O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

ABU RAYHON BERUNIY NOMIDAGI
TOSHKENT DAVLAT TEXNIKA UNIVERSITETI

«OLIY MATEMATIKA» FANIDAN MA'RUZALAR MATNI

3-QISM

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«Oliy matematika» fanidan ma'ruzalar matni. 3-qism. Boymurodov I., Qayumov E.Q., Eshov A.T., Yusupov A. – Toshkent, ToshDTU, 2007. – 201 b.

Mazkur ma'ruzalar matni ToshDTU ning mexanika, tog'-kon, neft va gaz, elektronika va avtomatika, energetika, iqtisod fakultetlarida texnika yo'nalishi bo'yicha bakalavrlar tayyorlash uchun «Oliy matematika» fanining namunaviy dasturiga ko'ra o'qilgan ma'ruzalar asosida qayta tuzilgan.

Abu Rayhon Beruniy nomidagi Toshkent davlat texnika universitetining ilmiy-uslubiy kengashi qarori asosida chop etildi.

TAQRIZCHILAR:

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2. ToshDTU «Oliy matematika» kafedrasining dotsenti f.-m.f.n., G.R. Abduraxmanov

KIRISH

Matematik tadqiqot usullari hozirgi zamon fan va texnikasida o'ziga xos muhim o'ringa ega. Hisoblash texnikasining rivojlanishi va uning inson faoliyatining barcha jabhalaridagi tatbiqi kengayishi bilan bog'liq bo'lishligi oqibatida, ayniqsa, matematikaning ahamiyati yanada oshdi. Bu esa muhandis mutaxassislarning matematik tayyorgarligi yuqori bo'lishligiga doir bo'lgan talabni yanada oshirish zaruriyatini vujudga keltiradi.

Matematika fani fundamental fanlardan biri bo'lib, bu har doim tabiiy va texnika fanlarining taraqqiy etishida muhim o'rin egallagan.

Matematika fanining taraqqiy etishida o'rta asr olimlaridan Muso al-Xorazmiy, Ahmad al-Farg'oniy, Abu Rayhon Beruniy, Mirzo Ulug'bek va boshqalar juda katta hissa qo'shganlar.

O'zbek matematiklarining matematika fani sohasidagi xizmatlarini yuqori baholab, O'zbekiston Respublikasi Prezidenti I.A.Karimovning "O'zbekiston XXI asr bo'sag'asida" asarida shunday deyiladi: **"Matematikaning "ehtimollar nazariyasi va matematik statistika", "differensial tenglamalar nazariyasi", "matematika-fizika tenglamalari", "funksional analiz" sohalari bo'yicha erishilgan natijalar respublikadan tashqarida ham ma'lum".**

Yuqorida aytilgan natijalarga ega bo'lishda o'zbek matematiklardan V.I. Romanovskiy, T.N. Qori-Niyoziy, T.A.Sarimsoqov, S.X.Sirojdinov, I.S.Artanix, M.S.Saloxiddinov, Sh.A.Achilov, T.A.Azlarov, Sh.A.Alimov, D.X.Xojiyev va boshqalarning xizmatlari nihoyatda kattadir.

Mamlakatimizda barcha yo'nalishlar bo'yicha bakalavrlar tayyorlash tizimiga o'tilgach, barcha fanlar bo'yicha o'quv rejalari va tipik dasturlarni davlat standartiga qo'yish zarurati tug'ildi. Shu bilan birga xalqaro ta'lim berish standartlarini qo'llash, ajdodlarimizning boy milliy meroslarini shu jarayonga jalb qilish kerak bo'ldi. Har bir yo'nalish bir necha mutaxassisliklarni o'z ichiga olgani uchun, mamlakatimizdagi barcha texnika oliy o'quv yurtlarida bakalavrlar tayyorlash bo'yicha oliy matematika fanidan o'zbek tilidagi darslik tayyorlash muammosi paydo bo'ldi. Tavsiya etilayotgan ushbu ma'ruzalar matni ana shu maqsadlarni ko'zda tutadi.

Bu ma'ruzalar matni ToshDTU «Oliy matematika» kafedrasi professor-o'qituvchilari tomonidan tayyorlanib, ToshDTUning ilmiy-uslubiy ishlar bo'yicha muhokama qilindi.

Ushbu ma'ruzalar matni oliy matematikaning barcha jabhalarini o'z ichiga olib, **har bir bob bo'yicha oliy matematikaning injenerlik ishiga tatbiqi ko'rsatilgan**. U oliy o'quv yurtlarida dastlabki ikki bosqichda 456 soatlik auditoriya darslariga mo'ljallangan 112 ta ma'ruzadan iborat. To'plamning mazkur 3-qismiga 3-semestrda o'qiladigan 28 ta ma'ruza kiritilgan.

57-58-ma'ruzalar

QATORLAR. ISHORASI NAVBATLASHUVCHI QATORLAR. LEYBNIS TEOREMASI.

1. Qator. Qatorning yig'indisi.

Elementlari sonlar (haqiqiy yoki kompleks) yoki funksiyalar bo'lgan

$$u_1, u_2, ..., u_n, ...$$

cheksiz ketma-ketlikni qaraymiz.

1-ta'rif. Ushbu

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

ifoda cheksiz qator deyiladi.

Kelishib olinganiga ko'ra qatorni belgilash uchun Σ belgisidan foydalaniladi, ya'ni (1) qatorni qisqacha

$$\sum_{n=1}^{\infty} u_n$$

ko'rinishda yozish mumkin. $u_1, u_2, ..., u_n, ...$ ketma-ketlikning elementlari qatorning hadlari deyiladi.

Agar qatorning hadlari sonlar (funksiyalar) dan iborat bo'lsa, qator sonli (funksional) qator deyiladi. Qatorning n–hadining ifodasiga qatorning umumiy hadi deyiladi.

1-misol. Ushbu $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$ qator sonli qatordir, uning umumiy hadi $\frac{1}{2^n}$ ga teng, bu qatorni qisqacha bunday yozish mumkin: $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

Biz avval sonli qatorlarni ko'rib chiqamiz.

2-ta'rif. (1) qatorning dastlabki n ta hadining yig'indisi

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

shu qatorning qismiy yig'indisi deyiladi.

Quyidagi qismiy yig'indilarni qaraymiz:

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\dots\dots\dots\dots\dots\dots$$
$$S_n = u_1 + u_2 + \dots + u_n$$

Agar $\lim_{n\to\infty} S_n = S$ chekli limit mavjud bo'lsa, u (1) qatorning yig'indisi deb ataladi va qator yaqinlashuvchi deyiladi.

Aks holda, ya'ni $\lim_{n\to\infty} S_n$ mavjud bo'lmasa, (1) qator uzoqlashuvchi deyiladi va uning yig'indisi bo'lmaydi.

Misol. Ushbu

$$a + aq + aq^2 + aq^3 + \dots + aq^{n-1} + \dots \qquad (2)$$

qatorni tekshiramiz.

Bu qator birinchi hadi a $(a \neq 0)$ va maxraji q bo'lgan geometrik progressiyadir. Geometrik progressiya dastlabki n ta hadining yig'indisi ($q \neq 1$ bo'lganda):

$$S_n = \frac{a - aq^n}{1 - q}$$

1) agar $|q| < 1$ bo'lsa, u holda $\lim_{n\to\infty} q^n = 0$, demak,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - aq^n}{1 - q} = \frac{a}{1 - q}$$

Demak, $|q| < 1$ bo'lganda (2) qator yaqinlashadi va uning yig'indisi

$$S = \frac{a}{1 - q}$$

2) agar $|q| > 1$ bo'lsa, u holda $\lim_{n\to\infty} q^n = \infty$ va shuning uchun

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - aq^n}{1-q} = \infty.$$

Shunday qilib, $|q| > 1$ da cheksiz geometrik progressiya uzoqlashuvchi qator hosil qiladi.

3) agar $q = 1$ bo'lsa, u holda

$$a + a + a + \cdots + a + \cdots$$

qator hosil bo'ladi, bu qatorning n–qismiy yig'indisi $S_n = na$ bo'ladi. Ravshanki,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty,$$

demak, bu holda qator uzoqlashadi.

4) agar $q = (-1)^n$ bo'lsa, (2) qator

$$a - a + a - a + \cdots$$

ko'rinishda bo'ladi. Bu holda

$$S_n = \begin{cases} 0, & n \text{ juft bo'lganda} \\ a, & \text{n toq bolganda} \end{cases}$$

Demak, S_n aniq limitga ega bo'lmaydi, qator uzoqlashadi.

Shunday qilib, cheksiz geometrik progressiya $|q|<1$ da yaqinlashuvchi va $|q| \geq 1$ bo'lganda uzoqlashuvchi qator ekan.

2. Qatorlar ustida sodda amallar

1-teorema. Agar

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

qator yaqinlashuvchi bo'lib, uning yig'indisi S ga teng bo'lsa, u holda

$$cu_1 + cu_2 + cu_3 + \cdots + cu_n + \cdots \qquad (2)$$

qator ham yaqinlashuvchi bo'ladi va uning yig'indisi cS ga teng bo'ladi, bunda c biror belgilangan o'zgarmas son.

Isboti. (1) va (2) qatorlarning n-qismiy yig'indilarini mos ravishda S_n va σ_n bilan belgilaymiz. U holda quyidagiga ega bo'lamiz:

$$\sigma_n = cu_1 + cu_2 + cu_3 + \cdots + cu_n = c(u_1 + u_2 + u_3 + \cdots + u_n) = cS_n,$$

Bundan $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty}(cS_n) = c\lim_{n\to\infty} S_n = cS$.

Shunday qilib, (2) qator yaqinlashuvchi va uning yig'indisi cS ga teng. Teorema isbotlandi.

2-teorema. Agar

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{3}$$

va

$$\vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots + \vartheta_n + \cdots \tag{4}$$

qatorlar yaqinlashsa va ularning yig'indilari mos ravishda $\overline{S}$ va $\overline{\overline{S}}$ ga teng bo'lsa, u holda

$$(u_1 + \vartheta_1) + (u_2 + \vartheta_2) + \cdots \tag{5}$$

va

$$(u_1 - \vartheta_1) + (u_2 - \vartheta_2) + \cdots \tag{6}$$

qatorlar ham yaqinlashadi va yig'indilari mos ravishda $\overline{S} + \overline{\overline{S}}$ va $\overline{S} - \overline{\overline{S}}$ ga teng bo'ladi.

Isboti. (5) qatorning yaqinlashishini isbotlaymiz. Uning n–qismiy yig'indisini σ_n bilan, (3) va (4) qatorlarning n–qismiy yig'indilarini mos ravishda $\overline{S}_n$ va $\overline{\overline{S}}_n$ bilan belgilab, quyidagini hosil qilamiz:

$$\sigma_n = (u_1 + \vartheta_1) + \cdots + (u_n + \vartheta_n) = (u_1 + u_2 + \cdots + u_n) +$$
$$+ (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n) = \overline{S}_n + \overline{\overline{S}}_n$$

Bu tenglikda $n \to \infty$ da limitga o'tsak:

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}(\bar{S}_n + \bar{\bar{S}}_n) = \lim_{n\to\infty}\bar{S}_n + \lim_{n\to\infty}\bar{\bar{S}}_n = \bar{S} + \bar{\bar{S}}$$

Shunday qilib, (5) qator yaqinlashadi va uning yig'indisi $\bar{S} + \bar{\bar{S}}$ ga teng.

(6) qatorning yaqinlashishi va uning yig'indisi $\bar{S} - \bar{\bar{S}}$ ga tengligi ham xuddı shuningdek isbot qilinadi.

3-teorema. Agar qator yaqinlashuvchi bo'lsa, u holda berilgan qatorga chekli sondagi hadlarni qo'shish yoki undan chekli sondagi hadlarni tashlab yuborishdan hosil bo'lgan qator ham yaqinlashuvchi bo'ladi.

Isboti. Ushbu

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

qator yaqinlashuvchi, uning yig'indisi S ga teng bo'lsin. (1) qator dastlabki n ta hadining yig'indisini S_n bilan belgilaymiz. k $(k < n)$ ta tashlab yuborilgan hadlar yig'indisini S_k bilan, qolgan $n-k$ ta hadlar yig'indisini σ_{n-k} bilan belgilaymiz.

Demak, $S_n = S_k + \sigma_{n-k}$, bunda S_k n ga bog'liq bo'lmagan chekli son, shu sababli:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(S_n + \sigma_{n-k}) = \lim_{n\to\infty} S_k + \lim_{n\to\infty}\sigma_{n-k}$$

Bundan: $\lim\limits_{n\to\infty} S_n = S_k + \lim\limits_{n\to\infty}\sigma_{n-k}$.

Shunday qilib, agar $\lim\limits_{n\to\infty} S_n$ mavjud bo'lsa, u holda $\lim\limits_{n\to\infty}\sigma_{n-k}$ ham mavjud bo'ladi. Demak, chekli sondagi hadlarni tashlab yuborishdan hosil qilingan qator ham yaqinlashadi.

Chekli sondagi hadlarni qo'shishdan hosil bo'lgan qatorning yaqinlashuvchi bo'lishi yuqoridagidek ko'rsatiladi.

3. Qator yaqinlashishining zaruriy sharti

Qator yaqinlashishining zaruriy sharti shunday shartki, u shu qatorning qismiy yig'indisi deyiladi.

Quyidagi qismiy yig'indilarni qaraymiz:

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n = u_1 + u_2 + \cdots + u_n$$

Agar $\lim\limits_{n\to\infty} S_n = S$ chekli limit mavjud bo'lsa, u (1) qatorning yig'indisi deb ataladi va qator yaqinlashuvchi deyiladi.

Aks holda, ya'ni $\lim\limits_{n\to\infty} S_n$ mavjud bo'lmasa, (1) qator uzoqlashuvchi deyiladi va uning yig'indisi bo'lmaydi.

Misol. Ushbu

$$a + aq + aq^2 + aq^3 + \cdots + aq^{n-1} + \cdots \qquad (2)$$

qatorni tekshiramiz.

Bu qator birinchi hadi a $(a \neq 0)$ va maxraji q bo'lgan geometrik progressiyadir. Geometrik progressiya dastlabki n ta hadining yig'indisi ($q \neq 1$ bo'lganda):

$$S_n = \frac{a - aq^n}{1-q}$$

1) agar $|q| < 1$ bo'lsa, u holda $\lim\limits_{n\to\infty} q^n = 0$, demak,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - aq^n}{1-q} = \frac{a}{1-q}$$

Demak, $|q| < 1$ bo'lganda (2) qator yaqinlashadi va uning yig'indisi

$$S = \frac{a}{1-q}$$

2) agar $|q| > 1$ bo'lsa, u holda $\lim\limits_{n\to\infty} q^n = \infty$ va shuning uchun

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{a-aq^n}{1-q} = \infty.$$

Shunday qilib, $|q|>1$ da cheksiz geometrik progressiya uzoqlashuvchi qator hosil qiladi.

3) agar $q=1$ bo'lsa, u holda

$$a+a+a+\cdots+a+\cdots$$

qator hosil bo'ladi, bu qatorning n–qismiy yig'indisi $S_n = na$ bo'ladi. Ravshanki,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty,$$

demak, bu holda qator uzoqlashadi.

4) agar $q=(-1)^n$ bo'lsa, (2) qator

$$a-a+a-a+\cdots$$

ko'rinishda bo'ladi. Bu holda

$$S_n = \begin{cases} 0, & n \text{ juft bo'lganda} \\ a, & \text{n toq bolganda} \end{cases}$$

Demak, S_n aniq limitga ega bo'lmaydi, qator uzoqlashadi.

Shunday qilib, cheksiz geometrik progressiya $|q|<1$ da yaqinlashuvchi va $|q|\geq 1$ bo'lganda uzoqlashuvchi qator ekan.

2. Qatorlar ustida sodda amallar

1-teorema. Agar

$$u_1+u_2+u_3+\cdots+u_n+\cdots \qquad (1)$$

qator yaqinlashuvchi bo'lib, uning yig'indisi S ga teng bo'lsa, u holda

$$cu_1+cu_2+cu_3+\cdots+cu_n+\cdots \qquad (2)$$

qator ham yaqinlashuvchi bo'ladi va uning yig'indisi cS ga teng bo'ladi, bunda c biror belgilangan o'zgarmas son.

Isboti. (1) va (2) qatorlarning n-qismiy yig'indilarini mos ravishda S_n va σ_n bilan belgilaymiz. U holda quyidagiga ega bo'lamiz:

$$\sigma_n = cu_1 + cu_2 + cu_3 + \cdots + cu_n = c(u_1 + u_2 + u_3 + \cdots + u_n) = cS_n,$$

Bundan $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty}(cS_n) = c \lim_{n\to\infty} S_n = cS$.

Shunday qilib, (2) qator yaqinlashuvchi va uning yig'indisi cS ga teng. Teorema isbotlandi.

2-teorema. Agar

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (3)$$

va

$$\vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots + \vartheta_n + \cdots \qquad (4)$$

qatorlar yaqinlashsa va ularning yig'indilari mos ravishda $\overline{S}$ va $\overline{\overline{S}}$ ga teng bo'lsa, u holda

$$(u_1 + \vartheta_1) + (u_2 + \vartheta_2) + \cdots \qquad (5)$$

va

$$(u_1 - \vartheta_1) + (u_2 - \vartheta_2) + \cdots \qquad (6)$$

qatorlar ham yaqinlashadi va yig'indilari mos ravishda $\overline{S} + \overline{\overline{S}}$ va $\overline{S} - \overline{\overline{S}}$ ga teng bo'ladi.

Isboti. (5) qatorning yaqinlashishini isbotlaymiz. Uning n–qismiy yig'indisini σ_n bilan, (3) va (4) qatorlarning n–qismiy yig'indilarini mos ravishda $\overline{S}_n$ va $\overline{\overline{S}}_n$ bilan belgilab, quyidagini hosil qilamiz:

$$\sigma_n = (u_1 + \vartheta_1) + \cdots + (u_n + \vartheta_n) = (u_1 + u_2 + \cdots + u_n) +$$
$$+ (\vartheta_1 + \vartheta_2 + \cdots + \vartheta_n) = \overline{S}_n + \overline{\overline{S}}_n$$

Bu tenglikda $n \to \infty$ da limitga o'tsak:

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}(\overline{S}_n + \overline{\overline{S}}_n) = \lim_{n\to\infty}\overline{S}_n + \lim_{n\to\infty}\overline{\overline{S}}_n = \overline{S} + \overline{\overline{S}}$$

Shunday qilib, (5) qator yaqinlashadi va uning yig'indisi $\overline{S} + \overline{\overline{S}}$ ga teng.

(6) qatorning yaqinlashishi va uning yig'indisi $\overline{S} - \overline{\overline{S}}$ ga tengligi ham xuddı shuningdek isbot qilinadi.

3-teorema. Agar qator yaqinlashuvchi bo'lsa, u holda berilgan qatorga chekli sondagi hadlarni qo'shish yoki undan chekli sondagi hadlarni tashlab yuborishdan hosil bo'lgan qator ham yaqinlashuvchi bo'ladi.

Isboti. Ushbu

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

qator yaqinlashuvchi, uning yig'indisi S ga teng bo'lsin. (1) qator dastlabki n ta hadining yig'indisini S_n bilan belgilaymiz. k $(k < n)$ ta tashlab yuborilgan hadlar yig'indisini S_k bilan, qolgan $n-k$ ta hadlar yig'indisini σ_{n-k} bilan belgilaymiz.

Demak, $S_n = S_k + \sigma_{n-k}$, bunda S_k n ga bog'liq bo'lmagan chekli son, shu sababli:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(S_n + \sigma_{n-k}) = \lim_{n\to\infty} S_k + \lim_{n\to\infty}\sigma_{n-k}$$

Bundan: $\lim\limits_{n\to\infty} S_n = S_k + \lim\limits_{n\to\infty}\sigma_{n-k}$.

Shunday qilib, agar $\lim\limits_{n\to\infty} S_n$ mavjud bo'lsa, u holda $\lim\limits_{n\to\infty}\sigma_{n-k}$ ham mavjud bo'ladi. Demak, chekli sondagi hadlarni tashlab yuborishdan hosil qilingan qator ham yaqinlashadi.

Chekli sondagi hadlarni qo'shishdan hosil bo'lgan qatorning yaqinlashuvchi bo'lishi yuqoridagidek ko'rsatiladi.

3. Qator yaqinlashishining zaruriy sharti

Qator yaqinlashishining zaruriy sharti shunday shartki, u bajarilmaganda qator uzoqlashadi.

Teorema. Agar qator yaqinlashsa, n cheksiz o'sib borganda uning n–hadi nolga intiladi.

Isboti. Faraz qilaylik,

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

qator yaqinlashsin, ya'ni $\lim_{n\to\infty} S_n = S$ tenglik o'rinli bo'lsin, bunda S qatorning yig'indisi (chekli son), lekin bu holda $\lim_{n\to\infty} S_{n-1} = S$ tenglik ham o'rinli, chunki $n \to \infty$ da $(n-1) \to \infty$. Oxirgi ikki tenglikni hadlab ayirib quyidagini hosil qilamiz:

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$$

yoki

$$\lim_{n\to\infty} (S_n - S_{n-1}) = 0$$

Lekin

$$S_n\text{-}S_{n-1}=u_n$$

Demak,

$$\lim_{n\to\infty} u_n = 0$$

Shuni isbotlash talab qilingan edi.

Misol.

$$\frac{1}{4}+\frac{2}{7}+\frac{3}{10}+\ldots+\frac{n}{3n+1}+\ldots$$

qator uzoqlashadi, chunki

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n}{3n+1}\right) = \frac{1}{3} \neq 0$$

$\lim_{n\to\infty} u_n = 0$ tenglik o'rinli bo'ladigan har qanday qator ham yaqinlashuvchi bo'lavermaydi. Bu shartning bajarilishi qator yaqinlashuvchi bo'lishi uchun zaruriy shart bo'lib, ammo yetarli shart emas, ya'ni qator umumiy hadining nolga intilishi bilan qatorning yaqinlashuvchi ekanligi kelib chiqavermaydi, qator uzoqlashuvchi ham bo'lishi mumkin.

Masalan, garmonik qator deb ataluvchi ushbu

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$$

qator

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n}=0$$

bo'lishiga qaramay, u uzoqlashadi. Buni isbot qilish maqsadida garmonik qator bir necha hadini quyidagidek guruhlab yozamiz

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+ \\ +\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+...+\frac{1}{16}\right)+\left(\frac{1}{17}+...\right. \tag{1}$$

Endi yordamchi qator tuzamiz, ya'ni har qaysi qavs ichidagi qo'shiluvchilarni ularning kichigi bilan almashtiramiz. Natijada

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+ \\ +\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+...+\frac{1}{16}\right)+\left(\frac{1}{32}+...\right. \tag{2}$$

Har qaysi qavs ichidagi qo'shiluvchilar yig'indisi kichiklashdi va 1/2 ga teng bo'ldi, ya'ni

$$S_n = 1+(n-1)\frac{1}{2}$$

Bundan limitga o'tsak

$$\lim_{n\to\infty} S_n = \infty .$$

Demak, garmonik qatorning yig'indisi albatta cheksizlikka intiladi. Shunday qilib, biz garmonik qatorning uzoqlashuvchi ekanini isbotladik.

4. Musbat hadli qatorlarni taqqoslash

Musbat hadli ikkita qator berilgan bo'lsin:

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

$$\vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots + \vartheta_n + \cdots \qquad (2)$$

Bu qatorlar uchun quyidagi teoremalar o'rinli:

1-teorema. Agar (1) qatorning hadlari (2) qatorning mos hadlaridan katta bo'lmasa, ya'ni

$$u_n \leq v_n \qquad (n=1,\ 2,\ \ldots) \qquad (3)$$

bo'lsa va (2) qator yaqinlashuvchi bo'lsa, u holda (1) qator ham yaqinlashuvchi bo'ladi.

Isboti. (1) va (2) qatorning qismiy yig'indilarini mos ravishda S_n va σ_n bilan belgilaymiz. (3) tengsizlikdan

$$S_n \leq \sigma_n \qquad (4)$$

ekanligi kelib chiqadi. (2) qator yaqinlashuvchi bo'lgani sababli

$$\lim_{n \to \infty} \sigma_n = \sigma$$

(1) va (2) qatorlar musbat hadli bo'lgani sababli $\sigma_n < \sigma$ ekani va (4) tengsizlikka asosan

$$S_n < \sigma$$

kelib chiqadi.

Shunday qilib, (1) musbat hadli qator qismiy yig'indilari ketma-ketligi chegaralangan va demak bu qator yaqinlashuvchi. Shu bilan birga bu qator yig'indisi (2) qator yig'indisidan katta bo'lmaydi.

1-misol. Ushbu qator

$$\frac{1}{2^2} + \frac{1}{3^3} + \ldots + \frac{1}{(n+1)^{n+1}} + \ldots$$

yaqinlashadi, chunki uning hadlari

$$\frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n+1}} + \ldots$$

qatorning mos hadlaridan kichik. Ammo keyingi qator yaqinlashadi, chunki bu qator maxraji $q=1/2$ ga teng bo'lgan geometrik progressiyadan iborat. Bu holda 1-teoremaga asosan, berilgan qator ham yaqinlashuvchi bo'ladi.

2-teorema. Agar (2) qatorning hadlari (1) qatorning mos hadlaridan kichik bo'lmasa, ya'ni

$$u_n \le v_n \qquad (n=1,\ 2,\ 3,\ \ldots) \qquad (3)$$

bo'lsa va (1) qator uzoqlashuvchi bo'lsa, u holda (2) qator ham uzoqlashuvchidir.

Isboti. (1) va (2) qatorlarning n–qismiy yig'indilarini mos ravishda S_n va σ_n bilan belgilaymiz. (3) tengsizliklardan

$$\sigma_n \ge S_n \qquad (5)$$

ekani kelib chiqadi. (1) qator uzoqlashuvchi va uning qismiy yig'indilari ortib borganligi sababli

$$\lim_{n\to\infty} S_n = \infty$$

Lekin (5) tengsizlikka asosan

$$\lim_{n\to\infty} \sigma_n = \infty$$

Demak, (2) qator uzoqlashuvchi. Teorema isbotlandi.

2-misol. Ushbu

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

qator uzoqlashuvchi, chunki uning hadlari, ikkinchi hadidan boshlab uzoqlashuvchi bo'lgan

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

garmonik qatorning mos hadlaridan katta.

1-izoh. Yuqorida isbotlangan 1- va 2-taqqoslash teoremalari faqat musbat hadli qatorlar uchun o'rinli. (1) va (2) qatorlarning ba'zi hadlari nollar bo'lgan hol uchun ham o'z kuchida qoladi. Ammo qatorning hadlari orasida manfiy sonlar bo'lsa, bu alomatlar to'g'ri bo'lmaydi.

2-izoh. Agar (3) tengsizliklar barcha $n=1,\ 2,\ 3,\ \ldots$ uchun emas, balki faqat $n \ge N$ uchun bajarila boshlasa, shu holdagina 1- va 2-teoremalar o'rinlidir.

5. Dalamber alomati

Teorema. Agar musbat hadli

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

qator (n+1) hadining n–hadiga nisbati $n\to\infty$ da chekli l limitga ega bo'lsa, ya'ni:

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$$

bo'lsa, u holda:

1) $l<1$ bo'lganda qator yaqinlashadi.
2) $l>1$ bo'lganda qator uzoqlashadi.

Isboti. 1) $l<1$ bo'lsin. $l<q<1$ munosabatni qanoatlantiruvchi q sonini qaraymiz (1-shakl). Limitning ta'rifidan va (2) munosabatdan biror N nomerdan boshlab, n-ning barcha qiymatlari uchun, ya'ni $n\leq N$ uchun

$$\frac{u_{n+1}}{u_n} < q \qquad (3)$$

tengsizlikning o'rinli bo'lishi kelib chiqadi. Haqiqatan, $\frac{u_{n+1}}{u_n}$ miqdor l ga intilganligi sababli $\frac{u_n}{u_{n+1}}$ miqdor bilan l son orasidagi ayirmani (biror N nomerdan boshlab), absolyut qiymati har qanday musbat q - l sondan kichik bo'ladi, demak

$$\left|\frac{u_{n+1}}{u_n} - l\right| < q - l.$$

Bu tengsizlikdan (3) tengsizlik kelib chiqadi. Bu tengsizlikni N nomerdan boshlab n ning turli qiymatlari uchun yozib quyidagilarni hosil qilamiz:

$$\begin{cases} u_{N+1} < qu_N \\ u_{N+2} < qu_{N+1} < q^2 u_N \\ u_{N+3} < qu_{N+2} < q^3 u_N \\ \dots\dots\dots\dots\dots\dots \end{cases} \qquad (4)$$

Endi quyidagi ikki qatorni tekshiramiz:

$$u_1+u_2+u_3+\ldots+u_N+u_{N+1}+u_{N+2}+\ldots \qquad (1)$$

$$u_N+qu_N+q^2u_N+\ldots \qquad (5)$$

(5) qator maxraji $q<1$ bo'lgan geometrik progressiyadir. Demak, (5) qator yaqinlashadi. (1) qatorning hadlari u_{N+1} dan boshlab (4) tengsizliklarga asosan (5) qatorning hadlaridan kichik. Musbat hadli qatorlarni taqqoslash haqidagi 1-teoremaga asosan (1) qator yaqinlashadi.

2) $l>1$ bo'lsin. U holda $\lim\limits_{n\to\infty}\frac{u_{N+1}}{u_N}=l$ tenglikdan, biror N nomerdan boshlab, ya'ni $n\geq N$ uchun $\left|\frac{u_{n+1}}{u_n}\right|>1$ yoki barcha $n\geq N$ uchun $u_{N+1}>u_N$ tengsizlikning bajarilishi kelib chiqadi. Lekin bu tengsizlik qatorning hadlari $N+1$ dan nomerdan boshlab o'sishini bildiradi. Shuning uchun qatorning umumiy hadi nolga intilmaydi. Demak, qator uzoqlashadi.

1-eslatma. Agar $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$ bo'lsa, u holda qator uzoqlashadi, chunki bu holda $\frac{u_{n+1}}{u_n}>1$ va $u_{N+1}>u_N$, ya'ni $\lim\limits_{n\to\infty}u_n\neq 0$ (zaruriy shart bajarilmaydi).

2-eslatma. Agar $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}$ mavjud va birga teng bo'lsa yoki mavjud bo'lmasa, u holda Dalamber alomati qatorning yaqinlashuvchi yoki uzoqlashuvchi ekanini aniqlash imkonini bermaydi.

Bu masalani hal qilish uchun boshqa alomatlardan foydalanish kerak.

1-misol. Ushbu

$$\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3^2}}+\frac{3}{\sqrt{3^3}}+\ldots+\frac{n}{\sqrt{3^n}}+\ldots$$

qatorning yaqinlashishi tekshirilsin.

Yechish. Tekshirilayotgan qatorda

$$u_n=\frac{n}{\sqrt{3^n}};u_{n+1}=\frac{n+1}{\sqrt{3^{n+1}}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)\sqrt{3^n}}{\sqrt{3^{n+1}}n}=\lim_{n\to\infty}\frac{n+1}{n\sqrt{3}}=\lim_{n\to\infty}\frac{n(1+1/n)}{n\sqrt{3}}=$$

$$\lim_{n\to\infty}\frac{1+1/n}{\sqrt{3}}=\frac{1}{\sqrt{3}}<1$$

Demak, qator yaqinlashuvchi.

2-misol. Ushbu

$$\frac{1!}{2}+\frac{2!}{2^2}+\frac{3!}{2^3}+\ldots+\frac{n!}{2^n}+\ldots$$

qatorning yaqinlashishi tekshirilsin.

Yechish. Bunda

$$u_n=\frac{n!}{2^n};u_{n+1}=\frac{n+1!}{2^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!}=\lim_{n\to\infty}\frac{n+1}{2^n}=\infty>1$$

Berilgan qator uzoqlashadi.

3-misol. Ushbu

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

qatorning yaqinlashishi tekshirilsin.

Bu garmonik qatordir. Unga Dalamber alomatini tatbiq etib, $u_n = \frac{1}{n}; u_{n+1} = \frac{1}{n+1}$ ekanligini, demak

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

bo'lishini topamiz. Demak, bu qatorni Dalamber alomatidan foydalanib yaqinlashuvchi yoki uzoqlashuvchi ekanligini ayta olmaymiz. Lekin biz bilamizki, garmonik qator uzoqlashuvchidir.

6. Koshi alomati

Teorema. Agar musbat hadli qator

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \quad (1)$$

uchun $\sqrt[n]{u_n}$ miqdor $n\to\infty$ da l chekli limitga ega bo'lsa, ya'ni $\lim_{n\to\infty} \sqrt[n]{u_n} = l$ bo'lsa, u holda

1) $l<1$ bo'lganda qator yaqinlashadi;
2) $l>1$ bo'lganda qator uzoqlashadi.

Isbot. 1) $l<1$ bo'lsin. $l<q<1$ munosabatni qanoatlantiruvchi sonni qaraymiz. Biror $n=N$ nomerdan boshlab

$$\left|\sqrt[n]{u_n} - l\right| < q - l$$

munosabat o'rinli bo'ladi, bundan

$$\sqrt[n]{u_n} < q$$

yoki hamma $n\geq N$ uchun $u_n<q^n$ ekanligi kelib chiqadi.

Endi ushbu ikki qatorni qarab chiqamiz:

$$u_1+u_2+u_3+\ldots+u_N+u_{N+1}+u_{N+2}+\ldots \quad (1)$$

$$q^N+q^{N+1}+q^{N+2}+\ldots \quad (2)$$

(2) qator yaqinlashadi, chunki u maxraji $q<1$ bo'lgan geometrik progressiyani tashkil qiladi. (1) qatorning hadlari u_N dan boshlab

(2) qatorning hadlaridan kichik. Bu holda musbat hadli qatorlarni taqqoslash haqidagi 1-teoremaga asosan (1) qator ham yaqinlashadi.

2) $l>1$ bo'lsin. U holda $n=N$ nomerdan boshlab $\sqrt[n]{u_n}>1$ yoki $u_n>1$ bo'ladi. Lekin qaralayotgan qatorning hamma hadlari u_N dan boshlab 1 dan katta bo'lsa, uning umumiy hadi nolga intilmaydi, demak qator uzoqlashuvchi.

Eslatma. Dalamber alomatidagi kabi, $l=1$ bo'lgan holda Koshi alomati ham qo'shimcha tekshirishni talab qiladi.

1-misol. Quyidagi qatorni yaqinlashuvchanlikka tekshiring.

$$\frac{1}{2}\cdot 2+\frac{1}{2^2}\left(\frac{3}{2}\right)^4+\ldots+\frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}+\ldots$$

Yechish. Bunda

$$u_n=\frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}$$

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}}=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^n=\frac{e}{2}>1$$

Qator uzoqlashuvchi.

2-misol. Ushbu

$$\frac{3}{2}+\left(\frac{9}{11}\right)^2+\ldots+\left(\frac{2n^2+1}{3n^2-1}\right)^n+\ldots$$

qator yaqinlashuvchanlikka tekshirilsin.

Yechish. Bunda

$$u_n=\left(\frac{2n^2+1}{3n^2-1}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n^2+1}{3n^2-1}\right)^n}=\lim_{n\to\infty}\frac{2n^2+1}{3n^2-1}=\lim_{n\to\infty}\frac{n^2(2+1/n^2)}{n^2(3-1/n^2)}=$$

$$=\lim_{n\to\infty}\frac{2+1/n^2}{3-1/n^2}=\frac{2}{3}<1$$

qator yaqinlashadi.

7. Qator yaqinlashishining integral alomati

Teorema. Agar

$$u_1+u_2+...+u_n+... \qquad (1)$$

qatorning hadlari musbat va o'smaydigan bo'lsa, ya'ni

$$u_1 \geq u_2 \geq ... \geq u_n \geq ...$$

va f(x) uzluksiz funksiya uchun

$$f(1)=u_1,\ f(2)=u_2,\ ...,\ f(n)=u_n,\ ...$$

tengliklar o'rinli bo'lsa, u holda:

1) agar $\int_1^\infty f(x)dx$ xosmas integral yaqinlashsa, (1) qator ham yaqinlashadi.

2) agar $\int_1^\infty f(x)dx$ xosmas integral uzoqlashuvchi bo'lsa, (1) qator ham uzoqlashadi.

Isbot. Yuqoridan $y=f(x)$ egri chiziq bilan chegaralangan, asoslari $x=1$ dan x=n gacha bo'lgan, (bunda n–ixtiyoriy butun musbat son) egri chiziqli trapetsiyani qaraymiz (2-shakl). Bu trapetsiyaga asoslari [1, 2], [2, 3], ..., [n-1, n] kesmalardan iborat ichki va tashqi zinasimon to'rtburchaklar chizamiz, bunda funksiyaning

$$u_2=f(2),\ u_3=f(3),\ ..,\ u_n=f(n)$$

qiymatlari ichki chizilgan to'rtburchaklarga

$$u_1=f(1),\ u_2=f(2),\ ...,\ u_{n-1}=f(n-1)$$

qiymatlari esa tashqi chizilgan to'rtburchaklarga balandlik bo'lib xizmat qiladi.

S_n-qatorning n-qismiy yig'indisi, $\overline{S}_n$-egri chiziqli trapetsiyaning yuzi, S_{ur}, S_{tr}- mos ravishda ichki va tashqi chizilgan zinasimon shakllarning yuzlari bo'lsin. Bu holda

$$S_n=u_1+u_2+\ldots+u_n$$

$\overline{S}_n = \int_1^n f(x)dx$ ekani ravshan.

Shakldan

$$S_{ur}<\overline{S}_n<S_{tr} \tag{2}$$

ekanligi kelib chiqadi, bunda

$$S_{u,r}=u_2+u_3+\ldots+u_n=S_n-u_1.$$
$$S_{t,r}=u_1+u_2+\ldots+u_{n-1}=S_n-u_n.$$

Shunday qilib, (2) tengsizlikni quyidagicha yozish mumkin:

$$S_n-u_1<\overline{S}_n<S_n-u_n$$

yoki

$$S_n-u_1<\int_1^n f(x)dx<S_n-u_n.$$

Bundan ikkita tengsizlikka ega bo'lamiz:

$$S_n<u_1+\int_1^n f(x)dx \tag{3}$$

$$S_n>u_n+\int_1^n f(x)dx \tag{4}$$

$f(x)$ funksiya musbat, shu sababli n ning ortishi bilan $\int_1^n f(x)dx$ integral ham kattalashib boradi.

Ikki holni qaraymiz:

1. Faraz qilaylik $\int_1^\infty f(x)dx$ xosmas integral yaqinlashsin, ya'ni

$$\int_1^\infty f(x)dx=\lim_{n\to\infty}\int_1^n f(x)dx=J$$

integral chekli songa teng bo'lsin. U holda $\int_1^n f(x)dx<J$ va (3)

tengsizlikdan har qanday n da $S_n<u_1+J$ ekanligi kelib chiqadi. Shunday qilib, bu holda S_n–qismiy yig'indilar ketma-ketligi chegaralangan va demak, (1) qator yaqinlashadi.

2. $\int_1^{\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'lsin, ya'ni

$$\int_1^{\infty} f(x)dx = \lim_{n\to\infty} \int_1^{\infty} f(x)dx = \infty$$

bo'lsin. Bu esa n o'sganda (4) tengsizlikka asosan S_n qismiy yig'indilar ketma-ketligi chegaralanmaganligi kelib chiqadi, ya'ni qator uzoqlashadi.

Misol. Umumlashgan garmonik qator deb ataluvchi ushbu

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+...$$

qatorning yaqinlashishi tekshirilsin.

Yechish. $f(x)=\dfrac{1}{x^p}$ ekanligi ravshan, bunda p–tayinlangan son. Quyidagi integralni qaraymiz:

$$\int_1^{\infty}\frac{dx}{x^p}=\frac{x^{-p+1}}{-p+1}\Bigg|_1^{\infty}=\frac{1}{1-p}\lim_{N\to\infty}(N^p-1),\ p\neq 1.$$

p ning turli qiymatlarida qatorning yaqinlashishi yoki uzoqlashishi haqida fikr yuritamiz.

Agar $p>1$ bo'lsa, $\int_1^{\infty}\dfrac{dx}{x^p}=\dfrac{1}{(p-1)}$ ya'ni integral chekli, shuning uchun qator yaqinlashadi, agar $p<1$ bo'lsa, $\int_1^{\infty}\dfrac{dx}{x^p}=\infty$ qator uzoqlashadi, agar $p=1$ bo'lsa, $\int_1^{\infty}\dfrac{dx}{x}=\ln x|_1^{\infty}=\infty$, ya'ni qator uzoqlashadi. Demak, umumlashgan garmonik qator faqat $p>1$ da yaqinlashuvchi bo'lib, $p\leq 1$ bo'lganda uzoqlashuvchi ekan.

8. Ishorasi navbatlashuvchi qatorlar. Leybnis teoremasi

Biz hozirgacha musbat hadli qatorlar bilan ish ko'rdik. Bu paragrafda hadlarning ishoralari navbatlashuvchi qatorlarni, ya'ni

$$U_1 - U_2 + U_3 - U_4 + \ldots + (-1)^{n+1} U_n + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} U_n \qquad (1)$$

ko'rinishdagi qatorni tekshiramiz. Bunda: U_1, U_2, ... U_n... musbat sonlar.

Leybnis teoremasi. Agar ishoralari navbatlashuvchi

$$U_1 - U_2 + U_3 - U_4 + \ldots + (-1)^{n+1} U_n + \ldots \qquad (U_n > 0) \qquad (1)$$

qatorning hadlari

$$U_1 > U_2 > U_3 > \ldots > U_N > \ldots \qquad (2)$$

va

$$\lim_{n \to \infty} U_n = 0 \qquad (3)$$

bo'lsa, (1) qator yaqinlashadi va uning yig'indisi musbat bo'lib, birinchi hadidan katta bo'lmaydi: $0 < S < U_1$.

Isbot: (1) qatorning birinchi $n=2m$ ta hadining yig'indisini qaraymiz:

$$S_{2m} = (U_1 - U_2) + (U_3 - U_4) + \ldots + (U_{2m-1} - U_{2m})$$

(2) shartdan qavs ichidagi har bir ifodaning musbat ekanligi kelib chiqadi. Demak, $S_{2m} > 0$ va m o'sishi bilan o'sadi. Endi bu yig'indini quyidagicha yozamiz:

$$S_{2m} = U_1 - (U_2 - U_3) - (U_4 - U_5) - \ldots - (U_{2m-2} - U_{2m-1}) - U_{2m} \qquad (2)$$

(2) shartga asosan qavslarning har biri musbat. Shuning uchun bu qavslarni U_1 dan ayirsak, U_1 dan kichik son hosil bo'ladi:

$$S_{2m} < U_1$$

shunday qilib m o'sganda S_{2m} ning o'sishini va yuqoridan chegaralanganligini aniqladik. Demak, u limitga ega, ya'ni

$$\lim_{m \to \infty} S_{2m} = S$$

shu bilan birga: $0 < S < U_1$. Endi n-toq bo'lgan holni, ya'ni $n=2m+1$ bo'lgan holni tekshiramiz:

$$S_{2m+1} = S_{2m} + U_{2m+1}$$

(3) shartga asosan

$$\lim_{m\to\infty} U_{2m+1} = 0 .$$

demak,

$$\lim_{n\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} U_{2m+1} = \lim_{m\to\infty} S_{2m} = s$$

shu bilan biz, *n* juft bo'lganda ham, *n* toq bo'lganda ham

$$\lim_{n\to\infty} S_n = S$$

ekanligini isbotladik. Demak (1) qator yaqinlashuvchi va $0<S<U_1$.

Agar ishoralari navbatlashuvchi (1) qator Leybnis teoremasi shartni qanoatlantirsa, u holda uning *n*-qoldig'i

$$R_n=\pm(U_{n+1}-U_{n+2}+U_{n+3}-U_{n+4}+\ldots)$$

absolyut qiymat bo'yicha tashlab yuborilgan hadlarning birinchisining modulidan katta bo'lmaydi, ya'ni $|R_n|\le U_{n+1}$ dan bo'ladi.

Demak, qatorning *S* yig'indisini S_n xususiy yig'indi bilan almashtirishda qo'l keladigan xato absolyut qiymati bo'yicha tashlab yuborilgan hadlarning birinchisidan katta bo'lmaydi.

Misol. Ushbu

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots$$

qatorining yaqinlashishi tekshirilsin.

Yechish. Qatorning hadlari absolyut qiymati bo'yicha kamayib boradi:

$$\frac{1}{2!}>\frac{1}{3!}>\frac{1}{4!}>\ldots \quad \text{va} \quad \lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{1}{n!}=0$$

qator yaqinlashuvchi.

9. O'zgaruvchan ishorali qatorlar

Agar qatorning hadlari orasida musbatlari ham, manfiylari ham bo'lsa, bunday qator o'zgaruvchan ishorali qator deyiladi.

Bizga o'zgaruvchan ishorali qator:

$$U_1+U_2+U_3+...+U_n+... \quad (1)$$
berilgan bo'lsin, bunda U_1, U_2, U_3, ..., U_n, ... sonlar musbat ham, manfiy ham bo'lishi mumkin.

Oldin biz ko'rib o'tgan ishoralari navbatlashuvchi qatorlar o'zgaruvchan ishorali qatorlarning xususiy holidir.

O'zgaruvchan ishorali qator yaqinlashishining yetarli shartini ko'ramiz.

1-teorema. O'zgaruvchan ishorali qator
$$U_1+U_2+U_3+...+U_n+... \quad (1)$$
hadlarning absolyut qiymatlaridan tuzilgan:
$$|U_1|+|U_2|+|U_3|+...+|U_n|+... \quad (2)$$
qator yaqinlashsa, berilgan o'zgaruvchan ishorali qator ham yaqinlashadi.

Isbot. (1) va (2) qatorlarning birinchi n ta hadlarining yig'indisi mos ravishda S_n va G_n bo'lsin.

S_n^+ berilgan qatorning S_n ta qismiy yig'indisi orasidagi hamma musbat hadlarning yig'indisi, S_n^- esa hamma manfiy hadlarning absolyut qiymatlarining yig'indisi bo'lsin, u holda
$$S_n=S_n^+ - S_n^-, \quad G_n=S_n^+ + S_n^-$$
Shartga ko'ra (2) qator yaqinlashuvchi, shuning uchun:
$$\lim_{n\to\infty} G_n = G$$
S_n^+ va S_n^- lar esa musbat va o'suvchi, shu bilan birga:
$S_n^+ \le G_n < G$ va $S_n^- \le G_n < G$ (chegaralangan) demak, ular ham limitga ega:
$$\lim_{n\to\infty} S_n^+ = S^+, \qquad \lim_{n\to\infty} S_n^- = S^-.$$
Demak, (1) o'zgaruvchan ishorali qator yaqinlashadi.

1- Misol. Ushbu
$$\frac{\sin\alpha}{1^2}+\frac{\sin 2\alpha}{2^2}+\frac{\sin 3\alpha}{3^2}+...+\frac{\sin n\alpha}{n^2}+... \quad (3)$$
o'zgaruvchan ishorali qatorning yaqinlashishi tekshirilsin, bunda α istalgan son.

Yechish. Berilgan qatorga mos bo'lgan

$$\left|\frac{\sin\alpha}{1^2}\right|+\left|\frac{\sin 2\alpha}{2^2}\right|+\left|\frac{\sin 3\alpha}{3^2}\right|+...+\left|\frac{\sin n\alpha}{n^2}\right|+... \qquad (4)$$

qatorni qaraymiz. Bu qatorni yaqinlashuvchi bo'lgan:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+... \qquad (5)$$

qator bilan solishtiramiz.

(4) qatorning hadlari (5) qatorning mos hadlaridan katta emas, shu sababli taqqoslashning 1-teoremasiga ko'ra (4) qator yaqinlashuvchi. Bu holda isbotlangan teoremaga ko'ra (3) qator ham yaqinlashuvchi.

Yuqorida isbot qilingan yaqinlashish alomati o'zgaruvchan ishorali qatorning yaqinlashishi uchun yetarligina bo'lib, zaruriy emasligini ko'ramiz. Shunday o'zgaruvchan ishorali qatorlar ham borki, ularning o'zlari yaqinlashuvchi bo'lsa ham, hadlarining absolyut qiymatlaridan tuzilgan qatorlar uzoqlashuvchi bo'ladi. Shu munosabat bilan o'zgaruvchan ishorali qatorlarning absolyut va shartli yaqinlashishi haqidagi tushunchani kiritish foydalidir.

Absolyut va shartli yaqinlashish

1-ta'rif. Ushbu o'zgaruvchan ishorali qator:

$$U_1+U_2+U_3+...+U_n+... \qquad (1)$$

hadlarining absolyut qiymatlaridan tuzilgan

$$|U_1|+|U_2|+|U_3|+...+|U_n|+... \qquad (2)$$

qator yaqinlashsa, berilgan (1) qator absolyut yaqinlashuvchi deyiladi.

2-ta'rif. Agar o'zgaruvchan ishorali (1) qator yaqinlashuvchi bo'lib, bu qatorning hadlari absolyut qiymatlaridan tuzilgan (2) qator uzoqlashuvchi bo'lsa, u holda berilgan o'zgaruvchan ishorali (1) qator shartli yoki absolyut yaqinlashuvchi qator deyiladi.

1-misol. Ushbu

$$\frac{\cos\frac{\pi}{4}}{3}+\frac{\cos\frac{3\pi}{4}}{3^2}+\frac{\cos\frac{5\pi}{4}}{3^3}+\ldots+\frac{\cos(2n-1)\frac{\pi}{4}}{3^n}+\ldots \qquad (3)$$

qatorning yaqinlashishi tekshirilsin.

Yechish. Berilgan qator bilan birga

$$\left|\frac{\cos\frac{\pi}{4}}{3}\right|+\left|\frac{\cos\frac{3\pi}{4}}{3^2}\right|+\left|\frac{\cos\frac{5\pi}{4}}{3^3}\right|+\ldots+\left|\frac{\cos(2n-1)\frac{\pi}{4}}{3^n}\right|+\ldots \qquad (4)$$

qatorni qaraymiz. Bu qatorni geometrik progressiya tashkil qiluvchi

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots+\frac{1}{3^n}+\ldots \qquad (5)$$

qator bilan taqqoslaymiz. (5) qator $q<1$ bo'lgani uchun yaqinlashuvchi.

(4) hadlari (5) qatorning mos hadlaridan katta emas, shu sababli qatorlarni taqqoslashning 1-teoremasiga asosan (3) qator ham yaqinlashuvchi.

2-misol. Ushbu

$$-1+\frac{1}{\sqrt{2}}+\ldots+(-1)^n\frac{1}{\sqrt{n}}+\ldots \qquad (6)$$

qatorning yaqinlashishi tekshirilsin.

Yechish. Berilgan qator bilan birga

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots \qquad (7)$$

qatorni qaraymiz. (6) qator Leybnis teoremasiga ko'ra yaqinlashuvchi, ya'ni

$$\lim_{n\to\infty}U_n=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$$

Lekin (6) qator hadlarining absolyut qiymatidan tuzilgan (7) qator uzoqlashuvchi, chunki $f(x)=\frac{1}{\sqrt{x}}$, $p=\frac{1}{2}\leq 1$ bo'lgan qatordir.

Bu holda berilgan (6) qator shartli yaqinlashuvchi bo'ladi.

Absolyut va shartli yaqinlashuvchi qatorlarning quyidagi xossalarini (isbotsiz) keltiramiz:

2-teorema. Agar qator absolyut yaqinlashuvchi bo'lsa, uning hadlarining o'rinlari ixtiyoriy ravishda almashtirilganda ham u absolyut yaqinlashuvchi bo'lib qolaveradi. Bu holda qatorning yig'indisi qator hadlarining yig'indisiga bog'liq bo'lmaydi.

Bu hodisa shartli yaqinlashuvchi qator uchun o'z kuchini yo'qotadi.

3-teorema. Agar qator shartli yaqinlashuvchi bo'lsa, u holda bu qator hadlarining o'rinlarini shunday almashtirib qo'yish mumkinki, natijada uning yig'indisi o'zgaradi, buning ustiga almashtirishdan keyin hosil bo'lgan qator uzoqlashuvchi qator bo'lib qolishi mumkin.

59-ma'ruza
FUNKSIONAL QATORLAR

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots.. \qquad (1)$$

ko'rinishdagi ifoda funksional qator deb ataladi. Uning har bir hadi x ga bog'liq funksiyadir. x ga har xil sonli qiymatlarni berib turli tuman sonli qatorlarni hosil qilish mumkin. Ularning ayrimlari yaqinlashuvchi, ayrimlari esa uzoqlashuvchi bo'ladi. Ta'rif. Funksional qatorni yaqinlashuvchi qatorga aylantiradigan x larning sonli qiymatlar to'plami uning yaqinlashish sohasi deyiladi.

Tabiiyki, yaqinlashish sohasida funksional qatorning yig'indisi x ga bog'liq bo'lgan birorta funksiyadan iborat bo'ladi. Shuning uchun funksional qatorning yig'indisi $S(x)$ orqali belgilanadi.

Masalan, $\sum_{n=1}^{\infty} \ln^n x$ qatorning yaqinlashish sohasi topilsin.

Yechish. Berilgan qator maxraji $q=\ln x$ ga teng bo'lgan cheksiz geometrik progressiyani ifodalaydi. Geometrik progressiya $|q|<1$ shart bajarilgandagina yaqinlashgani uchun, berilgan qator $|\ln x|<1$ ya'ni $-1<\ln x<1$ tengsizlik bajarilganda absolyut yaqinlashadi, demak, $e^{-1}<x<e$ tengsizlik berilgan qatorning yaqinlashish sohasini ifodalaydi. Shunday qilib (e^{-1},e) oraliqda berilgan qatorning yig'indisini

$$S(x) = \frac{\ln x}{1-\ln x}$$

formula yordamida hisoblaymiz.

(1) qatorning birinchi n ta hadining yig'indisini $S_n(x)$ deb belgilaylik. Agar bu qator yaqinlashsa va uning yig'indisi $S(x)$ ga teng bo'lsa, u holda

$$S(x)=S_n(x)+r_n(x)$$

tenglikni yozishimiz mumkin, bu yerda

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+u_{n+3}(x)+\ldots$$

kattalik (1) qatorning qoldig'i deyiladi. Qatorning yaqinlashish sohasida $\lim_{n\to\infty} S_n(x) = S(x)$ munosabat o'rinli, shuning uchun

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty}\left[S(x) - S_n(x)\right] = 0,$$

ya'ni yaqinlashuvchi qatorning qoldig'i $r_n(x)$, $n\to\infty$ da nolga intiladi.

1. Kuchaytirilgan qatorlar

Ta'rif

$$u_1(x)+u_2(x)+u_3(x)+\ldots+u_n(x) \qquad (1)$$

funksional qator berilgan bo'lsin. Agar shunday bir musbat hadli yaqinlashuvchi sonli qator

$$\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n+\ldots \qquad (2)$$

mavjud bo'lib,

$$|u_1(x)| \le \alpha_1, |u_2(x) \le \alpha_2|, \ldots |u_n(x)| \le \alpha_n \qquad (3)$$

shart bajarilsa, (1) funksional qatorni o'zining aniqlanish sohasida kuchaytirilgan qator deb ataladi.

Misol uchun:

$$\frac{\cos x}{1}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots+\frac{\cos nx}{n^2}+\ldots$$

qator $(-\infty,+\infty)$ oraliqda kuchaytirilgan qatordir. Haqiqatan ham x ning har qanday qiymatlari uchun.

$\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}$ (n=1,2,3,...) munosabat bajariladi va bu qator yaqinlashuvchidir, chunki $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ umumlashgan garmonik qator yaqinlashuvchi qator hisoblanadi.

Ta'rifdan ko'rinadiki, ma'lum bir sohada berilgan kuchaytirilgan qator o'sha sohaning har bir nuqtasida absolyut yaqinlashadi.

Undan tashqari kuchaytirilgan qator quyidagi muhim xossaga ega.

2. Veyershtrass teoremasi

$[a, b]$ kesmada

$$u_1(x)+u_2(x)+u_3(x)+\ldots+u_n(x)+\ldots$$

kuchaytirilgan qator berilgan bo'lib, $S(x)$ uning yig'indisi, $S_n(x)$

esa uning birinchi n ta hadining yig'indisi bo'lsin. U holda istalgan kichik musbat son $\varepsilon>0$ uchun shunday musbat son N topiladiki, hamma $n \geq N$ lar uchun $|S(x)-S_n(x)|<\varepsilon$ tengsizlik bajariladi.

Isbot. (2) qatorning yig'indisini G bilan belgilaymiz:

$$G=\alpha_1+\alpha_2+\alpha_3+...+\alpha_n+...$$

u holda $G=G_n+E_n$, bu yerda

$$G_n=\alpha_1+\alpha_2+...+\alpha_n,\ E_n=\alpha_{n+1}+\alpha_{n+2}+\alpha_{n+3}...$$

(2) qator yaqinlashuvchi qator, shuning uchun $\lim\limits_{n\to\infty} G_n = G$, demak $\lim\limits_{n\to\infty} E_n = 0$.

Endi (1) qatorni $S(x)=S_n(x)+r_n(x)$ ko'rinishda yozib olamiz, bu yerda

$$S_n(x)=u_1(x)+u_2(x)+u_3(x)+...+u_n(x);\ r_n(x)=u_{n+1}(x)+u_{n+2}(x)+...$$

(3) shartga ko'ra $|u_{n+1}(x)|\leq\alpha_{n+1}$, $|u_{n+2}(x)|\leq\alpha_{n+2}$,..., shuning uchun $|r_n(x)|\leq E_n$. Shunday qilib har qanday $x\in[a, b]$ lar uchun

$$|S(x)-S_n(x)|\leq\varepsilon_n$$

tengsizlik o'rinli. Lekin $\lim\limits_{n\to\infty} \varepsilon_n = 0$ edi, shuning uchun

$$|S(x)-S_n(x)|<\varepsilon_n.$$

Ta'rif. Veyershtrass teoremasiga bo'ysunadigan har qanday qator $[a, b]$ kesmada tekis yaqinlashuvchi qator deb ataladi.

Veyershtrass teoremasidan ko'rinadiki kuchaytirilgan qator tekis yaqinlashuvchi qatordir.

Misol. $1+x+x^2+...+x^{n-1}+...=\sum\limits_{n=1}^{\infty} x^{n-1}$ qator (0;1) intervalda tekis yaqinlashishga tekshirilsin.

Yechish. Berilgan qatorning yig'indisini va qoldig'ining modulini topaylik:

$$S(x)=\frac{1}{1-x},\ |r_n(x)|=|S(x)-S_n(x)|=\left|\frac{1}{1-x}-\frac{1-x^n}{1-x}\right|=\frac{x^n}{1-x}$$

Endi ixtiyoriy $\varepsilon>0$ uchun shunday N son topaylikki, $n\geq N$ da $|r_n(x)|<\varepsilon$, ya'ni $\frac{x^n}{1-x}<\varepsilon$ bajarilsin. Buning uchun oxirgi tengsizlikni n ga nisbatan yechamiz:

$$\frac{x^n}{1-x}<\varepsilon,\quad x^n<(1-x)\varepsilon,\quad n\ln x<(1-x)\varepsilon,\quad n>\frac{\ln(1-x)\varepsilon}{\ln x}$$

(tengsizlik teskarisiga o'zgardi, chunki (0,1) oraliqda $\ln x<0$)

$n\geq N$ munosabatga asosan N ni $N=\frac{\ln(1-x)\varepsilon}{\ln x}$ ko'rinishda qabul qilamiz.

Oxirgi formuladan ko'rinadiki N ε bilan x ga bog'liq. Shunday ekan $N(x)$ funksiyaning (0,1) intervaldagi xarakterini tekshirish qoladi. Bu funksiya ushbu intervalda chegaralanmagan, chunki $x\to 1$ da $\ln x\to 0$. Demak, $N\to\infty$, bu degani $|r_n(x)|<\varepsilon$ tengsizlikni bajaradigan N mavjud emas. Shuning uchun berilgan qator (0,1) intervalda tekis yaqinlashmaydi. Lekin har qanday $(0,\infty]$ yarim intervalda $(0<\delta<1)$ berilgan qator tekis yaqinlashadi, chunki $N(x)$ funksiya yarim intervalda chegaralangan.

3. Qatorlarni integrallash va differensiallash

1-teorema. $[a,b]$ kesmada kuchaytirilgan bo'lgan uzluksiz funksiyalarning quyidagi

$$u_1(x)+u_2(x)+u_3(x)+...+u_n(x)+.....$$

qatori berilgan va $s(x)$ bu qatorning yig'indisi bo'lsin. Bu holda $[a,b]$ kesmaga tegishli bo'lgan α dan x gacha chegarada $s(x)$ dan olingan integral berilgan qator hadlaridan shunday chegarada olingan integrallar yig'indisiga teng, ya'ni

$$\int_{\alpha}^{x}S(t)dt=\int_{\alpha}^{x}u_1(t)dt+\int_{\alpha}^{x}u_2(t)dt+\int_{\alpha}^{\lambda}u_3(t)dt+...+\int_{\alpha}^{x}u_n(t)dt+.....$$

2-teorema. Agar (a, b) kesmada hosilalari uzluksiz bo'lgan funksiyalardan tuzilgan

$$u_1(x)+u_2(x)+u_3(x)+...+u_n(x)+.....$$

qator shu kesmada $S(x)$ yig'indiga yaqinlashsa, va uning hadlarining hosilalaridan tuzilgan

$$u_1'(x)+u_2'(x)+u_3'(x)+...+u_n'(x)+.....$$

qator o'sha kesmada kuchaytirilgan bo'lsa, hosilalar qatorining yig'nidisi boshlang'ich qator yig'indisining hosilasiga teng, ya'ni

$$S'(x)=u_1'(x)+u_2'(x)+u_3'(x)+...+u_n'(x)+.....$$

bo'ladi.

DARAJALI QATORLAR. QATORLAR YORDAMIDA TAQRIBIY HISOBLASHLAR

Ta'rif. Hadlari darajali funksiyalardan iborat bo'lgan

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n \qquad (1)$$

ko'rinishdagi qator darajali qator deb ataladi.

Abel teoremasi. 1) Agar darajali qator $x=x_0\neq 0$ nuqtada yaqinlashsa, u holda bu qator $-|x_0|<x<|x_0|$ oraliqda absolyut yaqinlashadi; 2) Agar darajali qator $x=x_0'$ nuqtada uzoqlashsa, u holda bu qator $-|x_0'|>x$ va $x>|x_0'|$ oraliqlarda uzoqlashadi;

Isbot. 1) Teoremaning shartiga ko'ra

$$a_0+a_1x_0+a_2x_0^2+\ldots+a_nx_0^n+\ldots \qquad (2)$$

qator yaqinlashadi, demak $n\to\infty$ da $a_nx_0^n\to 0$, bu degani shunday bir musbat M soni mavjud bo'ldiki, qatorning hamma hadi absolyut qiymati bo'yicha M dan kichik bo'ladi. (1) qatorni

$$a_0 + a_1x_0\left(\frac{x}{x_0}\right) + a_2x_0^2\left(\frac{x}{x_0}\right)^2 + \ldots + a_nx_0^n\left(\frac{x}{x_0}\right)^n + \ldots \qquad (3)$$

ko'rinishda yozib olamiz va

$$|a_0| + |a_1x_0|\left|\frac{x}{x_0}\right| + |a_2x_0^2|\left|\frac{x}{x_0}\right|^2 + \ldots + |a_nx_0^n|\left|\frac{x}{x_0}\right|^n + \ldots \qquad (4)$$

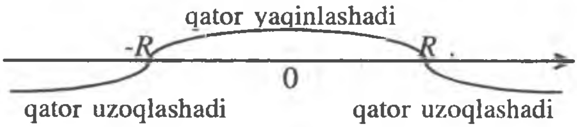
qatorni ko'raylik. Bu qatorning hadlari

$$M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + \ldots + M\left|\frac{x}{x_0}\right|^n + \ldots \qquad (5)$$

qatorning mos hadidan kichik. $|x|<|x_0|$ tengsizlik bajarilganda (5) qator maxraji $q=\left|\frac{x}{x_0}\right|<1$ ga teng bo'lgan cheksiz kamayuvchi geometrik progressiyani tashkil etadi, demak, yaqinlashadi. Shunday qilib, (5) qator yaqinlashgani uchun (4) qator ham yaqinlashadi, natijada (3) qator yoki (1) qator absolyut yaqinlashadi.

2) Endi teoremaning ikkinchi qismini ham isbot qilish unchalik qiyin emas: faraz qilaylik x_0^{I} nuqtada (1) qator uzoqlashsin. U holda $|x|>|x_0^{I}|$ tengsizlikni qanoatlantiruvchi har qanday x nuqtada ham qator uzoqlashadi. Demak, $-|x_0^{I}|>x$ va $x>|x_0^{I}|$ oraliqlarda (1) qator uzoqlashadi. Shunday qilib, teorema to'la isbot qilindi.

Ta'rif. Darajali qatorning yaqinlashish sohasi markazi koordinat boshida yotadigan intervaldan iboratdir. Darajali qatorning yaqinlashish intervali deb shunday $-R$ dan $+R$ gacha bo'lgan intervalga aytiladiki, bu intervalning ichida yotadigan har qanday x nuqtada qator absolyut yaqinlashadi, intervalning tashqarisida yotadigan istalgan x nuqtada esa uzoqlashadi.



R soni darajali qatorning yaqinlashish radiusi deb aytiladi. Intervalning oxirlarida (ya'ni $x=-R$ va $x=R$ nuqtalarida) berilgan qatorning yaqinlashishi va uzoqlashishi haqidagi savol har bir qator uchun alohida yechiladi.

1. Darajali qatorning yaqinlashish intervalini (yoki yaqinlashish radiusini) topish

(1) darajali qatorni

$$|a_0|+|a_1||x|+|a_2||x|^2+|a_3||x|^3+\ldots+|a_n||x|^n+\ldots \qquad (6)$$

ko'rinishda yozib olamiz. Bu qator musbat hadli qator bo'lgani uchun uning yaqinlashishini Dalamber alomatiga ko'ra aniqlaymiz. Faraz qilaylik, quyidagi limit mavjud bo'lsin:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|=L|x|$$

Unda agar $L|x|<1$ bo'lsa, ya'ni $|x|<1/L$ yoki $-1/L<x<1/L$ intervalda qator absolyut yaqinlashadi.

Agar $L|x|>1$ bo'lsa, ya'ni $|x|>1/L$ yoki $-1/L>x$ va $x>1/L$ intervallarda qator uzoqlashadi. Yaqinlashish radiusi

$$R=\frac{1}{L}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

formulaga ko'ra topiladi. Shunga o'xshab R ni Koshi alomatini qo'llab ham topish mumkin: $R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}$;

Misol. $\frac{x}{2}+\frac{2^2x^2}{2^2}+\frac{3^2x^3}{2^3}+\frac{4^2x^4}{2^4}+\ldots+\frac{n^2x^n}{2^n}+\ldots$ darajali qatorning yaqinlashish intervali topilsin.

Yechish. Bu yerda $a_n=\frac{n^2}{2^n}$, $a_{n+1}=\frac{(n+1)^2}{2^{n+1}}$, demak,

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n^2 2^{n+1}}{2^n(n+1)^2}=2$$

Javob. Berilgan darajali qatorning yaqinlashish intervali $-2<x<2$ tengsizlikdan iborat. Intervalning chegaralarida qator uzoqlashadi.

2. Teylor va Makloren qatorlari

Agar $y=f(x)$ funksiya $x=a$ nuqtaning atrofida $(n+1)$ tartibgacha hosilaga ega bo'lsa Teylor formulasi deb ataluvchi

$$f(x)=f(a)+\frac{x-a}{1!}f^{I}(a)+\frac{(x-a)^2}{2!}f^{II}(a)+\ldots+ \\ +\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x). \qquad (1)$$

formula bizga ma'lum, bu yerda

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)]$$

qoldiq had edi, $0<\theta<1$.

Agar $f(x)$ funksiya $x=a$ nuqtaning atrofida istalgan tartibgacha hosilaga ega bo'lsa, Teylor formulasidagi n istalgancha katta qilib olinishi mumkin. Faraz qilaylik $\lim\limits_{n\to\infty}R_n(x)=0$ bajarilsin, u holda (1) formulada $n\to\infty$ da limitga o'tib, o'ng tomonda qator hosil qilinadi va u Teylor qatori deb ataladi:

$$f(x)=f(a)+\frac{x-a}{1!}f^{I}(a)+\frac{(x-a)^2}{2!}f^{II}(a)+...+\frac{(x-a)^n}{n!}f^{(n)}(a)+... \quad (2)$$

(2) tenglik $\lim_{n\to\infty} R_n(x)\to 0$ bajarilgandagina o'rinlidir.

Agar Teylor qatorida a=0 desak uning xususiy ko'rinishi bo'lgan Makloren qatori hosil bo'ladi:

$$f(x)=f(0)+\frac{x}{1!}f^{I}(0)+\frac{x^2}{2!}f^{II}(0)+...+\frac{x^n}{n!}f^{(n)}(0)+... \quad (3)$$

Berilgan $f(x)$ funksiyani Teylor qatoriga yoyish uchun:

a) $f(x)$ funksiyaning barcha tartibdagi hosilalarining $x=a$ nuqtadagi qiymatlari hisoblanadi va Teylor qatorining yoyilmasiga olib borib qo'yiladi;

b) hosil bo'lgan qatorning yaqinlashish sohasi topiladi.

Misol. $f(x)=2^x$ funksiya x ning darajalari bo'yicha Teylor qatoriga yoyilsin.

Yechish. a) 2^x funksiyaning barcha tartibdagi hosilalarini x=0 nuqtadagi qiymatlarini topamiz:

$f(x)=2^x$, $\quad f(0)=1$;

$f^{I}(x)=2^x\ln 2$, $\quad f^{I}(0)=\ln 2$;

$f^{II}(x)=2^x\ln^2 2$, $\quad f^{II}(0)=\ln^2 2$;

. . . .

$f^{(n)}(x)=2^x\ln^n 2$, $\quad f^{(n)}(0)=\ln^n 2$;

. . . .

Endi topilgan qiymatlarni (3) ifodaga qo'yib, 2^x funksiya uchun x ning darajalari bo'yicha Teylor qatorini hosil qilamiz:

$$2^x=1+\frac{\ln 2}{1!}x+\frac{\ln^2 2}{2!}x^2+...+\frac{\ln^n 2}{n!}x^n+...$$

b) hosil bo'lgan qatorning yaqinlashish sohasini topamiz:

$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{\ln^n 2(n+1)!}{n!\ln^{n+1}2}=\infty$ dan ko'rinadiki topilgan qator x ning har qanday qiymatlarida yaqinlashadi.

3. Asosiy funksiyalar yoyilmasining jadvali:

$$1.\ e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots \qquad (|x|<\infty);$$

$$2.\ \sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots \qquad (|x|<\infty);$$

$$3.\ \cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad (|x|<\infty);$$

$$4.\ \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+x^4+\ldots \qquad (|x|<1);$$

$$5.\ (1+x)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n=$$

$$=1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots \quad (|x|<1)$$

(binomial m-istalgan haqiqiy son);

$$6.\ \ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots \qquad (-1<x\le 1);$$

$$7.\ arctgx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots \qquad (|x|\le 1);$$

4. Qatorlar yordamida taqribiy hisoblashlar

$f(x)$ funksiyaning x_0 nuqtadagi qiymatini taqribiy hisoblash uchun bu funksiya darajali qatorga yoyiladi va yoyilmadagi x lar o'rniga x_0 qiymat qo'yiladi. Shundan keyin $f(x_0)$ qiymatni kerakli aniqlikda hisoblash uchun qatorning zarur sondagi boshlang'ich hadlari olinadi. Masalan, arcsin1/10 ni hisoblash uchun arcsin$x$ funksiyani darajali qatorga yoyish (x ning darajalari bo'yicha) va undagi x lar o'rniga 1/10 qiymatni qo'yish kerak.

4-misol. $\sqrt[4]{e}$ 0,00001 aniqlikda hisoblansin.

Yechilishi. $e^x\approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+;$ yoyilmada

$x=1/4$ deb olamiz:

$$e^{1/4}\approx 1+\frac{1}{4}+\frac{1}{4^2\cdot 2!}+\frac{1}{4^3\cdot 3!}+\frac{1}{4^4\cdot 4!}+\dots$$

Ushbu hisoblashda $|x|<n+1$ lar uchun qilinadigan xatolik $|R_n|<\dfrac{|x|^{n+1}}{n!(n+1-|x|)}$ tengsizlikdan topiladi.

Agar $n=4$ deb, beshta hadni olsak ko'rilayotgan hisoblashdagi xatolik 0,00001 dan oshmaydi:

$$R_n<\frac{x^{4+1}}{4!(4+1-x)}=\frac{1}{4^5\cdot 4!(5-1/4)}<0{,}00001$$

5-misol. $\cos 1^\circ$ 0,0001 aniqlikda hisoblansin.

Yechilishi. Kosinusning taqribiy qiymatlarini hisoblashda

$$\cos x\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+(-1)^n\frac{x^{2n}}{(2n)!}$$

formuladan foydalaniladi.

Bunda qilinadigan xatolik

$$|R_{2n}|\le\frac{x^{2n+2}}{(2n+2)!}$$

tengsizlikdan topiladi.

Demak, $\cos 1^0=\cos\dfrac{\pi}{180}$ bo'lgani uchun kosinusning yoyilmasida $x=\dfrac{\pi}{180}$ deb birinchi ikkita hadni olsak,

$$\cos 1^0\approx 1-\frac{\pi^2}{180^2\cdot 2!}\approx 0{,}9998$$

hosil bo'ladi. Bunda qilingan xatolik nihoyatda kichikdir:

$$|R_2|\le\frac{\pi^4}{180^4\cdot 4!}<\frac{4^4}{180^4\cdot 4!}=\frac{1}{45^2\cdot 24}<0{,}0000001.$$

FURYE QATORLARI. TRIGONOMETRIK QATOR. MASALANING QO'YILISHI

Ushbu ko'rinishdagi

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+...=$$
$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad (1)$$

funksional qator trigonometrik qator deyiladi.

(1) da a_0, a_n va b_n (n=1, 2, 3, ...,) o'zgarmas sonlar trigonometrik qatorning koeffitsientlari deb ataladi.

Agar (1) qator yaqinlashsa, uning yig'indisi $f(x)$ davri 2π bo'lgan davriy funksiya bo'ladi, chunki $\sin nx$ va $\cos nx$ davri 2π bo'lgan davriy funksiyalardir. Shunday qilib:

$$f(x)=f(x+2\pi).$$

Endi quyidagi masalani qaraymiz: Davri 2π bo'lgan $f(x)$ davriy funksiya berilgan bo'lsin. Qanday shartlar bajarilganda $f(x)$ uchun berilgan funksiyaga yaqinlashuvchi trigonometrik qatorni topish mumkin? Faraz qilaylik, davri 2π bo'lgan $f(x)$ davriy funksiya $(-\pi;\ \pi)$ intervalda shu funksiyaga yaqinlashuvchi trigonometrik qatorni ifoda etsin, ya'ni shu qatorning yig'indisi bo'lsin

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad (2)$$

Faraz qilaylik, bu tenglikning chap tomonidagi funksiyadan olingan integral (2) qator hadlaridan olingan integrallarning yig'indisiga teng bo'lsin. Bu esa berilgan trigonometrik qatorning koeffitsientlaridan tuzilgan sonli qator absolyut yaqinlashganda, ya'ni

$$\left|\frac{a_0}{2}\right|+|a_1|+|b_1|+|a_2|+|b_2|+...+|a_n|+|b_n|+... \quad (3)$$

musbat qator yaqinlashganda bajariladi. Bu holda (1) qator kuchaytirilgan, demak uni $-\pi$ dan $+\pi$ gacha hadlab integrallash mumkin. Bundan a_0 koeffitsientini hisoblash uchun foydalanamiz. Agar (2) tenglikning ikkala qismini $-\pi$ dan $+\pi$ gacha integrallasak;

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nxdx + \int_{-\pi}^{\pi} b_n \sin nxdx \right)$$

va o'ng tomondagi integrallarni alohida hisoblab, quyidagilarni hosil qilamiz:

$$\int_{-\pi}^{\pi} \frac{a_0}{2}dx = \pi a_0$$

$$\int_{-\pi}^{\pi} a_n \cos nxdx = a_n \int_{-\pi}^{\pi} \cos nxdx = \frac{a_n \sin nx}{n}\Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} b_n \sin nxdx = b_n \int_{-\pi}^{\pi} \sin nxdx = -\frac{b_n \cos nx}{n}\Big|_{-\pi}^{\pi} = 0$$

Demak,

$$\int_{-\pi}^{\pi} f(x)dx = \pi a_0$$

bundan

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (4)$$

Qatorning qolgan koeffitsientlarini hisoblash uchun ba'zi bir aniq integrallarni qarab chiqamiz: Faraz qilaylik, n va k butun sonlar bo'lsin: agar $n \neq k$ bo'lsa:

$$\begin{cases} \int_{-\pi}^{\pi} \cos nx \cos kxdx = 0 \\ \int_{-\pi}^{\pi} \cos nx \sin kxdx = 0 \\ \int_{-\pi}^{\pi} \sin nx \sin kxdx = 0 \end{cases} \qquad (*)$$

agar $n=k$ bo'lsa

$$\begin{cases} \int_{-\pi}^{\pi} \cos^2 kx dx = \pi \\ \int_{-\pi}^{\pi} \cos nx \sin kx dx = 0 \\ \int_{-\pi}^{\pi} \sin^2 kx dx = \pi \end{cases} \qquad (**)$$

Masalan (*) dagi birinchi integralni hisoblaymiz:

$$\cos nx \cos kx = \frac{1}{2}[\cos(n+k)x + \cos(n-k)x]$$

bo'lgani uchun:

$$\int_{-\pi}^{\pi} \cos nx \cos kx dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+k)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-k)x dx = 0.$$

Demak (*) dagi qolgan formulalar shu tariqa hisoblanadi, (**) gruppadagi integrallar esa bevosita hisoblanadi. Endi yuqoridagilardan foydalanib (2) qatorning a_k va b_k koeffitsientlarini hisoblaymiz. Agar (2) qatorning ikkala qismini $\cos kx$ ga ko'paytirsak shuni hosil qilamiz:

$$f(x)\cos kx = \frac{a_0}{2}\cos kx + \sum_{n=1}^{\infty}(a_n \cos nx \cos kx + b_n \sin nx \cos kx) \quad (2')$$

Tenglikning o'ng tomonidagi hosil bo'lgan qator kuchaytirilgan qatordir, chunki uning hadlari absolyut qiymati bo'yicha (3) yaqinlashuvchi musbat qatorning hadlaridan katta bo'la olmaydi. Shuning uchun uni istalgan kesmada hadlab integrallash mumkin. Demak (2') tenglikni $-\pi$ dan $+\pi$ gacha integrallaymiz

$$\int_{-\pi}^{\pi} f(x)\cos kx dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos kx dx + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} \cos nx \cos kx dx +$$

$$+ b_n \int_{-\pi}^{\pi} \sin nx \cos kx dx).$$

(*) va (**) formulalarni e'tiborga olib tenglikning o'ng tomonidagi a_k koeffitsientli integraldan boshqa hamma integrallarning nolga teng ekanligini ko'rish mumkin.

Demak: $\int_{-\pi}^{\pi} f(x)\cos kxdx = a_k \int_{-\pi}^{\pi} \cos^2 kxdx = a_k\pi$ bundan:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx \qquad (5)$$

Endi agar (2) tenglikning ikkala qismini sinkx ga ko'paytirib, yana $-\pi$ dan $+\pi$ gacha integrallasak shularni hosil qilamiz:

$$\int_{-\pi}^{\pi} f(x)\sin kxdx = b_k \int_{-\pi}^{\pi} \sin^2 kxdx = b_k\pi$$

Bundan

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx \qquad (6)$$

Demak, yuqorida hosil qilingan (4), (5) va (6) formulalar bo'yicha aniqlangan koeffitsientlar $f(x)$ funksiyaning Furye koeffitsientlari deb ataladi, shunday koeffitsientli (1) trigonometrik qator esa $f(x)$ funksiyaning Furye qatori deyiladi.

Endi yuqorida qo'yilgan masalani yanada aniqroq bayon etamiz, demak berilgan funksiya uchun tuzilgan Furye qatori yaqinlashishi va tuzilgan bu Furye qatorining yig'indisi berilgan funksiyaning mos nuqtalaridagi qiymatlariga teng bo'lishi uchun bu funksiya qanday xossalarga ega bo'lishi kerak? Ana shu masalani hal qilish uchun biz $f(x)$ funksiyani Furye qatoriga yoyish uchun yetarli shartlarni beruvchi quyidagi teoremani keltiramiz:

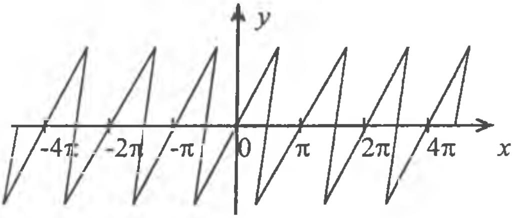
Teorema (Dirixle). Agar davri 2π bo'lgan $f(x)$ davriy funksiya $(-\pi, \pi)$ kesmada bo'lakli monoton va chegaralangan bo'lsa, bu funksiya uchun tuzilgan Furye qatori shu kesmaning hamma nuqtalarida yaqinlashadi.Hosil qilingan qatorning $S(x)$ yig'indisi $f(x)$ funksiyaning uzluksizlik nuqtalaridagi qiymatiga teng. $f(x)$ funksiyaning uzilish nuqtalarida qatorning yig'indisi funksiyaning o'ng va chap limitlarining o'rta arifmetik qiymatiga teng bo'ladi, ya'ni agar $x=s$ nuqta $f(x)$ funksiyaning uzilish nuqtasi bo'lsa, u vaqtda $S(x)|_{x=c} = \frac{f(c-0)+f(c+0)}{2}$.

Bu teoremadan ko'rinadiki, Furye qatori bilan tasvirlanuvchi funksiyalar sinfi ancha keng ekanligi kelib chiqadi. Berilgan teoremani isbotsiz qabul qilamiz.

1. Funksiyalarni Furye qatoriga yoyishga misollar

1-misol. Davri 2π бўлган $f(x)$ davriy funksiya quyidagicha aniqlangan. $f(x)=x \quad -\pi<x\leq\pi$.

Bu funksiya bo'lakli monoton va chegaralangan. Shuning uchun uni Furye qatoriga yoyish mumkin.



1-rasm.

Demak, yuqoridagi (4) formuladan:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\cdot\frac{x^2}{2}\bigg|_{-\pi}^{\pi} = 0$$

a_k ni topish uchun yuqoridagi (5) formulani tatbiq etib va bo'laklab integrallab, shuni hosil qilamiz:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx dx = \frac{1}{\pi}\left[x\frac{\sin kx}{k}\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}\sin kx dx = 0$$

b_k ni topish uchun (6) formulani qo'llaymiz:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx dx = \frac{1}{\pi}\left[-x\frac{\cos kx}{k}\Big|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx dx\right] = (-1)^{k+1}\frac{2}{k};$$

Shunday qilib quyidagi qator hosil bo'ldi:

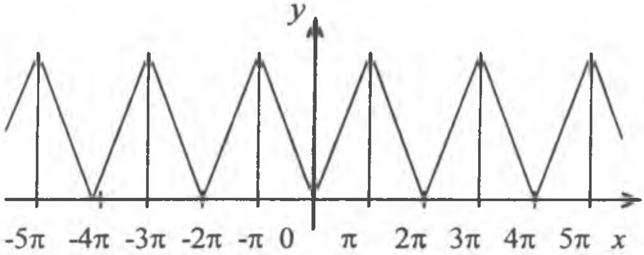
$$f(x) = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ... + (-1)^{k+1}\frac{\sin kx}{k} + ...\right]$$

Bu tenglik uzilish nuqtalaridan boshqa hamma nuqtalarda o'rinlidir. Qatorning har bir uzilish nuqtasidagi yig'indisi uning o'ngdan va chapdan limitlarining o'rta arifmetigi nolga tengdir.

2-misol. Davri 2π bo'lgan $f(x)$ davriy funksiya quyidagicha aniqlangan:

$-\pi \le x \le 0$ bo'lsa, $f(x)=-x$, $0<x<\pi$ bo'lsa, $f(x)=x$,

(ya'ni $f(x)=|x|$ bu funksiya ham $-\pi \le x \le \pi$ bo'lakli monoton va chegaralangandir.



2-rasm.

Uning Furye koeffitsientlarini aniqlaymiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)dx + \int_{0}^{\pi} xdx\right] = \pi.$$

$$a_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos kxdx + \int_{0}^{\pi} x\cos kxdx\right] =$$

$$= \frac{1}{\pi}\left[-\frac{x\sin kx}{k}\Big|_{-\pi}^{0} + \frac{1}{k}\int_{-\pi}^{0}\sin kxdx + x\frac{\sin kx}{k}\Big|_{0}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi}\sin kxdx\right] =$$

$$= \frac{1}{k\pi}\left[-\frac{\cos kx}{k}\Big|_{-\pi}^{0} + \frac{\cos kx}{k}\Big|_{0}^{\pi}\right] = \frac{2}{k^2\pi}(\cos k\pi - 1) = \begin{cases} k \text{ juft bolsa, } 0; \\ k \text{ toq bolsa, } -\dfrac{4}{k^2\pi}; \end{cases}$$

$$b_k = \frac{1}{k}\left[\int_{-\pi}^{0}(-x)\sin kxdx + \int_{0}^{\pi} x\sin kxdx\right] = 0$$

Shunday qilib, quyidagi qatorni hosil qilamiz:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots + \frac{\cos(2p+1)x}{(2p+1)^2} + \ldots\right]$$

Bu qator hamma nuqtalarda yaqinlashadi va uning yig'indisi berilgan funksiyaga teng.

DAVRIY FUNKTSIYALARNI FURYE QATORIGA YOYISH

Endi davri 2π bo'lgan $\psi(x)$ davriy funksiyaning λ har qanday son bo'lganda ham o'rinli bo'lgan quyidagi xossasini keltiramiz:

$$\int_{-\pi}^{\pi}\psi(x)dx=\int_{\lambda}^{\lambda+2\pi}\psi(x)dx$$

Haqiqatan agar $\psi(\xi-2\pi)=\psi(\xi)$ bo'lganligidan $x=\xi-2\pi$ deb olib, c va d har qanday bo'lganda ham ushbu tenglikni yoza olamiz:

$$\int_{c}^{d}\psi(x)dx=\int_{c+2\pi}^{d+2\pi}\psi(\xi-2\pi)d\xi=\int_{c+2\pi}^{d+2\pi}\psi(\xi)d\xi=\int_{c+2\pi}^{d+2\pi}\psi(x)dx$$

Agar $c=-\pi$, $d=\lambda$ deb olsak, quyidagi tenglik hosil bo'ladi:

$$\int_{-\pi}^{\lambda}\psi(x)dx=\int_{\pi}^{\lambda+2\pi}\psi(x)dx$$

shuning uchun:

$$\int_{\lambda}^{\lambda+2\pi}\psi(x)dx=\int_{\lambda}^{-\pi}\psi(x)dx+\int_{-\pi}^{\pi}\psi(x)dx+\int_{\pi}^{\lambda+2\pi}\psi(x)dx=$$

$$=\int_{\lambda}^{-\pi}\psi(x)dx+\int_{-\pi}^{\pi}\psi(x)dx+\int_{-\pi}^{\lambda}\psi(x)dx=\int_{-\pi}^{\pi}\psi(x)dx.$$

Demak, bu ko'rsatilgan xossa $\psi(x)$ davriy funksiyadan uzunligi funksiyaning davriga teng bo'lgan, istalgan kesma bo'yicha olingan integral doimo bitta va faqat bitta qiymatga teng bo'lishini bildiradi. Endi shu xossani geometrik jihatdan shunday tasvirlash mumkin. Yuqorida isbot qilinganiga ko'ra Furye koeffitsientlarini hisoblashda biz integrallash oralig'ini $(-\pi,\ \pi)$ ni $(\lambda,\ \lambda+2\pi)$ integrallash oralig'i bilan almashtirishimiz mumkinligi kelib chiqdi, bunda

$$a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)dx; a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nxdx; b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nxdx;$$

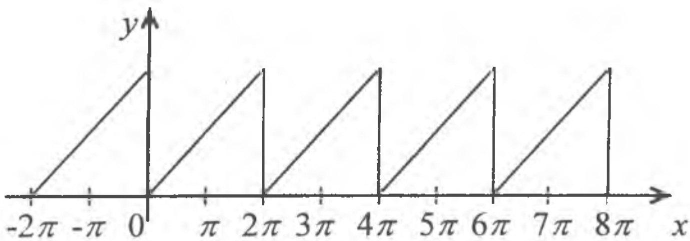
bunda λ-ixtiyoriy son.

Bundan esa shartga asosan $f(x)$ davriy funksiyaning davri 2π ekanligi kelib chiqadi.

Demak, $f(x)\cos nx$ va $f(x)\sin nx$ funksiyalar ham davri 2π bo'lgan davriy funksiyalar bo'ladi. Isbot qilingan xossa ba'zi hollarda koeffitsientlarni topish jarayonini juda soddalashtiradi, ana shuni konkret misolda ko'ramiz.

Misol. $0 \le x \le 2\pi$ kesmada $f(x)=x$ tenglik bilan berilgan davri 2π bo'lgan $f(x)$ davriy funksiyani Furye qatoriga yoying:

Bu funksiyaning grafigi:



1-rasm.

Demak, funksiya $[-\pi; \pi]$ kesmada ikkita formula bilan berilgan: $[-\pi; 0]$ kesmada $f(x)=x+2\pi$ va $[0; \pi]$ kesmada $f(x)=x$.

Ayni paytda $[0; 2\pi)$ kesmada bu funksiya sodda $f(x)=x$ formula bilan berilgan. Shuning uchun bu funksiyani Furye qatoriga yoyishda $\lambda=0$ deb (1) formuladan foydalanamiz.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = 2\pi;$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi} x\cos nxdx = \frac{1}{\pi}\left[\frac{x\sin x}{n} + \frac{\cos nx}{n^2}\right]\Bigg|_0^{2\pi} = 0;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{2\pi} x\sin nxdx = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]\Bigg|_0^{2\pi} = -\frac{2}{n};$$

Demak:

$$f(x)=\pi-2\sin x-\frac{2}{2}\sin 2x-\frac{2}{3}\sin 3x-\frac{2}{4}\sin 4x-\frac{2}{5}\sin 5x-...$$

Bu qator uzilish nuqtalaridan (ya'ni $x=0$, 2π, 4π,... nuqtalardan) boshqa barcha nuqtalarda berilgan funksiyani aniqlaydi. Bu nuqtalarda qatorning yig'indisi $f(x)$ funksiyaning o'ng va chap limitlari qiymatining yarim yig'indisiga (ya'ni bu holda π soniga) teng.

1. Juft va toq funksiyalar uchun Furye qatorlari

Juft va toq funksiyalarning ta'rifidan, agar $\psi(x)$ funksiya juft bo'lsa,

$$\int_{-\pi}^{\pi}\psi(x)dx=2\int_{0}^{\pi}\psi(x)dx$$

ekanligi kelib chiqadi.

Haqiqatan:

$$\int_{-\pi}^{\pi}\psi(x)dx=\int_{-\pi}^{0}\psi(x)dx+\int_{0}^{\pi}\psi(x)dx=\int_{0}^{\pi}\psi(-x)dx+\int_{0}^{\pi}\psi(x)dx=$$

$$=\int_{0}^{\pi}\psi(x)dx+\int_{0}^{\pi}\psi(x)dx=2\int_{0}^{\pi}\psi(x)dx.$$

chunki juft funksiyaning ta'rifiga ko'ra: $\psi(-x)=\psi(x)$.

Shuning singari φ(x) toq funksiya bo'lsa

$$\int_{-\pi}^{\pi}\varphi(x)dx=\int_{0}^{\pi}\varphi(-x)dx+\int_{0}^{\pi}\varphi(x)dx=-\int_{0}^{\pi}\varphi(x)dx+\int_{0}^{\pi}\varphi(x)dx=0$$

ekanligini isbot qilish mumkin.

Agar $f(x)$ toq funksiya Furye qatoriga yoyilsa, $f(x)\cos nx$ ko'paytma ham toq funksiya, $f(x)\sin nx$ esa juft funksiya bo'ladi. Demak,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0; \quad a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kxdx=0;$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin kx dx; \qquad (1)$$

ya'ni toq funksiyaning Furye qatori faqat sinuslarni o'z ichiga oladi.

Agar juft funksiya Furye qatoriga yoyilsa, $f(x)\sin kx$ ko'paytma toq, $f(x)\cos kx$ esa juft funksiya bo'ladi, demak,

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx; \quad a_k = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos kx dx;$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx = 0; \qquad (2)$$

ya'ni juft funksiyaning Furye qatori faqat kosinuslarni o'z ichiga oladi. Hosil qilingan (1) va (2) formulalar berilgan funksiya juft yoki toq bo'lgan hollarda Furye koeffitsientlarini topishda hisoblashlarni soddalashtirishga imkon beradi.

2. Davri $2l$ bo'lgan funksiyalar uchun Furye qatori

Agar $f(x)$ funksiya davri $2l$, umuman aytganda 2π dan farqli bo'lgan davriy funksiya berilgan bo'lsin. Uni Furye qatoriga yoyamiz. Buning uchun o'zgaruvchini

$$x = \frac{l}{\pi}t, \qquad (*)$$

formula bo'yicha almashtiramiz. U holda $f\left(\frac{lt}{\pi}\right)$ funksiya t ning davri 2π bo'lgan davriy funksiyasi bo'ladi. Uni $-\pi \le x \le \pi$ kesmada Furye qatoriga yoyish mumkin:

$$f\left(\frac{lt}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_k \cos kt + b_k \sin kt\right) \qquad (1)$$

bunda

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)dt,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\cos ktdt,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\sin ktdt,$$

Endi eski o'zgaruvchiga qaytib,

$$x = \frac{l}{\pi}t, \qquad t = x\frac{\pi}{l}, \quad dt = \frac{\pi}{l}dx.$$

quyidagilarni hosil qilamiz:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx,$$

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi kx}{l}dx, \tag{2}$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi kx}{l}dx,$$

Natijada (1) formula ushbu ko'rinishni oladi:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos\frac{k\pi}{l}x + b_k\sin\frac{k\pi}{l}x) \tag{3}$$

bundagi a_0, a_k, b_k koeffitsientlar (2) formulalar bo'yicha hisoblanadi. Bu esa davri $2l$ bo'lgan davriy funksiyaning Furye qatoridir.

Davri 2π bo'lgan davriy funksiyalardan hosil qilingan Furye qatorlari uchun o'rinli bo'lgan barcha teoremalar, biror $2l$ davrli davriy funksiyalardan hosil qilingan Furye qatorlari uchun o'rinli bo'ladi.

Misol. $[-l;\ l]$ kesmada $f(x)=|x|$ tenglik bilan berilgan $2l$ davrli f(x) davriy funksiya Furye qatoriga yoyilsin.

Yechish. Qaralayotgan funksiya juft bo'lgani uchun

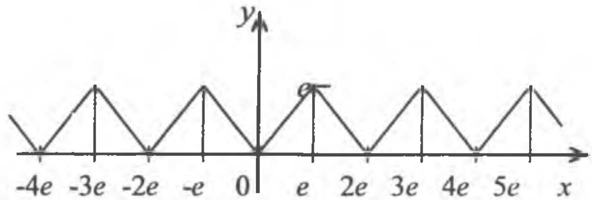
$$b_k = 0,\ a_0 = \frac{2}{l}\int_0^{l} xdx = l,\ \ a_k = \frac{2}{l}\int_0^{l} x\cos\frac{k\pi}{l}xdx =$$

$$=\frac{2l}{\pi^2}\int_0^{\pi} x\cos kx dx=\begin{cases} k \quad \text{juft}, \quad 0 \\ k \quad \text{toq}, \quad -\dfrac{4l}{\pi^2 k^2} \end{cases}$$

Demak, yoyilma quyidagi ko'rinishda bo'ladi:

$$|x|=\frac{l}{2}-\frac{4l}{\pi^2}\left[\frac{\cos\pi x/l}{1}+\frac{\cos 3\pi x/l}{3^2}+...+\frac{\cos(2p+1)\pi x/l}{(2p+1)^2}+...\right]$$

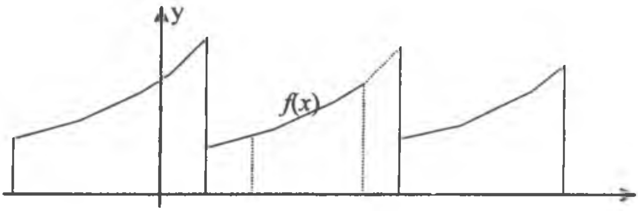
Berilgan funksiyaning grafigi quyidagi ko'rinishda bo'ladi:



2-rasm.

3. Davriy bo'lmagan funksiyalarni Furye qatoriga yoyish

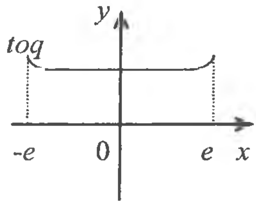
Faraz qilaylik, biror $[a, b]$ kesmada bo'lakli monoton bo'lgan $f(x)$ funksiya berilgan bo'lsin.



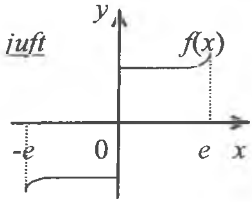
3-rasm.

Berilgan $f(x)$ funksiyani bu funksiya uzluksiz bo'lgan nuqtalarda Furye qatorining yig'indisi sifatida tasvirlash mumkin ekanligini ko'rsatamiz. Buning uchun $[a, b]$ kesmada $f(x)$ funksiya bilan mos tushadigan davri $2\mu\geq b-a$ bo'lgan qandaydir davriy bo'lakli monoton $f_1(x)$ funksiyani qaraymiz. $f_1(x)$ funksiyani Furye qatoriga yoyamiz. Bu qatorning yig'indisi $[a, b]$ kesmaning hamma nuqtalarida (uzilish nuqtalaridan boshqa) berilgan $f(x)$ funksiya bilan ustma-ust tushadi, ya'ni biz $f(x)$ funksiyani $[a, b]$ kesmada Furye qatoriga yoydik.

Endi quyidagi muhim holni qarab chiqamiz. Faraz qilaylik, $f(x)$ funksiya $[0, l]$ kesmada berilgan bo'lsin. Bu funksiyaning aniqlanish sohasini ixtiyoriy ravishda $[-l, 0]$ kesmada to'ldirib, (bo'lakli monotonligini saqlab), bu funksiyani Furye qatoriga yoyishimiz mumkin. Jumladan, agar berilgan funksiyaning aniqlanish sohasini $-l \le x < 0$ da $f(x)=f(-x)$ bo'ladigan qilib to'ldirsak, natijada juft funksiya hosil qilamiz.



4-rasm.



5-rasm.

Bu holda $f(x)$ funksiya juft holda davom ettirilgan deyiladi. Bu funksiya faqat kosinuslarni o'z ichiga olgan Furye qatoriga yoyiladi. Shunday qilib, biz $[0, l]$ kesmada berilgan $f(x)$ funksiyani kosinuslar bo'yicha yoydik. Agar $f(x)$ funksiyaning aniqlanish sohasini $-l \le x < 0$ da $f(x)=-f(-x)$ bo'ladigan qilib to'ldirsak, sinuslar bo'yicha yoyiladigan toq funksiyani hosil qilamiz, bu holda $f(x)$ funksiya toq holda davom ettirilgan deyiladi.

Demak, agar $[0, l]$ kesmada bo'lakli monoton $f(x)$ funksiya berilgan bo'lsa, uni kosinuslar bo'yicha ham, sinuslar bo'yicha ham Furye qatoriga yoyish mumkin.

Misol. $f(x)=x$ funksiyani $[0, \pi]$ kesmada sinuslar bo'yicha qatorga yoying.

Yechish. Bu funksiyani toq holda davom ettirib,

$$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

qatorni hosil qilamiz.

DIRIXLE INTEGRALI

Davri 2π bo'lgan $f(x)$ davriy funksiya Furye qatorining n-qismiy yig'indisini qaraymiz:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \quad \text{bunda}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt, a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt, b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt.$$

Bu ifodalarni $S_n(x)$ ning formulasiga qo'ysak:

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{k=1}^{n}\left[\frac{\cos kx}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt + \frac{\sin kx}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt\right]$$

yoki coskx va sinkx ni integral ostiga kiritamiz, chunki coskx va sinkx larni o'zgarmas deb qarash mumkin:

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} f(t)\cos kx \cos ktdt + \int_{-\pi}^{\pi} f(t)\sin kx \sin ktdt\right]$$

Endi $1/\pi$ ni qavsdan chiqarib va integrallar yig'indisini yig'indining integraliga almashtirib, quyidagini hosil qilamiz:

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{f(t)}{2} + \sum_{k=1}^{n}\left[f(t)\cos kx \cos kt + f(t)\sin kx \sin kt\right]\right\}dt,$$

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{n}\left[\cos kx \cos kt + \sin kx \sin kt\right]\right\}dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right\}dt \quad (1)$$

Endi o'rta qavs ichidagi ifodani quyidagicha almashtiramiz. Faraz qilaylik,

$$G_n(z) = 1/2 + \cos z + \cos 2z + \ldots + \cos nz$$

bo'lsin. U holda:

$$2G_n(z)\cos z=\cos z+2\cos z\cos z+2\cos z\cos 2z+\ldots+2\cos z\cos nz=\cos z+$$
$$+(1+\cos 2z)+(\cos z+\cos 3z)+(\cos 2z+\cos 4z)+\ldots+[\cos(n-1)z+$$
$$+\cos(n+1)z]=1+2\cos z+2\cos 2z+\ldots+2\cos(n-1)z+\cos nz+\cos(n+1)z;$$

yoki

$$2G_n(z)\cos z=2Gn(z)-\cos nz+\cos(n+1)z;$$

$$G_n(x)=\frac{\cos nz-\cos(n+1)z}{2(1-\cos z)}$$

lekin: $\cos nz-\cos(n+1)z=2\sin(2n+1)z/2\sin z/2$;

Demak: $G_n(x)=\dfrac{\sin(2n+1)z/2}{2\sin z/2}$.

Shunday qilib, (1) tenglikni yuqoridagilarga asosan shunday yozish mumkin:

$$S_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}dt$$

Integral ostidagi funksiya davriy funksiya bo'lgani uchun bu integral integrallash uzunligi 2π bo'lgan har qanday kesmada o'z qiymatini saqlaydi. Shunga asosan buni shunday yozish mumkin:

$$S_n(x)=\frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}dt$$

Endi $t-x=\alpha$, $t=x+\alpha$ deb olib, yangi α o'zgaruvchini kiritamiz. U vaqtda shu formulani hosil qilamiz:

$$S_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+a)\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}d\alpha \qquad (2)$$

Bu formulaning o'ng tomonidagi integral Dirixle integrali deb ataladi. Agar (2) dan $f(x)\equiv 1$ teng bo'lsa va $k>0$ $a_0=2$, $a_k=0$, $b_k=0$ bo'lganda $Sn(x)=1$ bo'ladi. Bundan quyidagi ayniyatga kelamiz.

$$1=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}d\alpha \qquad (3)$$

Demak, natijada biz Furye qatorining n-qismiy yig'indisini biror ixtiyoriy integral orqali ifodalovchi formulasini hosil qildik.

1. Berilgan nuqtada Furye qatorining yaqinlashishi

Faraz qilaylik, $f(x)$ funksiya $[-\pi; \pi]$ kesmada bo'lakli uzluksiz bo'lsin. Endi (3) ayniyatni $f(x)$ ga ko'paytirib va $f(x)$ ni integral ishorasi ostiga kiritib quyidagini hosil qilamiz:

$$f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}d\alpha \qquad (*)$$

Endi (*) tenglikning hadlarini yuqoridagi (2) tenglikning mos hadlaridan ayirib shuni hosil qilamiz:

$$S_n(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+a)-f(x)]\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}d\alpha$$

Demak, Furye qatorining $f(x)$ funksiyaning berilgan nuqtadagi qiymatiga yaqinlashishi $n\to\infty$ da o'ng tomondagi integralning nolga intilishiga bog'liq.

Oxirgi integralni $\sin(2n+1)\frac{\alpha}{2} = \sin n\alpha\cos\frac{\alpha}{2} + \cos n\alpha\sin\frac{\alpha}{2}$ formuladan foydalanib, ikki integralga ajratamiz.

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{\pi}^{\pi}[f(x+a) - f(x)]\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin n\alpha d\alpha +$$

$$+\frac{1}{2}\frac{1}{\pi}\int_{\pi}^{\pi}[f(x+a) - f(x)]\cos n\alpha\, d\alpha$$

Oxirgi tenglikning o'ng tomonidagi birinchi integralni uchta integralga ajratib yozamiz:

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{\pi}^{\pi}[f(x+a) - f(x)]\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin n\alpha d\alpha +$$

$$+\frac{1}{\pi}\int_{\pi}^{\delta}[f(x+a) - f(x)]\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin n\alpha d\alpha +$$

$$+\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+a) - f(x)]\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin n\alpha d\alpha +$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+a) - f(x)]\frac{1}{2}\cos n\alpha d\alpha$$

Faraz qilaylik, $\Phi_1(x) = \frac{f(x+a) - f(x)}{2}$ bo'lsin. $f(x)$ funksiya chegaralangan bo'lakli uzluksiz bo'lganligi uchun $\Phi_1(\alpha)$ ham α argumentning chegaralangan bo'lakli uzluksiz funksiyasi bo'ladi. Demak $n \to \infty$ da oxirgi integral nolga teng bo'ladi, chunki u shu funksiyaning Furye koeffitsientlaridir.

$$\Phi_2(\alpha) = [f(x+a) - f(x)]\frac{\cos\alpha/2}{2\sin\alpha/2}$$

Funksiya $-\pi \leq \alpha < -\delta$ va $\delta \leq \alpha \leq \pi$ bo'lganda chegaralangan hamda

$$|\Phi_2(\alpha)| \leq |M + M|\frac{1}{2\sin\delta/2}$$

bunda M soni $|f(x)|$ miqdorning yuqori chegarasi, va $\Phi_2(\alpha)$ funksiya bo'lakli uzluksizligidan $n\to\infty$ da ikkinchi va uchinchi integrallar ham nolga intiladi. Natijada shu tenglikka ega bo'lamiz

$$\lim_{n\to\infty}[S_n(x) - f(x)] = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x+a) - f(x)]\frac{\cos\alpha/2}{2\sin\alpha/2}\sin n\alpha d\alpha \quad (1)$$

O'ng tomondagi ifoda $-\delta \leq \alpha \leq \delta$ oraliqda integrallanadigan va integral $f(x)$ funksiyaning faqat $x-\delta$ dan $x+\delta$ gacha oraliqdagi qiymatlariga bog'liq bo'ladi. Shunday qilib, (1) tenglikdan muhim xulosaga kelamiz.

Berilgan x nuqtada Furye qatorining yaqinlashishi $f(x)$ funksiya x nuqtaning istalgancha kichik atrofidagi xarakteriga bog'liqdir.

2. Furye qatori yaqinlashishining ba'zi bir yetarli shartlari

Teorema. Agar x_0 nuqtaning atrofida $f(x)$ funksiyaning

$$\lim_{\alpha\to-0}\frac{f(x_0+\alpha) - f(x_0)}{\alpha} = k_1 \quad (1)$$

$$\lim_{\alpha\to+0}\frac{f(x_0+\alpha) - f(x_0)}{\alpha} = k_2 \quad (2)$$

chekli limitlari mavjud bo'lib, x_0 nuqtaning o'zida funksiya uzluksiz bo'lsa, bu nuqtada Furye qatori $f(x)$ funksiyaning mos qiymatiga yaqinlashadi.

Isboti. Buning uchun yuqoridagi $\Phi_2(\alpha)$ funksiyasini qaraymiz

$$\Phi_2(\alpha) = [f(x_0 + a) - f(x_0)]\frac{\cos\alpha/2}{2\sin\alpha/2}$$

Bunda $f(x)$ funksiya $[-\pi, \pi]$ kesmada bo'lakli uzluksiz va x_0 nuqtada uzluksiz bo'lgani uchun u x_0 nuqtaning biror $[x_0-\delta, x_0+\delta]$ atrofida uzluksizdir. Shuning uchun $\Phi_2(\alpha)$ funksiya $\alpha\neq0$ va $|\alpha|\leq\delta$ bo'lgan hamma nuqtalarda uzluksizdir. $\alpha=0$ bo'lganda $\Phi_2(\alpha)$ aniqlangan. Endi (1) va (2) shartlardan foydalanib: $\lim\limits_{\alpha\to 0-0}\Phi_2(\alpha)$ ni va $\lim\limits_{\alpha\to 0+0}\Phi_2(\alpha)$ ni topamiz.

$$\lim_{\alpha\to0-0}\Phi_2(\alpha) = \lim_{\alpha\to0-0}[f(x_0+\alpha)-f(x_0)]\frac{\cos\alpha/2}{2\sin\alpha/2} =$$

$$= \lim_{\alpha\to0-0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}\frac{\alpha/2}{\sin\alpha/2}\cos\alpha/2 =$$

$$= \lim_{\alpha\to0-0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}\lim_{\alpha\to0-0}\frac{\alpha/2}{\sin\alpha/2}\lim_{\alpha\to0-0}\cos\alpha/2 = k_1\cdot1\cdot1 = k_1$$

Shunday qilib, agar $\Phi_2(\alpha)$ funksiyani $\Phi_2(0)=k_1$ desak, u holda bu funksiya $[-\delta, 0]$ kesmada uzluksiz, demak, u chegaralangan. Shuning singari $\lim\limits_{\alpha\to 0+0}\Phi_2(\alpha) = k_2$ ni ham ko'rsatish mumkin. Bundan $\Phi_2(\alpha)$ funksiya $[0, \delta]$ kesmada chegaralangan va uzluksizdir. Shunday qilib $\Phi_2(\alpha)$ funksiya $[-\delta, \delta]$ kesmada chegaralangan va bo'lak-bo'lak uzluksizdir. Shularga asosan yuqoridagi tenglikdan:

$$\lim_{n\to\infty}[S_n(x_0) - f(x_0)] = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x_0+\alpha)-f(x_0)]\frac{\cos\alpha/2}{2\sin\alpha/2}\sin n\alpha d\alpha$$

$$\text{yoki } \lim_{n\to\infty}[S(x_0) \quad f(x_0)] = \lim_{n\to\infty}\frac{1}{\pi}\int_{\delta}^{\delta}\Phi_2(\alpha)\sin n\alpha d\alpha$$

Bundan o'ng tomondagi limitning qiymati nolga tengdir. Shuning uchun:

$$\lim_{n\to\infty}[S_n(x_0) - f(x_0)] = 0 \quad \text{yoki}$$

$$\lim_{n\to\infty}S_n(x_0) = 0$$

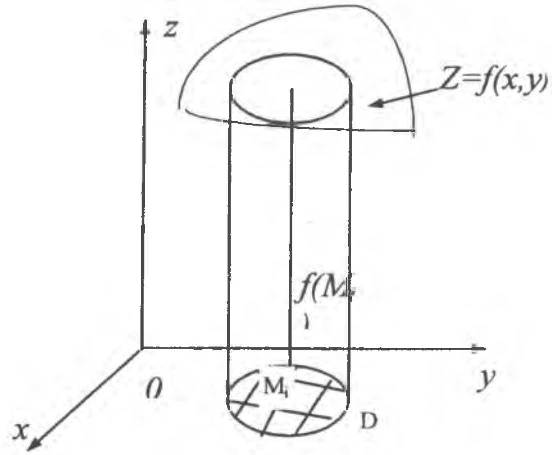
Demak, oxirgi ifoda yuqorida keltirilgan teoremaning isbotini beradi.

IKKI KARRALI INTEGRALGA KELTIRILADIGAN BA'ZI MASALALAR. IKKI KARRALI INTEGRALNING TA'RIFI. IKKI KARRALI INTEGRALNING XOSSALARI. O'RTA QIYMAT HAQIDAGI TEOREMA

1. Ikki karrali integralga keltiriladigan ba'zi masalalar

Geometriyada, mexanikada va fizikada shunday masalalar borki, ularni yechishda aniq integralning kuchi yetmaydi.

1-masala. Yasovchisi OZ o'qiga parallel, pastki asosi XOY tekisligida yotuvchi va $z = f(x,y)$ sirt bilan chegaralangan silindrsimon g'o'laning hajmini toping.



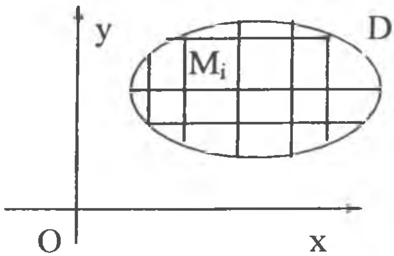
1-rasm.

Silindr pastki asosi D - sohani ixtiyoriy ravishda $D_1, D_2, ..., D_n$ bo'laklarga bo'lamiz. $D = \bigcup_{i=1}^{n} D_i$, bunda $D_1, D_2, ..., D_n$ sohalar umumiy ichki nuqtalarga ega bo'lmagan sohalardir. D_i $(i = \overline{1,n})$ sohaning yuzini Δs_i bilan belgilaymiz va shu sohada ixtiyoriy M_i nuqta olamiz. Pastki asosi D_i bo'lgan kichik silindrsimon g'o'laning hajmi $\upsilon_i \approx f(M_i)\cdot\Delta s_i$ taqribiy formula bilan aniqlanadi. Natijada qaralayotgan g'o'laning hajmi

$$\upsilon = \sum_{i=1}^{n}\upsilon_i \approx \sum_{i=1}^{n} f(M_i)\Delta s_i \qquad (1)$$

taqribiy formula bilan hisoblanadi. Bu masalaning aniq yechimiga keyinroq to'xtalamiz. (1) ning o'ng tomoni integral yig'indidir, lekin bizga tanish bo'lgan bir o'lchovli (aniq) integral hosil bo'ladigan yig'indi emas.

2-masala. Har bir (x,y) nuqtasidagi sirt zichligi $\rho=\rho(x,y)$ bo'lgan tekis figuraning (plastinkaning) massasini toping.



2-rasm.

Bu yerda yuqorida keltirilgan mulohazalarni takrorlaymiz. Plastinkaning D_i bo'lagida ixtiyoriy M_i nuqtani olib, D_i bo'lagining massasini m_i bilan belgilab, fizikadan ma'lum bo'lgan quyidagi taqribiy formulani yozamiz (2-rasm)

$$m_i \approx \rho(M_i)\Delta s_i \quad (i=1,2,...,n).$$

Butun plastinkaning massasi

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n}\rho(M_i)\Delta s_i \qquad (2)$$

taqribiy formula bilan hisoblanadi. (2) ning o'ng tomoni ham integral yig'indidir, lekin bunda ham bir o'lchovli (aniq) integral hosil bo'lmaydi. Massaning aniq qiymatini hisoblashga keyinroq to'xtaymiz.

2. Ikki karrali integralning ta'rifi

Biror D_i sohaning diametri deb shu D_i sohada yotuvchi ikkita nuqta orasidagi masofalarning aniq yuqori chegarasiga (eng kattasiga) aytiladi. D_i sohaning diametrini d_i bilan belgilasak, ta'rifdan $d_i = \sup\limits_{M_1 M_2 \in D_i} |M_1 M_2|$, bunda $M_1(x_1, y_1)$, bo'lsa,

$$|M_1 M_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

bo'ladi. Biz yuqorida silindrsimon g'o'laning hajmini taqribiy hisoblagan edik. Agar (1) integral yig'indi D_i sohalar diametrlarining eng kattasi nolga intilganda M_i nuqtalarning D_i sohalardan tanlanishiga va D sohaning $D_1, D_2, ..., D_n$ sohalarga bo'linish usuliga bog'liq bo'lmay yagona limitga intilsa, shu limitga $f(x,y)$ funksiyaning D soha bo'yicha ikki karrali integrali deyiladi va quyidagicha yoziladi:

$$\lim_{\substack{\max d_i \to 0 \\ i}} \sum_{i=1}^{n} f(M_i)\Delta s_i = \iint\limits_D f(x,y)dxdy \tag{3}$$

Ikki karrali integral yordamida (1) va (2) tengliklardagi υ va m larning aniq qiymatlarini yoza olamiz

$$\upsilon = \iint\limits_{D} f(M)ds = \iint\limits_D f(x,y)dxdy \tag{4}$$

$$m = \iint\limits_D \rho(M)ds = \iint\limits_D \rho(x,y)dxdy \tag{5}$$

Bunda (1) va (2) tengliklarning o'ng tomonlarida $\max_i d_i \to 0$ bo'lganda limitga o'tib, (3) formulani qo'lladik. Shunday qilib, ikki karrali integral - yasovchilari OZ o'qiga parallel, asoslarining biri XOY tekisligida, ikkincnisi esa $z = f(x, y)$ sirtda yotuvchi silindrsimon g'o'laning hajmini aniqlar ekan. Agar integral ostidagi funksiya D soha bilan ustma-ust tushgan plastinkaning sirt zichligini ifodalasa, u holda ikki karrali integral shu plastinkaning massasini aniqlar ekan, bunda $\rho(x, y) > 0, \quad (x, y) \in D$.

3. Ikki karrali integralning xossalari

1-xossa. D sohaning yuzini $D = S$ deb belgilasak, $\iint_D dxdy = S$ tenglik o'rinlidir.

Isboti. (4) da $f(x, y) = 1$ qo'yamiz, natijada $v = \iint_D dxdy$ hosil bo'ladi. Ikkinchi tomondan $\upsilon = 1 \cdot S_{acoc} = S$. Bularni taqqoslab, $\iint_D dxdy = S$ tenglikka ega bo'lamiz.

2-xossa. $a = const$ bo'lsa,

$$\iint_D af(x, y)dxdy = a\iint_D f(x, y)dxdy$$

o'zgarmas ko'paytuvchini integral belgisidan tashqariga chiqarish mumkin.

Isboti. (3) da $f(x, y)$ o'rniga $af(x, y)$ qo'yamiz va chap tomonga yig'indining va limitning xossalarini qo'llasak, 2-xossaning to'g'riligi kelib chiqadi.

3-xossa. Agar $\iint_D f(x, y)dxdy$ va $\iint_D g(x, y)dxdy$ integrallar mavjud bo'lsa, u holda

$$\iint_D [f(x,y) \pm g(x,y)]dxdy = \iint_D f(x,y)dxdy \pm \iint_D g(x,y)dxdy$$

tenglik o'rinlidir. Bu xossaning to'g'riligi ikki karrali integral ta'rifidan, yig'indi va limitning xossalaridan bevosita kelib chiqadi.

4-xossa. Agar $D = D_1 + D_2$ bo'lib, D_1 va D_2 sohalar umumiy ichki nuqtalarga ega bo'lmagan sohalar bo'lsa, u holda $\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$ tenglik o'rinli bo'ladi.

Bu xossa ham ikki karrali integralning ta'rifidan kelib chiqadi.

5-xossa. Agar $(x,y) \in D$ bo'lganda $f(x,y) \leq g(x,y)$ bo'lsa, u holda

$$\iint_D f(x,y)dxdy \leq \iint_D g(x,y)dxdy$$

tengsizlik o'rinli bo'ladi.

Isboti. Belgilash kiritamiz:

$$0 \leq g(x,y) - f(x,y) = \alpha(x,y). \qquad (4)$$

formulaga ko'ra, bundan $\iint_D \alpha(x,y)dxdy = \vartheta \geq 0$ ya'ni 5-xossaning to'g'riligi kelib chiqadi.

6-xossa. D sohada integrallanuvchi $f(x,y)$ funksiya uchun quyidagi tengsizlik o'rinlidir

$$\left| \iint_D f(x,y)dxdy \right| \leq \iint_D |f(x,y)|dxdy$$

Isboti. $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$ bo'lgani uchun 5-xossaga ko'ra

$$-\iint_D |f(x,y)dxdy| \leq \iint_D f(x,y)dxdy \leq \iint_D |f(x,y)|dxdy,$$

bundan $\left|\iint\limits_D f(x,y)dxdy\right| \le \iint\limits_D |f(x,y)|dxdy$ kelib chiqadi.

7-xossa. D sohada integrallanuvchi $f(x,y)$ funksiya shu sohada $m \le f(x,y) \le M$ tengsizlikni qanoatlantirsa, u holda

$$mS \le \iint\limits_D f(x,y)dxdy \le MS$$

tengsizlik o'rinli bo'ladi. Bu yerda S - D sohaning yuzi.

Bu xossaning isboti 1- va 5-xossalardan kelib chiqadi.

4. O'rta qiymat haqidagi teorema

$f(x,y)$ funksiya D yopiq sohada uzluksiz bo'lsa, u holda D sohada shunday $(x_0, y_0) \in D$ nuqta mavjudki, bunda

$$\iint\limits_D f(x,y)dxdy = f(x_0, y_0)\cdot S \qquad (6)$$

tenglik o'rinli bo'ladi.

Isboti. Yopiq sohada uzluksiz bo'lgan $f(x,y)$ funksiya shu sohada o'zining eng katta va eng kichik qiymatlariga erishadi. $\sup\limits_{(x,y)\in D} f(x,y) = M$, $\inf\limits_{(x,y)\in D} f(x,y) = m$ deb belgilab,

$$mS \le \iint\limits_D f(x,y)dxdy \le MS$$

(7-xossaga ko'ra) tengsizlikni hosil qilamiz. M va m sonlar orasida shunday $m < K < M$ son mavjudki, bunda $\iint\limits_D f(x,y)dxdy = KS$ tenglik bajariladi. $f(x,y)$ funksiya D sohada uzluksiz bo'lgani uchun shunday $(x_0, y_0) \in D$ nuqta mavjudki, $f(x_0, y_0) = K$ bo'ladi. Natijada

$$\iint\limits_D f(x,y)dxdy = f(x_0, y_0)\cdot S$$

kelib chiqadi.

IKKI KARRALI INTEGRALNI HISOBLASH

Biror D sohada uzluksiz bo'lgan $z=f(x, y)$ funksiyadan olingan ikki o'lchovli

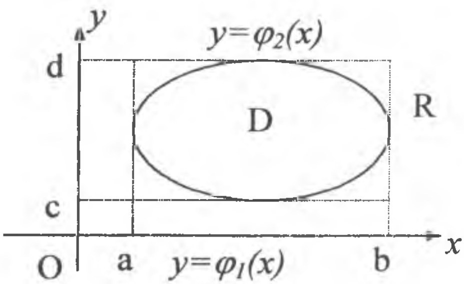
$$U = \iint_D f(x,y)dxdy$$

integralni hisoblash talab etilsin.

Ta'rif. Agar koordinata o'qlariga parallel bo'lib, D sohani kesib o'tuvchi to'g'ri chiziqlar Z konturni ikki nuqtada kesib o'tsa, D soha to'g'ri soha deyiladi.

Z yopiq kontur bilan chegaralangan D soha to'g'ri soha bo'lsin. Shaklga asosan yozamiz:

$$D: \begin{Bmatrix} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \end{Bmatrix}$$



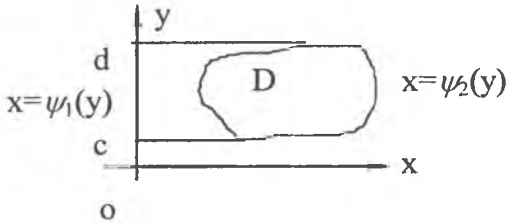
1-rasm

1-teorema. $z=f(x, y)$ uzluksiz funksiyaning D to'g'ri soha bo'yicha olingan ikki o'lchovli integrali funksiyaning o'sha D soha bo'yicha olingan ikki karrali integraliga teng, ya'ni:

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right) dx$$

Xuddi shuningdek, D to'g'ri sohani quyidagicha aniqlash mumkin:

$$D:\begin{Bmatrix} c \le y \le d \\ \psi_1(y) \le x \le \psi_2(y) \end{Bmatrix}$$



2-rasm.

U holda

$$\iint\limits_D f(x,y)dxdy = \int\limits_c^d \Big(\int\limits_{\psi_1(x)}^{\psi_2(x)} f(x,y)dx\Big)dy$$

tenglik o'rinlidir. Bundan

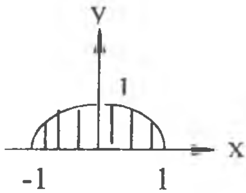
$$\int\limits_a^b \Big(\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\Big)dx = \int\limits_c^d \Big(\int\limits_{\psi_1(x)}^{\psi_2(x)} f(x,y)dx\Big)dy$$

ekanligi kelib chiqadi. Demak, karrali integrallarning integrallash tartibini o'zgartirish mumkin. Qaysi biri hisoblash uchun qulay bo'lsa, misollar ishlash vaqtida shunisidan foydalanamiz.

1-misol. Quyidagi integralning integrallash tartibi o'zgartirilsin:

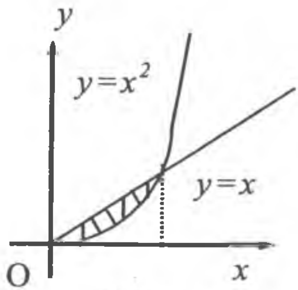
$$\int\limits_{-1}^{1} \Big(\int\limits_{0}^{\sqrt{1-x^2}} f(x,y)dy\Big)dx = \int\limits_{0}^{1} \Big(\int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\Big)dy$$

$$D:\begin{Bmatrix} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{Bmatrix} \qquad D:\begin{Bmatrix} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{Bmatrix}$$



3-rasm.

2-misol. $\iint\limits_D (x^2+2xy)dxdy$ karrali integralni hisoblang, bu yerda D soha $y=x^2$ va $y=x$ chiziqlar bilan chegaralangan.



4-rasm

Yechish. Rasmdan

$$a=0, b=1;\quad y_1(x)=x^2, y_2(x)=x;\quad y_1(x)\le y_2(x);\quad 0\le x\le 1$$

formulaga ko'ra

$$\iint\limits_D (x^2+2xy)dxdy=\int_0^1 dx\int_{x^2}^{x}(x^2+2xy)dy=\int_0^1 (x^2y+xy^2)\Big|_{x^2}^{x}dx=$$

$$=\int_0^1 (x^3+x^3-x^4-x^5)dx=\int_0^1 (2x^3-x^4-x^5)dx=$$

$$=\left(2\cdot\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{6}\right)\Big|_0^1=\frac{1}{2}-\frac{1}{5}-\frac{1}{6}=\frac{2}{15}.$$

2-teorema. Chegaralangan va yopiq D sohani OX o'qiga parallel to'g'ri chiziqlar bilan kesganda shu to'g'ri chiziqlarning istalgan biri D soha chegaralarini ikkitadan ortiq bo'lmagan nuqtalarida kesib o'tsin. D soha nuqtalarining koordinatalari $c \le y \le d$ va $\varphi_1(y) \le x \le \varphi_2(y)$ tengsizliklarni qanoatlantirsa, bu yerda $x=\varphi_1(y)$ va $x=\varphi_2(y)$ soha chegarasining tenglamalari, c va d lar esa D soha nuqtalarining eng kichik va eng katta ordinatalaridir, u holda D sohada uzluksiz $f(x, y)$ funksiyaning shu soha bo'yicha ikki karrali integrali quyidagi formula bilan aniqlanadi

$$\iint\limits_D f(x,y)dxdy = \int\limits_c^d dy \int\limits_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dxdy$$

3-misol. $\iint\limits_D (x+y^2+1)dxdy$ integralni hisoblang.

Bu yerda D: $y^2 = x - 1$, $y^2 = -x - 1$, $y = -2$, $y = 2$ chiziqlar bilan chegaralangan.

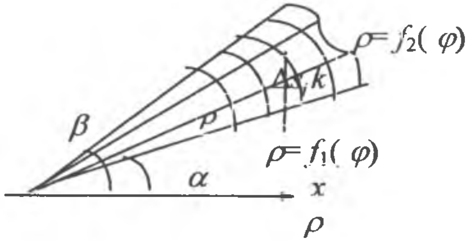
Yechish. OX o'qiga parallel to'g'ri chiziqlar D soha chegaralarini faqat ikkita nuqtada kesib o'tadi. Demak 2-teoremaga ko'ra

$$\iint\limits_D (x+y^2+1)dxdy = \int\limits_{-2}^{2} dy \int\limits_{-y^2-1}^{y^2+1} (x+y^2+1)dx =$$

$$= \int\limits_{-2}^{2} \left[\frac{x^2}{2} + (y^2+1)x \right] \Bigg|_{-y^2-1}^{y^2+1} dy = \int\limits_{-2}^{2} \left(2y^4+4y^2+2\right)dy =$$

$$= \left(2\frac{y^5}{5} + 4\frac{y^3}{3} + 2y \right)\Bigg|_{-2}^{2} = \frac{64}{5} + \frac{64}{5} + \frac{32}{3} + \frac{32}{3} + 4 + 4 = 54\frac{14}{15}.$$

2. Qutb koordinatalarida ikki karrali integral



5-rasm.

Qutb koordinatalari tekisligida biror D soha berilgan bo'lsin

$$D=\{\varphi,\rho|\alpha\le\varphi\le\beta, f_1(\varphi)\le\rho\le f_2(\varphi)\}$$

D sohani markazi qutb boshida bo'lgan konsentrik aylanalar va qutb boshidan chiqarilgan nurlar yordamida kichik bo'laklarga bo'lamiz

$$\alpha=\varphi_0<\varphi_1<\varphi_2<...<\varphi_n=\beta,\quad \rho_0<\rho_1<\rho_2<...<\rho_n$$

Geometriyadan ma'lumki, har bir Δs_{ik} yuzacha

$$\Delta s_{ik}=\tfrac{1}{2}(\rho_i+\Delta\rho_i)^2\cdot\Delta\varphi_k-\tfrac{1}{2}\rho_i^2\cdot\Delta\varphi_k=$$

$$=\tfrac{1}{2}(2\rho_i\Delta\rho_i+\Delta\rho^2{}_i)\Delta\varphi_k=(\rho_i+\tfrac{1}{2}\Delta\rho_i)\Delta\rho_i\Delta\varphi_k$$

yoki $\Delta s_{ik}=\rho_i^*\Delta\rho_i\Delta\varphi_k$, bu yerda

$$\rho_i<\rho_i^*<\rho_i+\Delta\rho_i \qquad (1)$$

formula bilan aniqlanadi (5-rasm).

D sohada uzluksiz $F(\varphi,\rho)$ funksiya uchun integral yig'indini yozamiz

$$\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}F(\varphi_k^*,\rho_i^*)\rho_i^*\Delta\rho_i\Delta\varphi_k \qquad (2)$$

(2) yig'indida $\max_i \Delta\rho_i \to 0$ va $\max_k \Delta\varphi_k \to 0$ bo'lganda limitga o'tsak, bu limit

$$\iint_D F(\varphi,\rho)\rho d\rho d\varphi$$

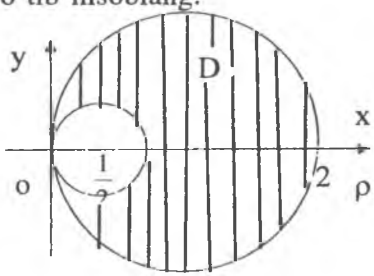
integralni beradi. Buni hisoblash usuli ham Dekart koordinatalarida berilgan holga o'xshash bajariladi

$$\iint_D F(\varphi,\rho)\rho d\rho d\varphi = \int_\alpha^\beta d\varphi \int_{f_1(\varphi)}^{f_2(\varphi)} F(\varphi,\rho)\rho d\rho \qquad (3)$$

Agar ikki karrali integral Dekart koordinatalarida berilsa, uni qutb koordinatalariga o'tib hisoblash quyidagi formula bilan bajariladi

$$\iint_D F(x,y)dxdy = \int_\alpha^\beta d\varphi \int_{f_1(\varphi)}^{f_2(\varphi)} F(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho \qquad (4)$$

4-misol. $\iint_D (x^2+y^2)dxdy,$ bu yerda D soha $x^2+y^2=x$ va $x^2+y^2=2x$ aylanalar bilan chegaralangan. Integralni qutb koordinatalariga o'tib hisoblang.



6-rasm.

Yechish. $x=\rho\cos\varphi, y=\rho\sin\varphi$ larni $x^2+y^2=x$ va $x^2+y^2=2x$ ga qo'ysak,

$$\begin{cases} \rho^2 = \rho\cos\varphi & \Rightarrow \quad \rho=\cos\varphi \\ \rho^2 = 2\rho\cos\varphi & \Rightarrow \quad \rho = 2\cos\varphi \end{cases}$$

kelib chiqadi, shakldan $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ bo'ladi. Natijada

$$\iint_D (x^2 + y^2)dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} \rho^3 d\rho = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho^4}{4}\Bigg|_{\cos\varphi}^{2\cos\varphi} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{15\cos^4\varphi}{4} d\varphi =$$

$$= \frac{15}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi = \frac{15}{8}\int_0^{\frac{\pi}{2}} (1 + 2\cos 2\varphi + \cos^2 2\varphi)d\varphi =$$

$$= \frac{15}{8}(\varphi + \sin 2\varphi)\Big|_0^{\frac{\pi}{2}} + \frac{15}{16}\int_0^{\frac{\pi}{2}} (1 + \cos 4\varphi)d\varphi =$$

$$= \frac{15\pi}{16} + \frac{15}{16}\left(\varphi + \frac{\sin 4\varphi}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{15\pi}{16} + \frac{15\pi}{32} = \frac{45\pi}{32}.$$

Bu integralning qiymati pastki asosi D soha va yuqoridan $z = x^2 + y^2$ sirt bilan chegaralangan silindrik g'o'laning hajmini beradi.

3. Ikki o'lchovli integralda o'zgaruvchilarni almashtirish

Oxy tekislikda Z yopiq kontur bilan chegaralangan D soha berilgan bo'lsin. O'zgaruvchi x, y lar yangi o'zgaruvchi u, v larning funksiyasi bo'lsin, ya'ni

$$x=\varphi(u, v), \quad y=\psi(u, v) \qquad (1)$$

$\varphi(u, v)$, $\psi(u, v)$ funksiya D_1 sohada bir qiymatli uzluksiz va uzluksiz xususiy hosilalarga ega bo'lsin.

Shunday qilib, (1) formula D va D_1 sohalarning nuqtalari orasida o'zaro bir qiymatli moslik o'rnatadi. D_1 sohani ixtiyoriy n ta yuzalarga bo'lamiz va har bir yuza Δs^1

$$\Delta s^1=\Delta u\Delta v \qquad (2)$$

formula bilan hisoblanadi. Δs^1 mos keladigan Δs yuza

$$\Delta s=\Delta x\Delta y \qquad (3)$$

formula bilan hisoblanadi.

Teylor formulasi va analitik geometriya kursidagi formuladan foydalanib, Δs va Δs^{I} orasidagi quyidagi munosabatni o'rnatish mumkin

$$\Delta s \approx |J|\, \Delta s^{I} \qquad (4)$$

Bu yerda J determinant $\varphi(u, v)$ va $\psi(u, v)$ funksiyalarning funksional determinanti bo'lib, u Yakobian deb ataladi.

Yakobian $J = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix}$ formula bilan hisoblanadi.

Agar D sohada uzluksiz bo'lgan $z=f(x, y)$ funksiya berilgan bo'lsa, u holda bu funksiya uchun tuzilgan integral yig'indilar quyidagi

$$\sum f(x,y)\Delta s = \sum F(u,v)\Delta s \qquad (5)$$

tenglik o'rinli bo'ladi. Bu yerda

$$F(u, v)=f(\varphi(u, v),\ \psi(u, v))$$

U holda (4) ga asosan (5) ni yozamiz.

$$\sum f(x,y)\Delta s = \sum F(u,v)\,|\,J\,|\,\Delta s^{I}$$

So'nggi ifodada diam$\Delta s' \to 0$ intilgandagi limitini topsak, quyidagi

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D_1} F(u,v)|J|dudv \qquad (6)$$

tenglik hosil bo'ladi. Bu esa ikki o'lchovli integralda o'zgaruvchilarni almashtirish formulasidir.

Xususan, agar x,y lar qutb koordinatalar sistemasida berilgan bo'lsa, ya'ni $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ bo'lsa, uning yakobianini hisoblaymiz.

$$J=\begin{vmatrix}\frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\rho}\\ \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\rho}\end{vmatrix}=\begin{vmatrix}-\rho\sin\varphi & \cos\varphi\\ \rho\cos\varphi & \sin\varphi\end{vmatrix}=-\rho\sin^2\theta-\rho\cos^2\theta=-\rho$$

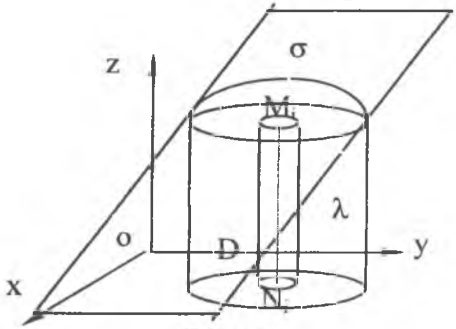
Demak, $|J|=\rho$, shuning uchun

$$\iint\limits_D f(x,y)dxdy=\iint\limits_{D_1} F(\varphi,\rho)\rho\, d\varphi\, d\rho$$

Bu formula yuqorida keltirilgan formula bilan mos tushadi.

Sirtning yuzini hisoblash

Fazoda λ egri chiziq bilan chegaralangan σ sirt berilgan bo'lsin. Sirtning tenglamasi $z=f(x, y)$ uzluksiz funksiya bilan aniqlangan bo'lib, $z=f(x, y)$ uzluksiz xususiy hosilalarga ham ega bo'lsin. Sirt yuzini hisoblash talab etilsin. Sirtning OXY tekislikdagi proyeksiyasi D to'g'ri soha bo'lsin. D sohani ixtiyoriy n ta $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ – yuzalarga bo'lamiz va har bir yuzadan $N_i(x_i, y_i)$ nuqtalarni tanlab olamiz. $N_i(x_i, y_i)$ nuqtaga fazoda $M_i(x_i, y_i, f(x_i, y_i))$ nuqta mos keladi.



7-rasm.

Analitik geometriya kursidan ma'lumki, sirtning har bir nuqtasiga urinma tekislik o'tkazish mumkin, uning tenglamasi

$$z - z_i = \frac{\partial f(x_i, y_i)}{\partial x}(x - x_i) + \frac{\partial f(x_i, y_i)}{\partial y}(y - y_i) \qquad (1)$$

ko'rinishga ega.

Fazodagi σ sirt proyeksiyalari Δs_i ga mos keluvchi $\Delta\sigma_i$ yuzalarga ajraladi va biz ularning yig'indisini qaraymiz.

$$\sigma_n = \sum_{i=1}^{n} \Delta\sigma_i \qquad (2)$$

$\Delta\sigma_i$-yuzalar diametrining eng kattasi nolga intilgandagi (2) yig'indi chekli limitga ega bo'lsa, shu limitni sirtning yuzi deymiz, ya'ni

$$\sigma = \lim_{diam\Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i \qquad (3)$$

Urinma tekislik va *Oxy* tekisliklar orasidagi burchakni γ_i deb belgilaymiz, u holda analitik geometriya kursidan ma'lumki,

$$\Delta s_i = \Delta\sigma_i \cos\gamma_i$$

yoki

$$\Delta\sigma_i = \frac{\Delta s_i}{\cos\gamma_i} \qquad (4)$$

(1) tenglamadan urinma tekislik tenglamasining yo'naltiruvchi kosinusi

$$\cos\gamma_i = \frac{1}{\sqrt{1 + \left(\frac{\partial f(x_i, y_i)}{\partial x}\right)^2 + \left(\frac{\partial f(x_i, y_i)}{\partial y}\right)^2}}$$

ga teng. Bundan

$$\Delta\sigma_i = \sqrt{1 + \left(\frac{\partial f(x_i, y_i)}{\partial x}\right)^2 + \left(\frac{\partial f(x_i, y_i)}{\partial y}\right)^2}\,\Delta s_i \qquad (5)$$

(5) ni (3) ga qo'yamiz.

$$\sigma = \lim_{diam\Delta\sigma_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\partial f(x_i, y_i)}{\partial x}\right)^2 + \left(\frac{\partial f(x_i, y_i)}{\partial y}\right)^2} \Delta s,$$

O'ng tomondagi ifodaning limiti ikki o'lchovli integralning integral yig'indisidir. Shu sababli yozamiz:

$$\sigma = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy \qquad (6)$$

Xuddi shuningdek, fazodagi σ sirtni *Oxz* yoki *Oyz* tekisliklarga pronyeksiyalasak, quyidagi sirt yuzini hisoblash formulalariga kelamiz

$$\sigma = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} dydz$$

$$\sigma = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} dxdz .$$

Ikki karrali integralning tatbiqi

1-masala. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipsning yuzi topilsin.

Yechish: *D* sohada ellips simmetrik bo'lgani uchun I-chorakdagi yuzi hisoblanib, 4 ga ko'paytiramiz. $0 \le x \le a$; $0 \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$.

Natijada

$$S = 4\iint_{D'} dxdy = 4\int_0^a dx \int_0^{a} dy = 4\frac{b}{a}\int_0^a \sqrt{a^2 x^2}\, dx = \left| \begin{array}{l} x = a\sin t \\ dx = a\cos t dt \\ 0 \le t \le \frac{\pi}{2} \end{array} \right| =$$

$$\frac{4b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2 t\,dt = 4ab\int_0^{\frac{\pi}{2}}\frac{1+\cos^2 t}{2}dt = 2ab\left(t+\frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = 2ab\cdot\frac{\pi}{2} = \pi ab.$$

Demak, $a=b$ bo'lsa, ellips doiraga aylanadi va yuza πa beradi. a-doira radiusi bo'ladi.

2-masala. $r = a(1+\cos\varphi)$ kardioidaning qutb boshiga nisbatan inersiya momentini toping.

Yechish: $I_o = \iint\limits_D \rho^2 ds$ formula bilan aniqlanadi. Bu yerda ds-yuza elementi bo'lib, $ds = \rho d\rho d\varphi$ ga teng.

Natijada $I_o = \iint\limits_D \rho^3 d\rho d\varphi$. Bu yerda

$0 \le \rho \le a(1+\cos\varphi)$

$-\pi \le \varphi \le \pi \;\; a(1+\cos\varphi)$. Bundan

$$I_o = \int_{-\pi}^{\pi} d\varphi \int_0 \rho^3 d\rho = \int_0^{2\pi} d\varphi\left(\frac{\rho^4}{4}\right)\Bigg|_0^{a(1+\cos\varphi)} =$$

$$= \frac{a^4}{4}\left\{\varphi\Big|_0^{2\pi} + 4\sin\varphi\Big|_0^{2\pi} + 3\int_0^{2\pi}(1+\cos^2\varphi)d\varphi + \right.$$

$$\left. + 4\int_0^{2\pi}(1-\sin^2\varphi)d(\sin\varphi) + \int_0^{2\pi}\left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi\right\} =$$

$$\frac{a^4}{4}\left\{2\pi + 3\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Bigg|_0^{2\pi} + 4\left(\sin\varphi - \frac{\sin 3\varphi}{3}\right)\Bigg|_0^{2\pi} + \right.$$

$$\left. + \frac{1}{4}\int_0^{2\pi}\left(1+2\cos 2\varphi + \frac{1+\cos 4\varphi}{2}\right)d\varphi\right\} =$$

$$= \frac{a^4}{4}\left\{8\pi + \frac{1}{4}(3\pi+0)\right\} = \frac{a^4}{4}\cdot\frac{35\pi}{4} = \frac{35\pi a^4}{16}.$$

3-masala. $z = 4 - x^2 - y^2$ sirt bilan chegaralangan jism hajmini hisoblang $(z \ge 0)$.

Yechish. Shartga ko'ra $4 - x^2 - y^2 \ge 0 \Rightarrow 4 \ge x^2 + y^2$. Agar

$z=0$ bo'lsa, $x^2+y^2=4$ bo'ladi. Demak, D soha

$$D=\{x,y\,|\,-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\},$$

natijada

$$V=4\iint(4-x^2-y^2)dxdy=4\int_0^2 dx\int_0^{\sqrt{4-x^2}}(4-x^2-y^2)dy=$$

$$=4\int_0^2\left(4y-x^2y-\frac{y^3}{3}\right)\Bigg|_0^{\sqrt{4-x^2}}dx=4\int_0^2\left[(4-x^2)^{3/2}-\frac{1}{3}(4-x^2)^{3/2}\right]dx=$$

$$=4\frac{2}{3}\int_0^2(4-x^2)^{3/2}dx=\left|\begin{array}{l}x=2\sin t\\ dx=2\cos t dt\\ 0\le t\le\frac{\pi}{2}\end{array}\right|=\frac{8}{3}\int_0^{\pi/2}(2\cos t)^3 2\cos t dt=$$

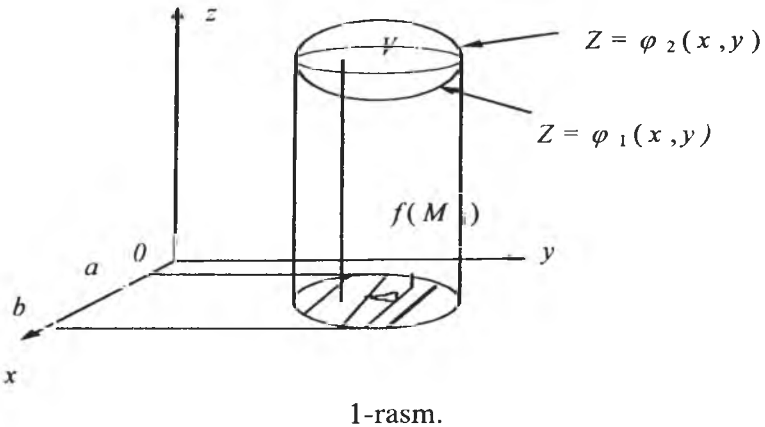
$$=\frac{8}{3}16\int_0^{\pi/2}\left(\frac{1+\cos 2t}{2}\right)^2dt=\frac{2\cdot 16}{3}\int_0^{\pi/2}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt=$$

$$=\frac{32}{3}\left(t+\sin 2t+\frac{1}{2}t+\frac{1}{2}\cdot\frac{\sin 4t}{4}\right)\Bigg|_0^{\pi/2}=\frac{32}{3}\left(\frac{3}{2}\cdot\frac{\pi}{2}+0\right)=8\pi.$$

UCH KARRALI INTEGRAL

Fazoda biror V soha (jism) berilgan bo'lsin. Shu soha chegarasi silliq yoki bo'lakli silliq sirt bo'lsin. V sohani ixtiyoriy ravishda v_1, $v_2,...,$ v_n bo'laklarga bo'lamiz. v_i soha hajmini Δv_i bilan belgilaymiz. Ikki karrali integralda qilingan barcha farazlar va shartlar bu yerda ham o'rinli deb hisoblaymiz. Natijada $f(x,y,z)$ funksiyaning V soha bo'yicha uch karrali integrali deb quyidagiga aytiladi:

$$\lim_{\max_i d_i \to 0} \sum_{i=1}^{n} f(\xi_i,\zeta_i,\eta_i)\Delta v_i = \iiint f(x,y,z)dxdydz \qquad (1)$$



1-rasm.

Bu yerda $(\xi_i,\zeta_i,\eta_i)\in v_i$, d_i esa v_i sohaning diametridir.

Agar V yopiq sohada $f(x,y,z)$ funksiya aniqlangan va uzluksiz bo'lsa, u holda shu funksiyaning V soha bo'yicha uch karrali integrali (1) mavjud va yagonadir.

Uch karrali integralni hisoblash

Faraz qilaylik, biror S yopiq sirt bilan chegaralangan v soha quyidagi shartlarni qanoatlantirsin:

1) V sohaning istalgan ichki nuqtasidan Oz o'qiga parallel qilib o'tkazilgan to'g'ri chiziq V soha chegarasini faqat ikkita nuqtada kesib o'tsin.

2) V soha Oxy tekisligiga proyeksiyalanadi va uning proyeksiyasi D soha to'g'ri soha bo'lsin. v soha pastdan $z=\varphi_1(x,y)$, yuqoridan $z=\varphi_2(x,y)$ sirtlar bilan chegaralangan bo'lsin (1-rasm), bunda

$$\varphi_1(x,y)\le\varphi_2(x,y),\quad x,y\in D,\quad D=\{x,y|a\le x\le b, y_1(x)\le y\le y_2(x)\}.$$

Agar V sohada $f(x,y,z)$ funksiya aniqlangan va uzluksiz bo'lsa, u holda (1) uch karrali integral mavjud bo'lib, quyidagicha hisoblanadi

$$\iiint\limits_V f(x,y,z)dxdydz=\int_a^b\left\{\int_{y_1(x)}^{y_2(x)}\left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)}f(x,y,z)dz\right]dy\right\}dx \qquad (2)$$

formulada z bo'yicha integrallashda x va y lar o'zgarmas son deb qaraladi, y bo'yicha integrallashda esa x o'zgarmas son deb qaraladi. (2) ni boshqacha ko'rinishda yozish ham mumkin:

$$\iiint\limits_V f(x,y,z)dxdydz=\int_a^b dx\int_{y_1(x)}^{y_2(x)}dy\int_{\varphi_1(x,y)}^{\varphi_2(x,y)}f(x,y,z)dz \qquad (3)$$

xususiy holda Z soha to'g'ri burchakli parallelepiped bo'lsa, ya'ni $a\le x\le b$, $c\le y\le d$, $p\le z\le q$, u holda (2) va (3) formulalar soddaroq ko'rinishni oladi:

$$\iiint\limits_V f(x,y,z)dxdydz=\int_a^b dx\int_c^d dy\int_p^q f(x,y,z)dz \qquad (4)$$

Misol. Uch karrali integralni hisoblang $\iiint\limits_V xyzdxdydz$, bu yerda V - to'g'ri burchakli parallelepiped $[a,b]\times[c,d]\times[p,q]$.

Yechish. (4) formulaga asosan

$$\iiint\limits_V xyzdxdydz = \int_a^b dx\int_c^d dy\int_p^q xyzdz = \int_a^b dx\int_c^d xydy\cdot\left.\frac{z^2}{2}\right|_p^q =$$

$$= \frac{q^2-p^2}{2}\cdot\int_a^b xdx\cdot\left.\frac{y^2}{2}\right|_c^d = \frac{d^2-c^2}{2}\cdot\frac{q^2-p^2}{2}\cdot\frac{b^2-a^2}{2} =$$

$$\frac{1}{8}(b^2-a^2)(d^2-c^2)(q^2-p^2).$$

Yuqorida keltirilgan ikki karrali integralning barcha xossalari uch karrali integral uchun ham o'rinlidir. D tekis soha o'rnida bu yerda V hajm olinadi.

Uch o'lchovli integralda o'zgaruvchilarni almashtirish

Fazoda x, y, z Dekart koordinatalari V sohani $x=\varphi_1(u, v, t)$, $y=\varphi_2(u, v, t)$, $z=\varphi_3(u, v, t)$, funksiyalar orqali u, v, t egri chiziqli koordinatalarning V^I sohasiga akslantirsin. Bunda V sohaning Δv hajm elementi V^I sohaning Δv^I hajm elementiga o'tadi. U holda $\lim\limits_{\Delta v^I\to 0}\frac{\Delta v}{\Delta v^I}=|J|$ bo'ladi. Ikki o'lchovli integralda bo'lganidek J Yakobiyan quyidagi formula bilan aniqlanadi.

$$J = \frac{D(x,y,z)}{D(u,v,t)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix}$$

Natijada V sohada uzluksiz bo'lgan $f(x, y, z)$ funksiyaning shu soha bo'yicha uch o'lchovli integrali quyidagicha aniqlanadi.

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V^1} f(\varphi_1(u,v,t),\varphi_2(u,v,t),\varphi_3(u,v,t))|J|dudvdt$$

Bu yerda $\varphi_1(u,v,t)$, $\varphi_2(u,v,t)$, $\varphi_3(u,v,t)$ lar V^1 sohada uzluksiz va birinchi tartibli xususiy hosilalarga ega bo'lgan funksiyalardir.

1. $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$ silindrik koordinatalarda Yakobiyan

$$J = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

bo'ladi.

Natijada

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V^1} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho\, d\varphi\, d\rho\, dz$$

kelib chiqadi.

2. $x=r\sin\varphi\cos\psi$, $y=r\sin\varphi\sin\psi$, $z=r\cos\varphi$ sferik koordinatalarda Yakobiyan

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \psi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \psi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \psi} \end{vmatrix} =$$

$$= \begin{vmatrix} \sin\varphi\cos\psi & r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \\ \sin\varphi\sin\psi & r\cos\varphi\sin\psi & r\sin\varphi\cos\psi \\ \cos & -r\sin\varphi & 0 \end{vmatrix} = r^2\sin\varphi$$

Uch o'lchovli integral quyidagiga teng bo'ladi.

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V'} f(r\sin\varphi\cos\psi, r\sin\varphi\sin\psi, r\cos\varphi)r^2\sin\varphi\, dr\, d\varphi\, d\psi$$

Silindrik va sferik koordinatalarda uch karrali integral. Karrali integrallarning geometriyaga tatbiqlari: tekis figuraning yuzi, jismning hajmi, jism massasi va og'irlik markazining koordinatalari.

1. Silindrik koordinatalarda uch karrali integral.

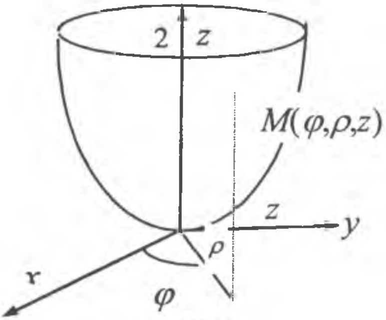
Nuqtaning Dekart va silindrik koordinatalari orasidagi munosabat $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$ edi. Bu tekislikdagi qutb koordinatalariga o'xshashdir. Ikki karrali integral uchun D sohaning elementar yuzasi $\Delta s_{ik} = \rho_i^*\Delta\rho_i\Delta\varphi_k$ edi, shunga o'xshash V sohaning elementar hajmi $\Delta\vartheta_{ikl} = \rho_i^*\Delta\rho_i\Delta\varphi_k\Delta z_l$ bo'ladi. $F(\varphi,\rho,z)$ funksiya uchun integral yig'indini tuzamiz va bu yig'indida $\max_i \Delta\rho_i \to 0$, $\max_k \Delta\varphi_k \to 0$, $\max_l \Delta z_l \to 0$ bo'lganda limitga o'tamiz. Shu limit mavjud bo'lsa, $F(\varphi,\rho,z)$ funksiyaning V sohada silindrik koordinatalardagi uch karrali integrali bo'ladi

$$\lim_{\substack{\max\limits_{i,k,l} \Delta\rho_i\to 0\\ \Delta\varphi_k\to 0\\ \Delta z_l\to 0}} \sum_{i,k,l=0}^{n-1} F(\varphi_k^*, \rho_i^*, z_l^*)\rho_i^*\Delta\rho_i\,\Delta\varphi_k\Delta z_l = \iiint_V F(\varphi,\rho,z)\rho\, d\rho\, d\varphi\, dz$$

Amaliyotda $f(x,y,z)$ funksiyaning V soha bo'yicha uch karrali integralini silindrik koordinatalarga o'tib hisoblash katta ahamiyatga ega. Bunda $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$ almashtirishlar orqali V soha V' sohaga o'tadi va integral quyidagicha aniqlanadi:

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V'} f(\rho\cos\varphi, \rho\sin\varphi, z)\rho d\rho d\varphi dz \quad (1)$$

Misol. $\iiint\limits_V (x^2+y^2)dxdydz$ Integralni silindrik koordinatalarga o'tib, hisoblang. Bu yerda V soha $x^2+y^2=2z$, $z=2$ sirtlar bilan chegaralangan. (2-rasm).



2-rasm.

Yechish. Sirt tenglamasidan $\rho^2=x^2+y^2=2z \Rightarrow z=\frac{\rho^2}{2}$.

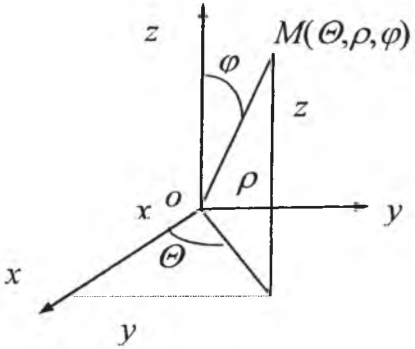
V' sohada: $0\le\varphi\le 2\pi$, $0\le\rho\le 2$, $\frac{\rho^2}{2}\le z\le 2$ o'rinli bo'ladi. Bularni (1) ga qo'yib hisoblaymiz

$$\iiint\limits_V (x^2+y^2)dxdydz=\iiint\limits_{V'}\rho^3 d\rho d\varphi dz=\int_0^{2\pi}d\varphi\int_0^2 d\rho\int_{\frac{\rho^2}{2}}^2\rho^3dz=2\pi\int_0^2\rho^3(2-\frac{\rho^2}{2})d\rho=$$

$$=2\pi\int_0^2(2\rho^3-\frac{1}{2}\rho^5)d\rho=2\pi(\frac{1}{2}\rho^4-\frac{1}{12}\rho^6)\Big|_0^2=2\pi(8-\frac{16}{3})=\frac{16\pi}{3}.$$

2. Sferik koordinatalarda uch karrali integral

Sferik koordinatalarda nuqtaning vaziyati uchta (θ, ρ, φ) sonlar bilan aniqlanadi. Ixtiyoriy M nuqtaning Dekart va sferik koordinatalari orasidagi munosabat quyidagicha bo'ladi (3-rasm):



3-rasm.

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi \qquad (2)$$

Bunda $0 \le \rho \le \infty, \quad 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi.$

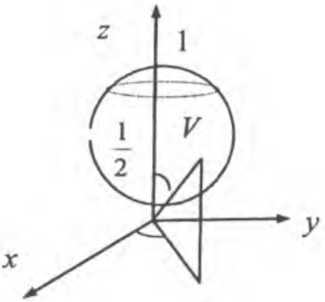
Sferik koordinatalarda Δv elementar hajm quyidagiga teng $\Delta v = \rho^2 \sin \quad \Delta\rho\Delta\theta\Delta$, bu yerda soddalik uchun indekslarni tashlab yubordik. Yuqorida ko'rilgan hollarga o'xshash integral yig'indi tuzib, limitga o'tsak, uch karrali integralning sferik koordinatalardagi ifodasi kelib chiqadi

$$\iiint_V f(x,y,z)dxdydz =$$

$$\iiint_{V'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\quad)\rho^2\sin\varphi d\rho d\theta d\varphi \qquad (3)$$

V sohadan V' sohaga (2) almashtirishlar orqali o'tiladi, (3) ning o'ng tomonini hisoblash esa Dekart koordinatalardagiga o'xshash bajariladi.

Misol. $\iiint_V \sqrt{x^2+y^2+z^2}\,dxdydz$ Integralni sferik koordinatalarga o'tib hisoblang. Bu yerda V soha $x^2+y^2+z^2=z$ sirt bilan chegaralangan.



4-rasm.

Yechish. Sirt tenglamasidan

$$x^2+y^2+z^2-z+\frac{1}{4}=\frac{1}{4} \quad \Rightarrow \quad x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4},$$

bu markazi $\left(0,\ 0,\ \frac{1}{2}\right)$ nuqtada va radiusi $R=\frac{1}{2}$ bo'lgan sferadir. x, y, z lar o'rniga (2) formulalardan qiymatlarini qo'ysak,

$$\rho^2=\rho\cos\varphi \quad \Rightarrow \quad \rho=\cos\varphi, \quad \sqrt{x^2+y^2+z^2}=\rho.$$

Shakldan $0\le\varphi\le\frac{\pi}{2}, \quad 0\le\theta\le 2\pi, \quad 0\le\rho\le\cos\varphi.$ Bularni (3) formulaga qo'yib hisoblaymiz

$$\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz=\iiint\limits_{V'}\rho^3\sin\varphi d\rho d\theta d\varphi=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\cos\varphi}\rho^3\sin\varphi d\rho=$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cdot\frac{\rho^4}{4}\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^4\varphi\sin\varphi d\varphi=-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^4\varphi d(\cos\varphi)=$$

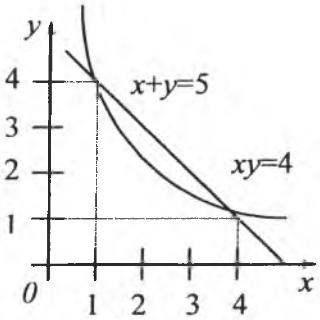
$$-\frac{\pi}{2}\cdot\frac{\cos^5\varphi}{5}\Big|_0^{\frac{\pi}{2}}=-\frac{\pi}{2}(0-\frac{1}{5})=\frac{\pi}{10}.$$

3. Tekis figuraning yuzi. Jismning hajmi

Ikki karrali integralning 1-xossasiga ko'ra $\iint_D dxdy = S$, bu yerda S – D sohaning yuzi. Biz 58-ma'ruzada yasovchilari OZ o'qiga parallel bo'lgan, pastki asosi D soha va yuqoridan $z = f(x, y)$ sirt bilan chegaralangan silindrsimon g'o'laning hajmini (4) formula yordamida hisoblashni ko'rgan edik ($f(x, y) \geq 0$). Jism silindrsimon bo'lishi shart emas, masalan piramida, shar, ellipsoid va hokazo bo'lishi mumkin. V jism hajmini uch karrali integral yordamida hisoblash formulasi quyidagicha bo'ladi:

$$V = \iiint_V dxdydz \qquad (4)$$

Misol. $xy = 4$ va $x + y = 5$ chiziqlar bilan chegaralangan sohaning yuzini toping.



5-rasm.

Yechish. $\begin{cases} x + y = 5 \\ xy = 4 \end{cases}$ sistemani yechib, chiziqlar kesishish nuqtalarining abssissalarini topamiz (5-rasm) $x_1 = 1, \quad x_2 = 4$.

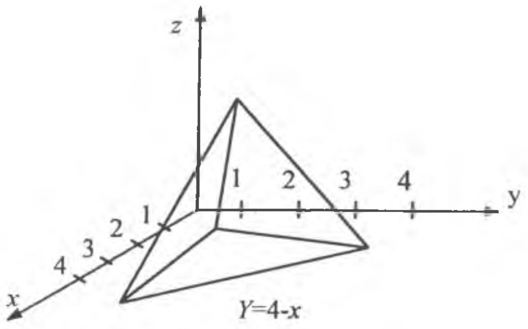
$$y_1(x) = \frac{4}{x}, \quad y_2(x) = 5 - x.$$

Natijada $D = \left\{ x, y \middle| 1 \le x \le 4, \frac{4}{x} \le y \le 5 - x \right\}$. Bundan

$$S = \iint_D dxdy = \int_1^4 dx \int_{\frac{4}{x}}^{5-x} dy = \int_1^4 (5 - x - \frac{4}{x})dx = \left(5x - \frac{x^2}{2} - 4\ln x \right)\Big|_1^4 =$$

$$= 20 - 8 - 4\ln 4 - 5 + \frac{1}{2} + 0 = 7{,}5 - 4\ln 4(kv.\ birlik).$$

Misol. $x - 1 = 0, \quad y - 1 = 0, \quad z = 0$ va $x + y + z = 4$ tekisliklar bilan chegaralangan jismning hajmini toping.



6-rasm.

Yechish. Qaralayotgan jism piramidadir. $x + y + z = 4$ tenglamada $z = 0$ qo'ysak, $x + y = 4$. $\begin{cases} y = 4 - x \\ y = 1 \end{cases} \Rightarrow x = 3$

Natijada $D = \{x, y | 1 \le x \le 3, 1 \le y \le 4 - x\}$ (6-rasm). Berilgan tenglamadan $z = f(x, y) = 4 - x - y \quad (z \ge 0)$. Bularni 58-ma'ruzadagi (4) ga qo'yamiz

$$V=\iint_D (4-x-y)dxdy=\int_1^3 dx\int_1^{4-x}(4-x-y)dy=\int_1^3\left(4y-xy-\frac{y^2}{2}\right)\Bigg|_1^{4-x}dx=$$

$$=\int_1^3\left[(4-x)^2-4+x-\frac{(4-x)^2}{2}+\frac{1}{2}\right]dx=\int_1^3\left[\frac{(4-x)^2}{2}+x-3{,}5\right]dx=$$

$$=\left(-\frac{(4-x)^3}{6}+\frac{x^2}{2}-3{,}5x\right)\Bigg|_1^3=\frac{13}{3}+4-7=\frac{4}{3}$$

Buni (4) formula orqali hisoblash ham mumkin

$$V=\int_1^3 dx\int_1^{4-x}dy\int_0^{4-x-y}dz=\int_1^3 dx\int_1^{4-x}(4-x-y)dy$$

bu oldingi integraldir.

4. Jismning massasi, og'irlik markazining koordinatalari

Biz 58-ma'ruzada tekis figura (plastinka)ning har bir $M(x,y)$ nuqtasidagi zichligi $\rho(x,y)$ $\rho(x,y)\geq 0$ bo'lsa, uning massasi (5) formula bilan aniqlanishini ko'rgan edik. Agar fazoviy V jismning istalgan $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)$ $(\rho(x,y,z)\geq 0)$ bo'lsa, u holda shu jism massasi

$$m=\iiint_V \rho(x,y,z)dxdydz \qquad (5)$$

tenglik bilan aniqlanadi. V jismni n ta bo'laklarga bo'lamiz v_i bo'lakning elementar hajmini Δv_i, tegishli massa elementini Δm_i bilan belgilaymiz, v_i ning diametrini d_i bilan belgilaymiz. Mexanikadan ma'lumki, agar Δv_i hajm yetarlicha kichik bo'lganda V jism og'irlik markazining koordinatalari quyidagi taqribiy formulalar bilan hisoblanadi

$$x_c \approx \frac{\sum_{i=1}^{n} x_i \Delta m_i}{\sum_{i=1}^{n} \Delta m_i}, \quad y_c \approx \frac{\sum_{i=1}^{n} y_i \Delta m_i}{\sum_{i=1}^{n} \Delta m_i}, \quad z_c \approx \frac{\sum_{i=1}^{n} z_i \Delta m_i}{\sum_{i=1}^{n} \Delta m_i}. \qquad (6)$$

Bu yerda $\Delta m_i = \rho(\xi_i, \zeta_i, \eta_i)\Delta v_i$, $(\xi_i, \zeta_i, \eta_i) \in v_i$ (6) ning o'ng tomonidagi kasrlarning surat va maxrajlari integral yig'indidir. Shu integral yig'indilarda $\max_i d_i \to 0$ bo'lganda limitga o'tsak, taqribiy formulalar aniq tenglikka aylanadi va quyidagi ko'rinishni oladi:

$$x_c = \frac{\iiint\limits_V x\rho(x,y,z)dxdydz}{\iiint\limits_V \rho(x,y,z)dxdydz},$$

$$y_c = \frac{\iiint\limits_V y\rho(x,y,z)dxdydz}{\iiint\limits_V \rho(x,y,z)dxdydz}, \qquad (7)$$

$$z_c = \frac{\iiint\limits_V z\rho(x,y,z)dxdydz}{\iiint\limits_V \rho(x,y,z)dxdydz}$$

Agar V jism o'rnida XOY tekisligida D sohani egallagan tekis figura (plastinka) qaralsa, shu figura og'irlik markazining koordinatalari (7) formulalarning xususiy holi sifatida kelib chiqadi. Plastinka $M(x,y)$ nuqtasining sirt zichligi $\rho(x,y)$ bo'lsa, og'irlik markazining koordinatalari:

$$x_c = \frac{\iint\limits_D x\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy}, \quad y_c = \frac{\iint\limits_D y\rho(x,y)dxdy}{\iint\limits_D \rho(x,y)dxdy} \qquad (8)$$

formulalardan aniqlanadi. Agar qaralayotgan jism va tekis figura bir jinsli bo'lsa, u holda (7) va (8) formulalarda

$\rho(x,y,z)=1$, $\rho(x,y)=1$ qo'yib hisoblanadi.

Misol. 2 karrali integral ostidagi $\rho(x,y)=x^2+2xy$ funksiya D tekis figuraning ixtiyoriy $M(x,y)$ nuqtasidagi sirt zichligi bo'lsin. Shu tekis figuraning massasi $m=\iint\limits_D (x^2+2xy)dxdy=\frac{2}{15}$.

$$\iint\limits_D x\rho(x,y)dxdy=\int_0^1 dx\int_{x^2}^{x}(x^3+2x^2y)dy=\int_0^1 (x^3y+x^2y^2)\Big|_{x^2}^{x}dx=$$

$$=\int_0^1 (2x^4-x^5-x^6)dx=\left(\frac{2}{5}x^5-\frac{x^6}{6}-\frac{x^7}{7}\right)\Bigg|_0^1=\frac{2}{5}-\frac{1}{6}-\frac{1}{7}=\frac{19}{210}.$$

$$\iint\limits_D y\rho(x,y)dxdy=\int_0^1 dx\int_{x^2}^{x}(x^2y+2xy^2)dy=\int_0^1\left(\frac{x^2y^2}{2}+\frac{2xy^3}{3}\right)\Bigg|_{x^2}^{x}dx=$$

$$\int_0^1\left(\frac{7x^4}{6}-\frac{x^6}{2}-\frac{2x^7}{3}\right)dx=\left(\frac{7}{30}x^5-\frac{x^7}{14}-\frac{x^8}{12}\right)\Bigg|_0^1=\frac{7}{30}-\frac{1}{14}-\frac{1}{12}=\frac{11}{140}.$$

Bularni (8) formulalarga qo'ysak, og'irlik markazining koordinatalari kelib chiqadi

$$x_c=\frac{19}{210}:\frac{2}{15}=\frac{19\cdot 15}{210\cdot 2}=\frac{19}{28},\quad y_c=\frac{11}{140}:\frac{2}{15}=\frac{11\cdot 15}{140\cdot 2}=\frac{33}{56}.$$

Misol. $F=\sqrt{x^2+y^2+z^2}$ funksiyani qaralayotgan jism ixtiyoriy $M(x,y,z)$ nuqtasidagi zichligi deb olamiz. Demak, $\rho(x,y,z)=\sqrt{x^2+y^2+z^2}$. Jism massasi

$$m=\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz=\frac{\pi}{10}.$$

$$\iiint\limits_V x\rho(x,y,z)dxdydz=\int_0^{2\pi}\cos\theta d\theta\int_0^{\frac{\pi}{2}}\sin^2\varphi d\varphi\int_0^{\cos\varphi}\rho^4 d\rho=$$

$$=\sin\theta\Big|_0^{2\pi}\cdot\int_0^{\frac{\pi}{2}}\sin^2\varphi\cdot\frac{\cos^5\varphi}{5}d\varphi=0,$$

chunki $\sin\theta\Big|_0^{2\pi}=0.$

$$\iiint\limits_V y\rho(x,y,z)dxdydz=\int_0^{2\pi}\sin\theta d\theta\int_0^{\frac{\pi}{2}}\sin^2\varphi d\varphi\int_0^{\cos\varphi}\rho^4 d\rho=$$

$$=-\cos\theta\Big|_0^{2\pi}\cdot\int_0^{\frac{\pi}{2}}\sin^2\varphi\cdot\frac{\cos^5\varphi}{5}d\varphi=0,$$

chunki $\cos\theta\Big|_0^{2\pi}=0.$

$$\iiint\limits_V z\rho(x,y,z)dxdydz=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi d\varphi\int_0^{\cos\varphi}\rho^4 d\rho=2\pi\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\frac{\cos^6\varphi}{5}d\varphi=$$

$$=-\frac{2\pi}{5}\int_0^{\frac{\pi}{2}}\cos^6\varphi d(\cos\varphi)=-\frac{2\pi}{5}\cdot\frac{\cos^7\varphi}{7}\Big|_0^{\frac{\pi}{2}}=\frac{2\pi}{35}.$$

Bularni (7) formulalarga qo'yib, og'irlik markazining koordinatalarini topamiz $x_c=y_c=0,\quad z_c=\frac{2\pi}{35}:\frac{\pi}{10}=\frac{2\pi\cdot 10}{35\pi}=\frac{4}{7}.$

Og'irlik markazi: $C(0,0,\frac{4}{7}).$

Inersiya momentlari

Massasi m bo'lgan $M(x,y,z)$ moddiy nuqtaning Ox, Oy, Oz koordinata o'qlariga nisbatan inersiya momentlari

$$I_x=(y^2+z^2)m,\quad I_y=(x^2+z^2)m,\quad I_z=(x^2+y^2)m \tag{9}$$

formulalar bilan aniqlanadi. Agar moddiy nuqta o'rniga $D=\{x,y|a\le x\le b,\varphi_1(x)\le y\le\varphi_2(x)\}$ tekis figura qaralsa, uning istalgan $M(x,y)$ nuqtasidagi sirt zichligi $\rho(x,y)$ bo'lsa, u holda (9) formulalarda $z=0$ va m o'rniga $dm=\rho(x,y)ds$ qo'yib (ds – D sohaning yuza elementi) D soha bo'yicha integrallasak, D tekis figuraning Ox va Oy o'qlariga nisbatan inersiya momentlari kelib chiqadi

$$I_x = \iint_D y^2 \rho(x,y)dxdy, \quad I_y = \iint_D x^2 \rho(x,y)dxdy \qquad (10)$$

(9) va (10) formulalardan foydalanib hajmi V va istalgan $M(x,y,z)$ nuqtasidagi zichligi $\rho(x,y,z)$ bo'lgan fazoviy jism uchun ham koordinata o'qlariga nisbatan inersiya momentlarini yoza olamiz

$$I_x = \iiint_V (y^2 + z^2)\rho(x,y,z)dxdydz,$$
$$I_y = \iiint_V (x^2 + z^2)\rho(x,y,z)dxdydz, \qquad (11)$$
$$I_z = \iiint_V (y^2 + x^2)\rho(x,y,z)dxdydz$$

Misol. $x^2 + y^2 = 2z$, $z = 2$ aylanma paraboloidning hamma nuqtalarida zichligi o'zgarmas bo'lib, $\rho(x,y,z) = 1$ bo'lsin. Shu paraboloid kesimining Oz o'qiga nisbatan inersiya momentini topamiz. (11) formulada $\rho(x,y,z) = 1$ qo'ysak,

$$I_z = \iiint_V (x^2 + y^2)dxdydz = \frac{16\pi}{3}.$$

Bu integralni hisoblagan edik.

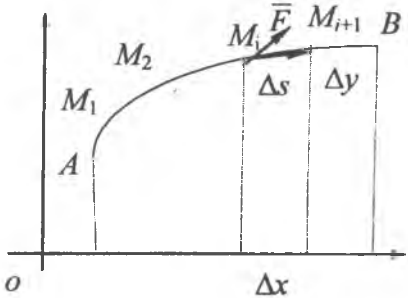
71-ma'ruza.

EGRI CHIZIQLI INTEGRALLAR

Tekislikda yo'nalishga ega bo'lgan l egri chiziq bo'yicha M moddiy nuqta $\vec{F}$ kuch ta'sirida harakatlansin. U holda $\vec{F}$ kuch M moddiy nuqtaning funksiyasi bo'ladi, ya'ni

$$\overline{F} = \overline{F}(M)$$

Egri chiziqning boshlang'ich nuqtasi A va oxirgi nuqtasi B bo'lsin. Moddiy nuqtaning $\vec{F}$ kuch ta'sirida A dan B gacha bajargan ishini hisoblash talab etilgan bo'lsin. Buning uchun AB egri chiziqni ixtiyoriy n ta bo'lakka $A = M_0, M_1, ..., M_i, M_{i+1}, ..., M_n = \text{B}$ bo'lamiz. Har bir $\overline{M_i M_{i+1}}$ vektorni $\overline{\Delta s_i}$ deb belgilaymiz.



1-rasm.

U holda $A_i \approx \overline{F_i}\,\overline{\Delta s_i}$ skalyar ko'paytma $\overline{M_i M_{i+1}}$ yoy bo'yicha bajargan ishning taqribiy qiymatini ifodalaydi. Barcha AB egri chiziq uchun bajargan ishning taqribiy qiymati quyidagi yig'indidan iboratdir.

$$A_n = \sum A_i \approx \overline{F_i}\,\overline{\Delta s_i}$$

Endi $\max \Delta s_i \to 0$ da

$$\lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} \overline{F_i}\,\overline{\Delta s_i} = \int_l F ds \tag{1}$$

Agar $\vec{F}$ vektorning o'qdagi proyeksiyalari $P(x, y)$, $Q(x, y)$ bo'lsa, $\overline{\Delta s}$ vektorning proyeksiyalari Δx, Δy bo'lsa, u holda

$$\overline{F_i \Delta s_i} = P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i$$

ga teng bo'ladi.

$$A = \sum_{i=1}^{n} \overline{F_i \Delta s_i} = \sum_{i=1}^{n} [P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i]$$

Endi Δx_i va Δy_i nolga intilganda, ya'ni

$$A = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_i \to 0}} \sum_{i=1}^{n} [P(x_i, y_i)\Delta x_i + Q(x_i, y_i)\Delta y_i] = \\ = \int_l P(x, y)dx + Q(x, y)dy \tag{2}$$

yoki

$$A = \int_{(A)}^{(B)} P(x, y)dx + Q(x, y)dy \tag{2'}$$

ko'rinishga keladi. Shunday qilib, egri chiziqli integral moddiy nuqtaning $\vec{F}$ kuch ta'sirida bajargan ishini aniqlar ekan.
(1) ga 1-tur egri chiziqli integral,
(2) ga 2-tur egri chiziqli integral deyiladi.

1-xocca. Egri chiziqli integral egri chiziqning shakliga va egri chiziqning yo'nalishiga bog'liqdir.

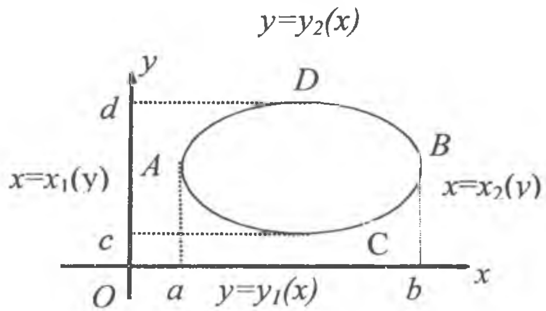
2-xossa. Agar $A\smile B$ egri chiziqni biror K nuqta bilan $A\smile B$ va $K\smile B$ bo'laklarga ajratsak, u holda

$$\int_{(A)}^{(B)} Pdx + Qdy = \int_{(A)}^{(K)} Pdx + Qdy + \int_{(K)}^{(B)} Pdx + Qdy$$

tenglik o'rinli bo'ladi.

Egri chiziqli integral yordamida yuza hisoblash

Tekislikda Z yopiq kontur bilan chegaralangan D to'g'ri soha berilgan bo'lsin. D sohaning yuzini hisoblaymiz. D sohaning OX o'qdagi proyeksiyasi $[a, b]$ kesma, OY o'qdagi proyeksiyasi esa $[c, d]$ kesma bo'lsin.



2-rasm.

D sohaning yuzini quyidagi formula bilan hisoblash mumkin. D: $\{a \le x \le b,\ y_1(x) \le y \le y_2(x)\}$

$$S = \int_a^b [y_2(x) - y_1(x)]dx = \int_a^b y_2(x)dx - \int_a^b y_1(x)dx \qquad (1)$$

Har bir integralni ko'rib chiqamiz.

$$\int_a^b y_2(x)dx = \int_{ADB} ydx = -\int_{BDA} ydx, \qquad \int_a^b y_1(x)dx = \int_{ACB} ydx$$

Bu tengliklarni (1) ifodaga qo'ysak,

$$S = -\int\limits_{BDA} ydx - \int\limits_{ACB} ydx = -\int\limits_{Z} ydx \qquad (2)$$

formulaga ega bo'lamiz. Bu yerda Z egri chiziq soat miliga qarshi yo'nalishga ega.

Xuddi shuningdek, D: $\{c \leq y \leq d,\ x_1(y) \leq x \leq x_2(y)\}$

$$S = \int\limits_{c}^{d} [x_2(y) - x_1(y)]dy = \int\limits_{c}^{d} x_2(y)dy - \int\limits_{c}^{d} x_1(y)dy = \int\limits_{Z} xdy \qquad (3)$$

ekanligi kelib chiqadi.

(2) va (3) tengliklardan quyidagi

$$S = \frac{1}{2}\int\limits_{Z} xdy - ydx$$

yuzani hisoblash formulasiga ega bo'lamiz.

Bu yerda Z soat miliga qarama-qarshi yo'nalgan.

Misol. $x=a\cos t$, $y=a\sin t$ aylananing yuzi hisoblansin. $dx=-a\sin tdt$, $dy=a\cos tdt$, $0 \leq t \leq 2\pi$

$$S = \frac{1}{2}\int\limits_{0}^{2\pi} (a\cos t \cdot a\cos tdt + a\sin t \cdot a\sin tdt) =$$

$$= \frac{1}{2}a^2\int\limits_{0}^{2\pi} dt = \frac{1}{2}a^2 2\pi = \pi a^2.$$

Grin formulasi

Grin formulasi biror D soha bo'yicha olingan ikki o'lchovli integral bilan shu sohani chegaralab turuvchi yopiq kontur bo'yicha olingan egri chiziqli integral orasidagi munosabatni ifodalaydi.

D soha $a \leq x \leq b$, va $y_1(x) \leq y \leq y_2(x)$ chiziqlar bilan chegaralangan to'g'ri soha bo'lsin (2-rasm). Quyidagi integralni hisoblaymiz.

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy \right) dx = \int_a^b P(x,y) \Big|_{y_1(x)}^{y_2(x)} dx =$$

$$= \int_a^b P(x, y_2(x))dx - \int_a^b P(x, y_1(x))dx$$

Birinchi integralni yozamiz.

$$\int_a^b P(x, y_2(x))dx = \int_{ADB} P(x,y)dx$$

ikkinchisi esa

$$\int_a^b P(x, y_1(x))dx = \int_{ACB} P(x,y)dx = -\int_{BCA} P(x,y)dx$$

So'nggi tengliklarni hisobga olib yozamiz.

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_{ADB} P(x,y)dx + \int_{BCA} P(x,y)dx = \int_Z P(x,y)dx \qquad (1)$$

Bu yerda Z soat mili yo'nalishi bo'yicha olingan. Xuddi shuningdek

$$\iint_D \frac{\partial Q}{\partial x} dxdy = -\int_Z Q(x,y)dy \qquad (2)$$

ekanligini keltirib chiqarish mumkin. (1) dan (2) ni ayirsak,

$$\iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dxdy = \int_Z Pdx + Qdy$$

formulaga ega bo'lamiz. Bu yerda Z soat mili bo'yicha yo'nalgan. Bu formula Grin formulasi deyiladi.

Egri chiziqli integralning integrallash yo'liga bog'liq bo'lmaslik sharti

Tekislikdagi biror D sohada uzluksiz va uzluksiz xususiy hosilalarga ega bo'lgan $P(x, y)$ va $Q(x, y)$ funksiyalar berilgan bo'lsin. D sohada yotgan A va B nuqtalarni tutashtiruvchi egri chiziq bo'yicha olingan egri chiziqli integralni qaraymiz (3-rasm).

$$\int_{(A)}^{(B)} Pdx + Qdy$$

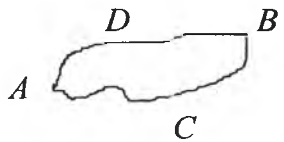
Qaralayotgan D sohada yotgan A va B nuqtalarni tutashtiruvchi ikki ACB va ADB ixtiyoriy egri chiziqni qaraymiz. Faraz qilaylik,

$$\int_{ACB} Pdx + Qdy = \int_{ADB} Pdx + Qdy \qquad (1)$$

$$\int_{ACB} Pdx + Qdy - \int_{ADB} Pdx + Qdy = 0$$

yoki

$$\int_{ACB} Pdx + Qdy + \int_{BDA} Pdx + Qdy = 0$$



3-rasm.

bundan

$$\int_{Z} Pdx + Qdy = 0 \qquad (2)$$

ekanligi kelib chiqadi.

Shunday qilib ixtiyoriy yopiq kontur bo'yicha olingan egri chiziqli integralning nolga teng bo'lishi kelib chiqadi. Egri chiziqli integral A va B nuqtalarning vaziyatiga bog'liq bo'lib, ularni tutashtiruvchi egri chiziqning shakliga bog'liq emas ekan.

Qanday shartlarda, ya'ni $P(x, y)$ va $Q(x, y)$ funksiyalar qanday shartlarni bajarganda yopiq kontur bo'yicha olingan egri chiziqli integral nolga teng bo'ladi. Bu savolga quyidagi teorema javob beradi.

Teorema (isbotsiz). Agar $P(x, y)$ va $Q(x, y)$ funksiyalar D sohaning barcha nuqtalarida uzluksiz va uzluksiz xususiy hosilalarga ega bo'lsa, u holda shu D sohadagi yopiq kontur bo'yicha olingan egri chiziqli integral nolga teng bo'lishi uchun, ya'ni $\int\limits_{z} P(x,y)dx + Q(x,y)dy = 0$ uchun D sohaning barcha nuqtalarida

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad (3)$$

tenglikning bajarilishi zarur va yetarlidir.

Sirt integrali

Fazoda v sohaning har bir nuqtasida uzluksiz bo'lgan $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ funksiyalar berilgan bo'lsin. Bu sohada λ fazoviy egri chiziq bilan chegaralangan σ sirt berilgan bo'lsin. Sirtning har bir nuqtasida

$$\overline{F}(x,y,z) = P(x,y,z)\overline{i} + Q(x,y,z)\overline{j} + R(x,y,z)\overline{k}$$

aniqlangan bo'lsin.

Fazoning har bir nuqtasida

$$\overline{n} = \cos(n,x)\overline{i} + \cos(n,y)\overline{j} + \cos(n,z)\overline{k}$$

birlik normal vektor berilgan bo'lib, nuqtalarning uzluksiz funksiyalari bo'lsin.

Fazoviy sirtni ixtiyoriy usul bilan $\Delta\sigma_i$ $(i = \overline{1,n})$ yuzalarga bo'lamiz. Har bir yuzadan bittadan nuqta olamiz va quyidagi yig'indini tuzamiz (4-rasm).

$$\sum_i \overline{F}(M_i)\overline{n}(M_i)\Delta\sigma_i \qquad (1)$$

Ta'rif. Agar *diam* $\Delta\sigma_i \to 0$ da (1) integral yig'indi chekli limitga intilsa, shu limitga sirt integrali deyiladi va

$$\iint_\sigma \overline{F}\overline{n}d\sigma$$

ko'rinishda yoziladi. Ta'rifga asosan,

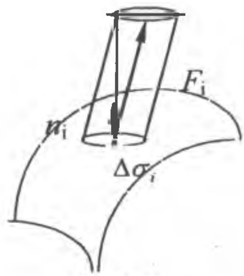
$$\lim_{diam\Delta\sigma_i \to 0} \sum_i \overline{F}(M_i)\overline{n}(M_i)\Delta\sigma_i = \iint_\sigma \overline{F}\overline{n}d\sigma \qquad (2)$$

bu yerda $\overline{F}\overline{n}$-shu vektorlarning skalyar ko'paytmasi. Endi quyidagi ko'paytmaning

$$\overline{F}_i\overline{n}\Delta\sigma_i = \overline{F}_i\Delta\sigma_i \cos(\overline{n}_i \hat{\ } \overline{F}_i) \qquad (3)$$

fizik ma'nosiga e'tibor beraylik.

(3) tenglik asosi $\Delta\sigma_i$ va balandligi $F_i \cos(\overline{n}_i \hat{\ } \overline{F})$ ga teng bo'lgan silindrning hajmiga tengdir. Agar $\overline{F}$ vektor σ sirtdan oqib o'tuvchi suyuqlikning tezligi bo'lsa, (3) ko'paytma $\Delta\sigma_i$ yuzadan vaqt birligida n_i vektor yo'nalishi bo'yicha oqib o'tuvchi suyuqlikning miqdoriga tengdir.



4-rasm.

Agar $\overline{F}$ vektorni suyuqlikning berilgan nuqtadagi oqish tezligi deb olsak, $\iint\limits_{\sigma} \overline{F}\overline{n}d\sigma$ ifoda vaqt birligida σ sirt orqali musbat yo'nalishda oqib o'tuvchi suyuqlikning umumiy miqdorini bildiradi.

Ta'rif. Sirt integrali σ sirt orqali o'tuvchi $\overline{F}$ vektor maydonning oqimi deb ataladi.

$\overline{F}\overline{n}$ -вектorlarning skalyar ko'paytmasi bo'lgani uchun quyidagicha yozamiz.

$$\iint\limits_{\sigma} \overline{F}\overline{n}d\sigma = \iint\limits_{\sigma} [P\cos(n,x) + Q\cos(n,y) + R\cos(n,z)]d\sigma$$

Geometriya kursidan ma'lumki,

$$\left.\begin{aligned} \Delta\sigma\cos(n,x) &= \Delta\sigma_{yz} \\ \Delta\sigma\cos(n,y) &= \Delta\sigma_{xz} \\ \Delta\sigma\cos(n,z) &= \Delta\sigma_{xy} \end{aligned}\right\}$$

tengliklar o'rinlidir. U holda

$$\iint\limits_{\sigma} \overline{F}\overline{n}d\sigma = \iint\limits_{\sigma} P\cos(n,x)\Delta\sigma + Q\cos(n,y)\Delta\sigma + R\cos(n,z)\Delta\sigma =$$
$$= \iint\limits_{\sigma} Pdydz + Qdxdz + Rdxdy \qquad (4)$$

formula kelib chiqadi.

(4) formulada σ fazoviy sirt bo'lib, integralni hisoblash tekis soha bo'yicha olingan ikki o'lchovli integralni hisoblashga keltiriladi.

Masalan, xususan

$$\iint\limits_{\sigma} R\cos(n,z)d\sigma$$

integralni qaraymiz.

σ sirt shunday bo'lsinki, OZ o'qqa parallel har qanday to'g'ri chiziq uni bitta nuqtada kesib o'tsin. Sirt tenglamasini

$z=f(x, y)$ sirtning OXY tekislikdagi proyeksiyasini D deb olamiz.

$$\iint_{\sigma} R(x,y,z)\cos(n,z)d\sigma = \pm\iint_{D} R(x,y,f(x,y,))dxdy$$

Shunday qilib, sirt bo'yicha integral D soha bo'yicha olingan integralga keltiriladi. Bu yerda $\cos(n, z)\geq 0$ bo'lsa, musbat, $\cos(n, z)<0$ bo'lsa, ikki o'lchovli integral oldida manfiy olinadi.

Xuddi shuningdek,

$$\iint_{\sigma} Pdydz, \qquad \iint_{\sigma} Qdxdz$$

integrallarni ham hisoblash mumkin.

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Xuddi shuningdek,

$$\iint_{\sigma} Pdydz, \qquad \iint_{\sigma} Qdxdz$$

integrallarni ham hisoblash mumkin.

MAYDON NAZARIYASI. SKALYAR MAYDON. VEKTOR MAYDON. VEKTOR CHIZIG'I, UNING DIFFERENSIAL TENGLAMASI

Ta'rif. Fazoning har bir nuqtasida qandaydir $u(M)=u(x, y, z)$ skalyar funksiya aniqlangan bo'lsa, bu sohaga skalyar maydon deyiladi.

Masalan, biror xona ichida temperaturani tarqalishini ko'raylik. Bu temperatura har xil nuqtada har xil bo'ladi, bu esa skalyar maydonga misol bo'la oladi.

M nuqta qaralayotgan sohaning ixtiyoriy nuqtasini ifodalasin, u esa skalyar miqdor bo'lsin, u holda maydonning ichida *M* ning har bir holatiga u miqdorning ma'lum qiymati mos kelsin. Bu shunday yoziladi:

$$u=u(M) \qquad (1)$$

Skalyar maydon statsionar va nostatsionar bo'lishi mumkin. Maydon statsionar bo'ladi, agar u vaqtga bog'liq bo'lmasa. Agar vaqtga bog'liq bo'lsa, u statsionar bo'lmaydi.

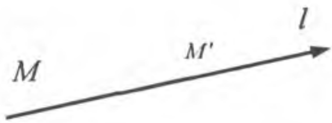
Agar fazoviy skalyar maydon $u(x, y, z)$ funksiya bilan aniqlanadigan bo'lsa, u holda uning sirtlarining hamma to'plamini

$$u(x, y, z)=C \qquad (2)$$

tenglama bilan ifodalash mumkin. Bu yerda *C* har xil sonli qiymatlarni qabul qiladi.

Ta'rif. (2) ko'rinishdagi tenglamaga $u=u(x, y, z)$ skalyar maydonning yuksaklik sirti deyiladi.

Fazoda ixtiyoriy $M(x, y, z)$ nuqta olib, bu nuqtadan ixtiyoriy l to'g'ri chiziq o'tkazamiz va uni aniq yo'naltiramiz. So'ngra shu chiziq ustida *M* nuqtaga yaqin yotgan M' nuqtani olib, ushbu (1-rasm)



1-rasm.

$$\frac{u(M)-u(M')}{MM'}$$

nisbatni tuzamiz. Bu nisbatning $M'\to M$ dagi limiti l yo'nalish bo'yicha $u(M)$ funksiyaning M nuqtadagi hosilasi deyiladi va $\frac{\partial u}{\partial l}$ bilan belgilanadi. Demak,

$$\lim_{M'\to M}\frac{u(M')-u(M)}{MM'}=\frac{\partial u}{\partial l} \qquad (3)$$

Bu ta'rifdan har bir nuqtada hosilalar soni cheksiz ko'p ekani ko'rinadi.

Ixtiyoriy yo'nalish bo'yicha olingan hosilani uchta o'zaro original x, y, z yo'nalishlar bo'yicha olingan hosilalar orqali ifodalash mumkin, chunonchi

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\frac{dx}{dl}+\frac{\partial u}{\partial y}\frac{dy}{dl}+\frac{\partial u}{\partial z}\frac{dz}{dl} \qquad (4)$$

lekin

$$\frac{dx}{dl}=\cos(\hat{l,x}),\quad \frac{dy}{dl}=\cos(\hat{l,y}),\quad \frac{dz}{dl}=\cos(\hat{l,z})$$

Agar bu kosinuslarni $\vec{l}$ yo'nalishdagi birlik vektorning koordinatalari deb qarasak, ya'ni

$$\vec{l}^{\,0}=\{\cos(l,\hat{x}),\cos(l,\hat{y}),\cos(l,\hat{z})\}$$

desak, u holda

$$\vec{l}^{\,0} = \frac{dx}{dl}\vec{i} + \frac{dy}{dl}\vec{j} + \frac{dz}{dl}\vec{k} \qquad (5)$$

va agar (4) dagi $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ larni qandaydir yangi vektorning koordinatalari deb qarasak, u vektorni *grad* deb belgilasak,

$$gradu = \left\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\} \qquad (6)$$

(4) shunday yozish mumkin

$$\frac{du}{dl} = (grad\, u, \vec{l}^{\,0}) \qquad (7)$$

yoki

$$grad\ u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k} \qquad (8)$$

Ta'rif. $u(x, y, z)$ funksiyaning M_0 nuqtadagi gradienti deb, Ox, Oy va Oz o'qlarga proyeksiyalari $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ va $\frac{\partial u}{\partial z}$ dan iborat bo'lgan vektorga aytiladi, ya'ni grad $u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}$.

Misol. $u=\ln(x^2+y^2+z^2)$ maydon uchun $M(0, 1, 0)$ nuqtadagi gradientni toping.

Yechish.

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2+z^2}; \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2+z^2}; \frac{\partial u}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial x}|_M = 0; \quad \frac{\partial u}{\partial y}|_M = 2; \quad \frac{\partial u}{\partial z}|_M = 0 \qquad gradu\,|_M = 2\bar{j}$$

Skalyar maydon gradientining asosiy xossalari

1) $grad\ C=0$ C - doimiy son.
2) $grad\ (u_1 \pm u_2) = grad\ u_1 \pm$ grad u_2
3) $grad\ Cu = C\ grad\ u$
4) $grad\ f(u) = f'(u)\ grad\ u$.
5) $grad\ (\varphi\psi) = \varphi grad\psi + \psi grad\varphi$

Vektor maydon

Ta'rif. Agar fazoning har bir nuqtasida $\bar{a}\,(x, y, z)$ vektor aniqlangan bo'lsa, u holda $\bar{a}\,(x, y, z)$ vektor maydon aniqlangan deyiladi. M nuqta qaralayotgan vektor maydonning ixtiyoriy nuqtasi bo'lsin. U holda nuqtaning har bir joylashishiga vektor funksiya $\bar{a}$ ning ma'lum bir qiymati mos keladi va shunday yoziladi:

$$\bar{a} = \bar{a}(M) \qquad (1)$$

Vektor maydon ikki yoki uch o'lchovli bo'lishi mumkin. Agar tekislikda bo'lsa $M(x,y)$, fazoda bo'lsa M(x,y,z) koordinatalar orqali aniqlanadi. Shuning uchun tekislikdagi vektor maydon $\bar{a} = \bar{a}\,(x, y)$, fazodagi esa $\bar{a} = \bar{a}\,(x, y, z)$ funksiyalar ko'rinishida beriladi. $\bar{a}$ vektor maydonni koordinata o'qlarini yo'nalishlari bo'yicha yoysak

$$\bar{a} = a_x(x,y,z)\bar{i} + a_y(x,y,z)\bar{j} + a_z(x,y,z)\bar{k} \qquad (2)$$

bo'ladi.

(1) vektor maydon yana quyidagi ko'rinishda berilishi mumkin.

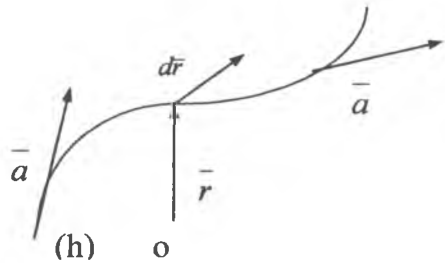
$\bar{a} = \bar{a}\,(\bar{r})$ bu yerda $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ -$M(x, y, z)$ nuqtaning shu maydonga qarashli radius vektori.

Biz ko'proq statsionar ya'ni vaqtga bog'liq bo'lmagan vektor maydon bilan ish ko'ramiz.

Vektor chizig'i va uning differensial tenglamasi

Ta'rif. Vektor maydonning vektor chizig'i deb shunday chiziqqa aytiladiki, uning har bir nuqtasida vektor maydon shu chiziqning urinmasi bo'ylab yo'nalgan bo'ladi.

Bu ta'rifdan kelib chiqib, statsionar vektor maydonning vektor chizig'ining differensial tenglamasini hosil qilish mumkin. Buning uchun $\bar{a}=\bar{a}(\bar{r})$ vektor maydon va L vektor chizig'ini olamiz (2-rasm).



2-rasm.

Agar $\bar{r}$ - L chiziqni nuqtasining radius vektori bo'lsa $d\bar{r}$ - L ning uyurmasi bo'ylab yo'nalgan bo'ladi. Bundan $\bar{a}$ va $d\bar{r}$ vektorlarning kollinearligi kelib chiqadi. Shuning uchun dx, dy, dz proyeksiyalarni $d\bar{r}$ vektorga kollinearligiga ko'ra va $a_x(x,y,z)$, $a_y(x,y,z)$, $a_z(x,y,z)$ larning shu maydonda proporsionalligi kelib chiqadi.

$$\frac{dx}{a_x}=\frac{dy}{a_y}=\frac{dz}{a_z} \qquad (1)$$

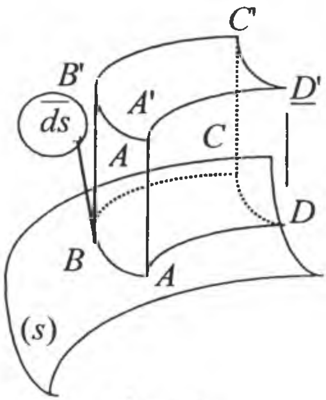
Biz (1) differensial tenglamalar sistemasini hosil qildik, ularni yechib vektor maydonning vektor chizig'ini topish mumkin.

74-ma'ruza

VEKTOR MAYDONNING OQIMI

$\overline{a}$ vektor harakatlanayotgan suyuqlik zarralarining tezligi bo'lsin.

Ta'rif. Vektor maydonning oqimi deb, birlik vaqt mobaynida tanlab olingan yo'nalish bo'yicha sirtdan oqib o'tuvchi suyuqlik miqdoriga aytiladi.



1-rasm.

Bu oqimni П bilan belgilaymiz, va uni aniqlash usulini qidiramiz. Buning uchun П ni $\overline{a}$ va S noma'lumlar orqali hisoblash mumkin. Bu maqsadda kichik $ABCD$ to'rtburchakni olamiz (1-rasm). Uning yuzi ds ga teng. Bu to'rtburchak kichik bo'lgani uchun uni tekislikda deb qarash mumkin. Endi $ABCDA^{l}B^{l}C^{l}D^{l}$ prizmani quramiz, uning qirrasi $\overline{a}$ vektor bilan ustma-ust tushadi. Qurilgan prizmaning asosining hamma nuqtalarida $\overline{a}$ vektor doimiy qiymatni qabul qiladi. U holda $ABCD$ maydondan birlik vaqt mobaynida oqib o'tuvchi suyuqlik miqdori $d\Pi$ $ABCDA^{l}B^{l}C^{l}D^{l}$ prizmani hajmiga teng. Bu hajmning S sirtning yo'naltiruvchi elementi $\overline{ds}$ ni $\overline{a}$ vektor maydoniga skalyar ko'paytmasi orqali aniqlanadi. Bu holda oqim P dP dan olingan sirt integraliga teng bo'ladi, ya'ni

$$\Pi = \oiint_{s} (\bar{a} \cdot d\bar{s}) \qquad (1)$$

Demak, berilgan ikki tomonli sirt orqali oqib o'tayotgan oqim tanlab olingan tomon bo'yicha sirtdan olingan sirt integraliga teng ekan. Bu yerda integral ostida maydon vektori maydon sirtining yo'naltiruvchi elementiga skalyar ko'paytmasi olinadi.

Divergensiya. Uning fizik ma'nosi. Solenoidal maydon

$\bar{a} = a(x,y,z)$ S-sohada harakatlanayotgan suyuqlik zarrachalarining tezligini ifodalovchi vektor bo'lsin. $\bar{a}$ vektor maydon va uning xususiy hosilalari S sohada aniqlangan va uzluksiz bo'lib - vektor maydon

$$\bar{a} = a_x(x,y,z)\bar{i} + a_y(x,y,z)\bar{j} + a_z(x,y,z)\bar{k}$$

koordinata o'qlarining yo'naltiruvchilari orqali yoyilgan bo'lsin.

Ma'lumki, S sohaning sirtidan oqib o'tayotgan $\bar{a}$ vektorning maydon oqimi $\oiint_{s} (\bar{a} d\bar{s})$ sirt integrali bilan hisoblanadi. Bu sirt integrali vektor ko'rinishida Ostrogradskiy formulasining chap tomonidir, ya'ni

$$\oiint_{s} (\bar{a} d\bar{s}) = \iiint_{V} \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) dxdydz \qquad (1)$$

bu yerda V - s bilan chegaralangan hajmdir. Ana shu formuladagi 3 karrali integral ostidagi ifoda vektor maydonining divergensiyasi (tarqalishi) deyiladi va $div\bar{a}$ deb yoziladi.

$$div\bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \qquad (2)$$

Bunga asosan, (1) formula ushbu ko'rinishni oladi.

$$\oint\!\!\!\oint_s (\bar{a}d\bar{s}) = \iiint_V div\bar{a}dv \qquad (3)$$

Bu oxirgi tenglik vektor maydonining asosiy va muhim tushunchalarini ifodalaydi, ya'ni vektor maydonining oqimi sohaning ichkarisida joylashgan divergensiyadan olingan 3 karrali integralga teng ekan. Agar oqimni $\oint\!\!\!\oint_s (\bar{a}d\bar{s})$ ni П bilan belgilasak va ushbu nisbatni qarasak,

$$\frac{\Pi}{V} = \frac{\oint\!\!\!\oint_s (\bar{a}d\bar{s})}{V}$$

bo'lib, (agar П>0) oqib kelayotgan suyuqlikning o'rtacha zichligini ifodalaydi, aksincha (agar П<0) bo'lsa, oqib chiqayotgan suyuqlikning zichligini ifodalaydi.

V sohani shu sohada qandaydir *M* nuqtaga siqib keltiramiz, u holda $\frac{\Pi}{V}$ limiti M nuqtadagi vektor maydonining $div\bar{a}(M)$ deyiladi va shunday yoziladi.

$$div\bar{a}(M) = \lim_{V \to M} \frac{\oint\!\!\!\oint_s \bar{a}d\bar{s}}{V}$$

Demak, berilgan nuqtadagi divergensiya biror sohaning sirti bo'yicha maydon oqimining shu soha hajmiga nisbatining sirt shu nuqtada cheksiz tortilgandagi limitiga teng ekan. Shuni eslatib o'tish kerakki, divergensiyaning ma'nosi vektor maydonining turiga bog'liqdir. Divergensiyaning gidrodinamik ma'nosi quyidagidan iborat: Suyuqlikning turg'un oqimi va uning zarrachalarining tezliklari maydoni $\bar{a}$ ni qaraymiz. Suyuqlik *V* sohadan oqib o'tayotgan bo'lsin. Bu sohani chegaralovchi *S* sirtni 2 ta S_1 va S_2 qismlarga bo'lamiz. S_1 dan suyuqlik kirayotgan bo'lsa, 2 qism S_2 dan suyuqlik chiqayotgan bo'lsin.

U holda yopiq S sirtdan o'tuvchi tezliklar maydoni oqimi ushbu qismlardan o'tuvchi oqimlar yig'indisidan iborat

$$\oint\!\!\!\oint_{s}(\bar{a}d\bar{s}) = \oint\!\!\!\oint_{s_1}(\bar{a}d\bar{s}_1) + \oint\!\!\!\oint_{s_2}(\bar{a}d\bar{s}_2) \qquad (4)$$

bo'ladi.

O'ng tomondagi birinchi integral (σ_1) sirtdan sohaga birlik vaqtda kirayotgan suyuqlik miqdorini bildiradi. Ikkinchisi esa (σ_2) sirt orqali V sohadan birlik vaqt ichida chiqib ketayotgan suyuqlik miqdorini bildiradi.

Demak, V sohani chegaralovchi σ sirtdan o'tayotgan suyuqlik tezliklari maydoni oqimi V sohadagi suyuqlik hajmining sarfiga teng ekan, ya'ni suyuqlikning birlik vaqtdagi hajmining ko'paytmasiga teng ekan.

Divergensiyaning (3) formuladagi ko'rinishiga asosan aytish mumkinki, suyuqlik tezligi maydonning divergensiyasi shu nuqtadagi birlik hajmga mos keluvchi suyuqlik sarfini bildirar ekan.

Agar tezliklar maydonining har bir nuqtasida divergensiya nolga teng bo'lsa, u holda suyuqlik kengaymaydi ham, siqilmaydi ham. Bunga misol sifatida oquvchi suvni qarash mumkin. Bizni o'rab turgan havo ham tezligi tovush tezligining yarmidan kichik o'zini siqilmaydigan suyuqlik kabi tutadi. Shu sababli kichik tezliklar aeromexanikasida divergensiya nolga teng deb qaraladi.

Havoning tezligi tovush tezligiga yaqin va undan oshiq bo'lganda havo oqimining bosimi qayta taqsimlanishga ulgurmaydi va havo o'zini siqiladigan gaz kabi tutadi.

Agar biz (3) formulaga e'tibor qilsak, uning chap tomoni V sohani chegaralovchi σ sirtdan o'tayotgan maydon oqimini bildirsa, o'ng tomoni V sohadagi maydon sarfini bildiradi. Demak, quyidagi Ostrogradskiy teoremasi o'rinlidir.

Teorema. Yopiq σ sirtdan o'tayotgan maydon oqimi shu sirt bilan chegaralangan V sohadagi maydon sarfiga teng.

Divergensiya xossalari

1. $div\,\vec{c}=0$ $\quad(\vec{c}=\text{const})$

Agar $\vec{c}=\{c_x,\ c_y,\ c_z\}$ bo'lsa, $div\,\vec{c}=\frac{\partial c_x}{\partial x}+\frac{\partial c_y}{\partial y}+\frac{\partial c_z}{\partial z}=0$

2. $div(m\bar{a})=m\,div\,\bar{a}$ (m-const)

Haqiqatan ham $m\bar{a}=\{ma_x,\ ma_y,\ ma_z\}$ bo'lsa, u holda

$$div(m\bar{a})=\frac{\partial(\mathrm{ma}_x)}{\partial x}+\frac{\partial(\mathrm{ma}_y)}{\partial y}+\frac{\partial(\mathrm{ma}_z)}{\partial z}=$$

$$m\left(\frac{\partial \mathrm{a}_x}{\partial x}+\frac{\partial \mathrm{a}_y}{\partial y}+\frac{\partial \mathrm{a}_z}{\partial z}\right)=m\,div\,\bar{a}$$

3. $div(\bar{a}+\bar{b})=div\,\bar{a}+div\,\bar{b}$

Haqiqatan ham agar $\bar{a}=\{a,\ a_y,\ a_z\}$, $\bar{b}=\{b_x,\ b_y,\ b_z\}$ bo'lsa,

$$\bar{a}+\bar{b}=\{a_x+b_x,\ a_y+b_y,\ a_z+b_z\}$$

demak,

$$div(\bar{a}+\bar{b})=\frac{\partial(\mathrm{a}_x+b_x)}{\partial x}+\frac{\partial(\mathrm{a}_y+b_y)}{\partial y}+\frac{\partial(\mathrm{a}_z+b_z)}{\partial z}=div\,\bar{a}+div\,\bar{b}$$

1-misol sifatida koordinata boshida joylashgan e nuqtaviy zaryadning elektr maydonini qaraylik

$$\bar{E}(M)=\mathrm{ke}\bar{r}/r^3=ke(x\bar{i}+y\bar{j}+z\bar{k})/r^3$$

bu maydonning divergensiyasi

$$div\bar{E}(M)=ke\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]=$$

$$=ke\left(\frac{r^2-3r^2r_x^{|}x}{r^5}+\frac{r^2-3r^2r_y^{|}y}{r^5}+\frac{r^2-3r^2r_z^{|}z}{r^5}\right)=$$

$$=ke(3r^2-3x^2-3y^2-3z^2)/r^5=3ke(r^2-(x^2+y^2+z^2))/r^5=$$

$$=3ke(r^2-r^2)/r^5=0 \qquad (r\neq 0)$$

uning fizik ma'nosi shundan iboratki, koordinata boshidan boshqa birorta ham nuqtada zaryad manbai joylashmagan koordinata boshida esa

$$\bar{a}+\bar{b}=\{a_x+b_x,\ a_y+b_y,\ a_z+b_z\}$$

demak,

$$div(\bar{a}+\bar{b})=\frac{\partial\ (a_x+b_x)}{\partial\ x}+\frac{\partial\ (a_y+b_y)}{\partial\ y}+\frac{\partial\ (a_z+b_z)}{\partial\ z}=div\,\bar{a}+div\,\bar{b}$$

2-misol. Quyidagi

$$\bar{A}=\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$$

radius vektor maydonining divergensiyasi topilsin.

Yechish: Bu holda $a_x=x$, $a_y=y$, $a_z=z$ va *div* formulasini qo'llasak,

$$div\bar{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3$$

demak, radius - vektor maydonining har bir nuqtasidan zichlik manbai 3 birlikka teng ekan.

3-misol. $\bar{a}=3x^2y\bar{i}+(x-y)\bar{j}+z^2\bar{k}$ vektor maydonining divergensiyasi topilsin.

Yechish: Mos o'zgaruvchilardan maydon proyeksiyasi bo'yicha xususiy hosilalarni topamiz

$$\frac{\partial\ a_x}{\partial x}=\frac{\partial\ (3x^2y)}{\partial x}=6xy;$$

$$\frac{\partial\ a_y}{\partial y}=\frac{\partial\ (x-y)}{\partial y}=-1;$$

$$\frac{\partial\ a_z}{\partial z}=\frac{\partial\ (z^2)}{\partial z}=2z;$$

Divergensiya formulasiga ko'ra

$$div\,\bar{a}=6xy-1+2z=6xy+2z-1.$$

Ostrogradskiy teoremasi

Bizga integral hisob kursidan ma'lumki,

$$\iint_S a_x dydz + a_y dzdx + a_z dxdy = \iiint_V \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) dxdydz \quad (1)$$

bu formulaga (teoremaga) Ostrogradskiy formulasi deyiladi. Bu formulaning o'ng va chap tomonida turgan integrallarni vektor formasida yozsak, shuningdek $\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}$ vektor maydonining proyeksiyasi sifatida $a_x(x, y, z)$, $a_y(x, y, z)$, $a_z(x, y, z)$ lar olingan. Ammo chap tomondagi integral $\bar{a}$ vektor maydonning S sirt bo'yicha oqimini bildiradi, o'ng tomondagi uch karrali integral ostidagi funksiya $\bar{a}$ maydonni tarqalishini bildiradi. (1) ni vektor formada quyidagicha yoziladi.

$$\oiint_s (\bar{a}d\bar{s}) = \iiint_V div\bar{a}dv$$

Yopiq sirt bo'yicha oqim sohaning maydon divergensiya bo'yicha olingan uch karrali integralga teng.

Ostrogradskiy teoremasi yopiq sirt bo'yicha oquvchi vektor oqimini hisoblashga imkon beradi.

Misol. Ostrogradskiy teoremasi yordamida birinchi oktantda joylashgan va koordinata tekisliklari bilan chegaralangan va ushbu sirt tenglamalari bilan berilgan $(y_2 - x_2 = 4 - z)$ $\bar{a} = (y - z)(\bar{i} + \bar{k})$ vektor maydonning oqimi topilsin.

Yechish. Berilgan maydon uchun

$$a_x = y - z, \; a_y = 0, \; a_z = y - z$$

$$\frac{\partial a_x}{\partial x} = 0 \qquad \frac{\partial a_y}{\partial y} = 0 \qquad \frac{\partial a_z}{\partial z} = -1$$

bo'ladi.

Demak, $div\bar{a} = -1$

Ostrogradskiy teoremasiga asosan

$$\Pi = \iiint\limits_V (-1)dxdydz$$

Bu integralni hisoblash uchun silindrik koordinatalarga o'tamiz

$$x=\rho\cos\varphi, \quad y=\rho\sin\varphi, \quad z=z$$

bundan $\Pi = -\int\limits_0^{\pi/2} d\varphi \int\limits_0^{2} \rho d\varphi \int\limits_0^{4-\rho^2} dz = -2\pi$.

Demak, $\Pi<0$, bu degan so'z berilgan sirtning ichiga kelayotgan oqim ichki manba ishlab chiqarayotgan oqimga nisbatan suyuqlikni ko'proq singdirayotgan ekan.

VEKTOR MAYDONNING SIRKULYATSIYASI

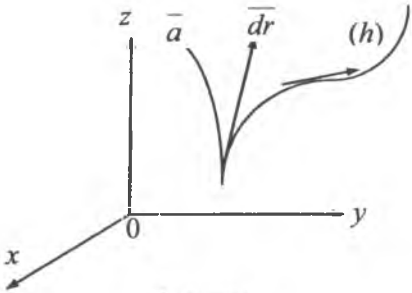
Yo'naltirilgan L chiziq bo'ylab $\overline{a}$ vektorning ta'sirida material zarrachalar harakatlanayotgan bo'lsin. $\overline{a}$ kuch ta'sirida bajarilayotgan A ishni hisoblash masalasini qo'yamiz. Bu elementar joyda $\overline{a}$ kuchni doimiy deb dA ishning elementini topamiz, u $\overline{a}$ kuchning yo'naltirilgan $d\overline{r}$ elementga skalyar ko'paytirilganiga teng

$$dA = (\overline{a},\ d\overline{r})$$

A ishni butun L chiziq bo'ylab qidirsak,

$$A = \int_L (\overline{a},\ d\overline{r})$$

bo'ladi.



1-rasm.

Demak, chap tomonda $\overline{a}$ vektor bo'yicha L chiziqning uzunligi bo'ylab olingan egri chiziqli integral turibdi.

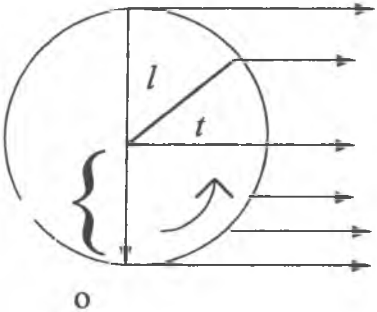
Ta'rif. Agar yo'naltirilgan L kontur yopiq bo'lsa, $\overline{a}$ ning fizik ma'nosidan qat'iy nazar shu kontur bo'yicha olingan egri chiziqli integral L kontur bo'ylab $\overline{a}$ vektor maydonning sirkulyatsiyasi deyiladi va shunday yoziladi

$$Ц = \oint_L (\bar{a},\ d\bar{r})$$

Agar $\bar{a}$ vektor kuch bo'lsa, u holda $Ц$ shu yopiq kontur bo'ylab shu kuchning bajargan ishini bildiradi.

Misol. $\bar{a} = ui$ vektor maydonning sirkulyatsiyasi $x^2+(y-b)^2=6^2$ aylananing konturi bo'ylab topilsin.

$$\bar{a} = y\bar{i}$$



2-rasm.

Yechish.

$$d\bar{r} = \bar{i}dx + \bar{j}dy + \bar{k}dz$$

$\bar{a} = ui$

$\bar{a}d\bar{r} = ydx$

$Ц = \oint_L (\bar{a},\ d\bar{r}) = \oint_L ydx$, L- aylana.

Tenglamani parametrik ko'rinishda yozib olamiz.

$$x=b\cos t, \quad y=b+b\sin t,$$

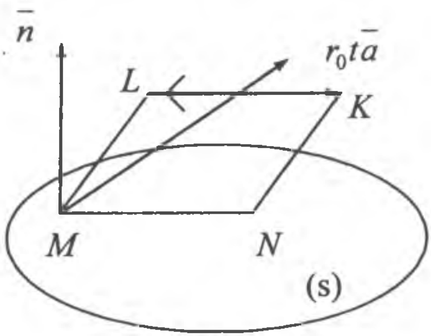
t- musbat o'tishdagi burchak bo'lib, u 0 dan 2π gacha o'zgaradi.

$$Ц = \oint_L (\bar{a},\ d\bar{r}) = \oint_L ydx = -\int_0^{2\pi} (b + b\sin t)b\sin t dt = -\pi b^2$$

Bu yerda minus ishora aylana $\bar{a} = ui$ kuch ta'sirida teskari yo'nalishi bo'ylab aylanadi degan ma'noni beradi.

Vektor maydonning rotori

Vektor maydonning rotori tushunchasini kiritamiz. Buning uchun $\overline{a}$ vektor maydonning ichkarisidan M nuqtani va fiksirlangan $\overline{n}$ vektorni tanlab olamiz (3-rasm).



3-rasm.

Ma'lumki, $0Z$ o'qi bo'yicha sirkulyatsiyaning zichligi

$$P_{0Z} = \lim_{Q\to 0}\frac{Ц}{Q} = \lim_{Q\to 0}\frac{\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)Q}{Q}$$

Bu yerda Q – $MNKL$ to'rtburchakning yuzi, yoki

$$P_{0Z} = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}.$$

Ammo P_{0Z} ta'rifga ko'ra $rot_{0Z}\overline{a} - rot\overline{a}$ ning $0Z$ o'qqa proyeksiyasi. Shunday qilib,

$$rot_{0Z} = \overline{a} = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}$$

$$rot_{0U}\bar{a} = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}$$

$$rot_{0X}\bar{a} = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}$$

Uchalasini qo'shsak proyeksiyadagi $rot\bar{a}$

$$rot\ \bar{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\bar{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\bar{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\bar{k} \quad (1)$$

Bu formulani qisqacha shunday yozish mumkin:

$$rot\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad (2)$$

Bu formula ishlatilganda birinchi satr elementlari bo'yicha yoyilib ikkinchi satr bilan uchinchi satr elementlarining «ko'paytmasi» esa, differensiallash ma'nosida tushunilishi lozim.

Vektor maydonining oqimini ya'ni (1) ni Stoks formulasi bilan solishtirsak

$$\oiint_s \left(\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z}\right)dydz + \left(\frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x}\right)dzdx + \left(\frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y}\right)dxdy = \\ = \oint_L a_x dx + a_y dy + a_z dz \quad (3)$$

ning o'ng tomoni $\bar{a}$ ning rotorini ifodalaydi, agar buni qisqacha vektor ko'rinishda yozsak, $\oiint_S (rot\bar{a}d\bar{s}) = \oint_L (\bar{a}d\bar{r})$ bo'ladi.

Ta'rif. $\bar{a}$ vektor maydonning M nuqtadagi rotori deb, istalgan $\bar{n}$ yo'nalishga proyeksiyasi M nuqtadagi (shu yo'nalish bilan bir xil bo'lgan) sirkulyatsiya zichligiga teng bo'lgan vektorga aytiladi:

$$rot\bar{a}\big|_M = \lim_{L\to M}\frac{\int_L \bar{a}_l\,dl}{S} \qquad (4)$$

$\bar{a}$ vektor maydon rotorining quyidagi xossalari mavjud:

1. $rot\bar{c} = 0$ $\bar{c}$ - doimiy vektor
2. $rot(\bar{a}_1 \pm \bar{a}_2) = rot\bar{a}_1 \pm rot\bar{a}_2$
3. $rot\, c\bar{a} = c\, rot\bar{a}$ c-doimiy skalyar
4. $rot(u\bar{a}) = u\, rot\bar{a} + [grad\ u\,\bar{a}]$
5. $rot(grad\ u) = 0$

Misol. $0Z$ o'q atrofida o'zgarmas $\vec{w}$ aylanma burchak tezligi bilan soat milining harakatiga teskari yo'nalishda aylanuvchi jism nuqtalarining tezligi, maydonning uyurmasi topilsin.

Mexanikadan ma'lumki, chiziqli tezlik $\vec{v} = [\vec{w}, \vec{r}]$, bunda $\vec{w}$ - burchak tezligi bo'lib, aylanish o'qi bo'ylab yo'nalgan, $\vec{r}$ -nuqtaning koordinatalar boshidan bo'lgan radius vektori.

Masala shartida aylanish o'qi z bo'lgani uchun w_x=0, w_y=0, w_z=w $\vec{r}\,(x, y, z)$

$$\vec{v} = [\vec{w}, \vec{r}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix}$$ bundan w_x=-wy, w_y=wx, w_z=0

yoki $\vec{v}$ ni ushbu formula orqali ifodalasak,

$$\vec{v} = \vec{v}_0 + \vec{w} \times \vec{\rho} \qquad (5)$$

bu yerda

$$\vec{\rho} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$$

$$\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$$

$$\vec{w} = w_x\vec{i} + w_y\vec{j} + w_z\vec{k}$$

ekanligini e'tiborga olsak, (5) vektorli formuladan $\vec{v}$ tezlik vektorining proyeksiyasining hisoblash formulasini hosil qilamiz

$$X = x_0 + w_y(z - z_0) - w_z(y - y_0),$$

$$Y = y_0 + w_z(x - x_0) - w_x(z - z_0),$$

$$Z = z_0 + w_x(y - y_0) - w_y(x - x_0).$$

Bularni e'tiborga olsak, $\vec{v}$ tezlik maydoni rotori

$$rot\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\vec{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\vec{k} =$$

$$= 2w_x\vec{i} + 2w_y\vec{j} + 2w_z\vec{k} = 2\vec{w}$$

bo'ladi.

Demak, qattiq jism tezligi maydonning ixtiyoriy nuqtadagi rotatsiyasi burchak tezligining ikkilanganiga teng ekan.

Stoks formulasi

$$\iint_s \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)dydz + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)dzdx +$$

$$+ \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)dxdy = \oint_L a_x dx + a_y dy + a_z dz \qquad (1)$$

bu formula integral hisobda ko'rsatilgan edi.

Bu formula S sirtning yuzasi bo'yicha olingan integralni yopiq chegarali shu sirt bo'yicha olingan egri chiziqli integral bilan bog'laydi.

a_x, a_y, a_z funksiyalar vektor maydonning koordinata o'qlariga proyeksiyalari deb qaraymiz. Bu esa (1) formulani vektor ko'rinishda yozishga imkon beradi. (1) formulaga e'tibor qilsak uning chap tomoni S sirtdan o'tuvchi ushbu

$$\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z};\quad \frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x};\quad \frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}$$

proyeksiyalarga ega bo'lgan vektorning uyurmasidir.

O'ng tomoni esa $\overline{a}$ vektorning L kontur bo'yicha sirkulyatsiyasini beradi. Demak, Stoks formulasi vektor ko'rinishda

$$\iint\limits_S (rot\overline{a}ds) = \oint\limits_L (\overline{a}d\overline{r})$$

Demak, vektor maydonning sirt bo'yicha rotori shu sirtning chegarasi bo'ylab hosil qilingan sirkulyatsiyaga teng ekan.

Solenoidal maydon

Ta'rif. Agar $\overline{a}$ vektor maydonning har bir nuqtasida divergensiya nolga teng bo'lsa $(div\overline{a}=0)$ $\overline{a}$ vektor maydonni solenoidal yoki quvurli maydon deyiladi. Solenoidal maydon siqilmaydigan maydondir. Ixtiyoriy $\overline{a}$ vektor maydoni berilgan bo'lsin. Bu maydon rotatsiyasi maydonini $rot\overline{a}=\overline{a}_0+\overline{a}$ deb belgilaymiz.

$$rot\overline{a}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\vec{i}+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\vec{j}+\left(\frac{\partial a_y}{\partial z}-\frac{\partial a_z}{\partial y}\right)\vec{k}$$

bo'ladi.

Bu maydon divergensiyasini hisoblaymiz

$$div rot \bar{a} = \frac{\partial^2 a_z}{\partial y \partial x} - \frac{\partial^2 a_y}{\partial x \partial z} + \frac{\partial^2 a_x}{\partial z \partial y} - \frac{\partial^2 a_z}{\partial x \partial y} + \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial z} = 0$$

Demak, ixtiyoriy vektor maydonning rotatsiya maydoni solenoidal maydon ekan.

$$div(rot\bar{a}) = \nabla(\bar{v} \times \bar{a}) = 0$$

Solenoidal maydon oqimi nolga tengdir.

$$\oiint_S \bar{a} d\bar{s} = 0$$

Misol. $\bar{a} = -y\bar{i} + x\bar{j}$ maydonning solenoidal ekanligini va uning vektor chiziqlarining yopiq ekanligini ko'rsating.

Yechish. Osonlik bilan ko'rsatish mumkinki,

$$div\bar{a} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0$$

demak. vektor maydon haqiqatan ham quvurli ekan.

Endi ushbu differensial tenglamani qaraymiz.

$$\frac{d\,y}{x} = \frac{d\,x}{-y}, \qquad xdx + ydy = 0$$

integrallasak, $x^2 + y^2 = c^2$ aylana hosil bo'ladi, demak maydonning vektor chiziqlari yopiq ekan.

GAMILTON OPERATORI

Gamilton operatori - nabla operatori yoki qisqacha nabla deb yuritiladi. Nabla ramziy shartli vektor bo'lib, koordinata o'qlaridagi proyeksiyalari orqali ifodalanadi.

$$\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

Nabla operatori yordamida yuqorida qaralgan *grad u*, div $\bar{a}$ va rot $\bar{a}$ birinchi tartibli operatorlar juda qulay yoziladi. ∇ vektorning *u* skalyarga ko'paytmasini topamiz.

$$\nabla u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}$$

Bu tenglikning o'ng tomoni *u* ning gradientini beradi. Demak,

$$grad\ u = \nabla u \qquad (1)$$

Endi ∇ orqali divergensiya va rotorni yozsak

$$\text{div } \bar{a} = (\nabla, \bar{a})$$

bo'ladi.

Bundan

$$(\nabla, \bar{a}) = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

va nihoyat

$$rot\ \bar{a} = [\nabla, \bar{a}].$$

Potensial vektor maydon

Ta'rif. Agar qandaydir skalyar maydon gradienti vektor maydon bo'lsa, vektor maydon potensial vektor maydon deyiladi. $\bar{a}$ ning potensial maydoni uchun shunday skalyar u maydon bo'lishi kerakki,

$$\bar{a} = grad\, u \qquad (1)$$

$$\bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k} \qquad (2)$$

(1) tenglikning o'ng tomonini ochib yozsak,

$$\bar{a} = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k} \qquad (3)$$

(2) bilan (3) ni solishtirsak,

$$a_x = \frac{\partial u}{\partial x},\ a_y = \frac{\partial u}{\partial y},\ a_z = \frac{\partial u}{\partial z} \qquad (4)$$

u funksiyaning to'la differensialini yozsak,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

yoki

$$du = a_x dx + a_y dy + a_z dz \qquad (5)$$

$\bar{a}$ maydonning potensialligidan (5) munosabat qandaydir $u=u(x, y, z)$ funksiyaning to'la differensialini bildiradi. Potensial maydonning u funksiyasi uning potensiali deb ataladi.

Potensial maydonning xossalari

1-xossa. $\bar{a}$ vektor maydon potensial bo'lishi uchun u uyurmasiz bo'lishi zarur va yetarli, yoki

$$rot\ \bar{a} \equiv 0 \qquad (6)$$

Isbot. Avval (6) shartning bajarilishi zaruriyligini isbot qilamiz. Faraz qilaylik maydon potensial bo'lsin. U holda ta'rifga ko'ra shunday u funksiya mavjudki,

$$\overline{a} = grad\ u$$

Shuningdek, $rot\ grad\ u \equiv 0$ bo'lgani uchun

$$rot\ \overline{a} = rot\ grad\ u \equiv 0$$

Demak, (6) shart bajariladi.

Endi yetarlilik shartini isbotlaymiz. Faraz qilaylik,

$$rot\ \overline{a} \equiv 0$$

yoki

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \equiv 0$$

Bundan esa

$$\begin{cases} \dfrac{\partial a_z}{\partial y} - \dfrac{\partial a_y}{\partial z} = 0 \\ \dfrac{\partial a_x}{\partial z} - \dfrac{\partial a_z}{\partial x} = 0 \\ \dfrac{\partial a_y}{\partial x} - \dfrac{\partial a_x}{\partial y} = 0 \end{cases} \qquad (7)$$

(7) dan ko'rinadiki shunday u funksiya mavjudki,

$$du = a_x dx + a_y dy + a_z dz$$

bo'ladi.

Bu degan so'z $\overline{a} = grad\ u$ bo'lib $\overline{a}$ maydonning potensialligi kelib chiqadi.

2-xossa. Bir tomonlama bog'lanishli soha potensial bo'lishi uchun ixtiyoriy yopiq kontur bo'yicha olingan sirkulyatsiya nolga teng bo'lishi kerak.

Isbot. Avvalo zaruriy shartni isbot qilamiz. Stoks formulasidan foydalansak,

$$\oint_L (\overline{a}, d\overline{r}) = \iint_S (rot\overline{a}, d\overline{s})$$

Bu yerda L -ixtiyoriy yopiq kontur, S – L ustiga tortilgan sirt.

Quyidagicha fikr yuritsak: faraz qilaylik maydon potensialli bo'lsin, unda (1) teoremaga asosan $rot\,\overline{a} \equiv 0$. Ammo Stoks formulasining o'ng tomoni nolga teng bo'ladi. Bundan chap tomonining ham nolga teng ekanligi kelib chiqadi.

Endi yetarli shartini isbotlaymiz. Buning uchun ixtiyoriy L kontur ($\overline{a}$ maydonning ichida joylashgan) uchun sirkulyatsiya $Ц$=0 deb faraz qilamiz va $Ц$ ning egri chiziqli mos integral orqali ifodalanishini hisobga olsak,

$$\oint_L a_x dx + a_y dy + a_z dz = 0$$

Bu degan so'z

$$\oint_L a_x dx + a_y dy + a_z dz$$

L konturning yo'liga bog'liq emas. Bundan shunday u funksiya mavjudki uning uchun $du = a_x dx + a_y dy + a_z dz$ ya'ni $\overline{a}$ maydon potensiali. Teorema isbot bo'ldi.

Laplas operatori

Faraz qilaylik, birinchi tartibli operatorlar $grad\,u$, $div\,\overline{a}$, $rot\,\overline{a}$ dagi funksiyalar u va a_x, a_y, a_z lar birinchi tartibli x, y va z bo'yicha xususiy hosilalarga ega bo'lsin. Bu talabni kuchaytiramiz, ya'ni yuqorida keltirilgan funksiyalar ikkinchi tartibgacha xususiy hosilalarga ega bo'lsin va mos ravishda ikkinchi tartibli operatsiyalar to'g'risidagi savolni qo'yamiz.

Biz besh xıl ikkinchi tartibli operatsiyalarni ko'rishimiz mumkin.

Nabla operatori yordamida uchta eng ko'p tarqalgan uchta operatsiyalarni qaraymiz.

$$div\ grad\ u = (\nabla;\ \nabla u)=(\nabla;\ \nabla)u;$$

grad $u=\nabla u$ va $div\,\overline{a}=(\nabla;\ \overline{a})$ dan foydalansak va u ko'paytuvchini skalyar ko'paytma belgisidan tashqariga chiqardik. Ammo

$$(\nabla;\nabla)=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

bo'ladi.

Natijada biz yangi skalyar operator hosil qildik. U Laplas operatori deb yuritiladi va ∇ delta harf bilan belgilanadi. Demak, gradientdan olingan divergensiya (skalyar maydonda) laplasiyanga ya'ni *div grad* $u = \nabla u$ teng bo'ladi.

Endi

$$\text{rot grad u} = [\nabla;\nabla u]= [\nabla;\nabla]u$$

va

$$grad\ u=\nabla u; \qquad rot\,\overline{a}=[\nabla;\ \overline{a}]$$

e'tiborga olsak, shuningdek $[\nabla;\ \nabla]=0$ vektorni vektor ko'paytmasi ya'ni kvadrati nolga teng. Shuning uchun skalyar maydonning gradientidan olingan rotor

$$rot\ grad\ u = 0$$

div rot a - operatsiyani qaraymiz

$$div\ rot\,\overline{a}=(\nabla,[\nabla,\overline{a}])$$

bo'ladi va vektorlarni vektor ko'paytmasining ta'rifiga ko'ra vektorlar ∇ va $[\nabla,\ \overline{a}]$ perpendikulyar, demak ularning skalyar ko'paytmasi nolga teng. Bundan esa

$$div\ rot\ \overline{a} = 0$$

Endi *grad div* $\overline{a}$ va *rot rot* $\overline{a}$ ikki operatsiyani o'quvchi kerak bo'ladigan hollarda bemalol mustaqil foydalanadi.

Bu natijada ushbu muhim uchta operatsiyani hosil qildik

$$div\ grad\ u = \nabla u \qquad (1)$$

bu yerda

$$\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

laplasiyan

$$rot\ grad\ u \equiv 0 \tag{2}$$

$$div\ rot\ \overline{a} \equiv 0. \tag{3}$$

KOMPLEKS O'ZGARUVCHINING FUNKSIYALARI NAZARIYASI

1. Kompleks sonlar va ular ustida amallar

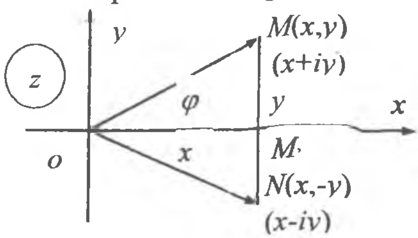
1-ta'rif. Kompleks son deb $x+iy$ ko'rinishdagi ifodaga aytiladi, bunda x va y - haqiqiy sonlar, i - mavhum birlik; $i=\sqrt{-1}$ kompleks sonlarni z harfi bilan belgilaymiz, ya'ni $z=x+iy$, x - kompleks sonning haqiqiy qismi, iy - kompleks sonning mavhum qismi, y - mavhum qismining koeffitsienti deyiladi. x va y lar quyidagicha belgilanadi:

$$x=Re\ z, \qquad y=Jm\ z.$$

2-ta'rif. Agar $x_1=x_2$, $y_1=y_2$ bo'lsa, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ - ikki kompleks son o'zaro teng, ya'ni $z_1=z_2$ deyiladi.

3-ta'rif. $z=x+iy$ va $\bar{z}=x-iy$ kompleks sonlar qo'shma kompleks sonlar deyiladi.

Kompleks sonlarning geometrik tasviri va trigonometrik formasini ko'ramiz. To'g'ri burchakli Dekart koordinatalar sistemasidagi har bir (x, y) nuqtaga bitta $x+iy$ kompleks sonni mos keltiraylik. Umuman shu usulda har bir kompleks songa tekislikda bitta nuqta mos keladi va aksincha tekislikdagi har bir nuqtaga bitta kompleks son mos keladi. Abssissa o'qi haqiqiy sonlarning geometrik o'rni, ordinata o'qi mavhum iy sonlarning geometrik o'rni bo'ladi. Shuning uchun abssissalar o'qi haqiqiy o'q, ordinatalar o'qi mavhum o'q deyiladi. Shunday tekislik z kompleks tekisligi deyiladi.



1-rasm.

Tekislikning har bir (x, y) nuqtasiga koordinatalar boshidan chiqqan, oxiri shu nuqtada bo'lgan vektorni mos keltirish mumkin. Shuningdek, har bir $x+iy$ kompleks songa koordinatalar x va y bo'lgan $\overrightarrow{OM}$ vektor mos keltiriladi.

1-rasmga asosan:

$$r=\sqrt{x^2+y^2},\quad tg\varphi=\frac{y}{x},\quad \varphi=arctg\frac{y}{x},\quad x=r\cos\varphi,\quad y=r\sin\varphi$$

Unda $z=x+iy=r\cos\varphi+ir\sin\varphi=r(\cos\varphi+i\sin\varphi)$, yoki

$$z=r(\cos\varphi+i\sin\varphi) \qquad (1)$$

bunda r-kompleks sonning moduli, ya'ni $r=|z|$, φ-uning argumenti $\varphi=\mathrm{Arg}z$. (Agar $-\pi<\mathrm{Arg}z\leq\pi$ bo'lsa, unda $\mathrm{Arg}z=\arg z$ bo'ladi $\arg z$-bosh argument deyiladi.) (1) formula - kompleks sonning trigonometrik formasi deyiladi.Agar Eyler formulasini $e^{i\varphi}=\cos\varphi+i\sin\varphi$ e'tiborga olsak, unda

$$z=re^{i\varphi} \qquad (2)$$

(2) kompleks sonning ko'rsatkichli formasi deyiladi.

1-misol. $z=1+i$ trigonometrik formaga keltiring.

Yechish: $x=1$, $y=1$, $r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$ $\mathrm{tg}\varphi=1 \Rightarrow$ $\varphi=\pi/4$. Demak,

$$z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right).$$

2-misol. $z=-1$ son trigonometrik formaga keltirilsin.

Yechish: $x=-1$, $y=0$, $r=\sqrt{x^2+y^2}=1$, $\mathrm{tg}\varphi=0$, $\varphi=\pi$, $z=\cos\pi+i\sin\pi$.

Kompleks sonlar ustidagi amallar

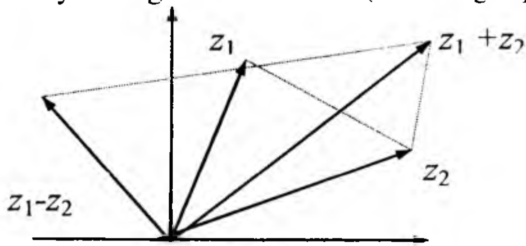
1) qo'shish va ayirish.

$$z_1=x_1+iy_1$$
$$z_2=x_2+iy_2$$

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2). \qquad (3)$$

Demak, kompleks sonlar qo'shilganda (ayrilganda) ularning haqiqiy qismlari alohida va mavhum qismlari alohida qo'shiladi (ayriladi).

Kompleks sonlarni qo'shish va ayirish vektorlar qo'shilishi va ayrilishiga mos bo'ladi. (2- rasmga qarang)



2-rasm.

$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ demak, ikkita nuqta orasidagi masofaga teng. $|z_2 - z_1|$ - kompleks sonlar ayirmasining moduli.

2) ko'paytirish va bo'lish.

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

a) $z_1 \cdot z_2 = (x_1 + iy_1)\ (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Agar kompleks sonlarni trigonometrik formada olsak, unda

$$z_1 \cdot z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1) \cdot r_2(\cos\varphi_2 + i\sin\varphi_2) = r_1r_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)] = r_1r_2[(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

yoki

$$z_1 \cdot z_2 = r_1r_2[(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)] \qquad (4)$$

Demak, kompleks sonlarni ko'paytirishda modullari ko'paytiriladi, argumentlari esa qo'shiladi.

$$z_1 = r_1e^{i\varphi_1},\ z_2 = r_2e^{i\varphi_2},\ z_1 \cdot z_2 = r_1r_2e^{i\varphi_1} \cdot e^{i\varphi_2} = r_1r_2e^{i(\varphi_1 + \varphi_2)}$$

b) $\frac{z_1}{z_2}=\frac{x_1+iy_1}{x_2+iy_2}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=$

$$=\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

Agar z_1 va z_2 trigonometrik formada berilgan bo'lsa, unda

$\frac{z_1}{z_2}=\frac{r_1e^{i\varphi_1}}{r_2e^{i\varphi_2}}=\frac{r_1}{r_2}e^{i(\varphi_1-\varphi_2)}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)]$ yoki

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}=[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)] \qquad (5)$$

Demak, kompleks sonlarni bo'lishda ularning argumentlari ayriladi, modullari bo'linadi.

3) darajaga ko'tarish va ildiz chiqarish.

a) $z=re^{i\varphi}$, kompleks sonining n - darajaga ko'taraylik

$z^n=(re^{i\varphi})^n=r^ne^{in\varphi}$, yoki

$$z^n=r^n(\cos n\varphi+i\sin n\varphi) \qquad (6)$$

Demak, trigonometrik formada berilgan kompleks sonni darajaga ko'tarishda modul shu darajaga ko'tariladi, argument darajaga ko'paytiriladi.

Agar (6) da $r=1$ bo'lsa $[r(\cos\varphi+i\sin\varphi)]^n=\cos n\varphi+i\sin n\varphi$ Muavr formulasi hosil bo'ladi.

b) $z=re^{i\varphi}$, kompleks sonining n darajali ildizi w bo'lsin, ya'ni

$\sqrt[n]{z}=w=\rho e^{i\psi}$, $z=w^n=\rho^n(\cos n\psi+i\sin n\psi)$,

$r(\cos\varphi+i\sin\varphi)=\rho^n(\cos n\psi+i\sin n\psi)\Rightarrow r=\rho^n$, $n\psi=\varphi+2k\pi$, $\rho=\sqrt[n]{z}$,

$\psi=\frac{\varphi+2k\pi}{n}$, ya'ni

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right] \qquad (7)$$

3-misol $\sqrt[3]{-8}=?$ $-8=8(\cos\pi+i\sin\pi)$, chunki $r=\sqrt{64}=8$, $\varphi=\pi$.

Yechish: $\sqrt[3]{-8}=2\left(\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3}\right)$ k=0, 1, 2,

bo'lganda $\sqrt[3]{-8}=\begin{cases}1+i\sqrt{3}\\-2\\1-i\sqrt{3}\end{cases}$.

2. Kompleks sonning logarifmi. Soha to'g'risida tushuncha. Jordan chizig'i. Stereografik proyeksiya

1. Kompleks sonning logarifmi

$z=2(\cos\varphi+i\sin\varphi)=re^{i\varphi}$ kompleks son berilgan bo'lsin. $z=re^{i(\varphi+2k\pi)}=re^{i\varphi}$

$\ln z=\ln(re^{i\varphi})=\ln r+i\varphi\ln e=\ln r+i\varphi$, ya'ni

$$\ln z=\ln r+i\varphi \qquad (1)$$

$$\ln z=\ln r+i\varphi+2k\pi i \qquad (2)$$

1-misol $z=-1$ ning logarifmini toping.

Yechish: $z=-1=\cos\pi+i\sin\pi$; $2=1$, $\varphi=\pi$

$$\ln z=\ln 1+i\pi=i\pi$$

$$\ln z=i\pi+2k\pi i=i\pi(1+2k),\ k=0,\ \pm1,\ \pm2,\ \ldots$$

2. Soha tushunchasi

Kompleks sonlar tekisligi (Z) da biror E to'plam berilgan bo'lsin.

1-ta'rif. z-nuqtaning kichik atrofi deb, markazi z nuqtada bo'lgan yetarli kichik radiusli doiraga tegishli nuqtalar to'plamiga aytiladi.

2-ta'rif. Agar z nuqtaning kichik atrofidagi barcha nuqtalar E to'plamga tegishli bo'lsa, z nuqta E to'plamning ichki nuqtasi deyiladi.

3-ta'rif. Agar z nuqtaning kichik atrofidagi nuqtalarning ba'zilari E ga tegishli, ba'zilari tegishli bo'lmasa, u E ning chegaraviy nuqtasi deyiladi.

3-rasmda z_1-ichki, z_2-chegaraviy, z_3-tashqi nuqtalardir.

3-rasm.

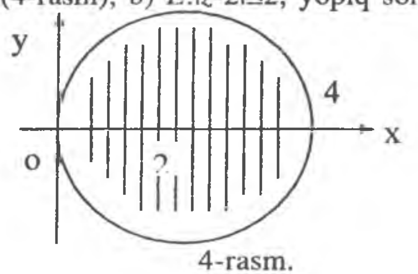
2-misol. a) E:$|z|<1$, $x^2+y^2<1$ - aylana ichki nuqtalari to'plami.

b) E:$|z|=1$, $x^2+y^2=1$ - aylana nuqtalari to'plami.

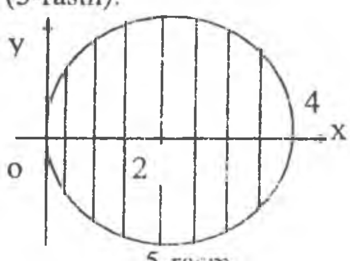
Agar quyidagi ikki shart bajarilsa:
1. E-to'plam faqat ichki nuqtalardan iborat bo'lsa, 2. E-to'plamning har qanday ikki nuqtasini birlashtiruvchi uzluksiz chiziqning barcha nuqtalari E ga tegishli bo'lsa, tekislikdagi nuqtalar to'plami (E) - soha deyiladi.

Agar soha chegarasidagi har qanday nuqta atrofida shu sohaning hech bo'lmaganda bitta nuqtasi mavjud bo'lsa, shu nuqta chegaraviy nuqta deyiladi. Chegaraviy nuqtalari o'ziga tegishli bo'lmagan E soha ochiq soha, chegaraviy nuqtalari o'ziga tegishli bo'lgan soha yopiq soha deyiladi.

3-misol. a) E:$|z-2|<2$, $|x+iy-2|<2$, $(x-2)^2+y^2<4$ - ochiq soha (4-rasm), b) E:$|z-2|\leq 2$, yopiq soha (5-rasm).



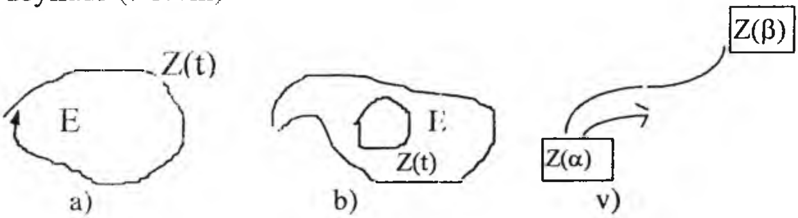
4-rasm.



5-rasm.

3. Jordan chizig'i

Haqiqiy t argumentli $x=x(t)$, $y=y(t)$ $(\alpha<t<\beta)$ uzluksiz funksiyalar berilgan bo'lsin. Ular tekislikdagi biror uzluksiz egri chiziqning parametrik tenglamasidan iborat bo'ladi. Agar (bu egri chiziqdagi) t-ning ikkita har xil qiymatiga har xil nuqtalar mos kelsa, ya'ni karrali nuqtalarga ega bo'lmasa bu chiziq Jordan chizig'i deyiladi yoki uzluksiz silliq chiziq deyiladi (v-rasm).

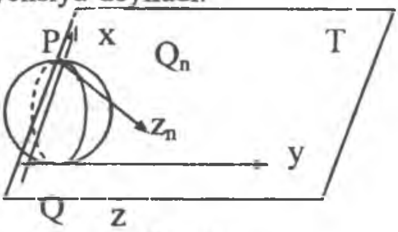


6-rasm.

Agar $z=x+iy$ ga $x=x(t)$, $y=y(t)$ ni qo'ysak $z=x(t)+iy(t)=z(t)$ $(\alpha\leq t\leq\beta)$ egri chiziq tenglamasi hosil bo'ladi. Bunda parametr t α dan β gacha o'zgarganda z nuqta Jordan chizig'ini chizadi. Agar $z(\alpha)=z(\beta)$ bo'lsa, chiziq yopiq chiziq deyiladi. Bitta yopiq Jordan chizig'i bilan chegaralangan soha bir bog'lamli (6 a-rasm), aks holda ko'p bog'lamli soha deyiladi (6 b-rasm)

4. Stereografik proyeksiya

Berilgan $z=x+iy$ kompleks sonni tekislikda nuqtaga mos keltirish mumkinligini ko'rgan edik. Endi har qanday kompleks sonni sferadagi nuqta bilan tasvirlash ham mumkinligini ko'rsatamiz. Buning uchun sferaning janubiy qutbini xoy tekislikning 0 markazi bilan ustma-ust qo'yamiz. Mana shu tekislikdagi $z=x+iy$ nuqtani P shimoliy qutb bilan to'g'ri chiziq orqali tutashtirsak, u chiziq sferani biror Q nuqtada kesib o'tadi. Hosil qilingan Q nuqta tekislikdagi z nuqtaning sferadagi aksi deyiladi. Shu usulda xoy tekislikning barcha z_n nuqtalarining ham sferadagi aksini topish mumkin, faqat P nuqtaning o'ziga tekislikdagi cheksiz uzoqlashgan $z=\infty$ nuqta mos keladi deb qabul qilinadi. xOy tekislikning va sferaning nuqtalarini yuqoridagidek bir qiymatli moslash stereografik proyeksiya deyiladi.



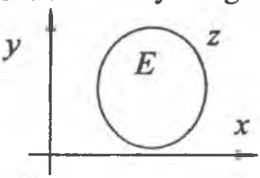
7-rasm.

3. Kompleks o'zgaruvchining funksiyalari va ularning aniqlanish sohasi

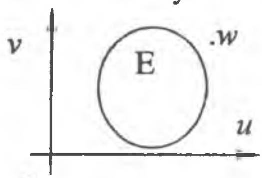
Biror (Z) kompleks tekisligida E kompleks $z = x + iy$ sonlar to'plami berilgan bo'lsin.

1-ta'rif. Agar E to'plamdan olingan har bir $z = x + iy$ songa biror qonun bo'yicha G dan olingan aniq bir $w = u + iv$ kompleks son mos kelsa, E to'plamda $w = f(z)$ funksiya berilgan deyiladi.

Bunda $z = x + iy$ argument, $w = u + iv$ esa funksiyadir. E to'plam $f(z)$ funksiyaning aniqlanish sohasi deyiladi.



8-rasm.



9-rasm.

2-ta'rif. Agar $z = x + iy$ ning har bir qiymatiga w ning birgina qiymati mos kelsa, $w = f(z)$ bir qiymatli, aks holda ko'p qiymatli funksiya deyiladi.

Masalan,

$w=z^2$, $w=\frac{1}{z}$, $w=2z^3$,... - bir qiymatli,

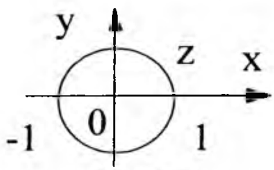
$w=\sqrt{z}$, $w=\sqrt[4]{z}$, $w=\frac{1}{\sqrt[3]{z-1}}$,... - ko'p qiymatli funksiyalardir.

Agar z ning qiymatlariga tegishli nuqtalarni *(Z)* tekisligida, w ning qiymatlariga tegishli nuqtalarni *(W)* tekisligiga joylashtirsak, *(Z)* tekisligidagi E to'plamdan olingan har bir z nuqta *(W)* tekisligidagi w nuqtaga mos keladi. Natijada E to'plamning aksi *(W)* tekislikka tushib, biror G to'plamni hosil qiladi. Bunga esa, $w=f(z)$ funksiya yordamida E to'plamni G to'plamga akslantirish deyiladi.

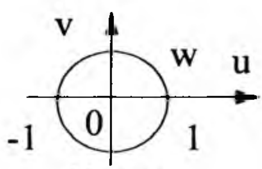
1-misol. $w=z^2$ funksiya yordami bilan *(Z)* tekislikdagi $|z|=1$ chiziqning *(W)* tekislikdagi aksi topilsin.

Yechish. $w=u+iv$, $w=z^2=(x+iy)^2=x^2-y^2+2xyi$

$u=x^2-y^2$, $v=2xy$, $u^2+v^2=(x^2-y^2)^2+(2xy)^2=(x^2+y^2)^2=|z|^4=1$



10-rasm.



11-rasm.

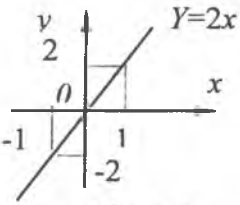
2-misol. $w=z^2$ funksiya yordami bilan *(Z)* tekisligidagi $y=kx$ to'g'ri chiziqning *(W)* tekislikdagi aksi topilsin.

Yechish: $y=kx$, $w=z^2$

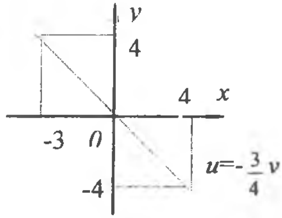
$u=x^2-y^2$, $v=2xy$, $y=kx$

$$\begin{cases} u=x^2-k^2x^2=x^2(1-k^2) \\ v=2x\cdot kx=2kx^2 \end{cases} \Rightarrow \quad \frac{u}{v}=\frac{1-k^2}{2k}; \quad u=\frac{1-k^2}{2k}\cdot v$$

Agar k=2 bo'lsa, u holda 12- va 13-rasmdagi to'g'ri chiziqlarga ega bo'lamiz.



12-rasm.



13-rasm.

4. Funksiyaning limiti va uzluksizligi

Biror E - kompleks sohada $w=f(z)$ funksiya berilgan bo'lib, $z_0 \in E$ nuqta berilgan bo'lsin.

1-ta'rif. Agar oldindan berilgan har qanday kichik $\varepsilon > 0$ son uchun shunday musbat $\delta(\varepsilon)>0$ sonni topish mumkin bo'lsaki, $|z-z_0|<\delta(\varepsilon)$ bo'lganda $|f(z)-A|<\varepsilon$ o'rinli bo'lsa, $f(z)$ funksiya A o'zgarmas songa intiladi deyiladi va quyidagicha yoziladi

$$\lim_{z \to z_0} f(z) = A \tag{1}$$

2-ta'rif. Agar oldindan berilgan har qanday kichik musbat $\varepsilon > 0$ son uchun shunday musbat $\delta > 0$ sonni topish mumkin bo'lsaki, bunda $|z-z_0|<\delta$ o'rinli bo'lganda, $|f(z)-f(z_0)|<\varepsilon$ tengsizlik o'rinli bo'lsa, $f(z)$ funksiya z_0 nuqtada uzluksiz deyiladi va quyidagicha yoziladi

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{2}$$

Bu geometrik jihatdan $w=f(z)$ funksiya z_0 nuqtada uzluksiz bo'lsa, (Z) tekisligidagi markazi z_0 nuqtada, radiusi $\delta(\varepsilon)$-ga teng bo'lgan doira nuqtalari, (w) tekislikdagi markazi w_0 nuqtada, radiusi ε bo'lgan doira nuqtalariga o'tishini ko'rsatadi.

3-ta'rif. Sohaning har bir nuqtasida uzluksiz bo'lgan funksiyalar shu sohada uzluksiz deyiladi.

Kompleks o'zgaruvchili funksiyaning limiti va uzluksizligi ta'riflari haqiqiy o'zgaruvchining limiti va uzluksizligi ta'rifiga o'xshash bo'lgani uchun uzluksiz funksiyaning xossalari, ular bilan bajariladigan amallar, ular haqidagi teoremalar va ularning isboti ham haqiqiy o'zgaruvchili funksiyalar isbotlari kabi bo'ladi.

Uzluksizlikni quyidagicha ham ta'riflash mumkin: $w(z)=f(z)$, $w_0=f(z)$, $z=x+iy$, $z_0=x_0+iy_0$, $\Delta x=x-x_0$, $\Delta y=y-y_0$, bo'lsa, $\Delta z=z-z_0=\Delta x+i\Delta y$ va $\Delta w=f(z)-f(z_0)$ funksiya orttirmasi bo'ladi.

4-ta'rif. Agar haqiqiy kichik musbat $\varepsilon>0$ uchun shunday $\delta(\varepsilon)>0$ son topish mumkin bo'lsa, $|\Delta z|<\delta(\varepsilon)$ bo'lganda $|\Delta W|<\varepsilon$ o'rinli bo'lsa, $f(z)$ funksiya z_0 nuqtada uzluksiz deyiladi va quyidagicha yoziladi

$$\lim_{\Delta z\to 0}\Delta W=0 \tag{3}$$

Agar $z=x_0+iy_0$ bo'lsa,

$$f(z)-f(z_0)=[u(x,y)-u(x_0,y_0)]+i[v(x,y)-v(x_0,y_0)]$$

bo'ladi va

$$|f(z)-f(z_0)|=\sqrt{[u(x,y)-u(x_0,y_0)]^2+[v(x,y)-v(x_0,y_0)]^2} \tag{4}$$

4-ta'rifdan quyidagi tengsizliklar kelib chiqadi

$$\left.\begin{aligned}&|[u(x,y)-u(x_0,y_0)]|\prec\varepsilon\\&|[v(x,y)-v(x_0,y_0)]|\prec\varepsilon\end{aligned}\right\} \tag{5}$$

Demak, $u(x,y)$ va $v(x,y)$ funksiyalar (x_0,y_0) nuqtada uzluksiz ekan.

1-misol. $w=z^2$ funksiya ixtiyoriy z_0 nuqtada uzluksizmi?

Yechish: $\Delta w=(z_0\Delta z)^2-z_0^2=2z_0\Delta z+(\Delta z)^2$

$$\lim_{\Delta z\to 0}\Delta W=\lim_{\Delta z\to 0}[2z_0\Delta z+(\Delta z)^2]=2z_0\lim_{\Delta z\to 0}\Delta z+\lim_{\Delta z\to 0}(\Delta t)^2=0$$

5. Asosiy elementar funksiyalar

1. Darajali funksiya: $w=z^n$

a) n - natural son bo'lsa, $n\in N$. $w=z^n=r^n e^{in\varphi}$;

b) $n=1/q$ - kasr son bo'lsa

$$w=\sqrt[q]{z}=\sqrt[q]{r}\left[\cos\frac{\varphi+2\pi k}{q}+i\sin\frac{\varphi+2\pi k}{q}\right],\quad k=0,\pm1,\pm2,...$$

q- ta ildizga ega.

2. Ko'rsatkichli funksiya:

Biz $a=e^z$ bo'lgan hol bilan ko'proq ish ko'ramiz, ya'ni $w=e^z=e^{x+iy}=e^x(\cos y+i\sin y)$ bundan:

1) $e^{z+2\pi i}=e^{x+iy+2\pi i}=e^x\cdot e^{i(y+2\pi)}=e^x[\cos(y+2\pi)+i\sin(y+2\pi)]=e^x\cdot e^{iy}=e^z$,

ya'ni $w=e^z$ funksiya $2\pi i$ sof mavhum davrli. Bu haqiqiy sonlar nazariyasidagi ko'rsatkichli funksiyadan farqli demakdir.

2) $e^{z_1+z_2}=e^{z_1}\cdot e^{z_2}$; 3) $e^{z_1-z_2}=e^{z_1}/e^{z_2}$; 4) $\left(e^z\right)^m=e^{zm}$ mos bo'ladi.

3. Logarifmik funksiya: $w=\ln z$

Logarifmik funksiya deb, ko'rsatkichli funksiyaga teskari bo'lgan $w=\ln z$ ushbu ko'rinishdagi funksiyaga aytiladi. agar $z=e^w$ bo'lsa, $w=\ln z$ bo'ladi.

$$w=\ln z=\ln(re^{i\varphi})=\ln r+i\varphi+2k\pi i \quad (k=0,\ \pm1,\ \pm2,\ ...\ )$$

bunda $\ln r+i\varphi$ - logarifmik funksiyaning bosh qismi deyiladi. Bulardan ko'rinadiki, kompleks o'zgaruvchining logarifmik funksiyasi ko'p qiymatli ekan. Kompleks o'zgaruvchining logarifmik funksiyasi ham haqiqiy o'zgaruvchining logarifmik funksiyasining ko'pgina xossalariga bo'ysunadi.

Masalan: 1) $\ln(z_1 z_2)=\ln z_1+\ln z_2$ 3) $\ln(z^n)=n\ln z+2k\pi i$

2) $\ln(z_1/z_2)=\ln z_1-\ln z_2$ 4) $\ln\sqrt[n]{Z}=\frac{1}{n}\ln Z$

2-misol. $3+4i$ ning logarifmini toping.

Yechish: $|z|=|3+4i|=\sqrt{9+16}=5$, $\arg z=\operatorname{arctg}\frac{4}{3}$

$$\ln(3+4i)=\ln 5+i\operatorname{arctg}\frac{4}{3}$$

$$\operatorname{Ln}(3+4i)=\ln 5+i\operatorname{arctg}\frac{4}{3}+2k\pi i \quad (k=0,\ \pm1,\ \pm2,\ \ldots)$$

4. Kompleks o'zgaruvchilarning trigonometrik funksiyalari

Ushbu $e^{iz}=\cos z+i\sin z$ va $e^{-iz}=\cos z-i\sin z$ Eyler formulalari berilgan bo'lsin. Bu formulalarni hadlab qo'shib va ayirib, quyidagi funksiyaning trigonometrik funksiyalarini aniqlaymiz.

$$w=\cos z=\frac{e^{iz}+e^{-iz}}{2},\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad tgz=\frac{\sin z}{\cos z}=i\frac{e^{-iz}-e^{iz}}{e^{iz}+e^{-iz}};$$

$ctgz=i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$; Kompleks o'zgaruvchilarning trigonometrik funksiyalari ham haqiqiy o'zgaruvchili funksiyalarning ko'pgina xossalariga bo'ysunadi. Bunda faqat kompleks son $\cos z$ va $\sin z$ funksiyalarining modullari birdan katta ham bo'lishi mumkin.

Masalan: $\sin i=\frac{e^{ii}-e^{-ii}}{2i}=\frac{e^{-1}-e}{2i}=i\frac{e^{2}-1}{2e}\approx 1{,}17i$

$$\cos i=\frac{e^{ii}+e^{-ii}}{2}=\frac{e^{-1}+e}{2}=\frac{e^{2}+1}{2e}\approx 1{,}54.$$

$|\sin i|>1$, $|\cos i|>1$

5. Teskari trigonometrik funksiyalar

Agar $z=\sin w$ trigonometrik funksiya berilgan bo'lsa, w - o'zgaruvchi unga teskari funksiya bo'lib, u z ning arksinusi deyiladi va bunday yoziladi w=Arcsinz. Xuddi shuningdek, $w=\operatorname{Arccos} z$, $w=\operatorname{Arctg} z$, $w=\operatorname{Arcctg} z$.

a) $z=\sin w=\dfrac{e^{iw}-e^{-iw}}{2i}=\dfrac{e^{2iw}-1}{2ie^{iw}}\Rightarrow e^{2(iw)}-2ize^{iw}-1=0$

$e^{iw}=y$ desak, unda $y^2-(2iz)y-1=0$

$$y_{1,2}=iz\pm\sqrt{(iz)^2+1}=iz\pm\sqrt{1-z^2},$$

$$e^{iw}=iz\pm\sqrt{1-z^2};\quad iwLne=Ln(iz\pm\sqrt{1-z^2})\Rightarrow$$

$$w=Arc\sin z=\frac{1}{i}Ln(iz\pm\sqrt{1-z^2})\Rightarrow \tag{6}$$

$$w=Arc\sin z=-iLn(iz\pm\sqrt{1-z^2})$$

Xuddi shuningdek,

$$w=Arc\cos z=-iLn(z\pm\sqrt{z^2-1}) \tag{7}$$

$$w=Arctgz=-\frac{i}{2}Ln\frac{1+iz}{1-iz} \tag{8}$$

$$w=Arcctgz=\frac{i}{2}Ln\frac{z-i}{z+i} \tag{9}$$

Teskari trigonometrik funksiyalar Ln - ga bog'liq bo'lganligi uchun ular ham ko'p qiymatli funksiyalardir.

3-misol. Arcsin2 ning barcha qiymatlarini hisoblang.

Yechish:

$$\text{Arcsin}2=-i\text{Ln}(2i\pm i\sqrt{3})=-i\text{Ln}[(2\pm\sqrt{3})i]=-i[\ln(2\pm\sqrt{3})+i\frac{\pi}{2}+2k\pi i]=$$

$$=\frac{\pi}{2}-i\ln(2\pm i\sqrt{3})+2k\pi \qquad (k=0,\ \pm1,\ \pm2,\ \dots)$$

4-misol. Arctg(1+2i) ning barcha qiymatlarini toping.

Yechish: $\text{Arctg}(1+2i)=\dfrac{1}{2i}\text{Ln}\dfrac{1+i(1+2i)}{1-i(1+2i)}=\dfrac{1}{2i}Ln\dfrac{i-1}{3-i}$

kasrning maxrajini komplekslikdan ozod qilib, uning moduli va argumentini topaylik:

$$\frac{i-1}{3-i}=-\frac{2}{5}+\frac{i}{5};\quad \left|\frac{i-1}{3-i}\right|=\left|-\frac{2}{5}+\frac{i}{5}\right|=\sqrt{\frac{4}{5}+\frac{1}{25}}=\frac{1}{\sqrt{5}}$$

$$\arg\left(-\frac{2}{5}+\frac{i}{5}\right)=\varphi tg=\frac{1}{5}\left(-\frac{2}{5}\right)=-\frac{1}{2};$$

$$\varphi=arctg\left(-\frac{1}{2}\right)=-arctg\frac{1}{2}=-arcctg2$$

$$Ln\left(-\frac{2}{5}+\frac{i}{5}\right)=\ln\frac{1}{\sqrt{5}}-iarcctg2+2k\pi$$

U holda Arctg (1+2i)= $k\pi-\frac{1}{2}arcctg2+\frac{i}{4}\ln 5$

6. Giperbolik funksiyalar

Kompleks o'zgaruvchilarning giperbolik funksiyalari ham haqiqiy o'zgaruvchilarning giperbolik funksiyalari kabi aniqlanadi.

$$shz=\frac{e^{z}-e^{-z}}{2};\quad chz=\frac{e^{z}+e^{-z}}{2};$$

$$thz=\frac{shz}{chz}=\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}};\quad cthz=\frac{e^{z}+e^{-z}}{e^{z}-e^{-z}};$$

Bunda *shz*, *chz* lar $2\pi i$ davrli, *shz*, *chz* lar πi davrli funksiyalar. Kompleks o'zgaruvchining giperbolik va trigonometrik funksiyalari orasida quyidagi bog'lanish mavjud.

shz=-*i*sin*iz*, *chz*=cos*iz*, *thz*=*i*tg*iz*, *cthz*=-*i*ctg*iz*.

Isboti:

$$-i\sin iz=-i\frac{e^{i(iz)}-e^{-i(iz)}}{2i}=-\frac{e^{-z}-e^{z}}{2}=\frac{e^{-z}-e^{z}}{2}=shz$$

5-misol. cos(1+*i*) ning qiymatini hisoblang.
Yechish:

$$\cos(1+i)=ch[-i(1+i)]=ch(1-i)=\frac{e^{1-i}+e^{-(1-i)}}{2}=\frac{1}{2}(e^{1-i}+e^{i-1})=$$

$$=\frac{1}{2}[e^1(\cos 1 - i\sin 1)+e^{-1}(\cos 1+i\sin 1)]=\cos 1\,\frac{e+e^{-1}}{2} - i\sin 1\,\frac{e-e^{-1}}{2};$$

6. Kompleks o'zgaruvchilar funksiyasining hosilasi

Agar E kompleks sohada $w=f(z)$ funksiya berilgan bo'lib va bu sohaning biror z_0 nuqtasidagi argument va funksiya orttirmalari quyidagicha bo'lsin: $\Delta z=z-z_0$, $\Delta w=f(z_0+\Delta z)-f(z_0)$.

1-ta'rif. Agar Δz har qanday yo'l bilan nolga intilganda $\frac{\Delta w}{\Delta z}$ nisbat faqat birgina aniq limitga intilsa, u limitning qiymati $f(z)$ funksiyaning z_0 nuqtadagi hosilasi deyiladi va w^1, $f^1(z_0)$, $\frac{dw}{dz}$ yoki $\frac{df}{dz}$ kabi belgilanadi, demak

$$f^1(z_0)=\lim\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \qquad (1)$$

yoki $w=f(z_0)=u(x, y)+iv(x, y)$; $\Delta w=\Delta u+i\Delta v$ bo'lgani uchun (1) ni quyidagicha yozish mumkin:

$$f^1(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y} \qquad (2)$$

chunki

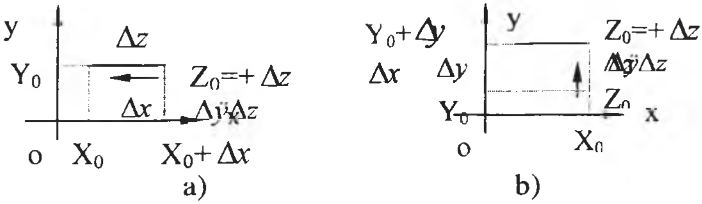
$$\Delta w=f(z+\Delta z)-f(z)=[u(x+\Delta x,\ y+\Delta y-u(x,\ y))]+$$
$$+i[v(x+\Delta x,\ y+\Delta y)-v(x,\ y)]=\Delta u+i\Delta v$$

bunda

$$\Delta u=u(x+\Delta x,\ y+\Delta y)-u(x,\ y)$$
$$\Delta v=v(x+\Delta x,\ y+\Delta y)-v(x,\ y)$$

2-ta'rif. Agar $w=f(z)$ funksiya z_0 nuqtada hosilaga ega bo'lsa, uni bu nuqtada differensiallanuvchi yoki monogen funksiya deyiladi.

1-ta'rifdan ko'rinadiki, agar $f(z)$ z_0 nuqtada hosilaga ega bo'lsa, (1) limitining qiymati Δz nolga qaysi yo'l bilan intilishiga bog'liq emas. Demak, biz $z_0+\Delta z$ nuqtani z_0 nuqtaga parallel holda ham intiltirishimiz mumkin. Bu holda $\Delta z=\Delta x$, $\Delta y=0$ bo'ladi (1 a)-rasm)



1-rasm.

$$f'(z_0)=\frac{\Delta u}{\Delta x}+i\frac{\Delta v}{\Delta x} \qquad (3)$$

Xuddi shuningdek, $z_0+\Delta z$ nuqta z_0 ga Oy ga parallel holda intiltirsak $\Delta x=0$, $\Delta z=i\Delta y$ bo'ladi va (2) dan ushbu kelib chiqadi (1 b-rasm)

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta v}{i\Delta y}=\lim_{\Delta y\to 0}\left(\frac{\Delta v}{\Delta y}-i\frac{\Delta u}{\Delta y}\right)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

$$f'(z_0)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \qquad (4)$$

(3) va (4) lardan ushbu tenglamalarni hosil qilish mumkin

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\Rightarrow\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\quad\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad (5)$$

(5)-ga Koshi-Riman shartlari deyiladi.

Teorema. $f(z)$ funksiya z_0 nuqtada differensiallanuvchi bo'lishi uchun $u(x, y)$, $v(x, y)$ funksiyalar z_0 da differensiallanuvchi va Koshi-Riman shartlarining bajarilishi uchun zarur va yetarlidir.

1-misol. $w=(x^2-y^2)+2xyi$ hosilaga egaligi yoki ega emasligini toping.

Yechish: $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$; $\frac{\partial v}{\partial x}=2y$, $\frac{\partial v}{\partial y}=2x$ Koshi-Riman shartlarini $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ tekshiramiz $2x=2x$; $-2y=-(2y)$. Demak, bu funksiya hosilaga ega.

$f'(z)=w'=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+i2y=2(x+iy)=2z$ yoki

$$f(z)=(x^2-y^2)+i2xy=(x+iy)^2=z^2$$
$$f'(z)=(z^2)'=2z$$

2-misol. $w=y+ix$ hosilaga ega ekanligini tekshiring.

Yechish: $u=y$, $v=x$, $\frac{\partial u}{\partial x}=0$, $\frac{\partial u}{\partial y}=1$, $\frac{\partial v}{\partial x}=1$, $\frac{\partial v}{\partial y}=0$.

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$ $\frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$ $(1\neq-1)$ bitta shart bajarilmagani uchun bu funksiya hosilaga ega emas.

Biz ko'rdikki, agar funksiyaning nuqtadagi hosilasini topish kerak bo'lsa, quyidagi 4 ta formulaning biridan foydalanish mumkin.

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$
$$f'(z)=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x},\quad f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}, \tag{6}$$

Lekin $f(z)$ funksiyaning haqiqiy va mavhum qismlari ajralmagan holda bo'lsa, bu formulalardan foydalanish noqulay bo'ladi. $f(z)$ ning hosilasiga matematik analizdagi haqiqiy o'zgaruvchili funksiyaning hosilasi qoidalarini qo'llash mumkin, ya'ni:

1. $c'=0$ 2) $z'=1$ 3) $[f_1(z)\pm f_2(z)]'=f'_1(z)\pm f'_2(z)$
4) $[cf(z)]'=cf'(z)$ 5) $[f^{(n)}(z)]'=nf^{n-1}(z)\cdot f'(z)$

3-ta'rif. Agar $f(z)$ funksiya E sohaning z_0 nuqtasida va uning atrofida ham differensiallanuvchi bo'lsa, u funksiya shu nuqtada analitik deyiladi.

4-ta'rif. Agar $f(z)$ funksiya E sohaning barcha nuqtalarida hosilaga ega bo'lsa, u funksiya E da analitik deyiladi.

5-ta'rif. $f(z)$ Funksiya analitik bo'lgan nuqtalar uning to'g'ri nuqtasi, analitik bo'lmagan nuqtalari esa maxsus nuqtalari deyiladi.

3-misol. $w=x^2+y^2+ixy^3$ funksiyaning analitik yoki analitik emasligi tekshirilsin.

Yechish: $u=x^2+y^2$, $v=xy^3$,

$$\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=2y,\quad \frac{\partial v}{\partial x}=y^3,\quad \frac{\partial v}{\partial y}=3xy^2$$

$$2x=3xy^2 \Rightarrow x(2-3y^2)=0;\ x=0,\ y=\sqrt{\frac{2}{3}}$$

$y^3=-2y \Rightarrow y(y^2+2)=0$; $y=0$, $y\in\varnothing \Rightarrow$ (0,0)- shu nuqtadagina hosila mavjud, boshqa nuqtada hosila yo'q, ya'ni funksiya analitik emas.

HOSILA ARGUMENTINING VA MODULINING GEOMETRIK MA'NOSI. KOMPLEKS O'ZGARUVCHINING FUNKSIYASIDAN OLINGAN INTEGRAL

1. Hosila argumentining va modulining geometrik ma'nosi

Biror E sohada $w=f(z)$ analitik funksiya berilgan bo'lsin. Bu funksiya E dan olingan biror aniq z_0 nuqtada $f'(z_0)\neq 0$ hosilaga ega bo'lsin. Bu funksiya yordami bilan (z) dagi z_0 nuqtani G dagi w_0 nuqtaga akslantirsak, z_0 dan o'tuvchi ixtiyoriy C_1, C_2 chiziqlar w_0 dan o'tuvchi Γ_1 va Γ_2 chiziqlarga akslanadi. Biz $f'(z_0)\neq 0$ hosilaning argumenti va modulining geometrik ma'nosini ko'raylik. Buning uchun $f'(z_0)$ kompleks sonni trigonometrik formaga keltiraylik.

$$f'(z_0)=R(\cos\phi+i\sin\phi),\ R=|f'(z_0)|\neq 0,\ \phi=\arg[f'(z_0)]$$

a) rasmdan va o'tilgan hosila ta'rifidan foydalansak:

$$F=\mathrm{Arg}(f'(z_0))=\arg\left(\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\right)=\lim_{\Delta z\to 0}\arg\frac{\Delta w}{\Delta z}=\lim_{\Delta w\to 0}\arg\Delta w-$$

$$-\lim_{\Delta z\to 0}\arg\Delta z=\lim_{\Delta w\to 0}\psi-\lim_{\Delta z\to 0}\tau=\psi_1-\tau_1;\quad F=\psi_1-\tau_1 \quad (1)$$

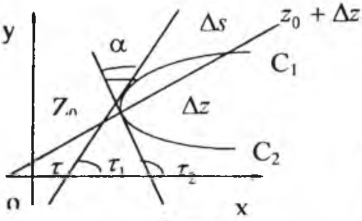
ya'ni C_1 ning aksi G_1 bo'lib, u ϕ burchakka burilar ekan. Xuddi shuningdek, C_2 ning aksi Γ_2 hosil bo'lib, u ham ϕ burchakka burilishini ko'rsatish mumkin, ya'ni

$$\phi=\psi_2-\tau_2 \quad (2)$$

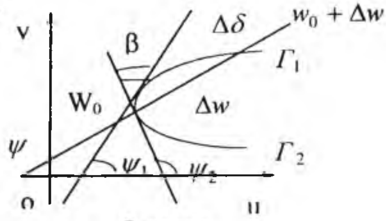
(1) va (2) larni tenglab ushbuni $\psi_1-\tau_1=\psi_2-\tau_2$ yoki $\psi_1-\psi_2=\tau_2-\tau_1$ bundan

$$\alpha=\beta \quad (3)$$

ekanligi kelib chiqadi.



2-rasm.



3-rasm.

Shunday qilib, $w=f(z)$ analitik funksiya yordami bilan E sohani G sohaga akslantirsak, z_0 nuqtadan o'tuvchi C_1, C_2 ... chiziqlarning hammasi G da bir xil $F=\arg[f'(z_0)]$ burchakka burilar ekan. C_1, C_2 chiziqlar orasida burchaklar o'zgarmay akslanadi, ya'ni $\alpha=\beta$ bo'ladi.

1-xossa. Analitik funksiya yordami bilan bajariladigan akslantirish hosila nolga teng bo'lmagan barcha nuqtalarda burchaklarni saqlash xossasiga ega

b) $R=|f'(z_0)|=\left|\lim\limits_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\right|=\lim\limits_{\Delta z\to 0}\left|\frac{\Delta w}{\Delta z}\right|=\lim\limits_{\Delta z\to 0}\frac{\Delta\delta}{\Delta S}$, yoki

$$R=\lim_{\Delta z\to 0}\frac{\Delta\delta}{\Delta S} \tag{4}$$

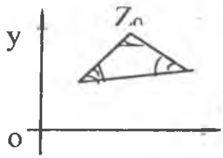
bu z_0 nuqtadan o'tuvchi har qanday C-chiziqning cho'zilish koeffitsienti deyiladi. Boshqacha aytganda $w=f(z)$ analitik funksiya yordamida bajariladigan akslantirish jarayonida istalgan kichik yoy z_0 nuqtaning kichik atrofida $R=|f'(z_0)|$ marta o'zgaradi. z_0 dan o'tuvchi istalgan barcha chiziqlar uchun cho'zilish koeffitsienti bir xil bo'ladi, chunki $z_0+\Delta z$ ning $z_0$ga intilishi ixtiyoriydir.

2-xossa. Analitik funksiya yordami bilan bajariladigan akslantirish $f'(z)$ hosila nolga teng bo'lmagan barcha nuqtalar o'zgarmas $R=|f'(z_0)|$ cho'zilishga ega.

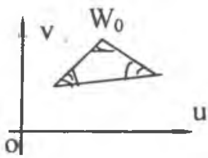
2. I va II tur konform akslantirishlar

1-ta'rif. Agar $w=f(z)$ analitik funksiya yordamidagi akslantirish natijasida z_0 nuqtadagi cho'zilish koeffitsienti va burchakning kattaligi ham o'zgarmasa u I tur konform akslantirish deyiladi.

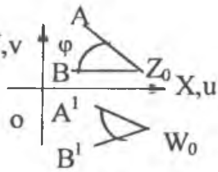
Misol uchun, bir uchi z_0 nuqtadan iborat istalgancha kichik uchburchakni $w=f(z)$ analitik funksiya yordamida akslantirsak, bir uchi w_0 nuqtadan iborat egri chiziqli uchburchakka ega bo'lamiz. Lekin ikkala uchburchakdagi mos burchaklar o'zaro teng bo'ladi (4-rasm).



4-rasm.



5- rasm.



6-rasm.

Endi *xOy* va *uOv* koordinata tekisliklarini birga joylashtiraylik, $w=f(z)=z=x-iy$ funksiya vositasi bilan $z=x-iy$ nuqtani akslantirsak, o'sha tekislikdagi *Ox* o'qqa simmetrik bo'lgan nuqta paydo bo'ladi. Masalan, z_0 nuqtadan chiqqan ikki to'g'ri chiziq va ular orasidagi φ burchakni akslantirsak, *Ox* ga nisbatan simmetrik bo'lgan ikki to'g'ri chiziqqa almashadi. φ burchakning aksi $(-\psi)$ ya'ni qarama-qarshi tomonga yo'nalgan bo'lib tushadi (5-rasm).

2-ta'rif. Agar $w=f(z)$ analitik funksiya yordamidagi akslantirish natijasida z_0 nuqtadagi cho'zilishi o'zgarmasa va burchakning ham kattaligi o'zgarmay, faqat yo'nalishi qarama-qarshisiga o'zgarsa, u II tur konform akslantirish deyiladi.

Shunday qilib, $w=f(z)$ analitik funksiyaga qo'shma bo'lgan $\overline{w}=\bar{f}(z)$ funksiya yordamida amalga oshiriladigan akslantirish II tur konform akslantirish bo'lar ekan. Biz ko'pincha I tur konform akslantirish bilan ish ko'ramiz. 1-misol. $w=z^2$ Funksiya yordamida bajariladigan akslantirishda ushbu nuqtalardagi ϕ burilish burchagi va *R*-cho'zilish koeffitsienti topilsin.

a) $z_0=\frac{1}{2}$ b) $z_0=1+i$

Yechish $w'=2z$; $f'(\frac{1}{2})=2\cdot\frac{1}{2}=1\neq 0$,

a) $1=\cos 0+i\cdot\sin 0$, $R=|f'(z_0)|=1$ ya'ni cho'zilishi ro'y bermaydi.

$\phi=\arg Z'\ (1/2)=0$, ya'ni burilmaydi.

b) $f'(1+i)=2(1+i)=2+2i=2\sqrt{2}\,(\cos\frac{\pi}{4}+i\cdot\sin\frac{\pi}{4}$, $R=|f'(1+i)|=2\sqrt{2}$ demak $2\sqrt{2}$ ga cho'ziladi.

$\phi=\arg Z'\ (1+i)=\frac{\pi}{4}$, demak chiziq $\frac{\pi}{4}$ ga buriladi.

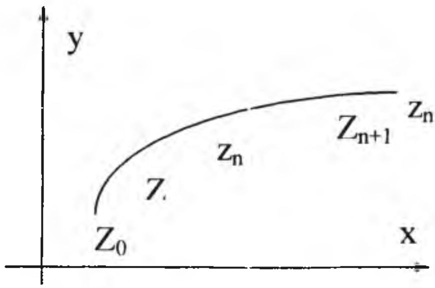
1. Integralning ta'rifi

Kompleks (z) tekisligidagi biror E sohada uzluksiz bir qiymatli

$$w=f(z)=u(x,\ y)+i\cdot v(x,\ y) \tag{1}$$

funksiya berilgan bo'lsin, u holda $f(z)$ funksiya E soha ichidan olingan ixtiyoriy Γ silliq chiziqda ham bir qiymatli va uzluksiz bo'ladi.

Bu chiziqning tenglamasi $z=f(t)$ $(\alpha\leq t\leq\beta)$ bo'lib, uning boshlang'ich nuqtasi z_0 va oxirgi nuqtasi z_n ya'ni $z_0=f(\alpha)$, $z_n=f(\beta)$ bo'lsin. Γ chiziqning yo'nalishini ikki xil aniqlash mumkin. Odatda t-parametrning oshib borishga mos yo'nalishini musbat Γ^+ yo'nalish, bunga teskari yo'nalishni Γ^- manfiy yo'nalish deb qabul qilinadi.



1-rasm.

Γ-chiziqni z_1, z_2, ... z_{n-1} nuqtalar orqali n-ta yoychalarga ajrataylik va har bir yoychada bittadan ixtiyoriy ξ_k nuqta olaylik va bu nuqtalardagi funksiyaning $f(\xi_k)$ qiymatlarini topaylik. Quyidagicha ko'paytmalarning yig'indisini tuzaylik:

$$S_n=f(\xi_1)\cdot\Delta x_1+f(\xi_2)\cdot\Delta x_2+\ldots+f(\xi_n)\cdot\Delta x_n=\sum_{k=1}^{n} f(\xi_K)\cdot\Delta x_K \qquad (2)$$

Bunda $\Delta z_k=z_k - z_{k-1}$ (k=1, ... n) va (2) ga integral yig'indi deyiladi. $\max|\Delta z_k|=\lambda$ deb belgilaylik.

Ta'rif. Agar λ-nolga intilganda (2) integral yig'indi aniq limitga ega bo'lsa va bu limitning qiymati Γ-ni qaysi usulda Δz_k-larga bo'lish usuliga va bu bo'lakchalarda ξ_k nuqtalarni tanlash usuliga bog'liq bo'lmasa, bu limitning qiymati $f(z)$ funksiyadan Γ chiziq bo'yicha olingan kontur integral deyiladi va quyidagicha yoziladi:

$$\int_{\Gamma} f(z)\cdot dz=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_K)\Delta z_k \qquad (3)$$

Γ-chiziq integrallash yo'li yoki konturi deyiladi. Agar $f(z)=u(x, y)+i\cdot v(x, y)$ ni e'tiborga olsak, bundagi $u(x, y)$ va $v(x, y)$ funksiyalar ham Γ-da uzluksiz bo'ladi va quyidagini yozish mumkin:

$$\int_{\Gamma} f(z)\cdot dz=\int_{\Gamma}(u+i\cdot v)\cdot(dx+i\cdot dy)=$$

$$=\int_{\Gamma}(u\cdot dx-v\cdot dy)+i\cdot(v\cdot dx+u\cdot dy) \qquad (4)$$

Bundan ko'rinadiki, kompleks o'zgaruvchilar funksiyasining integrali 2 ta haqiqiy o'zgaruvchilar funksiyasining egri chiziqli integrali ko'rinishiga keladi.

2. Integralning asosiy xossalari

1-xossa. O'zgarmas ko'paytuvchini integral belgisi tashqarisiga chiqarish mumkin:

$$\int_{\Gamma} a \cdot f(z) \cdot dz = a \cdot \int_{\Gamma} f(z) \cdot dz$$

Isboti. Haqiqatan ham agar a-o'zgarmas son bo'lsa, ushbu tenglik o'rinli:

$$\sum_{k=1}^{n} a \cdot f(\xi_K) \cdot \Delta z_K = a \cdot \sum_{k=1}^{n} f \cdot (\xi_K) \cdot \Delta z_K$$

bundan $\lambda \to 0$ da hadlab limitga o'tsak, 1-xossa isbot bo'ladi.

2-xossa. Chekli sondagi funksiyalar yig'indisidan olingan integral har bir qo'shiluvchi funksiyalardan olingan integrallar yig'indisiga teng, ya'ni:

$$\int_{\Gamma} [f_1(z) + f_2(z) + \ldots + f_n(z)]dt =$$

$$= \int_{\Gamma} f_1(z)dt + \int_{\Gamma} f_2(z)dt + \ldots + \int_{\Gamma} f_n(z)]dt$$

isboti 1- xossadagidek isbotlanadi.

3-xossa. Integrallar konturining yo'nalishi qarama-qarshisiga o'zgartirilsa, integral belgisi oldidagi ishora ham o'zgaradi, ya'ni:

$$\int_{\Gamma^-} f(z) \cdot dz = - \int_{\Gamma^+} f(z) \cdot dt$$

Isboti. Haqiqatan ham, integral yig'indi Γ ning musbat yo'nalishida olinsa $\Delta z_k = z_k - z_{k-1}$ ga teng, agar (manfiy) qarama-qarshi yo'nalishda olinsa $\Delta z_k = z_{k-1} - z_k = -(z_k - z_{k-1})$ ga teng bo'ladi. Shuning uchun $\sum_{k=1}^{n} f(\xi_K) \cdot (z_K - z_{K-1})$ va $\sum_{k=1}^{n} f(z_K) \cdot (z_{K-1} - z_K)$ yig'indilar faqat ishoralari bilan farq qiladi, demak limitlari ham faqat ishorasi bilan farq qiladi.

4-xossa. Agar Γ chiziqning uzunligi l bo'lib, uning barcha nuqtalarida $M>0$ son uchun $f(z)<M$ o'rinli bo'lsa, ushbu tengsizlik ham o'rinlidir:

$$\left|\int_{\Gamma} f(z)\cdot dz\right| \le M\cdot l$$ (Isbotsiz).

5-xossa. Agar $\Gamma = \Gamma_1 + \Gamma_2 + ... + \Gamma_n$ bo'lsa, ushbu tenglik o'rinlidir:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f_1(z)dz + \int_{\Gamma_2} f_2(z)dt + ... + \int_{\Gamma_n} f_n(z)]dz$$ (Isbotsiz).

3. Integralni hisoblash

$f(z)$ kompleks funksiyadan Γ-chiziq bo'ylab olingan integral (4) formulaga ko'ra haqiqiy o'zgaruvchidan olingan egri chiziqli integralni hisoblash uchun Γ-chiziqning tenglamasi parametrik holda berilgan bo'lishi kerak.

Γ-chiziqning parametrik tenglamalari $x=x(t)$, $y=y(t)$ $(\alpha \le t \le \beta)$ bo'lsin. Bu parametrik tenglamalarni kompleks shaklda yozsak, ya'ni:

$$z=z(t)=x(t)+i\cdot y(t),\ dt=[x'(t)+i\cdot y'(t)]dt,\ z(\alpha)=z_0,\ z(\beta)=z_n=z$$

ekanligi kelib chiqadi va $z(t)$ funksiya ham $[\alpha, \beta]$ segmentda uzluksiz bo'ladi. Bularni e'tiborga olsak, (5) ni quyidagicha yozish mumkin:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} u\cdot dx - v\cdot dy + i\cdot \int_{\Gamma} u\cdot dx + u\cdot dy = \int_{\alpha}^{\beta}[u\cdot x'(t) - v\cdot y'(t)]dt +$$

$$+ i\cdot \int_{\alpha}^{\beta}[v\cdot x'(t) + u\cdot y'(t)]dt$$

yoki

$$\int_{\Gamma} f(z)dz = \int_{\alpha}^{\beta}(u+i\cdot v)\cdot[x'(t)+i\cdot y'(t)]dt \qquad (6)$$

yoki

$$\int_{\Gamma} f(z)dz = \int_{\alpha}^{\beta} f\cdot[z(t)]z'(t)dt \tag{7}$$

Shunday qilib, $f(z)$ kompleks o'zgaruvchining funksiyasida Γ kontur bo'yicha olingan integralni hisoblash masalasi aniq integralni hisoblashga keltirildi.

Misol. $\int_{\Gamma} \mathrm{Re}\, z\cdot dz$ integral $|z|=1$, $0\leq \arg\cdot z\leq \pi$ yarim aylana bo'yicha hisoblansin.

Yechish: Γ-aylananing parametrik tenglamasi quyidagicha bo'ladi.

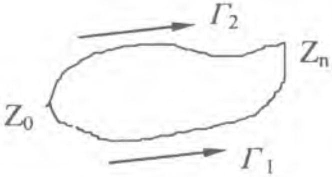
$x=\cos t$, $y=\sin t$ $(0\leq t\leq \pi)$; $\mathrm{Re}\, z$ bo'ladi

$$\int_{\Gamma} \mathrm{Re}\, z\cdot dz = \int_{0}^{\pi} x(x' + i\cdot y')dt = \int_{0}^{\pi}[\cos t(-\sin t + i\cdot\cos t)]dt =$$

$$= \int_{0}^{\pi}[-\frac{1}{2}\sin 2t + \frac{1}{2}\cdot i\cdot(1+\cos 2t)]dt = \left[\frac{1}{4}\cos 2t + \frac{i}{2}\left(t + \frac{1}{2}\sin 2t\right)\right]_{0}^{\pi} = \frac{\pi\cdot i}{2}$$

4. Yopiq kontur bo'yicha olingan integral. Koshi teoremalari

1. Ko'pgina hollarda $\int_{\Gamma} f(z)dz$ integralning qiymati ikki narsaga, ya'ni berilgan $f(z)$ funksiyaga va Γ-chiziqning formasiga bog'liq. Agar z_0, z_n nuqtalarni tutashtiruvchi ikki xil Γ_1 va Γ_2 chiziqlarni olsak, integralning qiymati ham umuman ikki turli bo'lishi, ba'zan esa teng bo'lib qolishi mumkin.



2-rasm.

Masalan, ushbu tenglik o'rinli bo'lsin:

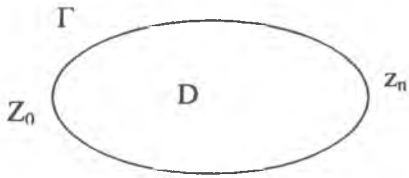
$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz, \qquad (8)$$

ya'ni integralning qiymati integrallash yo'liga bog'liq bo'lmasin, u holda (8) dan quyidagini yozish mumkin:

$$\int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0$$

yoki

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = \int_{\Gamma_1+\Gamma_2} f(z)dz = \oint_{\Gamma} f(z)dz = 0$$



3-rasm.

Demak, integralning qiymati integrallash yo'liga bog'liq bo'lmasligi uchun (uning z_0, z_n nuqtalarning o'rniga bog'liq bo'lishi uchun) shu nuqtalarni tutashtiruvchi yopiq kontur bo'yicha olingan integralning qiymati nolga teng bo'lishi kerak.

Qaysi shartlar bajarilganda integralning qiymati nolga teng yoki integrallash yo'liga bog'liq bo'lmasligiga quyidagi Koshi teoremasi javob beradi.

2. Bir bog'lamli soha uchun Koshi teoremasi

Teorema. Agar bir bog'lamli E sohada $f(z)$ funksiya analitik bo'lsa, Γ da yotuvchi har qanday Γ yopiq kontur bo'ylab $f(z)$ funksiyadan olingan integral nolga teng bo'ladi:

$$\oint_{\Gamma} f(z)dz = 0 \tag{9}$$

Isboti. $f(z)$ Funksiyaning hosilasi $f'(z)$ ham E da uzluksiz bo'lsin.

$$f'(z) = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \cdot \frac{\partial u}{\partial y}$$

bo'lgani uchun $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}$ lar ham D da uzluksiz bo'ladi, demak, $u(x, y)$, $v(x, y)$ larning ham uzluksizligi kelib chiqadi. U holda Grin formulasiga ko'ra quyidagini yozish mumkin:

$$\oint_{\Gamma} f(z)dz = \oint_{\Gamma}(u \cdot dx - v \cdot dy) + i \cdot \oint_{\Gamma}(v \cdot dx + u \cdot dy) =$$

$$= \oiint_{D}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \cdot dx \cdot dy$$

Koshi-Riman shartlariga asosan:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ bo'lgani uchun oxirgi tenglik nolga teng bo'ladi, ya'ni $\oint_{\Gamma} f(z)dz = 0$ Bu teorema birinchi marta mashhur fransuz matematigi Eduard Gursa (1858-1936) tomonidan isbotlangan.

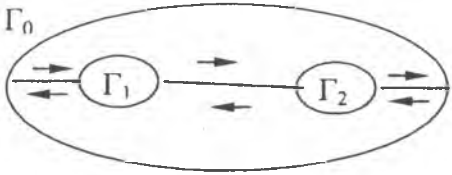
3. Ko'p bog'lamli soha uchun Koshi teoremalari

1-teorema. Agar ko'p bog'lamli yopiq $\overline{E}$ sohada $f(z)$ funksiya analitik bo'lsa, shu sohaning butun konturi bo'ylab musbat yo'nalishda $f(z)$ funksiyadan olingan integral nolga teng bo'ladi. Bu teorema quyidagicha ham (4-rasmga ko'ra) yoziladi:

$$\oint_{\Gamma_0} - \oint_{\Gamma_1} - \oint_{\Gamma_2} = 0$$

yoki

$$\oint_{\Gamma_0} f(z)dz = \oint_{\Gamma_1} f(z)dz + \oint_{\Gamma_2} f(z)dz \qquad (10)$$



4-rasm.

Teoremani isbotsiz qabul qilamiz.

2-teorema. Agar ko'p bog'lamli yopiq sohada $f(z)$ funksiya analitik bo'lsa, bu funksiyadan tashqi kontur bo'yicha olingan integral ichki konturlar bo'yicha olingan integrallar yig'indisiga teng.

Bu teoremaning isboti 1-teoremadan kelib chiqadi. Masalan, shaklga asosan quyidagi tenglikni yozish mumkin:

$$\oint_{\Gamma_0} f(z)dz = \oint_{\Gamma_1} f(z)dz + \oint_{\Gamma_2} f(z)dz \qquad (11)$$

Boshlang'ich funksiya va aniqmas integral

Ta'rif. Agar E sohaning barcha nuqtalarida $F'(z)=f(z)$ tenglik bajarilsa, u holda $F(z)$ funksiya berilgan $f(z)$ funksiyaning boshlang'ich funksiyasi deyiladi.

Haqiqiy o'zgaruvchilar sohasidagi kabi kompleks o'zgaruvchili $f(z)$ funksiyaning boshlang'ich funksiyasi uchun ham quyidagi teorema o'rinli.

Teorema. Agar E sohada $F(z)$ funksiya $f(z)$ ning boshlang'ich funksiyasi bo lsa, u holda $\Phi(z)=F(z)+C$ (bunda S-ixtiyoriy o'zgarmas) ham E da o'sha $f(z)$ funksiya uchun boshlang'ich funksiya bo'ladi va aksincha, agar $F(z)$ va $\Phi(z)$ lar $f(z)$ ning boshlang'ich funksiyalari bo'lsa, u holda barcha $z\in E$ lar uchun

$$\Phi(z)-F(z)\equiv C \quad (1)$$

bo'ladi.

Berilgan $f(z)$ funksiyaning hamma boshlang'ich funksiyalari aniqmas integral deyilib, ushbu

$$\int f(z)dz$$

simvol bilan belgilanadi. Demak (1) ga muvofiq

$$\int f(z)dz = F(z)+C = \Phi(z) \quad (2)$$

bunda

$$F^{|}(z) = f(z)$$

Endi funksiya bir bog'lamli E sohada analitik bo'lsin. E da z_0 va z nuqtalarni birlashtiruvchi Γ kontur bo'ylab integrallash talab qilinsa, unga to'g'ridan-to'g'ri haqiqiy sonlar nazariyasidagidek Nyuton-Leybnis formulasini qo'llash mumkin:

$$\int_{\Gamma} f(z)dz = F(z) - F(z_0). \quad (3)$$

1-misol. $\int z^n dz = \dfrac{z^{n+1}}{n+1} + C$.

Ratsional funksiyalarni integrallash

Biz ilgari

$$\oint_{\Gamma}(z-a)^n dz = 0$$

bo'lishini ko'rgan edik. Xususiy holda a=0 bo'lsa,

$$\oint_{\Gamma} z^n dz = 0.$$

Shularga asosan butun ratsional funksiyadan yopiq Γ kontur bo'ylab integral olishimiz mumkin.

a) $P(z)=a_0z^n+a_1z^{n-1}+...+a_{n-1}z+a_n$,

bunda a_0, a_1, ..., a_n koeffitsientlar o'zgarmas kompleks sonlardan iborat bo'lib, $z=x+iy$ - kompleks o'zgaruvchidir

$$\oint_{\Gamma} P(z)dz = a_0\oint_{\Gamma} z^n dz + a_1\oint_{\Gamma} z^{n-1}dz + ... + a_n\oint_{\Gamma} dz = 0.$$

Demak,

$$\oint_{\Gamma} P(z)dz = 0 \qquad (4)$$

b) Agar a nuqta Γ yopiq chiziq tashqarisida yotgan bo'lsa, $\oint_{\Gamma} \frac{dz}{(z-a)^n} = 0$ bo'lishini biz ko'rgan edik.

$z=a$ nuqta Γ ning ichida yotadi, deb faraz etaylik. Agar Γ ichida markazi $z=a$ dan iborat biror C aylana yasasak ikki bog'lamli soha hosil bo'lib, unda

$$f(z) = \frac{1}{(z-a)^n}$$

funksiya analitik bo'ladi. Shu sababli Koshi teoremasiga asosan tashqi va ichki konturlar bo'ylab olingan integrallar o'zaro teng bo'ladi, ya'ni

$$\oint_{\Gamma} \frac{dz}{(z-a)^n} = \oint_{C} \frac{dz}{(z-a)^n} \qquad (5)$$

$$\oint_{\Gamma} \frac{dz}{(z-a)^n} = \begin{cases} \text{agar} \quad n \neq 1 \quad \text{bo'lsa,} \quad 0 \\ \text{agar} \quad n = 1 \quad \text{bo'lsa,} \quad 2\pi i \end{cases} \qquad (6)$$

So'nggi formulaga asoslanib quyidagi oddiy kasr ratsional funksiyalardan integral olish qiyin emas.

$$R(z)=\frac{A_0}{(z-a)^n}+\frac{A_1}{(z-a)^{n-1}}+\ldots+\frac{A_{n-1}}{z-a}$$

Agar a nuqta Γ tashqarisida bo'lsa, har bir haddan olingan integral nolga teng bo'ladi. Agar a nuqta Γ ichida yotsa, u holda (6) ga asosan

$$\oint_\Gamma R(z)dz = A_0\oint_\Gamma \frac{dz}{(z-a)^n} + A_1\oint_\Gamma \frac{dz}{(z-a)^{n-1}} + \ldots + A_{n-1}\oint_\Gamma \frac{dz}{z-a} =$$

$$= A_0\cdot 0 + A_1\cdot 0 + \ldots + A_{n-1}\cdot 2\pi i = A_{n-1}\cdot 2\pi i,$$

ya'ni

$$\oint_\Gamma R(z)dz = A_{n-1}\cdot 2\pi i \qquad (7)$$

Mabodo ratsional funksiya ushbu

$$R(z)=\frac{P(z)}{Q(z)}$$

ko'rinishda berilgan bo'lsa, dastlab uni oddiy kasrlarga ajratib, so'ngra yuqoridagi metod bilan integrallash kerak.

Misol.

$$I=\oint_\Gamma \frac{(z+1)dz}{z^2(z-1)(z-3)},$$

bundagi Γ chiziq $|z|=2$ aylanadan iborat. Integral ishorasi ostidagi kasr-ratsional funksiyani quyidagicha oddiy kasrlarga ajratib olamiz.

$$\frac{(z+1)dz}{z^2(z-1)(z-3)}=\frac{1/3}{z^2}+\frac{7/9}{z}+\frac{-1}{z-1}+\frac{2/9}{z-3}.$$

U holda

$$I=\frac{1}{3}\oint_{\Gamma}\frac{dz}{z^2}+\frac{7}{9}\oint_{\Gamma}\frac{dz}{z}-\oint_{\Gamma}\frac{dz}{z-1}+\frac{2}{9}\oint_{\Gamma}\frac{dz}{z-3}$$

Bu integrallarning har birini tekshirib chiqaylik. (6) ga asosan

$$\oint_{\Gamma}\frac{dz}{z^2}=0, \qquad \oint_{\Gamma}\frac{dz}{z}=2\pi i, \qquad \oint_{\Gamma}\frac{dz}{z-1}=2\pi i.$$

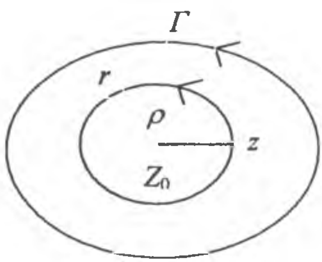
$z=3$ nuqta Γ aylananing tashqarisida bo'lgani uchun

$$\oint_{\Gamma}\frac{dz}{z-3}=0.$$

Demak, $I=\frac{7}{9}\cdot 2\pi i-2\pi i=\frac{-4}{9}\pi i.$

Koshining integral formulasi

Γ-chiziq bilan chegaralangan $\overline{E}$ yopiq sohada $f(z)$ analitik bo'lsin, u holda $\overline{E}$ ga tegishli har qanday $\overline{E}_1$ yopiq sohada ham $f(z)$ funksiya analitik bo'ladi. Sohaning ichida ixtiyoriy z_0 nuqta olaylik va bu nuqtani markaz qilib, ρ-radiusli aylana chizaylik. U holda Γ va γ lar bilan chegaralangan sohada $\frac{f(z)}{z-z_0}$ ham analitik bo'ladi, chunki $z\neq z_0$.



1-rasm.

Demak, (10) Koshi formulasiga asosan:

$$\oint_{\Gamma}\frac{f(z)}{z-z_0}d\xi=\oint_{\gamma}\frac{f(z)}{z-z_0}d\xi \qquad (1)$$

Har qanday $\varepsilon>0$ uchun shunday $\delta>0$ son mavjudki, γ-aylananing ixtiyoriy z nuqtasi uchun $|z-z_0|=\rho<\delta$, $|f(z)-f(z_0)|<\varepsilon$ tengsizliklar o'rinli. Demak,

$$\left|\oint_{\gamma}\frac{f(\xi)}{z-z_0}d\xi-\oint_{\gamma}\frac{f(z_0)}{z-z_0}d\xi\right|=\left|\oint_{\gamma}\frac{f(z)-f(z_0)}{z-z_0}d\xi\right|<2\pi\cdot\varepsilon$$

Ikkinchi tomondan

$$\oint_{\gamma}\frac{f(z_0)}{z-z_0}dz=f(z_0)\cdot\oint_{\gamma}\frac{dz}{z-z_0}=f(z_0)\cdot 2\pi\cdot i$$

Demak, $\lim\limits_{\rho\to 0}\oint_{\gamma}\frac{f(z)}{z-z_0}dz-f(z_0)\cdot 2\pi\cdot i=0$ bunda, agar (1) $\rho\to 0$ da hadlab limitga o'tsak, quyidagi tenglik hosil bo'ladi:

$$\lim_{\rho\to 0}\oint_{\gamma}\frac{f(\xi)}{z-z_0}dz=\lim_{\rho\to 0}\oint_{\gamma}\frac{f(z)}{z-z_0}dz=f(z_0)\cdot 2\pi\cdot i$$

Γ-chiziq bo'ylab olingan integral ρ-ga bog'liq bo'lmagani uchun limit belgisini tashlab yuborish mumkin, demak, ushbu tenglikni yozish mumkin:

$$f(z_0)=\frac{1}{2\cdot\pi\cdot i}\oint_{\Gamma}\frac{f(z)}{z-z_0}dz \qquad (2)$$

Bu Koshining integral formulasi deyiladi. Bu formulani E ko'p bog'lamli soha bo'lganida ham qo'llash mumkin, bunda $z_0-\widetilde{E}$ ichki nuqtasidir.

1-misol. $\oint_{\Gamma}\frac{z^2}{z-i}dz$ integral $|z-2i|=2$ aylana bo'ylab hisoblansin.

Yechish: Misolimizda $f(z)=z^2$, $z_0=i$.

Demak, Koshi integral formulasiga ko'ra:

$$\oint_{\Gamma} \frac{f(z)}{z - z_0} dz = 2\pi \cdot i \cdot f(z_0)$$

yoki

$$\oint_{\Gamma} \frac{z^2 dz}{z - i} = 2\pi \cdot i \cdot f(i) = 2\pi \cdot i(z^2)_i = -2\pi \cdot i$$

Koshi tipidagi integral. Yuqori tartibli hosilaning mavjudligi

1. Biz (2) Koshining integral formulasini keltirib chiqarishda $f(z)$ funksiya Γ bilan chegaralangan $\overline{E}$ yopiq sohada analitik bo'lishini talab qilgan edik. Agar bu ikki shartdan (yopiq, analitik) birortasi buzilsa, Koshi formulasi ham o'rinli bo'lmaydi. Γ- chiziq yopiq bo'lmasligi ham mumkin. Faraz qilaylik, $\varphi(\xi)$ funksiya shu Γ- chiziqda faqat uzluksiz bo'lsin, u holda agar biz Γ da yotmaydigan birorta z nuqta olsak, $\dfrac{\varphi(\xi)}{z - z_0}$ kasr shu chiziqning barcha nuqtalarida uzluksiz bo'ladi, chunki $z \neq z_0$. Shu sababli ushbu integral $\dfrac{1}{2\pi \cdot i} \int_{\Gamma} \dfrac{\varphi(\xi)}{\xi - z}$ tekislikdagi har bir z_0 (Γ- da yotmaydigan) nuqta uchun aniq qiymatga ega, ya'ni z ning funksiyasi

$$\Phi(z) = \frac{1}{2\pi \cdot i} \int_{\Gamma} \frac{\varphi(\xi)}{\xi - z} d\xi \tag{3}$$

2. Bu Koshi tipidagi integral deyiladi.

1-teorema. Koshi tipidagi integral bilan aniqlangan F(z) funksiya Γ- chiziqda yotmaydigan, har qanday chekli z nuqtada analitik bo'ladi. Shunday nuqtalarda funksiya barcha yuqori tartibli xossalarga ega bo'lib, ular ushbu formula orqali ifodalanadi:

$$\Phi^{n}(z)=\frac{n!}{2\pi\cdot i}\int_{\Gamma}\frac{\varphi(\xi)}{(\xi-z)^{n+1}}d\xi \qquad (4)$$

Teoremani isbotsiz qabul qilamiz.

2-teorema. Yopiq $\overline{E}$ sohada analitik bo'lgan har qanday $f(z)$ funksiya shu sohada istalgan tartibli hosilalarga ega bo'lib, ular ushbu formula bilan ifodalanadi:

$$f^{n}(z)=\frac{n!}{2\pi\cdot i}\int_{\Gamma}\frac{\varphi(\xi)}{(\xi-z)^{n+1}}d\xi \qquad (5)$$

Bu $f(z)$ funksiyaning yuqori tartibli hosilasini topish formulasidir.

82-ma'ruza

KOMPLEKS HADLI QATORLAR

1-ta'rif. Hadlari kompleks sonlardan iborat bo'lgan

$$\sum_{n=1}^{\infty} z_n \qquad (1)$$

qator kompleks hadli sonli qator deyiladi. Bunda $z_n=x_n+iy_n$ bo'lib, x_n, y_n lar haqiqiy sonlardir.

2-ta'rif. Barcha hadlari z ning funksiyasi bo'lgan $f_n(z)$ dan iborat

$$\sum_{n=1}^{\infty} f_n(z) \qquad (2)$$

qator funksional qator deyiladi.

3-ta'rif. Funksional qatorning xususiy ko'rinishi bo'lgan ushbu ko'rinishdagi qator

$$\sum_{n=1}^{\infty} C_n(z-a)^n \qquad (3)$$

darajali qator deyiladi.

Kompleks hadli qatorlarning tekshirishda haqiqiy sonli qatorlarni tekshirishdagi barcha xossalarni qo'llash mumkin, chunki kompleks hadli qatorlarni har doim ikkita haqiqiy sonli qatorlarning yig'indisi ko'rinishiga keltirish mumkin, ya'ni

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty}(x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n \qquad (4)$$

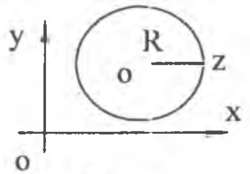
bunda $\sum_{n=1}^{\infty} x_n$ va $\sum_{n=1}^{\infty} y_n$ qatorlar haqiqiy sonli qatorlardir. Shuning uchun biz amaliyotda ko'proq ishlatiladigan z-z_0 ning darajalari bo'yicha yoyilgan (3) ko'rinishdagi qatorlarning ba'zi muhim xossalari bilan tanishamiz.

Ushbu

$$\sum_{n=1}^{\infty} C_n(z-a)^n = C_0(z-a)^0 + C_1(z-a)^1 + \\ + C_2(z-a)^2 + \ldots + C_n(z-a)^n + \ldots \quad (5)$$

ko'rinishdagi qatorni ba'zi xossalari bilan tanishaylik. Bunda $z=x+iy$, $a=\alpha+i\beta$, $C_n=a_n+ib_n$ kompleks sonlar bo'lib, x, y, α, β, a_n, b_n - lar haqiqiy sonlardir. Agar $z_0=0$ bo'lsa, ushbu qator hosil bo'ladi:

$$\sum_{n=1}^{\infty} C_n z^n = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \ldots + C_n z^n + \ldots \quad (6)$$



1-rasm.

4-ta'rif. Agar funksiya $f(z) = \sum_{n=1}^{\infty} C_n(z-a)^n$ ko'rinishda yozilgan bo'lsa, $f(z)$ funksiya z - a ning darajalari bo'yicha darajali qatorga yoyilgan deyiladi. Darajali qatorlarning yaqinlashish sohasi $|z-a|<R$ doiradan iborat bo'ladi.

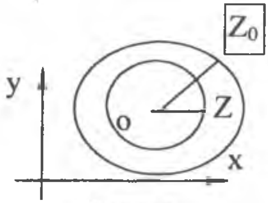
Teorema (Abel teoremasi). Agar $\sum_{n=1}^{\infty} C_n(z-a)^n$ darajali qator biror z - a nuqtada yaqinlashuvchi bo'lsa, u $|z-a|<|z_0-a|$ doiraning hamma nuqtalarida absolyut yaqinlashuvchi bo'ladi. Shuningdek, har qanday yopiq $|z-a|=r$ doirada ham tekis yaqinlashuvchidir, bunda $z<|z_0-a|$.

Teoremani isbotsiz qabul qilamiz.

Natija. Agar qator biror $z=z_0$ nuqtada uzoqlashuvchi bo'lsa, bu qator $|z-a|>|z_0-a|$ tengsizlikni qanoatlantiruvchi ixtiyoriy nuqtada uzoqlashuvchi bo'ladi.

Misol. $\sum_{n=1}^{\infty}\frac{z^n}{n^n}$ - Qator ixtiyoriy z nuqtada absolyut yaqinlashuvchi bo'ladi. Chunki z qanday bo'lmasin, biror n-nomerdan boshlab $\frac{|z|}{n}\le\frac{1}{2}$ qilib olish mumkin. U holda

$$\left|\frac{z^n}{n^n}\right|=\left(\frac{|z|}{n}\right)^n=\left(\frac{1}{2}\right)^n.$$



2-rasm.

Demak, berilgan qatorning barcha hadlari $\sum\frac{1}{2^n}$ cheksiz kamayuvchi geometrik progressiyaning mos hadlaridan katta emas, ya'ni berilgan qator ixtiyoriy z nuqtada yaqinlashuvchi.

Umuman (5) yoki (6) ko'rinishdagi darajali qatorlarning yaqinlashish doirasining radiusi Dalamber, Koshi-Adamar formulalari yordamida topiladi.

1. Koshi-Adamar formulasi:

$$R=\frac{1}{L};\quad L=\lim_{n\to\infty}\sqrt[n]{|C_n|} \tag{7}$$

2. Dalamber formulasi:

$$R=\frac{1}{L};\quad L=\lim_{n\to\infty}\frac{C_{n+1}}{C_n} \tag{8}$$

1-misol. $\sum_{n=1}^{\infty}\left(\frac{z}{in}\right)^n$ Qatorning yaqinlashish doirasi topilsin.

Yechish: Koshi-Adamar formulasini qo'llaymiz, bunda $C_n = \frac{1}{(in)^n}$

$$L = \lim_{n\to\infty} \sqrt[n]{|C_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{C_n}{(in)^n}\right|} = \lim_{n\to\infty} \frac{1}{(in)} = \lim_{n\to\infty} \frac{1}{n} = 0; \qquad R = \frac{1}{L} = \frac{1}{0} = \infty .$$

Demak, qator butun kompleks tekisligida yaqinlashuvchi.

2-misol. $\sum_{n=1}^{\infty} e^{in} z^n$ - qatorning yaqinlashish doirasi topilsin.

Yechish: Koshi-Adamar formulasiga ko'ra e^i=cos1+$i$sin1 Eyler formulasini e'tiborga olsak:

$$L = \lim_{n\to\infty} \sqrt[n]{|e^{in}|} = \lim_{n\to\infty} |e^{in}| = \lim_{n\to\infty} |\cos 1 + i\sin 1| =$$

$$= \sqrt{\cos^2 1 + \sin^2 1} = 1; \qquad R = \frac{1}{L} = 1;$$

Demak, qator birlik doiraning ichki nuqtalarida yaqinlashuvchi.

Teylor va Makloren qatorlari

Agar $w=f(z)$ funksiya biror nuqtaning atrofida analitik bo'lsa, uni z - a ga nisbatan musbat darajali qatorga quyidagicha yoyish mumkin:

$$F(z)=C_0+C_1(z-a)+C_2(z-a)^2+...+C_n(z-a)^n+... \qquad (9)$$

Bundagi C_0, C_1, C_2 ,... koeffitsientlar quyidagicha topiladi: avvalo $f'(z)$, $f''(z)$, ..., $f^n(z)$, ... larni topib, $z=a$ nuqtadagi qiymatlarini topsak ular quyidagicha bo'ladi:

$$C_0=f(a),\ C_1=\frac{1}{1.1}f'(a),\ C_2=\frac{1}{2.1}f''(a),\ ...\ ,\ C_1=\frac{1}{n.1}f^{n}(a),\ ... \qquad (10)$$

Bularni (9)-ga qo'ysak ushbu Teylor qatori hosil bo'ladi:

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+...+\frac{f^{(n)}(a)}{n!}(z-a)^n+.... \qquad (11)$$

Agar (11) da $a=0$ bo'lsa, quyidagi Makloren qatori kelib chiqadi:

$$f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\ldots+\frac{f^{(n)}(0)}{n!}z^n+\ldots. \quad (12)$$

Demak,

$$C_n=\frac{f^{(n)}(a)}{n!} \quad (13)$$

ekanligini ko'rish mumkin. Umuman (9) qatorning C_n koeffitsientlarini Koshining integral formulasidan foydalanib ham topish mumkin:

$$C_n=\frac{f^{(n)}(a)}{n!}=\frac{1}{2\pi i}\oint_{r}\frac{f(z)}{(z-a)^{n+1}}dz, \qquad n=0,\pm 1,\pm 2,\ldots. \quad (14)$$

Misol. $f(z)=\cos az$ funksiya $z=0$ nuqta atrofida Makloren qatoriga yoyilsin.

Yechish: $f'(z)=-a\sin az \quad f(0)=1$

$f''(z)=-a^2\cos az \quad f'(0)=0$

$f'''(z)=-a^3\sin az \; f''(0)=-a^2$

\- - - - - - - - - - $f'''(0)=0$

$f^{IV}(0)=a^4$

\- - - - - - - -

Misol. Funksiya nuqta atrofida Makloren qatoriga yoyilsin.

Yechish: Bularni (12) Makloren qatoriga qo'yamiz:

$$\cos az=1-\frac{(az)^2}{2!}+\frac{(az)^4}{4!}-\frac{(az)^6}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{(az)^{2n}}{(2n)!}.$$

3. Manfiy darajali qatorlar. Loran qatori

5-ta'rif. Agar kompleks hadli qator z-a ning manfiy darajalari bo'yicha yoyilgan bo'lsa, u manfiy darajali qator deyiladi, ya'ni

$$b_1(z-a)^{-1}+b_2(z-a)^{-2}+\ \dots\ +b_n(z-a)^{-n}+\dots=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n} \qquad (15)$$

Bu manfiy darajali qator deyiladi va uning yaqinlashish sohasi $|z-a|>r$ doira tashqarisidan iborat, bunda $r=\lim\limits_{n\to\infty}\left|\frac{b_n}{b_{n-1}}\right|$ yaqinlashish radiusi deyiladi.

Misol. $\sum\limits_{n=1}^{\infty} e^n (iz)^{-n}$ qatorning yaqinlashish radiusi topilsin.

Yechish:

$$r=\lim_{n\to\infty}\left|\frac{b_n}{b_{n-1}}\right|=\lim_{n\to\infty}\left|\frac{e^n i^n}{e^{n-1}i^{n-1}}\right|=e\cdot\frac{1}{|i|}=e;$$

$r=e$ - yaqinlashish radiusi. Demak, $|z|>e$ doira tashqarisida qator yaqinlashadi.

6-ta'rif. Ushbu ko'rinishdagi qator

$$\sum_{n=-\infty}^{+\infty}C_n(z-a)^n=\dots+C_{-n}(z-a)^{-n}+C_{-n-1}(z-a)^{-n-1}+\dots+C_{-2}(z-a)^{-2}+$$
$$+C_{-1}(z-a)^{-1}+C_0+C_1(z-a)+C_2(z-a)^2+\dots+C_n(z-a)^n+\dots \qquad (16)$$

Loran qatori deyiladi. Bunda

$$Q=\sum_{n=-\infty}^{-1}C_n(z-a)^n \qquad (17)$$

qatorning bosh qismi deyilib, $|z-a|>r$ da yaqinlashadi.

$$P=\sum_{n=0}^{+\infty}C_n(z-a)^n \qquad (18)$$

Loran qatorining to'g'ri qismi deyiladi va $|z-a|<R$ - da yaqinlashadi.

Teorema. Agar $f(z)$ funksiya $r<|z-a|<R$ halqada analitik va bir qiymatli bo'lsa, shu halqada bu funksiyani yagona usulda Loran qatoriga yoyish mumkin bo'lib, uning to'g'ri

qismi $|z-a|<R$ doirada, bosh qismi esa $|z-a|<r$ doiradan tashqarida yaqinlashuvchi bo'ladi, ya'ni

$$f(z)=\sum_{n=-\infty}^{+\infty}C_n(z-a)^n. \qquad (19)$$

Bundan ko'rinadiki $f(z)$ funksiyaning Loran qatori $r<|z-a|<R$ halqada yaqinlashuvchi bo'ladi.

Loran qatorining C_n koeffitsientlarini ham (14) Koshi formulasidan foydalanib topish mumkin. Ba'zan bu tipdagi integrallarni hisoblash murakkab bo'lgan hollarda sun'iy usullardan ham foydalaniladi.

1-misol. $f(z)=\dfrac{1}{z^2-3z+2}$ funksiya $1<|z|<2$ halqada Loran qatoriga n-ning darajalari bo'yicha yoyilsin.

Yechish:

$$f(z)=\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$$

ko'rinishga keltiramiz.

$$f_1(z)=-\frac{1}{z-1}=-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+...\right)=-\sum_{n=1}^{\infty}\frac{1}{z^n}.$$

Bunda $\left|\frac{1}{z}\right|<1$, $z>1$ yaqinlashish sohasi.

$$f_2(z)=\frac{1}{z-2}=-\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+...\right)=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$

Bunda $\left|\frac{z}{2}\right|<1$, $|z|<2$ yaqinlashish sohasi.

Demak, $f(z)=-\sum_{n=1}^{\infty}\frac{1}{z^n}-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$ - Loran qatori bo'lib, u

$1<|z|<2$ halqada yaqinlashuvchi bo'ladi.

2-misol. $f(z)=z^2\cdot\cos\frac{1}{z}$ funksiya $z=0$ nuqta atrofida Loran qatoriga yoyilsin.

Yechish: asosiy elementar kompleks funksiyalarning Teylor va Makloren qatorlari ham haqiqiy o'zgaruvchili elementar funksiyalarning Teylor va Makloren qatorlariga o'xshash ko'rinishda bo'ladi. Shuning uchun $\cos z = 1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\frac{1}{6!}z^6+\ldots$ dan foydalanamiz

$$f(z)=z^2\cos\frac{1}{z}=z^2\left(1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}+\frac{1}{6!}\frac{1}{z^6}+\ldots\right)=$$

$$=z^2-+\frac{1}{2!}+\frac{1}{4!z^2}-\frac{1}{6!z^4}+\ldots=-\frac{1}{2}+z^2+\frac{1}{4!z^2}-\frac{1}{6!z^4}+\ldots$$

Demak, $f(z)=-\frac{1}{2}+z^2+\frac{1}{4!z^2}-\frac{1}{6!z^4}-\frac{1}{8!z^6}+\ldots \qquad z\neq 0.$

FUNKSIYANING NOLLARI. MAXSUS NUQTALAR VA ULARNING TURLARI

I. Funksiyaning nollari

1-ta'rif. Agar $f(z)$ funksiya z_0 nuqtada analitik bo'lib, $f'(z_0)=f(z_0)=\ ...\ =f^{(n-1)}(z_0)=0$; $f^{(n)}(z_0)\neq 0$ bo'lsa, z_0 nuqta $f(z)$ funksiyaning n- tartibli noli deyiladi.

Teorema. Agar z_0 nuqtaning biror atrofida $f(z)=)z-z_0)^n\cdot\varphi(z)$ bo'lib, $\varphi(z)$ funksiya z_0 nuqtada analitik va $\varphi(z_0)\neq 0$ bo'lsa, z_0 nuqta $f(z)$-funksiyaning n- tartibli noli bo'ladi.

Misol. $f(z)=1+\cos z$ funksiyaning nollari va uning tartibi aniqlansin.

Yechish: $f(z)=0 \Rightarrow 1+\cos z=0$; $\cos z=-1$; $z_n=\pi+2\pi n$.

$f'(z)=-\sin z$; $f'(z_0)=f'(\pi+2\pi n)=0$.

$f''(z)=-\cos z$; $f''(z_0)=f''(\pi+2\pi n)=1\neq 0$.

Demak, $z_n=\pi+2\pi n$, $n=0;\ \pm1;\ \pm2;\ ...$ nuqtalar funksiyaning II tartibli nollaridir.

II. Maxsus nuqtalar va ularning turlari

2-ta'rif. Agar $w=f(z)$ funksiya z_0 nuqtada analitik bo'lsa, bu nuqta $f(z)$ funksiyaning to'g'ri nuqtasi, aks holda u nuqta maxsus nuqtasi deyiladi.

1-misol. $f(z)=\frac{1}{z}$ funksiyaning maxsus nuqtasi $z_0=0$ dan iborat, chunki uning $f'(z)=-\frac{1}{z^2}$ hosilasi $z_0=0$ nuqtada mavjud emas.

2-misol. $f(z)=\frac{1}{z^2+1}$ funksiyaning $z=\pm i$ nuqtalari maxsus nuqtalardir. Chunki uning $f'(z)=-\frac{2z}{(z^2+1)^2}$ hosilasi $z=\pm i$

nuqtalarda mavjud emas.

Maxsus nuqtalarning turlari ko'p bo'lib, ulardan amaliyotda ko'p uchraydiganlari ajralgan maxsus nuqtalardir.

Ajralgan maxsus nuqtalar

3-ta'rif. Agar $f(z)$ funksiya z_0 nuqtaning biror δ-atrofi $0<|z-z_0|<\infty$ da analitik bo'lib, z_0 nuqtaning o'zida analitik bo'lmasa, bu nuqta $f(z)$ funksiyaning ajralgan maxsus nuqtasi deyiladi.

Ajralgan maxsus nuqtalar uch xilga bo'ladi, ya'ni qutilib bo'ladigan maxsus nuqtalar, qutblar va muhim maxsus nuqtalar.

4-ta'rif. Agar $\lim\limits_{z\to z_0} f(z)=A$ chekli, aniq chekli limit mavjud bo'lsa, z_0 nuqta $f(z)$ funksiyaning qutilib bo'ladigan maxsus nuqtasi deyiladi.

4-misol. $f(z)=\frac{1}{z-1}\sin(z-1)$ funksiyaning $z=1$ qutilib bo'ladigan maxsus nuqtasidir, chunki $\lim\limits_{z\to 1}\frac{\sin(z-1)}{z-1}=1$.

1-teorema. $f(z)$ funksiyaning ajralgan maxsus nuqtasi z_0- qutilib bo'ladigan maxsus nuqta bo'lishi uchun uning $z-z_0$ ni darajalari bo'yicha yoyilgan Loran qatori bosh qismga ega bo'lmasligi zarur va yetarli, ya'ni

$$f(z)=C_0+C_1(z-z_0)+C_2(z-z_0)^2+\ \dots\ +C_n(z-z_0)^n+\ \dots\ .$$

Teoremani isbotsiz qabul qilamiz.

5-ta'rif. Agar $\lim\limits_{z\to z_0} f(z)=\infty$ bo'lsa, z_0 nuqta $f(z)$ funksiyaning qutb nuqtasi deyiladi.

5-misol. $f(z)=\frac{1}{z-2}$ funksiyaning $z=2$ oddiy qutb nuqtasidir, chunki $\lim\limits_{z\to 2}\frac{1}{z-2}=\infty$.

2-teorema. Berilgan $f(z)$ funksiyaning ajralgan maxsus nuqtasi qutb nuqta bo'lishi uchun u $\varphi(z)=\frac{1}{f(z)}$ funksiyaning noli bo'lishi zarur va yetarlidir.

6-ta'rif. $\varphi(z)$ funksiyaning z_0-nolining tartibi $f(z)$ ning z_0 qutbining tartibi deyiladi.

6-misol. $f(z)=\frac{1}{z^3(z^2+9)^2}$ - funksiyaning qutblari $\varphi(z)=z^3(z^2+9)^2$ funksiyaning nollaridan iborat, ya'ni:

$$\varphi(z)=z^3(z^2+9)^2=0 \quad \Rightarrow \quad \{z_1=0;\, z_2=3i;\, z_3=-3i\}.$$

Bunda $z=0$ III - tartibli, $z=3i$, $z=-3i$ lar funksiyaning II tartibli qutblari bo'ladi.

3-teorema. Berilgan $f(z)$ funksiyaning ajralgan maxsus nuqtasi z_0 n-tartibli qutb bo'lishi uchun uning z-z_0 ni darajalari bo'yicha yoyilgan Loran qatorining bosh qismi chekli n-ta haddan iborat bo'lishi zarur va yetarlidir, ya'ni:

$$f(z)=c_{-n}(z-z_0)^{-n}+c_{-n+1}(z-z_0)^{-(n-1)}+\ldots+c_{-1}(z-z_0)^{-1}+c_0+$$
$$+c_1(z-z_0)+c_2(z-z_0)^2+\ldots$$

Teoremani isbotsiz qabul qilamiz.

Misol. $f(z)=\frac{1}{z-\sin z}$ funksiyaning maxsus nuqtalari va ularning tartibi aniqlansin.

Yechish. $z=0$ maxsus nuqta bo'ladi. $\varphi(z)=z-\sin z$ funksiyaning nollarini topaylik, buning uchun uni qatorga yoysak:

$$\varphi(z)=z-\sin z=z-(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots)=$$

$$z^3(\frac{1}{3!}+\frac{z^2}{5!}+\frac{z^4}{7!}-\ldots)$$

Demak, $z=0$ $\varphi(z)$ ning 3-tartibli noli bo'ladi. Demak $z=0$ $f(z)$ funksiyaning 3-tartibli qutb nuqtasi ekan.

Ta'rif 7. Agar z_0 $f(z)$ ning ajralgan maxsus nuqtasi bo'lib, $\lim\limits_{z\to z_0} f(z)$ limit mavjud bo'lmasa, z_0-nuqta shu funksiyaning muhim maxsus nuqtasi deyiladi.

4-teorema. $f(z)$ funksiyaning ajralgan maxsus nuqtasi muhim maxsus nuqtasi bo'lishi uchun shu funksiyaning $z=z_0$ ni darajalari bo'yicha yoyilgan Loran qatorining bosh qismi cheksiz ko'p hadlarga ega bo'lishi zarur va yetarlidir, ya'ni:

$$f(z)=\sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n$$

Bu holda xususan Loran qatorining bosh qismi bo'lmasligi ham mumkin, ya'ni c_0, c_1, c_2, ..., c_n lar nolga teng bo'lishi ham mumkin, biroq $c_{-n}\neq 0$, $c_{-n+1}\neq 0$ va $c_{-1}\neq 0$ va hokazo.

Misol. $f(z)=e^{\frac{1}{z^2}}$ funksiyasining maxsus nuqtasi va uning tipi aniqlansin.

Yechish. $z=0$ $f(z)$ ning maxsus nuqtasidir.

a) $z=x$ haqiqiy sonlar o'qida olsak $f(z)=f(x)=e^{\frac{1}{x^2}}$; $x\to 0$, $f(x)\to\infty$.

b) $z=iy$ mavhum o'qda olsak $f(z)=f(iy)=e^{\frac{1}{(iy)^2}}=e^{-\frac{1}{y^2}}$; $y\to 0$, $f(iy)\to 0$.

Demak, $f(z)=e^{\frac{1}{z^2}}$ funksiya $z=0$ da aniq limitga ega emas, bu esa $z=0$ muhim maxsus nuqta degan so'zdir.

Chegirmalar va ularning tatbiqlari

Biror z_0 nuqta $f(z)$ funksiyaning ajralgan maxsus nuqtasi bo'lsin.

Ta'rif. Agar $f(z)$ funksiya $0<|z-z_0|<R$ halqada analitik bo'lsa, uning ajralgan z_0 nuqtaga nisbatan chegirmasi deb

ushbu $\frac{1}{2\pi i}\oint\limits_{\Gamma} f(z)dz$ integralning qiymatiga aytiladi va u Res$f(z_0)$- kabi belgilanadi. Demak,

$$\text{Res}f(z_0)=\frac{1}{2\pi i}\oint\limits_{\Gamma} f(z)dz \tag{1}$$

Bunda Γ - markazi z_0 nuqtada bo'lgan kichik radiusli aylana bo'lib, $f(z)$ funksiyaning analitik bo'lgan sohasiga tegishli va boshqa maxsus nuqtalarni o'z ichiga olmasligi kerak.

1. Agar bizga ma'lum bo'lgan $f(z)$ funksiyaning z_0 nuqta atrofida yoyilgan Loran qatorining koeffitsientlarini va (1) ifodani e'tiborga olsak ushbu tenglikni yozish mumkin.

$$c_{-1}=\frac{1}{2\pi i}\oint\limits_{\Gamma} f(z)dz=\text{Res}f(z_0), \tag{2}$$

ya'ni z_0 - ajralgan maxsus nuqtadagi chegirma Loran qatori yoyilmasidagi $c_{-1}(z-z_0)^{-1}$ hadining koeffitsienti c_{-1} dan iborat ekan.

2. Agar z_0 nuqta $f(z)$ funksiyaning to'g'ri nuqtasi yoki qutilib bo'ladigan maxsus nuqtasi bo'lsa, u holda Loran qatorining bosh qismi bo'lmaydi ya'ni $c_{-n}=0$ (n=-1,-2,-3,...) Bu holda (2) formulaga ko'ra $c_{-1}=\text{Res}f(z_0)=0$ bo'ladi.

Demak, to'g'ri nuqtada va qutilib bo'ladigan maxsus nuqtalarda chegirma nolga teng bo'ladi.

Teorema. Agar $f(z)$ funksiya Γ chiziq bilan o'ralgan E yopiq sohaning ajralgan maxsus nuqtalari $z_1, z_2, .., z_n$ lardan boshqa barcha nuqtalarda analitik bo'lsa, u holda $f(z)$ funksiyadan Γ chiziq bo'ylab olingan integralning qiymati Γ chiziq ichidagi barcha maxsus z_k nuqtalarga nisbatan olingan funksiya chegirmalari yig'indisining $2\pi i$ ga ko'paytirilganiga teng, ya'ni

$$\oint\limits_{\Gamma} f(z)dz=2\pi i\sum_{k=1}^{n}\text{Res}f(z_k) \tag{3}$$

Teoremani isbotsiz qabul qilamiz.

3. Agar z_0 nuqta $f(z)$ funksiyaning oddiy qutb (n=1) nuqtasi bo'lsa, z_0 nuqta atrofida yoyilgan Loran qatori quyidagi ko'rinishda bo'ladi:

$$f(z)=c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots \qquad (4)$$

Buni hadlab $z-z_0$ ga ko'paytirib $z\to z_0$ dagi limitini topsak, c_{-1} kelib chiqadi:

$$\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}[c_{-1}+c_0(z-z_0)+c_1(z-z_0)^2+\ldots]=c_{-1}$$

Demak, $c_{-1}=\lim_{z\to z_0}[(z-z_0)f(z)]$, ya'ni

$$\operatorname{Res}f(z_0)=\lim_{z\to z_0}[(z-z_0)f(z)]. \qquad (5)$$

4. Agar z_0 nuqta $f(z)$ funksiyaning n- tartibli qutbi bo'lsa, uning bu nuqtaga nisbatan Loran qatori quyidagi ko'rinishda bo'ladi:

$$f(z)=c_{-n}(z-z_0)^{-n}+c_{-n+1}(z-z_0)^{-(n-1)}+\ldots+c_{-1}(z-z_0)^{-1}+c_0+$$
$$+c_1(z-z_0)+c_2(z-z_0)^2+\ldots$$

Bundan c_{-1} koeffitsientni topish uchun tenglikni hadlab $(z-z_0)^n$ ga ko'paytirib (n-1) marta hosila olamiz, $z\to z_0$ dagi limitini topamiz. Natijada ushbu topiladi:

$$c_{-1}=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]$$

Demak, z_0 n- tartibli qutb bo'lsa, u nuqtadagi $f(z)$ funksiyaning chegirmasi quyidagi formuladan topiladi:

$$\operatorname{Res}f(z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] \qquad (6)$$

5. Agar $f(z)$ funksiya quyidagicha kasr shaklda berilgan bo'lib, $f(z)=\dfrac{\varphi(z)}{\psi(z)}$, z_0 uning oddiy qutbi bo'lsa, ya'ni

$\varphi(a)\neq 0$; $\psi(a)=0$; $\psi'(a)\neq 0$. Ya'ni z_0 $\psi(z)$ uchun oddiy nol.

(5) ga ko'ra

$$\operatorname{Res}f(z_0)=\lim_{z\to z_0}[(z-z_0)f(z)]=\frac{\lim\limits_{z\to z_0}\varphi(z)}{\lim\limits_{z\to z_0}\dfrac{\psi(z)-\psi(z_0)}{z-z_0}}=\frac{\varphi(a)}{\psi^{|}(a)},$$

ya'ni

$$\operatorname{Res}f(z_0)=\frac{\varphi(a)}{\psi^{|}(a)}. \tag{7}$$

6. Agar z_0 nuqta $f(z)$ funksiyaning muhim maxsus nuqtasi bo'lsa, shu funksiyaning z_0 nuqtadagi chegirmasini topish uchun u funksiyaning z_0 nuqta atrofida yoyilgan Loran qatorining c_{-1} koeffitsientini topish kifoya, haqiqatan bu holda $f(z)$ ning yoyilmasi

$$f(z)=\sum_{n=-\infty}^{+\infty}c_n(z-z_0)^n$$

kabi bo'ladi.

Bundan c_{-1} ni topish yetarli.

1-misol. $f(z)=\dfrac{z^3}{z+2}$ funksiyaning z=-2 oddiy qutbga ko'ra chegirmasi topilsin.

Yechish. (5) formulaga ko'ra

$$\operatorname{Res}f(-2)=\lim_{z\to-2}[(z+2)f(z)]=\lim_{z\to-2}[(z+2)\frac{z^3}{z+2}]=(-2)^3=-8.$$

2-misol. $\oint\limits_{|z|=2}\dfrac{z^2dz}{(z^2+1)(z+3)}$ integral chegirma (qoldiq) lardan foydalanib hisoblansin.

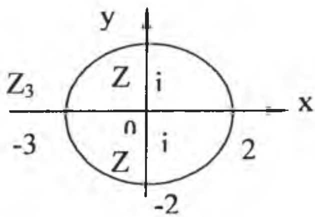
Yechish. $(z^2+1)(z+3)=0 \Rightarrow z_1=i,\ z_2=-i,\ z_3=-3$ maxsus nuqtalar bo'lib, $z=-3$ nuqta $|z|=2$ doiraga tegishli emas.

a) $z=i$ oddiy qutbdagi chegirma quyidagicha:

$$\operatorname{Re}sf(i) = \lim_{z\to i}\frac{z^2(z-i)}{(z-i)(z+i)(z+3)} = \frac{i^2}{2i(i+3)} = \frac{3i-1}{20}$$

b) z=-i oddiy qutb nuqta

$$\operatorname{Re}sf(-i) = \lim_{z\to -i}\frac{z^2(z+i)}{(z-i)(z+i)(z+3)} = -\frac{i^2}{2i(-i+3)} = \frac{1-3i}{20}$$



1-rasm.

c) $$\oint\limits_{|z|=2}\frac{z^2}{(z^2+1)(z+3)}dz = 2\pi i[\operatorname{Re}sf(i) + \operatorname{Re}sf(-i)] =$$

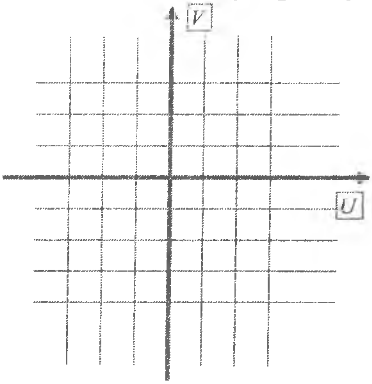
$$= 2\pi i(\frac{1+3i+1-3i}{20}) = \frac{\pi i}{5}.$$

TFKP ish kartografiyaga tatbiqi

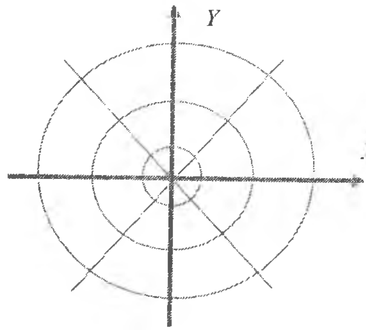
1-masala. Kartografiyadagi asosiy masalalardan biri sitrning biror qismini tekislikka akslantirishdan iborat. Bu akslantirish konform yoki masshtabli bo'lishi mumkin. 1569 yilda mashhur gollandiyalik kartograf Gerard Merkator (1512-1594) kartalarni kashf etdi. Bu kartada yer sharining parallellari va meridianlari to'g'ri chiziqlar ko'rinishda tasvirlangan va mos ravishda bular bir-biriga perpendikulyar bo'lgan. Buni Meokator nazariy jihatdan asoslab bera olmagan.

Hozirgi zamon fani nuqtai nazaridan Merkatorning bu proyeksiyasi konform akslantirishdan iborat. Yer sharidagi meridianlar va parallellar orasidagi burchaklar akslashtirish natijasida kartaga to'g'ri burchaklar bo'lib akslantiriladi.

$W = e^z$ funksiyaga teskari bo'lgan logarifmik funksiya koordinata boshidan chiqqan nurlar va konsentrik aylanalarni kesib to'rt burchaklari to'g'rito'rtburchaklari to'rga (setkaga) akslanadi (2-rasmga qarang).



2-rasm.



3-rasm.

Bu akslantirish ushbu formula yordamida ifodalanadi

$$W = \ln z \qquad (1)$$

bu yerda $\ln z = \ln|z| + i \arg z$ yoki

$$\ln z = \sqrt{x^2 + y^2} + arctg\,\frac{y}{x} \qquad (2)$$

har qanday $agr\, z = \alpha$ yoki $z = t + iat$ (bu yerda $a = tg\,\alpha$) ga tekislikda quyidagi to'g'ri chiziqlar mos keladi

$$W = \ln\sqrt{t^2 + a^2t^2} + i\alpha = \ln|t| + \ln\sqrt{1 + a^2} + i\alpha \qquad (3)$$

har qanday $|z| = r$ parallellarga W tekisligida ushbu to'g'ri chiziqlar mos keladi.

$$W = \ln tg\left(\frac{U}{2} + \frac{\pi}{4}\right) + it \qquad (4)$$

bu yerda $r = tg\left(\frac{U}{2} + \frac{\pi}{4}\right)$ (3) to'g'ri chiziq OU o'qiga, (4) esa

OV o'qiga parallel, ular bir-biriga perpendikulyar (3-rasmga qarang) Nolli meridian OV o'qiga akslanadi, ekvator esa OV o'qiga akslanadi.

Kengligi $\alpha = \pm\frac{\pi}{2}$ bo'lgan $\alpha = \pm\infty$ ga yoki cheksiz uzoqlashgan $z = \infty$ nuqtaga mos keladi.

Yuqorida ko'rilgan Merkator proyeksiyasi juda muhim bo'lgan xossaga ega.

Kemaning o'zgarmas yo'nalishi bo'yicha boshqarish, yana harakat yo'nalishi kompas ayirmasi bilan o'zgarmas burchak tashkil etsa, bu holda harakat loksodroma bo'yicha harakatlanadi (loksodroma - qiyshiq yurish).

To'la kompleks qarshilik tushunchasi

2-masala. Qarshilik R va $\hbar$ funksiyadan tuzilgan zanjirni ko'ramiz.

Rasmda ko'rsatilgan zanjirdan tok $j = I\sin(\omega t + \alpha)$ o'tadi deb faraz qilamiz. Zanjir uchlarida paydo bo'lgan potensiallar farqi $U = V\sin(\omega t + \varphi)$ ni topish kerak.

Bu holda:

$$U = h\frac{\partial j}{\partial t} + Rj$$

$U = Ve^{i\varphi}$, $j = \tau e^{i\omega t}$ deb qabul qilamiz.

Va bu yerda

$$U = Ve^{i\varphi},\ \tau = Ie^{i\alpha}.$$

Agar j ni (1) ifodaga qo'yib U ni topamiz.

$$U = (ih\omega + R)\tau = z\tau$$

Ushbu tenglamada U va T lar orasidagi proporsionallik koeffitsienti z kompleks son bo'lib, vaqtga bog'liq emas. z qarshilikning funksiyasi bo'lib, to'la komleks qarshiligi deyiladi.

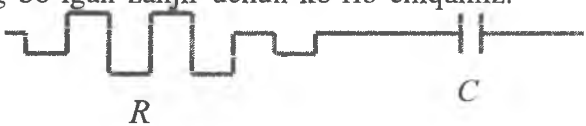
Potensiallar qarshiligi esa U ning mavhum qismi bilan aniqlanadi:

Demak, $\mathrm{Im}(U) = V\sin(\omega t + \varphi)$.

Bunda $V = I\,|\,z\,| = I\sqrt{R^2 + h^2\omega^2}$

$$\varphi = \alpha + \beta,\ tg\beta = \frac{h\omega}{R}.$$

Endi shu masalani R bilan ketma-ket ulangan va sig'imi C ga teng bo'lgan zanjir uchun ko'rib chiqamiz.



5-rasm.

Bu masala uchun tenglama quyidagicha bo'ladi.

$$U = Rj + \frac{1}{c}\int_0^t jdt$$

va

$$U = (R + \frac{1}{ic\omega})\tau = z\tau.$$

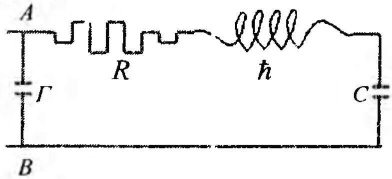
Demak, $U = V\sin(\omega t + \varphi)$ va bu yerda

$$V = I\,|\,z\,| = I\sqrt{R^2 + \frac{1}{c^2\omega^2}}\quad \varphi = \alpha + \beta,\ tg\beta = -\frac{1}{Rc\omega}$$

3-masala. Rasmda ko'rsatilgan zanjirning A va B nuqtalari orasidagi to'la komleks qarshilik topilsin.

Bu zanjir ikkita konturdan iborat bo'lib, birinchisi to'la kompleks qarshiligi $z_1 = -\frac{i}{\Gamma\omega}$ bo'lgan Γ sig'imli kondensator, ikkinchi esa, bir-biriga ketma-ket ulangan qarshiligi R, induksiya h va sig'imi C lardan tuzilgan bo'lsin. Ikkinchi konturning to'la komleks qarshiligi z_2 quyidagicha aniqlanadi.

$$z_2 = ih\omega - \frac{i}{c\omega} + R.$$



6-rasm.

Bu holda masalada ko'rilayotgan to'la kompleks qarshilik uchun.

$$z = \frac{z_1 z_2}{z_1 + z_2}$$

Ifoda o'rinli bo'ladi, ni

$$z(i\omega) = \frac{-\frac{i}{\Gamma\omega}\left(R + ih\omega - \frac{i}{c\omega}\right)}{R + ih\omega - i\left(\frac{1}{c} + \frac{1}{\Gamma}\right)\frac{1}{\omega}}$$

bunday

$$|z| = \frac{1}{h\varpi}\sqrt{\frac{R^2 + \left(h\omega - \frac{1}{c\omega}\right)^2}{R^2 + \left[h\omega - \left(\frac{1}{c} + \frac{1}{\Gamma}\right)\frac{1}{\omega}\right]^2}}$$

$$\varphi = arctg\frac{R}{\frac{1}{c\omega} - h\omega} - arctg\frac{h\omega - \left(\frac{1}{c} + \frac{1}{\Gamma}\right)\frac{1}{\omega}}{R}.$$

Chapligin va Jukovskiy formulalari

4-masala. Faraz qilaylik, zichligi ρ ga teng havoda tovushgacha bo'lgan o'zgarmas tezlik - V_∞ bilan samolyot qanoti harakatlanmoqda, yoki boshqacha qilib aytganda tinch holatda turgan qanotga, tezligi V_∞ ga teng bo'lgan havo oqimi ta'sir etmoqda.

Yasovchilari tezlik vektoriga perpendikulyar bo'lgan cheksiz silindr shaklidagi qanotni osmonga ko'taruvchi kuchni hisoblash kerak.

Agar S qanot kesimining konturi va $P(z)$ esa z nuqtadagi havo bosimi bo'lsa, bu holda S ning dz elementga $ip(z)dz$ kuch, butun kontur S ga esa

$$P = \int_S ip(z)dz$$

to'la kuch ta'sir etadi.

Bernulli formulasiga asosan $P(z) = A - \frac{\rho}{2}V^2$ tenglik o'rinlidir.

Bu yerda A-o'zgarmas miqdor. $v = |V|$, V -oqim tezligi vektori. Demak,

$$P = -\frac{\rho_i}{2}\int_S v^2 dz .$$

Konturning barcha nuqtalarida tezlik urinma bo'yicha yo'nalgan bo'lgani uchun.

$$V = S'(z) = ve^{i\varphi} \quad \varphi = \arg dz .$$

Agar $e^{-2i\varphi}$ $dz = d\bar{z}$ bo'lsa, bu holda

$$P = -\frac{\rho_i}{2}\int_S \overline{(f'(z))}^2 e^{-2i\varphi}dz = -\frac{\rho_i}{2}\int_S \overline{(f'(z))}^2 d\bar{z} .$$

Oxirgi tenglikdan, ko'taruvchi kuchning vektoriga qo'shma vektor P, quyidagicha aniqlanadi.

$$P = \frac{\rho_i}{2}\int_S (f'(z))^2 dz$$

Keltirilgan formula S.A. Chanligin tomonidan isbotlangan. Ushbu formuladan va quyidagi ifodadan

$$f'(z) = \overline{V}_\infty + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots$$

chegirmalar teoremasiga binoan

$$\overline{P} + 2\pi i \frac{\rho_i}{2}; \quad \frac{\Gamma v_\infty}{\pi i} = i\rho\Gamma\overline{V}_\infty$$

bo'ladi.

Demak,

$$P = -i\rho\Gamma V_\infty$$

bu yerda Γ maydonning sirkulytsiyasi. Olingan formulani N.E. Jukovskiy isbotlagan.

OLIY MATEMATIKADAN YAKUNIY YOZMA SAVOLLAR

Qatorlar

1. Sonli qatorlar. Xususiy yig'indi.
2. Musbat hadli qatorlarning yaqinlashishining zaruriy sharti.
3. Qatorlarni taqqoslash teoremalari.
4. Qator yaqinlashishining Dalamber alomati.
5. Qator yaqinlashishining integral alomati.
6. Ishoralari almashinuvchi qatorlar. Leybnis alomati.
7. Ishoralari o'zgaruvchi qatorlar. Absolyut va shartli yaqinlashish.
8. Funksional qatorlar. Qatorlarning yaqinlashish sohasi.
9. Qatorlarni differensiallash va integrallash.
10. Kuchaytirilgan va tekis yaqinlashuvchi qatorlar.
11. Darajali qatorlar. Abel teoremasi.
12. Yaqinlashish intervali va uni topish usullari.
13. Darajali qatorlarni differensiallash va integrallash.
14. (x-a) ning darajalari bo'yicha qatorlar.
15. Teylor va Makleron qatorlari.
16. Elementar funksiyalarni Makloren qatorlariga yoyish.
17. Binomial va $ln(1+x)$ funksiyalarni darajali qatorga yoyish.

Ikki o'lchovli integrallar

1) Ikki o'lchovli integral ta'rifi va uning geometrik ma'nosi
2) Ikki o'lchovli integral xossalari
3) D sohada f(x, y) funksiyaning ikki karrali integrali.
4) Ikki o'lchovli integral va ikki karrali integral orasidagi farq va ularning o'zaro teng bo'lishi shartlari
5) To'g'ri soha ta'rifi
6) Ikki karrali integral xossalari
7) Ikki o'lchovli integral yordamida soha yuzasini va jism hajmini hisoblash
8) Qutb koordinata sistemasida berilgan funksiya uchun ikki o'lchovli integralni hisoblash formulasi

9) D sohada uzluksiz xususiy hosilalarga ega bo'lgan z=f(x,y) funksiya uchun $\iint\limits_D \sqrt{1+z_x'^2+z_y'^2}\,dxdy$ ning geometrik ma'nosi

10) Ikki o'lchovli integral yordamida tekis shaklning statik inersiya momentlarini topish.

11) Ikki o'lchovli integral yordamida tekis shaklning og'irlik markazini topish

Uch o'lchovli integrallar

1) V fazoviy jism bo'yicha u=f(x,y,z) funksiyaning uch o'lchovli integrali va uning geometrik ma'nosi
2) V fazoviy jism bo'yicha u=f(x,y,z) funksiyaning uch karrali integrali va uni hisoblash
3) Uch o'lchovli integralni uch karrali integral yordamida hisoblash. To'g'ri soha
4) Silindrik va sferik koordinatalar sistemasida uch karrali integrallarni hisoblash
5) Uch karrali integral yordamida jism hajmini hisoblash
6) Uch karrali integralning mexanikaga tatbiqi

Egri chiziqli va sirt integrallari

1) Koordinatalar bo'yicha egri chiziqli integral va uning xossalari (1-tur egri chiziqli integral)
2) Yassi egri chiziq yoyi bo'yicha egri chiziq ta'rifi va uning xossalari (2-tur egri chiziq)
3) Parametrik tenglama bilan berilgan chiziq bo'yicha egri chiziqli integralni hisoblash formulasi
4) Egri chiziqli integral yordamida yassi figura yuzasini hisoblash formulasi
5) Grin formulasini keltirib chiqarish
6) Egri chiziqli integralning integrallash yo'liga bog'liq bo'lmasligi shartini keltirib chiqarish
7) Sirt integralining ta'rifi va xossalari
8) Sirt integralini hisoblash formulalari

Maydonlar nazariyasi asoslari

1) Skalyar maydon ta'rifi. Skalyar maydonning yuksaklik sirtlari va chiziqlari
2) Vektor maydon ta'rifi. Vektor maydonning vektor chiziqlari va vektor chiziqlarning differensial tenglamalari
3) Vektor maydonning chiziqli integrali. Vektor maydon sirkulyatsiyasi
4) Vektor maydon oqimi va uni hisoblash formulasi
5) Vektor maydon rotori va uni hisoblashga misol
6) Skalyar maydon gradiyenti. Gradiyent va biror vektor yo'nalishi bo'yicha olingan hosila orasidagi bog'liqlik
7) Vektor maydon divergensiyasi va uni hisoblashga misol
8) Stoks va Ostrogradskiy formulalarining koordinatalardagi va vektor ko'rinishi
9) Solenoidal va potensial maydon ta'rifi. Potensial ta'rifi va uni hisoblash formulasi
10) Gamilton va Laplas operatori. Garmonik funksiya ta'rifi
11) Laplas tenglamasi va bu tenglamani qanoatlantiruvchi funksiyalar

Kompleks o'zgaruvchili funksiyalar nazariyasi asoslari

1) Kompleks o'zgaruvchili funksiya ta'rifi, uning limiti va uzluksizligi
2) Asosiy elementar kompleks o'zgaruvchili funksiyalar
3) Kompleks o'zgaruvchili funksiya hosilasi ta'rifi. Koshi-Riman sharti
4) Analitik va garmonik funksiyalar orasidagi bog'liqlik. Analitik funksiyani uning haqiqiy (mavhum) qismidan foydalanib tiklash.
5) Kompleks o'zgaruvchili funksiya hosilasining argumenti va modulining geometrik ma'nosi
6) Kompleks o'zgaruvchili funksiyaning konturli integrali, uning asosiy xossalari
7) Kompleks o'zgaruvchili funksiyaning konturli integrali va haqiqiy o'zgaruvchili funksiyaning egri chiziqli integrali orasidagi bog'liqlik, aniq integralga keltirish

8) Koshi integrali (yopiq konturdan olingan integral haqida)
9) Ko'p bog'lamli soha uchun Koshi teoremasi
10) Boshlang'ich funksiya va aniqmas integral. Kompleks o'zgaruvchili ratsional funksiyalarni integrallash
11) Koshi integral formulasi
12) O'rta qiymat haqidagi teorema Modul uchun maksimum prinsipi.
13) Koshi tipidagi integrallar. Yuqori tartibli hosilalar.
14) Kompleks o'zgaruvchili funksiyalarning Teylor qatori. Asosiy kompleks o'zgaruvchili funksiyalarni Teylor qatoriga yoyish/
15) Kompleks o'zgaruvchili funksiyalarning Loran qatori. Loran koeffitsientlarini topish.
16) Kompleks o'zgaruvchili funksiyalarning nollari va qutblari. Ular orasidagi bog'liqlik.
17) Kompleks o'zgaruvchili funksiyalarning yakkalangan maxsus nuqtalari klassifikatsiyasi (to'g'rilanadigan, qutb va muhim maxsus nuqtalar).
18) Kompleks o'zgaruvchili funksiyalar chegirmalari. Chegirmalar haqida Koshi teoremasi.
19) Karrali va oddiy chekli qutbdagi chegirma
20) Chegirmalarni aniq integrallarni hisoblashga qo'llash

Qatorlar

1. Qatorning yaqinlashishini ko'rsating va yig'indisini toping: $\sum_{n=1}^{\infty}\frac{4^n-3^n+6^n}{12^n}$.
2. Qatorning yaqinlashishini ko'rsating va yig'indisini toping: $\sum_{n=1}^{\infty}\frac{2^n+9^n-3^n}{18^n}$.
3. Qatorning yaqinlashishini ko'rsating va yig'indisini toping: $\sum_{n=1}^{\infty}\frac{1}{(5n+4)(5n-1)}$.
4. Qatorning yaqinlashishini ko'rsating va yig'indisini toping: $\sum_{n=1}^{\infty}\frac{1}{(2n+3)(2n+5)}$.

5. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{(n+1)^2}{2^n\cdot n!}$.

6. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{n!}{6^n\cdot(n+2)!}$.

7. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{3n+1}{\sqrt{n\cdot 2^n}}$.

8. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{2n}{(n+3)!}$.

9. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\left(\frac{5}{8}\right)^n\cdot\left(\frac{1}{n-2}\right)^6$.

10. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\left(\frac{4n^2+5n-2}{7n^2+2n+1}\right)^n$.

11. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\left(\frac{2n+4}{2n+7}\right)^{n^2}$.

12. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\left(\arcsin\frac{1}{3^n}\right)^{2n}$.

13. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\left(arctg\frac{1}{7^n-5}\right)^n$.

14. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{2}{(n+7)\ln^2(n+7)}$.

15. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{\ln(n+2)}{(n+2)}$.

16. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{n}{n^2+7}$.

17. Yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}\frac{n^2}{n^3+5}$.

18. Absolyut va shartli yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln^2 n}$.

19. Absolyut va shartli yaqinlashishga tekshiring:
$\sum_{r=1}^{\infty}(-1)^{n-1}\frac{0,5^n}{(n-1)!}$.

20. Absolyut va shartli yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

21. Absolyut va shartli yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{3^n}$.

Funksional qatorlar

1.Yaqinlashish radiusini toping: $\sum_{n=1}^{\infty}\frac{(x-2)^n}{(2n-1)\cdot 2^n}$.

2. Yaqinlashish radiusini toping: $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n\cdot 3^n}$.

3. Yaqinlashish radiusini toping: $\sum_{n=1}^{\infty}\frac{(x+5)^n}{4^n}$.

4. Yaqinlashish radiusini toping: $\sum_{n=1}^{\infty}\frac{(x+2)^n}{3^n}$.

5. Yaqinlashish radiusini toping: $\sum_{n=1}^{\infty}\frac{x^{3n}}{8^n(n^2+2)}$.

6. $x_0=0$ nuqta atrofida darajali qatorga yoying: $y=\frac{e^x-2}{x}$.

7. $x_0=0$ nuqta atrofida darajali qatorga yoying: $y=\frac{\ln(1+x^2)}{x}$.

8. $x_0=0$ nuqta atrofida darajali qatorga yoying: $y=\frac{\arcsin x}{x}$.

9. Berilgan funksiyaning berilgan oraliqdagi Furye qatoriga yoyilgandagi berilgan koeffitsientini toping: $y= x^2$, $-2<x<2$, a_n-?

10. Berilgan funksiyaning berilgan oraliqdagi Furye qatoriga yoyilgandagi berilgan koeffitsientini toping: $y=-x$, $-1<x<1$, b_n-?

11. Berilgan funksiyaning berilgan oraliqdagi Furye qatoriga yoyilgandagi berilgan koeffitsientini toping: $y= x-3$, $-p<x<p$, a_n-?

Karrali va egri chiziqli integrallar

1) Integralni hisoblang: $\iint_D (x+y+3)dxdy$, bu yerda D soha x+y=2, x=0, y=0 chiziqlar bilan chegaralangan.

2) Integralni hisoblang: $\int_0^2 dx\int_0^1 (x^2+2y)dy$.

3) Integrallash tartibini o'zgartiring: $\int_{-2}^{2} dx \int_{0}^{\sqrt{4-x^2}} f(x, y)dy$.

4) Integralni hisoblang: $\iint\limits_D (x^2 + y)dxdy$, bu yerda D soha $y=x^2$, $y^2=x$ chiziqlar bilan chegaralangan.

5) Integralni hisoblang: $\iint\limits_D (x^3y^2dxdy$, bu yerda D soha $x^2+y^2=9$ chiziq bilan chegaralangan.

6) Integralni hisoblang: $\iint\limits_D x(\cos(x + y)dxdy$, bu yerda D soha $y=0$, $x=\pi$, $y=x$ chiziqlar bilan chegaralangan.

7) Integralni hisoblang: $\iint\limits_D xdxdy$, bu yerda D soha $y=x^2$, $y=2x$ chiziqlar bilan chegaralangan.

8) Integrallash tartibini o'zgartiring: $\int_0^4 dx \int_{x^2/2-3}^{2x-3} f(x, y)dy$.

9) Integralni hisoblang: $\iint\limits_D (x + y)dxdy$, bu yerda D soha $2x+y=1$, $2x+y=3$, $x-y=-1$, $x-y=2$ to'g'ri chiziqlar bilan chegaralangan.

10) Qutb koordinatalar sistemasiga o'tib integralni hisoblang: $\iint\limits_D (x^2 + y^2)dxdy$, bu yerda D soha $x^2+y^2=4x$ chiziq bilan chegaralangan.

11) $y=\sqrt{x}$, $y=2\sqrt{x}$, $x=4$ chiziqlar bilan chegaralangan soha yuzini toping.

12) $x^2+y^2=1$, $z=0$, $x+y+z=4$ sirtlar bilan chegaralangan jism hajmini toping.

13) $x=0$, $y=0$, $z=0$, $x=4$, $y=4$ tekisliklar va $z=1+x^2+y^2$ paraboloid bilan chegaralangan jism hajmini toping.

14) $x^2+y^2=R^2$ va $x^2+z^2=R^2$ silindrlar bilan chegaralangan jism hajmini toping.

15) $z=y^2/2$ silindr va $2x+3y=12$, $x=0$, $y=0$, $z=0$ tekisliklar bilan chegaralangan jism hajmini toping.

16) $\iiint\limits_V x^2y^2dxdydz$ ni hisoblang, bu yerda V jism $0 \le x \le 1$, $0 \le y \le x$ va $0 \le z \le xy$ tengsizliklar bilan aniqlanadi.

17) Integralni hisoblang $\iiint\limits_V \frac{dxdydz}{(1 + x + y + z)^2}$, bu yerda V jism $x=0$, $y=0$, $z=0$, $x+y+z=1$ tekisliklar bilan chegaralangan.

18) $y=x^2$, $y+z=4$, $z=0$ sirtlar bilan chegaralangan jism hajmini toping.

19) Integralni hisoblang: $\iint_D (x^2 + y)dxdy$, D: $y=x^2$, $x=y^2$.

20) Integralni hisoblang: $\iint_D xy^2 dxdy$, D: $y=x^2$, $y=2x$.

21) Integralni hisoblang: $\iint_D (x + y)dxdy$, D: $y^2=x$, $y=x$.

22) Integralni hisoblang: $\iint_D x^2 ydxdy$, D: $y=2-x$, $y=x$, $x\geq 0$.

23) Integralni hisoblang: D: $y^2=4x$, $x+y=3$, $y\geq 0$.

24) Berilgan chiziqlar bilan chegaralangan D soha yuzini toping: $y=6x^2$, $x+y=2$, $x\geq 0$.

25) Berilgan chiziqlar bilan chegaralangan D soha yuzini toping: $y^2=x+2$, $x=2$.

26) Berilgan chiziqlar bilan chegaralangan D soha yuzini toping: $y=x^2+1$, $x+y=3$, $y\geq 0$.

27) Berilgan sirtlar bilan chegaralangan V jism hajmini toping: $z=x^2+y^2$, $x+y=1$, $x\geq 0$, $y\geq 0$, $z\geq 0$.

28) Berilgan sirtlar bilan chegaralangan V jism hajmini toping: $z=x^2$, $x-2y+2=0$, $x+y-7=0$, $z\geq 0$.

29) Berilgan sirtlar bilan chegaralangan V jism hajmini toping: $y = \sqrt{x}$, $y=x$, $x+y+z=2$, $z\geq 0$.

30) Berilgan sirtlar bilan chegaralangan V jism hajmini toping: $y=1-x^2$, $x+y+z=3$, $y\geq 0$, $z\geq 0$.

31) Integralni hisoblang: $\iiint_V (2x^2 + 3y + z)\,dxdydz$, V: $2\leq x\leq 3$, $-1\leq y\leq 2$, $0\leq z\leq 4$.

32) Integralni hisoblang: $\iiint_V x^2 yzdxdydz$, V: $-1\leq x\leq 2$, $0\leq y\leq 3$, $2\leq z\leq 3$.

33) Integralni hisoblang: $\iiint_V (x + y + 4z^2)\,dxdydz$, V: $-1\leq x\leq 1$, $0\leq y\leq 2$, $-1\leq z\leq 1$.

34) Hisoblang: $\iiint_V (x^2 + y^2 + z^2)\,dxdydz$, V: $0\leq x\leq 3$, $-1\leq y\leq 2$, $0\leq z\leq 2$.

35) Egri chiziqli integralni hisoblang: $\int_{AB} (x^2 - 2xy)dx + (y^2 - 2xy)dy$, bu yerda AV - $y=x^2$ parabolaning A(-1, 1) dan V(1, 1) gacha yoyi.

36) Egri chiziqli integralni hisoblang: $\int\limits_{AB}(x^2+y^2)dx+2xydy$, bu yerda AV - $y=x^3$ kubik parabolaning A(0, 0) dan V(1, 1) gacha yoyi.

37) Egri chiziqli integralni hisoblang: $\oint\limits_{L}(x+2y)dx+(x-y)dy$, bu yerda L: x=2cost, y=2sint musbat yo'nalishidagi aylana.

38) Egri chiziqli integralni hisoblang: $\oint\limits_{L}(x^2y-x)dx+(y^2x-2y)dy$, bu yerda L: x=3cost, y=2sint musbat yo'nalishidagi ellips.

39) Egri chiziqli integralni hisoblang: $\oint\limits_{L}(xy-1)dx+x^2ydy$, bu yerda L - x=cost, y=2sint musbat yo'nalishli ellips.

40) Egri chiziqli integralni hisoblang: $\int\limits_{AB}(x^2-y^2)dx+xydy$, bu yerda AV - to'g'ri chiziqning A(1, 1) dan V(3, 4) gacha bo'lgan kesmasi.

41) Hisoblang: $\oint\limits_{L}(x^2+y^2)dl$, L: $x^2+y^2=4$ - aylana.

42) Hisoblang: $\int\limits_{AB}(4\sqrt[3]{x}-3\sqrt{y})dl$, bu yerda L - to'g'ri chiziqning A(-1, 0) dan V(0, 1) gacha kesmasi.

Kompleks o'zgaruvchili funksiyalar nazariyasi

1. $f(x)=e^x\cos y+ie^x\sin y$ funksiya differensiallanuvchimi?
2. f(z) differensiallanuvchi funksiyaning haqiqiy qismi $u(x, y)=x^2-y^2-x$ berilgan, bu yerda z=x+iy. f(z) ni toping.
3. f(z) funksiyaning mavhum qismi v(x, y)=x+y berilgan. Bu funksiyani toping.
4. $f(z)=(x^2+y^2)-2xyi$ funksiya differensiallanuvchimi?
5. $f(z)=(x^3-3xy^2)+i(3x^2y-y^3)$ funksiya differensiallanuvchi ekanligini ko'rsating va uning hosilasini toping.
6. Integralni hisoblang: $\int_{AB} f(z)dz$, bu yerda f(z)=(y+1)-xi, AB – $z_A=1$ ва $z_B=-i$ nuqtalarni birlashtiruvchi kesma.
7. Integralni hisoblang: $\int\limits_{AB} f(z)dz$, bu yerda $f(z)=x^2+y^2i$, AB – kesma (A=1+i ва B=2+3i).
8. Integralni hisoblang: $\int^{1+i} zdz$.

9. Integralni hisoblang: $\int\limits_{AB} z^2 dz$, bu yerda AV – to'g'ri chiziq kesmasi ($z_A=1$, $z_B=i$).

10. Integralni hisoblang: $\int\limits_{\gamma} \frac{dz}{z}$, bu yerda γ - aylana: $z=e^{ti}$.

11. $f(z)=z^5$ funksiyani z-i darajalari bo'yicha Teylor qatoriga yoying.

12. $f(z)=1/(2z-5)$ funksiyani z=0 nuqta atrofida z ning darajalari bo'yicha yoying.

13. $f(z)=\frac{1}{(z-1)(z-3)}$ funksiyani $1<|z|<3$ halqada z ning darajalari bo'yicha Loran qatoriga yoying.

14. $f(z)=\frac{z^4}{(z-2)^2}$ funksiyani z-2 ning darajalari bo'yicha Loran qatoriga yoying.

15. Funksiya chegirmasini toping: $f(z)=\frac{z}{(z^2-1)(z^2+1)^2}$.

16. Funksiya chegirmasini toping: $f(z)=\frac{z}{(z-1)(z-3)}$.

17. Funksiya chegirmasini toping: $f(z)=\frac{1}{z^2+4}$.

18. Funksiya chegirmasini toping: $f(z)=\frac{1}{z^2-2z+5}$.

19. $f(z)=\frac{z^2}{(z-2)^3}$. funksiya chegirmasini hisoblang.

20. $f(z)=\frac{1}{1-\cos z}$ funksiyaning chegirmasini z=0 qutbga nisbatan hisoblang.

21. $\int\limits_{\gamma} \frac{z^2}{(z^2+1)(z-2)} dz$, ni $|z|=3$ tenglama bilan berilgan γ aylana bo'yicha hisoblang

22. $\int\limits_{\gamma} \frac{z+1}{(z-1)(z-2)(z-3)} dz$, integralni z=1, z=2, z=3 qutblarni o'z ichida olgan γ yopiq kontur bo'yicha hisoblang.

23. Integralni $\int_{\gamma}\frac{dz}{z(z+2)(z+4)}$, γ - aylana bo'yicha hisoblang: 1) $|z|=1$; 2) $|z|=3$; 3) $|z|=5$.

24. Funksiya chegirmasini toping: $f(z)=\frac{z+i}{z-i}$.

25. Funksiya chegirmasini toping: $f(z)=\frac{z^2+1}{z^2-1}$.

26. Aniq integralni hisoblang: $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^3}$.

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