y)$ yechimga ega bo‘ladi. Bu yechim bizga ma’lum deb hisoblaymiz va keyingi o‘rinlarda shu yechimni ifodalovchi $v(x,y)$ funksiyadan foydalanamiz. Bularni va (48) tenglamani hisobga olib, (50) formulani soddalashtirishni davom ettiramiz:

$$\iint_D v(x,y)F(x,y)ds =$$

$$= \int_B^C \left(\left[\tfrac{1}{2}(vu_y - uv_y) + auv\right]dy + \left[\frac{1}{2}(vu_x - uv_x) + buv\right]dx\right) + I_{CA} + I_{BA}.$$

$I_{CA},\ I_{BA}$ integrallarda koordinatalaridan biri tayin ekanligidan foydalanamiz. $v(x,y)$ uchun (51) shartdan $x = x_0$ bo‘lsa, $v_y - av = 0$ bo‘lishini oson aniqlash mumkin. U holda

$$I_{CA} = \int_C^A \left[\frac{1}{2}(vu_y - uv_y) + auv\right]dy =$$

$$= \int_C^A \left[\frac{1}{2}(vu)_y - u(v_y - av)\right]dy = \frac{1}{2}(uv)\big|_A - \frac{1}{2}(uv)\big|_C.$$

Xuddi shunga o‘xshash, $y = y_0$ bo‘lganda (52) tenglikka asosan $u_x - bu = 0$ ekanligini inobatga olsak,

$$I_{BA} = \int_B^A \left[\frac{1}{2}(vu_x - uv_x) + buv\right] dx =$$

$$= \int_B^A \left[\frac{1}{2}(vu_x - u(v_x - bv)\right] dx = \frac{1}{2}(uv)\big|_A - \frac{1}{2}(uv)\big|_B$$

formula hosil bo‘ladi. Shunday qilib, (50) ifodani quyidagi ko‘rinishda yozish mumkin:

$$\iint_D v(x,y)F(x,y)ds = \int_B^C \Big(\left[\frac{1}{2}(vu_y - uv_y) + auv\right] dy -$$

$$- \left[\frac{1}{2}(vu_x - uv_x) + buv\right] dx\Big) + uv\big|_A - \frac{1}{2}(uv)\big|_C - \frac{1}{2}(uv)\big|_B.$$

Bundan esa, o‘ng tomondagi uv qo‘shiluvchining $A(x_0, y_0)$ nuqtadagi qiymatini ushbu

$$u(x_0,y_0)v(x_0,y_0) =$$

$$= -\int_B^C \Big(\left[\frac{1}{2}(vu_y - uv_y) + auv\right] dy - \left[\frac{1}{2}(vu_x - uv_x) + buv\right] dx\Big) +$$

$$\frac{1}{2}(uv)\Big|_C + \frac{1}{2}(uv)\Big|_B + \iint_D v(x,y)F(x,y)ds$$

tenglik bilan ifodalaymiz.

(51), (52) chegaraviy shatrlardan $v(x_0, y_0) = 1$ ekanligi kelib chiqadi.

U holda

$$u(x_0,y_0) = -\int_B^C \Big(\left[\frac{1}{2}(vu_y - uv_y) + auv\right] dy - \left[\frac{1}{2}(vu_x - uv_x) + buv\right] dx\Big) +$$

$$\frac{1}{2}(uv)\Big|_C + \frac{1}{2}(uv)\Big|_B + \iint_D v(x,y)F(x,y)ds \qquad (53)$$

hosil bo‘ladi. Bu $u(x_0, y_0)$ uchun yakuniy formuladir.

L_f konturda $u(x,y)$ funksiyaning xususiy hosilalarini (49) shartga asosan berilgan funksiyalar orqali ifodalash mumkinligini koʻrsatamiz. Buning uchun (49) tengliklarni

$$u(x, f(x)) = \phi(x, f(x)), \quad \frac{\partial u}{\partial n}(x, f(x)) = \varphi(x, f(x))$$

koʻrinishda yozib olamiz. L_f ga urinmaning birlik vektori $\overrightarrow{\tau}$ quyidagi koʻrinishga ega:

$$\overrightarrow{\tau} = \left(\frac{1}{\sqrt{1+(f'(x))^2}}, \frac{f'(x)}{\sqrt{1+(f'(x))^2}}\right)$$

Bundan

$$\frac{\partial u(x,y)}{\partial n} = \frac{\partial u}{\partial x}\frac{1}{\sqrt{1+(f'(x))^2}} + \frac{\partial u}{\partial y}\frac{f'(x)}{\sqrt{1+(f'(x))^2}}$$

ifodaga ega boʻlamiz.

$\frac{\partial u}{\partial \tau}$ ushbu

$$\frac{\partial}{\partial n}u(x, f(x)) = \frac{\partial u(x, f(x))}{\partial x} + \frac{\partial u(x, f(x))}{\partial y}f'(x) = \sqrt{1+(f'(x))^2}\frac{\partial u}{\partial \tau}(x,y).$$

tenglamadan topiladi. Ma'lumki,

$$\frac{\partial y}{\partial n} = (\overrightarrow{n}, gradu).$$

L_f chiziqqa oʻtkazilgan normalning, $\overrightarrow{\tau}$ vektorga ortogonal boʻlgan birlik vektori quyidagicha hisoblanadi:

$$\overrightarrow{n} = \left(\frac{f'(x)}{\sqrt{1+(f'(x))^2}}, -\frac{1}{\sqrt{1+(f'(x))^2}}\right).$$

Bundan

$$\frac{\partial u(x,y)}{\partial n} = \frac{\partial u}{\partial x}\frac{f'(x)}{\sqrt{1+(f'(x))^2}} - \frac{\partial u}{\partial y}\frac{1}{\sqrt{1+(f'(x))^2}}$$

formulani hosil qilamiz.

Yuqoridagilarga asoslanib, chegaraviy shartlardan L_f konturda $u(x,y)$ ning xususiy hosilalarini topish uchun

$$\frac{\partial}{\partial x}\phi(x, f(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}f'(x)$$

$$\psi(x,f(x))=\frac{\partial u}{\partial x}\frac{f'(x)}{\sqrt{1+(f'(x))^2}}-\frac{\partial u}{\partial y}\frac{1}{\sqrt{1+(f'(x))^2}}$$

tenglamalar sistemani olamiz. Uning determinanti hech qayerda 0 ga teng emas. Bundan kelib chiqadiki, $u_x(x,y)$, $u_y(x,y)$ lar mavjud va berilganlar orqali bir qiymatli topiladi. Shunday qilib, (53) formulaning o‘ng tomonidagi barcha funksiyalar aniqlandi. Uni hosil qilish uchun qo‘llaniladigan usul *Riman usuli* deyiladi.

E s l a t m a. (7) Dalamber formulasi (53) formulaning xususiy holidan iboratdir.

4.9 Tor tebranish tenglamasi uchun boshlang‘ich-chegaraviy masalalarni Furye usuli bilan yechish

4.9.1 Birinchi boshlangich-chegaraviy masala. Bir jinsli tenglama va bir jinsli chegaraviy shartlar. Xos son va xos funksiya

Xususiy hosilali differensial tenglamalar nazariyasida chegaraviy yoki aralash masalalarni yechishda eng ko‘p qo‘llaniladigan usullardan biri bu o‘zgaruvchilarni ajratish yoki Furye usuli hisoblanadi. Bu usulni bayon etishni ikkala uchi ham mahkamlangan tor tebranish tenglamasi uchun birinchi boshlang‘ich - chegaraviy masalani yechishdan boshlaymiz. Qolgan chegaraviy masalalar va aralash masalalar xuddi shunga o‘xshash tarzda yechiladi. Quyi- dagi

$$u_{tt}=a^2u_{xx} \tag{54}$$

tenglamaning bir jinsli

$$u(0,t)=0,\ u(l,t)=0 \tag{55}$$

chegaraviy va

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \tag{56}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topamiz. (54) tenglama chiziqli va bir jinsli bo'lganligi uchun, uning xususiy yechimlari yig'indisi ham tenglamaning yechimi bo'ladi. Ularning biror koeffitsientlar bilan yig'indisini izlanayotgan yechim sifatida qarash mumkin.

Dastlab, quyidagi yordamchi masalani qaraymiz:

$$u_{tt} = a^2 u_{xx}$$

tenglamaning aynan nolga teng bo'lmagan

$$u(x,t) = X(x)T(t) \tag{57}$$

ko'paytma ko'rinishidagi, bu yerda X, T lar bir o'zgaruvchili, mos ravishda x va t larning funksiyalari, bir jinsli

$$u(0,t) = 0, \;\; u(l,t) = 0 \tag{58}$$

chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Bu masalani echish uchun (57) ifodani (54) ga qo'yib

$$X''(x)T(t) = \frac{1}{a^2}X(x)T''(t)$$

yoki, $X(x)T(t)$ ga bo'lgandan so'ng

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} \tag{59}$$

tenglikni hosil qilamiz. (57) formula bilan aniqlangan funksiya (54) tenglamaning yechimi bo'lishi uchun, (59) tenglik erkli o'zgaruvchilarning barcha $0 < x < l, \; t > 0$ qiymatlarida bajarilishi shart. Bu tenglikning chap tomoni faqat x o'zgaruvchining, o'ng tomoni esa, faqat t o'zgaruvchining funksiyalaridir. Navbat bilan, ularning birini tayin qilib, ikkinchisini o'zgartirib, (59) tenglikning chap va o'ng tomonalari o'zgarmas songa teng ekanligiga ishonch hosil gilamiz:

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = -\lambda, \tag{60}$$

bunda o'zgarmas λ ni qulaylik uchun manfiy ishora bilan oldik.

(60) tengliklardan $X(x)$ va $T(t)$ funksiyalarni aniqlash uchun

$$X''(x) + \lambda X(x) = 0, \ X(x) \neq 0,$$

$$T''(x) + a^2\lambda T(x) = 0, \ T(t) \neq 0 \tag{61}$$

oddiy differensial tenglamalarga ega bo'lamiz. Shuningdek, (58) chegaraviy shartlardan

$$u(0, t) = X(0)T(t) = 0,$$

$$u(l, t) = X(l)T(t) = 0$$

kelib chiqadi. O'z navbatida, bu tengliklar (57) formula bilan aniqlangan $u(x, t)$ funksiya nolga teng bo'lishi uchun $X(x)$ funksiyaning

$$X(0) = X(l) = 0$$

shartlarni qanoatlantirishi lozimligi, aks holda $T(t) = 0$ bo'lishini ko'rsatadi. Shuning uchun, yuqoridagi chegaraviy shartlardan

$$X(0) = X(l) = 0$$

shartlarni hosil qilamiz. Shunday qilib, biz qo'yilgan chegaraviy masalani yechish jarayonida $X(x)$ funksiya uchun *Shturm-Liuvill masalasi* deb ataluvchi quyidagi masalaga keldik:

T a ' r i f. λ ning

$$X''(x) + \lambda X(x) = 0, \ X(0) = X(l) = 0 \tag{62}$$

masala notrivial yechimga ega bo'ladigan qiymatiga shu masalaning xos soni va unga mos notrivial yechimga esa λ xos songa mos keluvchi xos funksiya deyiladi. Umuman, differensial tenglamalarning xos sonlarini va unga mos keluvchi xos funksiyalarini topish masalasiga Shturm-Liuvill masalasi deb yuritiladi.

Bu masalaninig yechimini topish maqsadida λ ning manfiy, nolga teng va musbat qiymatli hollarini alohida qaraymiz.

1-hol. Faraz qilaylik, $\lambda < 0$ bo'lsin. Bu holda oddiy differensial tenglamalar kursidan bizga ma'lumki, (62) tengliklardagi ikkinchi tartibli oddiy differensial tenglamaning umumiy yechimi

$$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \tag{63}$$

ko'rinishda bo'ladi. Bunda $C_1,\ C_2$ - ixtiyoriy haqiqiy sonlar. Ularni shunday tanlaymizki, natijada (62) dagi chegaraviy shartlar o'rinli bo'lsin:

$$X(0) = C_1 + C_2 = 0,\ X(l) = C_1 e^{\sqrt{-\lambda}l} + C_2 e^{-\sqrt{-\lambda}l}$$

yoki

$$C_1 = -C_2,\ C_1\left(e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l}\right) = 0.$$

Qaralayotgan holda $\lambda < 0$ va $l > 0$ haqiqiy sonlar bo'lganligi uchun $e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l} \neq 0$. Demak, ikkinchi tenglamadan $C_1 = 0$ va birinchisidan esa $C_2 = C_1 = 0$ hosil bo'ladi. Demak, bularga asosan, $\lambda < 0$ bo'lganda (62) masala faqat nol yechimga ega bo'lar ekan, ya'ni bu holda Shtuurm-Liuvill masalasi xos son va xos funksiyaga ega emas ekan.

2-hol. Faraz qilaylik, $\lambda = 0$ bo'lsin. Bu holda (62) dagi ikkinchi tartibli oddiy differensial tenglama $X''(x) = 0$ ko'rinishda bo'lib, ununig umumiy yechimi

$$X(x) = C_1 x + C_2 \tag{64}$$

dan iborat bo'ladi. Bunda $C_1,\ C_2$ - ixtiyoriy haqiqiy sonlar. Ularni (62) dagi chegaraviy shartlardan tanlaymiz:

$$X(0) = C_2,\ X(l) = C_1 l + C_2 = 0$$

yoki

$$C_1 = C_2 = 0.$$

Demak, (64) ga asosan, $\lambda = 0$ holda ham (62) masala faqat nol yechimga ega bo'lib, Shturm-Liuvill masalasi xos son va xos funksiyaga ega bo'lmas ekan.

3-hol. Faraz qilaylik, $\lambda > 0$ bo'lsin. Bu holda differensial tenglamalar kursidan bizga ma'lumki, (62) dagi ikkinchi tartibli oddiy differensial tenglama ikkita qo‘shma kompleks xarakteristik ildizlarga ega bo‘lib, uning umumiy yechimi

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x \tag{65}$$

ko‘rinishda bo‘ladi. C_1 va C_2 o‘zgarmaslarni shunday tanlaymizki, (62) dagi chegaraviy shartlar o‘rinli bo‘lsin, ya’ni:

$$X(0) = C_1 = 0,\ C_2 \sin \sqrt{\lambda} l = 0.$$

$X(x) \neq 0$ ekanligidan $C_2 \neq 0$ bo‘ladi. Demak, bu sistemadan

$$\sin \sqrt{\lambda} l = 0$$

ekanligini hosil qilamiz. Bu sodda trigonometrik tenglamaning yechimi quyidagidan iborat:

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2,\ n \in \mathbb{Z}.$$

Shunday qilib, (63) masala faqat $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2,\ n \in \mathbb{Z}$ bo‘lgan holda aynan nolga teng bo‘lmagan (notrivial)

$$X_n(x) = C_n \sin \frac{n\pi}{l} x$$

yechimlarga ega bo‘lar ekan. Bunda C_n - ixtiyoriy haqiqiy sonlar. Demak, (62) Shturm-Liuvill masalasi uchun $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 > 0,\ n = 1, 2, \ldots,$ sonlar xos sonlar va

$$X_n(x) = \sin \frac{n\pi}{l} x$$

funksiyalar esa o‘zgarmas ko‘paytuvchi aniqligida olingan (uni biz birga teng deb olamiz) xos funksiyalar bo‘ladi. Bu xos funkiyalarning skalyar ko‘paytmasi uchun

$$(X_n(x), X_m(x)) = \int_0^l X_n(x) X_m(x) dx = \int_0^l \sin \frac{n\pi}{l} x \sin \frac{m\pi}{l} x dx =$$

$$= \frac{1}{2} \left[\int_0^l \cos \frac{(n-m)\pi}{l} x - \cos \frac{n+m\pi}{l} x dx \right] = \begin{cases} \frac{l}{2}, & n = m, \\ 0, & n \neq m. \end{cases}$$

tenglik o‘rinli. Shunday qilib, biz quyidagi teoremani isbotladik:

T e o r e m a. (62) Shturm-Liuvill masalasi faqat $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$ bo'lgandagina notrivial yechimga ega bo'lib, barcha xos sonlar musbat va har xil xos songa mos keluvchi xos funksiyalar o'zaro ortogonaldir.

Endi topilgan xos sonlarga to'g'ri keluvchi (61) ning ikkinchi tenglamasining yechimini topamiz. $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$ bo'lganda

$$T''(x) + a^2\lambda T(x) = 0,\ T(t) \neq 0$$

differensial tenglama (61) ning birinchi differensial tenglamasiga o'xshash bo'lganligi ($X(x)$ o'rnida $T(t)$ va λ o'rnida esa $a^2\lambda$ ishtirok qilyapti) uchun uning umumiy yechimi quyidagi ko'rinishda bo'ladi:

$$T_n(t) = A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at. \tag{66}$$

U holda, (54) torning erkin tebranish tenglamasining (55) bir jinsli chegaraviy shartlarni qanoatlantiruvchi xususiy yechimi (57), (65) va (66) larga asosan quyidagi ko'rinishda bo'ladi:

$$u_n(x,t) = X_n(x)T_n(t) = \left(A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x.$$

Berilgan (54) tenglama chiziqli va bir jinsli ikkinchi tartibli xususiy hosilali differensial tenglama bo'lganligi uchun bu xususiy yechimlarning yig'indisi ham (54) tenglamani va (55) chegaraviy shartlarni qanoatlantiradi:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{l}at + B_n \sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x. \tag{67}$$

Bu yechimdagi A_n va B_n koeffisientlarni shunday tanlaymizki, (56) boshlang'ich shartlar ham bajarilsin, ya'ni:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x = \varphi(x),\ u_t(x,0) = \sum_{n=1}^{\infty}\frac{n\pi}{l}aB_n \sin\frac{n\pi}{l}x = \psi(x) \tag{68}$$

bo'lsin. Bu sistemadan A_n va B_n koeffisientlarni topish uchun $[0,\ l]$ oraliqda aniqlangan har qanday uzluksiz differensiallanuvchi $f(x)$ funksiyani sinuslar (yoki kosinuslar) bo‘yicha Furye qatori deb ataluvchi trigonometrik qatorga yoyish mumkinligidan foydalanamiz, yani

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

bo‘lsin. Bunda $b_n,\ n \in N$ sonlarga $f(x)$ funksiyaning Furye koeffitsientlari deb ataladi va ular

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx$$

tenglik bilan aniqlanadi. Bundan foydalanib, (68) tenglamalardan A_n va B_n larni topish uchun uzluksiz differensiallanuvchi $\varphi(x)$ va $\psi(x)$ funksiyalarni Furye qatoriga yoyamiz va mos ravishda Furye koeffisientlarini yozamiz:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin \frac{n\pi}{l} x dx, \quad \varphi_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x dx, \tag{69}$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin \frac{n\pi}{l} x dx, \quad \psi_n = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi}{l} x dx. \tag{70}$$

(69) va (70) larni (68) ga qo‘yib, mos koeffitsientlarni tenglashtirish bilan A_n va B_n lar uchun quyidagi ifodalarni hosil qilamiz:

$$A_n = \varphi_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x dx, B_n = \frac{l}{\pi n a} \psi_n = \frac{2}{\pi n a} \int_0^l \psi(x) \sin \frac{n\pi}{l} x dx. \tag{71}$$

A_n va B_n larning (71) formulalar orqali topilgan bu qiymatlarini (67) ga qo‘yib, (54)-(56) bir jinsli tor tebranish tenglamasi uchun 1-tur chegaraviy masalaning formal ko‘rinishda yozilgan yechimini hosil qilamiz. Chunki, bu ko‘rinishda yozilgan (67) yechim cheksiz hadli qator bo‘lib, bu qator uzoqlashuvchi bo‘lishi yoki uning yig‘indisi differensiallanuvchan bo‘lmasligi mumkin. Bu holda (67) qator orqali ifodalangan funksiyani qaralayotgan masalaning yechimi deya olmaymiz. Shu maqsadda koeffitsientlari (71) formulalar bilan aniqlanuvchi (67) funksional qatorning va uni ikki marta differensiallash natijasida hosil bo‘lgan qatorlarning ma’lum bir shartlar bajarilganda tekis yaqinlashuvchi bo‘lishini ko‘rsatsak, (67) qator bilan aniqlangan funksiya haqiqatan ham (54)- (56) masalaning yechimidan iborat bo'ladi. Quyidagi teorema o'rinli:

T e o r e m a. Agar $\varphi(x)$ funksiya $[0,\ l]$ oraliqda ikki marta uzluksiz differensiallanuvchi bo'lib, uchinchi tartibli bo'laklari uzluksiz bo'lgan hosilaga ega bo'lsa, $\psi(x)$ esa uzluksiz differensiallanuvchi bo'lib, ikkiinchi tartibli bo'laklari uzluksiz bo'lgan hosilaga ega bo'lsa, hamda

$$\varphi(0) = \varphi(l) = 0,\ \psi(0) = \psi(l) = 0,\ \varphi''(0) = \varphi''(l) = 0 \tag{72}$$

kelishuvchanlik shartlari bajarilsa, u holda (57) qator bilan aniqlangan $u(x,t)$ funksiya ikkinchi tartibli uzluksiz hosilalarga ega bo'lib, (54) tenglamani, (55) chegaraviy va (56) boshlang'ich shartlarni qanoatlantiradi. Shu bilan birga (67) qatorni x va t bo'yicha ikki marta hadlab differensiallash mumkin bo'lib, hosil bo'lgan qatorlar ixtiyoriy chekli $t > 0$ da $0 \leq x \leq l$ oraliqda absolyut va tekis yaqinlashuvchi bo'ladi.

Isbot. (72) shartlar qanday kelib chiqqaniga to'xtalaylik. (72) ning birinchi ikkita shartlari $u(x,t)$ funksiyaning $x = 0,\ t = 0$ va $x = l,\ t = 0$ nuqtalarda uzluksizligidan (55) va (56) shartlarga asosan kelib chiqadi. (72) ning ikkinchi ikkita shartlari esa xuddi shu nuqtalarda $u_t(x,t)$ funksiyaning uzluksizligidan hosil bo'ladi. Uchinchi juft shartlarni esa quyidagicha asoslash mumkin: (54) tenglamada $t = 0$ deb,

$$u_{tt}\mid_{t=0} = a^2\varphi''(x)$$

tenglikni hosil qilamiz. (55) shartlarni ikki marta t bo'yicha differentsiallab,

$$u_{tt}\mid_{x=0} = u_{tt}\mid_{x=l} = 0$$

tengliklarga ega bo'lamiz. Bu yerda $t = 0$ deb, oldingi tenglikda $x = 0$ va $x = l$ desak, (56) ning uchinchi juftlik shartlarii kelib chiqadi. (55) formulalardagi integrallarning birinchisini uch marta, ikkinchisini esa ikki marta bo'laklab integrallaymiz. (56) shartlarga asosan, quyidagilarni hosil qilamiz:

$$A_n = -\frac{2l^2}{(\pi n)^3}\int\limits_0^l \varphi'''(x)\cos\frac{n\pi}{l}xdx,\ B_n = -\frac{2l^2}{(\pi n)^3}\int\limits_0^l \frac{\psi''(x)}{a}\sin\frac{n\pi}{l}xdx.$$

Ushbu

$$\alpha_n = \frac{2}{l}\int\limits_0^l \varphi'''(x)\cos\frac{n\pi}{l}xdx, \quad \beta_n = \frac{2}{l}\int_0^l \psi''(x)\sin\frac{n\pi}{l}xdx$$

belgilashlarni kiritamiz. U holda A_n va B_n koeffitsientlar α_n va β_n lar orqali quyidagicha ifodalabadi:

$$A_n = -\left(\frac{l}{\pi}\right)^3\frac{\alpha_n}{n^3}, \; B_n = -\left(\frac{l}{\pi}\right)^3\frac{\beta_n}{n^3}. \qquad (73)$$

α_n va β_n miqdorlar $\varphi'''(x)$ va $\frac{\psi''(x)}{a}$ funksiyalarning Furye koeffitsientlaridan iboratdir. Trigonometrik qatorlar nazariyasidan ma'lumki,

$$\sum_{n=1}^{\infty}\frac{|\alpha_n|}{n}, \; \sum_{n=1}^{\infty}\frac{|\beta_n|}{n}$$

qatorlar yaqinlashuvchi boʻladi. (73) ni (67) qatorga qoʻyamiz:

$$u(x,t) = -\left(\frac{l}{\pi}\right)^3\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\alpha_n\cos\frac{n\pi}{l}at + \beta_n\sin\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x.$$

Bu qator va uni ikki marta hadlab differensiallash natijasida hosil boʻlgan qatorlar uchun ushbu

$$C_1\sum_{n=1}^{\infty}\frac{|\alpha_n| + |\beta_n|}{n^3}, \; C_2\sum_{n=1}^{\infty}\frac{|\alpha_n| + |\beta_n|}{n^2}, \; C_3\sum_{n=1}^{\infty}\frac{|\alpha_n| + |\beta_n|}{n},$$

$C_1, \; C_2, \; C_3$ - oʻzgarmaslar, yaqinlashuvchi qatorlar mojoranta qator rolini oʻynaydi. Demak, (67) qator va uni ikki marta differensiallash natijasida hosil boʻlgan qatorlar absolyut va tekis yaqinlashuvchi boʻladi. Bundan (67) qatorning yigʻindisi hisoblangan funksiya oʻzining birinchi va ikkinchi tartibli hosilalari bilan birga uzluksiz ekanligi kelib chiqadi. Shu bilan teorema isbot boʻldi.

4.9.2 Bir jinsli torning majburiy tebranishi

Uchlari mahkamlangan torning tashqi kuchlar taʻsiridagi majburiy tebranishi masalasini qaraymiz.

Bir jinsli bo‘lmagan

$$u_{tt} - a^2 u_{xx} = f(x,t) \quad x \in (0,l), \quad t > 0 \tag{74}$$

tor tebranish tenglamasining (56) boshlang‘ich va bir jinsli (55) chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini topamiz. Bu yerda $f(x,t)$ funksiya torga ta‘sir qiluvchi tashqi kuchlar yig‘indisini ifodalab, $0 \leq x \leq l$, $t > 0$ sohada ikki marta uzluksiz differensiallanuvchi deb faraz etiladi.

Bu masalaning $u(x,t)$ yechimini quyidagi

$$u(x,t) = v(x,t) + w(x,t) \tag{75}$$

ko‘rinishda qidiramiz. Bu yerda $v(x,t)$ funksiya bir jinsli bo‘lmagan

$$v_{tt} = a^2 v_{xx} + f(x,t) \tag{76}$$

tor tebranish tenglamaning bir jinsli boshlang‘ich

$$v\,|_{t=0} = 0, \quad v_t\,|_{t=0} = 0 \tag{77}$$

va chegaraviy

$$v\,|_{x=0} = 0, \quad v\,|_{x=l} = 0 \tag{78}$$

shartlarni qanoatlantiruvchi yechimi;

$w(x,t)$ funksiya esa bir jinsli

$$w_{tt} = a^2 w_{xx} \tag{79}$$

tenglamaning quyidagi boshlang‘ich

$$w\,|_{t=0} = \varphi_0(x), \quad w_t\,|_{t=0} = \varphi_1(x) \tag{80}$$

va chegaraviy

$$w\,|_{x=0} = 0, \quad w\,|_{x=l} = 0 \tag{81}$$

shartlarini qanoatlantiruvchi yechimidan iborat.

Yuqoridagi masalalardan ko‘rinib turibdiki, $v(x,t)$ funksiya uchlari mahkamlangan bir jinsli torning $f(x,t)$ tashqi kuchlar ta’siridagi majburiy tebranishini, $w(x,t)$ esa shu torning erkin tebranishini ifodalaydi.(79)-(81) masalaning $w(x,t)$ yechimi oldingi paragrafda topildi. U (67) formula bilan ifodalanadi.

Shuning uchun bu yerda (76)-(78) masalaning $v(x,t)$ yechimini qurish yetarlidir.

Bu masalaning $v(x,t)$ yechimini ushbu

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}, \tag{82}$$

qator ko‘rinishda izlaymiz. Bu yerda $T_n(t)$ hozircha no‘malum funksiyalar. (82) qatorni chekli $t > 0$ uchun $[0, l]$ oraliqda yaqinlashuvchi va shu sohada x va t o‘zgaruvchilar bo‘yicha hadma-had differensiallash mumkin bo‘lsin. U holda (82) qator bir jisli chegaraviy shartlarni qanoatlantiradi.

Bu qatorni (77) boshlang‘ich shartlarga qo‘yib, $T_k(t)$ funksiyalar uchun quyidagi

$$T_n(0) = 0, \qquad T_n'(0) = 0, \qquad n = 1, 2, 3 \dots \tag{83}$$

shartlarni olamiz.

Endi $f(x,t)$ funksiyani $[0, l]$ oraliqda x o‘zgaruvchiga nisbatan sinuslar bo‘yicha Furye qatoriga yoyiladi deb faraz qilamiz ya’ni

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l} \tag{84}$$

bunda f_n koeffitsientlar quyidagi

$$f_n(t) = \frac{2}{l} \int_0^l g(x,t) \sin \frac{n\pi x}{l} \tag{85}$$

formula bilan aniqlanadi.

$T_n(t)$ funksiyalarni topish uchun (82) va (84) ifodalarni (76) tenglamaga qo‘yib, ushbu

$$\sum_{n=1}^{\infty} \left[T_n''(t) + \omega_n^2 T_n(t)\right] \sin \frac{k\pi x}{l} = f(x,t) \tag{86}$$

tenglamaga ega bo‘lamiz. Bu yerda $\omega_n = \frac{n\pi}{l}$.

(84) va (86) yoyilmalarni o‘zaro taqqoslab, n ning har bir qiymatida chiziqli o‘zgarmas koeffitsientli

$$T_n''(t) + \omega_n^2 T_n(t) = f_n(t), \qquad t > 0 \tag{87}$$

oddiy differensial tenglamalarga ega bo‘lamiz.

Demak, $T_n(t)$ funksiyalarni topish uchun ikkinchi tartibli (87) oddiy differensial tenglamani (83) boshlang‘ich shartlar bilan qaraymiz. Bu masalaning yechimini o‘zgarmasni variatsiyalash usuli yordamida topamiz. Bir jinsli (87) tenglamaning umumiy yechimi

$$T_n(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t,$$

bu yerda C_1, C_2 - ixtiyoriy o‘zgarmaslar.

Endi (37) tenglamaning umumiy yechimini

$$T_n(t) = C_1(t) \cos \omega_n t + C_2(t) \sin \omega_n t \tag{88}$$

ko‘rinishida izlaymiz.

(88) formuladagi noma’lum $C_1(t)$ va $C_2(t)$ funksiyalarni topish uchun, differensial tenglamalar kursidan ma‘lumki, bu funksiyalarga nisbatan ushbu

$$\begin{cases} C_1'(t) \cos \omega_n t + C_2'(t) \sin \omega_n t = 0, \\ -C_1'(t)\omega_n \sin \omega_n t + C_2'(t)\omega_n \cos \omega_n t = f_n(t) \end{cases}$$

tenglamalar sistemasiga ega bo‘lamiz. Bundan $C_1'(t)$ va $C_2'(t)$ funksiyalarni

$$C_1'(t) = -\frac{f_n(t)}{\omega_n} \sin \omega_n t, \quad C_2'(t) = \frac{f_n(t)}{\omega_n} \cos \omega_n t$$

topamiz va bu tenglamalarni integrallab, $C_1(t)$ va $C_2(t)$ funksiyalarni

$$C_1(t) = -\frac{1}{\omega_n} \int_0^t f_n(\tau) \sin \omega_n \tau d\tau + C_1^0,$$

$$C_2(t) = \frac{1}{\omega_n} \int_0^t f_n(\tau) \cos \omega_n \tau d\tau + C_2^0$$

ko‘rinishda aniqlaymiz. Bunda C_1^0 va C_2^0 - ixtiyoriy o‘zgarmaslar.

Topilgan $C_1(t)$ va $C_2(t)$ funksiyalarni (88) formulaga qo'yib, (87) tenglamaning umumiy yechimini

$$T_n(t) = \frac{1}{\omega_n}\int\limits_0^t g_n(\tau)\sin[\omega_n(t-\tau)]d\tau + C_1^0\cos\omega_n t + C_2^0\sin\omega_n t \qquad (89)$$

ko'rinishda topamiz. (83) boshlang'ich shatrlarni qanoatlantirib, (89) umumiy yechimdan $C_1^0 = C_2^0 = 0$ ekanligini olamiz.

Agar $f_n(t) \in C[0,T]$ bo'lsa, u holda (87) tenglamaning (83) boshlang'ich shartlarini qanoatlantiruvchi yechimi

$$T_n(t) = \frac{1}{\omega_n}\int\limits_0^t f_n(\tau)\sin[\omega_n(t-\tau)]d\tau \qquad (90)$$

formula bilan aniqlanadi.

Endi (82) qatorning yopiq $0 \le t \le T$, $0 \le x \le l$ sohada tekis yaqinlashuvchi ekanligini hamda uni x va t argumentlar bo'yicha ikki marta hadmahad differensiallash mumkin ekanligini ko'rsatamiz.

Agar $f(x,t)$ funksiya yopiq $0 \le t \le T$, $0 \le x \le l$ sohada uzluksiz, shu sohada x o'zgaruvchi bo'yicha uzluksiz ikki marta differensiallanuvchi va ixtiyoriy $t \in [0,T]$ uchun $f(0,t) = f(l,t) = 0$ bo'lsa, u holda (85) ifodani ikki marta bo'laklab integrallaymiz, natijada

$$f_n(t) = -\frac{2}{l}\left(\frac{l}{\pi n}\right)^2\int\limits_0^l f''_{xx}(x,t)\sin\frac{n\pi x}{l}dx = -\left(\frac{l}{\pi}\right)^2\frac{a_n(t)}{n^2} \qquad (91)$$

ifodani olamiz, bu yerda

$$a_n(t) = \frac{2}{l}\int\limits_0^t f''_{xx}(x,t)\sin\frac{n\pi x}{l}dx.$$

Uzluksiz funksiyalarning kvadradlaridan tuzilgan $a_n(t)$ funksional

$$\sum_{n=1}^{\infty} a_n^2(t) < \infty, \qquad (92)$$

Qator ixtiyoriy chekli $t > 0$ da Bessel tengsizligiga asosan yaqinlashuvchi bo‘ladi. Endi (91) ifodani (90) formulaga qo‘yamiz

$$T_n(t) = -\left(\frac{l}{\pi}\right)^3 \frac{1}{a} \int\limits_0^t \frac{a_n(t)}{n^3} \sin \omega_n (t-\tau) d\tau$$

Hosil bo‘lgan oxirgi ifodani esa (82) formulaga qo‘ysak,

$$v(x,t) = -\left(\frac{l}{\pi}\right)^3 \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{n^3} \int\limits_0^t a_k(\tau) \sin[\omega_n(t-\tau)] d\tau \sin\frac{n\pi x}{l} \tag{93}$$

ifodaga ega bo‘lamiz. Bu qatorning har bir hadi ixtiyoriy $t \in [0,T]$ bo‘lganda ushbu

$$\left(\frac{l}{\pi}\right)^3 \frac{T}{a} \sum_{n=1}^{\infty} \frac{|a_n(t_0)|}{n^3}$$

sonli qatorning har bir hadi bilan chegaralangan. Bu yerda $|a_n(t_0)| = \max\limits_{0\le t\le T} |a_n(t)|$, t_0 biror tayin nuqta.

Shuning uchun (93) qator $\{[0,T],[0,l]\}$ sohada absolyut va tekis yaqinlashuvchi qator bo‘ladi.

Endi (93) qatorni hadma-had ikki marta x va t o‘zgaruvchilari bo‘yicha differensiallaymiz, natijada ushbu

$$v_{xx} = \frac{l}{a\pi} \sum_{k=1}^{\infty} \frac{1}{k} \int\limits_0^t a_k(\tau) \sin[\omega_k(t-\tau)] d\tau \sin\frac{k\pi x}{l}, \tag{94}$$

$$v_{tt} = -\left(\frac{l}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} a_n(t)\tau \sin\frac{n\pi x}{l} +$$

$$+\frac{la}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int\limits_0^t a_n(\tau) \sin[\omega_n(t-\tau)] d\tau \sin\frac{n\pi x}{l} \tag{95}$$

qatorlarni olamiz. Ma’lumki, bu qatorlar $x \in [0,l]$, $t \in [0,T]$ da quyidagi qatorlar uchun

$$\frac{l}{a\pi} T \sum_{k=1}^{\infty} \frac{|a_n(t_0)|}{\pi},$$

$$\left(\frac{l}{\pi}\right)^2 T\sum_{n=1}^{\infty}\frac{|a_n(t_0)|}{n^2}+\frac{la}{\pi}T\sum_{n=1}^{\infty}\frac{|a_n(t_0)|}{n}$$

qatorlar mojoranta bo‘ladi. Bu qatorlarning yaqinlashishi (92) qatorning yaqinlashishidan va

$$2\frac{|a_n(t_0)|}{n}\leq\frac{1}{n^2}+a_n^2(t_0)$$

tengsizlikdan kelib chiqadi.

U holda (94) va (95) qatorlar $[0,l],[0,T]$ sohada absolyut va tekis yaqinlashuvchi bo‘ladi, bundan esa $v_{xx}(x,t)$ va $v_{tt}(x,t)$ hosilalarning bu sohada uzluksiz ekanligi kelib chiqadi.

Endi (94)va (95) ifodalarni (76) tenglamaga qo‘ysak, (82) formula bilan aniqlangan $v(x,t)$ funksiya bir jinsli bo‘lmagan tor tebranish tenglamasini qanoatlantirishiga ishonch hosil qilish mumkin.

Shunday qilib, quyidagi teoremani isbotladik:

T e o r e m a. Agar $\varphi(x)$ va $\psi(x)$ funksiyalar oldingi banddagi teorema shartlarini qanoatlantirsa va $f(x,t)$ funksiya yopiq $([0,l]\times[0,T])$ sohada uzluksiz, shu sohada ikkinchi tartibli hosilalarga ega bo‘lib, $f(0,t)=0$, $f(l,t)=0$ tengliklar o‘rinli bo‘lsa, u holda (74), (55), (56) masalaning yagona yechimi mavjud va bu yechim

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{n\pi x}{l}+$$

$$+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

formula bilan aniqlanadi. Bu yerda

$$T_n(t)=\frac{2}{n\pi}\int_0^t\sin\frac{n\pi}{l}(t-\tau)d\tau\int_0^l f(x,\tau)\sin\frac{n\pi x}{l}dx,$$

$$a_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx,$$

$$b_n=\frac{2}{n\pi a}\int_0^l\psi(x)\sin\frac{n\pi x}{l}dx.$$

4.9.3 Uchlari qo'zg'aluvchan torning majburiy tebranishi

Torning uchlari mahkamlanmagan bo'lib, ular biror qoida asosida harakatlansin va tor tashqi kuch ta'sirida tebranayotgan bo'lsin. U holda bu masala bir jinsli bo'lmagan

$$u_{tt} = a^2 u_{xx} + f(x,t)$$

(74) tor tebranish tenglamasining (56) boshlang'ich va ushbu chegaraviy

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad 0 \le t \le T \tag{96}$$

shartlarni qanoatlantiruvchi yechimini topish masalasiga keladi.

Qaralayotgan umumiy holdagi (74),(56),(96) masala yechimining mavjudligini bir jinsli chegaraviy shartli masalaga keltirib ifodalash mumkin.

Buning uchun $\mu_1(t)$ va $\mu_2(t)$ funksiyalarni $C^2[0,T]$ sinfdan deb talab qilamiz. U holda (96) chegaraviy shartlarni qanoatlantiruvchi quyidagi

$$z(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)]$$

yordamchi funksiyani kiritamiz, ya'ni

$$z(0,t) = \mu_1(t), \quad z(l,t) = \mu_2(t)$$

Endi (74) tenglamaning (56) boshlang'ich va (96) chegaraviy shartlarini qanoatlantiruvchi $u(x,t)$ yechimini

$$u(x,t) = v(x,t) + z(x,t)$$

ko'rinishda ishlaymiz, bu yerda $v(x,t)$-yangi noma'lum funksiya. Boshlang'ich va chegaraviy shartlarga asosan $v(x,t)$ funksiya uchun quyidagi bir jinsli chegaraviy

$$v\,|_{x=0} = u(x,t)\,|_{x=0} - z(x,t)\,|_{x=0} = \mu_1(t) - \mu_1(t) = 0,$$

$$v\,|_{x=l} = u(x,t)\,|_{x=l} - z(x,t)\,|_{x=l} = \mu_2(t) - \mu_2(t) = 0$$

va bir jinsli bo'lmagan boshlang'ich

$$v\mid_{t=0}=u\mid_{t=0}-z\mid_{t=0}=\varphi(x)-\mu_1(0)-[\mu_2(0)-\mu_1(0)]\frac{x}{2}=\overline{\varphi}(x),$$

$$v_t\mid_{t=0}=u_t\mid_{t=0}-z_t\mid_{t=0}=\psi(x)-\mu_1'(0)-[\mu_2'(0)-\mu_1'(0)]\frac{x}{2}=\overline{\psi}(x)$$

shartlarga ega bo'lamiz.

(90) tenglikka ko'ra noma'lum $v(x,t)$ funksiyaga nisbatan ushbu

$$v_{tt}-a^2v_{xx}=(u-z)_{tt}-a^2(u-z)_{xx}=$$

$$=u_{tt}-a^2u_{xx}-(z_{tt}-a^2z_{xx})=$$

$$=f(x,t)-\mu_1''(t)-[\mu_2''(t)-\mu_1''(t)]\frac{x}{l}=\overline{f}(x,t)$$

yoki

$$v_{tt}=a^2v_{xx}+\overline{f}(x,t)$$

tenglamani olamiz. Bu yerda

$$\overline{f}(x,t)=f(x,t)-\mu_1''(t)-[\mu_2''(t)-\mu_1''(t)]\frac{x}{l}.$$

Shunday qilib, biz $v(x,t)$ funksiyani topish uchun quyidagi masalaga keldik: bir jinsli bo'lmagan

$$v_{tt}=a^2v_{xx}+\overline{f}(x,t) \tag{97}$$

tor tebranish tenglamasining quyidagi boshlang'ich

$$v(x,t)\mid_{t=0}=\overline{\varphi}(x),\quad v_t(x,t)\mid_{t=0}=\overline{\psi}_1(x), \tag{98}$$

va bir jinsli chegaraviy

$$v(x,t)\mid_{x=0}=0,\quad v(x,t)\mid_{x=l}=0 \tag{99}$$

shartlarni qanoatlantiruvchi $v(x,t)$ yechimini toping.

Bu masalaning yechimini oldingi bandda batafsil o'rgandik. Agar $\overline{\varphi}(x)$, $\overline{\psi}(x)$ va $\overline{f}(x,t)$ funksiyalar oldingi banddagi teorema shartlarini qanoatlantirsa, (97)-(99) masalaning $C^2([0,l]\times[0,T])$ sinfga tegishli bo'lgan $v(x,t)$ yechimi mavjud va yagona bo'ladi.

Shunday qilib, (74), (56), (96) aralash masala yechimining mavjudligi va yagonaligi haqidagi quyidagi teorema o‘rinli:

T e o r e m a. Agar berilgan funksiyalar

$$\varphi(x) \in C^3[0,l], \quad \psi(x) \in C^2[0,l]; \quad \mu_1(t), \mu_2(t) \in C^2[0,T];$$

$$\{f(x,t), \quad f_x(x,t), \quad f_{xx}(x,t)\} \in C([0,l] \times [0,T])$$

bo‘lib, ular uchun uchbu

$$\varphi(0) = \mu_1(0), \quad \varphi(l) = \mu_2(0), \quad \varphi''(0) = \varphi''(l) = 0,$$

$$\psi(0) = \mu_1'(0), \quad \psi(l) = \mu_2'(0), \quad f(0,t) = \mu_1''(t), \quad f(l,t) = \mu_2''(t)$$

tengliklar o‘rinli bo‘lsa, u holda (74), (56), (96) aralash masala yechimi mavjud bo‘ladi.

4.9.4 Ikkinchi boshlang‘ich-chegaraviy masala

Ushbu

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0,l), \quad t > 0, \tag{100}$$

tor tebranish tenglamaning quyidagi

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \qquad 0 \le x \le l \tag{101}$$

boshlang‘ich va

$$u_x(0,t) = 0, \quad u_x(l,t) = 0, \qquad 0 \le t \le T \tag{102}$$

chegaraviy shartlarini qanoatlantiruvchi $u(x,t)$ yechimi topilsin.

Berilgan bir jinsli tor tebranish tenglamasining $u_x(0,t) = u_x(l,t) = 0$ chegaraviy shartlarini qanoatlantiruvchi $u(x,t)$ yechimini $u(x,t) = X(x)T(t)$ ko‘rinishda izlaymiz. Bundan $X(x)$ funksiya uchun ushbu

$$X'(0) = 0, \qquad X'(l) = 0$$

chegaraviy shartlarni olamiz.

Endi $u(x,t)$ ko'paytmani (100) tenglamaga qo'ysak,

$$X(x)T''(t) = a^2 X''(x)T(t)$$

tenglikka ega bo'lamiz.

Oxirgi tenglikni $a^2X(x)T(t) \neq 0$ ifodaga bo'lib,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2T(t)} = -\lambda$$

ni olamiz. Bundan esa $X(x)$ funksiyaga nisbatan

$$X''(x) + \lambda X(x) = 0 \tag{103}$$

$$X'(0) = 0, \quad X'(l) = 0 \tag{104}$$

Shturm-Liuvill masalaga, $T(t)$ funksiyaga nisbatan esa

$$T''(t) + a^2\lambda T(t) = 0, \qquad t > 0 \tag{105}$$

tenglamaga ega bo'lamiz.

(103) tenglamaning umumiy yechimi quyidagi

$$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}}x + C_2 e^{-\sqrt{-\lambda}}x, \quad \text{agar } \lambda < 0, \\ X(x) &= C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x, \quad \text{agar } \lambda > 0, \\ X(x) &= C_1 x + C_2, \quad \text{agar } \lambda = 0 \end{aligned}$$

ko'rinishda bo'ladi.

Agar $\lambda < 0$ bo'lsa, u holda $X(x) \equiv 0$ bo'lishini ko'rsatish qiyin emas. Agar $\lambda > 0$ bo'lsa, u holda yuqoridagi umumiy yechimdan

$$X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

kelib chiqadi. (104) chegaraviy shartlarga asosan $C_2 = 0$ yoki $X(x) = C_1\cos\sqrt{\lambda}x = 0$ kelib chiqadi. Bundan $X'(x) = -C_1\lambda\cos\sqrt{\lambda}x = 0$ bo'ladi va $X'(l) = 0$ chegaraviy shartga ko'ra $\sqrt{\lambda}l = n\pi$ yoki Shturm-Liuvill masalasi cheksiz ko'p

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad n = 0, 1, 2, \ldots \tag{106}$$

xos sonlar ega ekanligi kelib chiqadi. Bularga mos xos funksiyalar

$$X_n(x) = \cos\frac{n\pi}{l}x, \qquad n = 0, 1, 2, \ldots \tag{107}$$

bo‘ladi.

Agar $\lambda = 0$ bo‘lsa, u holda (103) tenglamaning umumiy yechimidan yuqoridagi kabi $C_1 = 0$ va $X(x) = C_2$ ekanligi kelib chiqadi, bundan esa $X'(l) = 0$ chegaraviy shart aynan bajariladi. Demak, (103), (104) Shturm-Liuvill masalasi uchun $\lambda = 0$ xos son va unga mos funksiya $X_0(x) = 1$ bo‘ladi, ya‘ni

$$\lambda_0 = 0, \qquad X_0(x) = 1.$$

(103), (104) masalaning (106) λ_k xos sonlarini (107) va xos funksiyalarini $n = 0$ bo‘lganda (107) ko‘rinishda, ya’ni

$$\lambda_0 = \left(\frac{\pi 0}{l}\right)^2 = 0, \qquad X_0(x) = \cos\frac{\pi 0}{l}x = 1$$

kabi yozish mumkin.

Demak, (103), (104) Shturm-Liuvill masalasi uchun

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \cos\frac{n\pi}{l}x, \qquad n = 0, 1, 2, \ldots$$

xos son va xos funksiyalarga ega bo‘ldik.

Endi (105) tenglamani qaraymiz. Bu tenglama $\lambda = \lambda_n$ bo‘lganda ham ma‘noga ega va

$$T_n''(t) + a^2\lambda_n T_n(t) = 0, \qquad t > 0 \tag{108}$$

tenglamalarni qaraylik. Agar $n = 0$ bo‘lsa, oxirgi tenglamaning umumiy yechimi

$$T_0(t) = A_0 + B_0 t$$

bo‘ladi, bu yerda $A_0,\ B_0$ - ixtiyoriy o‘zgarmaslar.

Agar $n = 1, 2, ...$ bo‘lsa, (108) tenglamaning umumiy yechimi

$$T_n(t) = A_n \cos\left(\frac{na\pi}{l}\right)t + B_n \sin\left(\frac{na\pi}{l}\right)t, \qquad t > 0$$

ko‘rinishda bo‘ladi, bunda A_n va B_n - ixtiyoriy o‘zgarmaslar.

Endi qaralayotgan (100)-(102) aralash masalaning yechimini

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

koʻrinishda izlaymiz, yaʻni

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{na\pi}{l}\right) t + B_n \sin\left(\frac{na\pi}{l}\right) t \right] \cos\left(\frac{n\pi}{l}\right) x, \tag{109}$$

Qaralayotgan masalaning boshlangʻich shartlariga koʻra

$$\varphi(x) = \sum_{n=0}^{\infty} X_n(x) T_n(0) = A_0 + \sum_{n=1}^{\infty} A_n X_n(x),$$

$$\psi(x) = \sum_{n=0}^{\infty} X_n(x) T_n'(0) = B_0 + \sum_{n=1}^{\infty} \frac{n\pi a}{l} B_n X_n(x)$$

boʻladi. Faraz qilaylik, $\varphi(x)$ va $\psi(x)$ funksiyalar kosinuslar boʻyicha Furye qatoriga yoyilsin, yaʻni

$$\varphi(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{l}\right), \ \psi(x) = \frac{\beta_0}{2} + \sum_{n=1}^{\infty} \beta_n \cos\left(\frac{n\pi x}{l}\right).$$

Bu yerda α_n va β_n koeffitsientlar

$$\alpha_n = \frac{2}{l} \int_0^l \varphi(x) \cos\left(\frac{n\pi x}{l}\right) dx, \ \beta_n = \frac{2}{n\pi a} \int_0^l \psi(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

koʻrinishda aniqlanadi.

Shunday qilib, Furye qatorlari uchun standart formulalardan foydalanib, (109) formuladagi A_n va B_n koeffitsientlar uchun quyidagi

$$A_n = \alpha_n = \frac{2}{l} \int_0^l \varphi(x) \cos\left(\frac{n\pi x}{l}\right) dx, \ \text{n=1, 2, ...},$$

$$B_n = \frac{l}{n\pi a} \beta_n = \frac{2}{n\pi a} \int_0^l \psi(x) \cos\left(\frac{n\pi x}{l}\right) dx, \ \text{n=1, 2, ...},$$

$$A_0 = \frac{\alpha_0}{2} = \frac{1}{l}\int\limits_0^l \varphi(x)dx,\ B_0 = \frac{\beta_0}{2} = \frac{1}{l}\int\limits_0^l \psi(x)dx$$

formulalarni olamiz. Endi topilgan A_n va B_n koeffitsientlarni (109) formulaga qo‘yib, (100)-(102) ikkinchi boshlang‘ich-chegaraviy masalaning $u(x,t)$ yechimini hosil qilamiz.

Ba‘zida yuqoridagi masalalarning yetarlicha silliq bo‘lmagan *umumlashgan yechim* deb ataluvchi $u(x,t)$ yechimini topishga to‘g‘ri keladi. Umumlashgan yechim tushunchasi turli tarzda kiritilishi mumkin. Yetarlicha uzluksiz differentsiallanuvchi funksiyalar ketma-ketligi yordamida u quyidagicha kiritiladi: faqaz qilaylik, $u_n(x,t)$ funksiyalar (54) tenglamaning (55) chegaraviy, $u_n(x,0) = \varphi_n(x)$ va $(u_t)_n(x,0) = \psi_n(x)$ boshlang‘ich shartlarni qanoatlantiruvchi klassik yechimlari va bu funksiyalar ketma-ketliklarining limitlari mos ravishda $u(x,t)$, $\varphi(x)$, $\psi(x)$ lar bo‘lsin. Ushbu

$$\lim_{n\to\infty}\int_0^l [\varphi_n(x) - \varphi(x)]^2 = 0,\quad \lim_{n\to\infty}\int_0^l [\psi_n(x) - \psi(x)]^2 = 0$$

munosabatlarni qanoatlantiruvchi yetarlicha silliq $u_n(x,t)$ funksiyalarning limiti $u(x,t)$ (silliq bo‘lishi shart emas) ga (54) - (56) masalaning umumlashgan yechimi deyiladi. U ham (67) qator bilan ifodalanadi. Eslatib o‘tamiz, yuqoridagi yaqinlashish 1-bobning 2-paragrafida kiritilgan o‘rta kvadratik ma’nosidagi yaqinlashishdir.

4.10 To‘g‘ri to‘rtburchakli membrana tebranish tenglamasi uchun aralash masalani yechish

Membrana $0 \le x_1 \le p,\ 0 \le x_2 \le q$ to‘g‘ri to‘rtburchakli G soha bilan ustma-ust tushsin. Uning kichik tebranishlari

$$\frac{\partial^2 u}{\partial t^2} = a^2 \triangle u,\quad \triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{110}$$

tenglamaning

$$u(x_1, x_2, 0) = \varphi(x_1, x_2),$$

$$\frac{\partial u(x_1, x_2, t)}{\partial t}\Big|_{t=0} = \psi(x_1, x_2),\ \ (x_1, x_2) \in G \tag{111}$$

boshlang‘ich va

$$u_{x_1=0} = 0, \quad u_{x_1=p} = 0, \quad u_{x_2=0} = 0, \quad u_{x_2=q} = 0 \tag{112}$$

chegaraviy shartlarni qanoatlantiruvchi yechimini topishdan iboratdir. Bu masalaning yechimini

$$u(x_1, x_2, t) = T(t)v(x_1, x_2) \tag{113}$$

ko‘rinishda izlab, $T(t)$ funksiyani aniqlash uchun

$$T''(t) + \mu^2 a^2 T(t) = 0 \tag{114}$$

tenglamaga, $v(x_1, x_2)$ uchun esa

$$\triangle v(x_1, x_2) + \mu^2 v(x_1, x_2) = 0 \tag{115}$$

tenglama va

$$v_{x_1=0} = 0, \quad v_{x_1=p} = 0, \quad v_{x_2=0} = 0, \quad v_{x_2=q} = 0 \tag{116}$$

chegaraviy shartlarga ega bo‘lamiz. (115) tenglamaga *Gelmgols tenglamasi* deyiladi.

(115), (116) masalaning xos sonlari va xos funksiyalarini topamiz. (115) tenglamada

$$v(x_1, x_2) = X_1(x_1)X_2(x_2)$$

deb, o‘zgaruvchilarni ajratamiz:

$$\frac{X_2'}{X_2} + \mu^2 = -\frac{X_1'}{X_1} = \mu_1^2.$$

Bundan quyidagi ikkita oddiy differesial tenglama kelib chiqadi:

$$X_1''(x_1) + \mu_1^2 X_1(x_1) = 0, \qquad X_2''(x_2) + \mu_2^2 X_2(x_2), \tag{117}$$

bu yerda

$$\mu_2^2 = \mu^2 - \mu_1^2. \tag{118}$$

(116) chegaraviy shartlarga asosan (117) ning birinchi tenglamasini

$$X_1(0) = X_1(p) = 0, \tag{119}$$

ikkinchisini esa

$$X_2(0) = X_2(q) = 0, \tag{120}$$

shartlar bilan yechish kerak.

(117) tenglamalarning umumiy yechimlari, ma'lumki,

$$X_1(x_1) = C_1 \cos(\mu_1 x_1) + C_2 \sin(\mu_1 x_1),$$

$$X_1(x_1) = C_3 \cos(\mu_2 x_2) + C_4 \sin(\mu_2 x_2)$$

ko'rinishga ega bo'ladi.

(119) va (120) chegaraviy shartlarga ko'ra $C_1 = C_3 = 0$ bo'lib, agar $C_2 = C_4 = 1$ deb hisoblasak,

$$X_1(x_1) = \sin(\mu_1 x_1), \quad X_2(x_2) = \sin(\mu_2 x_2)$$

tengliklar hosil bo'ladi, shu bilan birga

$$\sin(\mu_1 p) = \sin(\mu_2 q) = 0 \tag{121}$$

bo'lishi kerak.

(121) tenglamalardan μ_1 va μ_2 lar cheksiz ko'p

$$\mu_{1,m} = \frac{m\pi}{p}, \quad \mu_{2,n} = \frac{n\pi}{q}, \quad m, n = 1, 2, \dots$$

sonlarga ega bo'lishi kelib chiqadi. U holda, (118) tenglikdan μ^2 ning mos sonlarini hosil qilamiz:

$$\mu^2_{m,n} = \mu^2_{1,m} + \mu^2_{2,n} = \left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right)\pi^2. \tag{122}$$

Shunday qilib, (122) xos sonlarga (115), (116) chegaraviy masalaning

$$v_{m,n}(x_1, x_2) = \sin\left(\frac{m\pi x_1}{p}\right)\sin\left(\frac{n\pi x_2}{q}\right) \tag{123}$$

xos funksiyalari mos kelar ekan.

Agar (123) tenglik bilan aniqlangan xos funksiyalarni $\frac{2}{\sqrt{pq}}$ songa ko'paytirsak, bu funksiyalar ortonormallangan funksiyalarning sistemasini tashkil qiladi, ya'ni

$$\frac{4}{pq}\int\limits_0^p\int\limits_0^q \sin\left(\frac{m\pi x_1}{p}\right)\sin\left(\frac{n\pi x_2}{q}\right)\sin\left(\frac{m'\pi x_1}{p}\right)\sin\left(\frac{n'\pi x_2}{q}\right)dx_1dx_2 =$$

$$= \begin{cases} 1, & m = m', \quad n = n', \\ 0, & m \neq m' \ yoki \ n \neq n'. \end{cases}$$

Aytish joizki, (115) va (116) masalaning topilgan xos sonlari orasida karralilari bo'lishi mumkin, ya'ni shunday xos sonlarki, bularga bitta emas, bir nechta chiziqli bog'liq bo'lmagan (xos sonning karrasi qadar) xos funkiyalar mos keladi.

Endi (114) tenglamaga murojaat qilamiz. Har bir $\mu^2 = \mu^2_{mn}$ xos son uchun uning umumiy yechimi

$$T_{mn}(t) = A_{mn}\cos(a\mu_{mn}t) + B_{mn}\sin(a\mu_{mn}t) \qquad (124)$$

ko'rinishga ega bo'ladi, bunda A_{mn} va $B_{mn}-$ ixtiyoriy o'zgarmaslar. Shunday qilib, (110) tenglamaning (112) chegaraviy shartlarni qanoatlantiruvchi xususiy yechimlari (113), (123) va (124) larga asosan quyidagicha bo'ladi:

$$u_{mn}(x_1, x_2, t) =$$

$$= (A_{mn}\cos(a\mu_{mn}t) + B_{mn}\sin(a\mu_{mn}t))\sin\left(\frac{m\pi x_1}{p}\right)\sin\left(\frac{n\pi x_2}{q}\right).$$

Tabiiyki, bundan kelib chiqib, (110)-(111) masalaning yechimini

$$u(x_1, x_2, t) = \sum_{m,n=1}^{\infty}\Big(A_{mn}\cos(a\mu_{mn}t) +$$

$$+ B_{mn}\sin(a\mu_{mn}t)\Big)\sin\left(\frac{m\pi x_1}{p}\right)\sin\left(\frac{n\pi x_2}{q}\right) \qquad (125)$$

ko'rinishda qidiramiz.

Agar ikkilangan (125) qator va uni x_1, x_2 va t o'zgaruvchilar bo'yicha ikki marta differensiallashdan hosil bo'lgan qatorlar tekis yaqinlashuvchi bo'lsa, u holda uning yig'indisi (110) membrana tebranish tenglamasini va (112) chegaraviy shartlarni qanoatlantiradi. (111) boshlang'ich shartlarning bajarilishi uchun

$$u(x_1, x_2, 0) = \varphi(x_1, x_2) = \sum_{m,n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x_1}{p}\right) \sin\left(\frac{n\pi x_2}{q}\right),$$

$$\left.\frac{\partial u(x_1, x_2, t)}{\partial t}\right|_{t=0} =$$

$$= \psi(x_1, x_2) = \sum_{m,n=1}^{\infty} a\mu_{mn} B_{mn} \sin\left(\frac{m\pi x_1}{p}\right) \sin\left(\frac{n\pi x_2}{q}\right) \qquad (126)$$

tengliklar o'rinli bo'lishi zarurdir.

(126) formulalar $\varphi(x_1, x_2)$ va $\psi(x_1, x_2)$ funksiyalarning sinuslar bo'yicha ikkilangan Furye qatoriga yoyilmasidan. Bundan noma'lum A_{mn} va B_{mn} koeffitsientlar

$$A_{mn} = \frac{4}{pq} \int_0^p \int_0^q \varphi(x_1, x_2) \sin\left(\frac{m\pi x_1}{p}\right) \sin\left(\frac{n\pi x_2}{q}\right) dx_1 dx_2,$$

$$B_{mn} = \frac{4}{a\mu_{mn} pq} \int_0^p \int_0^q \psi(x_1, x_2) \sin\left(\frac{m\pi x_1}{p}\right) \sin\left(\frac{n\pi x_2}{q}\right) dx_1 dx_2 \qquad (127)$$

formulalar bilan aniqlanadi.

(127) bilan aniqlangan formulalarni (125) qatorga qo'yib, (110)-(112) masalaning yechimiga ega bo'lamiz.

Agar $\varphi(x_1, x_2)$ va $\psi(x_1, x_2)$ funksiyalar $0 \le x_1 \le p$, $0 \le x_2 \le q$ to'g'ri to'rtburchakdan toq ravishda butun Ox_1x_2 tekislikka x_1 bo'yicha $2p$ davr bilan, x_2 bo'yicha esa $2q$ davr bilan davom ettirilganidan so'ng to'rt marta uzluksiz differensiallanuvchi bo'lsa, u holda (125) qator va uni ikki marta hadlab differensiallagandan so'ng hosil bo'ladigan qatorlar tekis yaqinlashuchi bo'ladi. Bunga xuddi oldingi paragrafdagi kabi ishonch hosil qilish mumkin. Demak, bu holda (110)-(112) masalaning yechimi uchun Furye usuli to'la asoslangan bo'ladi.

4.11 Doiraviy membrana tebranish tenglamasi uchun aralash masalani yechish

Markazi koordinatalar boshida radiusi r bo'lgan va qirralari mahkamlangan doiraviy membrananing erkin tebranishini o'rganish (110) tenglamaning (111) boshlang'ich va

$$u\Big|_{x_1^2+x_2^2=r^2} = 0 \tag{128}$$

chegaraviysartlarni qanoatlantiruvchi yechimini topishga keladi. Bu masalani organishda

$$x_1 = \rho\cos\theta, \quad x_2 = \rho\sin\theta$$

tengliklar bilan aniqlanadigan (ρ, θ) qutb koordinatalariga o'tish qulaydir. Ikki o'lchovli Laplas operatori bu koordinatalarda

$$\triangle = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

ko'rinishga ega bo'ladi. Bunga asosan (110), (111) va (128) masala quyidagicha yoziladi:

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \tag{129}$$

$$u|_{t=0} = \varphi(\rho, \theta), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(\rho, \theta), \tag{130}$$

$$u|_{\rho=r} = 0. \tag{131}$$

Oldingi paragrafda bo'lgani kabi, bu masalani yechish uchun o'zgaruvchilarni ajratish usulidan foydalamiz:

$$u(\rho, \theta, t) = T(t)v(\rho, \theta).$$

Ravshanki, noma'lum $T(t)$ funksiya uchun

$$T''(t) + a^2\mu^2 T(t) = 0$$

oddiy differensial tenglama hosil bo'ladi. Uning umumiy yechimi

$$T(t) = C_1\cos(a\mu t) + C_2\sin(a\mu t)$$

ko'rinishga ega.

$v(\rho,\theta)$ funksiya uchun

$$\frac{\partial^2 v}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial v}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2}+\mu^2 v=0 \tag{132}$$

$$v|_{\rho=r}=0 \tag{133}$$

chegaraviy masalani hosil qilamiz. Bu masalaning yechimini uhbu

$$v(\rho,\theta)=R(\rho)\Theta(\theta)$$

ko'rinishda izlaymiz. Bu ifodani (132) tenglamaga qo'yib,

$$\frac{\Theta''(\theta)}{\Theta(\theta)}=-\frac{\rho^2 R''(\rho)+\rho R(\rho)+\mu^2\rho^2 R(\rho)}{R(\rho)}=\omega=const$$

tengliklarni hosil qilamiz. Bundan quyidagi ikkita oddiy differenial tenglamalarni olamiz:

$$\rho^2 R''(\rho)+\rho R(\rho)+\left(\mu^2\rho^2-\omega\right)R(\rho)=0, \tag{134}$$

$$\Theta''(\theta)+\omega\Theta(\theta)=0. \tag{135}$$

$R(\rho)$ funksiya $\rho=r$ da nolga teng va $\rho=0$ da esa chegaralangan bo'lishi kerak, ya'ni bundan

$$R(r)=0, \quad R(0)\neq\infty$$

chegaraviy shartlarga kelamiz. $v(\rho,\theta)$ funksiya bir qiymatli aniqlanishi uchun $v(\rho,\theta)=v(\rho,\theta+2\pi)$ bo1lishi, bundan esa (133) ga asosan $\Theta(\theta)$ funksiya 2π davrli

$$\Theta(\theta)=\Theta(\theta+2\pi)$$

davriy funksiya bo'lishi kelib chiqadi. Bu esa o'z navbatida (135) tenglamadagi o'zgarmas $\omega=n^2$, $n-$ ixtiyoriy butun son, bo'lishini ko'rsatadi. Demak, (135) tenglamaning umumiy yechimi

$$\Theta_n(\theta)=\alpha_n\cos(n\theta)+\beta_n\sin(n\theta),$$

bu yerda α_n, β_n- ixtiyoriy haqiqiy sonlar.

Endi (134) tenglamaga murojaat qilamiz. Bu Bessel tenglamasi bo‘lib, uning yechimi $\omega = n^2$ da

$$R_n(\rho) = D_n J_n(\mu\rho) + E_n N_n(\mu\rho)$$

ko‘rinishga ega bo‘ladi, bu yerda J_n- n indeksli Beccel, N_n- n indeksli Neyman funksiyalari (7-bobga qarang). $R(\rho)$ funksiya $\rho = r$ da nolga teng bo‘lib, $\rho = 0$ da chegaralangan bo‘lishi kerak. $N_n(\mu\rho)$ funksiya $\rho = 0$ da cheksizlikka aylanishi sababli, $E_n = 0$ deb hisoblashimiz zarur. Bundan $D_n J_n(\mu r) = 0,\ D_n \neq 0$ bo‘lgani uchun $J_n(\mu r) = 0$. $\mu r = \varrho$ belgilash kiritib, ϱ ni aniqlash uchun ushbu

$$J_n(\varrho) = 0$$

transendent tenglamaga kelamiz. Ma’lumki, bu tenglamaning cheksiz ko‘p musbat

$$\varrho_1^{(n)},\ \varrho_2^{(n)},\ \varrho_3^{(n)}, \ldots$$

ildizlari mavjud. Bu ildizlarga

$$\mu_{mn} = \frac{\varrho_m^{(n)}}{r},\ m = 1, 2, \ldots,\quad n = 0, 1, 2, \ldots$$

sonlar mos keladi. Y holda biz tekshirayotgan masalaning mos yechimlari ushbu

$$R_{mn}(\varrho) = J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)$$

ko‘rinishga ega bo‘ladi. U holda (132), (133) masalaning

$$\mu_{mn}^2 = \left(\frac{\varrho_m^{(n)}}{r}\right)^2$$

xos soniga ikkita chiziqli bo‘lmagan

$$J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\cos(n\theta),\quad J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\sin(n\theta),\ m = 1, 2, \ldots,\quad n = 0, 1, 2, \ldots$$

xos funksiyalar mos keladi.

Shunday qilib, yuqoridagilarga asosan (129) tenglamaning (131) chegaraviy shartni qanoatlantiruvchi cheksiz ko'p

$$u_{mn}(\rho,\theta,t) = \Big[\left(A_{mn}\cos\left(\frac{a\varrho_m^{(n)}t}{r}\right) + B_{mn}\sin\left(\frac{a\varrho_m^{(n)}t}{r}\right)\right)\cos(n\theta)+$$

$$+\left(C_{mn}\cos\left(\frac{a\varrho_m^{(n)}t}{r}\right) + D_{mn}\sin\left(\frac{a\varrho_m^{(n)}t}{r}\right)\right)\sin(n\theta)\Big] J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)$$

xususiy yechimlarini tuzish mumkin.

Endi (130) boshlang'ich sartlarni qanoatlantirish maqsadida ushbu ikkilik

$$u(\rho,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Big[\left(A_{mn}\cos\left(\frac{a\varrho_m^{(n)}t}{r}\right) + B_{mn}\sin\left(\frac{a\varrho_m^{(n)}t}{r}\right)\right)\cos(n\theta)+$$

$$+\left(C_{mn}\cos\left(\frac{a\varrho_m^{(n)}t}{r}\right) + D_{mn}\sin\left(\frac{a\varrho_m^{(n)}t}{r}\right)\right)\sin(n\theta)\Big] J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right) \tag{136}$$

qatorni yozamiz. Bu qatorning noma'lum koeffitsientlari (130) boshlang'ich sartlardan aniqlanadi. Haqiqatan, (136) qatorda $t = 0$ deb,

$$\varphi(\rho,\theta) = \sum_{m=1}^{\infty} A_{m0}J_0\left(\frac{\varrho_m^{(0)}\rho}{r}\right) + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} A_{mn}J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\right)\cos(n\theta)+$$

$$+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} C_{mn}J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\right)\sin(n\theta)$$

qatorni hosil qilamiz. Bu qator $\varphi(\rho,\theta)$ funksiyaning $(0, 2\pi)$ intervalda Furye qatoriga yoyilmasidir. Demak, Furye qatorlarining umumiy nazariyasiga ko'ra $\cos(n\theta)$ va $\sin(n\theta)$ funksiyalar oldidagi ifodalar Furye koeffitsientlaridan iborat bo'lishi kerak, ya'ni

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(\rho,\theta)d\theta = \sum_{m=1}^{\infty} A_{m0}J_0\left(\frac{\varrho_m^{(0)}\rho}{r}\right),$$

$$\frac{1}{\pi}\int_0^{2\pi}\varphi(\rho,\theta)\cos(n\theta)d\theta = \sum_{m=1}^{\infty} A_{mn}J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right),$$

$$\frac{1}{\pi}\int_0^{2\pi}\varphi(\rho,\theta)\sin(n\theta)d\theta=\sum_{m=1}^{\infty}C_{mn}J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right).$$

Bu tengliklardan ayonki, ular ixtiyoriy $\phi(\rho)$ funksiyaning Bessel funksiyalari bo'yicha

$$\phi(\rho)=\sum_{m=1}^{\infty}c_mJ_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)$$

yoyilmasidan iboratdir. c_n koeffitsientlar Bessel funksiyalarining ortogonalligidan foydalanib,

$$c_n=\frac{2}{r^2J_{n+1}^2\left(\varrho_m^{(n)}\right)}\int_0^r\rho\phi(\rho)J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)d\rho$$

formula bilan aniqlanadi.

Bu formulaga asosan

$$A_{m0}=\frac{1}{\pi r^2J_1^2\left(\varrho_m^{(0)}\right)}\int_0^r\int_0^{2\pi}\varphi(\rho,\theta)J_0\left(\frac{\varrho_m^{(0)}\rho}{r}\right)\rho d\rho d\theta, \qquad (137)$$

$$A_{mn}=\frac{2}{\pi r^2J_{n+1}^2\left(\varrho_m^{(n)}\right)}\int_0^r\int_0^{2\pi}\varphi(\rho,\theta)J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\cos(n\theta)\rho d\rho d\theta, \qquad (138)$$

$$C_{mn}=\frac{2}{\pi\varrho_m^{(n)}r^2J_{n+1}^2\left(\varrho_m^{(n)}\right)}\int_0^r\int_0^{2\pi}\varphi(\rho,\theta)J_n\left(\frac{\varrho_m^{(n)}\rho}{r}\right)\sin(n\theta)\rho d\rho d\theta. \qquad (139)$$

Xuddi shunga o'xhash, B_{m0}, B_{mn}, D_{mn} koeffitsientlarni ham topamiz. Buning uchun (137), (138) va (139) formulalarda $\varphi(\rho,\theta)$ ni $\psi(\rho,\theta)$ bilan almahtirib, mos ifodalarni $\frac{a\varrho_m^{(n)}}{r}$ ga bo'lish kerak. Koeffitsientlarning topilgan qiymatlarini (136) qatorga qo'yib, (110), (111), (128) masalaning yechimini hosil qilamiz.

4.12 Tor tebranish tenglamasi uchun Koshi va Gursa tipidagi masalalarni yechishning boshqa usullari

1-paragrafda bir jinsli tor tebranish tenglamasi uchun Koshi masalasini yechishda Dalamber usulidan foydalanildi. Dalamber yechimi yordamida Dyuamel prinsipini qo‘llab, bir jinsli bo‘lmagan tenglama uchun ushbu masalaning yechimi olindi. Ushbu paragrafda bu kabi masalalarning yechimini topishda yuqoridagilardan farqli ravishda boshqacha usul qo‘llaniladi. Bu usul tor tebranish tenglamasi uchun xarakteristik masalalarni tadqiq etish uchun qo‘llaniladigan usulga yaqindir. Malumki, xarakteristik masalada noma’lum funksiya xarakteristikalarning ustida beriladi va uni xarakteristik chiziqlar bilan chegaralangan sohada topish talab etiladi. Bu usul yordamida xarakteristik masalalarga yaqin bo‘lgan, aniqrog‘i noma’lum funksiya xarakteristik chiziqlarning va tekislikdagi Dekart koordinatalar sistemasidagi o‘qlarning birortasida berilgan holda uni bu to‘g‘ri chiziqlar orasidagi sohada topish masalalarini yechishda qulaydir.

1. Xarakteristik masala yoki Gursa masalasi. Bir jinsli cheksiz tor tebranish tenglamasini qaraymiz.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t > 0. \tag{140}$$

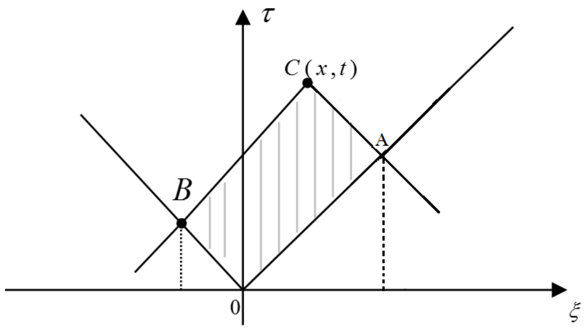
Ma’lumki, (140) tenglamaning $C(x,t)$ nuqtadan o‘tuvchi xarakteristikalari ikkita

$$\xi = x - t + \tau, \quad \xi = x + t - \tau$$

o‘zaro perpendikulyar bo‘lgan to‘g‘ri chiziqlardan iborat. Faraz qilaylik, (140) tenglamaning yechimi koordinatalar boshidan o‘tuvchi xarakteristikalalarning $t > 0$ sohadagi qismida ma’lum bo‘lsin

$$u\big|_{t=x} = \varphi(x), \quad u\big|_{t=-x} = \psi(x), \quad \varphi(0) = \psi(0). \tag{141}$$

Ma'lumki, (140) tenglamaning (141) shartlarini qanoatlantiruvchi klassik yechimini topish masalasiga xarakteristik masala yoki Gursa masalasi deyiladi.



12-chizma. D soha tasviri.

$C(x,t)$ nuqtadan chiquvchi $\xi = x - t + \tau$, $\xi = x + t - \tau$ xarakteristikalari $\xi = \tau$ va $\xi = -\tau$ chiziqlarni mos ravishda $A\left(\frac{x+t}{2}, \frac{x+t}{2}\right)$ va $B\left(\frac{x-t}{2}, -\frac{x-t}{2}\right)$ nuqtalarda kesadi. $O\tau\xi$ Dekart koordinatalar sistemasida pastdan $\tau = |\xi|$ va yuqoridan $\tau = t - |\xi - x|$ chiziqlar bilan chegaralangan sohani D orqali belgilaymiz, ya'ni

$$D = \left\{(\xi,\tau) : |\xi| < \tau < t - |\xi - x|\right\}.$$

(ξ,τ) o'zgaruvchilarga nisbatan yozilgan (140) tenglamani D soha bo'yicha integrallaymiz. Gauss–Ostragradskiy formulasini qo'llash natijasida

$$\int\limits_D \left[\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial\tau}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}\right)\right] d\xi d\tau = -\int\limits_{\partial D} \left[\frac{\partial u}{\partial\tau}d\xi + \frac{\partial u}{\partial\xi}d\tau\right] = 0$$

hosil bo'ladi, bu yerda ∂D orqali D sohaning chegarasi belgilangan. $\partial D = OA + AC + CB + BO$ ekanligidan oxirgi tenglik quyidagi ko'rinishni oladi:

$$\int\limits_{OA} \left[\frac{\partial u}{\partial\xi}d\tau + \frac{\partial u}{\partial\tau}d\xi\right] - \int\limits_{AC} \left[\frac{\partial u}{\partial\xi}d\tau + \frac{\partial u}{\partial\tau}d\xi\right] + \int\limits_{CB} \left[\frac{\partial u}{\partial\xi}d\tau + \frac{\partial u}{\partial\tau}d\xi\right] -$$

$$-\int\limits_{BO} \left[\frac{\partial u}{\partial\xi}d\tau + \frac{\partial u}{\partial\tau}d\xi\right] = 2u(A) - 2u(0) - 2u(C) + 2u(B) = 0.$$

Bu yerdan A, B, C, O nuqtalarning koordinatalarini hisobga olib,

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0)$$

Gursa masalasining yagona yechimini hosil qilamiz. Ravshanki, bu formula (33) formula bilan $t = t_0$, $x = x_0$ bo'lganda ustma-ust tushadi.

Bir jinsli bo'lmagan cheksiz tor tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in \mathbb{R},\ t > 0 \tag{142}$$

uchun (141) shartlarni qanoatlantiruvchi Gursa masalasining yechimi, ravshanki,

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0) + \frac{1}{2}\iint\limits_D f(\xi,\tau)d\tau d\xi$$

formula bilan beriladi. Bu tenglikdagi soha bo'yicha integralni takroriy integral ko'rinishida yozsak,

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0) + \frac{1}{2}\int\limits_{\frac{x-t}{2}}^{\frac{x+t}{2}} \int\limits_{|\xi|}^{t-|\xi-x|} f(\xi,\tau)d\tau d\xi \tag{143}$$

hosil bo'ladi. (143) formula masalalar yechishda qulaydir. Bu formula bilan aniqlangan $u(x,t)$ funksiya (140), (141) masalaning klassik yechimi bo'lishi uchun qaralayotgan sohada φ va ψ funksiyalar ikki marta, f funksiya esa bir marta uzluksiz differensiallanuvchi bo'lishi kerak.

2. Gursa tipidagi masala. Differensial tenglamalarning tadbiqiy masalalarini yechishda $x > 0,\ t > 0$ sohada qaralgan (140) tenglamaning

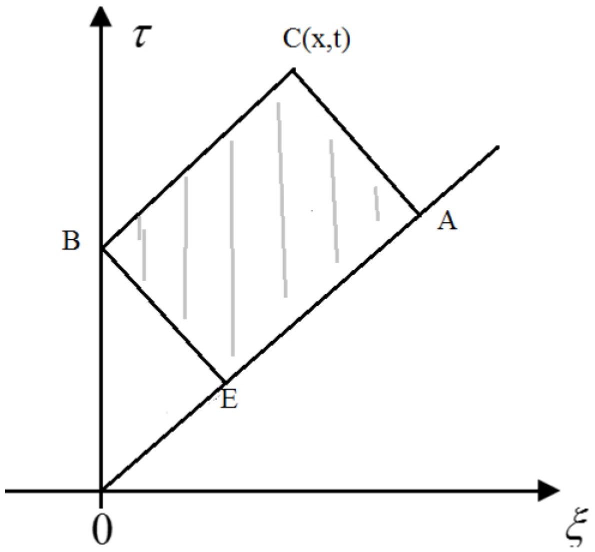
$$u\big|_{t=x} = \varphi(x), \quad u\big|_{x=0} = \psi(x), \quad \varphi(0) = \psi(0) \tag{144}$$

shartlarini qanoatlantiruvchi $x = 0,\ t = 0$ chiziqlar bilan chegaralangan burchak sohada yechimini topishga to'g'ri keladi.

Bu kabi masalalar ham yuqoridagi yuritilgan mulohazalarga asoslanib yechilishi mumkin.

$O\xi\tau$ tekislikda biror $C(x,t)$ $(t > x > 0)$ nuqtadan o'tuvchi (140) tenglamaning xarakteristikalari $\xi = \tau$ va $\xi = 0$ chiziqlarni mos ravishda $A\left(\frac{x+t}{2}, \frac{x+t}{2}\right)$ va $B(0, t-x)$ nuqtalarda kesib o'tadi. B nuqtadan chiquvchi $\xi = t - x - \tau$ xarakteristika $\xi = \tau$ chiziq bilan $E\left(\frac{t-x}{2}, \frac{t-x}{2}\right)$ nuqtada kesishadi EA, AC, CA va BE kesmalar bilan chegaralangan 13-chizmadagi to'g'ri to'rtburchakli sohani D_1 orqali belgilaymiz, ya'ni

$$D_1 = \left\{(\xi,\tau) : \frac{t-x}{2} + \left|\xi - \frac{t-x}{2}\right| < \tau < t - |\xi - x|\right\}.$$



13-chizma. D_1 soha tasviri.

(ξ,τ) o'zgaruvchilarga nisbatan yozilgan bir jinsli bo'lmagan (142) tenglamani D_1 soha bo'yicha integrallaymiz. Gauss–Ostrogradskiy formulasiga ko'ra quyidagiga ega bo'lamiz:

$$\int\limits_{D_1} \left[\frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}\right) - \frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial\tau}\right)\right] d\xi d\tau =$$

$$= \int\limits_{EA+AC+CB+BE} \left[\frac{\partial u}{\partial\xi} d\tau + \frac{\partial u}{\partial\tau} d\xi\right] = \int\limits_{D_1} f(\xi,\tau) d\xi d\tau. \qquad (145)$$

EA, AC, CA va BE kesmalar mos ravishda $\tau = \xi$, $\tau = x + t - \xi$, $\tau = t - x + \xi$, $\tau = t - x - \xi$ to'g'ri chiziqlar ustida yotadi. Shuning uchun EA da $d\tau = d\xi$, AC da $d\tau = -d\xi$, CB da $d\tau = d\xi$ va BE da $d\tau = -d\xi$ tengliklar bajariladi. Bularni hisobga olib, (145) tengliklardagi ikkinchi ifodani quyidagicha almashtirish mumkin:

$$\int\limits_{EA}\left[\frac{\partial u}{\partial\xi}d\xi+\frac{\partial u}{\partial\tau}d\tau\right]-\int\limits_{AC}\left[\frac{\partial u}{\partial\xi}d\xi+\frac{\partial u}{\partial\tau}d\tau\right]+\int\limits_{CB}\left[\frac{\partial u}{\partial\xi}d\xi+\frac{\partial u}{\partial\tau}d\tau\right]-$$

$$-\int\limits_{BE}\left[\frac{\partial u}{\partial\xi}d\xi+\frac{\partial u}{\partial\tau}d\tau\right]=2u(A)+u(E)-2u(C)+2u(B).$$

Olingan natijalarni (145) tengliklarning oxirgi ifodasi bilan tenglashtirib (142), (144) masalaning yechimini hosil qilamiz

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)-\varphi\left(\frac{x-t}{2}\right)+\psi(t-x)+\frac{1}{2}\iint\limits_{D_1}f(\xi,\tau)d\xi d\tau.$$

Bu formulani quyidagi ko‘rinishda ham yozish mumkin:

$$u(x,t)=\varphi\left(\frac{x+t}{2}\right)-\varphi\left(\frac{x-t}{2}\right)+\psi(t-x)+$$
$$+\frac{1}{2}\int\limits_{0}^{\frac{x+t}{2}}\int\limits_{\frac{t-x}{2}+\left|\xi-\frac{t-x}{2}\right|}^{t-|\xi-x|}f(\xi,\tau)d\xi d\tau.$$

3. Koshi masalasi. Endi Gauss–Ostrogradskiy formulasini qo‘llash orqali bir jinsli bo‘lmagan tor tebranish tenglamasi uchun Koshi masalasining yechimini to‘g‘ridan-to‘g‘ri hosil qilamiz.

(ξ,τ) o‘zgaruvchilarga nisbatan yozilgan cheksiz tor tebranish tenglamasi uchun quyidagi Koshi maslasini qaraymiz:

$$u_{\tau\tau}-u_{\xi\xi}=f(\xi,\tau),\quad \xi\in\mathbb{R},\ t>0 \tag{146}$$

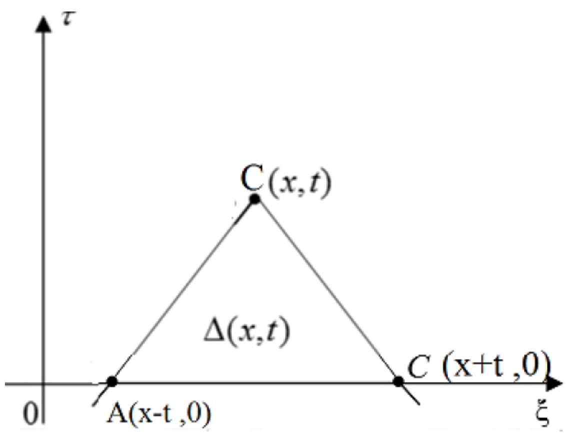
$$u(0,\xi)=\varphi(\xi),\quad u_\tau(0,\xi)=\psi(\xi),\quad \xi\in\mathbb{R}. \tag{147}$$

Berilgan funksiyalar $f(\tau,\xi)\in C^1(\mathbb{R}\times\{\tau>0\})$, $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$ shartlarni qanoatlantiradi deb faraz qilamiz. (146) tenglamaning (x,t) nuqtalaridan o‘tuvchi xarakteristikalari va $\tau=0$ to‘g‘ri chiziq bilan chegaralangan xarakteristik uchburchak deb ataluvchi $\Delta(x,t)=\{(\xi,\tau): x-(t-\tau)\le \xi\le x+(t-\tau),\quad 0<\tau<t\}$ soha (14-chizma) bo‘yicha (146) tenglama-

ning ikkala tomonini integrallaymiz. Natijada tenglamaning chap tomoni quyidagi koʻrinishni oladi:

$$\int\limits_{\Delta(x,t)} \left[\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial\tau}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}\right)\right] d\xi d\tau = -\int\limits_{\partial\Delta(x,t)} \left[\frac{\partial u}{\partial\tau}d\xi + \frac{\partial u}{\partial\xi}d\tau\right] =$$

$$= -\left[\int\limits_{AC}\left[\frac{\partial u}{\partial\tau}d\xi + \frac{\partial u}{\partial\xi}d\tau\right] + \int\limits_{CB}\left[\frac{\partial u}{\partial\tau}d\xi + \frac{\partial u}{\partial\xi}d\tau\right] + \int\limits_{BA}\left[\frac{\partial u}{\partial\tau}d\xi + \frac{\partial u}{\partial\xi}d\tau\right]\right].$$



14-chizma. $\Delta(x,t)$ soha tasviri.

AC kesmada $\tau = 0$, CB kesmada $\xi = x + t - \tau$ va BA kesmada $\xi = x - (t - \tau)$ ekanligini inobatga olsak, oxirgi tengliklardan

$$\int\limits_{\Delta(x,t)} \left[\frac{\partial}{\partial\tau}\left(\frac{\partial u}{\partial\tau}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}\right)\right] d\xi d\tau =$$

$$= -\left[\int\limits_{x-t}^{x+t} \frac{\partial u}{\partial\tau}\bigg|_{\tau=0} d\xi - \int\limits_0^t du(\tau, x+t-\tau) + \int\limits_0^t du(\tau, x-t+\tau)\right] =$$

$$= -\left[\int\limits_{x-t}^{x+t} \frac{\partial u}{\partial\tau}\bigg|_{\tau=0} d\xi - 2u(t,x) + u(0,x+t) + u(0,x-t)\right]$$

kelib chiqadi.

(146) tenglamaning oʻng qismi va (147) boshlangʻich shartlarni e'tiborga olib, oxirgi tengliklardan Dalamber formulasini hosil qilamiz:

$$u(t,x) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\xi)d\xi + \frac{1}{2}\iint\limits_{\Delta(x,t)} f(\tau,\xi)d\xi d\tau.$$

Bu formula (7) formula bilan $c = 1$ da ustma-ust tushadi.

5-Bob. Parabolik tenglamalar

Bu bobda parabolik tipdagi tenglamalar uchun Koshi, boshlang‘ich-chegaraviy masalalarining yechimi, fundamental yechim va uning xossalari haqida fikr yuritamiz.

5.1 Issiqlik o‘tkazuvchanlik tenglamasi. Masalalarning qo‘yilishi

Issiqlik o‘tkazuvchanlik

$$c(x)\rho(x)u_t = div\big(k(x)gradu\big) + F(x,t) \tag{1}$$

tenglamasi fazoviy va vaqt o‘zgarishlariga bog‘liq bo‘lgan $u(x,t)$ haroratni ifodalaydi. U parabolik tipdagi tenglamadir.

Chegaraviy shartlar. Fazoviy o‘zgaruvchilar tegishli bo‘lgan $\mathbb{R}^1 \equiv \mathbb{R}$, $\mathbb{R}^2$ yoki $\mathbb{R}^3$ - bir, ikki yoki uch o‘lchovli fazolardagi sohani D, uning chegarasini esa S orqali belgilaymiz. Eslatib o‘tamiz, soha - bog‘langan ochiq to‘plam. D soha chegaralangan deyiladi, agarda u chekli radiusga ega biror shar ($D \subset \mathbb{R}^3$), yoki chekli radiusli doira ($D \subset \mathbb{R}^2$) ichida yotsa, yoki chekli oraliqdan ($D \subset \mathbb{R}$) iborat bo‘lsa. S chegara silliq (bo‘lakli silliq) $D \subset \mathbb{R}^3$ soha uchun ikki yoqli sirt yoki $D \subset \mathbb{R}^2$ uchun silliq (bo‘lakli silliq) xos nuqtalarga ega bo‘lmagan va o‘zaro kesishmaydigan egri chiziqdan iborat deb hisoblanadi. $D \subset \mathbb{R}$ uchun S bitta yoki ikkita nuqtadan iborat. S ning har bir nuqtasida D sohaga nisbatan tashqariga yo‘naltirilgan $n, |n| = 1$ normal vektorni qaraymiz. U holda, n normal bo‘yicha $\frac{\partial}{\partial n}$ hosila aniqlangan bo‘ladi.

Masalan, l uzunlikka ega bo‘lgan sterjenni $D=\{x\in\mathbb{R}:0<x<l\}$ soha kabi beramiz. U holda,

$$S=\{x=0\}\cup\{x=l\},\ \frac{\partial}{\partial n}\bigg|_{x=0}=\frac{\partial}{\partial(-x)}\bigg|_{x=0},\ \frac{\partial}{\partial n}\bigg|_{x=l}=\frac{\partial}{\partial x}\bigg|_{x=l};$$

nur - $S=\{x=0\}$ chegaraga ega $D=\{x\in\mathbb{R}:0<x<+\infty\}$ cheksiz soha, $\frac{\partial}{\partial n}\Big|_{x=0}=\frac{\partial}{\partial(-x)}\Big|_{x=0};$

to‘g‘ri chiziq - $D=\{x\in\mathbb{R}\},\ S=\emptyset;$

doira - $D=\left\{(x,y)\in\mathbb{R}^2:\ 0\le x^2+y^2<r_0\right\}$ soha, $S=\{r=r_0\}$ - aylana, $\frac{\partial}{\partial n}\Big|_S=\frac{\partial}{\partial r}\Big|_{r=r_0};$

shar chegarasi $S=\{r=r_0\}$ sferadan iborat bo‘lgan

$$D=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:0\le x_1^2+x_2^2+x_3^2<r_0^2\right\}=$$
$$=\left\{0<r<r_0,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le\varphi\le 2\pi\right\}$$

to‘plam bilan aniqlangan soha, $\frac{\partial}{\partial n}\Big|_S=\frac{\partial}{\partial(r)}\Big|_{r=r_0}$ va boshqalar.

(1) tenglama D sohaning ichki nuqtalari uchun keltirib chiqarilgan. U S chegarada bajarilmasligi ham mumkin. Aniq issiqlik tarqalish jarayonini ajratish uchun S chegarada beriladigan qo‘shimcha shartlar zarur bo‘ladi. Ularning ba’zilari bilan tanishamiz. D sohaning chegarasida berilgan harorat ushlab turilsa, $u(x,t)=\mu(x,t),\ x\in S,\ t\ge 0$ shart beriladi, bu yerda $\mu(x,t)-$ ma’lum funksiya. Bunga birinchi chegaraviy shart yoki *Dirixle sharti* deyiladi.

Agar S chegarada issiqlik oqimi berilgan bo‘lsa, bu shart

$$\frac{\partial u(x,t)}{\partial n}=v(x,t),\ x\in S,\ t\ge 0$$

($v(x,t)-$ ma’lum funksiya) kabi yoziladi. Bu ikkinchi tur chegaraviy shart yoki *Neyman sharti* deb yuritiladi.

Shuningdek, uchinchi tur

$$\frac{\partial u(x,t)}{\partial n}+h(x)u(x,t)=\eta(x,t),\ x\in S,\ t\ge 0,$$

bu yerda $h(x)$ va $\eta(x,t)$ - ma’lum funksiyalar, chegaraviy shartlar ham beriladi.

M i s o l. l uzunlikka ega bo'lgan sterjenning uchlarida quyidagi chegaraviy shartlar qo'yilishi mumkin:

$u(0,t) = \mu_1(t), u(l,t) = \mu_2(t)-$ birinchi tur chegaraviy shartlar;

$u_x(0,t) = \nu_1(t), u_x(l,t) = \nu_2(t)-$ ikkinchi tur chegaraviy shartlar;

$u_x(0,t) - h_1 u(0,t) = \nu_1(t),\ u_x(0,t) + h_2 u(0,t) = \nu_2(t)$, bu yerda $h_1 = const > 0,\ h_2 = const > 0$ - uchinchi tur chegaraviy shartlar.

E s l a t m a. Bayon etilgan chegaraviy shartlar $u(x,t)$ funksiyaga nisbatan chiziqlidir. Murakkabroq fizik qonuniyatlarni ifodalovchi chiziqli bo'lmagan chegaraviy shartlar ham qaralishi mumkin.

5.2 Boshlang'ich-chegaraviy masalalar. Klassik yechim

O'zgarmas koeffitsientli

$$u_t = a^2 \Delta u + f(x,t), \quad u = u(x,t) \tag{2}$$

issiqlik o'tkazuvchanlik tenglamasi uchun masalalarni qaraymiz.

Ta'kidlash joizki, ularning asosiy xossalari (1) tenglama uchun qo'yilgan o'xshash masalalar uchun ham o'rinli bo'ladi. Fazoviy o'zgaruvchiga nisbatan qaralayotgan sohaning S chegarasida qo'yiladigan chegaraviy shartlar (2) tenglamaning yagona yechimini aniqlash uchun yetarli bo'lmasligi mumkin. Yana, $u(x,0) = \varphi(x),\ x \in \overline{D}$ boshlang'ich shartni ham qo'yamiz, bu yerda $\varphi-$ berilgan funksiya. Bu shart $t = 0$ vaqtda haroratning ma'lum ekanligini bildiradi.

D chegaralangan soha bo'lsin. Ochiq Q_T silindr deb,

$$Q_T = D \times (0,T] = \{(x,t) : x \in D,\ t \in (0,T]\}$$

sohani ataymiz.

$$\overline{Q}_T = \overline{D} \times [0,T] = \{(x,t) : x \in D,\ t \in [0,T]\}$$

yopiq silindr. Agar $T = +\infty$ bo'lsa, u holda $Q = D \times (0,+\infty)$.

Issiqlik o'tkazuvchnlik tenglamasi uchun boshlang'ich-chegaraviy masala quyidagicha qo'yiladi:

$$u_t = a^2\Delta u + f(x,t),\ (x,t) \in Q \tag{3}$$

tenglamaning

$$u(x,0) = \varphi(x),\ x \in \overline{D} \tag{4}$$

boshlang'ich va

$$\alpha\frac{\partial u(x,t)}{\partial u} + \beta u(x,t) = \chi(x,t),\ x \in S,\ t \in [0,+\infty), \tag{5}$$

$$\alpha + \beta > 0,\ \alpha \geq 0,\ \beta \geq 0$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechim topilsin.

$T > 0$ sonni tayin qilib, boshlang'ich-chegaraviy masalalarning $D \subset \mathbb{R}^3$ bo'lgan holi uchun klassik yechimlari ta'rifini beramiz. $D \subset \mathbb{R}^2$ va $D \subset \mathbb{R}$ hollar uchun ta'riflar o'xshash ravishda kiritiladi.

T a ' r i f. Birinchi boshlang'ich-chegaraviy masalaning $(\alpha \equiv 0, \beta \neq 0)$ klassik yechimi deb, quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyaga aytiladi:

1) $u(x,t)$ $\overline{Q}_T$ sohada uzluksiz;

2) $u(x,t)$ ochiq Q_T sohada uzluksiz $u_t,\ u_{x_1x_1},\ u_{x_2x_2},\ u_{x_3x_3},\ x = (x_1,x_2,x_3)$ hosilalarga ega va bu sohada (3) tenglamani qanoatlantiradi;

3) $u(x,t)$ $t = 0$ da berilgan (4) qiymatlarni qabul qiladi;

4) $u(x,t)$ funksiya $u(x,t) = \mu(x,t),\ x \in S,\ t \geq 0$ chegaraviy shartni qanoatlantiradi.

Ta'rifning 2-shartida qayd etilgan uzluksiz hosilalarga ega bo'lgan funksiyalar sinfi, odatda, $C^{2,1}(Q_T)$ kabi belgilanadi. Bundan

$$u(x,t) \in C(\overline{Q_T} \cap C^{2,1}(Q_T).$$

T a ' r i f. Ikkinchi boshlang'ich - chegaraviy masalaning $(\alpha \neq 0, \beta \equiv 0)$ klassik yechimi deb, quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiyaga aytiladi:

1) $u(x,t)$ $\overline{Q}_T$ sohada uzluksiz;

2) $u(x,t)$ $\overline{Q}_T$ sohada uzluksiz $u_{x_1},\ u_{x_2},\ u_{x_3},\ x = (x_1, x_2, x_3)$ hosilalarga ega ($t = 0$ da D sohaning ichki nuqtalaridan tashqari);

3) $u(x,t)$ ochiq Q_T sohada uzluksiz $u_t,\ u_{x_1x_1},\ u_{x_2x_2},\ u_{x_3x_3}$ hosilalarga ega va bu sohada (3) tenglamani qanoatlantiradi;

4) $u(x,t)$ $t = 0$ da berilgan (4) qiymatlarni qabul qiladi;

5) $u(x,t)$ berilgan

$$\frac{\partial u(x,t)}{\partial u} = \upsilon(x,t),\ x \in S,\ t \geq 0$$

chegaraviy shartni qanoatlantiradi.

Bunda

$$u(x,t) \in C(\overline{Q}_T \cap C^{1,0}(Q_T \cup \Gamma_T) \cap C^{2,1}(Q_T),$$

bu yerda $\Gamma_T = S \times (0,T)$.

T a ' r i f. Uchinchi boshlang‘ich - chegaraviy masalaning ($\alpha \neq 0$) klassik yechimi deb, oldingi ta’rifdagi 1)-4)- shartlarni hamda

$$\frac{\partial u(x,t)}{\partial u} + h(x)u(x,t) = \chi(x,t),\ x \in S,\ t \geq 0$$

chegaraviy shartni qanoatlantiruvchi $u(x,t)$ funksiya aytiladi.

Ta’riflardagi berilgan barcha $f, \varphi, \mu, \upsilon, h, \chi$ funksiyalar uzluksiz deb faraz qilinadi.

E s l a t m a. (3)-(5) boshlang‘ich-chegaraviy masalaning klassik yechimi mavjud bo‘lishining zaruriy sharti quyidagi (4) boshlang‘ich va (5) chegaraviy berilganlarning kelishuvchanlik shartidir:

$$\alpha\frac{\partial\varphi(x)}{\partial n} + \beta\varphi(x) = \chi(x,0),\ x \in S$$

Ko‘pincha, klassik yechimning mavjudligiga qo‘yilgan talablarni qanoatlantirmaydigan masalalarni tekshirishga to‘g‘ri keladi. Masalan, kelishuvchanlik shartlari bajarilmasligi mumkin. Bunday yechimlar umumlashgan yechim ma’nosida tushuniladi.

5.3 Maksimum prinsipi

Issiqlik o‘tkazuvchanlik tenglamasi yechiminig muhim xossasini ifodalovchi teoremani ko‘rib chiqamiz. D chegaralangan soha va $T > 0$ tayin son bo‘lsin.

T e o r e m a (maksimum prinsipi). Issiqlik o‘tkazuvchanlik

$$u_t = a^2 \Delta u, \ (x,t) \in Q_T$$

tenglamasining $\overline{Q}_T$ yopiq silindrdagi uzluksiz yechimi o‘zining maksimum qiymatini yoki $t = 0$ da, yoki D sohaning S chegarasida qabul qiladi.

Isbot.

$$A = max \left\{ \max_{x \in \overline{D}} u(x,0), \ \max_{x \in S, t \in [0,T]} u(x,t) \right\}$$

belgilashni kiritamiz. Barcha $(x,t) \in \overline{Q}_T$ nuqtalar uchun $u(x,t) \leq A$ ekanligini ko‘rsatish zarur. Teskarisini faraz qilamiz, shunday $(x_0, t_0) \in Q_T$ nuqta mavjud bo‘lib, bu nuqtada $u(x,t)$ funksiya o‘zining A dan katta bo‘lgan maksimal qiymatiga erishsin, ya’ni $u(x_0,t_0) = A + \varepsilon, \ \varepsilon > 0$. Yordamchi

$$v(x,t) = u(x,t) + \alpha(t_0 - t), \ \alpha > 0$$

funksiyani qaraymiz. Ravshanki,

$$v(x_0,t_0) = u(x_0,t_0) = A + \varepsilon$$

Endi $v(x,t)$ funksiyaning D sohaning S chegarasida yoki boshlang‘ich $t = 0$ vaqtdagi maksimal qiymatini baholaymiz:

$$max \left\{ \max_{x \in \overline{D}} v(x,0), \ \max_{x \in S, t \in [0,T]} v(x,t) \right\} \leq A + \alpha T < A + \frac{\varepsilon}{2},$$

agar $\alpha < \frac{\varepsilon}{2T}$.

Shunday qilib, $v(x,t)$ funksiyaning silindr chegarasidagi maksimal qiymati uning silindr ichidagi biror qiymatidan kichik. Bundan esa $v(x,t)$ funksiyaga silindr ichida maksimal qiymat beruvchi (x_1,t_1) nuqtaning mavjudligi kelib chiqadi, ya’ni $v(x_1,t_1) \geq v(x_0,t_0) = A + \varepsilon, \ (x_1,t_1) \in Q_T$. (x_1,t_1) maksimum nuqta bo‘lganligi sababli $v(x,t)$ funksiyaning birinchi tartibli hosilalari uchun

$$grad \ v(x_1,t_1) = 0, \ \frac{\partial v}{\partial t}\Big|_{t=t_1, x=x_1} \geq 0$$

$$\left(\frac{\partial v}{\partial t}\Big|_{t=t_1, x=x_1} = 0, \ agar \ t \neq T \ yoki \ \frac{\partial v}{\partial t}\Big|_{t=T, x=x_1} \geq 0 \right)$$

va ikkinchi tartibli hosilalar uchun

$$\Delta v(x_1, t_1) \le 0$$

munosabatlar o‘rinli. Bulardan $u(x,t)$ funksiyaga nisbatan quyidagilar kelib chiqadi:

$$u(x,t) = v(x,t) - \alpha(t_0 - t), \alpha > 0,$$

$$grad\ u(x_1, t_1) = grad\ v(x_1, t_1) = 0,$$

$$\Delta u(x_1, t_1) = \Delta v(x_1, t_1) \le 0,$$

$$\frac{\partial u}{\partial t}\Big|_{t=t_1, x=x_1} = \frac{\partial v}{\partial t}\Big|_{t=t_1, x=x_1} + \alpha \ge \alpha > 0.$$

Shunday qilib, Q_T sohaning ichida yotuvchi (x,t) nuqtada $\Delta u \le 0,\ u_t > 0$, ya’ni bu nuqtada $u(x,t)$ funksiya issiqlik o‘tkazuvchanlik tenglamasini qanoatlantirmaydi. Hosil bo‘lgan ziddiyat teorema isbotini yakunlaydi.

5.4 Ekstremum prinsipi

Bir jinsli issiqlik o‘tkazuvchanlik tenglamasi uchun minimum prinsipi ham o‘rinli.

T e o r e m a (minimum prinsipi). Issiqlik o‘tkazuvchanlik

$$u_t = a^2 \Delta u,\ (x,t) \in Q_T$$

tenglamasining $\overline{Q}_T$ yopiq silindrdagi uzluksiz yechimi o‘zining minimal qiymati yoki $t = 0$ da, yoki D sohaning S chegarasida qabul qiladi.

Isbot. $u_1(x,t) = -u(x,t)$ funksiya ham issiqlik o‘tkazuvchanlik tenglamasini qanoatlantiradi.

$u_1(x,t)$ funksiyaning maksimal qiymati $u(x,t)$ funksiya uchun minimal qiymat bo‘ladi. Bundan teoremaning isboti maksimum prinsipidan kelib chiqadi.

N a t i j a. Maksimum va minimum prinsiplaridan ekstremum prinsipi kelib chiqadi:

Issiqlik o‘tkazuvchanlik

$$u_t = a^2 \Delta u,\ (x,t) \in Q_T$$

tenglamasining $\overline{Q}_T$ yopiq silindrdagi uzluksiz $u(x,t)$ yechimining barcha qiymatlari uning soha chegarasidagi minimal va maksimal qiymatlari orasida yotadi:

$$\min\left\{\min_{x\in\overline{D}} u(x,0),\ \min_{x\in S, t\in[0,T]} u(x,t)\right\} \le u(x,t) \le$$
$$\le \max\left\{\max_{x\in\overline{D}} u(x,0),\ \max_{x\in S, t\in[0,T]} u(x,t)\right\}.$$

E s l a t m a. $u(x,t) = const$ funksiya issiqlik o‘tkazuvchanlik tenglamasini qanoatlantiradi va ekstremum prinsipiga zid emas.

M i s o l. $u(x,t) = x^2 + 2a^2t$ funksiya

$$Q_T = (x,t) : 0 < x < l,\ 0 < t \le T$$

sohada $u_t = a^2 u_{xx}$ tenglamani qanoatlantiradi va

$$\overline{Q}_T = (x,t) : 0 \le x \le l,\ 0 \le t \le T$$

yopiq sohada uzluksiz. Bu funksiya $\overline{Q}_T$ sohada o‘zining maksimum qiymatiga $x = l,\ t = T$ nuqtada, minimum qiymatga esa $x = 0,\ t = 0$ nuqtada erishadi.

Maksimum va minimum prinsipidan umumiyroq bo‘lgan

$$c\rho u_t = div(k(x) grad\ u) - qu,\ q > 0,\ c > 0,\ \rho > 0$$

tenglama uchun ham o‘rinli.

5.5 Birinchi boshlang‘ich-chegaraviy masala yechimining yagonaligi

Issiqlik o‘tkazuvchanlik tenglamasi uchun birinchi boshlang‘ich-chegaraviy masala ushbu

$$u_t = a^2\Delta + f(x,t),\ (x,t) \in Q \tag{6}$$

tenglamani va

$$u(x,0) = \varphi(x),\ x \in \overline{D}, \tag{7}$$

$$u(x,t) = \mu(x,t),\ x \in S,\ t \in \overline{\mathbb{R}}_+, \tag{8}$$

$$Q = D \times \mathbb{R}_+,\ \mathbb{R}_+ = (0, +\infty)$$

boshlang'ich-chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani topishdan iborat.

T e o r e m a. (6)-(8) masala yagona klassik yechimga ega.

Isbot. Faraz qilaylik, (6)-(8) masalaning ikkita $u_i(x,t),\ i = 1,2$ klassik yechimi mavjud bo'lsin. $T > 0$ - tayin son va $u_i(x,t) \in C\left(\overline{Q}_T\right) \cap C^{2,1}(Q_T),\ i = 1,2.$ U holda $v(x,t) = u_1(x,t) - u_2(x,t)$ funksiya

$$v_t = a^2 \Delta v,\ (x,t) \in Q_T,$$

$$v(x,0) = 0,\ x \in \overline{D},$$

$$v(x,t) = 0,\ x \in S,\ t \in [0,T],\ v \in C\left(\overline{Q}_T\right)$$

masalaning yechimi bo'ladi.

$v(x,t)$ funksiya uchun ekstremum prinsipi o'rinli ($u_i(x,t),\ i = 1,2$ funksiyalar bir jinsli tenglamani qanoatlantirgani uchun ekstremum prinsipini qo'llab bo'lmaydi). Bundan ekstremum prinsipiga asosan $0 \le v(x,t) \le 0$, ya'ni $v(x,t) = 0$.

5.6 Birinchi boshlang'ich-chegaraviy masala yechimining turg'unligi

Maksimum prinsipidan ixtiyoriy $T > 0$ uchun quyidagi teorema kelib chiqadi:

T e o r e m a. (6)-(8) masala yechimi boshlang'ich va chegaraviy shartlarga uzluksiz bog'liq.

Isbot. $u_i(x,t),\ i = 1,2$ funksiyalar ushbu

$$u_t = a^2 \Delta u + f(x,t),\ (x,t) \in Q_T,$$

$$u(x,0) = \varphi_i(x),\ x \in \overline{D},$$

$$u(x,t) = \mu_i(x,t),\ x \in S,\ t \in [0,T],\ i = 1,2$$

masalalarning yechimlari bo‘lsin.

Faraz qilaylik, ixtiyoriy $\varepsilon > 0$ soni uchun $| \varphi_1(x) - \varphi_2(x) | \leq \delta, \ x \in \overline{D}$ va

$|\mu_1(x,t) - \mu_2(x,t)| \leq \delta, \ x \in S, \ t \in [o,T]$ tengsizliklar o‘rinli bo‘lsin. $|u_1(x,t) - u_2(x,t)| \leq \varepsilon, \ (x,t) \in \overline{Q}_T$ bo‘lishini ko‘rsatamiz. $v(x,t) = u_1(x,t) - u_2(x,t)$ funksiyani qaraymiz. Ravshanki, $v(x,t)$ funksiya

$$v_t = a^2 \Delta v, (x,t) \in Q_t,$$

$$v(x,0) = \varphi_1(x) - \varphi_2(x), \ x \in \overline{D},$$

$$v(x,t) = \mu_1(x,t) - \mu_2(x,t), \ x \in S, \ t \in [o,T]$$

boshlang‘ich chegaraviy masalaning yechimi.

$v(x,t)$ funksiya boshlang‘ich va chegaraviy nuqtalarda $|v(x,0)| \leq \varepsilon, \ x \leq \overline{D}$ va $|v(x,t)| \leq \varepsilon, \ x \in S, \ t \in [0,T]$ shartlarni qanoatlantirgani uchun maksimum prinsipidan $|v(x,t)| \leq \varepsilon, \ x \in \overline{Q}_T$ tengsizlik kelib chiqadi.

(6)-(8) masalaning yechimi $f(x,t)$ funksiyaga nisbatan ham turg‘undir. Bu tasdiq maksimum va minimum prinsiplarini bir jinsli bo‘lmagan (6) tenglama uchun umumlashtiruvchi teoremalardan kelib chiqadi.

5.7 Issiqlik o‘tkazuvchanlik tenglamasi uchun Koshi masalasi

Chegaralanmagan sterjenda issiqlik tarqalishi.

Ma’lumki, chegaralanmagan bir jinsli sterjenda issiqlik tarqalish tenglamasi uchun Koshi masalasi quyidagicha qo‘yiladi: $\{x \in \mathbb{R}, t > 0\}$ sohada

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{9}$$

tenglamani va

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R} \tag{10}$$

boshlang‘ich shartni qanoatlantiruvchi $u(x,t)$ funksiya topilsin.

Avvalo (9) tenglamaning trivial bo‘lmagan

$$u(x,t) = X(x)T(t) \tag{11}$$

ko'rinishdagi xususiy yechimni topamiz.

(11) ni (9) ga qo'yib,

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 = const$$

yoki

$$T'(t) + a^2\lambda^2 T(t) = 0, \quad X''(t) + \lambda X(x) = 0$$

tenglamalarni olamiz va bularning yechimlari mos ravishda

$$T(t) = exp(-a^2\lambda^2 t), \quad X(x) = Acos\lambda x + Bsin\lambda x$$

bo'lib, A va B ixtiyoriy o'zgarmaslar λ ga bog'liq bo'lishi mumkin. U holda (11) ga ko'ra ixtiyoriy $A(\lambda)$ va $B(\lambda)$ lar uchun ham

$$u(x,t,\lambda) = [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]\, exp(-a^2\lambda^2 t) \tag{12}$$

ifoda (9) tenglamaning yechimi bo'ladi.

Agar (12) ni barcha λ lar bo'yicha jamlasak,

$$u(x,t) = \int_{-\pi}^{\pi} u(x,t,\lambda)d\lambda \tag{13}$$

va bu integral chekli bo'lib, uni t bo'yicha bir marta, x bo'yicha bir marta differensiallash mumkin bo'lsa, unda (13) funksiya (9) tenglamaning yechimi bo'ladi.

Endi $A(\lambda)$ va $B(\lambda)$ koeffitsientlarni shunday tanlaymizki, (13) funksiya (10) boshlang'ich shartni ham qanoatlantirsin, ya'ni

$$u(x,0) = \varphi(x) = \int_{\mathbb{R}} [A(\lambda)cos\lambda x + B(\lambda)sin\lambda x]\, d\lambda. \tag{14}$$

$\varphi(x)$ funksiya uchun Furye integralini yozamiz:

$$\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} \varphi(\xi)cos\lambda(\xi - x)d\xi =$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} \varphi(\xi)\, [cos\lambda\xi cos\lambda x + sin\lambda\xi sin\lambda x]\, d\xi.$$

Ko‘rinib turibdiki, (14) da A va B larni

$$A(\lambda) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \varphi(\xi)cos\lambda\xi d\xi,\ B(\lambda) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \varphi(\xi)sin\lambda\xi d\xi$$

kabi tanlasak, bu tenglik o‘rinli bo‘ladi.
So‘nggi ifodalarni (13) ga qo‘yib, quyidagi yechimni olamiz:

$$u(x,t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} d\lambda \int\limits_{\mathbb{R}} \varphi(\xi)cos\lambda(\xi - x)\exp(-a^2\lambda^2 t)d\xi =$$

$$= \frac{1}{\pi}\int\limits_0^\infty d\lambda \int\limits_{\mathbb{R}} \varphi(\xi)cos\lambda(\xi - x)exp(-a^2\lambda^2 t)d\xi = \tag{15}$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{R}} \varphi(\xi)d\xi \int\limits_0^\infty exp(-a^2\lambda^2 t)cos\lambda(\xi - x)d\lambda.$$

Ichki integralni hisoblaymiz. Buning uchun

$$a\lambda\sqrt{t} = z, \quad \lambda(\xi - x) = \mu z$$

almashtirishlar bajarib,

$$d\lambda = \frac{dz}{a\sqrt{t}}, \quad \mu = \frac{\xi - x}{a\sqrt{t}}$$

ekanligini inobatga olsak,

$$\int\limits_0^\infty exp(-a^2\lambda^2 t)cos\lambda(\xi - x)d\lambda = \frac{1}{a\sqrt{t}}\int\limits_0^\infty e^{-z^2} cos\mu z dz \tag{16}$$

tenglikka ega bo‘lamiz. (16) formulaning o‘ng tomonidagi integralni quyidagicha belgilaymiz:

$$\int\limits_0^\infty e^{-z^2} cos\mu z dz = I(\mu).$$

$I(\mu)$ integralni μ parametr bo‘yicha differensiallab va natijani bir marta bo‘laklab integrallasak,

$$I'(\mu) = -\int\limits_0^\infty z e^{-z^2} sin\mu z dz = -\frac{\mu}{2}\int\limits_0^\infty e^{-z^2} cos\mu z dz$$

yoki ushbu

$$I'(\mu) + \frac{1}{2}\mu I(\mu) = 0$$

oddiy differensial tenglama hosil bo'ladi. Bundan esa

$$I(\mu) = Cexp\left(-\frac{\mu^2}{4}\right).$$

C o'zgarmasni $C = I(0)$ tenglikdan aniqlaymiz. $I(0) = \int\limits_0^\infty e^{-z^2}dz$ va 1-bobning (27) formulasiga ko'ra $C = \frac{\sqrt{\pi}}{2}$.

Demak,

$$I(\mu) = \frac{\sqrt{\pi}}{2}exp\left(-\frac{\mu^2}{4}\right)$$

va (16) ga asosan

$$\int\limits_0^\infty exp(-a^2\lambda^2 t)cos\lambda(\xi - x)d\lambda = \frac{\sqrt{\pi}}{2a\sqrt{t}}exp\left[-\frac{(\xi - x)^2}{4a^2t}\right].$$

Nihoyat, (15) ga ko'ra, (9), (10) masalaning yechimini olamiz:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int\limits_{\mathbb{R}} \varphi(\xi)exp\left[-\frac{(\xi - x)^2}{4a^2t}\right]d\xi. \tag{17}$$

(17) ga *Puasson formulasi* deyiladi.

5.8 Koshi masalasi yechimining mavjudligi

T e o r e m a. $\varphi(x)$ uzluksiz va chegaralangan funksiya bo'lsin, ya'ni shunday N soni mavjudki, bunda $|\varphi(x)| \leq N$ barcha $x \in \mathbb{R}$ uchun. U holda (17) formula bilan aniqlangan $u(x,t)$ funksiya (9), (10) Koshi masalasining klassik yechimidir.

Isbot. Avvalo $u(x,t)$ funksiyanng mavjudligi va chegaralanganligini ko'rsatamiz. Buning uchun (17) integralda

$$z = \frac{\xi - x}{2a\sqrt{t}}, \quad \xi = 2a\sqrt{t}z + x$$

almashtirishni bajarib, integralni baholaymiz:

$$|u(x,t)| \leq \frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} \left|\varphi\left(2a\sqrt{t}z + x\right)\right| e^{-z^2} dz \leq \frac{N}{\sqrt{\pi}} \int\limits_{\mathbb{R}} e^{-z^2} dz \leq N.$$

Bundan Veyershtrass alomatiga koʻra (17) integral $-\infty < x < \infty$, $t > 0$ sohaning ixtiyoriy nuqtasida tekis yaqinlashadi va chegaralangan uzluksiz $u(x,t)$ funksiyani aniqlaydi.

Endi $-\infty < x < \infty$, $t > 0$ sohada $u(x,t)$ funksiyani t va x oʻzgaruvchilar boʻyicha istalgancha differensiallanuvchi ekanligi va barcha hosilalarni (17) tenglikni integral belgisi ostida differensiallash natijasida hosil qilish mumkinligini koʻrsatamiz. Masalan, $\frac{\partial u}{\partial t}$ hosilani tekshiramiz. (17) formulaning oʻng tomonini formal differensiallab, quyidagi ifodani hosil qilamiz:

$$\frac{1}{2a\sqrt{\pi t^3}} \int\limits_{\mathbb{R}} \varphi(\xi) \left[-\frac{1}{2} + \frac{(\xi - x)^2}{4a^2 t}\right] \exp\left[-\frac{(\xi - x)^2}{4a^2 t}\right] d\xi.$$

Bu integralning ixtiyoriy $(x,t) \in \overline{Q}_{\varepsilon,T} = \mathbb{R} \times [\varepsilon, T]$, $\varepsilon > 0$, $T > \varepsilon$ uchun tekis yaqinlashuvchi ekanini koʻrsatamiz. Yana yuqoridagi kabi $z = \frac{\xi - x}{2a\sqrt{t}}$ almashtirishni bajaramiz. Natijada hosil boʻlgan

$$\frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} \left[-\frac{1}{2} + z^2\right] \frac{\varphi\left(2a\sqrt{t}z + x\right)}{t} e^{-z^2} dz$$

integralni $\overline{Q}_{\varepsilon,T}$ sohada baholaymiz:

$$\left|\frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} \left[-\frac{1}{2} + z^2\right] \frac{\varphi\left(2a\sqrt{t}z + x\right)}{t} e^{-z^2} dz\right| \leq$$

$$\leq \frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} \left|-\frac{1}{2} + z^2\right| \frac{N}{t} e^{-z^2} dz \leq \frac{N}{\sqrt{\epsilon\pi}} \int\limits_{\mathbb{R}} \left|\frac{1}{2} + z^2\right| e^{-z^2} dz.$$

Bu tengsizliklardagi oxirgi integralning yaqinlashuvchi ekanligidan tekshirilayotgan integralning tekis yaqinlashuvchi boʻlishi kelib chiqadi. Bu esa $\frac{\partial u}{\partial t}$ hosilaning mavjudligi va uzluksizligini bildiradi.

Oʻxshash ravishda

$$u_{xx}(x,t) = \frac{1}{2a^3\sqrt{\pi t^3}} \int\limits_{\mathbb{R}} \varphi(\xi) \left[-\frac{1}{2} + \frac{(\xi - x)^2}{2a^2 t}\right] \exp\left[-\frac{(\xi - x)^2}{4a^2 t}\right] d\xi$$

hosilaning mavjudligi va uzluksizligi isbotlanadi. Bu hosilalarni (9) issiqlik o‘tkazuvchanlik tenglamasiga qo‘yib, u ayniyatga aylanishiga ishonch hosil qilamiz.

Endi $u(x,t)$ funksiyaning (10) boshlang‘ich shartni qanoatlantirishini ko‘r- satamiz. Yuqoridagi kabi, $z = \frac{\xi - x}{2a\sqrt{t}}$ almashtirishdan so‘ng

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}} \varphi\left(2a\sqrt{t}z + x\right) e^{-z^2} dz$$

ga ega bo‘lamiz. Bu integralning x, t lar bo‘yicha tekis yaqinlashuvchi ekanligidan va integral osti funksiyasining $t \in [0, T]$ bo‘yicha uzluksizligidan integral ostida limitga o‘tish mumkinligi kelib chiqadi:

$$\lim_{t \to +0} u(u,t) = \frac{\varphi(x)}{\sqrt{\pi}} \int\limits_{\mathbb{R}} e^{-z^2} dz = \varphi(x).$$

T a ’ r i f. Ushbu

$$U(x,t) = \frac{1}{2a\sqrt{\pi t}} exp\left[-\frac{x^2}{4a^2 t}\right] \tag{18}$$

funksiya (9) tenglamaning *fundamental yechimi* deyiladi. Bu funksiya (9), (10) Koshi masalasining Grin funksiya deb ham yuritiladi.

Eslatib o‘tamiz, (18) fundamental yechim 3-bobning 3-paragrafida n o‘lchovli issiqlik o‘tkazuvchanlik tenglamasi uchun keltirib chiqarilgan edi.

Agar boshlang‘ich shart $t = 0$ da emas, biror $t = \tau$ da berilsa, (18) formulada t ni $t - \tau$ bilan almashtirish lozim.

Endi fundamental yechimni qurishning boshqa bir usulini keltiramiz. (9) tenglamaning

$$u(x,0) = \delta(x - x_0) \tag{19}$$

boshlang‘ich shartni qanoatlantiruvchi Koshi masalasi yechimini qidiramiz. Bu yerda $\delta(x - x_0)-$ Dirakning delta-funksiyasi (1-bobga qarang). Qayd etish zarurki, (19) tenglik umumlashgan funksiyalar ma’nosida bajariladi. 1-bobdagi (9) formula bilan aniqlangan Dirakning delta-funksiyasi Furye integrali

$$\delta(x - x_0) = \frac{1}{\pi} \int\limits_0^{\infty} cos\lambda(x - x_0) d\lambda$$

ni e'tiborga olgan holda fundamental yechimni $H(x - x_0, t)$ orqali belgilab, uni

$$H(x - x_0, t) = \frac{1}{\pi} \int_0^\infty A_\lambda(t) cos\lambda(x - x_0) d\lambda$$

ko'rinishda izlaymiz. Bu funksiyani (9) tenglamaga qo'yib, A_λ ga nisbatan

$$A'_\lambda(t) + a^2\lambda^2 A_\lambda(t) = 0$$

differensial tenglamani hosil qilamiz. Uni yechib, (19) boshlang'ich shartga binoan $A_\lambda(0) = 1$ ekanligini inobatga olib,

$$A_\lambda(t) = exp(-a^2\lambda^2 t)$$

ni aniqlaymiz.

Shunday qilib,

$$H(x - x_0, t) = \frac{1}{\pi} \int_0^\infty exp(-a^2\lambda^2 t) \cdot cos\lambda(x - x_0) d\lambda =$$

$$= \frac{1}{2a\sqrt{\pi t}} \exp\left[-\frac{(x - x_0)^2}{4a^2 t}\right].$$

5.9 Ko'p o'zgaruvchili bo'lgan hol

Ushbu paragrafda ko'p o'zgaruvchili issiqlik o'tkazuvchanlik tenglamasi uchun qo'yilgan Koshi masalasining Grin funksiyasini quramiz.

L e m m a. Agar

$$u_t = a^2 \triangle u, \quad u(x, 0) = \varphi(x), \; x = (x_1, x_2, ..., x_n) \tag{20}$$

Koshi masalasida $\varphi(x)$ boshlang'ich funksiya

$$\varphi(x) = \varphi_1(x_1) \cdot \varphi_2(x_2) \cdot \varphi_3(x_3) \cdot ... \cdot \varphi_n(x_n) = \prod_{k=1}^n \varphi_k(x_k)$$

ko'rinishda bo'lsa, u holda

$$u(x, t) = \prod_{k=1}^n u_k(x_k, t) \tag{21}$$

funksiya (20) masalaning yechimi bo'ladi. Bu yerda $u_k(x_k,t)$ funksiyalar

$$\frac{\partial u_k}{\partial t}=a^2\frac{\partial^2 u_k}{\partial x_k^2},\quad u_k(x_k,0)=\varphi_k(x_k),\quad k=1,2,...,n$$

bir o'lchovli issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalalarining mos yechimlaridir.

Isbot. Lemma shartiga ko'ra

$$a^2\triangle\left(\prod_{k=1}^{n}u_k(x_k,t)\right)=$$

$$=a^2\sum_{k=1}^{n}\left(u_1\cdot u_2\cdot u_3\cdot ...\cdot u_{k-1}\cdot u_{k+1}\cdot ...\cdot u_n\cdot\frac{\partial^2 u_k}{\partial x_k^2}\right)=$$

$$=\sum_{k=1}^{n}\Bigg[u_1(x_1,t)\cdot u_2(x_2,t)\cdot u_3(x_3,t)\cdot ...\cdot u_{k-1}(x_{k-1},t)\cdot u_{k+1}(x_{k+1},t)\cdot ...\cdot u_n(x_n,t)\cdot$$

$$\cdot\left(a^2\frac{\partial^2 u_k}{\partial x_k^2}\right)\Bigg]=\frac{\partial}{\partial t}\left(\prod_{k=1}^{n}u_k(x_k,t)\right),$$

ya'ni (21) funksiya (20) masalaning yechimi bo'ladi, bu yerda $u_0(x_0,t)=1$ deb oldik.

Dirakning δ funksiyasining 1-bobdagi (16) formula bilan aniqlangan xossasiga ko'ra

$$\delta(x,\xi)=\prod_{k=1}^{n}\delta(x_k-\xi_k)$$

tenglik o'rinli. Lemmani ushbu

$$U_t(x,\xi,t)=a^2\triangle U(x,\xi,t),$$

$$U(x,\xi,0)=\delta(x-\xi)$$

maxsus masala uchun qo'llasak, ushbu

$$U(x,\xi,t)=\prod_{k=1}^{n}U(x_k-\xi_k,t)=\frac{1}{(2a\sqrt{\pi t})^n}\exp\left[-\frac{1}{4a^2t}\sum_{k=1}^{n}(x_k-\xi_k)^2\right]\quad(22)$$

Koshi masalasining Grin formulasi (yoki issiqlik o'tkazuvchilik tenglamasining fundamental yechimi) ni hosil qilamiz.

Bu funksiyaning ba'zi xossalarini keltirib o'tamiz. Soddalik uchun (22) da $n = 3$, $a = 1$ deylik ($a = 1$ ga t ni a^2t bilan almashtirib erishish mumkin). Unda (22) ning ko'rinishi quydagicha bo'ladi:

$$U(M, P; t-\tau) = \frac{1}{(2\sqrt{\pi(t-\tau)})^3} exp\left[-\frac{r^2}{4t}\right], \tag{23}$$

bu yerda $M = M(x, y, z)$, $\quad P = P(\xi, \eta, \zeta)$, $\quad M \neq P$, $\quad t \neq \tau$.

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}.$$

1-xossa. (23) funksiya $(x, y, z, t)-$ o'zgaruvchilari bo'yicha

$$\frac{\partial U}{\partial t} - \triangle U = 0$$

issiqlik o'tkazuvchanlik tenglamasini, (ξ, η, ζ, τ) - o'zgaruvchilar bo'yicha esa unga qo'shma bo'lgan

$$\frac{\partial U}{\partial t} + \triangle U = 0$$

tenglamani qanoatlantiradi.

Bu xossaning isboti (23) dan bevosita hisoblashlar orqali kelib chiqadi.

2-xossa. Ixtiyoriy $t > \tau$ uchun

$$\int\limits_{\mathbb{R}^3} U(M, P; t-\tau) d\xi d\eta d\zeta = 1$$

tenglik o'rinli.

Haqiqatan ham, sferik koordinatalar sistemasiga o'tib, so'ng bo'laklab integrallasak,

$$\int\limits_{\mathbb{R}^3} U(M, P; t-\tau) d\xi d\eta d\zeta =$$

$$= \int\limits_0^\infty r^2 dr \int\limits_0^\pi \int\limits_0^{2\pi} U(M, P; t-\tau) sin\theta d\theta d\varphi =$$

$$= \frac{1}{2(t-\tau)\sqrt{\pi(t-\tau)}} \int\limits_0^\infty r^2 \exp\left(-\frac{r^2}{4(t-\tau)}\right) dr =$$

$$= \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty rd\left(\exp\left(-\frac{r^2}{4(t-\tau)}\right)\right) =$$

$$= \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty \exp\left(-\frac{r^2}{4(t-\tau)}\right) dr =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp\left(-\frac{r^2}{4(t-\tau)}\right) d\left(\frac{r}{2\sqrt{t-\tau}}\right) =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-z^2)dz = 1$$

munosabatlarga ega bo'lamiz. Bu esa xossaning o'rinli ekanligini isbotlaydi.

3-xossa. Har qanday chekli (yoki cheksiz) Ω soha uchun quyidagi tenglik o'rinli:

$$\lim_{t\to\tau+0} \int_{\Omega} U(M,P;t-\tau)d\xi d\eta d\zeta = \begin{cases} 1, & \text{agar } M \in \Omega, \\ 0, & \text{agar } M \overline{\in} \Omega. \end{cases}$$

Isbot. Agar

$$\frac{x-\xi}{\sqrt{t-\tau}} = x_1, \quad \frac{y-\eta}{\sqrt{t-\tau}} = y_1, \quad \frac{z-\zeta}{\sqrt{t-\tau}} = z_1$$

almashtirishlarni kiritib, yangi o'zgaruvchiga o'tsak, integrallash sohasi Ω koordinata boshi $M(x,y,z)$ nuqtada bo'lgan biror Ω_1 sohaga o'tadi. Agar $M \in \Omega$ bo'lsa, Ω_1 soha butun $\mathbb{R}^3$ fazo bilan ustma-ust tushadi va 2-xossaga ko'ra integralning qiymati 1 ga teng bo'ladi. Agarda $M \overline{\in} \Omega$ bo'lsa, u holda Ω_1 sohaning nuqtalari $M(x,y,z)$ koordinata boshidan cheksiz uzoqlashadi va integralning qiymati nolga intiladi. Soddalik uchun bir o'lchovli holni qaraylik, ya'ni $\Omega = \{a < x < b\}$, $t > \tau$ bo'lsin. U holda $x = \xi + \sqrt{t-\tau} \cdot x_1$ almashtirish bajarsak,

$$\int_a^b U(x,\xi;t-\tau)d\xi = \frac{1}{2\sqrt{\pi(t-\tau)}} \int_a^b \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right] d\xi =$$

$$= \frac{1}{\pi} \int_{\frac{x-b}{\sqrt{t-\tau}}}^{\frac{x-a}{\sqrt{t-\tau}}} \exp\left(-\frac{x_1^2}{4}\right) d\left(\frac{x_1}{2}\right).$$

Bu tengliklardan esa $t \to \tau + 0$ da
1) $a < x < b$ bo‘lsa, $\frac{x-a}{\sqrt{t-\tau}} \to +\infty$, $\frac{x-b}{\sqrt{t-\tau}} \to -\infty$,
2) $x < a < b$ bo‘lsa, $\frac{x-a}{\sqrt{t-\tau}} \to -\infty$, $\frac{x-b}{\sqrt{t-\tau}} \to -\infty$,
3) $a < b < x$ bo‘lsa, $\frac{x-a}{\sqrt{t-\tau}} \to +\infty$, $\frac{x-b}{\sqrt{t-\tau}} \to +\infty$
munosabatlarni inobatga olsak, olgingi 2-xossaga ko‘ra isbotni yakunlaymiz.

4-xossa. Har qanday chekli yoki cheksiz Ω sohada uzluksiz va chegaralangan $\varphi(x, y, z)$ funksiya uchun

$$\lim_{t\to+0} \int\limits_{\Omega} U(M, P, t)\varphi(P)dP = \varphi(M), \quad dP = d\xi d\eta d\zeta$$

tenglik o‘rinli va bu limit har qanday $M(x, y, z) \in \Omega_1 \subset \Omega$ ga nisbatan tekis yaqinlashadi.

Isbot. Bu xossani isbotlash uchun quyidagi ayirmani qaraymiz:

$$\int\limits_{\Omega} U(M, P, t)\varphi(P)dP - \varphi(M) \int\limits_{\Omega} U(M, P, t)dP =$$

$$= \int\limits_{\Omega} U(M, P, t)[\varphi(P) - \varphi(M)]dP.$$

3-xossaga binoan bu ayirmaning ikkinchi hadi $\varphi(x, y, z)$ ga teng va 4-xossani isbotlash uchun ayirmaning $t \to +0$ da nolga intilishini ko‘rsatish lozim. Shu maqsadda $M(x, y, z)$ nuqtani δ kichik radiusli shar bilan o‘raymiz. Aytaylik, δ shunday kichikki, barcha $r < \delta$ lar uchun

$$|\varphi(\xi, \eta, \zeta) - \varphi(x, y, z)| \leq \frac{\varepsilon}{2}$$

tengsizlik bajarilsin. U holda

$$\int\limits_{\Omega} U(M, P; t)\,[\varphi(P) - \varphi(M)]\,dP =$$

$$= \int\limits_{r\leq\delta} U(M, P; t)\,[\varphi(P) - \varphi(M)]\,dP +$$

$$+ \int\limits_{r\geq\delta} U(M, P; t)\,[\varphi(P) - \varphi(M)]\,dP. \tag{24}$$

(20) tenglikning o‘ng tomonidagi birinchi integralni baholaymiz:

$$J_1 \leq \frac{\varepsilon}{2} \int\limits_{r \leq \delta} U(M, P; t) dP \leq \frac{\varepsilon}{2} \int\limits_{\mathbb{R}^3} U(M, P; t) dP = \frac{\varepsilon}{2}.$$

Berilgan $\varphi(x, y, z)$ funksiyaning chegaralanganligidan shunday N son topiladiki, $|\varphi(x, y, z)| \leq N$ bo‘ladi va

$$J_2 \leq \frac{2N}{l} \int\limits_{r \geq \delta} U(M, P; t) dP = \frac{N}{4\pi\sqrt{\pi} t^{\frac{3}{2}}} \int\limits_{r \geq \delta} \exp\left(-\frac{r^2}{4t}\right) dP.$$

Agar $\xi = \sqrt{t}\xi_1, \quad \eta = \sqrt{t}\eta_1, \quad \zeta = \sqrt{t}\zeta_1$ desak, $\frac{r^2}{t} = r_1^2$ bo‘lib,

$$J_2 \leq \frac{N}{4\pi^{\frac{3}{2}}} \int\limits_{r_1 \geq \frac{\delta}{\sqrt{t}}} \exp\left(-\frac{r^2}{4}\right) d\xi_1 d\eta_1 d\zeta_1 = \frac{N}{\sqrt{\pi}} \int\limits_{\frac{\delta}{\sqrt{t}}}^{\infty} r_1^2 \exp\left(-\frac{r_1^2}{4}\right) dr_1.$$

O‘z navbatida

$$\int\limits_{0}^{\infty} r_1^2 \exp\left(-\frac{r_1^2}{4}\right) dr_1$$

integralning yaqinlashuvchiligidan, yetarlicha kichik t lar uchun

$$\int\limits_{\frac{\delta}{\sqrt{t}}}^{\infty} r_1^2 \exp\left(-\frac{r_1^2}{4}\right) dr_1 \leq \frac{\varepsilon}{2}$$

tengsizlik o‘rinli bo‘ladi. Bundan esa 4-xossaning tasdig‘i kelib chiqadi.

Yuqorida keltirilgan xossalardan foydalanib, (20) Koshi masalasining yechimi

$$u(M, t) = \underbrace{\int\limits_{\mathbb{R}} \ldots \int\limits_{\mathbb{R}}}_{n \ ta} U(M, P; t)\varphi(P) dP \tag{25}$$

ko‘rinishda bo‘ladi, bu yerda $dP = d\xi_1 d\xi_2 \ldots d\xi_n$ shuningdek, bir jinsli bo‘lmagan

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(M, t) \tag{26}$$

tenglamaning $u(M,0)=0$ bir jinsli boshlang'ich shartni qanoatlantiruvchi yechimi

$$u(M,t)=\int_0^t \underbrace{\int_{\mathbb{R}}...\int_{\mathbb{R}}}_{n\ ta} U(M,P;t-\tau)f(P,\tau)dPd\tau \tag{27}$$

bo'ladi.

Tabiiyki, (26) tenglamaning bir jinsli bo'lmagan $u(M,0)=\varphi(M)$ boshlang'ich shartni qanoatlantiruvchi yechimi (25) va (27) funksiyalar yig'indisidan iborat bo'ladi.

5.10 Koshi masalasi yechimining yagonaligi

Ma'lumki, bir o'lchovli issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalasi $Q=\{(x,t): x\in\mathbb{R},\ 0<t<+\infty\}$ sohada qo'yiladi.

T e o r e m a. (9), (10) Koshi masalasining chegaralangan yechimi yagonadir.

Isbot. Faraz qilaylik, shunday $M>0$ soni mavjud bo'lsinki, bunda $|u(x,t)|\le M,\ (x,t)\in\overline{Q}=\{(x,t): x\in\mathbb{R},\ 0\le t<+\infty\}$. Teoremani teskarisini faraz qilish usuli bilan isbotlaymiz: (9), (10) masalaning chegaralangan ikkita $u_i(x,t),\ i=1,2$ yechimlari mavjud bo'lsin. $v(x,t)=u_1(x,t)-u_2(x,t)$ funksiyani kiritamiz. Ravshanki, $v(x,t)$ funksiya bir jinsli

$$v_t=a^2v_{xx},\quad (x,t)\in Q, \tag{28}$$

$$v(x,0)=0,\quad x\in\mathbb{R} \tag{29}$$

Koshi masalasining yechimi bo'ladi. $u_1(x,t)$ va $u_2(x,t)$ funksiyalarning chegaralanganligidan $v(x,t)$ ning ham chegaralanganligi kelib chiqadi:

$$|v(x,t)|=|u_1(x,t)-u_2(x,t)|\le|u_1(x,t)|+|u_2(x,t)|\le 2M,$$

bu yerda $|u_1(x,t)|\le M$ va $|u_2(x,t)|\le M$.

Shunday qilib, $v(x,t)$ funksiya (28), (29) Koshi masalasining $\overline{Q}$ sohada chegaralangan yechimi bo'ladi. $v(x,t)\equiv 0,\ (x,t)\in\overline{Q}$ ekanligini ko'rsatamiz.

Q yarim tekislikda $|x| = L, \quad t = T$ to‘g‘ri chiziqlarni tanlab, chegaralangan

$$Q_{L,T} = \{(x,t) : -L < x < L, \quad 0 < t \leq T\},$$

$$\left(\overline{Q}_{L,T} = \{(x,t) : -L \leq x \leq L, \quad 0 \leq t \leq T\}\right)$$

sohani qaraymiz. Shuningdek,

$$w_t = a^2 w_{xx}$$

isiqlik o‘tkazuvchanlik tenglamasini qanoatlantiruvchi

$$w(x,t) = \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

funksiyani kiritamiz. $t = 0$ da

$$w(x,0) = \frac{2Mx^2}{L^2} \geq |v(x,0)| = 0.$$

$|x| = L$ bo‘lsin. U holda

$$w(\pm L, t) = 2M + \frac{4Ma^2 t}{L^2} \geq 2M \geq v(\pm L, t).$$

$Q_{L,t}$ soha chegaralangan va $v(x,t)$, $w(x,t)$ funksiyalar bir jinsli issiqlik o‘tkazuvchanlik tenglamasining yechimlari bo‘lgani hamda soha chegarasida $|v(x,0)| \leq w(x,0)$, $|v(\pm L,t)| \leq w(\pm L,t)$ tengsizliklar bajarilgani uchun $v(x,t), w(x,t)$ funksiyalarga maksimum prinsipini qo‘llab, $|v(x,t)| \leq w(x,t)$, $(x,t) \in \overline{Q}_{L,T}$, yoki

$$-\frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right) \leq v(x,t) \leq \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

tengsizliklarni hosil qilamiz. Bu yerda $(x,t) \in \overline{Q}_{L,T}$ nuqtani tayinlab, $L \to +\infty$ da limitga o‘tsak, $\lim\limits_{L\to+\infty} v(x,t) = 0$. $v(x,t)$ funksiyaning L ga bog‘liq emasligi va (x,t) nuqta ixtiyoriy tanlanganligi sababli $\overline{Q}$ sohaning barcha nuqtalarida $v(x,t) \equiv 0$ bo‘ladi. Bu yerdan $u_1(x,t) \equiv u_2(x,t)$ ekanligi va demak, Koshi masalasi yechimining yagonaligi kelib chiqadi.

E s l a t m a. $u(x,t)$ funksiyaning fizik ma’nosi muhitning harorati ekanligini hisobga olsak, unga qo‘yilayotgan chegaralanganlik talabi tabiiydir.

E s l a t m a. Koshi masalasi yechimiga $|x| \to +\infty$ da qo‘yilayotgan chegaralanganlik shartini ancha yumshatish mumkin. Buning uchun butun $\mathbb{R}$ va $t \geq 0$ larda $|u(x,t)| \leq M \exp\left(Nx^2\right)$ tengsizlikni qanoatlantiruvchi funksiyalar sinfini kiritamiz, bu yerda M, N lar $u(x,t)$ funksiyaga bog‘liq ravishda aniqlanuvchi musbat o‘zgarmaslardir. Bu funksiyalar sinfida ham yagonalik teoremasi o‘rinli.

5.11 Koshi masalasi yechimining turg‘unligi

Bu paragrafda Koshi masalasi yechimining berilgan $\varphi(M)$, $f(M)$ funksiyalarning kichik o‘zgarishiga nisbatan turg‘unligini, ya‘ni yechim ham kichik o‘zgarishini ko‘rsatamiz.

T e o r e m a. Agar

$$u_t = a^2 \triangle u, \quad u(x,0) = \varphi_1(x), \tag{30}$$

$$v_t = a^2 \triangle v, \quad v(x,0) = \varphi_2(x), \quad x = (x_1, ..., x_n) \in \mathbb{R}^n \tag{31}$$

Koshi masalalarida barcha $x \in \mathbb{R}^n$ lar uchun

$$|\varphi_1(x) - \varphi_2(x)| \leq \varepsilon \tag{32}$$

bo‘lsa, u holda (30), (31) masalalarning yechimlari $u(x,t)$, $v(x,t)$ uchun barcha $x \in \mathbb{R}^n$, $t \geq 0$ qiymatlarda

$$|u(x,t) - v(x,t)| \leq \varepsilon$$

tengsizlik o‘rinli bo‘ladi.

Isbot. (30), (31) masalalarning (25) formula bilan ifodalangan mos yechimlarini yozib, (32) tengsizlik va 9-paragrafdagi 2-xossadan foydalanamiz. Natijada,

$$|u(x,t) - v(x,t)| \leq \int\limits_{\mathbb{R}^n} U(x,\xi,t)|\phi_1(x) - \phi_2(\xi)|d\xi \leq$$

$$\leq \varepsilon \int\limits_{\mathbb{R}^n} U(x,\xi,t)d\xi = \varepsilon,\ \xi = (\xi_1,...,\xi_n),\ d\xi = d\xi_1 \cdot ... \cdot d\xi_n$$

teoremaning isbotini olamiz.

T e o r e m a. Ixtiyoriy $\varepsilon > 0$, $T > 0$ berilgan sonlar uchun shunday $\delta = \delta(\varepsilon, T) > 0$ soni topiladiki,

$$u_t = a^2 \Delta u + f_1(x,t), \quad u(x,0) = 0, \tag{33}$$

$$v_t = a^2 \Delta v + f_2(x,t), \quad v(x,0) = 0 \tag{34}$$

Koshi masalalarida barcha $x \in \mathbb{R}^n$, $t \geq 0$ qiymatlar uchun

$$|f_1(x,t) - f_2(x,t)| < \delta(\varepsilon, T) \tag{35}$$

bo‘lsa, u holda $u(x,t),\ v(x,t)$ yechimlar uchun

$$|u(x,t) - v(x,t)| \leq \varepsilon \tag{36}$$

tengsizlik bajariladi.

Isbot. Ma’lumki, (33) va (34) masalalarning yechimlari (27) ko‘rinishda ifodalanadi. (35) tengsizlik va 9-paragrafdagi 2-xossadan foydalansak,

$$|u(x,t) - v(x,t)| \leq \int\limits_0^t \int\limits_{\mathbb{R}^n} U(x,\xi;t-\tau)|f_1(\xi,\tau) - f_2(\xi,\tau)|d\xi d\tau \leq$$

$$\leq \delta \int\limits_0^t \int\limits_{\mathbb{R}^n} U(x,\xi;t-\tau)d\xi d\tau = \delta \int\limits_0^t d\tau \leq \delta T$$

tengsizlikni olamiz. Agar $\delta = \frac{\varepsilon}{T}$ deb tanlasak, (36) tengsizlikni hosil qilamiz, ya‘ni teorema isbotlandi.

5.12 Koshi masalasi uchun Grin funksiyasini qurishning boshqa usullari

Biz bu paragrafda umumlashgan funksiyalar (1-bob) elementlaridan foydalanib, Koshi masalasi uchun Grin funksiyasi (fundamental yechim)ni qurishning ba'zi usullarini keltiramiz. Buning uchun, avvalo, quyidagi

$$u_t = a^2 u_{xx}, \tag{37}$$

$$u(x,0) = \varphi(x) = \theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0 \end{cases} \tag{38}$$

maxsus Koshi masalasini qaraymiz. Ma'lumki, $\theta(x)$ funksiyaga *Xevisayd funksiyasi* deyiladi. Bu masalaning ushbu

$$u(x,t) = f\left(\frac{x}{t^\alpha}\right) \tag{39}$$

ko‘rinishdagi *avtomodel yechim* deb ataluvchi yechimni izlaymiz. (39) ni (37) ga qo‘ysak,

$$\frac{a^2}{t^{2\alpha}} f''\left(\frac{x}{t^\alpha}\right) = \frac{\alpha x}{t^{\alpha+1}} f'\left(\frac{x}{t^\alpha}\right)$$

tenglikka ega bo‘lamiz. Bu tenglik $z = xt^{-\alpha}$ ga nisbatan ayniyat bo‘lishi uchun $\alpha = 0,5$ bo‘lishi kerak. U holda $f(z)$ ga nisbatan

$$f''(z) + \frac{z}{2a^2} f'(z) = 0 \tag{40}$$

tenglikni olamiz va (39), (38) larga ko‘ra

$$\lim_{z\to-\infty} f(z) = 0, \quad \lim_{z\to+\infty} f(z) = 1 \tag{41}$$

shartlarga ega bo‘lamiz.

(40) ni integrallab,

$$lnf'(z) = -\frac{z^2}{4a^2} + lnc, \quad c = const > 0$$

ni hosil qilamiz. Bundan esa

$$f(z) = c\int_{-\infty}^{z} exp\left(-\frac{\xi^2}{4a^2}\right) d\xi = 2ac\int_{-\infty}^{\frac{z}{2a}} exp(-y^2)dy$$

yechimlarni olamiz. Bu funksiya (41) shartning birinchisini qanoatlantiradi. Ikkinchi shartdan esa c ni aniqlash uchun foydalanamiz:

$$2ac\int_{-\infty}^{\infty} exp(-y^2)dy = 1$$

yoki

$$\int_{-\infty}^{\infty} exp(-y^2)dy = \sqrt{\pi}$$

ekanligidan $c = \frac{1}{2a\sqrt{\pi}}$ kelib chiqadi. Shunday qilib, (37), (38) masalaning yechimi

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2a\sqrt{t}}} exp(-y^2)dy$$

ko‘rinishda bo‘ladi. Bu yechimni quyidagicha ham yozish mumkin:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} exp(-y^2)dy + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{2a\sqrt{t}}} exp(-y^2)dy =$$

$$= \frac{1}{2}\left[1 + \Phi\left(\frac{x}{2a\sqrt{t}}\right)\right], \tag{42}$$

bu yerda

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} exp(-y^2)dy.$$

Bu integralga *xatolik integrali* deyiladi.

(42) funksiya (37) tenglamani qanoatlantirish bilan birga uzluksiz u_{xxx} va u_{xt} hosilalarga ham ega. Shuning uchun (37) tenglikni x bo‘yicha differensiallasak,

$$(u_x)_t = a^2(u_x)_{xx}$$

ayniyat hosil bo‘ladi olamiz. Bu esa (42) yechimning x bo‘yicha hosilasi ham (37) tenglama uchun yechim bo‘lishini ko‘rsatadi. Shuningdek, u $u_x(x,0) = \eta'(x)$ boshlang‘ich shartni ham qanoatlantiradi, bu yerda 1-bobning (17) formulasiga ko‘ra Xevisayd funksiyasining umumlashgan hosilasi $\eta'(x-\xi) = \delta(x-\xi)$ dan iborat.

Demak, (42) dan differensiallash yordamida ushbu

$$u_x(x-\xi;t) = U(x,\xi;t) = \frac{1}{2a\sqrt{\pi t}} \exp\left[-\frac{(x-\xi)^2}{4a^2t}\right]$$

Koshi masalasining Grin funksiyasi (yoki fundamental yechim)ni olamiz.

Endi faraz qilaylik, biror $U(x,t)$ funksiya

$$u_t = a^2 u_{xx}, \quad u(x,0) = \delta(x)$$

Koshi masalasining yechimi bo‘lsin. U holda

$$U_t \equiv a^2 U_{xx}, \quad U(x,0) \equiv \delta(x)$$

ayniyatlar (umumlashgan funksiyalar tengligi ma’nosida) bajariladi. Bu ayniyatlarga Furye almashtirishini qo‘llasak, va

$$g(\omega,t) = \int\limits_{\mathbb{R}} U(x,t) exp(-i\omega x) dx$$

belgilash kiritsak, $g(\omega,t)$ funksiyaga nisbatan

$$g'(t) + a^2\omega^2 g(t) = 0, \tag{43}$$

$$g(\omega,0) = 1 \tag{44}$$

masalani hosil qilamiz, bu yerda 1-bobning (22) formulasidan $x_0 = 0$ da foydalanildi.

Ma’lumki, (43), (44) masalaning yechimi

$$g(\omega,t) = exp(-a^2\omega^2 t)$$

bo‘lib, unga teskari Furye almashtirishini qo‘llab, 1-bobning (24) formulasiga asosan

$$\int\limits_{\mathbb{R}} exp(-\alpha^2\omega^2 + i\omega x) d\omega = \frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{x^2}{4\alpha^2}\right)$$

munosabatdan foydalansak,

$$U(x,t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} g(\omega,t)\exp(i\omega x) d\omega = \frac{1}{2a\sqrt{\pi t}} exp\left(-\frac{x^2}{4a^2 t}\right)$$

fundamental yechimni olamiz. Demak, $x = \xi$ nuqtada maxsuslikka ega bo‘lgan Grin funksiyasi $G(x,\xi;t) = U(x-\xi;t)$ bo‘ladi.

M i s o l. Ushbu $a^2 u_{xx} = u_t;\;\; u(x,0) = c_0 \exp(-x^2),\; c_0 = const$ Koshi masalasi yechilsin.

Bu masala yechimi

$$u(x,t) = c_0 \int_{\mathbb{R}} U(x-\xi;t)\exp(-\xi^2)d\xi$$

bo‘lib, unda $\alpha = \frac{x-\xi}{2a\sqrt{t}}$ almashtirish bajarsak,

$$u(x,t) = \frac{c_0}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-\alpha^2)\exp\left[-(x-2a\sqrt{t}\alpha)^2\right] d\alpha =$$

$$= \frac{c_0}{\sqrt{\pi}} \exp(-x^2) \int_{\mathbb{R}} \exp\left[-4a\alpha x\sqrt{t} - (4a^2t+1)\alpha^2\right] d\alpha =$$

$$= \frac{c_0 \exp(-x^2)}{\sqrt{\pi}} \exp\left(\frac{4a^2x^2t}{1+4a^2t}\right) \times$$

$$\times \int_{\mathbb{R}} \exp\left[-\left(\frac{2ax\sqrt{t}}{\sqrt{1+4a^2t}} + \alpha\sqrt{1+4a^2t}\right)^2\right] d\alpha$$

tengliklarga ega bo‘lamiz. Oxirgi integralda

$$\frac{2ax\sqrt{t}}{\sqrt{1+4a^2t}} + \alpha\sqrt{1+4a^2t} = \beta$$

almashtirishni amalga oshirib, sodda hisoblashlardan so‘ng, quyidagi yechimni olamiz:

$$u(x,t) = \frac{c_0}{\sqrt{\pi}} \exp\left(-\frac{x^2}{1+4a^2t}\right) \frac{1}{\sqrt{1+4a^2t}} \int_{\mathbb{R}} \exp(-\beta^2)d\beta =$$

$$= \frac{c_0}{\sqrt{1+4a^2t}} \exp\left(-\frac{x^2}{1+4a^2t}\right).$$

M i s o l. Bir jinsli bo‘lmagan $u_t - u_{xx} - u_{yy} = e^t$ tenglamaning $u(x,y,0) = \cos x \sin y$ boshlang‘ich shartni qanoatlantiruvchi yechimini toping.

(25), (27) formulalarga ko‘ra bu yechim quyidagi yig‘indi ko‘rinishida bo‘ladi:

$$u(x,y,t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} \cos\xi \sin\eta e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} d\xi d\eta +$$

$$+\frac{1}{4\pi}\int\limits_0^t\int\limits_{\mathbb{R}^2}\frac{e^\tau}{t-\tau}e^{-\frac{(x-\xi)^2+(y-\eta)^2}{t-\tau}}d\xi d\eta d\tau = u_1+u_2.$$

Birinchi integralda $\xi = x+2\sqrt{t}s$ va $\eta = y+2\sqrt{t}s$ almashtirishlar bajarib, sodda hisoblashlardan so‘ng, $u_1(x,y,t)$ uchun

$$u_1 = e^{-2t}\cos x\sin y$$

ifodani olamiz. Shuningdek, ikkinchi integralda ham $\xi = x+2\sqrt{t-\tau}s,\ \ \eta = y+2\sqrt{t-\tau}s$ almashtirishlar bajarib,

$$u_2 = \frac{1}{4\pi}\int\limits_0^t\frac{e^\tau}{t-\tau}4\pi(t-\tau)d\tau = e^t-1$$

ni va nihoyat,

$$u(x,y,t) = e^{-2t}\cos x\sin y + e^t - 1$$

yechimni hosil qilamiz.

5.13 Yarim chegaralangan sohalar uchun qo‘yilgan masalalar

Koshi masalasining yechimini topishdagi kabi yarim chegaralangan sohalarda qo‘yilgan masalalarni ham yechish mumkin.

Ushbu ikkita

1) $u_t = a^2u_{xx},\ \ x>0,\ \ t>0,$

$u(0,t)=0,\ t\geq 0,\ \ u(x,0)=\varphi(x),\ \ x\geq 0$

va

2) $u_t = a^2u_{xx},\ \ x>0,\ t>0,$

$u_x(0,t)=0,\ \ t\geq 0,\ \ u(x,0)=\varphi(x),\ \ x\geq 0$

mos ravishda birinchi va ikkinchi tur boshlang‘ich-chegaraviy masalalarni qaraymiz. Bu masalalarning Grin funksiyalarini mos ravishda $G_1(x,\xi;t)$ va

$G_2(x,\xi;t)$ orqali belgilasak, 1- va 2-masalalar yechimlari mos ravishda:

$$u(x,t)=\int_0^\infty G_1(x,\xi;t)\varphi(\xi)d\xi,$$

$$u(x,t)=\int_0^\infty G_2(x,\xi;t)\varphi(\xi)d\xi$$

formulalar bilan aniqlanadi.

1- va 2-masalalarning G_1 va G_2 Grin funksiyalarini qurish uchun Koshi masalasi yechimining quyidagi xossasidan foydalanamiz, ya'ni agar $u(x,t)$ - Koshi masalasining yechimi va $\varphi(x)$ boshlang'ich funksiya toq bo'lsa, u holda $u(0,t)=0$; agar $\varphi(x)$ juft funksiya bo'lsa, $u_x(0,t)=0$ bo'ladi.

Xuddi giperbolik tipdagi tenglamada ko'rganimizdek, bu yerda ham yuqoridagi xossalardan foydalanib,
1-masala uchun:

$$G_1(x,\xi;t)=\frac{1}{2a\sqrt{\pi t}}\left\{\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right)-\exp\left(-\frac{(x+\xi)^2}{4a^2t}\right)\right\},$$

2-masala uchun esa:

$$G_2(x,\xi;t)=\frac{1}{2a\sqrt{\pi t}}\left\{\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right)+\exp\left(-\frac{(x+\xi)^2}{4a^2t}\right)\right\}$$

- Grin funksiyalarini hosil qilamiz.

Agar bir jinsli bo'lmagan

$$u_t-a^2u_{xx}=f(x,t)$$

tenglama qaralsa, 1- va 2-masalalarning yechimlarini topish uchun mos ravishda

$$\int\limits_0^t\int\limits_0^\infty G_1(x,\xi;t-\tau)f(\xi,\tau)d\xi d\tau,$$

$$\int\limits_0^t\int\limits_0^\infty G_2(x,\xi;t-\tau)f(\xi,\tau)d\xi d\tau$$

ifodalarni bir jinsli tenglama uchun olingan yechimlarga qo'shib qo'yish yetarli.

5.14 Chegaralangan sterjenda issiqlik tarqalishi. Furye usuli

Ushbu paragrafdan boshlab chekli sohalarda parabolik tipdagi tenglamalar uchun qo‘yilgan masalalarni o‘zgaruvchilarni ajratish - Furye usuli bilan yechish haqida fikr yuritamiz.

5.14.1 Bir jinsli tenglama uchun bir jinsli chegaraviy shartli masala

Chekli l uzunlikka ega bo‘lgan bir jinsli sterjenda issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalani qaraylik: $\{0 < x < l, \quad t > 0\}$ sohada

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \tag{39}$$

tenglamaning

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \geq 0 \tag{40}$$

chegaraviy va

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq l \tag{41}$$

boshlang‘ich shartlarini qanoatlantiruvchi yechimi topilsin, bu yerda $\varphi(x)$ berilgan funksiya bo‘lib, $\varphi(0) = 0, \quad \varphi(l) = 0$.

Furye usuliga ko‘ra (39) tenglamaning trivial bo‘lmagan xususiy yechimini

$$u(x,t) = X(x)T(t) \tag{42}$$

ko‘rinishida izlaymiz va (42) ni (39) ga qo‘yib, ushbu

$$T'(t) + a^2 \lambda T(t) = 0, \tag{43}$$

$$X''(x) + \lambda X(x) = 0, \quad \lambda = const > 0 \tag{44}$$

oddiy differensial tenglamalarni olamiz.

O‘z navbatida $X(x)$ funksiyani aniqlash uchun (44) tenglamaning

$$X(0) = 0, \quad X(l) = 0 \tag{45}$$

shartlarini qanoatlantiruvchi yechimini topish, ya'ni xos sonlar haqidagi masalaga kelamiz. Ma'lumki, tor tebranish tenglamasidagi kabi, λ parametrning faqat

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3, \ldots$$

qiymatlaridagina (44), (45) masalaning trivial bo'lmagan

$$X_n(x) = \sin\frac{n\pi}{l}x, \quad n = 1, 2, 3, \ldots$$

yechimlari, ya'ni xos funksiyalari mavjud bo'ladi.
Shuningdek, (43) tenglamaning $\lambda = \lambda_n$ xos sonlarga mos yechimlari

$$T_n(t) = A_n \cdot \exp\left(-\left(\frac{an\pi}{l}\right)^2 t\right)$$

ko'rinishida bo'ladi, bu yerda A_n- ixtiyoriy o'zgarmas koeffitsientlar.

Shunday qilib, ixtiyoriy o'zgarmas A_n sonlar uchun

$$u_n(x,t) = A_n \exp\left[-\left(\frac{an\pi}{l}\right)^2 t\right] \sin\frac{n\pi}{l}x \tag{46}$$

funksiya (39) tenglamani va (40) shartlarni qanoatlantiradi. Ravshanki,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left[-\left(\frac{an\pi}{l}\right)^2 t\right] \sin\frac{n\pi}{l}x \tag{47}$$

qator ham (39) tenglamani va (40) shartlarni qanoatlantiradi.

Endi (47) qatorning (41) shartni qanoatlantirishini talab qilamiz, ya'ni

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} A_n sin\frac{n\pi}{l}x \tag{48}$$

bo'lsin. Bu qator berilgan $\varphi(x)$ funksiyaning $(0,l)$ oraliqdagi sinuslar bo'yicha Furye qatoriga yoyilmasini ifodalaydi. Shuning uchun A_n koeffitsientlar quyidagi ma'lum

$$A_n = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{n\pi}{l}\xi d\xi \tag{49}$$

formula bilan aniqlanadi.

Endi (47) qator (39), (40), (41) masalaning barcha tenglamalarini qanoatlantirishini ko'rsatamiz. Buning uchun esa, (47) qator bilan ifodalangan

$u(x,t)$ funksiyaning yetarlicha differensiallanuvchanligini va $\{0 < x < l,\ t > 0\}$ sohada tenglamani qanoatlantirib, $x = 0,\quad x = l,\quad t = 0$ chegaralarda uzluksiz ekanini koʻrsatish kerak.

Agar (47) qator yaqinlashuvchi boʻlib, uni x boʻyicha ikki marta, t boʻyicha bir marta hadma-had differensiallash mumkin boʻlsa, bu qator (39) tenglamani qanoatlantiradi, chunki (39) tenglama chiziqli boʻlgani uchun uning xususiy yechimlaridan tuzilgan qator ham yechim boʻladi.

Ixtiyoriy $t \geq t_1 > 0$ uchun quyidagi

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial t} u_n(x,t), \quad \sum_{n=1}^{\infty} \frac{\partial^2}{\partial x^2} u_n(x,t)$$

qatorlar tekis yaqinlashadi va ushbu

$$\left|\frac{\partial u_n}{\partial t}\right| \leq \left|-A_n \left(\frac{an\pi}{l}\right)^2 \exp\left[-\left(\frac{an\pi}{l}\right)^2 t\right] \sin\frac{n\pi}{l}x\right| <$$

$$< |A_n| \left(\frac{an\pi}{l}\right)^2 \exp\left[-\left(\frac{an\pi}{l}\right)^2 t\right]$$

tengsizliklar oʻrinli. Agar $|\varphi(x)| < M$ desak,

$$|A_n| \leq \frac{2}{l}\left|\int_0^l \varphi(\xi)\sin\frac{n\pi}{l}\xi d\xi\right| < 2M$$

boʻladi va $t \geq t_1$ uchun

$$\left|\frac{\partial u_n}{\partial t}\right| < 2M \left(\frac{an\pi}{l}\right)^2 \exp\left[-\left(\frac{an\pi}{l}\right)^2 t_1\right].$$

Xuddi shunga oʻxshash

$$\left|\frac{\partial^2 u_n}{\partial x^2}\right| < 2M \left(\frac{n\pi}{l}\right)^2 \exp\left[-\left(\frac{an\pi}{l}\right)^2 t_1\right]$$

baholarni olamiz.

Umuman olganda, quyidagi

$$\left|\frac{\partial^{i+j} u_n}{\partial x^j \partial t^i}\right| < 2M \left(\frac{n\pi}{l}\right)^{2i+j} a^{2i} \exp\left[-\left(\frac{an\pi}{l}\right)^2 t_1\right]$$

baholar o'rinli va ushbu

$$\sum_{n=1}^{\infty} Nn^q \exp\left[-\left(\frac{an\pi}{l}\right)^2 t_1\right] = \sum_{n=1}^{\infty} \alpha_n(t)$$

majorant qatorning yaqinlashishini tekshiramiz. Dalamber alomatiga ko'ra

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}(t)}{\alpha_n(t)}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \frac{\exp\left[-\left(\frac{a\pi}{l}\right)^2 (n^2+2n+1)\right]}{-\exp\left[-\left(\frac{a\pi}{l}\right)^2 n^2 t_1\right]} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \exp\left[-\left(\frac{n\pi}{l}\right)^2 (2n+1)t_1\right] = 0.$$

Bundan esa (47) qatorni ixtiyoriy $t \geq t_1 > 0$ uchun istalgancha hadma-had differensiallash mumkinligi kelib chiqadi. O'z navbatida yechimlar superpozitsiyasi prinsipiga ko'ra (47) qator bilan aniqlangan $u(x,t)$ funksiya (39) tenglamani qanoatlantiradi. t_1 ixtiyoriy bo'lgani uchun bu mulohazalar ixtiyoriy t>0 uchun ham o'rinli. Shunday qilib, agar $\varphi(x)$ funksiya uzluksiz va bo'lakli uzluksiz hosilaga ega bo'lib, $\varphi(0) = 0, \quad \varphi(l) = 0$ shartlar bajarilsa, u holda (47) qator ixtiyoriy $t \geq 0$ lar uchun uzluksiz funksiyani aniqlaydi. (47) qatorning (40) va (41) shartlarini qanoatlantirishi osongina ko'rsatiladi.

5.14.2 Bir jinsli tenglama uchun bir jinsli bo'lmagan masala

Endi (39) tenglamaning bir jinsli bo'lmagan

$$u(0,t) = \psi_1(t), \quad u(l,t) = \psi_2(t) \tag{50}$$

chegaraviy va (41) boshlang'ich shartlarni qanoatlantiruvchi yechimni topish masalasini qaraylik, bu yerda ham $\psi_1(t), \quad \psi_2(t)$ berilgan funksiyalar bo'lib, $\psi_1(0) = \varphi(0), \quad \psi_1(0) = \varphi(l)$ shartlar bajariladi.

(39), (50), (41) masala yechimini

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi}{l}x \tag{51}$$

qator ko‘rinishida izlaymiz.

Bu yerda

$$T_n(t)=\frac{2}{l}\int\limits_0^l u(x,t)\sin\frac{n\pi}{l}x \tag{52}$$

bo‘lib, uni ikki marta bo‘laklab integrallaymiz va (39), (50) larni e‘tiborga olsak,

$$T_n(t)=\frac{2}{n\pi}\left[u(0,t)-(-1)^n u(l,t)\right]-\frac{2l^2}{n^2\pi^2}\int\limits_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi}{l}xdx$$

$$=\frac{2}{n\pi}\left[\psi_1(t)-(-1)^n\psi_2(t)\right]-\frac{2l^2}{(an\pi)^2}\int\limits_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi}{l}xdx \tag{53}$$

tenglik hosil bo‘ladi.

Endi (52) tenglikni differensiallaymiz:

$$T_n'(t)=\frac{2}{l}\int\limits_0^l \frac{\partial u}{\partial t}(x,t)\sin\frac{n\pi}{l}xdx. \tag{54}$$

(43), (44) tenglamalardan $T_n(t)$ funksiyani aniqlash uchun quyidagi

$$T_n'(t)+\lambda_n T_n(t)=f_n(t) \tag{55}$$

bir jinsli bo‘lmagan oddiy differensial tenglamaga ega bo‘lamiz, bu yerda

$$\lambda_n=\left(\frac{an\pi}{l}\right)^2,\quad f_n(t)=\frac{2a^2n\pi}{l^2}\left[\psi_1(t)-(-1)^n\psi_2(t)\right].$$

(55) tenglama quyidagi umumiy yechimga ega:

$$T_n(t)=\left(c_n+\int\limits_0^t f_n(\tau)\exp(\lambda_n\tau)d\tau\right)\exp(-\lambda_n t). \tag{56}$$

(51) dan (41) shartga binoan

$$u(x,0)=\varphi(x)=\sum_{n=1}^{\infty}T_n(0)\sin\frac{n\pi}{l}x$$

bo'lib, bundan

$$T_n(0) = c_n = \frac{2}{l}\int\limits_0^l \varphi(x)\sin\frac{n\pi}{l}xdx \tag{57}$$

kelib chiqadi. Shunday qilib, (39), (50), (41) masalaning yechimi (51) qatordan iborat bo'lib, $T_n(t)$ koeffitsientlar (56) va (57) tengliklar bilan aniqlanadi.

5.14.3 Bir jinsli bo'lmagan tenglama uchun bir jinsli chegaraviy shartli masala

Bir jinsli bo'lmagan

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{58}$$

tenglamaning bir jinsli

$$u(x,0) = 0, \quad 0 \le x \le l, \tag{59}$$

$$u(0,t) = 0, \quad u(l,t) = 0 \tag{60}$$

shartlarni qanoatlantiruvchi yechimni topamiz. Bu yerda berilgan $f(x,t)$ funksiya x bo'yicha birinchi tartibli bo'lakli uzluksiz hosilaga ega hamda barcha $t > 0$ lar uchun $f(0,t) = f(l,t) = 0$ deb faraz qilamiz. (58)-(60) masala yechimini

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi}{l}x \tag{61}$$

ko'rinishida izlaymiz.

Faraz qilaylik, $f(x,t)$ funksiyaning x o'zgaruvchi bo'yicha Furye qatori mavjud bo'lsin, ya'ni

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{l}, \tag{62}$$

$$f_n(t) = \frac{2}{l}\int\limits_0^l f(x,t)\sin\frac{n\pi x}{l}dx. \tag{63}$$

(61) qatorini (58) ga qo'yib, (62) ni e'tiborga olsak,

$$\sum_{n=1}^{\infty}\left[T_n'(t)+\left(\frac{an\pi}{l}\right)^2 T_n(t)-f_n(t)\right]\sin\frac{n\pi x}{l}=0$$

tenglikni olamiz. Bundan esa

$$T_n'(t)+\alpha_n^2 T_n(t)=f_n(t) \tag{64}$$

tenglamaga ega bo'lamiz, bu yerda $\alpha_n=(an\pi)/l$. (61) ni (59) ga qo'yib, $T_n(0)=0$ boshlang'ich shartga va uni qanoatlantiruvchi

$$T_n(t)=\int\limits_0^t f_n(\tau)\exp\left(-\alpha_n^2(t-\tau)\right)d\tau$$

yechimga ega bo'lamiz. Bu ifodani (61) ga qo'yib, (58)-(60) masalaning

$$u(x,t)=\sum_{n=1}^{\infty}\left\{\int\limits_0^t f_n(\tau)\exp\left[-a^2(t-\tau)\right]d\tau\right\}\sin\frac{n\pi x}{l}$$

yechimini hosil qilamiz. Bu yechimda $f_n(\tau)$ funksiyalarning o'rniga (63) ifodani qo'yib, uni quyidagicha yozish mumkin:

$$u(x,t)=\int\limits_0^t\int\limits_0^l G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau, \tag{65}$$

bu yerda

$$G(x,\xi,t-\tau)=\frac{2}{l}\sum_{n=1}^{\infty}\exp\left[-a^2(t-\tau)\right]\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}.$$

Odatda $G(x,\xi;t)$ funksiya *manba funksiya* yoki *Grin funksiyasi* deb yuritiladi va u uchun

$$G\,|_{x=0}=0,\quad G|_{x=l}=0$$

shartlar bajariladi.

Agar boshlang'ich shart bir jinsli bo'lmasa, (65) yechimga (39)-(41) masala yechimini qo'shish yetarli.

Bir jinsli bo'lmagan masala.

Endi (58) tenglamaning (41) boshlang'ich va (50) chegaraviy shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik.

Agar $v(x,t)$ va $w(x,t)$ funksiyalar mos ravishda (39), (41), (50) va (58)-(60) masalalarning yechimlari bo'lsa, u holda $u(x,t) = v(x,t) + w(x,t)$ funksiya (58), (41), (50) masalaning yechimi bo'ladi.

Agar (58), (41), (50) masalada $u = v + w$ va

$$w = \psi_1(t) + \frac{x}{l}\left[\psi_1(t) - \psi_2(t)\right]$$

deb olsak, $v(x,t)$ ga nisbatan quyidagi

$$v_t = a^2 v_{xx} + \tilde{f}(x,t), \quad v(x,0) = \tilde{\varphi}(x), \quad v(0,t) = v(l,t) = 0$$

yechish o'rganilgan masalaga kelamiz. Bu yerda

$$\tilde{f}(x,t) = f(x,t) - w_t + a^2 w_{xx}, \quad \tilde{\varphi}(x) = \varphi(x) - \psi_1(0) - \frac{x}{l}(\psi_2(0) - \psi_1(0)).$$

5.15 To'g'ri to'rtburchakli sohada issiqlik tarqalishi haqidagi masala

Tomonlari uzunliklari p va q bo'lgan to'g'ri to'rtburchak shaklidagi yupqa plastinkada issiqlik tarqalish tenglamasi uchun boshlang'ich-chegaraviy masalani qaraylik: $\{0 < x < p, \quad 0 < y < q, \quad t > 0\}$ sohada

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{66}$$

tenglamaning

$$u|_{x=0} = u|_{x=p} = 0; \quad u|_{y=0} = u|_{y=q} = 0 \tag{67}$$

chegaraviy va

$$u(x,y,0) = \varphi(x,y), \quad 0 \le x \le p, \quad 0 \le y \le q \tag{68}$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,y,t)$ yechimi topilsin.

Furye usuliga asosan (66) tenglamaning trivial bo‘lmagan xususiy yechimini

$$u(x, y, t) = X(x)Y(y)T(t)$$

ko‘rinishida izlashimiz va $X(x)$, $Y(y)$, $T(t)$ funksiyalarni aniqlash uchun mos ravishda

$$X''(x) + \lambda^2 X(x) = 0,$$

$$Y''(y) + \mu^2 Y(y) = 0,$$

$$T'(t) + a^2(\lambda^2 + \mu^2)T(t) = 0$$

tenglamalarga ega bo‘lamiz, bu yerda α^2, μ^2 lar o‘zgarmas sonlar. Ravshanki, bu tenglamalarning umumiy yechimlari mos ravishda

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x,$$

$$Y(y) = c_3 \cos \mu y + c_4 \sin \mu y,$$

$$T(t) = A \exp\left[-a^2(\lambda^2 + \mu^2)t\right]$$

formulalar bilan ifodalanadi.

(67) shartlar bajarilishi uchun $c_1 = c_3 = 0$,

$$\lambda = \frac{m\pi}{p}, \quad \mu = \frac{n\pi}{q}, \quad (m, n = 1, 2, 3, \ldots)$$

deb olish kerak bo‘ladi.

Shunday qilib, (66) tenglamaning (67) shartlarni qanoatlantiruvchi yechimi quyidagicha bo‘ladi:

$$u_{mn}(x, y, t) = A_{mn} \exp\left[-a^2\pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right)t\right] \sin\frac{m\pi}{p}x \sin\frac{n\pi}{q}y.$$

Ushbu

$$u(x, y, t) = \sum_{m,n=1}^{\infty} A_{mn} \exp\left[-a^2\pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right)t\right] \sin\frac{m\pi}{p}x \sin\frac{n\pi}{q}y \qquad (69)$$

qatorni tuzamiz va unda $t = 0$ deb, (68) boshlang‘ich shartdan foydalanamiz:

$$\varphi(x, y) = \sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi}{p}x \sin\frac{n\pi}{q}y.$$

Bu esa $\varphi(x,y)$ funksiyaning ikki karrali Furye qatori bo'lib, A_{mn} quyidagicha aniqlanadi:

$$A_{mn} = \frac{4}{pq}\int_0^p\int_0^q \varphi(x,y)\sin\frac{m\pi}{p}x\sin\frac{n\pi}{q}ydxdy.$$

Bularni (69) qatorga qo'yib, qo'yilgan masalaning yechimini hosil qilamiz.

5.16 Boshlang'ich shartsiz masalalar

Boshlang'ich vaqtga nisbatan yetarlicha katta vaqtlardan keyin issiqlik tarqalish jarayonlari o'rganilsa, kuzatilayotgan vaqtdagi issiqlik tarqalishiga boshlang'ich shartning ta'siri deyarli bo'lmaydi. Bunday hollarda issiqlik o'tkazuvchanlik tenglamasi uchun barcha $t \to +\infty$ qiymatlarda faqat chegaraviy shartlarni qanoatlantiruvchi yechimni topish masalasi qo'yiladi. Agar sterjen chegaralangan bo'lsa, uning ikkala uchida ham chegaraviy shartlar beriladi. Yarim chegaralangan sterjen uchun esa bitta chegaraviy shart qo'yiladi.

Avvalo, yarim chegaralangan sterjen uchun birinchi chegaraviy masalani qaraylik: bir o'lchovli issiqlik o'tkazuvchanlik tenglamasining $x > 0$ sohada chegaralangan va

$$u(0,t) = \varphi(t)$$

chegaraviy shartni qanoatlantiruvchi $u(x,t)$ yechimi topilsin. Faraz qilaylik, $|u(x,t)| < M, \quad |\varphi(t)| < M$ bo'lsin. Tadbiqiy masalalarda ko'p uchraydigan

$$\varphi(t) = A\cos\omega t, \quad A = const, \quad \omega = const > 0 \qquad (70)$$

chegaraviy shartni olaylik. Bu masala fransiyalik matematik J. Furye tomonidan o'rganilgan. Agar issiqlik o'tkazuvchanlik tenglamasining kompleks $\phi(t) = A\exp(i\omega t)$ boshlang'ich shartni qanoatlantiruvchi yechimi topilgan bo'lsa, uning haqiqiy va mavhum qismlarining har biri alohida tenglamani hamda mos ravishda (70) va

$$\varphi(t) = A\sin\omega t$$

shartni qanoatlantiradi.

Shunday qilib, ushbu

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad v(0,t) = A\exp(i\omega t) \tag{71}$$

masalani qaraylik. Bu masala yechimini

$$v(x,t) = A\exp(\alpha x + \beta t) \tag{72}$$

ko'rinishida izlaymiz, bu yerda α va β hozircha noma'lum o'zgarmaslar.

(72) ni (71) ga qo'yib,

$$\alpha^2 = \frac{1}{a^2}\beta, \quad \beta = i\omega$$

munosabatlarni olamiz. Bundan

$$\alpha = \pm\sqrt{\frac{\omega i}{a^2}} = \pm\frac{\sqrt{\omega}}{a\sqrt{2}}(1+i)$$

bo'lib, $v(x,t)$ uchun esa quyidagi ifodani hosil qilamiz:

$$v(x,t) = A\exp\left[\pm\sqrt{\frac{\omega}{2a^2}}x + i\left(\omega t \pm \sqrt{\frac{\omega}{2a^2}}x\right)\right]. \tag{73}$$

O'z navbatida, bu yechimning haqiqiy qismi

$$v(x,t) = Rev(x,t) = A\exp\left(\pm\sqrt{\frac{\omega}{2a^2}}x\right)\cos\left(\pm\sqrt{\frac{\omega}{2}}\frac{x}{a} + \omega t\right) \tag{74}$$

issiqlik o'tkazuvchanlik tenglamasi va (70) shartni qanoatlantiradi. Yechimni chegaralanganligi va $\omega > 0$ ni inobatga olsak, qo'yilgan masalaning uzluksiz chegaralangan yechimini quyidagicha olish maqsadga muvofiq bo'ladi:

$$u(x,t) = A\exp\left(\pm\sqrt{\frac{\omega}{2}}\frac{x}{a}\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2}}\frac{x}{a}\right). \tag{75}$$

Endi chekli $0 < x < l$ soha uchun boshlang'ich shartsiz

$$u_t = a^2 u_{xx}, \quad u(0,t) = A\cos\omega t, \quad u(l,t) = 0 \tag{76}$$

masalani qaraymiz. Bu masala ham xuddi yuqoridagi kabi yechiladi. Chegaraviy shartlarni

$$\tilde{u}(0,t) = A\exp(-i\omega t), \quad u(l,t) = 0$$

kabi yozib olib, yechimni

$$\tilde{u}(x,t) = X(x)\exp(-i\omega t) \tag{77}$$

ko'rinishida izlaymiz. Bu ifodani (76) ga qo'yib, $X(x)$ funksiya uchun

$$X''(x) + \gamma^2 X(x) = 0, \quad X(0) = A, \quad X(l) = 0$$

masalani olamiz, bu yerda

$$\gamma = \sqrt{\frac{i\omega}{a^2}} = \sqrt{\frac{\omega}{2}}\frac{1+i}{a}.$$

Bundan esa $X(x)$ funksiyaning haqiqiy va mavhum qismlarini $X_1(x)$ va $X_2(x)$ kabi belgilasak,

$$X(x) = A\frac{\sin(l-x)}{\gamma l} = X_1(x) + iX_2(x)$$

bo'lib, (77) ga ko'ra

$$\overline{u}(x,t) = A\frac{\sin(l-x)}{\gamma l}\exp(-i\omega t).$$

Bu ifodaning haqiqiy qismi

$$u(x,t) = X_1(x)\cos\omega t + X_2(x)\sin\omega t$$

chekli soha uchun qo'yilgan (76) boshlang'ich shartlarsiz masalaning yechimi bo'ladi.

6-Bob. Elliptik tenglamalar

6.1 Garmonik funksiyalar. Grin formulalari va fundamental yechimlar

Ushbu paragrafda biz xususiy hosilali differensial tenglamalar kursida keng qo'llaniladigan Grin formulalari bilan tanishamiz. Grinning birinchi va ikkinchi formulalarini nisbatan umumiy bo'lgan elliptik tipdagi operator uchun keltirib chiqaramiz.

Faraz qilaylik, $\mathbb{R}^n$ da S yopiq sirt bilan chegaralangan D soha berilgan bo'lsin. Eslatib o'tamiz, agar sirtning har bir nuqtasida urinma tekislik (yoki normal) mavjud bo'lib, sirtda bir nuqtadan ikkinchi nuqtaga o'tganda bu urunma tekislik (normal) holati uzluksiz o'zgarsa, bunday sirtga silliq sirt deyiladi. Matematik tahlil kursidan ma'lumki, $\overline{D}$ da uzluksiz va D sohada uzluksiz differensiallanuvchi $F(x)$, $x \in \mathbb{R}^n$ vektor funksiya uchun Gauss-Ostrogradskiy formulasi o'rinli:

$$\int\limits_D divF(x)dx = \oint\limits_S \overrightarrow{n}F(s)ds. \tag{1}$$

Bu yerda $\overrightarrow{n}$ - D sohaga nisbatan tashqi bo'lgan S sirtga o'tkazilgan normalning birlik vektori, ds - sirt elementi.

Agar $u(x) = u(x_1, x_2, ..., x_n)$ funksiya chekli D sohada ikki marta uzluksiz differensiallanuvchi bo'lib, Laplas tenglamasini qanoatlantirsa, u chekli D sohada garmonik deyiladi.

Faraz qilaylik, $C^1\left(\overline{D}\right) \cap C^2(D)$ sinfga tegishli bo'lgan $u(x)$ va $v(x)$ funksiyalar berilgan bo'lsin. Ushbu

$$Lu = div\,(k\nabla u) - qu,$$

bu yerda $k(x)$ va $q(x)$ funksiyalar $\overline{D}$ da uzluksiz, $k(x)$ funksiya D sohada uzluksiz differensiallanuvchi, differensial operatorni qaraymiz. To‘g‘ridan - to‘g‘ri tekshiriladigan

$$v div\,(k\nabla u) = div\,(kv\nabla u) - k\nabla u\nabla v$$

tenglik va (1) formuladan foydalanib, Grinning birinchi formulasi deb ataluvchi

$$\int\limits_D vLudx = \oint\limits_S kv\frac{\partial u}{\partial \overrightarrow{n}}ds - \int\limits_D (k\nabla u\nabla v)\,dx - \int\limits_D qvudx \tag{2}$$

formulani hosil qilamiz. Bu formulani olishda $\overrightarrow{n}\nabla u = \frac{\partial u}{\partial \overrightarrow{n}}$ tenglikdan foydalanildi.

(2) formulada u va v funksiyalarning o‘rnini almashtirib,

$$\int\limits_D uLvdx = \oint\limits_S ku\frac{\partial v}{\partial \overrightarrow{n}}ds - \int\limits_D (k\nabla u\nabla v)\,dx - \int\limits_D quvdx \tag{3}$$

tenglikni ham yozish mumkin. (2) dan (3) ni ayirib,

$$\int\limits_D (vLu - uLv)\,dx = \oint\limits_S k\left(v\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial v}{\partial \overrightarrow{n}}\right)ds$$

Grinning ikkinchi formulasini hosil qilamiz.

Aytaylik, $k = 1$ va $q = 0$ bo‘lsin. U holda L operator Laplas operatori bilan ustma-ust tushadi, ya’ni $Lu \equiv \triangle u$. Ravshanki, Laplas operatori uchun Grin formulalari quyidagi ko‘rinishga ega bo‘ladi:

$$\int\limits_D v\triangle udx = \oint\limits_S v\frac{\partial u}{\partial \overrightarrow{n}}ds - \int\limits_D (\nabla u\nabla v)\,dx, \tag{4}$$

$$\int\limits_D (v\triangle u - u\triangle v)\,dx = \oint\limits_S \left(v\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial v}{\partial \overrightarrow{n}}\right)ds. \tag{5}$$

Endi Laplas tenglamasining fundamental yechimi deb ataluvchi maxsus yechimini uch va ikki o‘lchovli hollarda topamiz.

Faraz qilaylik, x_0 nuqta $D \subset \mathbb{R}^3$ sohaning tayin nuqtasi bo‘lsin. Laplas tenglamasining x_0 nuqtadan hisoblangan masofaga bog‘liq bo‘lgan yechimini qidiramiz. Buning uchun, markazi x_0 nuqtada bo'lgan sferik koordinatalar $(r,\ \theta,\ \varphi)$ sistemasini kiritamiz. U holda masala Laplas tenglamasining radial-simmetrik yechimi $u(r)$ ni topishga, ya'ni

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=0$$

tenglamani yechishga keladi. Bu tenglamaning umumiy yechimi

$$u(r)=\frac{C_1}{r}+C_2$$

ko'rinishga ega bo'ladi. Bu yerda $r=|x-x_0|$, $C_1,\ C_2$ - o'zgarmas sonlar.

$$v(x,x_0)=\frac{1}{|x-x_0|}$$

funksiyaga uch o'lchovli *Laplas operatorining fundamental yechimi* deyiladi. E'tibor bering, bu funksiya Laplas tenglamasini x_0 nuqtadan boshqa barcha nuqtalarda qanoatlantiradi, ya'ni u garmonik funksiyadir, x_0 nuqtada esa maxsuslikka ega.

Ikki o'lchovli holda x_0 nuqtani markaz qilib, qutb koordinatalar $(r,\ \varphi)$ sistemasini kiritamiz va Laplas tenglamasining $\sqrt{(x_1-x_{01})^2+(x_2-x_{02})^2}$ masofaga bog'liq yechimini topamiz. Bu holda Laplas tenglamasi

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=0$$

ko'rinishga ega va uning umumiy yechimi quyidagi tenglik bilan aniqlanadi:

$$u(r)=C_1\ln\frac{1}{r}+C_2.$$

$$v(x,x_0)=\ln\frac{1}{|x-x_0|}$$

funksiyaga ikki o'lchovli *Laplas operatorining fundamental yechimi* deyiladi.

6.2 C^2 sinf va garmonik funksiyalarning integral ifodasi

x_0 nuqta $D\subset\mathbb{R}^3$ sohaning ichki nuqtasi bo'lsin. Bu nuqtani markaz qilib, D sohaning ichida to'liq yotuvchi ε radiusli Σ_ε sferani chizamiz. $U_{x_0}^\varepsilon=$

$\{x: \ |x-x_0| \leq \varepsilon\}$ bo‘lsin. U holda D sohaning chegarasi S bilan Σ_ε sfera orasidagi soha $D \backslash U_{x_0}^\varepsilon$ kabi bo‘ladi. $D \backslash U_{x_0}^\varepsilon$ sohada ixtiyoriy

$$u(x) \in C^2(D) \cap C^1\left(\overline{D}\right)$$

funksiya va yuqorida kiritilgan $v(x, x_0)$ fundamental yechimga Grinning ikkinchi formulasi (5) ni qo‘llaymiz:

$$\int\limits_{D \backslash U_{x_0}^\varepsilon} (v \triangle u - u \triangle v)\, dx =$$

$$= \oint\limits_S \left(u \frac{\partial v}{\partial \overrightarrow{n}} - v \frac{\partial u}{\partial \overrightarrow{n}}\right) ds - \oint\limits_{\Sigma_\varepsilon} \left(u \frac{\partial v}{\partial \overrightarrow{n}} - v \frac{\partial u}{\partial \overrightarrow{n}}\right) ds. \tag{6}$$

Σ_ε sirtda

$$v|_{\Sigma_\varepsilon} = \frac{1}{\varepsilon}, \ \frac{dv}{d\overrightarrow{v}}|_{\Sigma_\varepsilon} = -\frac{d}{dr}\left(\frac{1}{r}\right)|_{r=\varepsilon} = \frac{1}{\varepsilon^2}$$

bo‘lganligidan, o‘rta qiymat to‘g‘risidagi teoremaga asosan

$$\oint\limits_{\Sigma_\varepsilon} \left(u \frac{\partial v}{\partial \overrightarrow{n}} - v \frac{\partial u}{\partial \overrightarrow{n}}\right) ds = \oint\limits_{\Sigma_\varepsilon} \left(u \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \frac{\partial u}{\partial \overrightarrow{n}}\right) ds =$$

$$= 4\pi\varepsilon^2 \left[\frac{1}{\varepsilon^2} u(x^\star) - \frac{1}{\varepsilon} \frac{\partial u(x^\star)}{\partial \overrightarrow{n}}\right],$$

bu yerda $x^\star \in \Sigma_\varepsilon$, tengliklarni hosil qilamiz. $D \backslash U_{x_0}^\varepsilon$ sohada $\triangle v = 0$ va $u(x) \in C^2(D) \cap C^1\left(\overline{D}\right)$ ekanligidan, (6) da $\varepsilon \to 0$ da limitga o‘tib,

$$\int\limits_D \frac{\triangle u}{|x-x_0|} dx =$$

$$= \oint\limits_S \left(u \frac{\partial}{\partial \overrightarrow{n}} \frac{1}{|x-x_0|} - \frac{1}{|x-x_0|} \frac{\partial u}{\partial \overrightarrow{n}}\right) ds - 4\pi u(x_0), \ x_0 \in D$$

formulani olamiz. Bundan,

$$u(x_0) = \frac{1}{4\pi} \oint\limits_S \left(\frac{1}{|x-x_0|} \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} \frac{1}{|x-x_0|}\right) ds -$$

$$-\frac{1}{4\pi} \int\limits_D \frac{\triangle u}{|x-x_0|} dx, \ x_0 \in D. \tag{7}$$

Bu formulada D soha bo‘yicha integral ikkinchi tur xosmas integral kabi tushuniladi. (7) ga C^2 sinf funksiyalarining integral ifodasi yoki *Grinning uchinchi formulasi* deyiladi.

Eslatib o‘tamiz, (7) formula x_0 nuqta D sohaning ichki nuqtasi bo‘lgan hol uchun keltirib chiqarildi. Agar x_0 nuqta D sohadan tashqarida joylashgan bo‘lsa, u holda $v(x,\ x_0) = \frac{1}{|x-x_0|}$ funksiya garmonik bo‘ladi va Grinning ikkinchi formulasiga ko‘ra

$$\oint\limits_S \left(\frac{1}{|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial}{\partial \overrightarrow{n}}\frac{1}{|x-x_0|}\right) ds = \int\limits_D \frac{\triangle u}{|x-x_0|}dx,\ x_0 \overline{\in} \overline{D}.$$

tenglik o‘rinli bo‘ladi.

Endi x_0 nuqta S sirtga tegishli bo‘lgan holni ko‘ramiz. Faraz qilamiz, S sirt x_0 nuqtada uzluksiz burchak koeffisientli urunma tekislikka ega bo‘lsin. Navbatdagi mulohazalar xuddi (7) formulani olishdagi kabi bo‘ladi. Grinning birinchi formulasini $D \backslash U^{\varepsilon}_{x_0}$ sohada $v(x,\ x_0) = \frac{1}{|x-x_0|}$ va

$$u(x) \in C^2(D) \cap C^1\left(\overline{D}\right)$$

funksiyalarga qo‘llaymiz. Hosil bo‘ladigan tenglikda sirt integrallari $S' \cup \Sigma'_{\varepsilon}$ chegara bo‘yicha olinadi, bunda Σ'_{ε} - S sferaning D soha ichidagi qismi, $S' = S \backslash S_{\varepsilon}$, S_{ε} - S sirtning $U^{\varepsilon}_{x_0}$ shar ichidagi qismi. Yetarlicha kichik ε larda Σ'_{ε} sirt markazi x_0 da va radiusi ε bo‘lgan yarim sferaga yaqin bo‘ladi. Natijada, (7) formulani olish jarayonidagi kabi, $\varepsilon \to 0$ da limitga o‘tib, 4π ko‘paytmani 2π bilan almashtirib,

$$u(x_0) = \frac{1}{2\pi}\oint\limits_S \left(\frac{1}{|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial}{\partial \overrightarrow{n}}\frac{1}{|x-x_0|}\right) ds -$$
$$-\frac{1}{2\pi}\int\limits_D \frac{\triangle u}{|x-x_0|}dx,\ x_0 \in S$$

tenglikni hosil qilamiz. Bu tenglikda sirt integrallari xosmas integrallar ma’nosida tushuniladi. Har uchala holni birlashtirib, Grinning uchinchi formulasini quyidagi ko‘rinishda yozamiz:

$$\frac{1}{4\pi}\oint\limits_S \left(\frac{1}{|x-x_0|}\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial}{\partial \overrightarrow{n}}\frac{1}{|x-x_0|}\right) ds - \frac{1}{4\pi}\int\limits_D \frac{\triangle u}{|x-x_0|}dx =$$

$$= \begin{cases} u(x_0), \text{ agar } x_0 \in D, \\ \frac{u(x_0)}{2}, \text{ agar } x_0 \in S, \\ 0, \text{ agar } \ x_0 \overline{\in} \overline{D}. \end{cases} \tag{8}$$

Grinning uchinchi formulasi sohaning ixtiyoriy ichki nuqtasida $u(x)$ silliq funksiyani soha chegarasida o‘zi va normal hosilasining qiymatlari hamda butun D sohada bu funksiyadan olingan Laplas operatorining qiymati orqali ifodalanadi. Garmonik funksiya uchun Grinning uchinchi formulasi oddiyroq ko‘rinishga ega. Masalan, $x_0 \in D$ uchun

$$u(x_0) = \frac{1}{4\pi} \oint\limits_S \left(\frac{1}{|x - x_0|} \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} \frac{1}{|x - x_0|} \right) ds. \tag{9}$$

Ikki o‘lchovli hol uchun Grin funksiyasi yuqoridagiga o‘xshash keltirib chiqariladi. U quyidagi ko‘rinishga ega:

$$\frac{1}{2\pi} \oint \left(\ln \frac{1}{|x - x_0|} \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} \ln \frac{1}{|x - x_0|} \right) ds - \frac{1}{2\pi} \int\limits_D \ln \frac{1}{|x - x_0|} \triangle u(x) dx =$$

$$= \begin{cases} u(x_0), \text{ agar } x_0 \in D, \\ \frac{u(x_0)}{2}, \text{ agar } x_0 \in S, \\ 0, \text{ agar } \ x_0 \overline{\in} \overline{D}. \end{cases}$$

Shuningdek, yuqoridagi kabi mulohaza yuritib, (7) Grinning uchinchi formulasini ixtiyoriy $n \geq 3$ uchun quyidagidek yozish mumkin:

$$u(x_0) = \frac{1}{\omega_n} \oint\limits_S \left(E(x,\ x_0) \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} E(x,\ x_0) \right) ds -$$

$$- \frac{1}{\omega_n} \int\limits_D E(x,\ x_0) \triangle u(x) dx,\ x_0 \in D \subset \mathbb{R}^n. \tag{10}$$

Bu yerda $\omega_n = 2\pi^{(n/2)} \Gamma\left(\frac{n}{2}\right)^{-1}$ - $\mathbb{R}^n$ da birlik sfera yuzi, Γ - Eylerning gamma-funksiyasi (7-bobning 2-paragrafiga qarang),

$$E(x, x_0) := \frac{1}{n-2} |x - x_0|^{2-n},\ n \geq 3. \tag{11}$$

$E(x, x_0)$ funksiyaga n ($n \geq 3$) o‘lchovli Laplas operatorining *fundamental yechimi* deyiladi. U $x \neq x_0$ lar uchun har ikkala argumenti bo‘yicha Laplas tenglamasini qanoatlantiradi. Haqiqatan, $x \neq x_0$ lar uchun hosilalarni hisoblab, quyidagilarga ega bo‘lamiz:

$$\frac{\partial^2 E}{\partial x_i^2} = -\left|x - x_0\right|^{-n} + n\left|x - x_0\right|^{-n-2}(x_i - x_{0i}).$$

Bu ifodalarni Laplas tenglamasiga qo‘yib

$$\triangle E = \sum_{i=1}^{n} \frac{\partial^2 E}{\partial x_i^2} = -n\left|x - x_0\right|^{-n} + n\left|x - x_0\right|^{-n-2}\sum_{i=1}^{n}(x_i - x_{0i}) = 0$$

ekanligiga ishonch hosil qilish mumkin. $E(x,\ x_0)$ funksiya simmetrik bo‘lganligi uchun, $x \neq x_0$ larda Laplas tenglamasini x_0 bo‘yicha ham qanoatlantiradi.

$n = 3$ da 7-bobning 2-paragrafidagi Eylerning gamma-funksiyaning $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ xossasidan $\omega_3 = 4\pi$ tenglik kelib chiqishini inobatga olsak, (10) dan (8) formula hosil bo‘lishini ko‘rish mumkin.

6.3 Garmonik funksiyalarning asosiy xossalari. O‘rta qiymat haqida teorema

Grin formulalaridan foydalanib garmonik funksiyalarning asosiy xossalarini keltirib chiqaramiz.

1-xossa. *Agar $u(x)$ funksiya D sohada garmonik bo‘lsa, u holda*

$$\oint_S \frac{\partial u}{\partial n} ds = 0, \tag{12}$$

bu yerda $S-$ D sohada to‘liq yotuvchi yopiq silliq sirt.

Haqiqatan ham, Grinning birinchi formulasini S sirt o‘rab turgan sohada u va $v = 1$ funksiyalar uchun qo‘llasak, (12) tenglik kelib chiqadi.

2-xossa (o‘rta qiymat haqidagi teorema). *Faraz qilaylik, $u(x)$ funksiya D sohada garmonik funksiya bo‘lsin. U holda*

$$u(x_0) = \frac{1}{4\pi R^2} \oint_{\Sigma_R} u(x) ds,$$

bu yerda Σ_R- D sohada yotuvchi, markazi x_0 nuqtada radiusi R bo'lgan sfera.

Isbot. (9) formulani markazi x_0 nuqtada va sirti Σ_R dan iborat bo'lgan shar uchun yozamiz:

$$u(x_0) = \frac{1}{4\pi} \oint\limits_{\Sigma_R} \left(\frac{1}{|x - x_0|} \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} \frac{1}{|x - x_0|} \right) ds.$$

$$\frac{1}{|x - x_0|}\bigg|_{x \in \Sigma_R} = \frac{1}{R}, \ \frac{\partial}{\partial \overrightarrow{n}} \frac{1}{|x - x_0|}\bigg|_{\Sigma_R} = \frac{\partial}{\partial R} \frac{1}{R}\bigg|_{R=r} = -\frac{1}{r^2}$$

ayniyatlardan, (12) tenglikka asosan

$$u(x_0) = \frac{1}{4\pi R^2} \oint\limits_{\Sigma_R} u(x) ds$$

formula hosil bo'ladi.

Shunday qilib, D sohada garmonik bo'lgan $u(x)$ funksiyaning x_0 nuqtadagi qiymati uning markazi shu nuqtada joylashgan D sohada yotuvchi ixtiyoriy S_r sferadagi o'rta qiymatiga teng.

D sohada garmonik va $\overline{D}$ sohada uzluksiz $u(x)$ funksiya uchun sfera D sohaning chegarasiga urungan holda ham o'rta qiymat haqidagi teorema o'rinli bo'ladi.

Haqiqatan ham, faraz qilaylik, r_0 radiusli Σ_{r_0} sfera D sohaning chegarasiga urunsin. U holda o'rta qiymat haqidagi teorema kichik radiusli $(r < r_0)$ Σ_r sfera uchun o'rinli:

$$u(x_0) = \frac{1}{4\pi r^2} \oint\limits_{\Sigma_r} u(x) ds.$$

Bu formulada $u(x) \in C\left(\overline{D}\right)$ ekanligidan foydalanib, $r \to r_0$ deb limitga o'tsak,

$$u(x_0) = \frac{1}{4\pi r_0^2} \oint\limits_{\Sigma_{r_0}} u(x) ds.$$

Ikki o'lchovli holda o'rta qiymat haqidagi teorema quyidagi formula bilan ifodalanadi:

$$u(x_0) = \frac{1}{2\pi r} \oint\limits_{C_r} udl,$$

bu yerda C_r- u funksiya garmonik bo‘lgan sohada yotuvchi, radiusi r va markazi x_0 nuqtada bo‘lgan aylana.

3-xossa. *D sohada garmonik funksiya shu sohada cheksiz differensiallanuvchidir.*

Bu tasdiqning isboti (9) ko‘rinishda yozilgan Grinning uchinchi formulasida $x_0 \in D$ lar uchun sirt integrallari xos integrallar bo‘lganligi va ularni x_0 nuqtaning koordinatalari bo‘yicha istalgan marta differensiallash mumkinligidan kelib chiqadi. Shuningdek, (9) formuladan D sohada garmonik funksiya sohaning barcha ichki nuqtalarida haqiqiy o‘zgaruvchili analitik funksiya ekanligi, ya’ni x_0 nuqtaning atrofida yaqinlashuvchi darajali qatorga yoyilishi kelib chiqadi. Bunda qatorning yaqinlashish radiusi x_0 nuqtadan D sohaning chegarasigacha bo‘lgan masofadan kichik bo‘ladi.

4-xossa (Maksimum prinsipi). *D sohada garmonik va $\overline{D}$ sohada uzluksiz $u(x)$ funksiya o‘zining maksimal va minimal qiymatlariga D sohaning chegarasida erishadi.*

Isbot. $u(x)$ funksiya $\overline{D}$ sohada uzluksiz bo‘lganligi uchun o‘zining maksimal qiymatiga shu sohada erishadi. Bu maksimal qiymat sohaning chegarasida erishilishini ko‘rsatamiz. Teskarisini faraz qilamiz: $u(x)$ funksiya maksimal qiymatga D sohaning biror x_0 ichki nuqtasida erishsin, ya’ni

$$u_o = \max_{x \in \overline{D}} = u(x_0) \geq u(x).$$

x_0 nuqtani markaz qilib, D sohada yotuvchi ρ radiusli sfera chizamiz va bu sfera uchun o‘rta qiymat haqidagi teoremaga ko‘ra

$$u(x_0) = \frac{1}{4\pi\rho^2} \oint\limits_{\Sigma_\rho} u(x)ds \leq \frac{1}{4\pi\rho^2} \oint\limits_{\Sigma_\rho} u(x_0)ds =$$

$$= u(x_0)\frac{1}{4\pi\rho^2} \oint\limits_{\Sigma_\rho} ds = u(x_0). \qquad (13)$$

Ravshanki, bu munosabatlar faqat

$$u(x)\Big|_{x\in\Sigma_\rho} \equiv u(x_0)$$

bo'lganda o'rinli. (13) dan

$$\oint\limits_{\Sigma_\rho} u(x)ds = 4\pi\rho^2 u(x_0)$$

tenglik kelib chiqadi. Faraz qilaylik, hech bo'lmaganda Σ_ρ sferaning bitta x_1 nuqtasida $u(x_1) < u(x_0)$ tengsizlik bajarilsin, u holda $u(x)$ ning uzluksizligidan bu tengsizlik x_1 nuqtaning Σ_ρ sferadagi biror atrofida ham bajariladi. Bundan

$$\oint\limits_{\Sigma_\rho} u(x)ds < 4\pi\rho^2 u(x_0)$$

kelib chiqadi. Bu esa yuqoridagi farazimizga zid. Demak, Σ_ρ sferada

$$u(x) = u(x_0).$$

ρ ixtiyoriy ekanligidan Σ_ρ sferani D sohaning chegarasiga urunadigan qilib tanlash mumkin. Urunish nuqtasini x_* bilan belgilaymiz. Aynan shu x_* nuqtada $u_0 = u(x_*)$ maksimal qiymatga erishiladi.

Yuqoridagi mulohazalarni $v = -u$ garmonik funksiyaga qo'llab, minimal qiymat D sohaning chegarasida erishilishi ko'rsatiladi.

6.4 Dirixle va Neyman masalalarining qo'yilishi hamda ular yechimlarining yagonaligi

$\mathbb{R}^n$ fazoda biror chekli sohani D orqali belgilab, uning chegarasi S bo'lakli silliq sirtdan iborat bo'lsin, deb faraz qilamiz. $\mathbb{R}^n\backslash\overline{D}$ ni D_1 bilan belgilab olamiz, yani $D_1 = \mathbb{R}^n\backslash\overline{D}$.

Dirixlening ichki masalasi. D sohada garmonik $\overline{D} = D\cup S$ da uzluksiz va

$$\lim_{x\to x_0} u(x) = \varphi(x_0), \quad x_0\in S, \quad x\in D$$

chegaraviy shartni qanoatlantiruvchi $u(x)$ funksiya topilsin.

Dirixlening tashqi masalasi. D_1 sohada garmonik shunday $u(x)$ funksiya topilsinki, u S da berilgan uzluksiz qiymatlarni qabul qilib, ya'ni

$$\lim_{x\to x_0} u(x) = \varphi(x_0), \qquad x_0 \in S, \qquad x \in D_1$$

$|x| \to \infty$ da $n > 2$ bo'lgan holda $|x|^{2-n}$ dan sekin bo'lmay nolga intilsin, $n = 2$ esa chekli limitga intilsin.

Neymanning ichki masalasi. D sohada garmonik, $D \cup S$ da o'zining birinchi tartibli hosilalari bilan birga uzluksiz bo'lgan $u(x)$ funksiya topilsinki, uning normal bo'yicha olingan hosilasi S da berilgan funksiyaga teng bo'lsin, ya'ni

$$\lim_{x\to x_0} \frac{\partial u}{\partial n} = \psi(x_0), \qquad x \in D, \qquad x_0 \in S$$

bu yerda $n - S$ ga o'tkazilgan normal.

Neymanning tashqi masalasi. D_1 sohada garmonik shunday $u(x)$ funksiya topilsinki, uning normal bo'yicha olingan hosilasi S da berilgan funksiyaga teng bo'lsin, ya'ni

$$\lim_{x\to x_0} \frac{\partial u}{\partial n} = \psi(x_0), \qquad x \in D, \qquad x_0 \in S$$

hamda funksiyaning o'zi cheksiz uzoqlashgan nuqtada: $n > 2$ bo'lgan holda nolga, $n = 2$ da esa chekli limitga intilsin.

Dirixlening ichki va tashqi masalalari bittadan ortiq yechimga ega bo'lmaydi. Haqiqatdan ham, bu masalalar bir xil chegaraviy shartlarni qanoatlantiruvchi ikkita u_1 va u_2 yechimlarga ega bo'lsin. U holda $u = u_1 - u_2$ funksiya ham garmonik bo'ladi va $u|_S = 0$ shartni qanoatlantiradi. Avval ichki masalani ko'ramiz. Maksimum prinsipiga ko'ra barcha D sohada $u = 0$ bo'ladi, demak, $u_1 = u_2$.

Endi tashqi masalani tekshiramiz. Avval $n > 2$ bo'lsin. Shartga asosan $u(x)$ funksiya D_1 sohada garmonik, shu bilan birga $u|_S = 0$ va x nuqta koordinata boshidan yetarli uzoqlikda joylashganda

$$|u(x) \leq \frac{C}{|x|^{n-2}}|, \quad C = const \tag{14}$$

tengsizlik o‘rinli bo‘lishi kerak. Bu Neymanning ichki masalasi yechimga ega bo‘lishi uchun zaruriy shartdir. Keyinchalik (14) ning yetarli shart ekanini ham ko‘rsatamiz. Agar fazoning o‘lchovi $n > 2$ bo‘lsa, u holda Neymanning tashqi masalasi yagona yechimga ega bo‘ladi. Bu fikrning to‘g‘riligiga ishonch hosil qilish uchun yuqorida kiritilgan chegaralari S va S_R dan iborat bo‘lgan D_R sohaga Gauss-Ostrogradskiy formulasini qo‘llaymiz:

$$\sum_{i=1}^{n} \int\limits_{D_R} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int\limits_{S} u\frac{\partial u}{\partial n} dS + \int\limits_{S_R} u\frac{\partial u}{\partial n} dS_R. \tag{15}$$

S_R sfera bo‘yicha olingan integralni baholaymiz. Yetarlicha katta bo‘lgan R lar uchun, ya’ni $R \to \infty$ da

$$\left| \int\limits_{S_R} u\frac{\partial u}{\partial n} dS_R \right| \le \frac{CC_1\omega_n}{R^{n-2}} \longrightarrow 0$$

munosabat o‘rinli bo‘ladi. U holda $\frac{\partial u}{\partial n}|_S = 0$ bo‘lgani uchun (15) formuladagi S bo‘yicha olingan integral nolga teng. Shunday qilib, $R \to \infty$ da

$$\sum_{i=1}^{n} \int\limits_{D_R} \left(\frac{\partial u}{\partial x_i}\right)^2 dx \longrightarrow 0$$

Demak, $\frac{\partial u}{\partial x_i} = 0,\ i = 1, 2, \cdots, n\ u = const$. Ammo $|x| \to \infty$ da $u \to 0$ ekanligidan $u = 0$ kelib chiqadi. Agar $n = 2$ bo‘lsa, Neymanning tashqi masalasi o‘zgarmas son aniqligida topiladi. Bu holda ham xuddi yuqoridagidek $u = const$ tenglikka ega bo‘lamiz. Cheksiz uzoqlashgan nuqtada $n = 2$ hol uchun garmonik funksiya chegaralangan bo‘lishidan, yuqoridagi fikrimizning to‘g‘riligiga ishonch hosil qilamiz.

6.5 Kelvin almashtirishi

Markazi koordinatalar boshida radiusi r ga teng bo‘lgan sferani Σ^r orqali belgilaymiz.

Agar x va y nuqtalar sfera markazidan o‘tuvchi nurda yotib, $|x| \cdot |y| = r^2$ tenglik o‘rinli bo‘lsa, u holda bu nuqtalar Σ^r sferaga nisbatan *qo‘shma* yoki *simmetrik nuqtalar* deyiladi.

Simmetrik nuqtalarning dekart koordinatalari quyidagi munosabatlar bilan bog'langan bo'ladi:

$$y = \frac{r^2}{|x|^2}x, \quad x = \frac{r^2}{|y|^2}y \quad |x| = \sqrt{x_1^2 + \ldots + x_n^2} \tag{16}$$

yoki

$$y_i = \frac{r^2}{|x_i|}x_i, \quad x_i = \frac{r^2}{|y|^2}y_i.$$

(16) almashtirishga *inversiya almashtirishi* deyiladi.

Agar $u(x)$ funksiya biror n-o'lchovli sohada garmonik bo'lsa, u holda ravshanki, $u(ax + b) = u(ax_1 + b_1, \ldots, ax_n + b_n)$ funksiya ham garmonik bo'ladi, ya'ni soha parallel ko'chirilganda va o'xshashlik markazi ixtiyoriy x_0 nuqtada bo'lgan o'xshashlik almashtirishida funksiyaning garmonikligi saqlanadi. Inversiya almashtirishida funksiyaning garmonikligi saqlanib qoladimi, degan savol tug'iladi.

Umuman aytganda, $n \geq 2$ bo'lganda inversiya almashtirishning o'zi funksi- yaning garmoniklik xossasini saqlab qolmaydi.

Soddalik uchun $r = 1$ deb olamiz. Masalan, $n = 3$ da $u(x) = \frac{1}{|x|}$ funksiya $x \neq 0$ bo'lgan nuqtalarda garmonik. $x = \frac{y}{|y|^2}$ almashtirish natijasida hosil bo'lgan

$$u\left(\frac{y}{|y|^2}\right) = \frac{1}{\frac{|y|}{|y|^2}} = |y|$$

funksiya esa garmonik bo'lmaydi. Biroq inversiya almashtirishining bu kamchiligini shu almashtirishdan so'ng funksiyani $|y|^{n-2}$ ga ko'paytirib bartaraf etish mumkin.

T a ' r i f. Ushbu

$$v(y) = \frac{1}{|y|^{n-2}}u\left(\frac{y}{|y|^2}\right), \quad n \geq 3 \tag{17}$$

funksiya $u(x)$ funksiyaning *Kelvin almashtirishi* deyiladi.

Kelvin t e o r e m a s i. Agar $u(x)$ funksiya $D \subset \mathbb{R}^n$ sohada garmonik bo'lsa, (17) formula bilan aniqlangan funksiya D sohadan uning birlik sferaga nisbatan inversiyasi natijasida hosil bo'lgan D_0 sohada garmonik bo'ladi.

Isbot. D_1 soha o'zining chegarasi bilan D_0 da to'la yotuvchi soha, D_2 esa D_1 ning inversiya almashtirishi natijasida hosil bo'lgan soha bo'lsin. U holda

D_2 soha D da oʻzining chegarasi bilan toʻla yotadi. D_2 sohaning chegarasini σ orqali belgilab olamiz. (10) formulaga asosan

$$u(x) = \frac{1}{(n-2)\omega_n} \int\limits_{\sigma} \left(\frac{1}{|x-\xi|^{n-2}} \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n} \frac{1}{|x-\xi|^{n-2}} \right) d\sigma_\xi .$$

(17) Kelvin almashtirishiga koʻra

$$v(y) = \frac{1}{|y|^{n-2}} u\left(\frac{y}{|y|^2}\right) = \frac{1}{(n-2)\omega_n} \times$$

$$\times \int\limits_{\sigma} \left[\frac{1}{|y|^{n-2} \left| \frac{y}{|y|^2} - \xi \right|^{n-2}} \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n} \frac{1}{|y|^{n-2} \left| \frac{y}{|y|^2} - \xi \right|^{n-2}} \right] d_\xi \sigma. \quad (18)$$

Ushbu

$$\frac{1}{|y|^{n-2} \left| \frac{y}{|y|^2} - \xi \right|^{n-2}}$$

ifoda ξ ga nisbatan garmonik funksiya, y boʻyicha ham garmonik funksiya boʻladi. Haqiqatan ham,

$$|y| \left| \frac{y}{|y|^2} - \xi \right| = \left| \frac{y}{|y|} - \xi|y| \right| = \left[\left(\frac{y}{|y|} - \xi|y| \right)^2 \right]^{\frac{1}{2}} =$$

$$\left[1 - 2\xi y - |y|^2|\xi|^2\right]^{\frac{1}{2}} = \left[\left(\frac{\xi}{|\xi|} - y|\xi| \right)^2 \right]^{\frac{1}{2}} =$$

$$\left| \frac{\xi}{|\xi|} - y|\xi| \right| = |\xi| \left| \frac{\xi}{|\xi|^2} - y \right|.$$

Shunday qilib,

$$\frac{1}{|y|^{n-2} \left| \frac{y}{|y|^2} - \xi \right|^{n-2}} = \frac{1}{|\xi|^{n-2} \left| \frac{\xi}{|\xi|^2} - y \right|^{n-2}}.$$

Bu ifoda $\xi \neq \frac{y}{|y|^2}$ boʻlganda garmonik funksiyadir, $\xi \in \sigma$ da bu tengsizlik oʻrinli boʻladi.

Demak, (18) formula bilan aniqlangan $v(y)$ funksiya y boʻyicha garmonik funksiyadir.

Agar $n = 2$ bo‘lsa,

$$v(y) = u\left(\frac{y}{|y|^2}\right) = u\left(\frac{y_1}{|y|^2}, \frac{y_2}{|y|^2}\right)$$

funksiya D_0 sohada garmonik bo‘ladi. Bevosita hisoblash natijasida bunga ishonch hosil qilish mumkin.

Kelvin teoremasidan foydalanib, odatda, garmonik funksiyaning cheksiz uzoqlashgan nuqta atrofida ta’rifi beriladi.

Agar

$$v(y) = |y|^{2-n} u\left(\frac{y}{|y|^2}\right) = |x|^{n-2} u(x)$$

funksiya $y = 0$ nuqtada $\lim\limits_{y\to 0} v(y)$ tarzda qo‘shimcha aniqlangan bo‘lib, $y = 0$ nuqta atrofida garmonik bo‘lsa, u holda $u(x)$ funksiya cheksiz uzoqlashgan nuqta atrofida (ya‘ni yetarlicha katta radiusli yopiq $|x| \leq R$ shartdan tashqarida) garmonik deb aytiladi.

6.6 Shar uchun Dirixle masalasi

Faraz qilaylik, U_0^r markazi koordinata boshida radiusi r bo‘lgan n-o‘lchovli shar va Σ_r bu shar sirti bo‘lsin. Σ_r da

$$u|_{\Sigma_r} = f(x), \qquad x \in \Sigma_r$$

berilgan qiymatni qabul qiluvchi U_0^r ning ichki nuqtalarida garmonik $u(x)$, $x = (x_1, ..., x_n)$ funksiyani topish talab etilsin. Bu masala yechimini mavjud va kerakli shartlarni qanoatlantiradi degan faraz bilan uni berilgan f funksiya orqali ifodalaydigan formulani topamiz. So‘ngra esa topilgan formula haqiqatan ham masala yechimi bo‘lishini ko‘rsatamiz.

Shunday qilib, yuqoridagi masalaning $u(x) \in C^2(\overline{U_0^r})$ yechimi mavjud bo‘lsin. Bu yechimning integral shaklini, (10) formulaga ko‘ra, quyidagicha yozamiz:

$$u(x) = \frac{1}{\omega_n} \oint\limits_S \left(E(\xi,\ x) \frac{\partial u}{\partial \overrightarrow{n}} - u \frac{\partial}{\partial \overrightarrow{n}} E(\xi,\ x) \right) ds_\xi \qquad (19)$$

x sharning ichki nuqtasi bo'lib, unga Σ_r sferaga nisbatan simmetrik joylashgan nuqtani x' orqali belgilaymiz. Simmetrik bo'lgan x va x' nuqtalar sharning markazidan o'tuvchi bitta nurda yotib, ular uchun

$$|x| \cdot |x'| = r^2$$

tenglik o'rinli. ξ nuqta Σ_r sferada o'zgargani uchun $|x' - \xi| \neq 0$. Endi

$$v(\xi) = E(\xi,\ x')$$

funksiyani qaraylik. Ma'lumki, bu funksiya har qanday (x' nuqtani o'z ichiga olmagan) sohada garmonik, shu jumladan U_0^r sharda ham.

Garmonik bo'lgan u va v funksiyalarga Grin formulasini qo'llasak,

$$\int\limits_S \left[E(\xi,\ x') \frac{\partial u(\xi)}{\partial \vec{n}} - u(\xi) \frac{\partial}{\partial \vec{n}} E(\xi,\ x') \right] ds_\xi = 0 \tag{20}$$

tenglikni olamiz. Dekart koordinatalar sistemasida koordinatalar boshini O, x, x', ξ nuqtalarning o'rnini mos ravishda M, M', P harflar bilan belgilab, bu nuqtalarni tutashtirib, hosil bo'lgan uchburchaklar uchun $\Delta OMP \sim \Delta OM'P$ munosabatni olamiz. Bundan esa

$$\frac{|MP|}{|M'P|} = \frac{|OM|}{r}$$

yoki

$$\frac{|x - \xi|}{|x' - \xi|} = \frac{|x|}{r} \tag{21}$$

tengliklar hosil bo'ladi. (21) dan

$$\frac{1}{|x - \xi|} = \frac{r}{|x||x' - \xi|}$$

yoki

$$\frac{1}{|x - \xi|^{2-n}} = \left(\frac{r}{|x|} \right)^{n-2} \frac{1}{|x' - \xi|^{2-n}}$$

kelib chiqadi. Bu esa (18) va (19) dagi birinchi integral ostidagi ifodalar ξ nuqtaga bog'liq bo'lmagan $(a/|x|)^{n-2}$ ko'paytuvchi bilan farq qilishini bildiradi.

Endi (20) tenglikni

$$\frac{1}{(n-2)\omega_n}\left(\frac{r}{|x|}\right)^{n-2}$$

ga ko‘paytirib, uni (19) dan ayirsak,

$$u(x)=\frac{1}{(n-2)\omega_n}\times$$

$$\times\int\limits_{\Sigma_r} f(\xi)\left[\left(\frac{r}{|x|}\right)^{n-2}\frac{\partial}{\partial\overrightarrow{n}}\left(\frac{1}{|x'-\xi|^{n-2}}\right)-\frac{\partial}{\partial\overrightarrow{n}}\left(\frac{1}{|x-\xi|^{n-2}}\right)\right]ds_\xi \quad (22)$$

izlanayotgan yechimni beruvchi formulani olamiz. Bu formulani yanada soddaroq ko‘rinishga keltirish mumkin.

Tashqi normallar radiuslar bilan ustma-ust tushgani uchun

$$\cos(\overrightarrow{n},x_k)=\frac{\xi_k}{r},\qquad \frac{\partial}{\partial\overrightarrow{n_\xi}}=\frac{\partial\xi_k}{\partial r}\cdot\frac{\partial}{\partial\xi_k},$$

$$\frac{\partial}{\partial\xi_k}|x-\xi|=\frac{\xi_k-x_k}{|x-\xi|},\qquad \frac{\partial}{\partial\xi_k}|x-\xi|=\frac{\xi_k-x'_k}{|x-\xi|},$$

bu yerda x_k $(k=\overline{1,n})$ va x'_k $(k=\overline{1,n})$ mos ravishda M va M' nuqtalarning, ξ_k $(k=\overline{1,n})$ esa P nuqtaning koordinatalari.

Bulardan foydalanib, (22) integral ostidagi ikkinchi hadni hisoblaymiz:

$$\frac{\partial}{\partial\overrightarrow{n_\xi}}\left(\frac{1}{|x-\xi|^{n-2}}\right)=-(n-2)\frac{\xi_k}{r}\cdot\frac{1}{|x-\xi|^{n-1}}\cdot\frac{\partial}{\partial\xi_k}|x-\xi|=$$

$$=-\frac{(n-2)}{r|x-\xi|^n}\cdot\xi_k(\xi_k-x_k)=-\frac{(n-2)}{r|x-\xi|^n}(r^2-\xi_k x_k). \quad (23)$$

Xuddi shunga o‘xshash

$$\frac{\partial}{\partial\overrightarrow{n_\xi}}\left(\frac{1}{|x'-\xi|^{n-2}}\right)=-\frac{(n-2)}{r|x'-\xi|^n}(r^2-\xi_k x'_k).$$

Bu ifodani $(r|x|)^{n-2}$ ga ko‘paytiramiz va yuqorida olingan

$$\frac{r}{|x||x'-\xi|}=\frac{1}{|x-\xi|}$$

munosabatdan foydalanib,

$$\left(\frac{r}{|x|}\right)^{n-2}\frac{\partial}{\partial\overrightarrow{n_\xi}}\left(\frac{1}{|x'-\xi|^{n-2}}\right)=-\frac{n-2}{r|x-\xi|^n}\left(|x|^2-\xi_k x'_k\frac{|x|^2}{r^2}\right)$$

tenglikka ega bo‘lamiz.

M va M' nuqtalar koordinata boshidan chiquvchi bitta nurda yotgani uchun

$$x_k = x'_k \frac{OM}{OM'} = x'_k \frac{|x|^2}{|x'||x|} = x'_k \frac{|x|^2}{r^2},$$

bundan esa

$$\left(\frac{r}{|x|}\right)^{n-2} \frac{\partial}{\partial \overrightarrow{n_\xi}} \left(\frac{1}{|x'-\xi|^{n-2}}\right) = -\frac{n-2}{r|x-\xi|^n}\left(|x|^2 - \xi_k x_k\right). \tag{24}$$

Nihoyat (23) va (24) ni (22) ga qo‘yib,

$$u(x) = \frac{1}{\omega_n r} \int\limits_{\Sigma^r} f(\xi) \frac{r^2 - |x|^2}{|x-\xi|^n} ds_\xi \tag{25}$$

Puasson formulasini hosil qilamiz, bu yerda $|x| \geq r$ bo‘lib,

$$\frac{r^2 - |x|^2}{r|x-\xi|^n}$$

ifodaga *Puasson yadrosi* deyiladi.

Eslatib o‘tamiz, yuqoridagi farazimizga ko‘ra har qanday $u \in C^2(\overline{U_0^r})$ garmonik funksiya uchun (25) Puasson formulasi o‘rinli.

Endi Puasson yadrosining ba’zi xossalarini keltiramiz.

1. *Puasson yadrosi manfiy emas.* Shuningdek, u $M \neq P$ va $|x| = r$ da nolga teng.

2. *U_0^r sharning ichki nuqtalarida Puasson yadrosi garmonik funksiya bo‘ladi.* Buni ko‘rsatamiz. Agar M nuqta U_0^r sharning ichida bo‘lsa, $|x-\xi| \neq 0$ bo‘lib, Puasson yadrosi istalgan tartibli hosilaga ega. Uning Laplas tenglamasini qanoatlantirishini ko‘rsatamiz. Leybnits formulasiga ko‘ra,

$$\frac{\partial^2}{\partial x_k^2}\left(\frac{r^2-|x|^2}{r|x-\xi|^n}\right) = \frac{1}{r|x-\xi|^n}\frac{\partial^2}{\partial x_k^2}\left(r^2 - |x|^2\right) +$$

$$2\frac{\partial(r^2-|x|^2)}{\partial x_k} \cdot \frac{\partial}{\partial x_k}\left(\frac{1}{r|x-\xi|^n}\right) + (r^2-|x|^2)\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r|x-\xi|^n}\right).$$

Bu yerda

$$\frac{\partial|x|}{\partial x_k}=\frac{x_k}{|x|},\quad \frac{\partial}{\partial x_k}|x-\xi|=\frac{x_k-\xi_k}{|x-\xi|}$$

larni e'tiborga olib, k bo'yicha yig'sak

$$\Delta\left(\frac{r^2-|x|^2}{r|x-\xi|^n}\right)=\frac{2n}{|x-\xi|^n}\left[-1+\frac{1}{|x-\xi|^2}(r^2+|x|^2-2x_k\xi_k)\right]=0,$$

chunki $|x-\xi|^2=\left(\overrightarrow{MP},\overrightarrow{PM}\right)=r^2+|x|^2-2\left(\overrightarrow{OM},\overrightarrow{OP}\right)=r^2+|x|^2-2\xi_kx_k$.

3. *Ushbu*

$$\frac{1}{\omega_n}\int\limits_{\Sigma^r}\frac{r^2-|x|^2}{|x-\xi|^n}ds_\xi=1,\qquad |x|<r \tag{26}$$

tenglik o'rinli. Buning isboti sharda garmonik bo'lib, uning chegarada 1 ga teng bo'lgan funksiya uchun yagonalik teoremasi va (25) formuladan kelib chiqadi.

4. Endi Σ^r sferada uzluksiz bo'lgan f funksiya uchun Puasson integrali bilan ifodalangan $u(x)$ yechim sharda garmonik bo'lib,

$$u|_{\Sigma_r}=f(x),\qquad x\in\Sigma_r$$

shartni qanoatlantirishini ko'rsatamiz.

Faraz qilaylik, $u(x)$ U_0^r sharning ichida (25) Puasson formulasi bilan aniqlangan bo'lsin. Unda ma'lumki, bu funksiya sharning ichki nuqtalarida istalgan tartibli hosilaga ega va u U_0^r da

$$\Delta u=\frac{1}{\omega_n r}\int\limits_{\Sigma^r}f(\xi)\Delta\left(\frac{r^2-|x|^2}{|x-\xi|}\right)ds_\xi=0$$

garmonik bo'ladi.

Yuqoridagi chegaraviy shartning bajarilishini ko'rsatish uchun x ichki nuqta Σ_r sirt ustidagi biror x_0 nuqtaga sohaning ichki tarafidan intilsin deb faraz qilamiz. (26) ni $f(x_0)$ ga ko'paytirib (26) dan ayiramiz. Natijada

$$u(x)-f(x_0)=\frac{1}{\omega_n r}\int\limits_{\Sigma^r}[f(x)-f(x_0)]\,\frac{r^2-|x|^2}{|x-\xi|^n}ds_\xi \tag{27}$$

tenglikni hosil qilamiz.

$f \in C(\Sigma_r)$ bo‘lgani uchun istalgancha kichik $\varepsilon > 0$ son olib, x_0 nuqtaning shunday σ sferik atrofini olamizki, barcha $\xi \in \sigma$ uchun

$$|f(\xi) - f(x_0)| < \frac{\varepsilon}{2}$$

bo‘lsin. Agar σ ning radiusini δ desak, $\Sigma^r \backslash \sigma$ sohada $|\xi - x_0| \geq \delta$ bo‘ladi. $u(x) - f(x_0)$ ayirmani baholash maqsadida (27) integralni σ va $\Sigma_r \backslash \sigma$ sohalar bo‘yicha ikkiga ajratib yozamiz:

$$u(x) - f(x_0) = \frac{1}{\omega_n r} \int\limits_{\sigma} [f(x) - f(x_0)] \frac{r^2 - |x|^2}{|x - \xi|^n} ds_\xi +$$

$$+\frac{1}{\omega_n r} \int\limits_{\Sigma_r \backslash \sigma} [f(x) - f(x_0)] \frac{r^2 - |x|^2}{|x - \xi|^n} ds_\xi = I_1 + I_2.$$

Birinchi integralni baholaymiz:

$$|I_1| \leq \frac{1}{\omega_n r} \int\limits_{\sigma} \frac{r^2 - |x|^2}{|x - \xi|^n} |f(\xi) - f(x_0)| ds_\xi <$$

$$< \frac{\varepsilon}{2\omega_n r|} \int\limits_{\sigma} \frac{r^2 - |x|^2}{|x - \xi|^n} ds_\xi < \frac{\varepsilon}{2\omega_n r} \int\limits_{\Sigma_r} \frac{r^2 - |x|^2}{|x - \xi|^n} ds_\xi = \frac{\varepsilon}{2},$$

ya‘ni, biz I_1 integral uchun M nuqtaning holatiga bog‘liq bo‘lmagan baho oldik.

I_2 integralni esa x va x_0 nuqtalarning yetarlicha yaqinligi hisobiga kichik qilishimiz mumkin. Bu nuqtalarni shunday yaqin olamizki, $|x - x_0| < \delta/2$ bo‘lsin. Unda

$$|x - \xi| \geq |x - x_0| - |\xi - x_0| \geq \frac{\delta}{2},$$

bundan esa

$$\frac{1}{|x - \xi|} < \frac{2}{\delta},$$

shuningdek,

$$]\frac{r^2 - |x|^2}{r|x - \xi|^n} = \frac{(r - |x|)(r + |x|)}{r|x - \xi|^n} < \frac{2^{n+1}(r - |x|)}{\delta^n}.$$

f funksiya yopiq sohada uzluksiz bo'lgani uchun $|f| < N = const$ bo'ladi va

$$|f(\xi) - f(x_0)| < 2N.$$

Bulardan esa

$$|u(x) - f(x_0)| < \frac{\varepsilon}{2} + |I_2| < \frac{\varepsilon}{2} + \frac{2^{n+1}N(r-|x|)}{\omega_n \delta^n} \int\limits_{\Sigma^r \setminus \sigma} ds_\xi <$$

$$< \frac{\varepsilon}{2} + \frac{2^{n+2}r^{n-1}N}{\delta^n}(r - |x|).$$

Endi $h > 0$ sonini shunday tanlaymizki,

$$\frac{2^{n+2}r^{n-1}}{\delta^n} \cdot Nh < \frac{\varepsilon}{2}$$

tengsizlik o'rinli bo'lsin. Agar $|x-x_0| < h$ bo'lsa, unda $r-|x| = |x_0|-|x| \leq |x - x_0| < h$ va $|u(x) - f(x_0)| < \varepsilon$ bo'ladi. Bundan esa ixtiyoriy $x_0 \in \Sigma^r$ uchun

$$\lim_{x \to x_0} u(x) = f(x_0)$$

kelib chiqadi. Shu bilan shar uchun Dirixle masalasining yechimi (25) formula orqali ifodalanishi to'liq asoslandi.

$n = 3$ va $n = 2$ bo'lgan hollarda (25) Puasson formulasi sferik va qutb koordinatalar sistemasida mos ravishda quyidagicha yoziladi:

$$u(x_1, x_2, x_3) = \frac{r}{4\pi} \int\limits_0^\pi d\psi \int\limits_0^{2\pi} f(\xi) \frac{r^2 - |x|^2}{r^2 - 2r|x|\cos\gamma + |x|^2} \sin\psi d\varphi,$$

$$f(\xi) = f(\xi_1, \xi_2, \xi_3), \qquad |x|\cos\gamma = (x, \xi),$$

$$\xi_1 = r\sin\psi\cos\varphi,\ \ \xi_2 = r\sin\psi\cos\varphi,\ \ \xi_3 = r\cos\psi;$$

$$u(x_1, x_2) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(r\cos\varphi, r\sin\varphi) \frac{r^2 - |x|^2}{r^2 - 2r|x|\cos(\psi - \varphi) + |x|^2} d\varphi, \quad (28)$$

$$x_1 = |x|\cos\psi, \quad x_2 = |x|\sin\psi, \quad \xi_1 = r\cos\varphi, \quad \xi_2 = r\sin\varphi.$$

6.7 O‘rta qiymat haqidagi teoremaga teskari teorema. Chetlashtiriladigan maxsuslik to‘g‘risidagi teorema

Markazi ixtiyoriy x_0 nuqtada radiusi r bo‘lgan sharni $U_{x_0}^r$ orqali belgilaymiz. $\Sigma_r - U_{x_0}^r$ sharning chegarasi. $u(x) \in C(D)$ funksiyani qaraymiz. Agar $U_{x_0}^r \subset D$ shar uchun $u(x)$ funksiya

$$u(x_0) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\Sigma_r} u(\xi) ds_\xi$$

shartni qanoatlantirsa, u holda bu funksiya D da garmonik bo‘ladi.

Buni isbotlash uchun $U_{x_0}^r$ sharda garmonik bo‘lib, $\overline{U_{x_0}^r}$ da uzluksiz va Σr da $u(x)$ qiymatini qabul qiluvchi $v(x)$ funksiyani kiritamiz. Bunday funksiyaning mavjudligi Puasson formulasidan kelib chiqadi. Ravshanki, $v(x) - u(x)$ ayirma $U_{x_0}^r$ sharda uzluksiz bo‘lib, Σ_r da aynan nolga teng. Shu bilan birga bu ayirma uchun o‘rta qiymat haqidagi teorema o‘rinli. Unga ko‘ra $v(x) - u(x)$ ayitma o‘zgarmas sondan farqli bo‘lsa, ekstremum prinsipiga asosan u $U_{x_0}^r$ sharning ichki nuqtalarida maksimum va minimum qiymatga ega bo‘lmaydi.

Biroq Σ_r da $v(x) - u(x) = 0$ bo‘lgani uchun $U_{x_0}^r$ ning barcha nuqtalarida $v(x) - u(x) = 0$, yoki $v(x) = u(x)$ bo‘ladi. Shunday qilib, $u(x)$ funksiya $U_{x_0}^r$ da garmonikdir. x_0 nuqta ixtiyoriy bo‘lganligi uchun $u(x)$ funksiyaning D sohada garmonik ekanligi kelib chiqadi.

Chetlashtiriladigan maxsuslik to‘g‘risidagi t e o r e m a.

Agar $u(x) = u(x_1, x_2, ..., x_n)$ funksiya D sohaning x_0 nuqtasidan tashqari barcha nuqtalarida garmonik bo‘lib, bu nuqtaning biror atrofida chegaralangan bo‘lsa, u holda $U(x)$ funksiyaning x_0 nuqtasidagi qiymatini shunday aniqlash mumkinki, natijada, u D sohaning barcha nuqtalarida garmonik bo‘ladi.

Teoremani isbotlash uchun D sohada yotuvchi $U_{x_0}^\varepsilon$ sharni qaraymiz. Σ_ε esa $U_{x_0}^\varepsilon$ sharning chegarasi bo‘lsin. Puasson formulasiga asosan $U_{x_0}^\varepsilon$ da garmonik va Σ_ε da $u(x)$ bilan ustma-ust tushadigan $u_1(x)$ funksiyani aniqlaymiz.

Buning uchun markazi x_0 nuqtada radiusi $\delta < \varepsilon$ bo‘lgan $U_{x_0}^{\delta}$ sharni chizamiz, Σ_δ bu sharning chegarasi bo‘lsin. $U_{x_0}^{\varepsilon}$ ning $U_{x_0}^{\delta}$ shardan tashqari qismini $U_{x_0}^{\delta,\varepsilon}$ orqali belgilaymiz, ya’ni $U_{x_0}^{\delta,\varepsilon} = U_{x_0}^{\delta} \backslash U_{x_0}^{\varepsilon}$. Ravshanki, $v(x) = u(x) - u_1(x)$ funksiya Σ_ε da nolga teng va δ ga bog‘liq bo‘lmagan shunday o‘zgarmas $C > 0$ sonni ko‘rsatish mumkinki, bunda

$$\max_{x \in U_{x_0}^{\delta,\varepsilon}} |v(x)| \leq C$$

bo‘ladi. $n = 2$ da

$$v_\delta(x) = C \frac{\ln |x - x_0| - \ln \varepsilon}{\ln \delta - \ln \varepsilon}$$

va $n \geq 3$ da

$$v_\delta(x) = C \frac{\varepsilon^{2-n} - |x - x_0|^{2-n}}{\varepsilon^{2-n} - \delta^{2-n}}$$

funksiyalarni kiritamiz.

Bevosita tekshirib, bu funksiyalarning $U_{x_0}^{\delta,\varepsilon}$ sohada garmonik ekanligiga ishonch hosil qilish mumkin ($n = 2$ holda shar va sfera o‘rniga mos ravishda doira va aylana tushiniladi). Shu bilan birga, agar $x \in \Sigma_\varepsilon$, ya’ni $|x - x_0| = \varepsilon$ bo‘lsa, $v_\delta(x) = 0$ va $x \in \Sigma_\delta$, ya’ni $|x - x_0| = \delta$ bo‘lsa, $v_\delta(x) = C$ bo‘ladi. Ekstremum prinsipiga asosan, $U_{x_0}^{\delta,\varepsilon}$ sohada

$$v_\delta(x) - v(x) \geq 0, \quad v_\delta(x) + v(x) \geq 0$$

tengsizliklar o‘rinli. Bundan

$$|v(x)| \leq v_\delta(x).$$

Demak, $\delta \to 0$ deb limitga o‘tsak, $U_{x_0}^{\varepsilon}$ ning barcha nuqtalarida (ehtimol x_0 dan tashqari) $v(x) = 0$ bo‘ladi. $v(x_0) = 0$ deb hisoblab, $U_{x_0}^{\varepsilon}$ da $v(x) = 0$ чки $u(x) = u_1(x)$ bo‘lishiga erishamiz.

Teoremaning isbotidan ko‘rinadiki, $u(x)$ funksiyaning x_0 nuqta atrofida chegaralanganlik shartini biroz yumshatish mumkin. Masalan, $u(x)$ funksiya $x \to x_0$ da ushbu

$$u(x) = O(|x - x_0|^{2-n}), \quad n \geq 3 \;\; (O(\ln |x - x_0|), \; n = 2)$$

shartni qanoatlantirsa ham teorema o'z kuchini saqlab qoladi, ya'ni $u(x)$ funksiya x_0 maxsus nuqta atrofida

$$\frac{1}{|x-x_0|^{2-n}}, \quad n \geq 3 \ \left(\ln\frac{1}{|x-x_0|}, \ n=2\right)$$

ifodaga nisbatan sekinroq o'ssa ham teorema tasdig'i o'z kuchida qoladi. Haqiqatan, yuqorida kiritilgan $v(x)$ funksiya Σ_ε da nolga teng. Shartga ko'ra $|x-x_0|^{n-2}u(x)$ ifoda x_0 nuqtaning kichik atrofida yetarlicha kichik, $u_1(x)$ funksiya bu atrofda chegaralangan bo'lgani uchun $|x-x_0|^{n-2}u_1(x)$ ham yetarlicha kichik bo'ladi. Shu sababli

$$|v(x)|_{x\in\Sigma_\delta} < \left.\frac{\varepsilon(\delta)}{|x-x_0|^{n-2}}\right|_{x\in\Sigma_\delta}$$

tenglikka ega bo'lamiz, bu yerda $\varepsilon(\delta)$ miqdor $\delta \to 0$ da nolga intiladi. Ekstremum prinsipiga asosan $v(x)$ funksiyaning qiymati $\frac{\varepsilon(\delta)}{|x-x_0|^{n-2}}$ ning qiymatidan katta bo'la olmaydi. Agar x_0 nuqtadan farqli biror nuqtada $v(x) \neq 0$ deb faraz qilsak, darhol qarama-qarshilikka kelamiz, chunki $\frac{\varepsilon(\delta)}{|x-x_0|^{n-2}}$ ni bu nuqtada δ ni yetarli kichik tanlab istalgancha kichik qilib olish mumkin. Shunday qilib, $u(x_0) = u_1(x_0)$ deb olsak, $u(x)$ funksiya $u_1(x)$ bilan $U_{x_0}^\varepsilon$ ning barcha nuqtalarida ustma-ust tushadi.

6.8 Garnak tengsizligi va teoremalari. Liuvill teoremasi

Faraz qilamiz, $u(x,y)$ funksiya markazi (x_0,y_0) radiusi R ga teng $(x-x_0)^2+(y-y_0)^2 \leq R^2$ doirada garmonik bo'lib, doiraning ichki nuqtalarida $u(x,y) \geq 0$ bo'lsin. Unda $0 \leq \rho < R$ sohada quyidagi *Garnak tengsizligi* deb ataluvchi munosabat o'rinli:

$$\frac{R-\rho}{R+\rho}u(x_0,y_0) \leq u(x_0+\rho cos\alpha, y_0+\rho sin\alpha) \leq \frac{R+\rho}{R-\rho}u(x_0,y_0).$$

Bu tengsizlikni isbotlash uchun

$$\frac{R-\rho}{R+\rho} = \frac{R^2-\rho^2}{(R+\rho)^2} \leq \frac{R^2-\rho^2}{(R-\rho)^2} = \frac{R+\rho}{R-\rho}$$

tengsizlik va qutb koordinatalarida yozilgan (28)

$$u(x_0 + \rho cos\alpha, y_0 + \rho sin\alpha) =$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} u(x_0 + Rcos\varphi, y_0 + Rsin\varphi)\frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho cos(\varphi - \alpha)}d\varphi$$

Puasson formulasidan foydalanamiz. Shartga ko‘ra,

$$u(x_0 + Rcos\varphi, y_0 + Rsin\varphi) = f(\varphi) \geq 0.$$

Bundan esa

$$\frac{R - \rho}{R + \rho}\frac{1}{2\pi}\int\limits_0^{2\pi} f(\varphi)d\varphi \leq u(x_0 + \rho cos\alpha, y_0 + \rho sin\alpha) \leq$$

$$\leq \frac{R + \rho}{R - \rho}\frac{1}{2\pi}\int\limits_0^{2\pi} f(\varphi)d\varphi.$$

O‘rta qiymat haqidagi teoremaga ko‘ra esa

$$\frac{1}{2\pi}\int\limits_0^{2\pi} f(\varphi)d\varphi = u(x_0, y_0).$$

Bulardan Garnak tengsizligining o‘rinli ekani kelib chiqadi.

Shunga o‘xshash ko‘rsatish mumkinki, $n-$ o‘lchovli fazodagi $\rho < R$ sharda manfiy bo‘lmagan $u(M)$ garmonik funksiya ushbu

$$R^{n-2}\frac{R - \rho}{(R + \rho)^{n-1}}u(M_0) \leq u(M) \leq R^{n-2}\frac{(R + \rho)}{(R - \rho)^{n-1}}u(M_0)$$

Garnak tengsizligi o‘rinli, bu yerda $\rho = |MM_0|$, $M = M(x_1, x_2, \cdots, x_n)$, $M_0 = M\left(x_1^0, x_2^0, \cdots, x_n^0\right)$.

Garnakning birinchi t e o r e m a s i. Agar chekli D sohada garmonik, $\overline{D}$ da uzluksiz funksiyalarning $\{u_i(x)\}$ $(i = 1, 2, \cdots)$ ketma-ketligi D ning chegarasi S da tekis yaqinlashuvchi bo‘lsa, u holda

1) $\{u_i(x)\}$ ketma-ketlik $\overline{D}$ yopiq sohada tekis yaqinlashuvchi bo‘ladi;

2) $u(x) = \lim\limits_{i\to\infty} u_(x)$ limit funksiya D da garmonik bo‘ladi;

3) D sohaning ixtiyoriy yopiq $\overline{D}_1$ qismida $u_k(x)$ funksiyalar istalgan tartibdagi hosilalarining ketma-ketligi limiti $u(x)$ funksiyaning mos tartibli hosilasiga tekis yaqinlashadi.

Isbot. $\{u_i(x)\}$ ketma-ketlik S da tekis yaqinlashuvchi bo'lgani uchun ixtiyoriy $\varepsilon > 0$ uchun shunday yetarlicha katta N natural soni topiladiki, $i \geq N$ lar va ixtiyoriy natural p son uchun

$$|u_{i+p}(x) - u_i(x)| < \varepsilon \tag{29}$$

bo'ladi. $u_{i+p}(x)$, $u_i(x)$ funksiyalar garmonik bo'lgani sababli, ularning ayirmasi ham D sohada garmonik, $\overline{D}$ da uzluksiz bo'ladi. Ekstremum prinsipiga asosan (29) tengsizlik barcha D sohada bajariladi. Bu esa $\{u_i(x)\}$ ketma-ketlikning yopiq $\overline{D}$ sohada tekis yaqinlashuvchanligini ko'rsatadi. Demak, $\overline{D}$ da aniqlangan va uzluksiz $u(x) = \lim\limits_{i\to\infty} u_i(x)$ limit funksiya mavjud. D da to'la yotuvchi markazi biror x_0 nuqtada radiusi R bo'lgan Q_R shar olamiz. Puasson formulasiga asosan $x \in Q_R$ nuqtalar uchun

$$u_i(x) = \int\limits_{\partial Q_R} u_i(\xi)K(x,\xi)dS_R, \quad K(x,\xi) = \frac{1}{\omega_n R}\frac{R^2 - |x - x_0|^2}{|\xi - x|^n}. \tag{30}$$

$\{u_i(x)\}$ ketma-ketlik tekis yaqinlashuvchi bo'lgani sababli, avvalgi tenglikda $i \to \infty$ da limitga o'tib, limit funksiya $u(x)$ uchun

$$u(x) = \int\limits_{\partial Q_R} u(\xi)K(x,\xi)dS_R \tag{31}$$

formulaga ega bo'lamiz. Bu formula $u(x)$ ning Q_R da garmonik funksiya ekanligini ko'rsatadi. Bundan Q_R ixtiyoriy shar bo'lgani uchun $u(x)$ ning barcha D sohada garmonik funksiya ekanligi kelib chiqadi. (30), (31) tengliklarning o'ng tomonini x nuqtaning funksiyasi sifatida Q_R sharning ichida istalgancha differensiallash mumkin. Ushbu

$$\left|\frac{\partial^l u(x)}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}} - \frac{\partial^l u_i(x)}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}}\right| \leq \int\limits_{\partial Q_R} |u(\xi) - u_i(\xi)| \left|\frac{\partial^l K(x,\xi)(x)}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}}\right| dS_R,$$

bunda $l = l_1 + \cdots + l_n$, tengsizlikdan hosilalar ketma-ketligining $i \to \infty$ da x ga nisbatan bu nuqtaning biror atrofida, masalan, markazi x_0 nuqtada radiusi Q_R sharning radiusidan ikki marta kichik bo'lgan sharda tekis yaqinlashuvchanligi kelib chiqadi. Agar $\bar{D}'$ soha D da to'la yotuvchi yopiq soha bo'lsa, u holda Geyne-Borel lemmasiga asosan $\bar{D}'$ ni chekli sondagi sharlar bilan qoplash mumkin. Bunga asosan hosilalar ketma-ketligining $\bar{D}'$ da tekis yaqinlashuvchi bo'lishiga ishonch hosil qilamiz.

Garnakning ikkinchi t e o r e m a s i. Agar D sohada garmonik funksiyalarning $\{u_i(x)\}$ ketma-ketligi monoton o'suvchi bo'lib,
sohaning kamida bitta nuqtasida yaqinlashuvchi bo'lsa , u holda bu ketma-ketlik D sohaning barcha nuqtalarida biror $u(x)$ garmonik funksiyaga yaqinlashadi. Shu bilan birga ixtiyoriy yopiq $\overline{D} \subset D$ sohada yaqinlashish tekis bo'ladi.

Isbot. Teoremaning shartiga ko'ra $u_1(x) \leq u_2(x) \leq \cdots \leq u_i(x) \leq \cdots$ va bu ketma-ketlik $x_0 \in D$ nuqtada yaqinlashuvchi bo'lsin. Markazi x_0 nuqtada bo'lgan R radiusli $\overline{Q_R} \subset D$ sharni olamiz. $x \in Q_R$ nuqtalar uchun

$$0 \leq u_{i+p}(x) - u_i(x) \leq \frac{R^{n-2}(R+r)}{(R-r)^{n-1}}\left[u_{i+p}(x_0) - u_i(x_0)\right]$$

Garnak tengsizligi o'rinli bo'ladi. Bu tengsizlikdan markazi x_0 nuqtada bo'lgan biror sharda, masalan, $R/2$ radiusli yopiq sharda $\{u_i(x)\}$ ketma-ketlikning tekis yaqinlashuvchanligi kelib chiqadi. D sohaning ixtiyoriy x nuqtasi atrofida $\{u_i(x)\}$ ketma-ketlikning tekis yaqinlashuvchi bo'lishini ko'rsatish uchun x_0 nuqtani x nuqta bilan D da to'la yotuvchi uzluksiz egri chiziq bilan tutashtirib, maksimum prinsipining isbotidagi mulohazalarni yuritamiz. Xuddi yuqoridagidek, x nuqtani o'z ichiga olgan sharda $\{u_i(x)\}$ ketma-ketlik tekis yaqinlashuvchi bo'ladi. U holda Geyny-Borel lemmasiga asosan bu ketma-ketlikning $\bar{D}'$ sohada tekis yaqinlashuvchi bo'lishi kelib chiqadi.

Garnak tengsizligining yana bir tadbiqi quyidagi *Liuvill teoremasi* hisoblanadi:

T e o r e m a. *Har qanday chekli sohada garmonik bo'lgan funksiya quyidan va yuqoridan chegaralangan bo'lsa, u o'zgarmasdir.*

Agar u(M) funksiya garmonik va $u(M) \leq N = const$ bo'lsa, $-u(M)$ ham garmonik va $-u(M) \geq -N$ bo'ladi. Demak, garmonik funksiya $u(M) \geq m$ quyidan chegaralangan holni qarash yetarli. Umuman, $m > 0$ deb olish mumkin (agar bunday bo‘lmasa, $u(M)$ funksiyaga yetarlicha katta musbat sonni qo‘shib $m > 0$ ga erishish mumkin).

M nuqtani tayin qilib, markazi koordinata boshida bo‘lgan shunday r radiusli U_0^r shar chizamizki, M nuqta uning ichida yotsin. $u(M)$ funksiya ixtiyoriy chekli sohada garmonik bo‘lgani uchun, jumladan, U_0^r sharda ham garmonik va (25) formulaga asosan

$$u(M) = \frac{1}{\omega_n} \int\limits_{\Sigma^r} \frac{r^2 - |x|^2}{r|x-\xi|^n} u(\xi) ds_\xi$$

Puasson formulasi o‘rinli, bu yerda $\Sigma^r -$ U_0^r shar sirti.

Ravshanki, $r - |x| < |x - \xi| < r + |x|$ va $u(M) > m > 0$ dan

$$\frac{r-|x|}{r(r+|x|)^{n-1}} \frac{1}{\omega_n} \int\limits_{\Sigma^r} u(\xi) ds_\xi \leq u(x) \leq$$

$$\leq \frac{r+|x|}{r(r-|x|)^{n-1}} \frac{1}{\omega_n} \int\limits_{\Sigma^r} u(\xi) ds_\xi. \tag{32}$$

O‘rta qiymat haqidagi teoremaga ko‘ra

$$u(0) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\Sigma^r} u(\xi) ds_\xi.$$

Bundan esa (32) tengsizlik ushbu

$$r^{n-2} \frac{r-|x|}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}} u(0)$$

ko‘rinishni oladi (Garnak tengsizligi). Radiusni $r \to \infty$ deb

$$u(0) \leq u(x) \leq u(0)$$

tengsizlikni olamiz. Bundan $u(x) = u(0) = const$ kelib chiqadi. Teorema isbotlandi.

6.9 Shar uchun Dirixlening tashqi masalasi

Faraz qilamiz, Ω soha Σ_r sfera bilan chegaralangan markazi koordinatalar boshida r radiusli U_0^r sharning tashqi qismi bo‘lsin, ya’ni $\Omega = \mathbb{R}^n$ va Ω da garmonik bo'lib, Σ_r sirtda

$$u|_{\Sigma_r} = f \tag{33}$$

qiymatni qabul qiluvchi $u(x)$ funksiyani topish talab qilinsin.

Bu masalaning yechimi

$$u(x) = \frac{1}{\omega_n} \int\limits_{\Sigma^r} \frac{|x|^2 - r^2}{r|x-\xi|^n} f(\xi) ds_\xi, \qquad |x| > r \tag{34}$$

Puasson formulasi bilan berilishini ko'rsatamiz, bu yerda ham $x \in \Omega$, $\xi \in \Sigma_r$.

Xuddi 6-paragrafdagi kabi bu yerda ham ko'rsatish mumkinki, (34) formula bilan aniqlangan $u(x)$ funksiya $x\bar{\in}\Sigma_r$ bo'lganda istagancha tartibli hosilalarga ega va u Laplas tenglamasini qanoatlantiradi. (34) funksiyani ∞ da tekshiramiz. Ma'lumki, $|x - \xi| \geq |x| - r$. Bundan

$$|u(x)| \leq C \frac{|x| + r}{(|x| - r)^{n-1}}, \qquad C = \frac{1}{\omega_n r} \int\limits_{\Sigma_r} f(\xi) ds_\xi.$$

Bizni $|x|$ ning katta qiymatlarida qiziqtirgani uchun $|x| > 2r$ deb olamiz. U holda $r < |x|/2, \quad |x| - r > |x|/2$ bo'lib,

$$|u(x)| < \frac{2^n}{|x|^{n-2}} C$$

va $u(x)$ funksiya Ω da garmonikdir.

Endi ixtiyoriy $x_0 \in \Sigma_r$ tayin nuqta uchun

$$\lim_{x \to x_0} u(x) = f(x_0) \tag{35}$$

limit tenglikni ko'rsatamiz. Buning uchun (34) integralni $f(x) = 1$ da hisoblaymiz. x nuqtaning Σ_r sferaga nisbatan simmetrik x' nuqtasini olamiz. Unda

$$|x|^2 = \frac{|x|^2}{|x'|^2}, \frac{1}{|x-\xi|} = \frac{1}{|x'-\xi|} \frac{r}{|x|}$$

bo'lib, Puasson yadrosini quyidagicha yozamiz:

$$\frac{|x|^2 - r^2}{r|x-\xi|^n} = \left(\frac{r}{|x|}\right)^{n-2} \frac{r^2 - |x'|^2}{r|x' - \xi^n}.$$

x' nuqta Σ_r sferaning ichkarisida yotadi va (26) formulaga koʻra

$$\frac{1}{\omega_n}\int\limits_{\Sigma_r}\frac{|x|^2-r^2}{r|x-\xi|^n}ds_\xi=\left(\frac{r}{|x|}\right)^{n-2}\frac{1}{\omega_n}\int\limits_{\Sigma_r}\frac{|x'|^2-r^2}{r|x'-\xi|^n}ds_\xi=\left(\frac{r}{|x|}\right)^{n-1}.$$

Bu tenglikni $f(x_0)$ ga koʻpaytirib, (34) dan ayiramiz:

$$u(x)-\left(\frac{r}{|x|}\right)^{n-2}f(x_0)=\frac{1}{\omega_n}\int\limits_{\Sigma_r}\frac{|x|^2-r^2}{r|x-\xi|^n}[f(\xi)-f(x_0)]ds_\xi.$$

6-paragrafdagi mulohazalarni takrorlab, quyidagiga ega boʻlamiz:

$$\lim_{x\to x_0}\left[u(x)-\left(\frac{r}{|x|}\right)^{n-2}f(x_0)\right]=0.$$

Bundan esa $x\to x_0$ da

$$|u(x)-f(x_0)|\le\left|u(x)-\left(\frac{r}{|x|}\right)^{n-2}f(x_0)\right|+|f(x_0)|\left[1-\left(\frac{r}{|x|}\right)\right]\to 0$$

boʻladi. Shu bilan (35) tenglik isbotlandi.

6.10 Doiraning tashqarisi va halqada Laplas tenglamasi uchun chegaraviy masalalar

Oʻzgaruvchilarni ajratish usuli bilan doirada Laplas tenglamasining yechimini quramiz. Doira markazini koordinatalar boshi qilib, qutb (ρ,φ) koordinatalarini kiritamiz. Ushbu koordinatalarda $n=2$ uchun yozilgan Laplas

$$\triangle u=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=0,\quad 0\le\rho<a \tag{36}$$

tenglamasining yechimini

$$u(\rho,\varphi)=R(\rho)\Phi(\varphi)\neq 0 \tag{37}$$

koʻrinishda qidiramiz. (37) ni (36) qoʻyib va oʻzgaruvchilarni ajratib,

$$\frac{\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)}{R(\rho)}=-\frac{\Phi''(\varphi)}{\Phi(\varphi)}=\lambda, \tag{38}$$

$\lambda = const$, tengliklarga ega bo'lamiz. (36) tenglama $0 \leq \rho < a$ doirada bajarilishini inobatga olsak, $u(\rho, \varphi)$ funksiya φ bo'yicha 2π davrli va bu doirada chegaralangan bo'lishi kerak.

(37) dan $R(\rho)$ va $\Phi(\varphi)$ funksiyalar uchun tenglamalar olamiz. Dastlab $\Phi(\varphi)$ uchun

$$\Phi'' + \lambda\Phi = 0, \ 0 \leq \varphi \leq 2\pi$$

tenglamani davriylik

$$\Phi(\varphi) = \Phi(\varphi + 2\pi)$$

sharti bilan qaraymiz. Demak, $\Phi(\varphi)$ uchun davriylik sharti bilan Shturm-Liuvill masalasi hosil bo'ldi. Uning yechimi (4-bobning 9-paragrafiga qarang)

$$\Phi(\varphi) = \Phi_n(\varphi) = \begin{cases} \cos n\varphi, \\ \sin n\varphi, \end{cases} \lambda = \lambda_n = n^2, \ n = 0, 1, ...$$

(38) dan λ_n ning topilgan qiymatlarini inobatga olib, $R(\rho)$ funksiya uchun

$$\rho^2 R'' + \rho R' - n^2 R = 0 \tag{39}$$

tenglamani hosil qilamiz. Bu Eyler tenglamasi bo'lib, uning umumiy yechimi quyidagi ko'rinishga ega:

$$R(\rho) = R_n(\rho) = C_1\rho^n + C_2\rho^{-n}, \ n \neq 0,$$

$$R_0(\rho) = C_1 + C_2 \ln \rho, \ n = 0, \tag{40}$$

$C_1, \ C_2-$ o'zgarmaslar. Yechim $0 \leq \rho < a$ larda chegalangan bo'lishi uchun (40) ning birinchi formulasiga ko'ra

$$R_n(\rho) = C_1\rho^n, n = 0, 1, ...$$

Shunday qilib, Laplas tenglamasining $0 \leq \rho < a$ doirada chegaralangan yechimi quyidagi funksiyalar sistemasidan iborat ($C_1 = 1$ deb olindi):

$$u_n(\rho, \varphi) = \rho^n \begin{cases} \cos n\varphi, \\ \sin n\varphi, \end{cases} \lambda = \lambda_n = n^2, \ n = 0, 1, ... \tag{41}$$

Laplas tenglamasining umumiy yechimi bu xususiy yechimlardan hosil qilingan cheksiz

$$u(\rho,\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\rho^n\left(A_n\cos n\varphi+B_n\sin n\varphi\right) \tag{42}$$

qator koʻrinishda boʻladi.

Laplas tenglamasining doiradan tashqarida ($\rho > a$) yechimini hosil qilish uchun (37) tenglamaning doiradan tashqaridagi sohasida chegaralangan xususiy yechimlarini tanlash lozim. Ular

$$u_n(\rho,\varphi)=\frac{1}{\rho^n}\begin{cases}\cos n\varphi,\\ \sin n\varphi,\end{cases}\quad \lambda=\lambda_n=n^2,\ n=0,1,... \tag{43}$$

koʻrinishga ega. Shuning uchun Laplas tenglamasining doiradan tashqaridagi sohada, cheksizlikda chegaralangan yechimi quyidagi qator koʻrinishida yozilishi mumkin:

$$u(\rho,\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\frac{1}{\rho^n}\left(A_n\cos n\varphi+B_n\sin n\varphi\right). \tag{44}$$

Endi chegaraviy masalalarni yechishga oʻtamiz. Doira uchun ichki masalani qaraymiz:

$$\triangle u=0,\ 0\le\rho<a, \tag{45}$$

$$P(u)=\left(\alpha\frac{\partial u}{\partial\rho}+\beta u\right)\Big|_{\rho=a}=f(\varphi),\ |\alpha|+|\beta|\neq 0. \tag{46}$$

(45), (46) masala yechimini (42) qator koʻrinishda yozish mumkin, bunda nomaʼlum koeffitsientlar chegaraviy shartdan topiladi.

Doirada Laplas tenglamasining birinchi, ikkinchi va uchinchi chegaraviy masalalarini alohida yozamiz.

1. *Dirixle masalasi:* $u\Big|_{\rho=a}=f(\varphi)$,

$$u(\rho,\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^2\left(A_n\cos n\varphi+B_n\sin n\varphi\right). \tag{47}$$

2. *Neyman masalasi:* $\left.\frac{\partial u}{\partial \rho}\right|_{\rho=a} = f(\varphi)$,

$$u(\rho,\varphi) = \sum_{n=1}^{\infty} \frac{\rho^n}{na^{n-1}} \left(A_n \cos n\varphi + B_n \sin n\varphi\right) + C. \tag{48}$$

3. *Uchinchi chegaraviy masalasi* $\left.\left(\frac{\partial u}{\partial \rho} + u\right)\right|_{\rho=a} = f(\varphi)$,

$$u(\rho,\varphi) = \frac{A_0}{2h} + \sum_{n=1}^{\infty} \frac{\rho^n}{(n+ah)a^{n-1}} \left(A_n \cos n\varphi + B_n \sin n\varphi\right). \tag{49}$$

(47)-(49) formulalarda A_n, B_n koeffitsientlar chegarada berilgan $f(\varphi)$ funksiya orqali

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi d\varphi,\ B_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin n\varphi d\varphi,$$
$$n = 0,\ 1,\ 2, ...$$

tengliklar yordamida aniqlanadi. (48) dagi $C-$ ixtiyoriy o'zgarmas. Eslatib o'tamiz, Neymanning ichki masalasi yechimi

$$\int_0^{2\pi} f(\varphi)d\varphi = 0$$

bo'lganda mavjud va ixtiyoriy o'zgarmas aniqligida topiladi. Doiraning tashqarisi uchun chegaraviy masala ham xuddi shunga o'xshash yechiladi. Uning yechimini qurish uchun (43) xususiy yechimlardan foydalanish kerak.

$a \le \rho \le b$ halqada chegaraviy masalani batafsilroq o'rganamiz. Aniqlik uchun

$$\triangle u = 0, \qquad a < \rho < b,$$
$$\left.u\right|_{\rho=a} = f_1(\varphi), \qquad \left.u\right|_{\rho=b} = f_2(\varphi) \tag{50}$$

Dirixle masalasini tanlaymiz. Bu holda (40) yechimlarning har ikkalasidan ham foydalanish zarur. Biroq har bir n uchun quyidagi shartlarni qanoatlantiruvchi (39) tenglama $R_n^{(a)}(\rho)$, $R_n^{(b)}(\rho)$ maxsus yechimlarining fundamental sistemasini qurish qulay:

$$R_n^{(a)}(a) = 0, \qquad R_n^{(b)}(b) = 0, \qquad n = 0, 1, \dots . \tag{51}$$

Bu yechimlar sifatida

$$R_n^{(a)}(\rho) = \frac{\rho^{2n} - a^{2n}}{\rho^n}, \qquad R_n^{(b)}(\rho) = \frac{b^{2n} - \rho^{2n}}{\rho^n}, \quad n = 1, 2, \ldots,$$

$$R_0^{(a)}(\rho) = \ln\frac{\rho}{a}, \qquad R_0^{(b)}(\rho) = \ln\frac{b}{\rho}$$

funksiyalarni olish mumkin. U holda (50) masala yechimini ushbu

$$u(\rho,\varphi) = \frac{A_0}{2}\frac{R_0^{(a)}(\rho)}{R_0^{(a)}(b)} + \frac{C_0}{2}\frac{R_0^{(b)}(\rho)}{R_0^{(b)}(a)} +$$

$$+\sum_{n=1}^{\infty}\frac{R_n^{(a)}(\rho)}{R_n^{(a)}(b)}(A_n\cos n\varphi + B_n\sin n\varphi) +$$

$$+\sum_{n=1}^{\infty}\frac{R_n^{(b)}(\rho)}{R_n^{(b)}(a)}(C_n\cos n\varphi + D_n\sin n\varphi)$$

ko‘rinishda yozamiz. Bu formulada $\rho = a$ deb, (50) masalaning birinchi chegaraviy sharti va (51) dan foydalanib, C_n va D_n larni topamiz:

$$C_n = \frac{1}{\pi}\int_0^{2\pi} f_1(\varphi)\cos n\varphi d\varphi, \qquad D_n = \frac{1}{\pi}\int_0^{2\pi} f_1(\varphi)\sin n\varphi d\varphi.$$

Shunga o‘xshash, A_n, B_n koeffitsientlar $\rho = b$ da ikkinchi chegaraviy shartdan aniqlanadi:

$$C_n = \frac{1}{\pi}\int_0^{2\pi} f_2(\varphi)\cos n\varphi d\varphi, \qquad D_n = \frac{1}{\pi}\int_0^{2\pi} f_2(\varphi)\sin n\varphi d\varphi.$$

Halqa uchun boshqa chegaraviy masalalar ham yuqoridagi kabi yechiladi.

6.11 To‘g‘ri to‘rtburchak uchun Dirixle masalasi

Laplas tenglamasi uchun to‘g‘ri to‘rtburchakda qo‘yilgan chegaraviy masalalar ham o‘zgaruvchilarni ajratish usuli bilan yechilishi mumkin. Masalan, ushbu

$$\Delta u = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{52}$$

$$u|_{x=0} = \varphi_1(y), \quad u|_{x=a} = \varphi_2(y), \tag{53}$$

$$u|_{y=0} = \psi_1(x), \quad u|_{y=b} = \psi_2(x) \tag{54}$$

Dirixle masalasini qaraylik. Bu yerda φ_1, φ_2, ψ_1, ψ_2 funksiyalar to‘g‘ri to‘rtburchakning uchlarida $\varphi_1(0) = \psi_1(0)$, $\varphi_1(b) = \psi_2(0)$, $\varphi_2(0) = \psi_1(a)$, $\varphi_2(a) = \psi_2(b)$ shartlarni qanoatlantiradi.

(52)-(54) masalani ikkiga ajratamiz. Bunda har bir masala (x, y) o‘zgaruvchilarning biri bo‘yicha bir jinsli chegaraviy shartlarga ega bo‘lsin. Bu masalalar yechimlarini

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

ko‘rinishda yozamiz, bu yerda $u_1(x, y)$ va $u_2(x, y)$ funksiyalar mos ravishda

$$\begin{array}{ll} \Delta u_1 = 0, & \Delta u_2 = 0, \\ u_1|_{x=0} = u_1|_{x=a} = 0, & u_2|_{y=0} = u_2|_{y=b} = 0, \\ u_1|_{y=0} = \psi_1(x), & u_2|_{x=0} = \varphi_1(y), \\ u_1|_{y=b} = \psi_2(x), & u_2|_{x=a} = \varphi_2(y). \end{array}$$

Dirixle masalalarining yechimlari.

Avvalo, $u_1(x, y)$ funksiya uchun masalani ko‘rib chiqamiz. Buning uchun dastlab Laplas tenglamasining

$$u(x, y) = X(x)Y(y) \neq 0 \tag{55}$$

ko‘rinishdagi

$$u|_{x=0} = u|_{x=a} = 0 \tag{56}$$

x bo‘yicha bir jinsli chegaraviy shartlarni qanoatlantiruvchi yechimini quramiz. (55) ifodani Laplas tenglamasiga qo‘yib, o‘zgaruvchilarni ajratgandan so‘ng, $X(x)$, $Y(y)$ funksiyalarga nisbatan

$$X'' + \lambda X = 0, \qquad 0 < x < a, \tag{57}$$

$$Y'' - \lambda Y = 0, \qquad 0 < y < b \tag{58}$$

tenglamalarni olamiz. (56) shartlarni inobatga olsak, $X(x)$ uchun ushbu

$$X'' + \lambda X = 0, \qquad 0 < x < a,$$

$$X(0) = X(a) = 0, \qquad X(x) \neq 0$$

Shturm-Liuvill masalasiga kelamiz. Uning yechimi quyidagi funksiyalardan iborat (4-bobning 9-paragrafiga qarang):

$$X = X_n = \sin\sqrt{\lambda_n}x, \qquad \lambda_n = \left(\frac{\pi n}{a}\right)^2, \quad n = 1, 2, \ldots.$$

$\lambda = \lambda_n$ da (58) tenglamaning umumiy yechimi

$$Y(y) = Y_n(y) = Ash\sqrt{\lambda_n}y + Bsh\sqrt{\lambda_n}(b-y)$$

ko‘rinishda yoziladi. Shunday qilib, Laplas tenglamasining xususiy yechimlari

$$u_n(x,y) = \left\{A_n sh\sqrt{\lambda_n}y + Bsh\sqrt{\lambda_n}(b-y)\right\}\sin\sqrt{\lambda_n}x \quad n = 1, 2, \ldots \tag{59}$$

hosil bo‘ldi. u_1 funksiya uchun masalaning yechimini (59) funksiyalar sistemasi bo‘yicha qator ko‘rinishda yozamiz:

$$u_1(x,y) = \sum_{n=1}^{\infty}\left\{A_n\frac{sh\sqrt{\lambda_n}y}{sh\sqrt{\lambda_n}b} + B_n\frac{sh\sqrt{\lambda_n}(b-y)}{sh\sqrt{\lambda_n}b}\right\}\sin\sqrt{\lambda_n}x, \tag{60}$$

bu yerda A_n, B_n koeffitsientlar y o‘zgaruvchi bo‘yicha chegaraviy shartlardan quyidagi tengliklar bilan aniqlanadi:

$$B_n = \frac{2}{a}\int_0^a \psi_1(x)\sin\sqrt{\lambda_n}x\,dx, \quad A_n = \frac{2}{a}\int_0^a \psi_2(x)\sin\sqrt{\lambda_n}x\,dx. \tag{61}$$

Demak, $u_1(x,y)$ uchun masala yechimi (60), (61) formulalar bilan aniqlanadi.

Shunga o‘xshash ravishda $u_2(x,y)$ funksiya uchun qo‘yilgan masala ham yechiladi. Uning yechimi

$$u_2(x,y) = \sum_{n=1}^{\infty}\left\{C_n\frac{sh\sqrt{\lambda_n}x}{sh\sqrt{\lambda_n}a} + D_n\frac{sh\sqrt{\lambda_n}(a-x)}{sh\sqrt{\lambda_n}a}\right\}\sin\sqrt{\lambda_n}y \tag{62}$$

ko'rinishda bo'ladi, bu yerda $\lambda_n = \left(\frac{\pi n}{b}\right)^2$,

$$D_n = \frac{2}{b}\int_0^b \psi_1(y)\sin\sqrt{\lambda_n}ydy, \quad C_n = \frac{2}{a}\int_0^b \psi_2(y)\sin\sqrt{\lambda_n}ydy.$$

Shunday qilib, (52)-(54) masalaning yechimi

$$u = u_1(x,y) + u_2(x,y)$$

ko'rinishda bo'lib, u_1 va u_2 funksiyalar mos ravishda (60) va (62) formulalar bilan aniqlanadi.

6.12 Shar uchun Dirixle masalasining Grin funksiyasi

(10) formulaga ko'ra D sohaga garmonik bo'lgan va $D \cup S$ da birinchi tartibli hosilalari bilan uzluksiz bo'lgan $u(x)$ funksiya uchun

$$u(x_0) = \frac{1}{\omega_n}\oint_S \left(E(x,x_0)\frac{\partial u}{\partial \overrightarrow{n}} - u\frac{\partial}{\partial \overrightarrow{n}}E(x,x_0)\right)ds, \ x_0 \in D \subset \mathbb{R}^n \tag{63}$$

tenglik o'rinli.

T a ' r i f. *Laplas tenglamasi uchun D sohada Dirixle masalasining Grin funksiyasi deb quyidagi shartlarni qanoatlantiruvchi $G(x,x_0)$ $(x, x_0 \in D \cup S)$ funksiyaga aytiladi:*

$$G(x,x_0) = E(x,x_0) + g(x,x_0),$$

bu yerda $E(x,x_0)$ (11) tenglik bilan aniqlangan Laplas tenglamasining fundamental yechimi, $g(x,x_0)$ funksiya shunday tanlanadiki, bunda u har ikkala argumenti bo'yicha garmonik va $x \in S$ yoki $x_0 \in S$ bo'lganda $G(x,x_0) = 0$ shart bajariladi.

Ushbu paragrafda, qulaylik uchun, yechim qidirilayotgan nuqta sifatida x va sohaning ichida yoki uning chegarasida o'zgaradigan nuqta uchun y

harflarini ishlatamiz. Agar (63) tenglikni $u(x)$ - Laplas tenglamasi uchun Dirixle masalasining yechimi va $G(x,y)$ Grin funksiyasi uchun qo'llasak,

$$u(x) = -\frac{1}{\omega_n} \oint_S \frac{\partial}{\partial \overrightarrow{n}} G(x,y)\varphi(x)ds, \tag{64}$$

bu yerda $\varphi(x)$ - S da berilgan haqiqiy o'zgaruvchili uzluksiz funksiya. Demak, (64) formula Grin funksiyasi ma'lum bo'lganda ushbu

$$\Delta u = 0,\ x \in D,\ \lim_{x \to \xi} = \varphi(\xi),\ x \in D,\ \xi \in S$$

Laplas tenglamasi uchun D sohada Dirixle masalasining yechimini beradi.

Lemma. *Laplas tenglamasi uchun $|x| < 1$ birlik sharda Dirixle masalasining Grin funksiyasi ushbu*

$$G(x,y) = E(x,y) - E\left(|x|y, \frac{x}{|x|}\right) \tag{65}$$

ko'rinishga ega.

Isbot. Haqiqatan ham,

$$\left| |x|y - \frac{x}{|x|} \right| = \left[|x|^2|y|^2 - 2(x,y) + 1 \right]^{\frac{1}{2}} = \left| |y|x - \frac{y}{|y|} \right| = |y| \left| x - \frac{y}{|y|^2} \right| =$$

$$= |x| \left| y - \frac{x}{|x|^2} \right|,\ (x,y) = \sum_{i=1}^n x_i y_i$$

tengliklar bajarilganligi uchun

$$g(x,y) = -E\left(|x|y, \frac{x}{|x|}\right)$$

funksiya $|x| < 1$, $|y| < 1$ lar uchun har ikkala x va y o'zgaruvchilar bo'yicha garmonik funksiyadir. Shu bilan birga $|y| = 1$ lar uchun

$$|y - x| = \sqrt{|x|^2 - 2xy + 1} = \left| |y|x - \frac{y}{|y|} \right| = \left| |x|y - \frac{x}{|x|} \right|$$

tengliklar o'rinli. Bundan esa (65) formula bilan aniqlangan $G(x,y)$ funksiyaning Grin funksiyasi xossalarini qanoatlantirishi kelib chiqadi.

6.13 Chegaraviy masalalarni potensiallar yordamida yechish

Ushbu paragrafda Laplas tenglamasi uchun Dirixle va Neyman masalalari potensiallar yordamida integral tenglamalarni yechishga keltiriladi.

6.13.1 Oddiy va ikkilangan qatlam potensiallari. Hajm potensiali

Grinning uchinchi formulasi yetarlicha silliq chegaraga ega bo‘lgan va chegaralangan $D \subset \mathbb{R}^3$ sohada $u(x) \in C^2(D) \cap C^1(\overline{D})$ sinf funksiyalarining integral ifodasini aniqlaydi:

$$u(x) = \frac{1}{4\pi} \oint\limits_S \left[\frac{1}{|x-\xi|} \frac{\partial u(\xi)}{\partial n} - u(\xi) \frac{\partial}{\partial n} \left(\frac{1}{|x-\xi|} \right) \right] dS - \frac{1}{4\pi} \int\limits_D \frac{\Delta u(\xi)}{|x-\xi|} d\xi, \tag{66}$$

bu yerda $x \in D$.

(66) formula uch xil integralni o‘z ichiga olgan:

$$I_1 = \oint\limits_S \frac{\mu(\xi)}{|x-\xi|} dS, \tag{67}$$

$$I_2 = \oint\limits_S \nu(\xi) \frac{\partial}{\partial n} \left(\frac{1}{|x-\xi|} \right) dS, \tag{68}$$

$$I_3 = \int\limits_D \frac{\rho(\xi)}{|x-\xi|} d\xi. \tag{69}$$

Bu integrallarning har biri aniq fizik ma’noga ega. $(67), (68), (69)$ integrallar mos ravishda oddiy qatlam, ikkilangan qatlam, hajm potensiallari deyiladi. μ, ν, ρ funksiyalar esa shu potensiallarni aniqlovchi zichliklar deb ataladi. $(67) - (69)$ lar x ga parametr sifatida bog‘liq bo‘lgan integrallardir. Bu integrallarga, shuningdek, Nyuton potensiallari ham deyiladi. Shu bilan bir vaqtda 2-paragrafdadgi ikki o‘lchovli hol uchun yozilgan Grinning uchinchi formulasidagi integrallarga mos ravishda logorifmik potensiallar ham kiritiladi. Bu potensiallar quyidagi koʻrinishga ega:

$$J_1 = \oint\limits_C \mu(\xi) ln \frac{1}{|x-\xi|} dl, \tag{70}$$

$$J_2 = \oint\limits_C \nu(\xi) \frac{\partial}{\partial n} ln \frac{1}{|x-\xi|} dl, \tag{71}$$

$$J_3 = \int\limits_D \rho(\xi) ln \frac{1}{|x-\xi|} d\xi, \; x = (x_1, x_2), d\xi = d\xi_1 d\xi_2. \tag{72}$$

Bu yerda $(67) - (69), (70) - (72)$ integrallardan Dirixle

$$\Delta u(x) = 0, x \in D,$$

$$u|_{x \in S} = f(x); \; f(x) \in C(S), \tag{73}$$

Neyman

$$\Delta u(x) = 0, \; x \in D,$$

$$\frac{\partial u}{\partial n}\Big|_{x \in S} = g(x); \; g(x) \in C(S) \tag{74}$$

masalalar yechimlarining mavjudligini koʻrsatishda foydalaniladi.

Bunda Dirixle va Neyman masalalarining yechimi zichligi Fredgolm ikkinchi tur integral tenglamalarini qanoatlantiruvchi mos ravishda ikkilangan va oddiy qatlam potensiallaridan iborat boʻlar ekan. (69) (yoki (72)) potensial D sohada Puasson tenglamasini qanoatlantiradi. Shuning uchun ham Puasson tenglamasining xususiy yechimini berilgan $\rho(x)$ zichlik bilan (69) (yoki (72)) koʻrinishda qidirish qulaydir.

Avvalo, uch oʻlchovli hol uchun

$$I_3 = \int\limits_D \frac{\rho(\xi)}{|x-\xi|} d\xi$$

hajm potensialini qaraymiz.

Fizika kursidan ma'lumki, $\frac{1}{|x-\xi|}$ funksiya $x \neq \xi$ nuqtalarda aniqlangan boʻlib, ξ nuqta atrofida jamlangan birlik nuqtaviy zaryadni aniqlaydi. Agar

D sohada hajm zichligi $\rho(\xi)$ dan iborat zaryad uzluksiz ravishda taqsimlangan bo‘lsa, u holda superpozitsiya prinsipiga ko‘ra bu zaryad tomonidan hosil qilinadigan potensial (69) integral bilan ifodalanadi.

Hajm potensialini regulyar umumlashgan funksiya sifatida qarab, agar $\rho(\xi)$ kvadrati bilan integrallanuvchi bo‘lsa, u

$$\Delta I_3 = -4\pi\rho$$

Puasson tenglamasining umumlashgan yechimi bo‘lishini ko‘rsatamiz. Ma’lumki, hajm potensialini finit umumlashgan ρ funksiya va Laplas operatorining $\frac{1}{|x|}$ fundamental yechimining yig‘masi deb qarash mumkin (3-bobga qarang).

$$I_3 = \int\limits_D \frac{\rho(\xi)}{|x-\xi|}d\xi = \frac{1}{|x|} * \rho$$

Yig‘mani differensiallash qoidasi va umumlashgan funksiyalar ma’nosida bajariluvchi

$$\Delta\frac{1}{|x-\xi|} = -4\pi\delta(x-\xi)$$

tenglikni inobatga olsak,

$$\Delta I_3 = \Delta\left(\frac{1}{|x|} * \rho\right) = \left(\Delta\frac{1}{|x|}\right) * \rho = -4\pi(\delta * \rho) = -4\pi\rho.$$

$\rho(\xi)$ funksiyaga nisbatan kuchliroq talablar qo‘yilganda hajm potensiali ma’lum bir silliqlik xossalariga ega bo‘lgan klassik funksiyadan iborat bo‘ladi. Xususan, ρ chegaralangan va integrallanuvchi funksiya bo‘lsa, hajm potensiali barcha fazoda uzluksiz differensiallanuvchi funksiya bo‘ladi. Agar ρ o‘zining birinchi tartibli xususiy hosilalari bilan uzluksiz bo‘lsa, hajm potensiali ρ funksiya uzluksiz differensialga ega bo‘lgan barcha nuqtalarida

$$\Delta I_3 = -4\pi\rho$$

Puasson tenglamasini qanoatlantiradi.

Ikki o‘lchovli holda $\rho(\xi)$ funksiyaga nisbatan xuddi yuqoridagi talablar bilan (72) logarifmik potensial $\Delta J_3 = -2\pi\rho$ tenglamaning yechimi bo‘ladi.

6.13.2 Parametrga bog‘liq bo‘lgan integrallar

Keyinchalik zarur bo‘ladigan parametrga bog‘liq bo‘lgan ikkinchi tur xosmas integrallar haqida ma’lumotlarni keltiramiz.

$$V(x) = \int\limits_D F(x,\xi)f(\xi)d\xi \tag{75}$$

ko‘rinishdagi xosmas integrallarni qaraymiz, bu yerda $F(x,\xi)$ - $x \neq \xi$ da uzluksiz va $x = \xi$ da chegaralanmagan funksiya, $f(\xi)$ - chegaralangan funksiya.

T a ’ r i f. Agar ixtiyoriy $\varepsilon > 0$ soni uchun $\delta(\varepsilon) > 0$ son mavjud bo‘lib,

$$\left| \int\limits_{D_{\delta(\varepsilon)}} F(x,\xi)f(\xi)d\xi \right| \leq \varepsilon$$

tengsizlik ixtiyoriy $x \in \cup_{x_0}^{\delta(\varepsilon)}$ nuqta va $D_{\delta(\varepsilon)} \in \cup_{x_0}^{\delta(\varepsilon)}$ soha uchun bajarilsa, bu yerda $\cup_{x_0}^{\delta(\varepsilon)}$ - markazi x_0 nuqtada radiusi $\delta(\varepsilon)$ bo‘lgan shar, (75) integral x_0 nuqtada tekis yaqinlashuvchi deyiladi.

T e o r e m a. x_0 nuqtada tekis yaqinlashuvchi (75) integral bu nuqtada uzluksiz funksiyadir.

Isbot. Ixtiyoriy $\varepsilon > 0$ soni uchun $|x - x_0| \leq \delta(\varepsilon)$ bo‘lganda $|V(x) - V(x_0)| \leq \varepsilon$ tengsizlik bajariladigan $\delta(\varepsilon) > 0$ sonining mavjudligini ko‘rsatamiz. Biror $D_1 \subset D$ sohani tanlaymiz, bunda $x_0 \in D_1$. (75) integralni $V(x) = V_1(x) + V_2(x)$ ko‘rinishda yozib olamiz, bu yerda

$$V_1(x) = \int\limits_{D_1} F(x,\xi)f(\xi)d\xi, \; V_2(x) = \int\limits_{D_2} F(x,\xi)f(\xi)d\xi, \; D_2 = D \setminus D_1.$$

$|V(x_0) - V(x)|$ ayirmaning modulini baholaymiz:

$$|V(x_0) - V(x)| \leq |V_2(x_0) - V_2(x)| + |V_1(x_0) - V_1(x)| \, .$$

(75) integralning x_0 nuqtada tekis yaqinlashuvchanligidan shunday $\delta_1(\varepsilon) > 0$ son mavjudki, $|x - x_0| \leq \delta_1(\varepsilon)$ lar uchun $|V_1(x_0)| \leq \frac{\varepsilon}{3}$ va $|V_1(x)| \leq \frac{\varepsilon}{3}$ bo‘ladi.

$x_0 \notin D_2$ ekanligi uchun $V_2(x)$ xos integraldir, demak, u x_0 nuqtada uzluksiz. Bundan shunday $\delta_2(\varepsilon)$ sonining mavjudligi va $|x - x_0| \leq \delta_2(\varepsilon)$ lar uchun

$|V_2(x_0) - V_2(x)| \leq \frac{\varepsilon}{3}$ bo‘lishi kelib chiqadi. $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$ bo‘lsin. U holda $|x - x_0| \leq \delta(\varepsilon)$ lar uchun

$$|V(x_0) - V(x)| \leq \varepsilon.$$

Bu esa $V(x)$ integralning x_0 nuqtada uzluksizligini bildiradi.

E s l a t m a. Bu teorema nafaqat hajm integrallari uchun, balki sirt va egri chiziqli integrallar uchun ham o‘rinli. Bu faktdan biz keyinchalik foydalanamiz.

6.13.3 Sirt potensiallari

Odatda ikki xil sirt potensiallari qaraladi: Oddiy va ikkilangan qatlam potensiallari.

(67) formula bilan berilgan oddiy qatlam potensialining fizik ma’nosini zichligi $\mu(x)$ dan iborat S sirtda taqsimlangan zaryad tomonidan hosil qilingan potensial deb talqin qilish mumkin.

(68) formula bilan aniqlangan ikkilangan qatlam potensiali uchun S sirt nuqtalarida $\frac{1}{|x-\xi|}$ funksiyaning normal hosilasini hisoblab, uni

$$I_2(x) = \int\limits_S \nu(\xi) \frac{cos\varphi}{|x - \xi|^2} ds$$

ko‘rinishida yozish mumkin, bu yerda $\varphi - S$ sirt ξ nuqtasidagi ichki normal va $\overrightarrow{\xi x}$ vektor orasidagi burchak.

Ikkilangan qatlam potensialining fizik ma’nosini ifodalash uchun $\xi_0 \in S$ nuqtada S sirtga o‘tkazilgan normalda yotuvchi ξ_1 va ξ_2 nuqtalarga qo‘yilgan qarama-qarshi ishorali $-e$ va$+e$ nuqtaviy zaryadlar tomonidan hosil qilinayotgan potensialni qaraymiz. Zaryadlar joylashgan nuqtalar S sirtning turli tomonlarida, aniqlik uchun ξ_1 nuqta S sirtning ichki normalida joylashgan bo‘lsin. Ma’lumki, bu potensialning ixtiyoriy $x(x \neq \xi_1, x = \xi_2)$ nuqtadagi qiymati

$$W_0(\xi_1, \xi_2, x) = -e\left(\frac{1}{|\xi_2 - x|} - \frac{1}{|\xi_1 - x|}\right)$$

formula bilan aniqlanadi. ξ_1 va ξ_2 nuqtalar orasidagi masofani d bilan belgilaymiz: $d = |\xi_1 - \xi_2|$. $\nu_0 = ed$ miqdorni doimiy saqlagan holda, e zaryad miqdorini oshirib, ξ_1 va ξ_2 nuqtalarni ξ_0 nuqtaga intiltiramiz. $e = \frac{\nu_0}{d}$ bo'lgani uchun, $d \to 0$ deb limitga o'tib,

$$\lim_{\xi_1,\xi_2 \to \xi_0} W_0(\xi_1, \xi_2, x) = W_0(\xi_o, x) = \nu_0 \frac{\partial}{\partial n} \frac{1}{|\xi_0 - x|}$$

formulani hosil qilamiz.

Bu formuladan foydalanib, (68) integral yadrosining fizik talqinini berish mumkin: ξ_0 nuqtada joylashib, bu nuqtada S sirtga o'tkazilgan tashqi normal bo'ylab yo'nalgan va ν_0 momentga ega bo'lgan dipol tomonidan ξ_0 dan boshqa nuqtalarda hosil qilinadigan potensial.

Ikkilangan qatlam potensiali esa zichligi $\nu(\xi)$ bo'lgan dipolli moment taqsimotiga ega ikki tomonlama zaryadlangan S sirt hosil qiluvchi potensialdan iboratdir.

Endi sirt potensiallarining xossalarini tekshirishga o'tamiz.

Agar x nuqta S sirtga (yoki C egri chiziqga) tegishli bo'lmasa, potensiallar ixtiyoriy tartibli hosilalarga ega va ularni integral ostida differensiallab, hisoblash mumkin. Shuningdek, ular garmonik funksiyalardir:

$$\Delta I_i(x) = 0, \ i = 1, 2, \ x \notin S,$$

ikki o'lchovli holda

$$\Delta J_i(x) = 0, \ i = 1, 2, \ x \notin C.$$

Qayd etish joizki, chegaralangan sirtli potensiallar uchun $x \to \infty$ da quyidagi munosabatlar o'rinli:

$$I_1(x) = O\left(\frac{1}{|x|}\right), \ I_2(x) = O\left(\frac{1}{|x|^2}\right), \ |x| \to \infty,$$

$$J_2(x) = O\left(\frac{1}{|x|}\right), \ |x| \to \infty.$$

Ikki o'lchovli holda oddiy qatlam

$$J_1(x) = \oint_C \mu(\xi) ln \frac{1}{|x - \xi|} dl$$

potensiali, umuman aytganda, x ning cheksiz katta qiymatlarida chegaralanmagan va cheksizlikda $\ln|x|$ kabi o'sadi. Agarda uning zichligi

$$\int\limits_C \mu(\xi)dl = 0$$

shartni qanoatlantirsa, $|x| \to \infty$ da $J_1(x) = O\left(\frac{1}{|x|}\right)$ bo'ladi.

Haqiqatan ham, x va ξ o'zgaruvchilarga mos ravishda (r, φ) va (ρ, α) qutb koordinatalari sistemasini kiritamiz. U holda

$$ln|x-\xi| = ln\sqrt{r^2+\rho^2-2r\rho cos(\varphi-\alpha)} =$$

$$= \frac{1}{2}ln\left\{r^2\left(1-2\frac{\rho}{r}cos(\varphi-\alpha)+\frac{\rho^2}{r^2}\right)\right\} =$$

$$= lnr - \frac{\rho}{r}cos(\varphi-\alpha) + O\left(\frac{1}{r^2}\right)$$

munosabatlar $r \to \infty$ bo'lganda bajariladi. So'nggi tengliklarda $y \to 0$ bo'lganda $ln(1+x) = x + O(x^2)$ ekanligidan foydalanildi. Demak,

$$J_1(x) = O\left(\frac{1}{|x|}\right), \ |x| \to \infty.$$

Keyinchalik, sirt potensiallarining xossalarini uch o'lchovli hol uchun tekshiramiz. Ikki o'lchovli holda tekshirishlar o'xshash ravishda amalga oshiriladi. Shu sababli bu hol uchun yakuniy natijalarni yozamiz.

6.13.4 Oddiy qatlam potensialining uzluksizligi

$x \in S$ bo'lsa, potensiallar xosmas integralardan iborat bo'lib, ularni uzluksizlikka tekshirish kerak bo'ladi. S silliq sirt bo'lsin, ya'ni uning har bir nuqtasida uzluksiz normal mavjud.

T e o r e m a. Silliq sirtda berilgan uzluksiz va chegaralangan zichlikka ega ikkilangan qatlam potensiali barcha fazoda uzluksizdir.

Isbot. Potensialning S sirtdan tashqarida uzluksiz funksiya bo'lishini ko'rsatgan edik. Teorema shartlari bajarilganda ikkilangan qatlam potensialining S sirtda uzluksizligi va uning S sirtdan tashqaridagi qiymatlari uzluksiz ravishda S sirtdagi qiymatlariga taqalishini ko'rsatamiz. Buning uchun isbot qilingan tekis yaqinlashuvchi xosmas integralning xossasiga ko'ra

$$I_1(x) = \int\limits_S \mu(\xi) \frac{ds}{|x-\xi|}, \ |\mu| \leq A = const$$

integralning S sirtda tekis yaqinlashishini ko'rsatish yetarli.

$x_0 \in S$ - ixtiyoriy nuqta. Markazi x_0 nuqtada radiusi δ ga teng bo'lgan Σ sferani qaraymiz. S_1 orqali S sirtning Σ sfera ichida yotuvchi qismini belgilaymiz. $S_2 = S \setminus S_1$ bo'lsin. U holda

$$I_1(x) = \int\limits_{S_1} \mu \frac{ds}{|x-\xi|} + \int\limits_{S_2} \mu \frac{ds}{|x-\xi|} = I_1^1(x) + I_2^2(x)$$

Ixtiyoriy $\varepsilon > 0$ soni uchun $|x - x_0| \leq \delta$ shartni qanoatlantiruvchi barcha x larda

$$\left| \int\limits_{S_1} \mu(\xi) \frac{ds}{|x-\xi|} \right| \leq \varepsilon$$

tengsizlik bajariladigan $\delta > 0$ sonining mavjudligini ko'rsatamiz.

Koordinatalar boshi $x_0 = (x_{01}, x_{02}, x_{03})$ nuqtada bo'lgan, x_3 o'q x_0 nuqtada S sirtga o'tkazilgan tashqi normal bo'yicha yo'nalgan dekart koordinatalar sistemasini kiritamiz. $\xi = (\xi_1, \xi_2, \xi_3)$ $-$ S_1 sirtda o'zgaruvchi nuqta. Bu koordinatalar sistemasida S_1 sirtning $x_1 O x_2$ tekislikka proyeksiyasi S_1' bo'lsin, $ds = \frac{d\xi_1 d\xi_2}{cos\gamma}$, bu yerda γ $-$ ξ nuqtada normal bilan x_3 o'q orasidagi burchak. $I_1^1(x)$ ni baholaymiz:

$$\left| I_1^1(x) \right| \leq A \int\limits_{S_1} \frac{ds}{|x-\xi|} = A \int\limits_{S_1'} \frac{d\xi_1 d\xi_2}{cos\gamma \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2}} \leq$$

$$\leq A \int\limits_{S_1'} \frac{d\xi_1 d\xi_2}{cos\gamma \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}}.$$

Markazi $x' = (x_1, x_2, 0)$ nuqtada va radiusi 2δ bo'lgan va S_1' sohani o'z ichiga oluvchi doirani $K_{2\delta}$ orqali belgilaymiz. Ravshanki, u holda

$$\left| I_1^1(x) \right| \leq 2A \int\limits_{K_{2\delta}} \frac{d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}}$$

Markazi x' nuqtada joylashgan qutb (ρ, φ) koordinatalar sistemasini kiritib,

$$\left|I_1^1(x)\right| \leq 2A \int_0^{2\delta} \int_0^{2\pi} \frac{\rho d\rho d\varphi}{\rho} = 8\pi A\delta$$

tengsizlikka ega bo‘lamiz. Bu esa δ sonini $\delta(\varepsilon) \leq \frac{\varepsilon}{8\pi A}$ shartdan tanlash mumkinligini ko‘rsatadi.

Shunday qilib, δ_1 sirt bo‘yicha integral x ga nisbatan x_0 nuqtada tekis yaqinlashadi va u bu nuqtada uzluksiz funksiyani aniqlaydi. Bundan esa oddiy qatlam $I_1(x)$ potensialining S sirtda uzluksizligi kelib chiqadi. Teorema isbotlandi.

Ikki o‘lchovli holda oddiy qatlam $J_1(x)$ potensialining uzluksizligi yuqoridagiga o‘xshash tarzda ko‘rsatiladi.

Ikkilangan qatlam potensiallarining mavjud bo‘lishi uchun silliq S sirtga kuchliroq shartlar qo‘yiladi. Quyida *Lyapunov sirtlari* deb ataluvchi sirtlar sinfini kiritamiz.

T a’ r i f. S sirt *Lyapunov sirti* deyiladi, agarda quyidagi shartlar bajarilgan bo‘lsa:

1. S sirtning har bir nuqtasida normal (yoki urinma sirt) aniqlangan.

2. Shunday $d > 0$ soni mavjudki, bunda S sirtning ξ nuqtasida o‘tkazilgan normalga parallel to‘g‘ri chiziqlar S sirtning markazi ξ nuqtada radiusi d bo‘lgan sferaga tegishli qismini bittadan ortiq nuqtada kesmaydi.

3. S sirt x va ξ nuqtalarida o‘tkazilgan normallar orasidagi $\gamma(x, \xi) = (\widehat{n_x, n_\xi})$ burchak

$$\gamma(x, \xi) \leq A|x - \xi|^\delta, A = const > 0,$$

shartni qanoatlantiradi. Bunda x nuqta S sirtning markazi ξ da radiusi d bo‘lgan sferaga ichida qismiga tegishli.

Endi ikkilangan qatlam potensiallarini tekshirishga o‘tamiz. Potensiallarni Lyapunov sirtlarida berilgan deb hisoblaymiz.

$x_0 \in S$ biror nuqta bo‘lsin. Markazi x_0 nuqtada, z o‘q shu nuqtada S sirtga o‘tkazilgan tashqi normal bo‘ylab yo‘nalgan mahalliy dekart (x_1, x_2, z) koordinatalar sistemasini kiritamiz. Bunda x_1Ox_2 tekislik S sirtga x_0 nuqtada o‘tkazilgan urinma tekislik bilan ustma-ust tushadi. S Lyapunov sirti bo‘lganligi sababli shunday $\rho_0 > 0$ mavjudki bunda $\rho = \sqrt{x_1^2 + x_2^2} < \rho_0$ uchun S sirtning tenglamasini

$$z = z(x_1, x_2)$$

ko‘rinishida yozish mumkin, bu yerda $z(x_1, x_2)-$ bir qiymatli uzluksiz differensiallanuvchi funksiya. (x_1, x_2) nuqtaning ko‘rsatilgan atrofida $z = z(x_1, x_2)$ funksiya va uning birinchi tartibli hosilalarining baholarini olamiz. ξ nuqtada S sirtga o‘tkazilgan n_ξ normalning $\rho < \rho_0$ uchun yo‘naltiruvchi kosinuslari ushbu

$$cos\alpha = \frac{z_{x_1}}{\sqrt{1 + z_{x_1}^2 + z_{x_2}^2}},\ cos\beta = \frac{z_{x_2}}{\sqrt{1 + z_{x_1}^2 + z_{x_2}^2}},\ cos\gamma = \frac{z_{x_3}}{\sqrt{1 + z_{x_1}^2 + z_{x_2}^2}}$$

tengliklar bilan ifodalanadi. n_ξ vektorning x_1Ox_2 tekislikka proyeksiyasini $n_{\xi'}$ bilan, α' va β' orqali esa $n_{\xi'}$ vektorning mos ravishda x_1 va x_2 o‘qlar bilan tashkil qilgan burchaklarini belgilaymiz:

$$cos\alpha = cos\alpha' sin\gamma,\ cos\beta = sin\alpha' sin\gamma.$$

Koordinatalar sistemasining tanlanilishiga ko‘ra

$$z(x_{01}, x_{02}) = 0,\ z_{x_1}(x_{01}, x_{02}) = 0,\ z_{x_2}(x_{01}, x_{02}) = 0$$

bo‘lishidan $\rho_0 > 0$ ni yetarlicha kichik tanlash mumkinki, $\rho < \rho_0$ lar uchun

$$cos\gamma = \frac{z_{x_3}}{\sqrt{1 + z_{x_1}^2 + z_{x_2}^2}} > \frac{1}{2}$$

bo‘ladi.

U holda Lyapunov sirtlarining 3-shartiga asosan

$$|cos\alpha| \le sin\gamma \le A|x_0 - \xi|,\ |cos\beta| \le A|x_0 - \xi|^\delta, \tag{76}$$

$$|z_{x_1}| = \left|\frac{cos\alpha}{cos\gamma}\right| \le 2|cos\alpha| \le 2A|x_0 - \xi|^\delta. \tag{77}$$

Shunga o‘xshash

$$|z_{x_2}| \le 2A|x_0 - \xi|^\delta. \tag{78}$$

x_0 nuqtaning atrofida $z(x_1, x_2)$ funksiya uchun, Teylor formulasidan foydalanib quyidagini olamiz:

$$z(x_1, x_2) = z(0,0) + x_1 z_{x_1}(\overline{x_1}, \overline{x_2}) + x_2 z_{x_2}(\overline{x_1}, \overline{x_2}),$$

bu yerda $0 \leq \overline{x_1} \leq x_1$, $0 \leq \overline{x_2} \leq x_2$. Bundan

$$|z(x_1, x_2)| \leq \rho|z_{x_1}| + \rho|z_{x_2}| \leq 4A|x_0 - \xi|^{1+\delta}. \qquad (79)$$

T e o r e m a. Chegaralangan $\nu(\xi)$ zichlikka ega bo‘lgan ikkilangan qatlam

$$I_2(x) = \int\limits_S \nu(\xi) \frac{cos\varphi}{|x - \xi|^2} ds \qquad (80)$$

potensiali $x \in S$ nuqtalarda yaqinlashuvchi xosmas integraldir.

Isbot. $x \in S$ ixtiyoriy nuqta bo‘lsin. Bu nuqta atrofida (80) ning integral osti funksiyasini baholaymiz. Yuqoridagi kabi qo‘zg‘almas koordinatalar sistemasini kiritamiz, ξ nuqta (ξ_1, ξ_2, ξ_3) koordinatalarga ega. U holda

$$cos\varphi = \frac{\xi_1}{|x - \xi|} cos\alpha + \frac{\xi_2}{|x - \xi|} cos\beta + \frac{\xi_3}{|x - \xi|} cos\gamma.$$

Bundan, (76)-(79) baholarni hisobga olib,

$$|cos\varphi| \leq |cos\alpha| + |cos\beta| + \frac{|\xi_3|}{|x - \xi|} \leq$$

$$\leq A|x - \xi|^\delta + A|x - \xi|^\delta + 4A|x - \xi|^\delta = 6A|x - \xi|^\delta. \qquad (81)$$

tengsizliklarni hosil qilamiz.

Teorema shartiga ko‘ra $\nu(\xi)$ funksiya chegaralangan, $|\nu(\xi)| \leq const$. Bu va (81) bahoga ko‘ra x nuqta atrofida $\xi \in S$ nuqtalar uchun (80) ning integral osti funksiyasi quyidagicha baholanadi:

$$\left| \nu \frac{cos\varphi}{|x - \xi|^2} \right| \leq \frac{6AC}{|x - \xi|^{2-\delta}} \qquad (82)$$

Bu esa (81) xosmas integralning $x \in S$ nuqtalarda yaqinlashuvchi ekanligini bildiradi. (82) baho S sirtning ixtiyoriy x nuqtasida o‘rinli bo‘lgani sababli u (80) integralning ixtiyoriy $x_0 \in S$ nuqtada x ga nisbatan tekis yaqinlashishini va S sirtda uzluksizligini ta’minlaydi. Tekislikda ikkilangan qatlam potensiali sirt uchun 1)-3) larga o‘xshash shartlar bilan aniqlanuvchi Lyapunov egri chiziqlarida yotuvchi nuqtalar uchun mavjud.

6.13.5 Ikkilangan qatlam potensialining uzilishi. Gauss integrali. Teles burchak

Oddiy qatlam potensialidan farqli o'laroq ikkilangan qatlam potensiali butun fazoda uzluksiz emas. U qaralayotgan sirtda uzilishga ega.

Buni ko'rsatish uchun, avvalo, zichligi $\nu_0 = const$ bo'lgan ikkilangan qatlam

$$I_2(x) = \nu_0 \int_S \frac{\partial}{\partial n} \frac{1}{|x-\xi|} ds \tag{83}$$

potensialini qaraymiz. $\nu = 1$ bo'lganda (83) integral *Gauss integrali* deyiladi. S ni yopiq sirt deb hisoblaymiz. U holda (83) integral osongina hisoblanadi. Buning uchun (8) Grinning uchinchi formulasida $u \equiv v_0$ deb,

$$\nu_0 \int_S \frac{\partial}{\partial n} \frac{1}{|x-\xi|} ds = \begin{cases} -4\pi\nu_o, & x \in D, \\ -2\pi\nu_o, & x \in S, \\ 0, & x \notin D \cup S, \; n = 3. \end{cases} \tag{84}$$

Ikkilangan qatlam potensialining $\xi_0 \in S$ nuqtadagi qiymatini $I_2^0(\xi_0)$ bilan belgilaymiz, ya'ni bu uning $\xi_0 \in S$ nuqtadagi to'g'ri qiymati. $I_2^i(\xi_0) - I_2(x)$ potensialning sirt ichkarisidan ξ_0 nuqtada hisoblangan limit qiymati, ya'ni

$$I_2^i(\xi_0) = \lim_{x \to \xi_0 \in S} I_2(x), \; x \in D;$$

$I_2^l(\xi_0) - I_2(x)$ potensialning ξ_0 da sirt tashqarisidan hisoblangan limiti:

$$I_2^l(\xi_0) = \lim_{x \to \xi_0 \in S} I_2(x), \; x \notin D \cup S.$$

O'zgarmas ν_0 zichlikka ega bo'lgan potensial (84) formulaga ko'ra bo'lakli o'zgarmas funksiyadir. (84) formulani

$$I_2^i(x) = I_2^0(x) + 2\pi\nu_0,$$

$$I_2^l(x) = I_2^0(x) - 2\pi\nu_0 \tag{85}$$

ko'rinishda yozish mumkin.

Endi zichligi uzluksiz $\nu(\xi)$ funksiya bo‘lgan ikkilangan qatlam potensialini qaraymiz va bu holda ham (85) ga o‘xshash formulalar o‘rinli ekanligini ko‘rsatamiz.

$\xi_0 \in S$ ixtiyoriy nuqta bo‘lsin. Ikkilangan qatlam potensialini quyidagi ko‘rinishda yozamiz:

$$I_2(x) = \oint_S \frac{cos\varphi}{|x-\xi|^2} ds =$$

$$= \oint_S [\nu(\xi) - \nu(\xi_0)] \frac{cos\varphi}{|x-\xi|} ds + \oint_S \nu(\xi_0) \frac{cos\varphi}{|x-\xi|} ds = \overline{I}_2(x) + \widetilde{I}_2(x). \quad (86)$$

Bu yerda ikkinchi qo‘shiluvchi zichligi $\nu(\xi_0)$ o‘zgarmas bo‘lgan ikkilangan qatlam potensiali va uning xossalari ma’lum. Birinchi qo‘shiluvchining ξ_0 nuqtada uzluksiz ekanligini ko‘rsatamiz. Buning uchun $\overline{I}_2(x)$ integralning x parametrga nisbatan ξ_0 nuqtada tekis yaqinlashuvchi bo‘lishini ko‘rsatish yetarli. Ixtiyoriy $\varepsilon > 0$ sonini olamiz. $\nu(\xi)$ funksiyaning ξ_0 nuqtada uzluksizligidan ixtiyoriy $\varepsilon > 0$ uchun S sirtda ξ_0 nuqtaning shunday K_1 atrofi mavjudki, $\xi \in K_1$ lar uchun

$$|\nu(\xi) - \nu(\xi_0)| \leq \varepsilon.$$

$\overline{I}_2(x)$ integralni K_1 bo‘yicha baholaymiz:

$$\left| \int_{K_1} [\nu(\xi) - \nu(\xi_0)] \frac{cos\varphi}{|x-\xi|^2} ds \right| \leq \varepsilon \int_{K_1} \frac{cos\varphi}{|x-\xi|^2} ds.$$

(84) ga ko‘ra yetarlicha kichik K_1 va ixtiyoriy x lar uchun

$$\int_{K_1} \frac{cos\varphi}{|x-\xi|^2} ds \leq B = const$$

tengsizlik o‘rinli. Bundan

$$\left| \int_{K_1} [\nu(\xi) - \nu(\xi_0)] \frac{cos\varphi}{|x-\xi|^2} ds \right| \leq \varepsilon B$$

kelib chiqadi. Bu esa $\overline{I}_3(x)$ integralning ξ_0 nuqtada tekis yaqinlashuvchi ekanligini bildiradi. Demak, $\overline{I}_2$ funksiya ξ_0 nuqtada uzluksizdir.

Shunday qilib, ikkilangan qatlam potensialining ξ_0 nuqtada uzilishi (86) formulaga asosan ikkinchi $\widetilde{I}_2(x)$ qo'shiluvchi bilan aniqlanadi.

(86) formulada $x \to \xi_0$ deb, limitga o'tamiz. Oldingi belgilashlarni saqlab va o'zgarmas zichlikka ega potensialning xossalaridan foydalanib,

$$I_2^i(\xi_0) = \overline{I}_2(\xi_0) + \widetilde{I}_2^i(\xi_0) = \overline{I}_2(\xi_0) + \widetilde{I}_2^0(\xi_0) - 2\pi\nu(\xi_0) = I_2^0(\xi_0) - 2\pi\nu(\xi_0),$$

$$I_2^l(\xi_0) = \overline{I}_2(\xi_0) + \widetilde{I}_2^l(\xi_0) = \overline{I}_2(\xi_0) + \widetilde{I}_2^0(\xi_0) + 2\pi\nu(\xi_0) = I_2^0(\xi_0) + 2\pi\nu(\xi_0)$$

munosabatlarni hosil qilamiz, bu yerda $I_2^0(\xi_0) = \overline{I}_2(\xi_0) + \widetilde{I}_2^0(\xi_0) - I_3(x)$ ning $\xi_0 \in S$ dagi to'g'ri qiymati.

Demak, ikkilangan qatlam potensiali sirtni kesib o'tganda uzilishga ega va bu uzilish miqdori

$$I_2^i(\xi) = I_2^0(\xi) - 2\pi\nu(\xi), \tag{87}$$

$$I_2^l(\xi) = I_2^0(\xi) + 2\pi\nu(\xi), \tag{88}$$

$$I_2^l(\xi) - I_2^i(\xi) = 4\pi\nu(\xi) \tag{89}$$

formulalar bilan aniqlanadi. (89) tenglik ikkilangan qatlam potensialining $\xi \in S$ nuqtadagi sakrashini ifodalaydi.

Agar S sirt yopiq bo'lmasa, quyidagicha yo'l tutiladi. S sirt S^* bilan shunday to'ldiriladiki, bunda $S \cup S^*$ yopiq Lyapunov sirti bo'ladi. S^* da $\nu(\xi)$ zichlikni 0 deb qo'shimcha aniqlaymiz. Ravshanki, bajarilgan tekshirishlar $\nu(\xi)$ funksiya uzluksiz bo'lgan barcha nuqtalarida o'rinli bo'ladi.

Tekislikda ikkilangan qatlam potensiallari yuqoridagilarga o'xshash ravishda tekshiriladi. Bu holda potensialning C egri chiziq nuqtalarida uzilishiga oid kattaliklar

$$J_2^i(\xi) = J_2^0(\xi) - \pi\nu(\xi),$$

$$J_2^l(\xi) = J_2^0(\xi) + \pi\nu(\xi),$$

$$J_2^l(\xi) - J_2^i(\xi) = 2\pi\nu(\xi)$$

formulalar bilan aniqlanadi.

Bo'laklari silliq, umuman olganda, yopiq bo'lmagan va unda normalning musbat yo'nalishi aniqlangan S sirtni qaraymiz. Bu sirtning nuqtasini ξ va bu nuqtada S sirtga o'tkazilgan normalni n orqali belgilaymiz. $x \in \mathbb{R}^3$

$(x\bar{\in}S)$ nuqta shunday nuqta bo‘lsinki, bunda undan ixtiyoriy $\xi \in S$ nuqtalarga o‘tkazilgan radius-vektor n normal bilan o‘tkir, hech bo‘lmaganda to‘gri burchak tashkil qilsin, ya’ni $\cos(n, r) \geq 0$ bo‘lsin. x nuqtadan S sirtning barcha nuqtalariga radius-vektorlar o‘tkazamiz. Bu radius-vektorlar bu sirt bilan va sirtning chetki nuqtalariga kelib tushgan radius-vektorlar hosil qilgan K konik (konussimon) sirt bilan chegaralangan sohani qoplaydi.

Agar S yopiq sirt bo‘lsa, x nuqta sirtning ichida yotishi kerak (aks holda (n, r) burchak o‘tmas bo‘lishi mumkin) va K soha S sirtning ichidan iborat bo‘ladi. x nuqtani markaz qilib ixtiyoriy ρ radiusli sfera chizamiz. K konusning ichida yotgan bu sfera qismining yuzasini σ_ρ bilan belgilaymiz. Ushbu

$$\omega_x(S) = \frac{\sigma_\rho}{\rho^2}$$

nisbat ρ ga bog‘liq bo‘lmaydi va u *teles burchak* deyiladi. Bu burchak ostida x nuqtadan S sirt ko‘rinadi. Yuritilgan mulohazalarda S da $\cos(n, r) \leq 0$ bo‘lishi ham mumkin. Bu holda teles burchak deb, yuqoridagi nisbat teskari ishora bilan olingan miqdorga aytiladi.

E s l a t m a. (84) formulani ixtiyoriy S sirt uchun umumlashtirish mumkin. Agar $x\bar{\in}S$ bo‘lsa, Gauss integrali teles burchakka teng bo‘ladi (sirt tomonining ishorasini inobatga olgan holda).

6.13.6 Oddiy qatlam potensiali normal hosilasining uzilishi

Oddiy qatlam potensiali S sirtdan tashqarida barcha tartibli uzluksiz hosilalariga ega. Uning normal hosilasini S sirt atrofida tekshiramiz. $\mu(\xi)$ zichlikni sirtda uzluksiz funksiya deb hisoblaymiz.

$\xi \in S$ ixtiyoriy nuqta bo‘lsin, $n_l(\xi_0)$ orqali bu nuqtadagi tashqi normalni belgilaymiz. Oddiy qatlam $I_1(x)$ potensialning $n_l(\xi_0)$ yo‘nalishdagi

$$\frac{\partial I_1(x)}{\partial n_l} = (n_l(\xi_0), gradI_1(x)) = \int\limits_S \mu(\xi)\left(n_l(\xi_0)grad_x, \frac{1}{x-\xi}\right) ds \qquad (90)$$

hosilasini qaraymiz va uni $x \to \xi_0$ da tekshiramiz. Faqat x nuqta ξ_0 ga $n_l(\xi_0)$ normal bo‘ylab intilgan holni chuqurroq o‘rganamiz.

$|x-\xi|$ funksiyaning faqat x va ξ nuqtalar koordinatalari sistemasiga bog‘liqligini inobatga olib, (90) formulani quyidagi ko‘rinishda yozish mumkin:

$$\frac{\partial I_1(x)}{\partial n_l} = -\int\limits_S \mu(\xi)\left(n_l(\xi_0), grad_\xi \frac{1}{|x-\xi|}\right) ds =$$

$$= \int\limits_S \mu(\xi)\left([n_l(\xi)-n_l(\xi_0)]\,, grad_\xi \frac{1}{|x-\xi|}\right) ds -$$

$$-\int\limits_S \mu(\xi)\left(n_l(\xi), grad_\xi \frac{1}{|x-\xi|}\right) ds, \tag{91}$$

bu yerda $n_l(\xi) - S$ sirtga ξ nuqtaga o‘tkazilgan tashqi birlik normal. (91) da

$$grad_x \frac{1}{|x-\xi|} = -grad_\xi \frac{1}{|x-\xi|}$$

tenglikdan foydalanildi. (91) ning ikkinchi integrali

$$I_2(x) = \int_S \mu(\xi) \frac{\partial}{\partial n_\xi} \frac{1}{|x-\xi|} ds$$

ga teng bo‘lib, u xossalari o‘rganilgan zichligi $\mu(\xi)$ bo‘lgan ikkilangan qatlam potensialidir. Shuning uchun

$$\frac{\partial I_1(x)}{\partial n_l} = \int\limits_S \mu(\xi)[n_l(\xi)-n_l(\xi_0)] grad_\xi \frac{1}{|x-\xi|} ds - I_2(x) = I_1^0(x) - I_2(x). \tag{92}$$

$I_1^0(x)$ ning ξ_0 nuqtada uzluksiz ekanligini ko‘rsatamiz. Buning uchun integral osti funksiyani ξ_0 nuqta atrofida baholaymiz. $|\mu(\xi)| < C = const$ ekanligidan

$$f = \left|\mu(\xi)[n_l(\xi)-n_l(\xi_0)] grad_\xi \frac{1}{|x-\xi|}\right| \le C\,|n_l(\xi)-n_l(\xi_0)|\,\frac{1}{|x-\xi|^2}$$

tengsizlikni hosil qilamiz. Ushbu

$$|n_l(\xi)-n_l(\xi_0)| = \sqrt{(n_l(\xi)-n_l(\xi_0))(n_l(\xi)-n_l(\xi_0))} =$$

$$= \sqrt{2-2\cos\gamma} = 2\left|\sin\frac{\gamma}{2}\right|,$$

bu yerda $\gamma - \xi$ va ξ_0 nuqtalardagi normallar orasidagi burchak, formulani va Lyapunov sirti uchun bajariluvchi $\gamma < A|\xi - \xi_0|^\delta$ tengsizlikni inobatga olsak,

$$f \leq \frac{2c\left|\sin\frac{\gamma}{2}\right|}{|x-\xi|^2} \leq \frac{\gamma c}{|x-\xi|^2} \leq \frac{Ac|\xi-\xi_0|^\delta}{|x-\xi|^2}$$

baholarga ega bo‘lamiz. Bu esa x nuqta ξ ga $n_l(\xi)$ normal bo‘yicha intilganda $I_1^0(x)$ integralning ξ_0 nuqtada uzluksizligini ta’minlaydi. Demak, $I_1^0(x)$ integral ξ_0 nuqtada uzluksiz.

Shunday qilib, (92) formulaga ko‘ra, $\frac{\partial I_1(x)}{\partial n_l}$ funksiyaning uzilishlari ikkinchi $I_2(x)$ integral bilan aniqlanadi. Ikkilangan qatlam potensiali uchun olingan (87), (88) formulalardan foydalanib, (92) dan quyidagilarni olamiz:

$$\lim_{x\to\xi_0,\ x\in D}\frac{\partial I_1(x)}{\partial n_l} =$$

$$= I_1^0(\xi_0) - I_2^i(\xi_0) = I_1^0(\xi_0) - I_2^0(\xi_0) + 2\pi\mu(\xi_0) = \left(\frac{\partial I_1(\xi)}{\partial n_l}\right)^0 + 2\pi\mu(\xi_0),$$

bu yerda $\left(\frac{\partial I_1(\xi)}{\partial n_l}\right)^0$ – normal hosilaning ξ_0 nuqtadagi to‘g‘ri qiymati. Bularga o‘xshash tarzda

$$\lim_{x\to\xi_0, x\in D}\frac{\partial I_1(x)}{\partial n_l} = \left(\frac{\partial I_1(\xi)}{\partial n_l}\right)^0 - 2\pi\mu(\xi_0).$$

$\left(\frac{\partial I_1(x)}{\partial n_l}\right)_i$ va $\left(\frac{\partial I_1(x)}{\partial n_l}\right)_l$ lar mos ravishda oddiy qatlam potensiali normal hosilalari ichki va tashqi limit qiymatlari bo‘lsin. Agar sirt yopiq bo‘lmasa, sirtning ichki va tashqi tomonlari tashqi normalning tanlanishi bilan aniqlanadi. Kiritilgan belgilashlar yordamida oddiy qatlam potensiali normal hosilasining uzilish xossalari quyidagi formulalar bilan yozilishi mumkin:

$$\left(\frac{\partial I_1(\xi)}{\partial n_l}\right)_i = \left(\frac{\partial I_1(\xi)}{\partial n_l}\right)^0 + 2\pi\mu(\xi), \tag{93}$$

$$\left(\frac{\partial I_1(\xi)}{\partial n_l}\right)_l = \left(\frac{\partial I_1(\xi)}{\partial n_l}\right)^0 - 2\pi\mu(\xi), \tag{94}$$

$$\left(\frac{\partial I_1(\xi)}{\partial n_l}\right)_i - \left(\frac{\partial I_1(\xi)}{\partial n_l}\right)_l = 4\pi\mu(\xi). \tag{95}$$

Ichki normal bo'yicha hosila uchun olingan (93), (94) formulalar faqat ikkinchi qo'shiluvchilarning ishoralari bilan farq qiladi. (90) formuladan oddiy qatlam potensialining normal hosilasi uchun quyidagi ifodani olish mumkin:

$$\frac{\partial I_1(\xi)}{\partial n_l} = \int\limits_S \mu(\xi)\frac{\cos\psi}{|x-\xi|^2}ds,$$

bu yerda $\psi - x$ nuqta yotuvchi S sirtga ξ_0 nuqtada o'tkazilgan $n_l(\xi_0)$ normal va $\overrightarrow{x\xi}$ vektor orasidagi burchak.

Tekislikda ham oddiy qatlam potensial normal hosilasi C egri chiziqdan o'tishda uzilishga ega. Bu hol uchun uzilish miqdorini aniqlovchi formulalar quyidagi ko'rinishga ega:

$$\left(\frac{\partial J_1(\xi)}{\partial n_l}\right)_i = \left(\frac{\partial J_1(\xi)}{\partial n_l}\right)^0 + \pi\mu(\xi),$$

$$\left(\frac{\partial J_1(\xi)}{\partial n_l}\right)_l = \left(\frac{\partial J_1(\xi)}{\partial n_l}\right)^0 - \pi\mu(\xi),$$

$$\left(\frac{\partial J_1(\xi)}{\partial n_l}\right)_i - \left(\frac{\partial J_1(\xi)}{\partial n_l}\right)_l = 2\pi\mu(\xi).$$

Sirt integrallarining tekshirilgan xossalari $\mu(\xi)$ va $\nu(\xi)$ funksiyalar S sirtda (yoki S egri chiziqda) chegaralangan va uzluksiz bo'lgan hol uchun olindi. Bu shartlarni yumshatish mumkin. Chuqurroq tekshiruvlar $\mu(\xi)$ va $\nu(\xi)$ lar S da kvadrati bilan integrallanuvchi ($\mu(\xi),\ \nu(\xi) \in L_2(S)$) funksiyalar bo'lgan holda ham sirt potensialining asosiy xossalari o'rinli bo'lishini ko'rsatadi. Bunga mos sirt integrallari S dan tashqarida garmonik funksiyalar bo'lib, ularning limit qiymatlari uchun olingan ifoda ham S ning qariyb hamma qiymatlarida (bajarilmaydigan nuqtalar to'plami 0 nol o'lchamga ega) bajariladi.

6.13.7 Fredgolm integral tenglamalari haqida

Xususiy hosilasi differensial tenglamalar uchun chegaraviy masalalarni yechishda integral tenglamalardan keng foydalaniladi. Bu yerda ular Laplas tenglamasi uchun chegaraviy masalalarni yechishda qo'llaniladi.

Ushbu paragrafda ikkinchi tur Fredgolm integral tenglamalari nazariyasining asosiy teoremalari isbotsiz keltiriladi. Quyidagi birinchi tur Fredgolm integral tenglamasini qaraymiz:

$$u(x) = \lambda \int_D K(x,\xi)u(\xi)d\xi + f(x). \tag{96}$$

Bunda $D-$ n o'lchovli chekli soha. $K(x,\xi)$ yadroni haqiqiy funksiya deb faraz qilinadi. $K^*(x,\xi) = K(\xi,x)$ funksiya *qo'shma yadro,*

$$v(x) = \lambda \int_D K(x,\xi)v(\xi)d\xi + g(x) \tag{97}$$

integral tenglama esa *qo'shma integral tenglama* deyiladi. Keyinchalik, $K(x,\xi)$ yadro yoki o'z argumentlarining uzluksiz funksiyasi yoki qutbli funksiya, ya'ni u ushbu

$$K(x,\xi) = \frac{H(x,\xi)}{|x-\xi|^\alpha}, \ \alpha < n$$

ko'rinishga ega deb hisoblanadi, bu yerda $H(x,\xi)-$ uzluksiz funksiya. Agar $\alpha < \frac{n}{2}$ bo'lsa, $K(x,\xi)$ yadro kuchsiz qutbli deyiladi.

λ soni $K(x,\xi)$ yadroning *xos soni* deyiladi, agarda bir jinsli

$$u(x) = \lambda \int_D K(x,\xi)u(\xi)d\xi \tag{98}$$

integral tenglamaning trivual (ya'ni nol) bo'lmagan yechimi mavjud bo'lsa, har bir λ xos songa mos keluvchi (95) tenglamaning yechimiga *xos funksiya* deyiladi.

Noldan farqli haqiqiy, simmetrik, uzluksiz yoki qutbli yadro hech bo'lmaganda bitta xos songa ega bo'ladi. Xos sonlar to'plami sanoqli bo'lib, quyilish nuqtalarga ega emas. Agar xos sonlar chekli bo'lsa, u holda $K(x,\xi)$ aynigan yadrodir. Xos sonning rangi deb bu songa mos keluvchi chiziqli bog'liq bo'lmagan xos funksiyalarning soniga aytiladi. Xos sonning rangi cheklidir.

Bir jinsli bo'lmagan integral tenglamaning yechilishi masalalari *Fredgolm tenglamalari (Fredgolm alternativalari)* bilan hal qilinadi.

Fredgolmning birinchi t e o r e m a s i. Agar λ soni $K(x,\xi)$ yadroning xos soni bo‘lmasa (ya’ni bir jinsli (98) tenglama nol yechimga ega), u holda bir jinsli bo‘lmagan(96) va unga qo‘shma (97) tenglamalar ixtiyoriy uzluksiz $f(x), g(x)$ funksiyalar uchun yagona yechimga ega.

Fredgolmning ikkinchi t e o r e m a s i. Agar λ soni $K(x,\xi)$ yadroning xos soni bo‘lsa, u holda u qo‘shma yadroning ham xos soni bo‘ladi va ularning rangi bir xil.

Fredgolmning uchinchi t e o r e m a s i. Agar λ soni $K(x,\xi)$ yadroning xos soni bo‘lsa, u holda bir jinsli bo‘lmagan (96) tenglama yechimga ega emas yoki bittadan ortiq yechimga ega. (96) tenglamaning bir qiymatli yechilishi uchun uning o‘ng qismidagi $f(x)$ funksiya λ xos soniga mos keluvchi qo‘shma yadroning barcha xos funksiyalariga ortogonal bo‘lishi zarur va yetarli.

Qayd etish lozimki, Fredgolm teoremalari integral tenglamalar $L^2(D)$ fazoda qaralganda ham o‘rinli.

6.13.8 Dirixlening ichki masalasi uchun integral tenglama

Dirixlening (73) ichki masalasini qaraymiz. S ni Lyapunov sirti deb hisoblaymiz.

Bu masala yechimini ikkilangan qatlam

$$u(x) = \oint\limits_S \nu(\xi) \frac{\partial}{\partial n} \frac{1}{|x-\xi|} ds \tag{99}$$

potensial ko‘rinishida izlaymiz. (99) integralning yadrosi avvaldek S ga ξ nuqtadagi tashqi n normal bo‘yicha Laplas operatori fundamental yechimining hosilasidan iborat. Ma’lumki, $\nu(\xi)$ zichlik uzluksiz funksiya bo‘lganda (99) funksiya D sohada Laplas tenglamasini qanoatlantiradi. Endi $\nu(\xi)$ funksiyani shunday aniqlash lozimki, bunda

$$\lim_{x \to \xi_0, x \in D} u(x) = f(\xi_0), \ \xi_0 \in S \tag{100}$$

limit munosabat o‘rinli bo‘lsin, ya’ni chegaraviy shart ham bajarilsin. $\overline{D}$ yopiq sohada uzluksiz yechimni olish uchun S sirtdagi yechimning chegaraviy qiymatlari sifatida ikkilangan qatlam (99) potensialining soha ichkarisidan hisoblangan limit $u_i(\xi_0)$ qiymatlarini qarash zarur. (100) chegaraviy shartdan, ikkilangan qatlam potensialining (87) xossasini inobatga olib,

$$\int\limits_S \nu(\xi)\frac{\partial}{\partial n}\frac{1}{|\xi-\xi_0|}ds - 2\pi\nu(\xi_0) = f(\xi_0),\ \xi_0 \in S$$

tenglamaga ega bo'lamiz. Buni ushbu

$$\nu(\xi_0) - \frac{1}{2\pi}\oint \nu(\xi_0)\frac{\partial}{\partial n}\frac{1}{|\xi-\xi_0|}ds = \frac{1}{2\pi}f(\xi_0),\ \xi \in S \qquad (101)$$

ikkinchi tur Fredgolm integral tenglamasi ko'rinishida yozib olamiz. (101) tenglama qutb

$$K(\xi_0,\xi) = \frac{\partial}{\partial n}\frac{1}{|\xi-\xi_0|}$$

yadroga ega bo'lib, (81) ga asosan,

$$|K(\xi_0,\xi)| \le \frac{A}{|\xi-\xi_0|^{2-\delta}},\ \delta > 0$$

baho o'rinli.

Agar (101) tenglamaning $\nu(\xi)$ yechimi mavjud bo'lsa, uni (99) formulaga qo'yib, (73) Dirixlening ichki masalasining klassik yechimini olamiz. Shunday qilib, Dirixlening ichki masalasi yechilishi (101) integral tenglamaning yechilishi masalasiga olib kelindi.

Ixtiyoriy uzluksiz $f(x)$ funksiya uchun (101) tenglama yagona yechimga ega ekanligini ko'rsatamiz. Qutb yadroli Fredgolm integral tenglamalari uchun Fredgolm teoremalari o'rinli. Fredgolmning birinchi teoremasiga ko'ra, (101) tenglamaning bir qiymatli yechilishi uchun unga mos

$$\nu(\xi_0) - \frac{1}{2\pi}\oint\limits_S \nu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}ds = 0$$

bir jinsli tenglama faqat nol yechimga ega bo'lishini ko'rsatish yetarli. Fredgolmning ikkinchi teoremaga ko'ra, dastlabki va unga qo'shma

$$\mu(\xi_0) - \frac{1}{2\pi}\oint\limits_S \mu(\xi)\frac{\partial}{\partial n_{\xi_0}}\frac{1}{|\xi-\xi_0|}ds = 0 \qquad (102)$$

integral tenglamalar bir xil xos sonlarga ega va ularning rangi ustma-ust tushadi. Bu yerda (102) tenglamani tekshirish osonroq.

(102) tenglamaning faqat nol yechimga ega bo‘lishini ko‘rsatamiz. Teskarisini faraz qilamiz: (102) tenglama $\mu_0(\xi) \neq 0$ yechimga ega bo‘lsin. Zichligi $\mu_0(\xi)$ bo‘lgan oddiy qatlam

$$I_1(x) = \oint\limits_S \mu_0(\xi) \frac{ds}{|x-\xi|} \tag{103}$$

potensialini qaraymiz. D_i va $D_l - D$ sohaninig mos ravishda ichki va tashqi qismlari bo‘lsin. $v(x)$ funksiya D_i da garmonik va D_l da $|x| \to \infty$ bo‘lganda nolga tekis yaqinlashadi. $v(x)$ funksiya normal hosilasining S dagi limit qiymatini qaraymiz. (94) formulaga ko‘ra

$$\left(\frac{I_1(x)}{\partial n_l}\right)_l = \oint\limits_S \mu_0(\xi) \frac{\partial}{\partial n_\xi} \frac{1}{|\xi - \xi_0|} ds - 2\pi\mu_0(\xi_0),\ \xi_0 \in S$$

$\mu_0(\xi)$ funksiya (102) tenglamaning yechimi ekanligidan S sirt nuqtalari uchun

$$\left(\frac{I_1(x)}{\partial n_l}\right)_l = 0$$

tenglik bajarilishi kelib chiqadi. Shunday qilib, $I_1(x)$ funksiya Neymanning bir jinsli tashqi

$$\triangle I_1(x) = 0,\ x \in D_l$$

$$\frac{\partial I_1(x)}{\partial n_l}|_S = 0,\ I_1(x) \rightrightarrows 0,\ |x| \to \infty \tag{104}$$

masalasi yechimi bo‘lar ekan. Neymanning tashqi masalasi yechimi yagona bo‘lgani uchun uch o‘lchovli holda (104) masala faqat

$$I_1(x) \equiv 0,\ x \in D_l \cup S$$

nol yechimga ega bo‘ladi. $I_1(x)$ funksiya (103) formula bilan aniqlangan oddiy qatlam potensiali sifatida S sirtni kesib o‘tishda uzluksiz. Shuning uchun bu funksiya D_i sohada ushbu

$$\triangle I_1(x) = 0,\ x \in D_i,$$

$$I_1(x)|_S = 0$$

Dirixlining bir jinsli masalasining yechimi bo‘ladi. Dirixlening ichki masalasi yechimining yagonaligiga ko‘ra $I_1(x) \equiv 0,\ x \in D_i \cup S$. Shunday qilib, butun fazoda $I_1(x) \equiv 0$. (95) formuladan foydalanib,

$$\mu_0(\xi) = 0,\ \xi \in S$$

bo‘lishini topamiz. Bu esa farazimizga zid. Demak, (102) tenglama faqat nol yechimga ega ekan. U holda Fredgolmning birinchi tenglamasiga binoan bir jinsli bo‘lmagan (101) integral tenglama ixtiyoriy uzluksiz f funksiya uchun yagona yechimga ega.

Bu yerdan ixtiyoriy $f \in C(S)$ lar uchun Dirixlening ichki (73) masalasi yagona klassik yechimga ega bo‘lishi kelib chiqadi.

Shunday qilib, quyidagi teorema o‘rinli.

T e o r e m a. Dirixlening ichki (73) masalasi ixtiyoriy uzluksiz f funksiya uchun yagona yechimga ega.

Ma’lumki, Dirixle masalasi uchun Grin funksiyasini qurish chegaraviy masalasini maxsus chegaraviy shart uchun yechishgaga ekvivalent. Bu yerdan Lyapunov sirtlari bilan chegaralangan sohalar uchun Dirixle masalasining Grin funksiyasi mavjudligi kelib chiqadi.

Shu bilan birga, yuqoridagi tekshirishlar jarayonida yo‘l-yo‘lakay quyidagi tasdiq ham isbotlandi:

T a s d i q. Agar uzluksiz zichlikka ega oddiy qatlam potensiali D_i yoki D_l sohalarda nolga teng bo‘lsa, uning zichligi S sirtda nolga teng.

6.13.9 Neymanning tashqi masalasi uchun integral tenglama

(74) masalani D_e soha uchun qaraymiz. Qo’shimcha ravishda $x \to \infty$ da $u \rightrightarrows 0$ bo’lishini ($u(x)$ funksiyaning nolga tekis yaqinlashishi) talab etamiz. Bu masala yechimini(67) oddiy qatlam $I_1(x)$ potensiali ko‘rinishida izlaymiz. Ma’lumki, $\mu(\xi)$ zichlik uzluksiz funksiya bo‘lganda $I_1(x)$ potensial D_e da

garmonik funksiya va cheksizlikda nolga tekis intiladi. $\mu(\xi)$ zichlikni

$$\lim_{x\to\xi_0, x\in D_e} \frac{\partial u(x)}{\partial n_e} = f(\xi_0),\ \xi_0 \in S \tag{105}$$

tenglamaning yechimi sifatida aniqlaymiz.(94) formulani inobatga olib, (105) dan

$$\oint_S \mu(\xi)\frac{\partial}{\partial_{n_{\xi_0}}}\frac{1}{|\xi-\xi_0|}ds - 2\pi\mu_0(\xi_0) = f(\xi_0)$$

yoki

$$\mu(\xi_0)_{\frac{1}{2\pi}} \oint_S \mu(\xi)\frac{\partial}{\partial_{n_{\xi_0}}}\frac{1}{|\xi-\xi_0|}ds = -\frac{1}{2\pi}f(\xi_0),\ \xi_0 \in S \tag{106}$$

integral tenglamani hosil qilamiz. (106) tenglamani yechib, olingan yechimni (67) ga olib borib qo‘ysak, (74) Neymanning tashqi masalasi klassik yechimiga ega bo‘lamiz. Sunday qilib, (74) Neymanning tashqi masalasi yechilishi ham integral tenglamaning yechilishiga olib kelindi. (106) integral tenglama Fredgolmning birinchi teoremasiga muvofiq ixtiyoriy uzluksiz $f(\xi)$ funksiya uchun yagona yechimga ega. Chunki unga mos keluvchi bir jinsli (102) tenglama yechimi, oldin isbot etilganiga ko‘ra, noldan iborat. Shunday qilib, quyidagi teorema isbotlandi.

Т е о р е м а. (74) Neymanning tashqi masalasi ixtiyoriy uzluksiz f funksiya uchun klassik yechimga ega.

6.13.10 Neymanning ichki va Dirixlening tashqi masalalari uchun integral tenglamalar

Oldingi paragraflardagi tekshirishlar shuni ko‘rsatdiki, Dirixlening ichki va Neymanning tashqi masalalari uchun yozilgan tenglamalar qo‘shma bo‘lar ekan. Tabiiyki, bu masalalarni bir vaqtning o‘zida o‘rganish mumkin va bunday holni Neymanning ichki va Dirixlening tashqi masalalari holida ham kutish mumkin.

(74) masalani D sohaning ichki D_i qismi uchun qaraymiz. Masala yechimini (67) oddiy qatlam $I_1(x)$ potensiali ko‘rinishida izlaymiz. U holda $\mu(\xi)$ zichlik (93) formuladan aniqlanadi. Unga ko‘ra $\mu(\xi)$ funksiya

$$\mu(\xi_0) + \frac{1}{\pi}\oint_S \mu(\xi)\frac{\partial}{\partial n_{\xi_0}}\frac{1}{|\xi-\xi_0|}ds = \frac{1}{2\pi}f(\xi_0),\ \xi_0 \in S \tag{107}$$

integral tenglamani qanoatlantiradi. (107) tenglamaning yechimini (67) ga qo'yib, Neymanning ichki masalasi yechimini hosil qilamiz. Eslatib o'tamiz, (74) masala D_i soha uchun hamma vaqt ham yagona yechimga ega bo'lavermaydi. Yagona yechim mavjud bo'lishi uchun

$$\oint_S f(\xi)d\xi = 0 \tag{108}$$

tenglik bajarilishi zarur.

Shu bilan bir vaqtda (73) masalani D sohaning tashqi D_e qismi uchun qaraymiz. Chegaraviy shartga qo'shimcha ravishda $|x| \to \infty$ da $u(x) \rightrightarrows 0$ bo'lishini ham talab etamiz. Bu masala yechimini quyidagi ko'rinishda qidiramiz:

$$u(x) = -\oint_S \nu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|x-\xi|}ds + \frac{\alpha}{|x-y|}, \tag{109}$$

Bu yerda $\alpha = const$, $y = (y_1, y_2, y_3) \in D_i$ - biror tayin nuqta, $|x-y|$ - x va y nuqtalar orasidagi masofa . Yechim ikkilangan qatlam va D_i sohadagi nuqtaviy zaryad potensiallari yig'indisi tarzida ifodalangan, bu yerda α miqdor keyinroq aniqlanadi.

Potensialning $\nu(\xi)$ zichligi

$$\lim_{x\to\infty,\ x\in D} u(x) = f(\xi_0),\ \xi_0 \in S$$

chegaraviy shartdan topiladi. (88) ni inobatga olib, (109) formuladan

$$-\oint_S \nu(\xi)\frac{\partial}{\partial n_{\xi_0}}\frac{1}{|\xi_0-\xi|}ds + 2\pi\nu(\xi_0) + \frac{\alpha}{|\xi_0-y|} = f(\xi_0)$$

tenglikka ega bo'lamiz. Bundan $\nu(\xi)$ funksiya uchun

$$\nu(\xi_0) + \frac{1}{2\pi}\oint_S \nu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi_0-\xi|}ds = \frac{1}{2\pi}\left[f(\xi_0) - \frac{\alpha}{|\xi_0-y|}\right],\ \xi_0 \in S \tag{110}$$

integral tenglamani olamiz. (107) va (110) lar qo'shma integral tenglamalar. Shuning uchun ularning yechilishini bir vaqtda tekshirish mumkin. Bir jinsli

$$\nu(\xi_0) + \frac{1}{2\pi}\oint_S \nu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}ds = 0,\ \xi_0 \in S \tag{111}$$

integral tenglamani qaraymiz. (84) formulaga ko'ra $\xi_0 \in S$ lar uchun

$$\frac{1}{2\pi}\oint\limits_S \frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0}ds = -1.$$

Shuning uchun, ravshanki, $\nu(\xi) = \nu_0 = const$ - (111) tenglamaning yechimi. Bu esa $\lambda = 1$ ning

$$K(\xi_0,\xi) = \frac{1}{2\pi}\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}$$

yadro uchun xos son bo'lishi va unga $\nu = \nu_0 = const$ xos funksiya mos kelishini bildiradi. Fredgolmning ikkinchi teoremasiga ko'ra $\lambda = 1$ soni

$$K^*(\xi_0,\xi) = \frac{1}{2\pi}\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}$$

qo'shma yadroning ham xos soni bo'ladi. Shuning uchun, avvalo, bu xos sonning rangini hisoblaymiz. $\lambda = 1$ xos sonning rangi birga teng ekanligini ko'rsatamiz. Buning uchun bir jinsli

$$\mu(\xi_0) + \frac{1}{2\pi}\oint\limits_S \mu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}ds = 0 \tag{112}$$

integral tenglama bitta xos funksiyaga ega ekanligini ko'rsatish yetarli. $\mu_0(\xi)$ funksiya (112) tenglamaning xos funksiyasi bo'lsin. Zichligi $\mu_0(\xi)$ ga teng bo'lgan oddiy qatlam

$$v(x) = \oint\limits_S \mu(\xi_0)\frac{1}{|x-\xi|}ds$$

potensialini tuzamiz. $\mu_0(\xi)$ funksiya (112) tenglamani qanoatlantirishi sababli

$$\left(\frac{\partial v}{\partial n_e}\right)_i = \oint\limits_S \mu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}ds + 2\pi\mu_0(\xi_0) \equiv 0$$

munosabatlar o'rinli. Demak, $v(x)$ funksiya Neymanning bir jinsli ichki

$$\Delta v(x) = 0,\ x \in D_i,\ \left.\frac{\partial v(x)}{\partial n_e}\right|_S = 0$$

masalasining yechimi bo'ladi. Shuning uchun $v(x) = C = const,\ x \in D \cup S$. Uzluksiz zichlikka ega bo'lgan oddiy qatlam potensiallarining D_i yoki D_e

sohada nolga teng bo'lishidan uning zichligi S da nolga teng bo'lishi kelib chiqishini hisobga olsak, $C \neq 0$ bo'ladi, aks holda $\mu_0 \equiv 0$ bo'lar edi.

(112) tenglamaning $\lambda = 1$ xos songa mos keluvchi noldan farqli $\tilde{\mu_0}(\xi)$ yechimi, ya'ni $\mu_0(\xi)$ bilan chiziqli bog'liq bo'lmagan funksiya mavjud bo'lsin. U holda oldingiga o'xshash ravishda

$$\tilde{v}(x) = \oint_S \tilde{\mu_0}(\xi)\frac{1}{|x-\xi|}ds \equiv \tilde{C} = const \neq 0, \ x \in D \cup S$$

kelib chiqadi. Ushbu

$$v_1(x) = \frac{\tilde{C}}{C}v(x) - \tilde{v}(x) = \oint_S \left(\frac{\tilde{C}}{C}\mu_0 - \tilde{\mu_0}\right)\frac{ds}{|x-\xi|} \tag{113}$$

funksiyani qaraymiz. Qurilishiga ko'ra, bu funksiya $v_1(x) \equiv 0, \ x \in D_i \cup S$ va (113)ga asosan $v_1(x)$ funksiya oddiy qatlam potensiali bo'lgani uchun uning zichligiga S da nolga teng, ya'ni

$$\frac{\tilde{C}}{c}\mu_0(\xi) - \tilde{\mu_0}(\xi) \equiv 0, \ \xi \in S.$$

Bu esa $\mu_0(\xi)$ va $\tilde{\mu_0}(\xi)$ funksiyalar chiziqli bog'liq ekanligini bildiradi. Demak, $\lambda = 1$ xos sonning rangi birga teng. (112) tenglamaning $\tilde{\mu_0}(\xi)$ xos fuksiyasini

$$\tilde{v(x)} = \oint_S \tilde{\mu_0}(\xi)\frac{1}{|x-\xi|}ds \equiv 1, \ x \in D_i \cup S \tag{114}$$

shartdan tanlaymiz. Zichligi $\mu_0(\xi)$ bo'lgan bunday potensialga *Roben potensiali* deyiladi. Shunday qilib, (112) tenglama yagona $\mu_0(\xi)$ xos funksiyaga ega, qo'shma bir jinsli

$$\nu(\xi_0) + \frac{1}{2\pi}\oint_S \nu(\xi)\frac{\partial}{\partial n_\xi}\frac{1}{|\xi-\xi_0|}ds = 0, \ \xi_0 \in S \tag{115}$$

tenglama esa yagona $\nu = \nu_0 = const$ xos funksiyaga ega, bunda $\nu_0 \equiv 1$ deb hisoblash mumkin. Bu yerdan esa, Fredgolmning uchinchi teoremasiga asosan, bir jinsli bo'lmagan (107) tenglamaning yechilish

$$\oint_S f(\xi) \cdot 1 \cdot ds = 0$$

sharti kelib chiqadi. Shunga o‘xshash, (110) tenglama uchun

$$\oint\limits_S \left[f(\xi) - \frac{\alpha}{|\xi - y|}\right] \mu_0(\xi) ds = 0 \tag{116}$$

shartni hosil qilamiz. (108) shart bajarilganda (107) tenglama yechimi quyidagi ko‘rinishga ega bo‘ladi:

$$\mu(\xi) = \overline{\mu}(\xi) + c\mu_0(\xi), \tag{117}$$

bu yerda $\overline{\mu}(\xi)$ - bir jinsli bo‘lmagan (107) tenglamaning biror yechimi, c - ixtiyoriy o‘zgarmas. (117) ni (107) ga qo‘yib, Neymanning ichki masalasi

$$u(x) = \oint\limits_S \overline{\mu}(\xi) \frac{ds}{|x - \xi|} + c \oint\limits_S \mu_0 \xi \frac{ds}{|x - \xi|}$$

yechimini yoki (114) ga asosan

$$u(x) = \oint\limits_S \overline{\mu}(\xi) \frac{ds}{|x - \xi|} + c \tag{118}$$

yechimni hosil qilamiz. Shunday qilib, (108) shart nafaqat Neymanning ichki masalasi yechilishinining zaruriy sharti, balki yetarli sharti ham ekan. Bu masala (108) shartni qanoatlantiruvchi ixtiyoriy uzluksiz f funksiya uchun (118) ko‘rinishidagi yechimga ega. Endi (116) shartni qaraymiz. Uni

$$\alpha \oint\limits_S \mu_0(\xi) \frac{ds}{|\xi - y|} = \oint\limits_S \mu_0(\xi) f(\xi) d\xi$$

ko‘rinishda yozib olamiz. $y \in D_i$ ekanligidan (114) tenglikka ko‘ra

$$\oint\limits_S \mu_0(\xi) \frac{ds}{|\xi - y|} \equiv 1$$

Shuning uchun

$$\alpha = \oint\limits_S \mu_0(\xi) f(\xi) d\xi \tag{119}$$

(110) tenglama ixtiyoriy uzluksiz f lar uchun yechimga ega. Biroq bu yechim yagona emas va u quyidagi ko‘rinishga ega:

$$\nu(\xi) = \overline{\mu}(\xi) + c_0, \tag{120}$$

bu yerda $\overline{\mu}(\xi)$ - (110) tenglamaning biror yechimi, c_0 - ixtiyoriy o‘zgarmas. (120) ni (109) ga qo‘yib, Dirixlening tashqi masalasining

$$u(x) = -\oint\limits_S \bar{\nu}(\xi)\frac{\partial}{\partial n}\frac{1}{|x-\xi|}ds - c_0\oint\limits_S \frac{\partial}{\partial n}\frac{1}{|x-\xi|}ds + \frac{\alpha}{|x-y|} \qquad (121)$$

yechimini hosil qilamiz. $x \in D_e$ ekanligidan (121) formula

$$U(x) = -\oint_(S)\bar{\nu}(\xi)\frac{\partial}{\partial n}\frac{1}{|x-\xi|}ds + \frac{\alpha}{|x-y|}$$

ko‘rinishni qabul qiladi. α soni (119) tenglik yordamida aniqlanadi. Shunday qilib, Dirixlening tashqi masalasi ixtiyoriy uzluksiz f funksiyalar uchun yagona yechimga ega.

Demak, quyidagi teoremalar isbotlandi.

T e o r e m a. Dirixlening (73) tashqi masalasi ixtiyoriy uzluksiz f funksiyalar uchun klassik yechimga ega.

T e o r a m a. Neymanning (74) ichki masalasi (108) shartni qanoatlantiruvchi ixtiyoriy uzluksiz f funksiyalar uchun yagona yechimga ega.

Biz faqat Laplas tenglamalari uchun chegaraviy masalalarni yechishning integral tenglamalar usulini bayon qildik. Puasson tenglamasi uchun chegaraviy masalalar o‘xshash tarzda yechiladi.

Misol uchun, Puasson tenglamasi uchun Dirixlening ichki

$$\Delta u = -F,\ x \in D_i = D,\ u|_S = f(x),\ f(x) \in C(S) \qquad (122)$$

masalasini qaraymiz.

$v(x)$ orqali zichligi $F(\xi)$ bo‘lgan hajm potensialini belgilaymiz:

$$v(x) = \frac{1}{4\pi}\int_D \frac{F(\xi_0)}{|x-\xi|}dv.$$

$u(x)$ funksiya o‘rniga

$$u(x) = v(x) + \omega(x)$$

formula bilan $\omega(x)$ funksiyani kiritamiz. U holda (122) dan hajm potensialining xossasini inobatga olib, $\omega(x)$ uchun chegaraviy

$$\Delta\omega(x) = 0,\ x \in D,\ \omega|_S = \tilde{f}(x),\ \tilde{f}(x) = f(x) - v(x)\Big|_{x\in S} \tag{123}$$

masalani olamiz.

Shunday qilib, (122) masala (123) Laplas tenglamasi uchun chegaraviy masalaga olib kelindi. O'xshash ravishda Puasson tenglamasi uchun boshqa chegaraviy masalalar ham Laplas tenglamasi uchun mos chegaraviy masalalarga keltiriladi.

6.14 Xususiy hosilali differensial tenglamalar yechimlari silliqligining xususiyati to'g'risida

1. Parabolik va elliptik tenglamalar bo'lgan hol.

5-bobning 8-paragrafida $\varphi(x) = u(x,0)$ funksiya uzluksiz va chegaralangan bo'lganda (9) issiqlik o'tkazuvchanlik tenglamasi uchun (10) Koshi masalasi $u(x,t)$ yechimining ixtiyoriy tartibdagi hosilalarining mavjudligi ko'rsatilgan edi.

Ushbu bobning 3-paragrafidagi 3-xossadan ma'lumki, D sohada garmonik $u(x)$ funksiya D sohada barcha o'zgaruvchilar bo'yicha ixtiyoriy tartibdagi hosilalarga ega bo'ladi. Shu bilan birga, D sohada garmonik funksiya shu sohada analitik funksiya bo'ladi, ya'ni absolyut yaqinlashuvchi darajali qator bilan ifodalanadi.

$u(x)$ funksiya D sohada analitik funksiya ekanligini ko'rsatish uchun bu sohada to'la yotuvchi ixtiyoriy sharda uning analitikligini ko'rsatish kifoyadir.

Bunga ishonch hosil qilish uchun $n = 2$ bo'lgan hol bilan cheklanamiz. Koordinata boshini D sohada joylashgan deb hisoblab, qutb koordinatalar sistemasida yozilgan (28) Puasson formulasida $\psi - \varphi = \theta$, $\frac{|x|}{r} = \rho$ belgilashlar kiritamiz. U holda, Puasson formulasining yadrosi ushbu

$$\frac{r^2 - |x|^2}{r^2 + |x|^2 - 2r|x|\cos(\psi - \varphi)} = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos\theta}$$

ko'rinishda yoziladi. Bu yadro $\rho < 1$ bo'lganda darajali qatorga yoyiladi. Haqiqatdan ham,

$$\frac{1-\rho^2}{1+\rho^2-2\rho\cos\theta} = -1 + 2Re\frac{1}{1-\rho e^{i\theta}} = -1 + 2Re\sum_{k=0}^{\infty}\rho^k e^{ik\theta} =$$

$$= -1 + 2\sum_{k=0}^{\infty}\rho^k\cos k\theta = 1 + 2\sum_{k=1}^{\infty}\rho^k\cos k(\psi-\varphi). \qquad (124)$$

(124) qator $\rho < 1$ da absolyut yaqinlashuvchi bo'ladi. Bunga asosan (28) Puasson formulasi

$$u(x) = \frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\alpha}\right)\left[1 + 2\sum_{k=1}^{\infty}\left(\frac{|x|}{r}\right)^k\cos k(\psi-\varphi)\right]d\varphi. \qquad (125)$$

ko'rinishga ega bo'ladi. (125) formulada $|x|\cos\psi,\ |x|\sin\psi$ larni x_1 va x_2 lar bilan almashtirib, $u(x)$ funksiyaning x_1, x_2 o'zgaruvchilar bo'yicha darajali qatorga yoyilmasini hosil qilamiz. Ravshanki, bu qator $|x| < r$ doirada absolyut yaqinlashadi. Shu bilan Puasson formulasidan $|x| < r$ doirada Laplas tenglamasi uchun Dirixle masalasining yechimi $u(x)$, $|x| = r$ aylanada berilgan $f(x)$ chegaraviy shartlarning qiymatlari faqat uzluksiz bo'lganda ham analitik ekanligi kelib chiqadi.

2. Giperbolik tenglamalar bo'lgan hol. 4-bobning 6-paragrafida ko'rilgan to'lqin tenglamari uchun Koshi, Gursa va boshqa masalalar uchun avvalgi bandda bayon qilingan fikrlar to'g'ri bo'lmaydi. Masalan, to'lqin tenglamasi uchun Koshi masalasining yechimi $u(x,t)$ Kirxgof formulasi bilan aniqlanadi. Bu formulada berilgan $\varphi(x)$ va $\psi(x)$, $f(x,t)$ funksiyalar mos ravishda uch va ikki marta uzluksiz differensiallanuvchi bo'lgandagina $u(x,t)$ yechim ikki marta uzluksiz differensialanuvchi bo'ladi.

Yoki tor tebranish tenglamasi uchun Gursa masalasining yechimi 4-bobdagi (33) formula bilan ifodalanadi. Bu formulada soddalik uchun $x_0 = y_0 = 0$ desak, (33) formula

$$u(x,t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0) \qquad (126)$$

ko'rinishda yoziladi.

Bu formuladan ko'rinadiki, $u(x,t)$ *yechimning silliqlik tartibi berilgan* $\varphi(x)$ *va* $\psi(x)$ *funksiyalarning silliqlik tartibi bilan bir xil bo'ladi,* ya'ni bu masala izlanayotgan $u(x,t)$ yechimning k tartibdagi hosilalarining mavjud bo'lishi uchun berilgan $\varphi(x)$ va $\psi(x)$ funksiyalarning k tartibli hosilalarining mavjudligini talab qilishga to'g'ri keladi.

Agar $\varphi(x)$ yoki $\psi(x)$ funksiya $x = 0$ nuqtada uzilishga ega bo'lsa, (126) formulaga asosan, $u(x,t)$ funksiya ham $x + t = 2\xi$ yoki $x - t = 2\xi$ xarakteristikalar bo'yicha uzilishga ega bo'ladi, yani $\varphi(x)$ va $\psi(x)$ funksiyalarning uzilishi $u(x,t)$ to'lqinning tor tebranishi tenglamasining xarakteristikalari bo'yicha uzilishiga sabab bo'ladi.

7-Bob. Maxsus funksiyalar

7.1 Eyler integrallari

7.1.1 Beta-funksiya (I-tur Eyler integrali)

Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx, \ a>0, \ b>0 \tag{1}$$

xosmas integralni qaraylik.

a va b sonlarning quyidagi hollarini ko‘ramiz:

1) $0<a<1$, $b\geq 1$ bo‘lganda $x=0$ maxsus nuqta; 2) $a\geq 1$, $0<b<1$ bo‘lganda $x=1$ maxsus nuqta; 3) $0<a<1$, $0<b<1$ bo‘lganda $x=0$ va $x=1$ nuqtalar maxsus nuqtalar bo‘ladi.

Binobarin, (1) parametrga bog‘liq bo‘lgan chegaralanmagan funksiyaning xosmas integralidir.

T a’ r i f. (1) integral beta-funksiya yoki I-tur Eyler integrali deb ataladi va $B(a,b)$ kabi belgilanadi, demak

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \quad (a>0, b>0).$$

Shunday qilib, $B(a,b)$ funksiya $\mathbb{R}^2$ fazodagi $M=\{(a,b)\in\mathbb{R}^2 : \ a\in(0,+\infty), \ b\in(0,+\infty)\}$ to‘plamda berilgandir.

Endi $B(a,b)$ funksiyaning xossalarini o‘rganamiz.

X o s s a. (1)

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integral ixtiyoriy

$$M_0 = \big\{(a,b) \in \mathbb{R}^2 : \ a \in (a_0, +\infty), \ b \in (b_0, +\infty)\big\} \ (a_0 > 0, \ b_0 > 0)$$

to‘plamda tekis yaqinlashuvchi bo‘ladi.

Isbot. Tekis yaqinlashuvchanlikka tekshirish uchun $B(a,b)$ integralni

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx + \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx$$

ko‘rinishda yozib olamiz.

Ravshanki, $a > 0$ bo‘lganida $\int_0^{1/2} x^{a-1}dx$ integral yaqinlashuvchi, $b > 0$ bo‘lganda esa $\int\limits_{1/2}^{1} (1-x)^{b-1}dx$ integral yaqinlashuvchi bo‘ladi. Parametr a ning $a \geq a_0$ $(a_0 > 0)$ qiymatlari va ixtiyoriy $b > 0$, $x \in (0, 1/2]$ lar uchun

$$x^{a-1}(1-x)^{b-1} \leq x^{a_0-1}(1-x)^{b_0-1} \leq 2x^{a_0-1}$$

tengsizliklar o‘rinli. Veyershtrass alomatidan foydalanib,

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx$$

integralning tekis yaqinlashuvchanligini topamiz. Shuningdek, parametr b ning $b \geq b_0\,(b_0 > 0)$ qiymatlari va ixtiyoriy $a > 0$, $x \in \left(\frac{1}{2}, 1\right]$ lar uchun

$$x^{a-1}(1-x)^{b-1} \leq x^{a_0-1}(1-x)^{b_0-1} \leq 2(1-x)^{b_0-1}$$

bo‘ladi. Yana Veyershtrass alomatidan foydalanib,

$$\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx$$

integralning tekis yaqinlashuvchan ekanligiga ishonch hosil qilamiz.

Demak,

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integral $a \geq a_0 > 0$ va $b \geq b_0 > 0$ bo‘lganida ya’ni,

$$M_0 = \{(a,b) \in \mathbb{R}^2 : a \in (a_0, +\infty), b \in (b_0, +\infty)\} \ (a_0 > 0, b_0 > 0)$$

to‘plamda tekis yaqinlashuvchi bo‘ladi.

E s l a t m a. $B(a,b)$ ning $M = \{(a,b) \in \mathbb{R}^2 : \ a \in (0, +\infty), \ b \in (0, +\infty)\}$ to‘plamda notekis yaqinlashuvchiligini ko‘rish qiyin emas.

X o s s a. $B(a,b)$ funksiya M to‘plamda uzluksiz funksiyadir, ya’ni $B(a,b) \in C(M)$.

Haqiqatdan ham,

$$B(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx$$

integralning M_0 to‘plamda tekis yaqinlashuvchi bo‘lishidan va integral ostidagi funksiyaning ixtiyoriy $(a,b) \in M$ da uzluksizligidan $B(a,b)$ funksiya

$$M = \{(a,b) \in \mathbb{R}^2 : a \in (0, +\infty), b \in (0, +\infty)\}$$

to‘plamda uzluksiz bo‘lishi kelib chiqadi.

X o s s a. Ixtiyoriy $(a,b) \in M$ lar uchun $B(a,b) = B(b,a)$, ya’ni beta-funksiya simmetrikdir.

Darhaqiqat, $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ integralda $x = 1-t$ almashtirish bajarib,

$$B(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx = \int\limits_0^1 t^{b-1}(1-t)^{a-1}dt = B(b,a)$$

bo‘lishini topamiz.

X o s s a. $B(a,b)$ funksiya quydagicha ham ifodalanadi:

$$B(a,b) = \int\limits_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}}dt \tag{2}$$

Haqiqatdan ham, agar (1) integralda $x = \frac{t}{1+t}$ almashtirish bajarilsa, u holda

$$B(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx =$$

$$= \int\limits_0^{+\infty} \left(\frac{t}{t+1}\right)^{a-1} \left(1 - \frac{t}{1+t}\right)^{b-1} \frac{dt}{(1+t)^2} = \int\limits_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt$$

tengliklar xossaning o'rinli ekanini ko'rsatadi.

Xususan, $b = 1 - a$ $(0 < a < 1)$ bo'lganda esa

$$B(a, 1-a) = \int\limits_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{sin a\pi}. \qquad (3)$$

(3) formulada $a = 1/2$ deb, quyidagini topamiz:

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

X o s s a. Ixtiyoriy

$$(a, b) \in M_1 = \left\{(a, b) \in \mathbb{R}^2 : \ a \in (0, +\infty), \ b \in (1, +\infty)\right\}$$

lar uchun

$$B(a, b) = \frac{b-1}{a+b-1} B(a, b-1) \qquad (4)$$

tenglik o'rinli.

(1) ni bo'laklab integrallaymiz:

$$B(a, b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1} dx = \int\limits_0^1 (1-x)^{b-1} d\left(\frac{x^a}{a}\right) =$$

$$= \frac{1}{a} x^a (1-x)^{b-1}\Big|_0^1 + \frac{b-1}{a} \int\limits_0^1 x^a (1-x)^{b-2} dx =$$

$$= \frac{b-1}{a} \int\limits_0^1 x^a (1-x)^{b-2} dx \ \ (a > 0, \ b > 0).$$

Agar

$$x^a (1-x)^{b-2} =$$

$$= x^{a-1} \left[1 - (1-x)\right] (1-x)^{b-2} = x^{a-1}(1-x)^{b-2} - x^{a-1}(1-x)^{b-1}$$

ekanligini etiborga olsak, u holda

$$\int_0^1 x^a(1-x)^{b-1}dx = \int_0^1 x^{a-1}(1-x)^{b-2}dx - \int_0^1 x^{a-1}(1-x)^{b-1}dx =$$

$$= B(a,b-1) - B(a,b)$$

bo‘lib, natijada

$$B(a,b) = \frac{b-1}{a}\left[B(a,b-1) - B(a,b)\right]$$

tenglik bajariladi. Bu tenglikdan esa

$$B(a,b) = \frac{b-1}{a+b-1}B(a,b-1) \qquad (a>0,\ b>1)$$

bo‘lishini topamiz.

Xuddi shunga o‘xshash, ixtiyoriy

$$(a,b) \in M_2 = \left\{(a,b)\in\mathbb{R}^2:\ a\in(1,+\infty),\ b\in(0,+\infty)\right\}$$

lar uchun

$$B(a,b) = \frac{a-1}{a+b-1}B(a-1,b) \qquad (a>1,\ b>0)$$

bo‘ladi.

Xususan, $b = n \in \mathbb{N}$ bo‘lganda

$$B(a,b) = B(a,n) = \frac{n-1}{a+n-1}B(a,n-1)$$

bo‘lib, (4) formulani takror qo‘llash hisobiga quyidagini topamiz:

$$B(a,n) = \frac{n-1}{a+n-1}\cdot\frac{n-2}{a+n-2}\cdots\frac{1}{a+1}B(a,1).$$

Ko‘rinib turibdiki,

$$B(a,1) = \int_0^1 x^{a-1}dx = \frac{1}{a}.$$

Demak,

$$B(a,n) = \frac{1\cdot 2\cdot 3\cdots(n-1)}{a(a+1)(a+2)\cdots(a+n-1)}. \qquad (5)$$

Agar (5) da $a = m (m \in N)$ bo'lsa, u holda

$$B(m,n) = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{m(m+1)(m+2)\cdots(m+n-1)} = \frac{(n-1)!(m-1)!}{(m+n-1)!}$$

tenglik bajariladi.

7.1.2 Gamma-funksiya (II-tur Eyler integrali) va uning xossalari

Ushbu

$$\int_0^{+\infty} x^{a-1}e^{-x}dx \tag{6}$$

xosmas integralni qaraylik. a ning $a < 1$ qiymatlarida, $x = 0$ nuqta integral ostidagi funksiyaning maxsus nuqtasi bo'ladi, chunki $x \to +0$ da integral ostidagi funksiya cheksizga intiladi. Demak, bu holda (6) integral ham cheksiz oraliq bo'yicha chegaralanmagan funksiyadan olingan xosmas integral ekan. Bu integralni ikki qismga

$$\int_0^{+\infty} x^{a-1}e^{-x}dx = \int_0^1 x^{a-1}e^{-x}dx + \int_1^{+\infty} x^{a-1}e^{-x}dx$$

ajratib, ularning har birini alohida-alohida yaqinlashuvchanlikka tekshiramiz.

Birinchi

$$\int_0^1 x^{a-1}e^{-x}dx$$

integralda, integral ostidagi funksiya uchun

$$\frac{1}{e} \cdot \frac{1}{x^{1-a}} \le x^{a-1}e^{-x} \le \frac{1}{x^{1-a}} \quad (0 < x \le 1)$$

tengsizlik o'rinli bo'ladi.

Ushbu

$$\int_0^1 \frac{1}{x^{1-a}}dx$$

integral $1-a<1$, ya'ni $a>0$ da yaqinlashuvchi, $1-a\geq 1$, ya'ni $a\leq 0$ da uzoqlashuvchi. Endi $\int_1^{+\infty} x^{a-1}e^{-x}dx$ integralni yaqinlashuvchilikka tekshiramiz.

Bunda,

$$\lim_{x\to+\infty}\frac{x^{a-1}e^{-x}}{\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{x^{a+1}}{e^x}=0$$

o'rinli ekanligini ko'rishimiz mumkin.

Ushbu $\int_1^{+\infty}\frac{1}{x^2}dx$ integral yaqinlashuvchi bo'lganligidan, $\int_1^{+\infty} x^{a-1}e^{-x}dx$ integral ham yaqinlashuvchidir. Shunday qilib, $\int_1^{+\infty} x^{a-1}e^{-x}dx$ integral a ning ixtiyoriy qiymatida yaqinlashuvchi. Natijada berilgan $\int_0^{+\infty} x^{a-1}e^{-x}dx$ integralning $a>0$ da yaqinlashuvchi bo'lishini topamiz.

T a' r i f. (6) integral Gamma-funksiya yoki II-tur Eyler integrali deb ataladi va $\Gamma(a)$ kabi belgilanadi. Demak,

$$\Gamma(a)=\int\limits_0^{+\infty} x^{a-1}e^{-x}dx.$$

Shunday qilib, $\Gamma(a)$ funksiya $(0;+\infty)$ da berilgan. Endi $\Gamma(a)$ funksiyaning xossalarini o'rganaylik.

X o s s a. (6) integral

$$\Gamma(a)=\int\limits_0^{+\infty} x^{a-1}e^{-x}dx.$$

ixtiyoriy $[a_0,b_0]$ $(0<a_0<b_0<+\infty)$ oraliqda tekis yaqinlashuvchi bo'ladi.

Isbot. (6) integralni quydagicha ikki qismga ajratib,

$$\int\limits_0^{+\infty} x^{a-1}e^{-x}dx=\int\limits_0^{1} x^{a-1}e^{-x}dx+\int\limits_1^{+\infty} x^{a-1}e^{-x}dx$$

ularning har birini alohida-alohida tekis yaqinlashuvchanlikka tekshiramiz.

Agar $a_0\,(a_0>0)$ sonni olib, parametr a ning $a\geq a_0$ qiymatlari qaralsa, unda barcha $x\in(0,1]$ uchun $x^{a-1}e^{-x}\leq\frac{1}{x^{1-a_0}}$ bo'lib, Veyershtrass alomatiga asosan

$$\int\limits_0^{1} x^{a-1}e^{-x}dx$$

integral tekis yaqinlashuvchi bo'ladi.

Agar b_0 $(b_0 > 0)$ sonni olib, parametr a ning $a \le b_0$ qiymatlari qaraladigan bo'lsa, unda barcha $x \ge 1$ lar uchun

$$x^{a-1}e^{-x} \le x^{b_0-1}e^{-x} \le \left(\frac{b_0+1}{e}\right)^{b_0+1} \cdot \frac{1}{x^2}$$

bo'lib,

$$\int\limits_1^{+\infty} \frac{1}{x^2}dx$$

integralning yaqinlashuvchanligidan, yana Veyershtrass alomatiga ko'ra

$$\int\limits_0^{+\infty} x^{a-1}e^{-x}dx$$

integralning tekis yaqinlashuvchi bo'lishini kelib chiqadi. Shunday qilib,

$$\Gamma(a) = \int\limits_0^{+\infty} x^{a-1}e^{-x}dx$$

integral $[a_0, b_0]$ $(0 < a_0 < b_0 < +\infty)$ da tekis yaqinlashuvchi bo'ladi.

E s l a t m a. $\Gamma(a)$ ning $(0, +\infty)$ da notekis yaqinlashuvchiligini ko'rish qiyin emas.

X o s s a. $\Gamma(a)$ funksiya $(0, +\infty)$ da uzluksiz hamda barcha tartibdagi uzluksiz hosilalarga ega va $n-$tartibli hosilasi

$$\Gamma^n(a) = \int\limits_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx$$

ga teng.

Ixtiyoriy $a \in (0, +\infty)$ nuqtani olaylik. Unda shunday $[a_0, b_0]$ $(0 < a_0 < b_0 < +\infty)$ oraliq topiladiki, bunda $a \in [a_0, b_0]$ bo'ladi. Ravshanki,

$$\Gamma(a) = \int\limits_0^{+\infty} x^{a-1}e^{-x}dx$$

integral ostidagi $f(x,a) = x^{a-1}e^{-x}$ funksiya

$$M = \{(x,a) \in \mathbb{R}^2 : \ x \in (0,+\infty),\, a \in (0,+\infty)\}$$

to‘plamda uzluksiz funksiyadir. (6) integral esa (yuqorida isbot etilganiga ko‘ra) $[a_0,\ b_0]$ kesmada tekis yaqinlashuvchi. U holda, $\Gamma(a)$ funksiya $[a_0,\ b_0]$ da shu bilan bir qatorda a nuqtada uzluksiz bo‘ladi. (6) integral ostidagi $f(x,a) = x^{a-1}e^{-x}$ funksiya

$$f'_a(x,a) = x^{a-1}e^{-x}\ln x$$

hosilasining M to‘plamda uzluksiz funksiya ekanligini payqash qiyin emas.

Endi

$$\int\limits_0^{+\infty} f'_a(x,a)dx = \int\limits_0^{+\infty} x^{a-1}e^{-x}\ln x dx$$

integralni $[a_0, b_0]$ da tekis yaqinlashuvchi bo‘lishini ko‘rsatamiz. Ushbu

$$\int\limits_0^{+\infty} x^{a-1}e^{-x}\ln x dx$$

integral ostidagi $x^{a-1}e^{-x}\ln x$ funksiya uchun $0 < x \le 1$ da

$$\left|x^{a-1}e^{-x}\ln x\right| \le x^{a_0-1}\left|\ln x\right|$$

tengsizlik o‘rinlidir. $\psi_1 = x^{a_0/2}\left|\ln x\right|$ funksiya $0 < x \le 1$ da chegaralanganligi va $\int\limits_0^1 x^{\frac{a_0}{2}-1}dx$ integralning yaqinlashuvchiligidan $\int\limits_0^1 x^{a_0-1}\left|\ln x\right|dx$ ning ham yaqinlashuvchi bo‘lishini va Veyershtrass alomatiga ko‘ra qaralayotgan

$$\int\limits_0^1 x^{a-1}e^{-x}\left|\ln x\right|dx$$

integralning tekis yaqinlashuvchi bo‘lishiga ishonch hosil qilamiz.

Shunga o‘xshash quyidagi

$$\int\limits_0^{+\infty} x^{a-1}e^{-x}\ln x dx$$

integralda, integral ostidagi $x^{a-1}e^{-x}\ln x$ funksiya uchun barcha $x \geq 1$ da

$$x^{a-1}e^{-x}\ln x \leq x^{b_0-1}e^{-x}\ln x < x^{b_0}e^{-x} \leq \left(\frac{b_0+2}{e}\right)^{b_0+2} \cdot \frac{1}{x^2}$$

bo'lib, $\int\limits_0^{+\infty} \frac{dx}{x^2}$ integralning yaqinlashuvchanligidan, yana Veyershtrass alomatiga ko'ra, $\int\limits_1^{+\infty} x^{a-1}e^{-x}dx$ ning tekis yaqinlashuvchiligi kelib chiqadi. Demak, $[a_0, b_0]$ da

$$\int\limits_0^{+\infty} x^{a-1}e^{-x}\ln x dx$$

integral tekis yaqinlashuvchidir. Unda

$$\Gamma'(a) = \left(\int\limits_0^{+\infty} x^{a-1}e^{-x}dx\right)' = \int\limits_0^{+\infty} (x^{a-1}e^{-x})'dx = \int\limits_0^{+\infty} x^{a-1}e^{-x}\ln x dx$$

bo'ladi va $\Gamma'(a)$ $[a_0, b_0]$ da shu bilan birga a nuqtada uzluksizdir.

Xuddi shu yo'l bilan $\Gamma(a)$ funksiyaning ikkinchi, uchinchi va hokazo tartibdagi hosilalarining mavjudligi, uzluksizligi hamda

$$\Gamma^{(n)}(a) = \int\limits_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx, \qquad n = 1, 2, 3, ...$$

bo'lishi ko'rsatildi.

X o s s a. $\Gamma(a)$ funksiya uchun ushbu

$$\Gamma(a+1) = a \cdot \Gamma(a), \quad a > 0$$

formula o'rinli.

Haqiqatan ham,

$$\Gamma(a) = \int\limits_0^{+\infty} x^{a-1}e^{-x}dx = \int\limits_0^{+\infty} e^{-x}d\left(\frac{x^a}{a}\right)$$

integralni bo'laklab integrallasak,

$$\Gamma(a) = e^{-x} \cdot \frac{x^a}{a}\Big|_0^{+\infty} + \int\limits_0^{+\infty} \frac{x^a}{a} \cdot e^{-x}dx = \frac{1}{a}\Gamma(a+1)$$

hosil bo‘lib, undan

$$\Gamma(a+1) = a\Gamma(a) \tag{7}$$

bo‘lishi kelib chiqadi.

Bu formula yordamida $\Gamma(a+n)$ ni topish mumkin. Darhaqiqat, (7) formulani takror qo‘llab,

$$\Gamma(a+2) = \Gamma(a+1)\cdot(a+1)$$

$$\Gamma(a+3) = \Gamma(a+2)\cdot(a+2)$$

$$\Gamma(a+4) = \Gamma(a+3)\cdot(a+3)$$

....

$$\Gamma(a+n) = \Gamma(a+n-1)\cdot(a+n-1)$$

bo‘lishini, bulardan esa

$$\Gamma(a+n) = (a+n-1)(a+n-2)\cdots(a+2)(a+1)a\Gamma(a)$$

ekanligini topamiz. Xususan, $a = 1$ bo‘lganda

$$\Gamma(n+1) = n(n-1)\cdots 2\cdot 1\cdot\Gamma(1)$$

bo‘ladi. Agar $\Gamma(1) = \int_0^{+\infty} e^{-x}dx = 1$ bo‘lishini e’tiborga olsak, unda

$$\Gamma(n+1) = n!$$

ekanligi kelib chiqadi.

Yana (7) formuladan foydalanib, $\Gamma(2) = \Gamma(1) = 1$ ekanini ko‘rish qiyin emas.

Shunday qilib, $\Gamma(a)$ funksiya $(0,+\infty)$ oraliqda berilgan bo‘lib, shu oraliqda istalgan tartibdagi hosilaga ega. Bu funksiyaning $a = 1$ va $a = 2$ nuqtadagi qiymatlari bir-biriga tengligi uchun unga matematik tahlil fanidan ma’lum bo‘lgan Roll teoremasini qo‘llaymiz. Demak, Roll teoremasiga ko‘ra, shunday $a^*(1 < a^* < 2)$ topiladiki, $\Gamma'(a^*) = 0$ bo‘ladi. Ixtiyoriy $a \in (0,+\infty)$ da

$$\Gamma''(a) = \int_0^{+\infty} x^{a-1}e^{-x}\ln^2 x dx > 0$$

bo‘lishi sababli, $\Gamma'(a)$ funksiya $(0, +\infty)$ oraliqda qat’iy o‘suvchi bo‘ladi. Demak, $\Gamma'(a)$ funksiya $(0, +\infty)$ da a^* nuqtadan boshqa nuqtalarda nolga aylanmaydi, ya’ni

$$\Gamma'(a) = \int_0^{+\infty} x^{a-1}e^{-x}\ln x dx = 0$$

tenglama $(0, +\infty)$ oraliqda a^* dan boshqa yechimga ega emas. U holda

$$0 < a < a^* \text{ da } \Gamma'(a) < 0,$$

$$a^* < a < +\infty \text{ da } \Gamma'(a) > 0$$

bo‘ladi. Demak, $\Gamma(a)$ funksiya a^* nuqtada minimumga ega. Uning minimum qiymati $\Gamma(a^*)$ ga teng.

Taqribiy hisoblash usuli bilan $a^* = 1,4616...$ va $\Gamma(a^*) = min\Gamma(a) = 0,8856...$ bo‘lishi topilgan.

$\Gamma(a)$ funksiya $a > a^*$ da o‘suvchi bo‘lganligi sababli $a > n+1,\ n \in \mathbb{N}$ bo‘lganda $\Gamma(a) > \Gamma(n+1) = n!$ bo‘lib, undan

$$\lim_{n\to\infty} \Gamma(a) = +\infty$$

bo‘lishini ko‘rish mumkin.

Ikkinchi tomondan, $a \to +0$ da $\Gamma(a+1) \to \Gamma(1) = 1$ hamda $\Gamma(a) = \frac{\Gamma(a+1)}{a}$ ekanligidan

$$\lim_{a\to+0} \Gamma(a) = +\infty$$

kelib chiqadi.

7.1.3 Beta va gamma funksiyalar orasidagi bog‘lanish

Quyida $B(a, b)$ va $\Gamma(a)$ funksiyalar orasidagi bog‘lanishni ifodalovchi formulani keltiramiz.

Ma’lumki, $\Gamma(a)$ funksiya $(0, +\infty)$ da, $B(a, b)$ funksiya esa $\mathbb{R}^2$ fazodagi

$$M = \{(a, b) \in \mathbb{R}^2 :\ a \in (0, +\infty),\ b \in (0, +\infty)\}$$

to‘plamda berilgan.

T e o r e m a. Ixtiyoriy $(a, b) \in M$ uchun

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

formula o‘rinlidir.

Isbot. Ushbu

$$\Gamma(a+b) = \int_0^{+\infty} x^{a+b-1} e^{-x} dx \quad (a > 0, b > 0)$$

integralda $x = (1+t)y, \ t > 0$ formula bilan o‘zgaruvchini almashtirib, quyidagiga ega bo‘lamiz:

$$\Gamma(a+b) = \int_0^{+\infty} (1+t)^{a+b-1} y^{a+b-1} e^{-(1+t)y} (1+t) dy =$$

$$= (1+t)^{a+b} \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy.$$

Oxirgi tenglikning chap va o‘ng tomonlarini noldan farqli $(1+t)^{a+b}$ ifodaga bo‘lib,

$$\frac{\Gamma(a+b)}{(1+t)^{a+b}} = \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy$$

formulani hosil qilamiz. Bu tenglikning ikkala tomonini t^{a-1} ga ko‘paytirib, natijani $(0, +\infty)$ oraliq bo‘yicha integrallaymiz. Natijada

$$\Gamma(a+b) \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt = \int_0^{+\infty} \left[\int_0^{+\infty} y^{a+b-1} e^{-((1+t)y} dy \right] t^{a-1} dt$$

tenglikni olamiz. (2) formulaga ko‘ra

$$\int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt = B(a;\, b)$$

bo‘lishini e’tiborga olsak, unda

$$\Gamma(a+b) \cdot B(a, b) = \int_0^{+\infty} \left[\int_0^{+\infty} y^{a+b-1} e^{-((1+t)y} dy \right] t^{a-1} dt \tag{8}$$

bo‘ladi. Endi (8) tenglikning o‘ng tomonidagi integral $\Gamma(a)\Gamma(b)$ ga teng ekanligini ko‘rsatamiz. Buning uchun, avvalo, bu integralda integrallash tartibini o‘zgartirish mumkinligini ko‘rsatamiz. Xosmas integrallar tartibini o‘zgartirish haqidagi teorema shartlari qanoatlantirishini tekshiramiz.

Dastlab $a > 1,\ b > 1$ bo‘lgan holni ko‘ramiz. $a > 1,\ b > 1$ da, ya’ni

$$\big\{(a,b) \in \mathbb{R}^2 :\ a \in (1,+\infty),\, b \in (1,+\infty)\big\}$$

to‘plamda integral ostidagi

$$f(t,y) = y^{a+b-1}t^{a-1}e^{-(1+t)y}$$

funksiya ixtiyoriy $(t,y) \in \big\{(t,y) \in \mathbb{R}^2 : t \in [0,+\infty), y \in [0,+\infty)\big\}$ da uzluksiz bo‘lib, $f(t,y) = y^{a+b-1}t^{a-1}e^{-(1+t)y} \geq 0$.

Ushbu

$$\int\limits_0^{+\infty} f(t,y)dy = \int\limits_0^{+\infty} y^{a+b-1}t^{a-1}e^{-(1+t)y}dy$$

integral t o‘zgaruvchining $[0,+\infty)$ oraliqda uzluksiz funksiyasi bo‘ladi, chunki

$$\int\limits_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dy = \Gamma(a+b)\cdot\frac{t^{a-1}}{(1+t)^{a+b}}.$$

$$\int\limits_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dt = \Gamma(a)\cdot y^{b-1}e^{-y}$$

formula o‘rinli ekanligidan ushbu

$$\int\limits_0^{+\infty} f(t,y)dt = \int\limits_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dt$$

integralning y o‘zgaruvchi bo‘yicha $[0,+\infty)$ oraliqda uzluksiz funksiya bo‘lishi kelib chiqadi. Nihoyat, yuqoridagi (8) munosabatga ko‘ra

$$\int\limits_0^{+\infty}\left[\int\limits_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dy\right]dt$$

integral yaqinlashuvchi. U holda

$$\int_0^{+\infty}\left[\int_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dt\right]dy$$

integral ham yaqinlashuvchi bo‘lib,

$$\int_0^{+\infty}\left[\int_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dy\right]dt =$$

$$= \int_0^{+\infty}\left[\int_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dt\right]dy$$

tenglik bajariladi. O‘ng tomondagi integralni quyidagicha soddalashtiramiz:

$$\int_0^{+\infty}\left[\int_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dy\right]dt =$$

$$= \int_0^{+\infty}\left[\int_0^{+\infty} t^{a-1}y^{a+b-1}e^{-(1+t)y}dt\right]dy =$$

$$= \int_0^{+\infty} y^{a+b-1}e^{-y}\left[\int_0^{+\infty} t^{a-1}e^{-ty}dt\right]dy =$$

$$= \int_0^{+\infty} y^{a+b-1}e^{-y}\frac{1}{y^a}\left[\int_0^{+\infty} (ty)^{a-1}e^{ty}d(ty)\right]dy =$$

$$= \int_0^{+\infty} y^{b-1}e^{-y}\Gamma(a)dy = \Gamma(a)\Gamma(b). \tag{9}$$

Natijada (8) va (9) munosabatlardan

$$\Gamma(a+b)B(a,b) = \Gamma(a)\Gamma(b),$$

ya’ni,

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \tag{10}$$

bo‘lishi kelib chiqadi. Biz bu formulani $a > 1$, $b > 1$ bo‘lgan hol uchun isbotladik, endi umumiy holni ko‘ramiz.

Faraz qilaylik, $a > 0$, $b > 0$ bo‘lsin. U holda isbot qilingan (10) formulaga ko‘ra

$$B(a+1, b+1) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}. \tag{11}$$

Shuningdek, $\Gamma(a)$, $B(a,b)$ funksiyalar uchun quyidagi tengliklar o‘rinli:

$$B(a+1, b+1) = \frac{a}{a+b+1}B(a, b+1) = \frac{a}{a+b+1}\frac{b}{a+b}B(a,b),$$

$$\Gamma(a+1) = a\Gamma(a),\ \Gamma(b+1) = b\Gamma(b),$$

$$\Gamma(a+b+2) = (a+b+1)\Gamma(a+b+1) = (a+b+1)(a+b)\Gamma(a+b)$$

Natijada, (11) formula quyidagi

$$\frac{a \cdot b}{(a+b)(a+b+1)}B(a,b) = \frac{a \cdot \Gamma(a) \cdot b \cdot \Gamma(b)}{(a+b)(a+b+1)\Gamma(a+b)}$$

ko‘rinishga keladi. Bu esa (10) formula $a > 0$, $b > 0$ da ham o‘rinli ekanligini ko‘rsatadi.

N a t i j a. Ixtiyoriy $a \in (0,1)$ uchun

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi} \tag{12}$$

bo‘ladi.

Haqiqatdan ham, (10) formulada $b = 1 - a$ deyilsa, unda

$$B(a, 1-a) = \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)}$$

bo‘lib, (3) va $\Gamma(1) = 1$ munosabatga muvofiq,

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}\ (0 < a < 1).$$

Odatda, (12) formula keltirish formulasi deyiladi. Xususan, (12) da $a = \frac{1}{2}$ deb olsak, unda

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

bo‘lishini ko‘ramiz.

N a t i j a. Ushbu

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a),\ \ a>0$$

formula o'rinli.

Dastlab (10) munosabatda $a=b$ deb hisoblab,

$$B(a,a)=\frac{\Gamma(a)\Gamma(a)}{\Gamma(2a)}$$

bo'lishiga ishonch hosil qilamiz. So'ngra,

$$B(a,a)=\int_0^1[x(1-x)]^{a-1}\,dx=\int_0^1\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx=$$

$$=2\int_0^{\frac{1}{2}}\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx=2\int_0^{\frac{1}{2}}\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx$$

tengliklarning oxirgi integralida $\frac{1}{2}-x=\frac{1}{2}\sqrt{t}$ almashtirish bajarib,

$$B(a,a)=2\int_0^1\left[\frac{1}{4}(1-t)\right]^{a-1}\frac{1}{4}t^{-\frac{1}{4}}dt=$$

$$=\frac{1}{2^{2a-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{a-1}dt=\frac{1}{2^{2a-1}}B\left(\frac{1}{2},a\right)$$

ga ega bo'lamiz. Natijada,

$$\frac{\Gamma^2(a)}{\Gamma(2a)}=\frac{1}{2^{2a-1}}B\left(\frac{1}{2},a\right)$$

formula hosil bo'ladi. Yana (10) formulaga ko'ra

$$B\left(\frac{1}{2},a\right)=\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma(a)}{\Gamma\left(\frac{1}{2}+a\right)}=\sqrt{\pi}\frac{\Gamma(a)}{\Gamma\left(\frac{1}{2}+a\right)} \qquad (13)$$

bo'lib, (13) munosabatdan

$$\frac{\Gamma(a)}{\Gamma(2a)}=\frac{1}{2^{2a-1}}\cdot\sqrt{\pi}\frac{1}{\Gamma\left(a+\frac{1}{2}\right)}$$

ekanligi kelib chiqadi. Demak,

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a). \qquad (14)$$

Odatda, (14) formula *Lejandr formulasi* deyiladi.

7.2 Bessel funksiyasi

7.2.1 Bessel tenglamasi

Ushbu

$$y'' + \frac{1}{x}y' + \left(1 - \frac{v^2}{x^2}\right)y = 0 \tag{15}$$

tenglamaga *Bessel tenglamasi* deyiladi. Matematik fizikaning ba'zi masalalarini o'zgaruvchilarni ajratish usuli (Furye usuli) bilan yechishda Bessel tenglamasiga kelinadi. (15) tenglamaga ekvivalent bo'lgan quyidagi tenglamalar ham uchrab turadi:

$$x^2y'' + xy' + (x^2 - v^2)y = 0, \tag{16}$$

$$(xy')' + (x - \frac{v^2}{x})y = 0. \tag{17}$$

(15) Bessel tenglamasining har qanday nol bo'lmagan yechimiga *silindrik funksiya* deyiladi.

7.2.2 Bessel funksiyasi

v indeksli Bessel funksiyasi

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+v} \tag{19}$$

qator bilan aniqlanadi, bunda $\Gamma(z)-$ Eylerning gamma-funksiyasi.

Qulaylik uchun Bessel funksiyasini quyidagicha ko'rinishda yozib olamiz:

$$J_v(z) = \left(\frac{x}{2}\right)^v f_v\left(\frac{x^2}{4}\right),$$

bunda

$$f_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(k+v+1)\Gamma(k+1)}. \tag{20}$$

Dalamber alomatiga ko'ra (20) qator barcha $|z| < R$, $|v| \leq N$ larda tekis yaqinlashuvchi, bunda R, N -ixtiyoriy musbat sonlar. Ko'rinib turibdiki, (20) qatorning har bir hadi butun funksiyalardan iborat bo'lib, bu funksiyalar barcha kompleks tekislikda analitikdir. Bundan tashqari, (20) qatorning har bir hadini tayin v ga mos z o‘zgaruvchining butun funksiyasi deb qarasak bo‘ladi yoki tayin z ga mos v o‘zgaruvchining butun funksiyasi desak ham bo‘ladi. U holda $f_v(z)$ ham komleks o‘zgaruvchili butun funksiyadan iborat bo‘ladi.

$J_v(x)$ funksiyaning (17) tenglamani qanoatlantirishini ko‘rsatamiz. Buning uchun (19) qatorni hadma-had differensiallaymiz va hosil bo‘lgan ifodalarnini soddalashtiramiz. Natijada

$$xJ_v'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k+v)}{\Gamma(k+v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+v},$$

$$x\frac{d}{dx}(xJ_v'(x)) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k+v)^2}{\Gamma(k+v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+v}$$

ifodalarni hosil qilamiz. U holda

$$x\frac{d}{dx}(xJ_v'(x)) - v^2 J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k[(2k+v)^2 - v^2]}{\Gamma(k+v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+v} =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k 4k(k+v)}{\Gamma(k+v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+v}. \tag{21}$$

Gamma - funksiyaning xossasiga asosan

$$\Gamma(k+v+1) = (k+v)\Gamma(k+v),\ \Gamma(k+1) = k\Gamma(k)$$

ekanligidan foydalansak, (21) quyidagicha soddalashadi:

$$x\frac{d}{dx}(xJ_v'(x)) - v^2 J_v(x) = 4\sum_{k=1}^{\infty} \frac{(-1)^k}{\Gamma(k+v)\Gamma(k)} \left(\frac{x}{2}\right)^{2k+v}.$$

$k-1=m$ belgilash kiritib,

$$x\frac{d}{dx}(xJ_v'(x)) - v^2 J_v(x) = 4\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\Gamma(m+v+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+v+2} =$$

$$= -x^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+v+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+v} = -x^2 J_v(x)$$

ayniyatlarni hosil qilamiz. Bu esa $J_v(x)$ funksiya Bessel tenglamasini qanoatlantirishini koʻrsatadi.

(15) tenglamada v ni $-v$ bilan almashtirsak, u holda $J_{-v}(x)$ funksiya ham (15) tenglamaning yechimi boʻladi.

$$J_{-v}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k-v+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k-v} \tag{22}$$

Koʻrinib turibdiki, (19) va (22) funksiyalarning har biri v ning butun boʻlmagan qiymatlarida $x = 0$ nuqta atrofida oʻzini turlicha tutadi:

$$J_v(x) = \frac{x^v}{2^v\Gamma(v+1)}[1 + O(x^2)], \ x \to 0, \tag{23}$$

$$J_{-v}(x) = \frac{2^v}{x^v\Gamma(1-v)}[1 + O(x^2)], \ x \to 0. \tag{24}$$

Bu funksiyalarning birinchisi $x = 0$ nuqta atrofida chegaralangan, ikkinchisi esa bu nuqta atrofida chegaralanmagan. Shuning uchun v ning butun boʻlmagan qiymatlarida $J_v(x)$ va $J_{-v}(x)$ funksiyalarning har biri oʻzaro bogʻliq boʻlmagan holatda (15) tenglama yechimlarining fundamental sistemasini tashkil etadi. Yuqorida keltirilganlarni hisobga olib, (15) tenglamaning yechimini butun boʻlmagan v lar uchun $J_v(x)$ va $J_{-v}(x)$ funksiyalarning chiziqli kombinatsiyasi koʻrinishida ifodalash mumkin:

$$y(x) = C_1 J_v(x) + C_2 J_{-v}(x).$$

7.3 Bessel funksiyasining asosiy xossalari va rekkurrent formulalar

Quyidagi lemma Bessel funksiyasi asosiy xossalarini ifoda etib, "Xususiy hosilali differensial tenglamalar" uchun turli masalalarni echishda keng qoʻllaniladi.

L e m m a. Bessel funksiyalari uchun quyidagi munosabatlar oʻrinli:

$$J_{-n}(x) = (-1)^n J_n(x), \ n \in \mathbb{Z}, \tag{25}$$

$$\frac{d}{dx}\left(\frac{J_v(x)}{x^v}\right) = -\frac{J_{v+1}(x)}{x^v}, \tag{26}$$

$$\frac{d}{dx}(x^v J_v(x)) = x^v J_{v-1}(x), \tag{27}$$

$$J_{v+1}(x) = \frac{v}{x}J_v(x) - J_v'(x), \tag{28}$$

$$J_{v-1}(x) = \frac{v}{x}J_v(x) + J_v'(x), \tag{29}$$

$$J_{v+1} + J_{v-1}(x) = -2J_v'(x), \tag{30}$$

$$J_{v+1}(x) - J_{v-1}(x) = -2J_v'(x), \tag{31}$$

$$\int x^{v+1} J_v dx = x^{v+1} J_{v+1}(x) + C, \tag{32}$$

$$\int xJ_0(x)dx = xJ_1(x) + C, \tag{33}$$

$$\int x^2 J_1(x)dx = -x^2 J_0(x) + 2xJ_1(x) + C, \tag{34}$$

$$\int x^3 J_0(x)dx = 2x^2 J_0(x) + (x^3 - 4x)J_1(x) + C, \tag{35}$$

$$\int xJ_v(\alpha x)J_v(\beta x)dx = \frac{\beta x J_v(\alpha x)J_v'(\beta x) - \alpha x J_v'(\alpha x)J_v(\beta x)}{\alpha^2 - \beta^2} + C, \tag{36}$$

bunda $\alpha \neq \beta$,

$$\int xJ_v^2(\alpha x)dx = \frac{1}{2}\left[x^2(J_v'(\alpha x))^2 + \left(x^2 - \frac{v^2}{x^2}\right)(J_v(\alpha x))^2\right] + C, \tag{37}$$

$$\int\limits_0^{x_0} xJ_v\left(\frac{\mu_k x}{x_0}\right)J_v\left(\frac{\mu_m x}{x_0}\right)dx = 0, \;\; k \neq m, \tag{38}$$

bunda μ_m va μ_k- lar quyidagi tenglamaning musbat ildizlari:

$$\alpha J_v(\mu) + \beta\mu J_v'(\mu) = 0.$$

Isbot. (25) ni isbotlaymiz. Umumiy holda $n > 0$ holatni ko‘raylik va (22) qatorda $v = n$ bo‘lsin. (22) qatorning dastlabki n ta hadi uchun gamma-funksiya xossasiga asosan $\frac{1}{\Gamma(k-n+1)} = 0$, bunda $k = 0, 1, 2..., n-1$, ekanligidan,

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k-n+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k-n}$$

bo‘ladi. Bu yig‘indida $k - n = m$ belgilash kiritish natijasida

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2(m+n)-n} =$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} = (-1)^n J_n(x).$$

(26) formulani tekshirish uchun (19) ning chap va o‘ng tomonlariga $\frac{1}{\nu}$ ni ko‘paytiramiz. Natijada

$$\frac{J_v(x)}{x^v} = \left(\frac{1}{2}\right)^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k}$$

ni hosil qilamiz. Bu qatorning ikkala tomonini differensiallasak, u holda

$$\frac{d}{dx}\left[\frac{J_v(x)}{x^v}\right] = \left(\frac{1}{2}\right)^{v} \sum_{k=1}^{\infty} \frac{(-1)^k k}{\Gamma(k+1)\Gamma(k+v+1)} \left(\frac{x}{2}\right)^{2k-1} =$$

$$= \left(\frac{1}{2}\right)^{v} \sum_{k=1}^{\infty} \frac{(-1)^k}{\Gamma(k)\Gamma(k+v+1)} \left(\frac{x}{2}\right)^{2k-1}.$$

Oxirgi yig‘indi indeksida $k - 1 = m$ almashtirish bajargandan so‘ng ifoda quyidagi ko‘rinishni oladi:

$$\frac{d}{dx}\left[\frac{J_v(x)}{x^v}\right] = \left(\frac{1}{2}\right)^{v} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\Gamma(m+1)\Gamma(m+v+2)} \left(\frac{x}{2}\right)^{2m+1} =$$

$$= -\left(\frac{1}{2}\right)^{v} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+v+2)} \left(\frac{x}{2}\right)^{2m+1}. \tag{39}$$

(26) formulaning o‘ng tomonidagi ifoda ushbu yoyiladi

$$\frac{J_{v+1}(x)}{x^v} = \left(\frac{1}{2}\right)^{v} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+v+2)} \left(\frac{x}{2}\right)^{2m+1} \tag{40}$$

qatorga yiyiladi. (39) va (40) formulalardan (26) kelib chiqadi.

Quyidagi tengliklar (27) munosabatning bajarilishini ko‘rsatadi:

$$\frac{d}{dx}\left(x^v J_v(x)\right) = \frac{d}{dx}\left(x^v \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+v+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k+v}\right) =$$

$$= vx^{v-1}\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+v+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k+v} +$$

$$+x^v \sum_{k=0}^{\infty} \frac{(-1)^k \frac{(2k+v)}{2}}{\Gamma(k+v+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k+v-1} =$$

$$= x^v \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+v)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k+v-1} = x^v J_{v-1}(x).$$

(26) dan (28) tenglik kelib chiqadi. Haqiqatan, (26) tenglikning chap tomonini ko‘paytmaning hosilasi

$$\frac{J_v'(x)}{x^v} - \frac{vJ_v(x)}{x^{v+1}} = -\frac{J_{v+1}(x)}{x^v}$$

ko‘rinishida yozib olamiz. Hosil bo‘lgan tenglikni x^v ga ko‘paytirib,

$$J_v'(x) - \frac{vJ_v(x)}{x} = -J_{v+1}(x)$$

ifodani hosil qilamiz.

Bevosita (26) dan (29) formula kelib chiqadi.

(28) va (29) tengliklarni o‘zaro qo‘shish natijasida (30) munosabat kelib chiqadi.

(28) va (29)tengliklarning birini ikkinchisidan ayirib, (31) munosabatni olamiz.

(27) formuladan foydalanib, ushbu

$$\int x^{v+1} J_v(x)dx$$

integralni hisoblaymiz.

(27) da v ni $v+1$ ga almashtiramiz va

$$x^{v+1}J_v(x) = \frac{d}{dx}(x^{v+1}J_{v+1}(x))$$

tenglikka kelamiz. Bundan esa,

$$\int x^{v+1} J_v(x)dx = \int \frac{d}{dx}(x^{v+1} J_{v+1}(x))dx = x^{v+1} J_{v+1}(x) + C.$$

(33) integral (32) integralning xususiy holi bo‘lib, u (32) da $v = 0$ deb olsak hosil bo‘ladi.

Endi (34) tenglikning o‘rinli ekanligini ko‘rsatamiz. (32) integral $v = 1$ da quyidagicha ko‘rinish oladi:

$$\int x^2 J_1(x)dx = x^2 J_2(x) + C$$

$v = 2$ indeksli Bessel funksiyasi uchun (30) rekkurent formuladan foydalanib, Bessel funksiyasining indeksini pasaytiramiz va

$$J_2(x) = -J_0(x)\frac{2}{x} + J_1(x)$$

ni hosil qilamiz. U holda

$$\int x^2 J_1(x)dx = -x^2 J_0(x) + 2xJ_1(x) + C$$

(34) bilan ustma-ust tushadi.

Ushbu

$$\int x^3 J_0(x)dx$$

integralni hisoblash uchun (27) formuladan $v = 0$ bo‘lgan holda foydalanamiz:

$$xJ_0(x) = \frac{d}{dx}\left[xJ_1(x)\right].$$

U holda hosil bo‘lgan

$$\int x^3 J_0(x)dx = \int x^2 \frac{d}{dx}[xJ_1(x)]dx$$

integralni bo‘laklab integrallash natijasida

$$\int x^3 J_0(x)dx = x^3 J_1(x) - 2\int x^2 J_1(x)dx$$

ga kelamiz. Bu ifodaga (34) formulani qo‘llab, isbotlanishi talab etilayotgan tenglikni hosil qilamiz.

(36) tenglikni isbotlashga kirishamiz. Buning uchun ushbu

$$\frac{d}{dx}(xy') + \left(\alpha^2 x - \frac{v^2}{x}\right) y = 0 \tag{41}$$

tenglamani qaraymiz. Bu tenglamadagi noma'lum funksiyaning argumentini

$$\xi = \alpha x, \ \alpha > 0$$

formula bilan yangi erkli o'zgaruvchiga almashtiramiz. U holda bu funksiya $y(x) = \widetilde{y}(\xi)$ ko'rinishda yozilsin. Hosil bo'lgan funksiyani (41) tenglamaga qo'yish uchun avval bu funksiyaning hosilasini topamiz:

$$y'(x) = \widetilde{y}'(\xi)\alpha, \ \ xy'(x) = \xi\widetilde{y}'(\xi), \ \ \frac{d}{dx}(xy'(x)) = \frac{d}{d\xi}(\xi(\widetilde{y}'(\xi)))\alpha.$$

Bularni (41) ga qo'yib, ushbu

$$\frac{d}{d\xi}(\xi\widetilde{y}'(\xi)) + \left(\xi - \frac{v^2}{\xi}\right) \widetilde{y} = 0 \tag{42}$$

Bessel tenglamasiga kelamiz. (42) Bessel tenglamasining yechimlaridan biri

$$\widetilde{y}(\xi) = J_v(\xi)$$

dan iborat. Bunda oldingi o'zgaruvchiga o'tadigan bo'lsak, $y(x) = J_v(\alpha x)$ (41) tenglamaning yechimini topamiz. Natijada $J_v(\alpha x)$ funksiya (42) tenglamani qanoatlantiradi va undan quyidagi munosabat kelib chiqadi:

$$\frac{d}{dx}\left(x\frac{dJ_v(\alpha x)}{dx}\right) + \left(\alpha^2 x - \frac{v^2}{x}\right) J_v(\alpha x) \equiv 0.$$

Ravshanki, quyidagi munosabat esa $J_v(\beta x)$ uchun o'rinli:

$$\frac{d}{dx}\left(x\frac{dJ_v(\beta x)}{dx}\right) + \left(\beta^2 x - \frac{v^2}{x}\right) J_v(\beta x) \equiv 0$$

Yuqoridagi tengliklardan birinchisini $J_v(\beta x)$ ga, ikkinchisini $J_v(\alpha x)$ ga ko'paytirib, birini ikkinchisidan hadma-had ayiramiz. Natijada quyidagi tenglik hosil bo'ladi:

$$J_v(\beta x)\frac{d}{dx}\left(x\frac{dJ_v(\alpha x)}{dx}\right) - J_v(\alpha x)\frac{d}{dx}\left(x\frac{dJ_v(\beta x)}{dx}\right) -$$

$$-(\alpha^2-\beta^2)xJ_v(\alpha x)J_v(\beta x)=0.$$

Bu tenglikning har ikkala tomonini integrallab,

$$\int J_v(\beta x)\frac{d}{dx}\left(x\frac{dJ_v(\alpha x)}{dx}\right)dx-\int J_v(\alpha x)\frac{d}{dx}\left(x\frac{dJ_v(\beta x)}{dx}\right)dx-$$

$$-(\alpha^2-\beta^2)\int xJ_v(\alpha x)J_v(\beta x)dx=C$$

formulani olamiz, bu yerda C - ixtiyoriy o‘zgarmas. Birinchi ikkita integralni bo‘laklab integrallaymiz:

$$J_v(\beta x)x\frac{dJ_v(\alpha x)}{dx}-\int x\frac{dJ_v(\alpha x)}{dx}\frac{dJ_v(\beta x)}{dx}dx-J_v(\alpha x)x\frac{dJ_v(\alpha x)}{dx}+$$

$$+\int x\frac{dJ_v(\alpha x)}{dx}\frac{dJ_v(\beta x)}{dx}dx-(\alpha^2-\beta^2)\int xJ_v(\alpha x)dx=CJ_v(\beta x).$$

Ravshanki, bu munosabatlardan (36) tenglik kelib chiqadi.

(37) formulani isbotlash uchun (36) tenglikda $\beta\to\alpha$ deb, limitga o‘tamiz. Ko‘rinib turibdiki, chap tomondagi ifodaning limiti

$$\int xJ_v^2(\alpha x)dx \tag{43}$$

ga teng bo‘ladi. O‘ng tomondagi ifodaning limitini topamiz. Payqash qiyin emaski, bunda o‘ng tomondagi ifoda $\frac{0}{0}$ ko‘rinishidagi aniqmaslikdan iborat. Limitni hisoblash uchun Lopital qoidasidan foydalanamiz. Natijada quyidagi ifodaga kelamiz:

$$\lim_{\beta\to\alpha}\frac{\beta xJ_v(\alpha x)J_v'(\beta x)-\alpha xJ_v'(\alpha x)J_v(\beta x)}{\alpha^2-\beta^2}=$$

$$=\lim_{\beta\to\alpha}\frac{\frac{d}{d\beta}\left[\beta xJ_v(\alpha x)J_v'(\beta x)-\alpha xJ_v'(\alpha x)J_v(\beta x)\right]}{-2\beta}=$$

$$=-\frac{1}{2\alpha}\lim_{\beta\to\alpha}\left[xJ_v(\alpha x)J_v'(\beta x)+\beta x^2J_v(\alpha x)J_v''(\beta x)-\alpha x^2J_v'(\alpha x)J_v'(\beta x)\right]. \tag{44}$$

$J_v(z)$ funksiya (16) tenglamani qanoatlantirishini hisobga olsak, ushbu

$$(\beta x)^2J_v''(\beta x)=-\beta xJ_v'(\beta x)+(v^2-(\beta x)^2)J_v(\beta x)$$

tenglik o‘rinli bo‘ladi. $J_v''(\beta x)$ ifodani (44) ga qo‘ysak, quyidagi munosabatlar o‘rinli bo‘ladi:

$$\lim_{\beta\to\alpha}\frac{\beta x J_v(\alpha x)J_v'(\beta x)-\alpha x J_v'(\alpha x)J_v(\beta x)}{\alpha^2-\beta^2}=$$

$$=-\frac{1}{2\alpha}\lim_{\beta\to\alpha}\Big\{xJ_v(\alpha x)J_v'(\beta x)-\alpha x^2 J_v'(\alpha x)J_v'(\beta x)+$$

$$+\frac{1}{\beta}J_v(\alpha x)\left[-\beta x J_v'(\beta x)+(v^2-(\beta x)^2)J_v(\beta x)\right]\Big\}=$$

$$=\frac{1}{2\alpha}\lim_{\beta\to\alpha}\left[\alpha x^2 J_v'(\alpha x)J_v'(\beta x)+\frac{(\beta x)^2-v^2}{\beta}J_v(\alpha x)J_v(\beta x)\right]=$$

$$=\frac{1}{2}\left[x^2(J_v'(\alpha x))^2+\left(x^2-\frac{v^2}{\alpha^2}\right)J_v^2(\alpha x)\right]. \qquad (45)$$

(43) va (45) dan (37) munosabat kelib chiqadi.

Nihoyat (38) formulaning o‘rinli ekanligini ko‘rsatamiz. (36) tenglikdan

$$\int xJ_v\left(\frac{\mu_k x}{r_0}\right)J_v\left(\frac{\mu_m x}{r_0}\right)dx=$$

$$=\left.\frac{\frac{\mu_m}{r_0}xJ_v\left(\frac{\mu_k x}{r_0}\right)J_v'\left(\frac{\mu_m x}{r_0}\right)-\frac{\mu_k}{r_0}xJ_v'\left(\frac{\mu_k x}{r_0}\right)J_v\left(\frac{\mu_m x}{r_0}\right)}{\left(\frac{\mu_k x}{r_0}\right)^2-\left(\frac{\mu_m x}{r_0}\right)^2}\right|_0^{r_0}=$$

$$=\frac{\mu_m J_v(\mu_k)J_v'(\mu_m)-\mu_k J_v'(\mu_k)J_v(\mu_m)}{\mu_k^2-\mu_m^2}r_0^2 \qquad (46)$$

lar kelib chiqadi. Bu tengliklarda μ_k va μ_m lar $\alpha J_v(\mu)+\beta\mu J_v'(\mu)=0$ tenglamaning musbat ildizlari ekanigini hisobga olsak,

$$\alpha J_v(\mu_k)+\beta\mu_k J_v'(\mu_k)=0,$$

$$\alpha J_v(\mu_m)+\beta\mu_m J_v'(\mu_m)=0 \qquad (47)$$

tengliklar bajarilishi shart. Bu yerda $\alpha,\ \beta$ lar bir vaqtning o‘zida nolga teng emas, ya’ni $\alpha^2+\beta^2\neq 0$. U holda (47) sistema determinanti nolga teng bo‘ladi:

$$0=\begin{vmatrix} J_v(\mu_k) & \mu_k J_v'(\mu_k) \\ J_v(\mu_m) & \mu_m J_v'(\mu_m)\end{vmatrix}=\mu_m J_v(\mu_k)J_v'(\mu_m)-\mu_k J_v'(\mu_k)J_v(\mu_m).$$

Bu esa, (46) bilan birga (38) tenglikning o‘rinli ekanligini ko‘rsatadi.

7.3.1 Neyman funksiyasi

Ushbu
$$N_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v},\ v \neq n \tag{48}$$
formula bilan aniqlanadigan funksiyaga v indeksli Neyman funksiyasi deyiladi. Koʻrinib turibdiki, Neyman funksiyasi (15) tenglamaning yechimi boʻladi, chunki bu funksiya $J_v(x)$ va $J_{-v}(x)$ funksiyalarning chiziqli kombinatsiyasidan tuzilgan. $v > 0$ boʻlgan hol uchun Bessel va Neyman funksiyalarini oʻzaro bogʻliqsiz ekanligini koʻrsataylik. Buning uchun x ning yetarlicha kichik qiymatlarida bu funksiyalarning asimtotik formulalarini keltirish yetarli. Bessel funksiyasi uchun bu (23) formula bilan ifodalanadi. Neyman funksiyasi uchun bu (23), (24), (48) lardan kelib chiqadi:
$$N_v(x) \sim \frac{J_{-v}(x)}{\sin\pi v} \sim \frac{2^v}{x^v \sin\pi v\Gamma(1-v)},\ x\to 0.$$
Bu holda $x = 0$ nuqta atrofida Bessel funksiyasi chegaralangan, Neyman funksiyasi esa chegaralanmagan boʻladi. Bunday funksiyalar oʻzaro chiziqli bogʻliq boʻlmaydi. Shuning uchun v ning butun boʻlmagan qiymatlarida (15) tenglamaning yechimini quyidagicha yozishimiz mumkin:
$$y(x) = C_1 J_v(x) + C_2 N_v(x). \tag{49}$$
Agar (48) formulaning oʻng tomonida $v = n$ bunda $n = 0, 1, 2, ...$ deb qarasak, u holda bu bizga $\frac{0}{0}$ koʻrinishidagi aniqmaslikni beradi, chunki $\cos\pi n = (-1)^n,\ \sin\pi n = 0,\ J_{-n}(x) = (-1)^n J_n(x)$. Bu holda biz Lopital qoidasidan foydalanamiz va n indeksga koʻra Neyman funksiyasini
$$N_n(x) = \lim_{v\to n} N_v(x)$$
limit yordamida aniqlaymiz. Lopital qoidasidan foydalangan holda $N_n(x)$ Neyman funksiyasining koʻrinishini aniqlab olaylik:
$$N_n(x) = \lim_{v\to n}\frac{J_v(x)\cos\pi v - J_{-v}(x)}{\pi\cos\pi v} =$$
$$= \lim_{v\to n}\frac{\frac{\partial J_v(x)}{\partial v}\cos\pi v - J_v(x)\pi\sin\pi v - \frac{\partial J_{-v}(x)}{\partial v}}{\pi cos\pi v} =$$

$$= \frac{1}{\pi}\left(\frac{\partial J_v(x)}{\partial v}\bigg|_{v=n} - (-1)^n \frac{\partial J_{-v}(x)}{\partial v}\bigg|_{v=n}\right). \tag{50}$$

Faraz qilaylik, $n \geq 1$ bo'lsin. (19) va (22) qatorlarning v o'zgaruvchi bo'yicha xususiy hosilalarini hisoblaymiz:

$$\frac{\partial J_v}{\partial v} = \left(\frac{x}{2}\right)^v \ln\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+v+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} +$$

$$+\left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k+v+1} =$$

$$= \ln\left(\frac{x}{2}\right) J_v(x) + \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k+v+1},$$

$$\frac{\partial J_{-v}}{\partial} = -\left(\frac{x}{2}\right)^{-v} \ln\left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k-v+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} -$$

$$-\left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-v+1} =$$

$$= -\ln\left(\frac{x}{2}\right) J_{-v}(x) - \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k-v}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-v+1}.$$

Oxirgi formulada $v = n$ deb olsak,

$$\frac{\partial J_v}{\partial v}\bigg|_{v=n} =$$

$$= \ln\left(\frac{x}{2}\right) J_n(x) + \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k+n+1}, \tag{51}$$

$$\frac{\partial J_{-v}}{\partial v}\bigg|_{v=n} =$$

$$= -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1}. \tag{52}$$

Hosil bo'lgan (52) formuladagi qatorda dastlabki n hadini ajratib olamiz va (25) formulalarni qo'llaymiz. Natijada quyidagi ifodaga kelamiz:

$$\frac{\partial J_{-v}}{\partial v}\bigg|_{v=n} =$$

$$= -\ln\left(\frac{x}{2}\right) J_{-n}(x) - \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(-1)^k}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} \frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1} -$$

$$- \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} \frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1}.$$

Ushbu yig‘indida $k - n = m$ deb belgilash kiritamiz:

$$\frac{\partial J_{-v}}{\partial v}\bigg|_{v=n} = -(-1)^n \ln\left(\frac{x}{2}\right) J_n(x) -$$

$$- \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(-1)^k}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} \frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1} -$$

$$- \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^{m+n}}{\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m} \frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=m+1}. \qquad (53)$$

Gamma-funksiya xossalariga ko‘ra

$$\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1} = (-1)^{k-n+1}\Gamma(n-k), \qquad (54)$$

$$\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k-n+1} = -\frac{\Gamma'(t)}{\Gamma^2(t)}\bigg|_{t=m+1} =$$

$$= -\frac{1}{\Gamma(m+1)}\left(-C + \sum_{p=1}^{m}\frac{1}{p}\right), \quad m = 1, 2, 3, ...,$$

$$\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=1} = C, \qquad (55)$$

$$\frac{d}{dt}\left(\frac{1}{\Gamma(t)}\right)\bigg|_{t=k+n+1} = \frac{1}{\Gamma(k+n+1)}\left(-C + \sum_{p=1}^{k+n}\frac{1}{p}\right) \qquad (56)$$

formulalar o‘rinli. (56) ni (51) ga va (54), (55) larni (53) ga qo‘ygandan so‘ng quyidagi ifodalarga kelamiz:

$$\frac{\partial J_v}{\partial v}\bigg|_{v=n} = \ln\left(\frac{x}{2}\right) J_n(x) + C J_n(x) - \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{\Gamma(k+1)\Gamma(k+n+1)} \sum_{p=1}^{k+n}\frac{1}{p},$$

$$\frac{\partial J_{-v}}{\partial v}\bigg|_{v=n} = -(-1)^n \ln\left(\frac{x}{2}\right) J_n(x) + (-1)^n \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} -$$

$$-(-1)^n CJ_n(x) + (-1)^n \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{\Gamma(m+1)\Gamma(m+n+1)} \sum_{p=1}^{m} \frac{1}{p}.$$

Bu tengliklarni (50) ga qo‘ysak, quyidagi formula hosil bo‘ladi:

$$N_n(x) = \frac{1}{\pi}\Bigg\{ 2(\ln\frac{x}{2} + C)J_n(x) - \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} -$$

$$- \left(\frac{x}{2}\right)^{n} \frac{1}{\Gamma(n+1)} \sum_{p=1}^{n} \frac{1}{p} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{\Gamma(k+n+1)\Gamma(k+1)} \left[\sum_{p=1}^{k+n} \frac{1}{p} + \sum_{p=1}^{k} \frac{1}{p}\right]\Bigg\}. \tag{57}$$

(57) funksiyada $n = 0$ deb olsak, $N_0(x)$ funksiya ushbu

$$N(x) = \frac{2}{\pi}\left\{\left(\ln\frac{x}{2} + C\right) J_0(x) - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{\Gamma^2(k+1)} \sum_{p=1}^{k} \frac{1}{p}\right\} \tag{58}$$

ko‘rinishda topiladi. (57) va (58) lardan $x \to 0$ dagi Neymanning asimtotik funksiyasi kelib chiqadi.

$$N_n(x) \sim -\left(\frac{x}{2}\right)^{-n} \frac{\Gamma(n)}{\pi}, \quad n \geq 1,$$

$$N_0(x) \sim \frac{2}{\pi} \ln\frac{x}{2}.$$

Ko‘rinib turibdiki, Neyman funksiyasi $x = 0$ nuqta atrofida chegaralanmagan, shuning uchun v ning ixtiyoriy qiymatida (15) Bessal tenglamasining umumiy yechimini (49) ko‘rinishda ifodalash mumkin.

Foydalanilgan adabiyotlar

[1] Арсенин В.Я. Методы математической физики и специальные функции. М.: "Наука". 2004. -432 с.

[2] Бицадзе А.В. Уравнения математической физики. М.: Наука. 1976. -296 с.

[3] Владимиров В.С. Уравнения математической физики. М.: Изд-во "Наука". 1981. - 512 с.

[4] Жўраев Т.Ж., Абдиназаров С. Математик физика тенгламалари. Тошкент. ЎзМУ. 2003. - 332 б.

[5] Захаров Е.В., Дмитриева И.В. Орлик С.И. Уравнения математической физики, Москва, Издельский центр "Академия". 2010, - 320 с.

[6] Курант Р. Уравнения с частными производными. М.: Изд-во "Мир". 1964. -830 с.

[7] Михлин С.Г. Курс математической физики. СПб. "Ланъ". 2002. -576 с.

[8] Салоҳиддинов М.С. Математик физика тенгламалари. Тошкент. "Ўзбекистон", 2002. - 445 б.

[9] Свешников А.Г., Боголюбов А.Н., Кравцов В.В. Лекции по математической физики. М.: Изд-во МГУ. 1993. - 352 с.

[10] Тихонов А.Н., Самарский А.А. Уравнения математической физики. М.: Изд-во МГУ. 2004. - 798 с.

[11] Zikirov O.S. Matematik fizika tenglamalari. Toshkent. "Fan va texnologiya". 2017. - 320 b.

[12] Lawrence C.Evans. Partial differential equations. -American Mathematical Society, second edition, 2010. - 749 p.

[13] Marcelo R. Ebert, Michael Reissig. Methods for Partial Differentia Equations: Qualitative Properties of Solutions, Phase Space Analysis, Semilinear Models. -Springer International Publishing, 2018. - 456 p.

[14] Victor Henner, Tatyana Belozerova, Mikhail Khenner. Ordinary and partial differential equations. -Taylor & Francis Croup, 2013. - 644 p.

[15] J.Robert Buchanan, Zhoude Shao. A frist course in partial differential equations, World Scientific, 2017. - 624 p.

[16] Hunter J.K. Notes on Partial Differential Equations. -University of California at Davis. 2014. - 242 p.

[17] Miersemann E.Partial Differential Equations, Lecture Notes. -Leipzig University. 2012. - 205 p.

[18] Moore J.D. Introduction to Partial Differential Equation. -University of California, CA. 2012. - 169 p.

[19] Pinchover Y and Rubinstein J. An introduction to Partial Differential Equations. -Cambridge University Press. 2005. - 385 p.

[20] Strauss W.S. Partial Differential Equations, An introduction. -John Wiley and Sons, Inc. NJ USA, 2008. - 466 p.