YULDASHEVA. A., PIRMATOV. SH.

OLIY MATEMATIKA

III-QISM

TOSHKENT

O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

YULDASHEV A., PIRMATOV SH.

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III-QISM

O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi tomonidan talabalar uchun o‘quv qo‘llanma sifatida tavsiya etilgan

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O‘quv qo‘llanma “Oliy matematika” ning asosiy bo‘limlaridan bo‘lgan kompleks o‘zgaruvchili funksiyalar nazariyasi, operatsion hisob nazariyasi elementlari, matematik fizika tenglamalari, ehtimollar nazariyasi va matematik statistika bo‘limlariga bag‘ishlangan. O‘quv qo‘llanmada har bir mavzuga doir misol va masalalarning izohli yechimi chizmalari bilan keltirilgan. Bundan tashqari mustaqil yechish uchun har bir mavzuga bir nechtadan masala va misollar berilgan.

O‘quv qo‘llanma Davlat ta’lim talablari asosida yozilgan bo‘lib, yetarli darajada adabiyotlar bilan boyitilgan.

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KIRISH

Mazkur o'quv qo'llanma tasdiqlangan "Oliy matematika" fanining o'quv dasturi asosida tayyorlangan. Ma'lumki, texnika ta'lim yo'nalishlarida kompleks o'zgaruvchili funksiyalar nazariyasi, operatsion hisob nazariyasi elementlari, matematik fizika tenglamalari va ehtimollar nazariyasi va matematik statistikani o'rganish katta ahamiyatga ega hisoblanadi.

Chegaralangan vaqt ichida talabalar katta hajmda va turli matematik g' oyalarni o'rganish lozim, lekin adabiyotlar orasida oliy texnika o'quv yurtlari talabalari uchun qisqa va tushunarli qilib yozilgan lotin imlosida o'quv qo'llanmalar mavjud emas. O'quv qo'llanma oliy o'quv yurti talabalariga mo'ljallangan. Bu esa talabalarni mustaqil o'zlashtirishlariga juda katta yordam beradi.

Qo'llanma kompleks sonlar va kompleks argumentli funksiyalar, elementar funksiyalar va ularning xossalari, kompleks argumentli funksiyalarning hosilasi, integrali va chegirmalar nazariyasi, operatsion hisob elementlari original va tasvir, originallarni differensiallash va integrallash, Laplas almashtirishning qoidalari, ba'zi elementar funksiyalarning tasvirlari, operatsion usullarni differensial tenglamalar va ularning sistemalarini yechishga tadbiq etish, matematik fizika tenglamarining asosiy tiplari, tor tebranshlari tenglamasi, torning majburiy tebranishi, ehtimollar nazariyasi va matematik statistika elementlari va yana boshqa mavzularni o'z ichiga oladi.

Mualliflar tomonidan tavsiya etilgan ushbu o'quv qo'llanma, "Oliy matematika" ning yuqorida qayd etilgan bo'limlarini mustaqil o'zlashtirish imkonini beradi.

I-BOB. KOMPLEKS O'ZGARUVCHILI FUNKSIYALAR NAZARIYASI

1.1. Kompleks sonlar haqida tushuncha

Kompleks son tushunchasi yuqori darajali algebraik tenglamalarni, jumladan, kvadrat tenglamalarni yechish jarayonida yuzaga kelgan.
Ushbu

$$ax^2 + dx + c = 0 \qquad (1.1)$$

kvadrat tenglamani ko'raylik, bunda $a \neq 0, b, c$ lar haqiqiy sonlar. Ma'lumki (1.1) tenglamaning ildizlari:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1.2)$$

Agar (1.1) tenglamaning diskreminanti $b^2 - 4ac < 0$ bo'lsa u holda (1.2) dan ikkita kompleks ildiz kelib chiqadi.

Aniqlik uchun ushbu xususiy holni ko'rib chiqaylik:

$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \qquad (1.3)$$

Bu tenglamaning ildizlari

$$x_1 = -\sqrt{-4}, \qquad x_2 = \sqrt{-4}, \qquad (1.4)$$

lardan iborat bo'lib haqiqiy sonlar to'plamida tenglama yechimga ega emas.

Shuning uchun $i = \sqrt{-1}$ ko'rinishdagi mavhum i birlik kiritamiz. U holda (1.3) tenglama ildizlari

$$x_1 = -2\sqrt{-1} = -2i, \qquad x_2 = 2\sqrt{-1} = 2i,$$

ko'rinishda yoziladi. (1.2) formulaga muvofiq ushbu kvadrat

$$x^2 + 4x + 5 = 0 \qquad (1.5)$$

tenglama ildizlari

$$x_1 = -2 - i \quad x_2 = -2 + i \qquad (1.6)$$

lardan iborat bo'lib, ular kompleks sonlarga misol bo'ladi.

1.1-ta'rif. Kompleks son deb, $\alpha = a + ib$ ko'rinishidagi songa kompleks sonning alfebraik ko'rinishi aytiladi, bu yerda a va b lar ixtiyoriy haqiqiy sonlar bo'lib, a ga α kompleks sonning haqiqiy

ib ga mavhum qısmı deyıladı. b–ga mavhum qısmınmg koeffitsiyenti deyiladi va $a = \mathrm{Re}\,\alpha$, $b = Jm\alpha$ ko'rinishda yoziladi.

realis – lotincha termin bo'lib, haqiqiy *imaginarus* – mavhum degan so'zdir.

1.1-misol. $\alpha = -3+4i$ kompleks son berilgan bo'lsa $\mathrm{Re}\,a = R(-3+4i) = -3$, $Jm\alpha = Jm(-3+4i) = 4$.

Shunday qilib, umuman kompleks songa ma'lum tartibda olingan ikkita haqiqiy sonning (a,b) sistemasi sifatida qarash mumkin. Xususiy holda, agar $b=0$ bo'lsa, u holda $\alpha = a+0i$. Kompleks son haqiqiy a songa teng deb qabul qilinadi.

Demak, haqiqiy son kompleks sonning xususiy xoli bo'ladi. Agar $a=0$ bo'lsa, $\alpha = 0+ib = ib$ – ga teng bo'lib unga sof mavhum son deyiladi $b \neq 0$.

Ikki kompleks sonning mos ravishda haqiqiy va mavhum qismlar bir – birlariga teng bo'lsa ular o'zaro teng deyiladi, ya'ni $\alpha = a_1 + ib_1$ va $\beta = a_2 + b_2 i$ kompleks sonlar teng bo'ladi, agarda $a_1 = a_2$, $b_1 = b_2$ bo'lsa $\alpha = \beta$ kelib chiqadi.

Kompleks sonlar tartiblanmagan, ya'ni ixtiyoriy ikkita kompleks son uchun katta – kichiklik tushunchasi kiritilmagan.

Ushbu $\alpha = a+ib$ va $\bar{\alpha} = a-ib$ ko'rinishdagi kompleks sonlar o'zaro qo'shma kompleks sonlar deyiladi.

1.2-misol. $\alpha = \frac{5}{4}+\frac{3}{5}i$, $\bar{\alpha} = \frac{5}{4}-\frac{3}{5}i$ o'zaro qoshma kompleks sonlardir.

1.2. Kompleks sonlar ustida amallar

1. Kompleks sonlarni qo'shish va ayirish. $\alpha = a+ib$ va $\beta = c+id$ komleks sonlar berilgan bo'lsin. Bu kompleks sonlarni qo'shish yoki ayirish uchun haqiqiy qismi haqiqiy qismiga, mavhum qismi mavhum qismiga qo'shiladi yoki haqiqiy qismi haqiqiy qismidan, mavhum qismi mavhum qismidan ayriladi. Ya'ni

$$\alpha \pm \beta = (a+ib) \pm (c+id) = (a \pm b) + (c \pm d)i \qquad (1.7)$$

1.3-misol. $\alpha = 4-6i$ va $\beta = \frac{5}{4}+\frac{3}{2}i$ kompleks sonlar berilgan bo'lsin.

$$a+\beta = (4-6i)+\left(\frac{5}{4}+\frac{3}{2}i\right),$$

$$\alpha+\beta=(4-6i)+\left(\frac{5}{4}+\frac{3}{2}i\right)=\frac{21}{4}-\frac{9}{2}i,$$

$$\alpha-\beta=(4-6i)-\left(\frac{5}{4}+\frac{3}{2}i\right)=\frac{11}{4}-\frac{15}{2}i.$$

2. Kompleks sonlarni ko‘paytirish va bo‘lish. Kompleks sonlarni ko‘paytirish va bo‘lishda $i^2=-1, i^3=-i, i^4=1, i^5=i$ va hokazo $i^{4k+1}=i$, $i^{4k+2}=-1$, $i^{4k+3}=-i,..$ $i^{4k}=1$, $k=0,1,2.....$ ekanligi e’tiborga olinadi.

Ikkita kompleks sonni ko‘paytirish uchun ko‘phadini ko‘phadga ko‘paytirish kabi ko‘paytiriladi. $\alpha=a+ib, \beta=c+id$ kompleks sonlarni ko‘paytirsak

$$\alpha\cdot\beta=(ac-bd)+i(ad+bc) \tag{1.8}$$

bo‘ladi.

α va β kompleks sonlarning α/β nisbati deb, shunday kompleks son $\gamma=x+iy-$ ga aytiladiki, u $\alpha=\beta\gamma$ ya’ni $a+ib=(c+id)(x+iy)$ tenglikni qanoatlantirsin.

$a+ib=(c+id)(x+iy)$ dan x va y larni topamiz.

$$a+ib=(cx-dy)+i(dx+cy)\Rightarrow\left.\begin{matrix}a=cx-dy\\ b=dx+cy\end{matrix}\right\}\Rightarrow x=\frac{ac+bd}{c^2+d^2}, y=\frac{bc-ad}{c^2+d^2} \text{ bo‘ladi.}$$

$$\frac{\alpha}{\beta}=\frac{a+ib}{c+id}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}=x+iy \tag{1.9}$$

Agar $\alpha=a+ib$ va $\bar{\alpha}=a-ib$ o‘zaro qo‘shma sonlar berilgan bo‘lsa $\alpha\cdot\bar{\alpha}$ va $\alpha+\bar{\alpha}$ ifodalar $\alpha\cdot\bar{\alpha}=a^2+b^2$, $\alpha+\bar{\alpha}=2a$ ko‘rinishidagi haqiqiy sonlardan iborat bo‘ladi.

1.4-misol. $z_1=2+3i$ va $z_2=1-2i$ kompleks sonlar berilgan bo‘lsa, $\frac{z_1}{z_2}$ ni toping.

Yechilishi. $\frac{z_1}{z_2}$ kasrning maxrajini z_2 kompleks sonning qo‘shmasiga ko‘paytiramiz

$$\frac{z_1}{z_2}=\frac{2+3i}{1-2i}=\frac{(1+2i(2+3i)}{(1-2i)(1+2i)}=\frac{2-6}{1^2+2^2}+i\frac{3+4}{1^2+2^2}=-\frac{4}{5}+i\frac{7}{5}.$$

1.5-misol. Ushbu tenglamani yeching: $(1+i)x+(2-3i)y=3-5i$

Yechilishi. Dastlab qavislarni ochib tenglamaning chap tomonida haqiqiy va mavhum qisimlarni ajratamiz: $\begin{cases} x+2y=3 \\ x-3y=-5 \end{cases}$

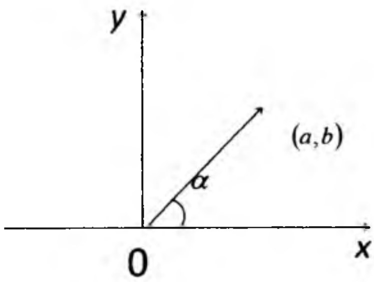
Bu sistemani yechib, $y=\frac{8}{5}$ $x=-\frac{1}{5}$ larni topamiz.

1.6-misol. Ushbu $\frac{\alpha}{\beta}=\frac{1+itg\alpha}{1-itg\alpha}$ kasrni soddalashtiring.

Yechilishi.

$$\frac{\alpha}{\beta}=\frac{1+itg\alpha}{1-itg\alpha}=\frac{(1+itg\alpha)^2}{1+tg^2\alpha}=\frac{1+2itg\alpha-tg^2\alpha}{1+tg^2\alpha}=\frac{1-tg^2\alpha}{1+tg^2\alpha}+i\frac{2tg\alpha}{1+tg^2\alpha}=\cos 2\alpha+i\sin 2\alpha.$$

3. Kompleks sonlarning geometrik tasviri va kompleks tekislik. To‘g‘ ri burchakli xoy Dekart koordinatalar sistemasida, uning absissalar o‘qida $\alpha=a+ib$ kompleks sonning haqiqiy qismi a-ni, ordinata o‘qiga mavhum qismining koeffitsiyenti b–ni joylashtiramiz, natijada tekislikda (a,b) nuqtaga ega bo‘lamiz.



1.1 - chizma. $\alpha=a+ib$ **kompleks sonnig geometrik tasviri.**

Ana shu nuqtani $\alpha=a+ib$ kompleks sonning geometrik tasviri deb qabul qilinadi (1.1- chizma). Shunga ko‘ra “kompleks α son” deyish o‘rniga “α – nuqta” ham deyiladi va aksincha.

Shunday qilib, tekislikning barcha nuqtalari to‘plami bilan birga barcha kompleks sonlar to‘plami orasida o‘zaro bir qiymatli moslik o‘rnatiladi. Agar $b=0$ bo‘lsa, $\alpha=a$ haqiqiy songa tegishli nuqta absisiyalar o‘qida yotadi, shu sababli ox – haqiqiy o‘q deb ataladi. Agar $a=0, b\neq 0$ bo‘lsa $\alpha=ib$ mavhum songa tegishli nuqta ordinatalar o‘qida yotadi, shu sababli oy - mavhum o‘q, xoy tekislik esa kompleks tekislik deyiladi.

Misollarni yeching.

1.1. $(3+2i)^3$

1.2. $\dfrac{(1+3(i)^2-(2+(i)^3(1+i)^3)}{1+(2i)^3+(1+2i)^2(3+2i)^2}+2i-5$

1.3. $\dfrac{(3-2i)^2}{(4+3i)^3}+(4-2i)^3$

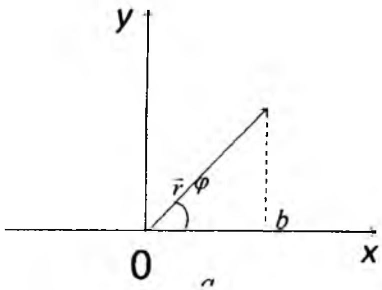
1.4. Ushbu $(3-4i)x(7+4i)y=2-5i$ tenglamadan x va y larning haqiqiy qiymatlarini toping.

1.5. Ushbu bikvadrat tenglamani yeching: $x^4+30x^2+289=0$.

4. Kompleks sonning trigonometrik shakli. $\alpha=a+ib$ kompleks sonning ikki xil geometrik ma'nosi bor:

Tekislikda (a,b) nuqtani tasvirlaydi, $(0,0)$) nuqta bilan (a,b) nuqtani tutashtiruvchi vektorni tasvirlaydi (1.2–chizma).

Ma'lumki, tekislikda har bir kompleks songa bitta nuqta va aksincha, har bir nuqtaga bitta kompleks son mos keladi. 1.2 – chizmadan quyidagilarni ko'rish mumkin:



1.2-chizma. $\alpha=a+ib$ **kompleks sonnig moduli va argumenti.**

1.2-chizmadan

$$a=r\cos\varphi,\quad b=r\sin\varphi,\quad r=\sqrt{a^2+b^2},\quad tg\varphi=\frac{b}{a},\quad \varphi=arctg\frac{b}{a} \qquad (1.10)$$

r – kompleks son α – ni uzunligini ifodalaydi va $r-\alpha$ sonning moduli $r=|\alpha|$, φ – burchakni α – ning argumenti deyiladi.

Shakldan ko'rinadiki, $0 \le r < +\infty$, $0 \le \varphi \le 2\pi$ ba'zan $-\pi < \varphi < \pi$ chegaralar ham ishlatilib, ikkita chegara ham bir maqsadga olib keladi. (1.10) – ga muvofiq α kompleks son ushbu

$$\alpha = a + ib = r(\cos\varphi + i\sin\varphi) \qquad (1.11)$$

ko'rinishga ega bo'lib, u kompleks sonning trigonometrik shakli deyiladi.

1.7-misol. $\alpha = 1 + i\sqrt{3}$ sonni trigonometrik shaklga keltiring.

Yechilishi. $a = 1, b = \sqrt{3},\ r = |a| = |1 + i\sqrt{3}| = \sqrt{1+3} = \sqrt{4} = 2$,

$$tg\varphi = \frac{\sqrt{3}}{1} = \sqrt{3},\ \varphi = arctg\sqrt{3} = \frac{\pi}{3},\quad \alpha = 1 + \sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}).$$

1.8-misol. $\alpha = \sqrt{3} - i$ sonni trigonometrik shaklga keltiring.

Yechilishi. $a = \sqrt{3};\ b = -1.$ $(a.b)$ nuqta IV chorakda joylashgan bo'lib, $r = 2$,

$$\varphi = arctg(-\frac{1}{\sqrt{3}}) = 2\pi - arctg\frac{1}{\sqrt{3}} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

bo'ladi. Demak,

$$\alpha = |\sqrt{3} - i| = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$$

1.9-misol. $\alpha = -1$ sonni trigonometrik shaklga keltiring.

Yechilishi. $a = -1, b = 0,\quad r = \sqrt{(-1)^2 + o^2} = 1,\quad \varphi = arctg\frac{0}{-1} = \pi.$

Demak, $\alpha = -1 = \cos\pi + i\sin\pi$.

5. Trigonometrik ko'rinishda berilgan sonlarni ko'paytirish, darajaga ko'tarish.

Quyidagi

$$\alpha = r_1(\cos\varphi_1 + i\sin\varphi)\ \ va\ \ \beta = r_2(\cos\varphi_2 + i\sin\varphi_2) \qquad (1.12)$$

ko'rinishdagi kompleks sonlar berilgan bo'lsin.

Trigonometrik shaklda berilgan kompleks sonlarni o'zaro ko'paytirish va darajaga ko'tarish quyidagicha bajariladi:

$$\alpha\cdot\beta = r_1(\cos\varphi_1 + i\sin\varphi_1)\cdot r_2(\cos\varphi_2 + i\sin\varphi_2) =$$

$$= r_1\cdot r_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1 i\sin\varphi_2)] + i(\sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1) = \qquad (1.13)$$

$$= r_1\cdot r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

ya'ni ikki kompleks sonning ko'paytmasi yana kompleks son bo'lib uning moduli ko'paytuvchilar modullarining ko'paytmasiga, argument ko'paytuvchilar argumentlarining yig' indisiga tengdir.

Quyidagi

$$\alpha_1 = r_1(\cos\varphi_1 + i\sin\varphi_1),$$
$$\alpha_2 = r_2(\cos\varphi + i\sin\varphi_2),$$
$$\alpha_3 = r_3(\cos\varphi_3 + i\sin\varphi_3),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_n = r_n(\cos\varphi_n + i\sin\varphi_n).$$

kompleks sonlar berilgan bo'lsin. Agar $\alpha_1 = \alpha_2 = \alpha_3 = \ldots\ldots\alpha_n$ bo'lsa ularning modullari va argumentlari ham o'zaro teng bo'ladi:

$$r_1 = r_3 = r_3 = \ldots\ldots r_n = r,$$
$$\varphi_1 = \varphi_3 = \varphi_3 = \ldots\ldots\varphi_n = \varphi.$$

Demak, (1.13) – ga asosan

$$\alpha^n = [r(\cos\varphi + i\sin\varphi]^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad (1.14)$$

Agar daraja ko'rsatkichi manfiy bo'lsa ham (1.14) formula o'z kuchini saqlaydi. Haqiqatdan ham

$$[r(\cos\phi + i\sin\phi)]^{-n} = \frac{1}{r^n(\cos\phi + i\sin\phi)^n} = \frac{1}{r^n}\cdot\frac{1}{(\cos\phi + i\sin\phi)^n} =$$
$$= r^{-n}\frac{\cos n\phi - i\sin n\phi}{\cos^2 n\phi + \sin^2 n\phi^n} = r^{-n}\left[\cos(-n\phi) + i\sin(-n\phi)\right].$$

Demak,

$$(r(\cos\varphi + i\sin\varphi))^n = r^{-n}[\cos(-n\varphi) + i\sin(-n\varphi].\qquad (1.15)$$

1.10-misol. $\alpha = (\sqrt{3} + i)^5$ hisoblansin.

Yechilishi.

$$\alpha_1 = \sqrt{3} + i = \sqrt{4}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

(1.14) formulaga asosan

$$\alpha = \alpha_1^5 = (\sqrt{3} + i)^5 = \left(2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\right)^5 = 2^5(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$$

1.11-misol. $\alpha = (1+i)^{21}$ ni hisoblang.

Yechilishi.

$$a_1 = 1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$$\alpha=\alpha_1^{21}=\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{21}=2^{10}\sqrt{2}\left(\cos\frac{21\pi}{4}+i\sin\frac{21\pi}{4}\right)=$$

$$=1024\sqrt{2}\left[\cos\left(5\pi+\frac{\pi}{4}\right)+i\sin\left(5\pi+\frac{4}{\pi}\right)\right]=1024\sqrt{2}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=-1024(1+i).$$

Quyidagi ifodalarni hisoblang.

1.6. $\frac{(1+i\sqrt{3})^3}{(\sqrt{3}-i)^2}+(1-i)(\sqrt{3}+i)$;

1.7. $\left(\frac{1-ictg\varphi}{1+ictg\varphi}\right)^3$;

1.8. $(1-i)(1+i)(\cos-i\sin\varphi)$;

1.9. $\left(\frac{\sqrt{3}+i}{1}\right)^2-(\sqrt{3}-(i)^3$.

6. Muavr formulasi. Ma'lumki,

$$\alpha^n=r^n(\cos n\varphi+i\sin n\varphi)$$

formuladagi $r-$ ixtiyoriy musbat sonlar bo'lgani uchun xususiy holda, $r=1$ deb faraz qilsak

$$\alpha^n=(\cos\varphi+i\sin\varphi)^n=\cos n\varphi+i\sin n\varphi$$

formula hosil bo'ladi. Bu formulaga Muavr formulasi deyiladi.

$(\cos\varphi+i\sin\varphi)^n$ - ni Nyuton binomi formulasiga asosan ochib chiqsak va i –larni darajalarini e'tiborga olib, o'sha tenglikning ikkala tomonidagi kompleks sonlarning haqiqiy va mavhum qismlarini tenglashtirib ushbu

$$\cos n\varphi=\cos^n\varphi-c_n^2\cos^{n-2}\varphi\sin^2\varphi+c_n^4\cos^{n-4}\varphi\sin^4\varphi-c_n^6\cos^{n-6}\varphi\cdot\sin^6\varphi+..\quad(1.16)$$

$$\sin n\varphi=n\cos^{n-1}\varphi\sin\varphi-c_n^3\cos^{n-3}\varphi\sin^3\varphi+c_n^5\cos^{n-5}\varphi\sin^5\varphi-......\quad(1.17)$$

formulaga ega bo'lamiz. Xususiy holda $n=2$ desak,

$$\cos 2\varphi=\cos^2\varphi-\sin^2\varphi,$$

$$\sin 2\varphi=2\cos\varphi\cdot\sin\varphi.$$

1.12-misol. $\cos 3\varphi$ va $\sin 3\varphi$ larni $\sin\varphi$ va $\cos\varphi$–larning darajalari bo'yicha yoyilmasi topilsin.

Yechilishi. (1.16) va(1.17) formulalarga asosan

$$\cos 3\varphi = \cos^3\varphi - 3\cos\varphi\cdot\sin^2\varphi,$$
$$\sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi.$$

7. Kompleks sondan ildiz olish. Agar $\alpha = a + ib$ kompleks sonning n–darajali ildizlarini anqlash lozim bo'lsa, dastlab uni trigonometrik shaklga keltirib olish zarur. Agar $\alpha = \beta^n$ bo 'lsa, β – son α – ning n – darajali ildizi deyilib, $\beta = \sqrt[n]{\alpha}$ ko 'rinishda yoziladi. β, α larni triganometrik ko 'rinishda yozamiz:

$$\alpha = r(cos\varphi + isin\varphi)\ ,\beta = \rho(cos\theta + isin\theta)\ .$$

U holda

$$\sqrt[n]{\alpha} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \rho(\cos\theta + i\sin\theta) \qquad (1.18)$$

θ va ρ larni topamiz . (1.18) ni n – darajaga oshiramiz, natijada

$$r(\cos\varphi + i\sin\varphi) = \rho^n(\cos n\theta + i\sin n\theta) \qquad (1.19)$$

hosil qilamiz.

(1.19) ikki tomonidagi kompleks sonlarning haqiqiy va mavhum qisimlarini tenglashtirsak,

$$\left.\begin{array}{l}\rho^n\cos n\theta = r\cos\varphi,\\ \rho^n\sin n\theta = r\sin\varphi,\end{array}\right\} \qquad (1.20)$$

hosil bo'ladi, bundan

$$\rho^n = r \Rightarrow \rho = \sqrt[n]{r} \qquad (1.21)$$

$$\cos n\theta = \cos\varphi = \cos(\varphi + 2k\pi) \Rightarrow n\theta = \varphi + 2k\pi \Rightarrow \theta = \frac{\varphi + 2k\pi}{n};\ k = 0,1,2.. \qquad (1.22)$$

(1.22) – dan aniq ko'rinib turibdiki, θ cheksiz ko'p qiymatlarga ega ekan.

Demak,

$$\sqrt[n]{\alpha} = \sqrt[n]{r}\left(\cos\frac{(\varphi + 2k\pi)}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)\cdot k = 0,1... \qquad (1.23)$$

1.13-misol. $\sqrt[4]{-2}$ -ni hisoblang.

Yechilishi. Dastlab ildiz ostidagi kompleks sonni trigonometrik shaklga keltirib olamiz: $-2 = 2(\cos\pi + i\sin\pi)$, u holda

$$\alpha_k = \sqrt[4]{2}\left(\cos\frac{(\pi + 2k\pi}{4}\right) + i\sin\frac{\pi + 2k\pi}{4} = \sqrt[4]{2}\left(\cos\frac{(1 + 2k)\pi)}{4} + i\sin\frac{(1 + 2k)\pi}{4}\right),$$

$$k = 0,1,2...$$

Misol uchun, $k = 0,1,2$ bo'lsin

$$\alpha_0=\sqrt[4]{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\quad \alpha_1=\sqrt[4]{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\alpha_2=\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$

1.14-misol. $\sqrt[3]{\frac{\sqrt{3}+i}{1-i\sqrt{3}}}$ ni hisoblang.

Yechilishi. Dastlab ildiz ostidagi kasrni trigonometrik shaklga keltirib olamiz:

$$\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right),\quad 1-i\sqrt{3}=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right),\ 1-i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right).$$

Bulardan

$$\frac{\sqrt{3}+i}{1-i\sqrt{3}}=\frac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{5}\right)}=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\cdot\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=$$

$$=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i$$

$$a_k=\sqrt[3]{\frac{\sqrt{3}+i}{1-i\sqrt{3}}}=\cos\frac{\left(\frac{3\pi}{2}+2\pi k\right)}{3}+i\sin\frac{\left(\frac{3\pi}{2}+2\pi k\right)}{3},\ k=0,1,2...$$

Ushbu misollarni yuqoridagi usul bilan yeching.

1.10. $\alpha=\sqrt[3]{1}$, 1.11. $\alpha=\sqrt[4]{1}$, 1.12. $\alpha=\sqrt[3]{1-i}$, 1.13. $\alpha=\sqrt[4]{\sqrt{3}+i\sqrt{3}}$

1.14. $\alpha=\sqrt[4]{(3+i)(1-3i/(4-2c))}$, 1.15. $\alpha=\sqrt[3]{(3+3i)(\sqrt{2}-2i}$

8. Kompleks sonning logorifmi. Elementar metematikada faqat musbat sonlarning logorifimlarini tekshirish bilan chegaralanadi. Kompleks argumentli funksiyalar nazariyasidan ma'lumki, noldan farqli boshqa har qanday kompleks sonning, shu jumladan, manfiy sonning logorifimi mavjud.

Berilgan

$$e^w=z \qquad (1.24)$$

tenglikni qanoatlantiruvchi har qanday w son z – sonning e asosli logorifii deyilib, quyidagicha yoziladi:

$$w=\ln z \qquad (1.25)$$

(1.24) dan ko'rinib turibdiki w – ning o'rniga har qanday son qo'ysak ham e^w hech vaqt 0 – ga teng bo'lmaydi.

Agar $w = u + iv, z = a + ib = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$ (bu yerda $e^{i\varphi} = \cos\varphi + i\sin\varphi -$ Eyler formulasi) ko'rinishda yozsak, (1.24) dan ushbu Eyler

$$e^{u+iv} = re^{i\varphi} \to e^{u} \cdot e^{iv} = re^{iv} \to e^{u} = r \to u = \ln r,\ e^{iv} = e^{i\varphi}, \quad v = \varphi + 2k\pi$$

Demak,

$$w = u + iv = \ln r + i(\varphi + 2k\pi)$$

yoki (1.25) ga asosan

$$Lnz = \ln r + i(\varphi + 2k\pi), \tag{1.26}$$

bunda $r = |z| = \sqrt{a^2 + b^2}$, $\varphi = \arg z$, $\varphi + 2k\pi = Argz$. Shuning uchun bazan (1.26) tenglik

$$Lnz = \ln|z| + iArgz \tag{1.27}$$

ko'rinishda ham yoziladi. Agar $k = 0$ bo'lsa, (1.26) dan $\ln r + i\varphi$ ni hosil qilamiz, unga logorifimning bosh qiymati deyiladi va $\ln z$ orqali belgilanadi.

$$\ln z = \ln|z| + i\varphi \tag{1.28}$$

(1.28) ni e'tiborga olib (1.27) formulani quyidagicha yozish mumkin:

$$Lnz = \ln z + 2k\pi i\ , k = 0, \pm 1, \pm 2$$

1.15-misol. (-4) - ning logorifimini toping.

Yechilishi. $|z| = |-4| = 4$, $\varphi = arctg\dfrac{0}{-4} = \pi$, $\ln r = \ln 4$.

$Ln(-4) = \ln 4 + (\pi + 2k\pi)i$

$Ln(-4) = \ln 4 + i(\pi + 2k\pi) = \ln 4 + i\pi(1 + 2k)$, bosh qiymati $\ln 4 + \pi i$.

1.16-misol. $1 + i\sqrt{3}$ ning logorifimini hisoblang.

Yechilishi. $z = 1 + i\sqrt{3}, |z| = 2$

$$\varphi = arctg\sqrt{3} = \frac{\pi}{3},\quad z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$Ln(1 + \sqrt{3}i) = \ln 2 + i\pi(2k + \frac{1}{3})$$

9. Umumiy daraja. Bu bo'limda kompleks sonni ixtiyoriy kompleks darajaga ko'tarish masalasi bilan shug' ullanamiz. (1.26) formuladan foydalanib, istalgan

$$w = z^{\alpha},\qquad z = x + iy \neq 0,\qquad \alpha = a + ib \tag{1.29}$$

darajani hisoblash formulasini keltirib chiqarish mumkin. Ma'lumki, har qanday $z = x + iy \neq 0$ sonni quyidagicha yozish mumkin. $z = e^{\ln z}$.

Buning ikkala tomonini α darajaga ko'taramiz

$$w = z^{\alpha} = e^{\alpha\, Lnz} = e^{\alpha(\ln r + i(\varphi + 2k\pi))} = e^{\alpha(\ln r + i\varphi)} \cdot e^{2k\pi i}, k = 0, \pm 1, \pm 2, \ldots$$

$k = 0$ da darajaning bosh qiymati

$$w_0 = e^{\alpha(\ln r + i\varphi)}. \qquad (1.30)$$

Demak, umumiy formula

$$w = z^{\alpha} = w_0 \cdot e^{2k\pi i}. \qquad (1.31)$$

1.17-misol. $w = (-1+i)^i$ - darajani hisoblang.

Yechilishi.

$$z = -1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}), \quad r = \sqrt{2}, \quad \varphi = \frac{3\pi}{4}$$

Demak,

$$w_0 = e^{i(\ln\sqrt{2} + \frac{3\pi}{4}i)} \qquad w = e^{\frac{-3\pi}{4}} e^{i\ln\sqrt{2}} e^{i2k\pi i} = e^{-\left(2k+\frac{3}{4}\right)\pi} \cdot e^{i\ln\sqrt{2}}$$

1.18-misol. $w = (-1-i)^{1+i}$ ni hisoblang.

Yechilishi. Ma'lumki,

$$z = -1 - i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \quad r = \sqrt{2}, \quad \varphi = \frac{5\pi}{4}.$$

Darajaning bosh qiymati

$$w_0 = e^{\alpha(\ln r + i\varphi)} = e^{(1+i)(\ln\sqrt{2} + i\frac{5\pi}{4})} = e^{(\frac{1}{2}\ln 2 - \frac{5\pi}{4}) + i\left(\frac{1}{2}\ln 2 + \frac{5\pi}{4}\right)},$$

$$w = w_0 \cdot e^{2k\pi i a} = w_0 \cdot e^{(1+i)2k\pi i} = w_0 \cdot e^{-2k\pi + 2k\pi i} = e^{\frac{1}{2}\ln 2 - \frac{5\pi}{4} - 2k\pi + i\left(\frac{1}{2}\ln 2 + \frac{5\pi}{4} + 2k\pi\right)} =$$

$$= \sqrt{2}e^{-\frac{5\pi}{4} - 2k\pi}\left(\cos(\frac{1}{2}\ln 2 + \frac{5\pi}{4} + 2k\pi) + i\sin(\frac{1}{2}\ln 2 + \frac{5\pi}{4} + 2k\pi)\right).$$

Mustaqil yechish uchun misollar

1.16. $w = (1-i)^{-1+i}$;

1.17. $w = (3+4i)^i$;

1.18. $w = (2+2i)^{+1+i}$;

1.19. $w = (1-\sqrt{3}i)^{-1+i}$;

1.20. $w = (-1-i)^i$;

1.21. $w = (1+i)^{1+i}$;

1.22. $w=(\sqrt{3}-i)^{1+i}$;

1.23. $w=(3+i\sqrt{3})^{-1+i}$;

1.24. $w=(5+4i)^{1-2i}$;

1.25. $w=(-\sqrt{3}+3i)^{2+i}$;

1.26. $w=\left(\frac{1-i}{\sqrt{3}}\right)^{1-i}$;

1.27. $w=(3-4i)^{1+i}$;

1.28. $w=\left(1+c0s\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{6}$;

1.29. $w=\left(1+\frac{\cos\pi}{4}+i\sin\frac{\pi}{4}\right)^{4}$;

1.30. $w=(-3-4i)^{1+i}$;

1.31. $w=\sqrt[3]{-4+3i}$;

1.32. $w=\left(\frac{1+i}{\sqrt{2}}\right)^{1+i}$;

1.33. $w=\left(\frac{-1+i}{\sqrt{2}}\right)^{1-i}$;

1.34. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{1+i}$;

1.35. $w=\left(\frac{1+i}{\sqrt{2}}\right)^{1+i}$;

1.36. $w=\left(\frac{-1+i}{\sqrt{2}}\right)^{1-i}$;

1.37. $w=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{1+i}$;

1.38. $w=(1+i)^{8}\cdot(1-\sqrt{3i})^{6}$;

1.39. $w=\sqrt[4]{-16}$;

1.40. $w=\sqrt[3]{-25}$;

1.41. $w=\frac{(-1+i)^{4}}{(\sqrt{-3}+i)^{5}}$;

1.42. $w=\left(\frac{i^{13}+2}{i^{23}+1}\right)^{2}$.

Nazorat savollari.

1. Kompleks son ta'rifi.
2. Kompleks sonning moduli va argumenti ta'rifi.
3. Kompleks sonning algebraik, triganometrik va ko'rsatkichli shaklini keltirib chiqaring.
4. Kompleks sonlar ustida amallarni misol misollarda ko'rsating.
5. Muavr formulasi.
6. Kompleks sondan ildiz chiqarishni misollar bilan tushuntirib bering.

II-BOB. KOMPLEKS O'ZGARUVCHAN FUNKSIYA

2.1. Soha tushunchasi. Jordan chizig' i.

Kompleks sonlarning biror E to'plami berilgan bo'lin. E kompleks (z) tekislikdagi nuqtalar to'plamidan iborat.

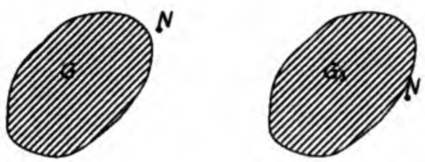
2.1- ta'rif. M nuqtani ε atrofi ($\varepsilon > 0$) deb, markazi M nuqtada bo'lgan ε radiusli doiraga aytiladi.

2.2-ta'rif. Agar M nuqta o'zining ε atrofi bilan E to'plamga tegishli bo'lsa, uni E ning ichki nuqtasi deyiladi.

2.3 -ta'rif. Agar E to'plam faqat ichki nuqtalardan iborat bo'lsa va E to'plamning har qanday ikkinchi nuqtasuni shunday siniq chiziq, bilan birlashtirish mumkin bo'lsinki, o'sha chiziqning barcha nuqtalari E to'plamga tegishli bo'lsa, bunday nuqtalar to'plamiga soha deyiladi. Soha odatda E, G, Z harflar bilan belgilanadi.

Agar tekislikda biror G soha berilgan bo'lsa, u holda (z) tekislikning barcha nuqtalari G - ga nisbatan ikki sinfga bo'linadi.

Brirnchi sinf deb G ning barcha nuqtalarini, ikkinchi sinf deb tekislikning G ga tegishli bo'lmagan nuqtalarini qabul qilinadi.



2.1-chizma. a)N nuqtaning G sohaga tegishliligi.
b) N nuqtaning G sohaga tegishli emasligi.

a)shaklda N nuqta G sohaga nisbatan tashqi nuqta deyiladi.
b)shaklda N nuqta G sohaning chegaraviy nuqtasi deyiladi.
G sohadan va uning chergasidan iborat bo'lgan nuqtalar to'plami yopiq soha deyiladi va $\overline{G}$ ko'rinishda belgilanadi.

Jordan chizig' i

Bizga t argumentli $x(t), y(t)$ haqiqiy va uzluksiz funksiyalar berigan bo'lib, $\alpha \le t \le \beta$ bo'lsin. U holda quyidagi ikkita tenglama

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \alpha \le t \le \beta$$

tekislikdagi uzluksiz egri chiziqni ifodalaydi. Agar t – parametrning ikki xil qiymatiga ($t=\alpha$ va $t=\beta$ qiymatlaridan boshqa) egri chiziqning ikki turli nuqtasi mos kelsa, u holda egri chiziq o'z – o'zini kesmaydi, ya'ni karrali nuqtaga ega bo'lmaydi, ana shunday xususiyatga ega bo'lgan egri chiziq Jordan chizig' i yoki uzluksiz chiziq deb aytiladi.

Agar biz kompleks o'zgaruvchi miqdorni $z=x+iy$ orqali belgilasak $z(t)=x(t)+iy(t)$hosil bo'lib, egri chiziqni birgina tenglama bilan ifoda qilish ham mumkin, ya'ni $z=z(t)$ $(\alpha \le t \le \beta)$. Parametr t berilgan $[\alpha,\beta]$ - segmentida o'sib borishi bilan z nuqta Jordan chizig' ini chizib uning boshlang' ich nuqtasi $z(\alpha)$ va oxirgi nuqtasi $z(\beta)$ lardan iborat bo'ladi. Shu bilan chiziqda ma'lum yo'nalishi ham belgilanib qoladi. Odatda bu yo'nalishni musbat yo'nalish deb olinadi.

Agar $z(\alpha)=z(\beta)$ bo'lsa, egri chiziqning boshlang' ich va oxirgi nuqtalari birlashib ketadi. Ya'ni yopiq chiziq hosil bo'ladi. Bu holda Jordan chizig' i tekislikni ikki qismga ajratadi. Ulardan biri ypiq chiziq ichidagi qismi bo'lsa, ikkinchisi–tekislikning qolgan barcha nuqtalaridan iboratdir.

Jordanning yopiq chizig' i ichida yotgan sohaning ajoyib xususiyati shundan iboratki, shu sohada har qanday yopiq uzluksiz chiziq olsak ham uning ichi o'sha berilgan sohaga tegishli bo'lib qoladi.

Shunday xususiyatga ega bo'lgan har qanday soha bir bog' lamli, aks holda, ko'p bog' lamli soha deb ataladi. Sohalarni ko'pincha tengsizliklar orqali berish mumkin.

Agar tekislikdagi chizmaning tenglamasi

$$y = f(x) \quad (a \le x \le b) \tag{2.1}$$

ko'rinishda berilgan bo'lsa, $z=x+iy$ va $\bar{z}=x-iy$ lardan $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$ qiymatlarni (2.1) ga qo'yib

$$\frac{z-\bar{z}}{2i}=f\left(\frac{z+\bar{z}}{2}\right) \qquad (2.2)$$

kompleks ko'rinishdagi tenglama keltiriladi.

2.1-misol. Ushbu $|z-a|<R$ tengsizlik qanday sohani hosil qiladi?

Yechilishi. Bunda $\alpha=a+ib, z=x+iy,\ z-\alpha=(x-a)+i(y+b)$ –ni topamiz.

Demak,

$$|z-\alpha|=|(x-a)+i(y-b)|=\sqrt{x-a)^2+(y-b)^2}<R,$$

bundan

$$(x-a)^2+(y-b)^2<R^2,$$

bu esa markazi (a,b) nuqtada joylashgan va radiusi R–ga teng bo'lgan aylana ichidagi nuqtalardan iborat.

2.2-misol. Ushbu $\operatorname{Re} z^2=a^2$ tenglamani qanoatlantiruvchi nuqtalar to'plami qanday chiziqni aniqlaydi?

Yechilishi.

$$z^2=(x+iy)^2=x^2-y^2+2xyi,$$

$$\operatorname{Re} z^2=x^2-y^2=a^2.$$

Bu teng yonli giperbolaning tenglamasidir.

2.3-misol. Ushbu

$$|z+i|=|z+2|$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi. Ma'lumki,

$$|z+i|=|x+iy+i|=|x+i(y+1)|=\sqrt{x^2+(y+1)^2},$$

$$|z+2|=|x-2+iy|=\sqrt{(x-2)+y^2}.$$

Bularga asosan

$$\sqrt{x^2+(y+1)^2}=\sqrt{(x-2)^2+y^2}\Rightarrow x^2+y^2+2y+1=x^2-4x+4+y^2$$

$$\Rightarrow 4x+2y-3=0$$

to'g' ri chiziq tenglamasi kelib chiqadi.

2.4-misol. Ushbu

$$\arg(z+i)=\frac{\pi}{4}$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi. Ma'lumki

$$z+i=x+i(y+1),\ tg\alpha=\frac{y+1}{x}=tg\frac{\pi}{4},\ y+1=x$$

$y-x+1=0$, to'g' ri chiziq $z_0=-i$ nuqtadan o'tib ox o'qining musbat yo'nalishi bilan $\pi/4$ burchak hosil qiladi.

2.5-misol. Ushbu

$$\begin{cases} 3\le|z+1-2i|\le 4 \\ \frac{\pi}{2}\le\arg z\le\pi \end{cases}$$

tengsizliklarni qanoatlantiruvchi nuqtalar to'plamini aniqlang.

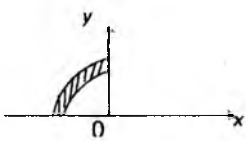
Yechilishi.

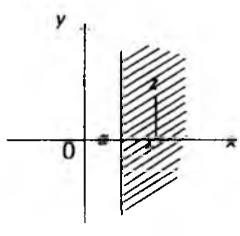
$$|z+1-2i|=|(x+1)+i(y-2)|=\sqrt{(x+1)^2+(y-2)^2}$$

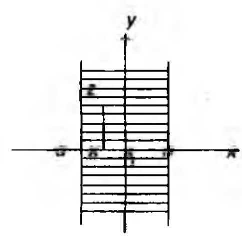
U holda

$$3\le\sqrt{(x+1)^2+(y-2)^2}\le 4\Rightarrow 9\le(x+1)^2+(y-2)^2\le 16$$

kelib chiqadi. Bu esa radiuslari $r=3$, R=4 va markazi $z_0=-1+2i$ nuqta bo'lgan konsentrik aylanalar orasidagi xalqani ifodalaydi. Shartaga asosan $\frac{\pi}{2}\le\varphi\le\pi$ bo'lgani uchun xalqani II chi chorakda yotgan qismi olinadi (2.2-chizma).







2.2-chizma. $\begin{cases} 3\le|z+1-2i|\le 4 \\ \frac{\pi}{2}\le\arg z\le\pi \end{cases}$ **tasviri. 2.3-chizma.** $\mathrm{Re}\,z=x\ge a$ **tasviri. 2.4-chizma.** $-a<\mathrm{Re}\,z<a$ **tasviri.**

2.6-misol. Ushbu $\mathrm{Re}\,z=x\ge a$ tekislik qanday sohani ifodalashini aniqlaymiz, bunda $a>0$.

Yechilishi. Demak, tengsizlik $x=a$ to‘g‘ ri chiziqning o‘ng tomonini ifoda qilib chegara ham shu to‘plamga kiradi (2.3–chizma).

2.7-misol. Ushbu $-a<\mathrm{Re}z<a$ tengsizliklar qanday sohani ifodalaydi?

Yechilishi. Bu $x=-a$ va $x=a$ to‘g‘ ri chiziqlar orasidagi sohani bildiradi. (2.4 – chizma)

2.8-misol. Ushbu $|z|\le 2+\mathrm{Re}\,z$ tengsizlik qanday sohani ifodalaydi?

Yechilishi. $|z|=|x+iy|=\sqrt{x^2+y^2}$, $\mathrm{Re}z=x$ bo‘lgani uchun

$$\sqrt{x^2+y^2}\le 2+x \Rightarrow x^2+y^2\le 4+4x+x^2 \Rightarrow y^2\le 4+4x$$

Bu esa parabola ichidagi nuqtalar to‘plamidan iborat bo‘lib chegaraviy nuqtalari ham sohaga tegishli. (2.5 – chizma)

2.9-misol. Ushbu $\varphi_1<\arg z<\varphi_2,\ 3\le \mathrm{Re}\,z\le 5$ tensizliklarga tegishli nuqtalar to‘plami aniqlansin, bunda

$$\frac{-\pi}{2}\le\varphi_1\le 0, \qquad 0\le\varphi_2<\frac{\pi}{2}$$

Yechilishi. Ma’lumki, z ning Dekart koordinatalari Sistema-sidagi ko‘rinishi $z=x+iy$dan, qutb koordinatalari sistemesidagi ko‘rinishi esa $z=r(\cos\varphi+i\sin\varphi)$ dan iborat. Shu sababli

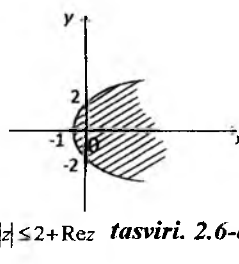
$$\arg z=\varphi\ , \quad \mathrm{Re}z=x.$$

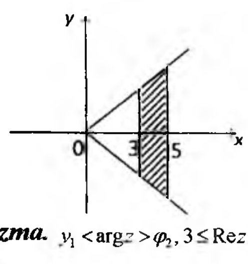
Berilgan tensizliklarga asosan:

a) $\varphi_1<\varphi<\varphi_2$ lar tengsizlikning $\varphi=\varphi_1$ *va* $\varphi=\varphi_2$ nurlar orasidagi bir qismini bildiradi;

b) $3\le x\le 5$esa $x=3$ va $x=5$ to‘g‘ ri chiziqlar orasidir.

Mana shularga asosan biz izlayotgan to‘plam 2.6–chizmada ko‘rsatilgan trapetsiya ichidagi nuqtalar to‘plamidan iborat bo‘lib, terapetsiyaning parallel tomonlari bu to‘plamga kiradi.





2.5-chizma. $|z|\le 2+\mathrm{Re}z$ ***tasviri. 2.6-chizma.*** $y_1<\arg z>\varphi_2, 3\le \mathrm{Re}z\le 5$***tasviri.***

2.10-misol. Ushbu

$$2z\bar{z} + (2+i)z + (2-i)\bar{z} = 2$$

tenglama qanday chiziqni ifodalaydi?

Yechilishi. Ma'lumki,

$$2z\bar{z} = 2(x+iy)(x-iy) = 2(x^2+y^2),$$

$$(2+i)(x+iy) + (2-i)(x-iy) = 4x - 2y$$

Shularga asosan berilgan tenglama

$$2(x^2+y^2) + 2(2x-2y) = 2$$

$$(x+1)^2 + (y-\tfrac{1}{2})^2 = \frac{5}{4}$$

ko'rinishdan iborat bo'lib, markazi $(-1; \tfrac{1}{2})$ nuqtada joylashgan radiusi $r = \frac{\sqrt{5}}{2}$ teng bo'lgan aylana tenglamasini ifodalaydi.

2.11-misol. Aylananing

$$x^2 + y^2 + 2x - 2y = 0$$

tenglamasini kompleks shaklga keltiring.

Yechilishi. Ma'lumki,

$$x^2 + y^2 = (x+iy)(x-iy) = z \cdot \bar{z},$$

$$2x = z + \bar{z}, \qquad 2y = \frac{z-\bar{z}}{i} = i(\bar{z} - z).$$

Demak,

$$z\bar{z} + z + \bar{z} - i(\bar{z} - z) = 0 \Rightarrow z\bar{z} + (z+i)z + (1-i)z = 0.$$

Mustaqil ishlash uchun misollar.

2.1. $1 < \operatorname{Im}(i - z) < 3$;

2.2. 2.2. $\arg(z + 2i) = \frac{\pi}{3}$;

2.3. $\operatorname{Im}(z^2 - z) = 2 - \operatorname{Im} z$;

2.4. $|z| - \operatorname{Im} z = 6$;

2.5. $|z-1| < |z-i|$;

2.6. $1 < \operatorname{Im} iz < 3$;

2.7. $\operatorname{Im}\left(\frac{1}{z}\right) < -\frac{1}{2}$;

2.8. $\left|\frac{z-3}{z-2}\right| \geq 1$;

2.9. $-\frac{3\pi}{2} < \arg(z-1+i) < \frac{2\pi}{3}$;

2.10. $\operatorname{Re} z^2 = 5$;

2.11. $\operatorname{Re}\frac{1}{z} = 5$.

2.12. $\operatorname{Im}\frac{1}{z} = 5$;

2.13. $\operatorname{Im}\frac{z-2}{z-3} = 0$, $\operatorname{Re}\frac{z-2}{z-3} = 0$;

2.14.2.14. $|z-2| - |z+2| > 3$;

2.15. $\operatorname{Re}\frac{z-4}{z+i} = 0$; $\operatorname{Im}\frac{z+4}{z+i} = 0$;

2.16. $|z-2i| = |z-3+2i|$;

2.17. $|z-4+3i| = |z+2-3i|$;

2.18. $\frac{\pi}{3} < \arg z < \frac{\pi}{2}$;

2.19. $\frac{\pi}{3} < Argz < \frac{2\pi}{3}$;

2.20. $|z-2+3i| < 3$;

2.21. $1 < |z-1-i1| < 3$;

2.22. $1 < |z-2i| < 2$;

2.24. $|z-i| < 5$;

2.25. $|z+3i| \geq 4$ Z;

2.26. $\operatorname{Re} z + Ymz < 1$;

2.27. $|z-3-4i| = 2$;

2.28. $|z-2| + |z+2 = 5|$;

2.29. $\operatorname{Im} z = a$;

2.30. $\operatorname{Im} z^2 = a$

2.2. Ketma-ketliklar va ularning limitlari

Bu paragrafda kompleks sonlardan tuzilgan ketma-ketliklarni o'rganamiz.

Agar $1,2,3,\ldots,n$ natural sonlarning har biriga bittadan kompleks son mos kelsa u holda quyidagi

$$z_1, z_2, z_3 \ldots z_n \qquad (2.3)$$

cheksiz ketma ketlik hosil bo'ladi. Bundagi z_n kompleks son natural n songa mos keladi $(n=1,2,3,.....)$. Endi $z_n = x_n + iy_n$ bo'lib, x_n va y_n - lar haqiqiy sonlardan iborat bo'lgani uchun (2.3) – ga asosan quyidagi ikkita haqiqiy ketma – ketlikni tuzib
olish mumkin:

$$x_1, x_2, x_3 ...... x_n \tag{2.4}$$

va

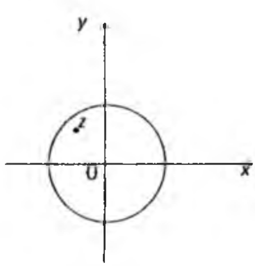
$$y_1, y_2, y_3 ...... y_n \tag{2.5}$$

Aksincha, agar (2.4) , (2.5) ketma –ketlik berilgan bo'lsa ularga asoslanib (2.3) – ni tuzish mumkin. Buning uchun (2.5) da hadlarni i – songa ko'paytirib (2.4) dagi mos hadlarga qo'shish kifoya.

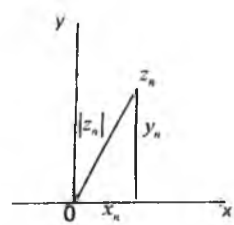
Berilgan (2.3) ketma – ketlikni qisqacha $\{z_n\}$ ko'rinishda yozish qabul qilingan. Agar $\{z_n\}$ ketma - ketlikning har bir hadining moduli biror musbat sondan kichik bo'lsa, ya'ni shunday $M > 0$ son mavjud bo'lsaki, barcha $\{z_n\}$ lar uchun

$$|z_n| < M \ (n = 1,2,3......) \tag{2.6}$$

bo'lsa, u holda ketma – ketlik chegaralangan deyiladi. (2.6) tengsizlik esa berilgan ketma – ketlikning barcha elementlari markazi koordinatalar boshida bo'lgan M radiusli doira ichida joylashganligini bildiradi. Demak, $\{z_n\}$ ning chegaralangan bo 'lishi uchun $z_1, z_2, ..., z_n, ...$ nuqtalarning hammasi biror chekli radiusli doira ichiga joylashmog' i kerak (2.7-chizma).



2.7-chizma. Chekli radiusli aylana.



2.8-chizma. To'g' ri burchakli uchburchak

$|x_n|$ va $|y_n|$ lar uchburchak katetlarining, $|z_n|$ esa uchburchak gipotenuzasining uzunliklari ekani 2.8–chizmadan ko'rinib turibdi. Shu sababli quyidagi tengsizliklarga ega bo'lamiz:

$$|x_n| \le |z_n|, \ |y_n| \le |z_n| \ \text{va} \ |z_n| \le |x_n| + |y_n|$$

Bu tengsizliklardan quyidagi xulosalar kelib chiqadi.

a)Agar $\{z_n\}$ ketma –ketlik chegaralangan bo 'lsa, u holda $\{x_n\}$ va $\{y_n\}$lar ham chegaralangan bo 'ladi.

b)Aksincha, agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar chegaralangan bo 'lsa, u holda $\{z_n\}$ ham chegaralangan bo'ladi.

Haqiqattan ham,

$$|x_n| < M_1, |y_n| < M_2$$

tengsizliklardan

$$|z_n| \le |x_n| + |y_n| < M_1 + M_2 = M$$

kelib chiqadi.

2.12-misol. Ushbu ketma – ketlik berilgan:

$$1 - i, \frac{1}{2} - i\frac{1}{2}, \ \frac{1}{3} - \frac{1}{3}i, \ldots, \frac{1}{n} - \frac{1}{n}i, \ldots$$

$$z_n = \frac{1}{n} - \frac{1}{n}i, \quad |z_n| = \left|\frac{1}{n} - \frac{1}{n}i\right| = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{2}{n} \le \sqrt{2} \Rightarrow |z_n| \le \sqrt{2}$$

Demak, berilgan ketma – ketlik chegaralngan ekan. Bundan kelib chiqadigan ushbu

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

$$-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}, \ldots$$

ketma ketliklar ham chegaralangandir, haqiqatdan ham,

$$|x_n| = |y_n| = \left|\pm\frac{1}{n}\right| = \frac{1}{n} \le 1 \quad n = 1, 2 \ldots$$

Ketma – ketliklarning limiti.

$\{z_n\}$ ketma – ketlik berilgan bo'lsin.

2.4-ta'rif. $\forall \varepsilon > 0. \exists N(\varepsilon)$ sonni topish mumkin bo'lsaki, $n > N(\varepsilon)$ bo'lganda

$$|z_n - z_0| < \varepsilon$$

tengsizlik o'rinli bo'lsa, u holda z_0 son berilgan ketma – ketlikning limiti (chekli) deyilib,

$$\lim_{n\to\infty} z_n = z_0 \text{ yoki } z_n \to z_0$$

kabi yoziladi.

2.1-teorema. Har qanday yaqinlashuvchi ketma – ketlik chegaralangandir.

2.3. Kompleks o'zgaruvchining funksiyasi

Agar har qanday kompleks sonning biror E to'plamga tegishli yoki tegishli emasligini ko'rsatadigan usul bizga ma'lum bo'lsa, u holda kompleks sonlarning E – to'plami berilgan deyiladi.

Misol uchun E to'plam $|z|<2$ doira ichidagi barcha $z=x+iy$ nuqtalardan iborat bo'lsa u holda $\frac{\sqrt{2}}{2}+\frac{2}{\sqrt{3}}i$ son E to'plamga tegishli, chunki

$$\left|\frac{\sqrt{2}}{2}+\frac{2}{\sqrt{3}}i\right| = \sqrt{\frac{2}{4}+\frac{4}{3}} = \sqrt{\frac{11}{6}} < 2.$$

$2+\frac{\sqrt{3}}{2}i$ son E to'plamga tegishli emas, chunki

$$\left|2+\frac{\sqrt{3}}{2}i\right| = \sqrt{4+\frac{3}{4}} = \sqrt{19}/2 > 2.$$

ya'ni $2+\frac{\sqrt{3}}{2}i$ nuqta doira tashqarisida yotadi.

2.5-ta'rif. Agar E to'plamdan olingan har bir $z=x+iy$ songa biror qonun bo'yicha tayin bir $w=u+iv$ kompleks son mos kelsa, u holda E to'plamda funksiya berilgan deyiladi va $w=f(z)$ ko'rinishda yoziladi.

Demak, bu ta'rifdan ko'rinadiki, $z=x+iy$ argument, ya'ni erkli o'zgaruvchi, $w=u+iv$ esa uning funksiyasidir. E to'plamdan har bir son z- argumentining qiymatidan iborat bo'lib, u to'plam $w=f(z)$ funksiyasining aniqlanish (berilishi) sohasi deyiladi. Agar z-ning har bir qiymatiga w - ning birgina qiymati mos kelsa, $w=f(z)$ bir qiymatli, aks holda ko'p qiymatli funksiya deyiladi.

Masalan, $w=\frac{1}{z}, w=5z^2, w=z^2-3z$ funksiyalar bir qiymatli bo'lib,

$$w=\sqrt{z}, w=\sqrt[3]{z^2}, w=\frac{1}{\sqrt[3]{Z}}, w\frac{1}{\sqrt{z+2}}$$

esa ko‘p qiymatli funksiyalardir. Ma’lumki, agar z berilgan bo‘lsa, x va y - lar berilgan bo‘ladi. $w=f(z)$ berilgan bo‘lsa, u va v lar berilgan demakdir. Ravshanki u va v lar ham x va y larning funksiyalaridir:

$$w = f(z) = u(x,y) + iv(x,y). \tag{2.7}$$

Shunday qilib,

$$\left.\begin{aligned} u &= u(x,y), \\ v &= v(x,y), \end{aligned}\right\} \tag{2.8}$$

ya’ni bitta $w=f(z)$ munosabat (2.8) ikkita munosabatga ekvivalentdir. Biz kelgusida funksiyani quyidagi ikki xil ko‘rinishda ham ishlatamiz:

$$w=f(z) \text{ va } w=u(x,y)+iv(x,y), (z=x+iy)$$

Agar bizga kompleks o‘zgaruvchili elementar funksiyalari berilgan bo‘lsa, ularni birinchi ko‘rinishdan ikkinchi ko‘rinishga sodda amallar yordami bilan o‘tkazib, $f(z)$ funksiyaning haqiqiy va mavhum qisimlarini ajratish mumkin.

2.13-misol. $w=z^3$ berilgan bo‘lsin, bundan

$$u+iv=w=z^3=(x+iy)^3=x^3-3xy^2+3x^2iy-iy^3=x^3-3xy^2+i(3x^2y-y)^3,$$

$$u=x^3-3xy^2; v=3x^2y-y^3$$

hosil bo‘ladi. Berilgan bu funksiyaning aniqlanish sohasi E to‘la kompleks tekislikdan iboratdir.

2.14-misol. $w=\frac{1}{z}$ berilgan bo‘lsin, bundan

$$w=\frac{1}{z}=\frac{1}{x+iy}=\frac{x-iy}{(x+iy)(x-iy)}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$

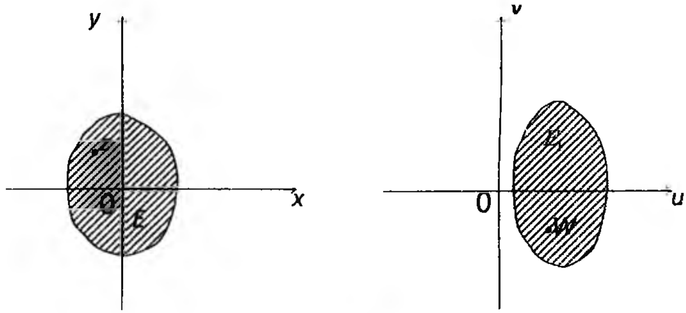
kelib chiqadi, bunda

$$u=\frac{x}{x^2+y^2}, v=-\frac{y}{x^2+y^2}$$

ga teng. Bu funksiyaning aniqlanish sohasi tekkislikning, noldan boshqa hamma nuqtalardan iborat, chunki. $w=\frac{1}{z}$ da $z\neq 0$.

Har bir kompleks sonnga tekislikda aniq bitta nuqtaning mos kelishidan foydalanib, kompleks o‘zgaruvchi funksiyaning geometrik ma’nosini aniqlaymiz. Buning uchun z ning qiymatlariga tegishli nuqtalarini (z) tekislikka, $w=f(z)$ funksiyaning qiymatlariga tegishli nuqtalarni (w) tekislikka joylaymiz. U holda (z) tekislikdagi

E to‘plamdan olingan har bir z nuqtaga w tekislikdagi biror w nuqta mos keladi. Natijada E to‘plamning aksi (w) tekislikka tushib biror E_1 to‘plamni hosil qiladi (2.9 – chizma).



2.9-chizma. E to‘plamni E_1 to‘plamga akslantirish.

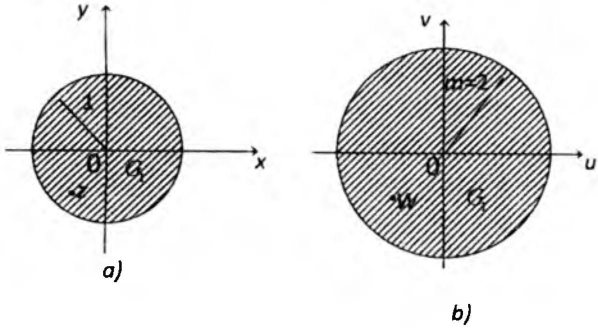
Shunday qilib, $w=f(z)$ funksional munosabat yordami bilan (z) tekislikdagi E to‘plamni (w) tekislikdagi E_1 to‘plamga ko‘chirar ekanmiz. Bu esa E to‘plamni E_1 to;plamga aks ettirish (yoki akslantirish) deyiladi.

Agar $w=f(z)$ funksiya E to‘plamda bir qiymatli funksiya bo‘lsa va E ning ikkita turli nuqtasiga E_1 ning hamma vaqt ikkita turli nuqtalari mos kelsa, (ya’ni $z_1 \neq z_2$, $f(z_1) \neq f(z_2)$), u holda akslantirish o‘zaro bir qiymatli, $f(z)$ esa E sohada bir yaproqli funksiya deyiladi.

Odatda E_1 to‘plam E ning aksi (пробрази), E esa E_1 ning asli (obrazi) deyiladi.

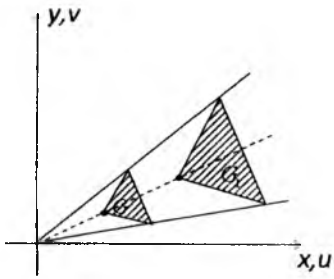
2.15-Misol. Ushbu $w=mz(m>0)$ funksiya vositasi bilan bajaraladigan akslantirishni tekshiraylik.

Yechilishi. E deb $|z|\leq 1$yopiq birlik doirani olaylik, bu esa bizga G_1 to‘plamni beradi. U holda $|w|=|mz|=m|z|\leq m$ ga ega b’lamiz, bu esa $\overline{G_1}$ ni tasviri m radiusli doiradan iboratdir (2.10-chizma a). Biz 2.10 b) – chizmada $m=2$ deb oldik.



2.10-chizma. $w = mz (m > 0)$ ***funksiya yordamida bajarilgan akslantirishlar.***

Xususiy holda $|z| = 1$ aylana образи $|w| = m$ aylanadan iborat bo‘ladi:



2.11-chizma. $w = mz (m > 0)$ ***funksiya yordamida uchburchakni akslantirish.***

Agar G_1 soha sifatida 2.11-chizmadagi uchburchakni olsak uning tasvirini topish uchun quyidagicha muloxaza qilamiz. Dastlab $z = x + iy$ va $w = u + iv$ sonlarni trigonometrik shaklda yozib olamiz:

$$\left.\begin{aligned} z &= r(\cos\varphi + i\sin) \\ w &= \rho(\cos\theta + i\sin\theta) \end{aligned}\right\} \quad (2.9)$$

Uni $w = mz$ ga qo‘yamiz

$$mz = w = \rho(\cos\theta + i\sin\theta) = mr(\cos\varphi + i\sin\varphi).$$

Bundan

$$\rho = mr, \quad \theta = \varphi \quad (2.10)$$

hosil bo‘ladi.

Endi z va w sonlarning har biriga bittadan vektor mos keladi deb qaraymiz. U holda (2.10) ga ko'ra z ning tasvirini topish uchun z ga tegishli vektorlarning r uzunligini m marta cho'zish kerak. Lekin vehtor burilmaydi, chunki $\theta=\varphi$ (qulay bo'lishi uchun xoy va uow koordinatalar sistemalarini bitta qilib oldik).

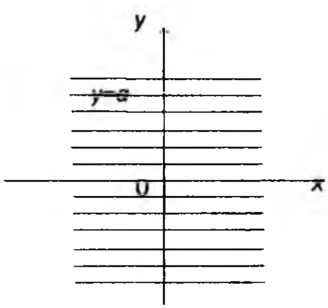
Agar $0<m<1$ bo'lsa, G uchburchakning aksi G_1 kichrayib qolishi ayon.

2.16-misol. Berilgan $w=z^2$ funksiya yordami bilan (z) tekislikdagi ox o'qqa parallel bo'lgan $y=a$ chiziqlar oilasining (2.12–chizma) (w) tekislikdagi aksi qanday chiziqlardan iborat bo'lishi aniqlansin (a- haqiqiy son).

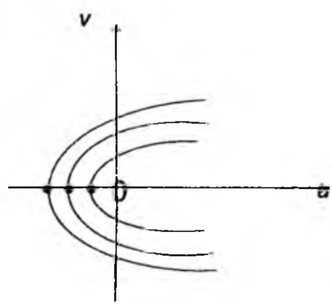
Yechilishi. Ma'lumki, $z=x+iy, w=u+iv$, va $w=z^2$ dan

$$u=x^2-y^2, v=2yx,$$

(2.10) tengliklar kelib chiqadi. Bularga $y=a$ - ni qo'yib chiqsak, $u=x^2-a^2, v=2ax \Rightarrow u=\frac{v^2}{4a^2}-a^2$ kelib chiqadi. Parametr a - ga turli qiymatlar berib haqiqiy o'qqa simmetrik parabolalar oilasini hosil qilish mumkin (2.13 – chizma). Demak, (z) tekislikdagi $y=a$ to'g'ri chiziqlar oilasi $w=z^2$ (w) tekislikdagi $u=\frac{v^2}{4a^2}-a^2$ parabolalar oilasiga akslantiradi. $y=a$ chiziqni akslantirish.



2.12 – chizma. $w=z^2$ funksiya yordamida



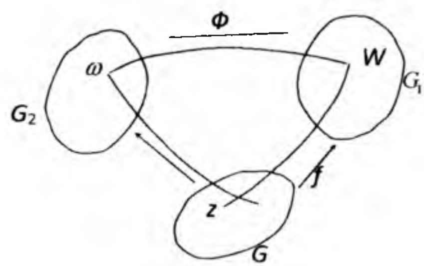
2.13– chizma. Parabolalar oilasi.

2.4. Murakkab va teskari funksiyalar

Berilgan $w=f(z)$ funksiya vositasi bilan uning aniqlanish sohasi Gning biror G_1 sohaga aks ettirilishi bizga ma'lum. Bundan tashqari G_1 sohada $\omega=\phi(w)$ berilgan bo'lsa uning yordami bilan G_1 ham biror G_2 sohaga aks ettirilgan bo'ladi (2.14-chizma). U holda

$$\omega=F(z)=\phi[f(z)] \qquad (2.11)$$

murakkab funksiya deyiladi, uning vositasida G_1 soha G_2 ga akslantiriladi.



2.14-chizma. Murakkab funksiyaning sohalarni akslantirilishi.

2.17-misol. $\omega=w^3$, $w=\frac{1}{z}$ funksiyalar berilgan bo'lsin. Bunda ω murakkab funksiya bo'lib,

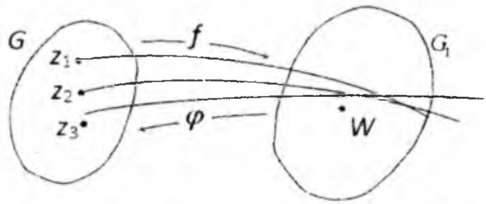
$$\omega=w^3=\frac{1}{z^3}=z^{-3}.$$

Faraz qilaylik, G sohani G_1 sohaga akslantiruvchi $w=f(z)$ funksiya berilgan bo'lsin. Umuman olganda ayrim hollarda G sohadagi bir necha nuqtalarga G_1 sohada bitta mos kelib qolishi ham mumkin.

Aksincha, agar $z=\varphi(w)$ funksiya yordami bilan G_1 sohadagi har bir w nuqta G sohaga aks ettirilganda u nuqtaga $w=f(z)$ vositasi bilan w da z nuqtalar mos kelsa, u holda

$$z=\varphi(w) \qquad (2.12)$$

funksiyaga berilgan $w=f(z)$ funksiyaga nisbatan teskari funksiya deyiladi (2.15-chizma).



2.15-chizma. w nuqtaning teskari funksiya yordamida aksi.

Misollar.

1. To'g' ri funksiya $w=\frac{z}{z+i}(z\neq i)$berilgan. Buni z ga nisbatan yechsak

$$z=\frac{iw}{1-w}\quad(w\neq 1)$$

teskari funksiya hosil bo'ladi. Bu misolda ikkala funksiya ham bir qiymatlidur. Shu sababli

$$\varphi[f(z)]=\frac{i(\frac{z}{z+i})}{1-\frac{z}{z+i}}=\frac{iz}{z+i-z}=z.$$

2. $w=\frac{z^3}{3}$to'g' ri funksiya berilgan. Buni z ga nisbatan yechsak $z=\sqrt[3]{3w}$ teskari funksiya kelib chiqadi. Bu misolda to'g' ri funksiya bir qiymatli bo'lib, teskari funksiya to'ri qiymatli funksiyadir.
3. Berilgan $w=\sqrt[3]{z}$ uch qiymatli bo'lib, buning teskari $z=w^3$ funksiyasi bir qiymatlidir.

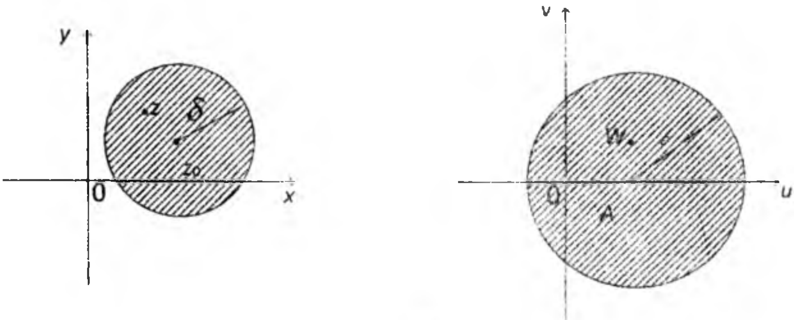
2.5. Funksiyaning limiti va uzluksizligi

Tekislikdagi biror G sohada aniqlangan bir qiymatli $W=f(z)$ funksiya berilgan bo'lib, z_0 nuqta G ning limit nuqtasi bo'lsin. Biz $f(z)$ funksiyaning z o'zgaruvchi z_0 (qo'zg' almas) nuqtaga intilgandagi intilishini topish bilan shug' ullanamiz.

2.6-ta'rif. $\forall(\varepsilon>0)\ \exists\delta(\varepsilon)>0\ \forall|z-z_0|<\delta$ tengsizlikni qanoatlantiruvchi hamma $z(z\neq z_0)$ lar uchun $|f(z)-A|<\varepsilon$ tengsizlik o'rinli bo'lsa, A-son $f(z)$ funksiyaning $z\to z_0$ dagi limiti deyiladi va

$$\lim_{z\to z_0} f(z)=A \qquad (2.13)$$

ko'rinishida yoziladi (2.16-chizma).



2.16-chizma. *z_0 nuqtaning δ va $w_0=f(z_0)$ nuqtaning ε atroflari.*

$A=B+iC$, $f(z)=u(x,y)+iv(x,y)$ va $z_0=x_0+iy_0$ bo'lsin

U holda (2.13) kompleks munosabatning quyidagi ikki haqiqiy munosabatlarga ekvivalent ekanligini ko'rish mumkin:

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)=B,\quad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y)=C$$

$\lim\limits_{z\to\infty} f(z)=A$ yoki $\lim\limits_{z\to z_0} f(z)=\infty$ ko'rinishdagi limitlarga ham yuqorida keltirilgan 2.1-ta'rifga o'xshash ta'rif berish mumkun.

$w=f(z_0)$ funksiya G sohada aniqlangan bir qiymatli funksiya va $z_0\in G$ bo'lsin. U holda $w_0=f(z_0)$ aniq qiymatga ega bo'ladi.

2.7-ta'rif. Agar $\forall(\varepsilon>0)\exists(\delta>0)$ topish mumkin bo'lsaki $|z-z_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha z lar uchun

$$|f(z)-f(z_0)|<\varepsilon \qquad (2.14)$$

tengsizlik o'rinli bo'lsa, u holda $w=f(z)$ funksiya $z=z_0$ nuqtada uzluksiz deyiladi.

Uzluksizlikning ortirmalar bo'yicha ta'rifi quyidagicha:

$z-z_0=\Delta z$, $f(z)-f(z_0)=\Delta w$ desak, $\lim\limits_{\Delta z\to 0}\Delta w=0$ bo'ladi.

2.2-teorema. $f(z)=u(x,y)+iv(x,y)$, $z_0=x_0+iy_0$ da uzluksiz bo'lishi uchun $u(x,y)$ va $v(x,y)$ funksiyalar (x_0,y_0) da uzluksiz bolishi zarur va yetarlidir.

2.3-teorema. (sohaning saqlanishi prinsipi) Agar $w=f(z)$ funksya D sohada bir yaproqli va uzluksiz bo'lsa, u holda bu funksiya D sohada akslantirgan D_1 – to'plam ham soha bo'ladi va $z=\varphi(w)$ D_1 da uzluksiz bo'ladi.

Misol. 1) $w=z^2, z_0=x_0+iy_0$ - qo'zg' almas nuqta

$$\Delta w=(z_0+\Delta z)^2-z_0^2=2z_0\Delta z+\Delta z^2.$$

$$\lim_{\Delta z\to 0}\Delta w=\lim_{\Delta z\to 0}\left[2z_0\Delta z+(\Delta z)^2\right]=0.$$

Demak, $w=z^2$ funksiya tekislikning har qanday z nuqtasida uzluksizdir.

2) $w=z^{-z}$, 3) $w=z^3$; 4) $w=\frac{1}{z}$ misollarda funksiya uzluksizligini tekshiring.

2.6. Oddiy transcendent funksiyalar

1. Ko'rsatgichli funksiya. Ixtiyoriy $z=x+iy$ kompleks son uchun e^z ko'rsatgichli funksiyani ushbu

$$w=e^z=e^x(\cos y+i\sin y)$$

ko'rinishida yozish mumkin. Agar $y=0$ bo'lsa, $e^z=e^x$ bo'ladi. $x=0$ da esa Eyler formulasi

$$e^z=e^{iy}=\cos y+i\sin y$$

hosil bo'ladi. $w=e^z$ funksiyaning xossalari:

$$e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}\ ;\quad \frac{e^{z_1}}{e^{z_2}}=e^{z_1-z_2}$$ haqiqatdan ham

$$e^{z_1}\cdot e^{z_2}=e^{x_1+x_2}(\cos(y_1+y_2)+i\sin(y_1+y_2))=e^{x_1+x_2}\cdot e^{i(y_1+y_2)}=$$

$$=e^{x_1+iy_1}\cdot e^{x_2+iy_2}=e^{(x_1+iy_1)+(x_2+iy_2)}=e^{z_1+z_2},$$

e^z davriy funksiya bo'lib, uning davri $2\pi i$ ga teng. Haqiqatdan ham

$$e^{z+2k\pi i}=e^z\cdot e^{2k\pi i}=e^z(\cos 2k\pi+i\sin 2k\pi)=e^z$$

Misollar.

1) $1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=\sqrt{2}e^{\frac{\pi}{4}i}$

2) $e^{-2i}=\cos 2-i\sin 2$

3) $1=\cos 2\pi+i\sin 2\pi=e^{2\pi i}$

4) $-1=\cos\pi+i\sin\pi=e^{\pi i}$

5) $1+\sqrt{3}i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2e^{\frac{\pi}{3}i}$

2. $w=Lnz$ funksiya ko‘rsatkichli funksiyaga teskari: $e^{w}=z\ (z\neq 0)$ $z=re^{i\varphi}$, $w=u+iv$ desak, $e^{w}=e^{u}\cdot e^{iv}=re^{i\varphi}$ bundan $e^{u}=r$, $e^{iv}=e^{i\varphi}$, $v=\varphi+2\pi k$. $e^{u}=r$ dan $u=\ln r$ kelib chiqadi

$$w=Lnz=\ln r+i(\varphi+2\pi k)=\ln|z|+i(\arg z+2\pi k)=\ln|z|+iArgz.$$

Bu formuladan ko‘rinadiki, kompleks argumentli logarifmik funksiya ko‘p qiymatli. $k=0$ logarifmning bosh qiymatini $\ln z$ deb belgilasak $\ln z=\ln|z|+i\varphi$. U holda $Lnz=\ln z+2k\pi i$ deb yoza olamiz.

Agar z haqiqiy musbat son bo‘lsa, u holda $\arg z=0$ bo‘lib, $\ln z=\ln|z|$. Haqiqiy musbat son logarifmining bosh qiymati bu sonning oddiy natural logarifmi bilan ustma-ust tushadi.

2.18-misol. $z=i$ bo‘lsa, $|z|=1;\arg z=\frac{\pi}{2}$.

Yechilishi. $\ln z=\ln 1+i\frac{\pi}{2}=i\frac{\pi}{2}$.

$$Lni=\ln i+2k\pi i=i\frac{\pi}{2}+2k\pi i=(\frac{\pi}{2}+2k\pi)i.$$

Kompleks argumentli logarifmik funksiya haqiqiy argumentli logarifmning ma’lum xususiyatlariga ega.

$$\ln(z_1\cdot z_2)=\ln z_1+\ln z_2;\ \ln\left(\frac{z_1}{z_2}\right)=\ln z_1-\ln z_2;\ \ln z^n=n\ln z;\ \ln\sqrt[n]{z}=\frac{1}{n}\ln z$$

Haqiqatdan ham

$$\ln(z_1\cdot z_2)=\ln(z_1\cdot z_2)+iArg(z_1\cdot z_2)=\ln(|z_1|\cdot|z_2|)+i(Argz_1+Argz_2)=$$
$$=(\ln|z_1|+iArgz_1)+(\ln|z_2|+iArgz_2)=\ln z_1+\ln z_2$$

3. Trigonometrik funksiya.

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i};\quad \cos z=\frac{e^{iz}+e^{-iz}}{2};\quad tgz=\frac{\sin z}{\cos z};\quad ctgz=\frac{\cos z}{\sin z}$$

$z=x$ bo‘lsa,

$$\sin z=\frac{e^{ix}-e^{-ix}}{2i}=\frac{1}{2i}(\cos x+i\sin x-(\cos x-i\sin x))=\sin x.$$

$\sin z$, $\cos z$ funksiyalarning davri 2π, tgz va $ctgz$ funksiyalarning davri π ga teng bo'ladi. Haqiqatdan ham

$$\sin(z+2\pi)=\frac{1}{2i}\left[e^{i(z+2\pi)}-e^{-i(z+2\pi)}\right]=\frac{1}{2i}\left[e^{iz+2\pi}-e^{-iz-2\pi}\right]=\frac{e^{iz}-e^{-iz}}{2i}=\sin z$$

Qolganlarini ham o'rinli ekanligini isbot qilish mumkin. Hamma trigonometrik ayniyatlar ham o'rinlidir. Masalan, $\sin^2 z+\cos^2 z=1$. Haqiqatdan ham

$$\sin^2 z+\cos^2 z=\left(\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)\right)^2+\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2=$$

$$=\frac{e^{2iz}-2e^{iz}e^{-iz}+e^{-2iz}}{-4}+\frac{e^{2iz}+2e^{iz}e^{-iz}+e^{-2iz}}{4}=$$

$$=\frac{-e^{2iz}+2e^{0}e^{-2iz}-2e^{-2iz}+2\cdot e^{0}+e^{2iz}}{4}=\frac{4}{4}=1.$$

Lekin bu funksiyalarning modullari birdan katta bo'lishi mumkin.

$$\cos i=\frac{1}{2}(e^{-1}+e)=1.5430,$$

$$\sin i=\frac{1}{2i}(e^{-1}-e)=1.17520i$$

ya'ni

$$|\sin z|\geq 1,\quad |\cos z|\geq 1,$$

$$z=\sin w\Rightarrow w=\arcsin z,$$

$$z=\sin w=\frac{e^{iw}-e^{-iw}}{2i}\Rightarrow e^{2iw}-2ize^{iw}-1=0,$$

$$e^{iw}=iz\pm\sqrt{1-z^2}\Rightarrow iw=Ln(iz\pm\sqrt{1-z^2})\Rightarrow w=\frac{1}{i}Ln(iz+\sqrt{1-z^2}).$$

Demak,

$$Arc\sin z=\frac{1}{i}Ln(iz+\sqrt{1-z^2}),\qquad (2.15)$$

bunda $\sqrt{1-z^2}$ ildizning algebraik qiymati olinadi. Mana shu usul bilan osongina

$$Arc\cos z=\frac{1}{i}Ln(z+\sqrt{z^2-1})\qquad (2.16)$$

formulani chiqarish mumkin.

Xuddi shuningdek, $Arctgz$, $Aarcctgz$- ga ham quyidagi formulani chiqarish mumkin.

$$z = tgw \Rightarrow w = Arctgz,$$

$$z = tgw = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}},$$

$$Arctgz = \frac{1}{2i}Ln\frac{1+iz}{1-iz}. \qquad (2.17)$$

Ildiz ikki qiymatli, logarifm ko'p qiymatli bo'lgani uchun bu funksiya ham ko'p qiymatli.

Xuddi shu usul bilan

$$Arcctgz = \frac{1}{2i}Ln\frac{z+i}{z-i} = \frac{1}{2i}Ln\frac{1+iz}{1-iz} \qquad (2.18)$$

2.19-misol. $ArcSin\, i$ ning barcha qiymatlari hisoblansin.

Yechilishi. (2.15) formulaga $z = i$ ni qo'yamiz.

$Arcsin i = \frac{1}{i}Ln(-1 \pm \sqrt{2})$ usbu $t_1 = -1 + \sqrt{2}, t_2 = -1 - \sqrt{2}$ belgilashlarni kiritamiz. U holda $Lnz = \ln z + 2k\pi i = \ln|z| + i\varphi + 2k\pi i$ ga asosan:

$$Lnt_1 = Ln(-1 + \sqrt{2}) = \ln|-1 + \sqrt{2}| + i\varphi_1 + 2k\pi i = \ln(-1 + \sqrt{2}) + 2k\pi i,$$

$$Lnt_2 = Ln(-1 - \sqrt{2}) = \ln|-1 - \sqrt{2}| + 2k\pi i = \ln(1 + \sqrt{2}) + \pi i + 2k\pi i;$$

chunki t-ga tegishli vektor OX o'qining musbat tomonida joylashganligi uchun $\varphi_1 = 0$, t_2 esa chap tomonida joylashganligi $\varphi_2 = \pi$ bo'ladi. Shunday qilib,

$$Arcsin i = 2k\pi - i\ln(\sqrt{2} - 1), \; Arcsin I = (2k+1)\pi - i\ln(\sqrt{2} + 1)$$

$k = 0, \pm 1, \ldots$ chunki, $k = 0$ bo'lganda $Arcsin i$ ning bosh qiymati $\arcsin i$ ni hosil qiladi.

2.20-misol. $Arctg(1 \pm 2i)$ ning barcha qiymatlarini toping.

Yechilishi. (2.17) formulaga $z = 1 + 2i$ ni qo'yib chiqamiz, ya'ni

$$Arctg(1+2i) = \frac{1}{2i}Ln\frac{1+i(1+2i)}{1-i(1+2i)} = \frac{1}{2i}\ln\frac{i-1}{3-i},$$

kasrga maxrajning qo'shmasiga ko'paytirib uning moduli va argumentini topaylik:

$$\frac{i-1}{3-i} = \frac{(i-1)(3+i)}{(3-i)(3+i)} = \frac{3i - i - 3 - 1}{9+1} = \frac{2i-4}{10} = -\frac{2}{5} + i\frac{1}{5},$$

$$\left|-\frac{2}{5} + i\frac{1}{5}\right| = \sqrt{\frac{4}{25} + \frac{1}{25}} = \frac{1}{\sqrt{5}}, \qquad \arg(-\frac{2}{5} + \frac{1}{5}i) = \varphi,$$

$$tg\varphi = \frac{1}{5} : (-\frac{2}{5}) = -\frac{1}{2} \Rightarrow \varphi = arctg(-\frac{1}{2}) = -arctg\frac{1}{2}$$

$$Ln(-\frac{2}{5}+i\frac{1}{5})=\ln\frac{1}{\sqrt{5}}-iarctg2+2k\pi i$$

U holda

$$Arctg(1+2i)=k\pi-\frac{1}{2}arctg2+\frac{i}{4}\ln5$$

2.21-Misol. $Arccos(-i)$ ning barcha qiymatlarini hisoblang.
Yechilishi. (2.16) formulaga $z=-i$ qo'yib

$$Arc\cos(-i)=iLn(-i\pm i\sqrt{2})=\frac{1}{i}Ln(-1\pm\sqrt{2})i$$

Ko'rinib turibdiki,

$$t_1=-1+\sqrt{2}>0 \qquad \varphi_1=\frac{\pi}{2}+2k\pi$$

$$t_2=-1-\sqrt{2}<0 \qquad \varphi_2=-\frac{\pi}{2}+2k\pi,\ (k=0,\pm1\pm2......)$$

Shu sababli

$$Ln(it_1)=Ln(\sqrt{2}-1)i=\ln(\sqrt{2}-1)+\frac{\pi}{2}i+2k\pi i,$$

$$Ln(it_2)=Ln(-\sqrt{2}-1)i=\ln(\sqrt{2}+1)-\frac{\pi}{2}i+2k\pi i,$$

demak, izlanayotgan qiymatlar ikkita bo'lib, ular quyidagilardan iborat ekan.

$$(2k+\frac{1}{2})\pi-i\ln(\sqrt{2}-1),\ (2k-\frac{1}{2})\pi-i\ln(\sqrt{2}+1),$$

$$k=0,\pm1\pm2......$$

$k=0$ desak $Arccos(-i)$ ning bosh qiymati, $arccos(-i)$ kelib chiqadi.

Mustaqil yechish uchun misollar.

Quyidagilarni barcha qiymatlarini hisoblang.

2.31. $Arctg5i$;

2.32. $Arc\sin(\sqrt{2}-i)$;

2.33. $Arcctg(2i)$

2.34. $Arc\sin\frac{1}{2}$;

2.35. $Arc\cos\frac{1}{2}$;

2.36. $Arc\cos2$;

2.37. $Arc\sin2$;

2.38. $Arctg 2i$;

2.39. $Arctg\frac{i}{3}$;

2.40. $Arc\sin\frac{\pi i}{3}$.

Giperbolik funksiyalar. Kompleks argumentli giperbolik funksiyalar deb quyidagilarni qabul qilamiz.

$$shz=\frac{e^{z}-e^{-z}}{2},\quad chz=\frac{e^{z}+e^{-z}}{2}$$

$$thz=\frac{shz}{chz},\qquad cthz=\frac{chz}{shz},$$

bunda $z=x+iy$ kompleks o'zgaruvchi. shz, chz funksiyalarni davri $2\pi i$, thz, $cthz$ funksiyalarni davri πi.

Kompleks sohada trigonometrik funksiyalar bilan giperbolik funksiyalar orasida ajoyib munosabatlar bor, ya'ni

$\sin z=\frac{e^{iz}-e^{-iz}}{2i}$ da z o'rniga (iz) ni qo'ysak quyidagiga ega bo'lamiz:

$$\sin(iz)=\frac{e^{i^2z}-e^{-i^2z}}{2i}=-\frac{e^{z}-e^{-z}}{2i}=\frac{i}{2}(e^{z}-e^{-z})=ishz\Rightarrow shz=-i\sin(iz)$$

Xuddi shu usulda $chz=\cos iz,\ thz=itgiz,\ cthz=ictgiz$ funksiyalarni keltirib chiqarish mumkin. Mana shu usul bilan quyidagi ayniyatlarni isbotlash mumkin

$$ch^2z-sh^2z=1;\quad sh2z=2shzchz.$$

Teskari giperbolik funksiyalar. Giperbolik funksiyalar oldingi paragraflardan bizga ma'lum bo'ib, ular quyidagilardan iborat:

$$z=shw=\frac{e^{w}-e^{-w}}{2},\ z=chw=\frac{e^{w}+e^{-w}}{2},$$
$$z=thw=\frac{e^{w}-e^{-w}}{e^{w}+e^{-w}},\ z=cthz=\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}},\qquad (2.19)$$

bu yerda w–argument, z–funksiya. Agar $w=U+iV$ ni funksiya, $z=x+iy$–ni argumenti deb faraz qilsak (2.19) dan quyidagilar kelib chiqadi.

$$w=Arcshz,\ w=Arcchz,\ w=Acthz,\ w=Arccthz\qquad (2.20)$$

$z=x+iy$ bo'lganda (2.20) dagi funksiyalarni logarifmik funksiya orqali ifodalash mumkin.

1) $w=Arcshz$ ga doir formulani keltirib chiqarish uchun (2.19) ga murojaat qilamiz:

$$\frac{e^{w}-e^{-w}}{2}=z\Rightarrow e^{w}-e^{-w}=2z\Rightarrow e^{2w}-2ze^{w}-1=0.$$

$\Rightarrow e^{w}=z\pm\sqrt{z^{2}+1}$ ni ikkala tomonini logarifmlaymiz.

$$w=Ln(z\pm\sqrt{z^{2}+1})=Arcshz \tag{2.21}$$

Xuddi shunday usul bilan

$$Arcchz=Ln(z\pm\sqrt{z^{2}-1}),\ Arcthz=\frac{1}{2}Ln\frac{1+z}{1-z},\quad Arccthz=\frac{1}{2}Ln\frac{z+1}{z-1}$$

hosil qilish mumkin.

2.22-misol. *Arcshi* ning barcha qiymatlarini aniqlang.

Yechilishi. (2.21) –ga ko'ra

$$Arcshi=Ln(i+\sqrt{i^{2}+1})=Lni$$

$$i=\frac{\cos\pi}{2}+i\sin\frac{\pi}{2},\varphi=\pi/2,r=|i|=1,Ln1=0$$

logarifmning formulasiga muvofiq

$$Arcshi=Lni=\left(2k+\frac{1}{2}\right)\pi i;\ \ k=0\pm1\pm2,\ldots..$$

2.23-misol. Ushbu

$$Sinz+\cos z=i$$

tenglamani yeching.

Yechilishi. Ma'lumki,

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}\ \text{va}\ \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

u holda

$$\frac{e^{iz}-e^{-iz}}{2i}+\frac{e^{iz}+e^{-iz}}{2}=i$$

bundan

$$(1+i)e^{2iz}+2e^{iz}-1+i=0$$

$e^{iz}=t$ deb belgilab

$$(1+i)t^{2}+2t-1+i=0$$

tenglamani hosil qilamiz.

$$t_{1,2}=\frac{-1\pm\sqrt{1+2}}{1+i}=\frac{-1\pm\sqrt{3}}{1+i}=\frac{(1\pm\sqrt{3})(1-i)}{2}$$

$$e^{iz}=\frac{1}{2}(1-i)(-1\pm\sqrt{3})\Rightarrow iz=\frac{\ln(1-i)(-1\pm\sqrt{3})}{2}+i\varphi 1{,}2+2k\pi,x\in z$$

$$z_1=i\ln\frac{(-1+\sqrt{3})(1-i)}{2}+\varphi_1,\ \ \varphi_1=\frac{\arg\left(\left(-1+\sqrt{3}\right)(1-i)\right)}{2}$$

Mustaqil yechish uchun misollar

2.41. $shiz=i$;

2.42. $\cos z=2$;

2.43. $sh(\ln x+\frac{\pi}{3}i)$;

2.44. $ch3z=i$;

2.45. $e^{z^2}=1$;

2.46. $ch(\ln x+\frac{\pi}{3}i)$;

2.47. $\sin z=3$;

2.48. $\sin z=\pi i$;

2.49. $\cos(2+i)$;

2.50. $sh(2+i)$;

2.51. $|z|+2z=3-4i$;

2.52. $|z|-2z=3i+6$;

2.53. $|z|+z=1+i$;

2.54. $\sin z=\cos z$;

2.55. $\sin z=ishz$;

2.56. $\sin 2z-\cos 2z=1$;

2.57. $|z|-z=2+i$;

2.58. $|z|-z=2+4i$;

2.59. $z^2=-1+i\sqrt{3}$;

2.60. $sh(-3+i)$;

2.61. shi;

2.62. chi.

2.7. Hosila. Koshi – Riman sharti. Analitik funksiyalar.

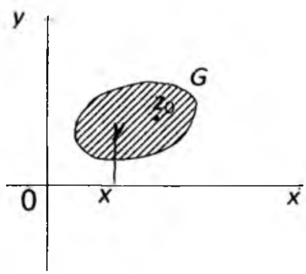
Kompleks tekislikdagi biror G – sohada aniqlangan bir qiymatli $w = f(z)$ funksiya berilgan bo'lib, $z_0 \in G$ bo'lsin (2.17–chizma). Ma'lumki, argument z ning orttirmasi $\Delta z = z - z_0$ bo'lib,funksiyaning unga mos orttirmasi esa $\Delta w = \Delta f(z_0) = f(z_0 + \Delta z) - f(z_0)$ bo'ladi.

Agar $w = u(x, y) + iv(x, y)$ ko'rinishda berilgan bo'lsa (2.18–chizma), ularni quyidagicha yozish mumkin:

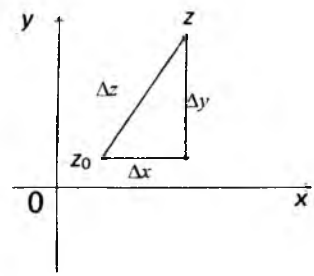
$$\Delta z = \Delta x + i\Delta y\,,\quad \Delta w = \Delta u + i\Delta v\,,$$

$$\Delta x = x - x_0\,,\quad \Delta y = y - y_0\,,\quad \Delta U = U(x_0 + \Delta x, y_0 + \Delta y) - U(x_0 y_0)\,,$$

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0 y_0)\,.$$



2.17 – chizma. z_0 *nuqtaning G ga tegishliligi.*



2.18- chizma. $w = f(z)$ *funksiyaning ortirmalari.*

Ma'lumki, z nuqta G sohaga tegishli bo'lib, tayinlangan z_0 nuqtaga ixtiyoriy ravishda intilishi mumkin.

2.6-ta'rif. Agar Δz har qanday yo'l bilan nolga intilganda ham $\frac{\Delta w}{\Delta z}$ nisbat aniq birgina chekli limitga intilsa, o'sha limit $f(z)$ funksiyaning z_0 nuqtadagi hosilasi bo'lib, bu quyidagicha yoziladi:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \qquad (2.22)$$

$f(z)$ funksiyaning differensiallanuvchanligidan shu nuqtada uning uzluksizligi kelib chiqadi, aksinchasi har doim o'rinli bo'lavermaydi.

Bu ta'rif matematik analiz kursidagi hosila ta'rifiga o'xshash bo'lgani sababli undagi asosiy formulalar bu joyda o'z kuchini saqlaydi.

1. $z' = c' = 0$, (c – const).
2. $z = c \; \frac{dz}{dz} = z' = 1$.
3. $(z^n)' = nz^{n-1}$.
4. $(z \neq 0)\,(\frac{1}{z})' = -\frac{1}{z^2}$.
5. $(z^{-n})' = \frac{-n}{z^{n+1}}$, $z \neq 0$.
6. $(\sqrt{z})' = \frac{1}{2\sqrt{z}}$.
7. $(e^z)' = e^z$.
8. $(e^{mz})' = me^{mz}$.
9. $(\ln z)' = \frac{1}{z}$, $z \neq 0$.
10. $(\sin z)' = \cos z$.
11. $(\cos z)' = -\sin z$.
12. $(tgz)' = \frac{1}{\cos^2 z} = 1 + tg^2 z$.
13. $(ctgz)' = -\frac{1}{\sin^2 z} = -(1 + ctg^2 z)$.

Misol. $w = \operatorname{Re} z = x$, bu funksiya ixtiyoriy nuqtada uzluksiz lekin hech qanday nuqtada differensiallanuvchi emas.

Haqiqatdan ham,

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z} = \frac{\Delta x + x - x}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i\Delta y}$$

$\Delta x \to 0$ da $\Delta z = 0 + i\Delta y$ bo'lib, u holda $\frac{\Delta w}{\Delta z} \to 0$, $\Delta y \to 0$ da $\Delta z = \Delta x + i0$ bo'lib unda $\frac{\Delta w}{\Delta z} \to 1$.

Demak, $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ mavjud emas, shuning uchun $w = \operatorname{Re} z$ funksiya hosilaga ega emas

Agar funksiya $w=u+iv$ formada berilgan bo'lsa, undan hosila olish uchun quyidagi to'rtta formuladan biridan foydalansa bo'ladi.

$$w'=f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x} \qquad (2.23)$$

2.2-teorema. Agar $w=f(z)=u(x,y)+iv(x,y)$ funksiya $z=x+yi$ nuqtada hosilaga ega bo'lsa, u holda $u(x,y);v(x,y)$ funksiyalar

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad (2.24)$$

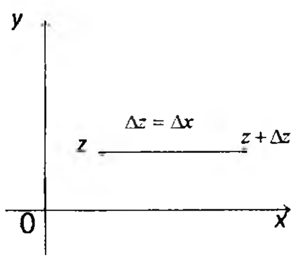
shartni qanoatlantiruvchi birinchi tartibli xususiy hosilalarga ega bo'ladi.

Isbot.

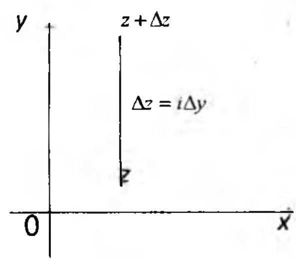
$$f'(z)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta u+i\Delta v}{\Delta z}$$

Bu limit Δz – ning qanday qilib nolga intilishiga bog' liq bo'lmagani uchun ikkita yo'nalishni tanlaymiz:

1) Ox o'qiga parallel bo'lgan to'g' ri chiziq bo'yicha, u holda $\Delta y=0$ bo'ladi, bundan $\Delta z=\Delta x$ (2.19 – chizma).



2.19– chizma. $\Delta y=0,\ \Delta z=\Delta x$ ***bo'gan hol.***



2.20–chizma. $\Delta x=0,\ \Delta z=\Delta y$ ***bo'gan hol.***

2) Oy o'qiga parallel bo'lgan to'g' ri chiziq bo'yicha, u holda $\Delta x=0$ bo'ladi, bundan $\Delta z=i\Delta y$ (2.20 – chizma).

Birinchi holdan:

$$f'(z)=\lim_{\Delta x\to 0}\frac{\Delta u+i\Delta v}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+i\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\,. \qquad (2.25)$$

Ikkinchi holdan:

$$f'(z)=\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta v}{i\Delta y}=\frac{1}{i}\lim_{\Delta y\to 0}\frac{\Delta u}{\Delta y}+\lim_{\Delta y\to 0}\frac{\Delta v}{\Delta y}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}. \quad (2.26)$$

(2.24) va (2.25) dan

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y},$$

bundan

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad (2.27)$$

(2.27) shart Eyler – Dalamber yoki Kashi – Riman shartlari deyiladi. (2.27) shart yetarli shart ham hisobanadi. Kompleks argumentli funksiyalardan olingan xususiy $\frac{\partial w}{\partial z}$, $\frac{\partial w}{\partial \bar{z}}$ hosilalar w – ning haqiqiy va mavhum qisimlarining hususiy hosilalari orqali ifodalanadi:

$$\frac{\partial w}{\partial z}=\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial w}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

(2.27) – Koshi – Riman sharti esa $\frac{\partial w}{\partial \bar{z}}=0$ ekvivalentdir.

2.24-misol. $w=z^2$ funksiyaning hosilasmi toping.

Yechilishi. Hosilalar jadvalidan foydalanib $w'=(z^2)'=2z$ - ni hosil qilamiz.

Buning to‘g‘ riligini (2.23) formulalarning biri orqali tekshirib ko‘raylik:

$$w=z^2=(x+iy)^2=x^2-y^2+2ixy;\ u=x^2-y^2,\quad v=2xy$$

$$u'_x=2x,\quad u'_y=-2y,\quad v'_x=2y.\quad v'_y=2x$$

bulardan esa $u'_x=v'_y=2x;\quad u'_y=-v'_x=-2y$

Ya’ni (2.27) shartlar bajariladi, demak hosila mavjud. Endi o‘sha hosilani quyidagicha topamiz:

$$w'=u'_y+iv'_x=2x+i2y=2(x+iy)=2z$$

2.3-teorema. Agar $u(x,y)$ va $v(x,y)$ funksiyalar (x,y) nuqtada (2.27) shartni qanoatlantiruvchi uzluksiz, birinchi tartibli xususiy hosilalarga ega bo‘lsa, u holda $f(z)$ funksiya $z=x+iy$ nuqtada differensiallanuvchi bo‘ladi.

2.7-ta’rif. Agar $f(z)$ funksiya $z=x+iy$ nuqtada va uning biror atrofida differensiallanuvchi bo‘lsa, funksiya o‘sha nuqtada analitik deyiladi.

2.8-ta'rif. Sohaning hamma nuqtalarida analitik bo'lgan funksiya shu sohada analitik bo'ladi.

2.9-ta'rif. (z) tekislikning qaysi nuqtalarida $w=f(z)$ funksiya analitik bo'lsa, o'sha nuqtalar funksiyaning to'g' ri (regulyar) nuqtalari deyiladi. Funksiyaning aniqlanish sohasining ba'zi qiymatlarida analitik bo'lmasa, shu nuqtalarga funksiyaning maxsus nuqtalari deyiladi.

2.25-misol. $w=\frac{1}{z}=f(z)$ funksyaning $\pm i,\ 1+i,\ -1+3i$ nuqtalardagi hosilasini toping.

Yechilishi.

$$f'(z)=\left(\frac{1}{z}\right)'=-\frac{1}{z^2}\ ,\ f(\pm i)=1$$

$$f'(1+i)=\frac{-1}{(1+i)^2}\ ;\ f'(-1+3i)=-\frac{1}{(-1+3i)^2}$$

o'ng tomonini hisoblash o'quvchiga havola. $f(z)=\frac{1}{z}$ da $z=0$ qiymat qabul qilolmaydi. Shuning uchun $z=0$ nuqtadan boshqa barcha nuqtalarda analitik funksiya ekan.

2.26-misol. $f(z)=z\cdot\bar{z}$ funksiyani hosilasini toping.

Yechilishi.

$$f(z)=z\cdot\bar{z}=(x+iy)(x-iy)=x^2+y^2$$

$$u(x,y)=x^2+y^2,\ v=0,\ u'_x=2x,\ u'_y=2y,\ v'_x=0,\ v'_y=0$$

Koshi – Riman shartlariga muvofiq

$$2x=0\Rightarrow x=0,\ 2y=0\Rightarrow y=0;\quad z=0$$

$$f'(0)=(u'_x+iv'_y)=(2x+i0)=0$$

Demak, berilgan funksiya birgina $z=0$ nuqtada hosilaga ega bo'lib, boshqa nuqtalarda hosilaga ega emas, demak, analitik ham emas.

2.8. Garmonik funksiyalar haqida tushuncha

Kompleks argumentli funksiyaning ikki hil shaklda yozilishi mumkin:

$$w=f(z)=u(x,y)+iv(x,y), \tag{2.28}$$

bu yerda

$$u(x,y)=\mathrm{Re}\,w=\mathrm{Re}\,f(z)\quad \text{va}\quad (x,y)=\mathrm{Im}\,w=\mathrm{Im}\,f(z) \tag{2.29}$$

Ma'lumki, ushbu

$$\Delta w=\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}=0 \qquad (2.30)$$

Laplas tenglamasini qanoatlantiradigan har qanday ikki argumentli $\omega=\omega(x,y)$ funksiya garmonik funksiya deyiladi[4]. Garchi $u(x,y)$ va $v(x,y)$ funksiyalar garmonik bo'lsada, (2.28) funksiya analitik bo'lmay qolishi mumkin. Agar funksiya analitik bo'lsa, u holda (2.30) ni qanoatlantiruvchi $u(x,y)$ va $v(x,y)$ lar o'zaro qo'shma garmonik funksiyalar deyiladi. Uning uchun Koshi – Riman shartlari bajarilishi kerak.

Agar noma'lum analitik funksiyaning haqiqiy yoki mavhum qismi ma'lum bo'lsa, u holda analitik funksiyaning o'zini topish mumkin.

Ma'lumki, haqiqiy o'zgaruvchi funksiyalarda $x\in(a,b)$ kesmaning hamma nuqtalarida $F'(x)=f(x)$ bo'lsa, u holda $F(x)$ funksiya $f(x)$ funksiyaning boshlang' ich funksiyasi ekanligi bizga ma'lum. Kompleks o'zgaruvchili funksiyalarga ham xuddi shunday, ya'ni $F'(z)=f(z)$ tenglik o'rinli bo'lsa, $F(z)$ analitik funksiya $f(z)$ analitik funksiya uchun boshlang' ich funksiya deyiladi va $F(z)=\int f(z)dz+C$ kabi yoziladi. (2.26) formuladan:

$$f(z)=\int_{z_0}^{z}(u'_x-iu'_y)dz=\int_{z_0}^{z}(v'_y+iv'_x)dz$$

2.27-misol. Analitik funksiyalarning haqiqiy qismi

$$u=\frac{x}{x^2+y^2} \qquad \text{va} \qquad f(\pi)=\frac{1}{\pi}$$

berilgan bo'lib o'zini topish talab qilinadi.

Yechilishi.

$$u'_x=\frac{y^2-x^2}{(x^2+y^2)^2} \qquad u'_y=-\frac{2xy}{(x^2+y^2)^2}$$

$$f'(z)=\frac{y^2-x^2}{(x^2+y^2)^2}+i\frac{2xy}{(x^2+y^2)^2}=\frac{(y-ix)^2}{(x^2+y^2)^2}$$

$y-ix=i(x-iy)=i\bar{z}$ va $x^2+y^2=z\cdot\bar{z}$ u holda

$$f'(z)=-\frac{(i\bar{z})^2}{(z\bar{z})^2}=-\frac{(\bar{z})^2}{z^2(\bar{z})^2}=-\frac{1}{z^2}$$

Bundan

$$f(z)=-\int\frac{1}{z^2}dz+C=\frac{1}{z}+C,\quad f(\pi)=\frac{1}{\pi}+C=\frac{1}{\pi}\Rightarrow C=0.$$

Demak,

$$f(z)=\frac{1}{z}.$$

2.28-misol. Analitik funksiyaning mavhum qismi

$$v(x,y)=arctg\frac{y}{x};\ (x>0),\quad f(1)=0$$

bo'lsa, berilgan funksiyaning o'zini toping.

Yechilishi.

$$v'_x=-\frac{y}{x^2+y^2},\qquad v'_y=\frac{x}{x^2+y^2}$$

$$f'=v'_y-iv'_x=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=\frac{x-iy}{x^2+y^2}=\frac{\bar{z}}{z\bar{z}}=\frac{1}{z}$$

Uni integrallasak:

$$f(z)=\int\frac{1}{z}dz+C=\ln z+C.$$

$$f(1)=\ln 1+C=0\Rightarrow C=0,\qquad f(z)=\ln z.$$

Agar noma'lum analitik funksiyaning haqiqiy yoki mavhum qismi berilgan bo'lsa uning o'zini yana quyidagicha aniqlash mumkin. Uning uchun Koshi– Riman (Eyler Dalamber) shartlaridan foydalanishga to'g'ri keladi.

2.29-misol. Funksiyaning haqiqiy qismi va qo'shimcha shart berilgan:

$$u(x,y)=2e^x\cos y,\ f(0)=2$$

Yechilishi. Koshi–Riman

$$u'_x=v'_y,\ v'_x=-u'_y$$

Shartlardan foydalanib, u ma'lum bo'lgani sababli v-ni topish mumkin:

$$v'_y=u'_x=(2e^x\cos y)'_x=2e^x\cos y,$$

bundan y bo'yicha integrallasak

$$2e^x\int\cos y dy+\varphi(x)=2e^x\sin y+\varphi(x)$$

Tenglikning ikki tomondan x bo'yicha xususiy hosila olib yuqoridagi shartlarning ikkinchisidan foydalanamiz, u holda

$$v'_x=2e^x\sin y+\varphi'(x)=-u'_y=(-2e^x\cos y)'_y=2e^x\sin y,$$

bundan

$$\varphi'(x) = 0 \Rightarrow \varphi(x) = C.$$

Shunday qilib,

$$v = 2e^x \sin y + C.$$

Bularga asosan

$$w = f(z) = u + iv = 2e^x \cos y + i2e^x \sin y + iC = 2e^{x+iy} + iC = 2e^z + iC$$

$$f(0) = 2 = 2 + iC \qquad iC = 0, \Rightarrow C = 0 \qquad f(z) = 2e^z.$$

2.30-misol. Qutb koordinatalari sistemasida funksiyaning haqiqiy qismi berilgan:

$$u(r,\varphi) = r\varphi\cos\varphi + r\ln r\sin\varphi.$$

Analitik funksiyani toping.

Yechilishi. Buning uchun Koshi – Rimanning qutb koordinatalari orqali ifoda qilingan ushbu shartlaridan foydalanamiz:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \varphi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \varphi}$$

Berilganiga ko‘ra:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi} = -\frac{1}{r}(r\cos\varphi - \varphi r\sin\varphi + r\ln r\cos\varphi) = \varphi\sin\varphi - (1+\ln r)\cos\varphi$$

$$\frac{\partial v}{\partial \varphi} = r\frac{\partial u}{\partial r} = r(\varphi\cos\varphi + \ln r\sin\varphi + \sin\varphi) = r[\varphi\cos\varphi + (1+\ln r)\sin\varphi]$$

Bularning birinchisini r bo‘yicha integrallaymiz:

$$v = r\varphi\sin\varphi - \cos\varphi\int(1+\ln r)dr + F(\varphi) = r\varphi\sin\varphi - r\ln r\cos\varphi + F(\varphi).$$

Tenglikning ikki tomonidan φ bo‘yicha hosila olsak:

$$\frac{\partial v}{\partial \varphi} = r\sin\varphi + r\varphi\cos\varphi + r\ln r\sin\varphi + F'(\varphi) = r[\varphi\cos\varphi + (1+\ln r)\sin\varphi]$$

Bundan

$$F'(\varphi) = 0 = F(\varphi) = C.$$

Demak,

$$v = r\varphi\sin\varphi - r\ln r\cos\varphi + C$$

$$w = f(z) = u + iv = (r\varphi\cos\varphi + r\ln r\sin\varphi) + i(r\varphi\sin\varphi - r\ln r\cos\varphi + c) =$$

$$= r\varphi(\cos\varphi + i\sin\varphi) - ir\ln r(\cos\varphi + i\sin\varphi) + ic = r(\varphi - i\ln r)e^{i\varphi} + ic.$$

Chunki Eyler formulasiga muvofiq:

$$\cos\varphi + i\sin\varphi = e^{i\varphi}.$$

Mustaqil yechish uchun misollar

2.63. $v(x,y) = 2xy + y$;

2.64. $v(x,y) = e^x \sin y$;

2.65. $v(x,y) = e^x \sin y,\ f(0) = 1$;

2.66. $u(x,y) = \dfrac{x}{x^2 + y^2},\ f(\pi) = \dfrac{1}{5}$;

2.67. $v(x,y) = x^2 - y^2 - xy$;

2.68. $u(x,y) = x^2 - y^2 + 5x - y - \dfrac{y}{x^2 - y^2}$;

2.69. $v(x,y) = -\dfrac{y}{x^2 + y^2} + 2x$;

2.70. $u(x,y) = x^2 - y^2 + 2y$;

2.71. $v(x,y) = -2\sin 2x sh 2y + y,\ f(0) = 2$;

2.72. $v(x,y) = -\dfrac{1}{2}(x^2 - y^2) + 2xy$;

2.74. $v(x,y) = 3 + x^2 - y^2 - \dfrac{y}{2(x^2 + y^2)}$;

2.75. $v(x,y) = \ln(x^2 + y^2) + x - 2y$;

2.76. $u(x,y) = 2arctg\dfrac{x}{y} + 5$;

2.78. $u(x,y) = \dfrac{x}{x^2 + y^2} - 2y$;

2.79. $v(x,y) = \ln(x^2 + y^2) - x^2 + y^2$;

2.80. $u(x,y) = (x^2 + y^2)e^x$;

2.81. $u(x,y) = e^x(x\cos x - y\sin y) - \dfrac{y}{x^2 + y^2}$;

2.82. $v(x,y) = 2e^x \sin y$;

2.83. $u(x,y) = x^3 - 3xy^2$;

2.84. $v(x,y) = 3xy^2 - x^3$;

2.85 $u(x,y) = \dfrac{1}{2(x^2 + y^2)}$,

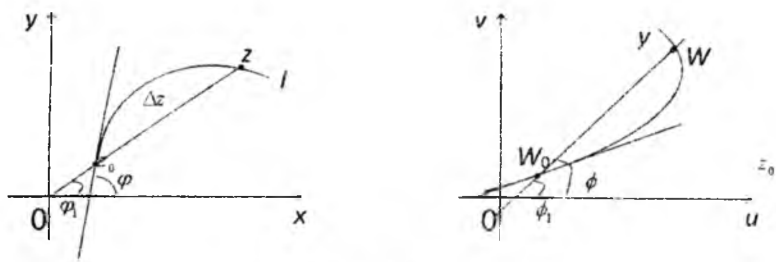
2.86. $u(x,y) = \dfrac{1}{2}(x^2 + y^2)$;

2.87. $u(x,y) = 2xy + 3$.

2.9. Hosila argumenti va modulning geometrik ma'nosi. Konform akslantirish.

$w=f(z)$ funksiya G sohada analitik bo'lib, shu sohaga tegishli z_0 nuqtada $f'(z_0)\neq 0$ bo'lsin. $w=f(z)$ funksiya (z) tekislikdagi z_0 nuqtani (w) tekislikdagi w_0 nuqtaga akslantirsin.

z_0 nuqtadan qandaydir l egri chiziqni o'tkazamiz. Bunda $w=f(z)$ funksiya l egri chiziqni (w) tekislikdagi w_0 nuqtadan o'tuvchi L egri chiziqqa akslantiradi. l egri chiziqda ixtiyoriy $z=z_0+\Delta z$ nuqtani olamiz. Bu nuqta L egri chiziq ustidagi $w=w_0+\Delta w$ nuqtaga akslanadi.



2.21-chizma.

Hosilaning ta'rifiga ko'ra

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{z\to \Delta z}\frac{w-w_0}{z-z_0}=f'(z_0) \qquad (2.31)$$

(2.31) dan ushbu munosabatni hosil qilish mumkin.

$$|f'(z_0)|=\lim_{z\to z_0}\left|\frac{w-w_0}{z-z_0}\right| \qquad (2.32)$$

z_0 nuqtada $f(z)$ funksiyaning analitikligidan (2.31) hamda (2.32) limitlar z ni $z_0$ga qanday intilishiga bog'liq emasligi ravshandir. Demak, (2.32) limit z_0 nuqtadan chiquvchi hamma yo'nalishlarda bir xil bo'ladi. Shuning uchun, $w=f(z)$ akslantirishda $|f'(z_0)|$ ni geometrik nuqtai nazardan z_0 nuqtadagi chuzilish koeffitsiyenti deb qarash mumkin va buni $k=|f'(z_0)|$ deb yoziladi.

Agar $|f'(z_0)|>1$ bo'lsa, $f(z)$ akslantirishda z_0 nuqtaning kichik atrofida nuqtalar orasidagi masofa kattalashadi, natijada tekislik cho'ziladi, agar $|f'(z_0)|<1$ bo'lsa, akslantirish tekislikni siqadi.

Shunday qilib, analitik funksiya yordamida bajariladigan akslantirish, $f(z_0)$nolga teng bo'lmagan barcha nuqtalarda o'zgarmas cho'zilishga ega bo'lar ekan.

(2.31) munosabatdan

$$Argf'(z_0)=\lim_{z\to z_0} Arg\frac{w-w_0}{z-z_0}=\lim_{z\to z_0}[Arg(w-w_0)-Arg(z-z_0)] \qquad (2.32')$$

$\Delta w=w-w_0$ va $\Delta z=z-z_0$ haqiqiy o'qlar bilan tashkil etgan burchaklari

$$\phi'=Arg(w-w_0),\quad \varphi'=Arg(z-z_0)$$

ko'rinishga ega.

l va L egri chiziqlarning mos ravishda z_0 va w_0 nuqtalariga o'tkazilgan urinmalar haqiqiy o'qlar bilan tashkil etgan burchaklari φ va ϕ bo'lsin. Bu holda $z\to z_0$ da $\varphi'\to\varphi$, $\phi'\to\phi$ bo'ladi.

(2.32') dan

$$Argf'(z_0)=\phi-\varphi \qquad (2.33)$$

(2.33) dan

$$\phi=Argf'(z_0)+\varphi$$

munosabatga ega bo'lamiz.

Shunday qilib, L egri chiziqning w_0 nuqtasiga o'tkazilgan urinmaning yo'nalishini hosil qilish uchun l egri chiziqning z_0 nuqtasiga o'tkazilgan urinmani $Argf'(z_0)$ burchakka burash kerak ekan. $f(z)$ funksiyani z_0 nuqtada analitik bo'lganligidan $Argf'(z_0)$ burchak z_0 nuqtasidan o'tuvchi barcha l egri chiziqlar uchun, $Argf'(z_0)$ ni $w=f(z)$ akslantirishda z_0 nuqta atrofida buralishi deyiladi.

Natijada egri chiziqning z_0 nuqtasidan o'tuvchi barcha urinmalar $w=f(z)$ akslantirishda hamda $f'(z_0)\neq 0$ bo'lganda $Argf'(z_0)$ burchakka burilar ekan.

Demak, analitik funksiya yordamida bajariladigan akslantirish $f'(z)$ hosila nolga teng bo'lmagan hamma nuqtalarda burchaklarni saqlash xossasiga ega ekan.

2.31-misol. $w = z^2 + 2z$ da tekislikning qysi qismi siqiladi va qaysi qismi cho‘ziladi.

Yechilishi. $|2z+2|<1 \Rightarrow |z+1|<\frac{1}{2}$, bu akslantirishda $|z+1|<\frac{1}{2}$ doiraning ichki qismi siqiladi, tashqi qismi cho‘ziladi.

Ma’lum bo‘ldiki, $f'(z_0) \neq 0$ shartda har qanday analitik akslantirish har bir z_0 nuqtada cho‘zilish o‘zgarmasligi va burchaklar konservatizmi xususiyatiga ega. U holda z tekislikdagi har qanday yetarli darajada kichik figura $w = f(z)$ analitik akslantirishda w tekislikdagi cheksiz kichik miqdor aniqligida o‘xshash figura o‘tadi. Haqiqatdan ham burchaklar konservatizmi xususiyatiga asosan ikki figuraning mos burchaklari bir hil, mos tomonlari kattaliklarining nisbati cheksiz kichik miqdor aniqligida faqat bitta $|f'(z_0)|$ songa teng. Shunday qilib, har bir nuqtaning yetarli darajada kichik atrofida $(f(z) \neq 0)$ analitik akslantirish o‘xshashlik akslantirishi bo‘ladi. Burchaklar konservatizimi va cho‘zilishning o‘zgarmaslik xususiyatiga ega bo‘lgan akslantirish I – tur konform akslantirish deyiladi.

Agar funksiya yordami bilan akslantirish natijasida z nuqtadagi cho‘zilish o‘zgarmasa va burilish burchagining ham kattaligi o‘zgarmay, faqat yo‘nalishi qarama qarshisiga o‘zgarsa, u II – tur konform akslantirish deyiladi.

Agar $f(z)$ funksiya analitik bo‘lsa, $w = \overline{f(z)}$ II – tur konform akslantirish bo‘ladi.

Nazorat savollari.

1. Soha tushunchasi ta’rifi.
2. Kompleks sonlar ketma-ketligi va uning xossalarini keltiring.
3. Kompleks o‘zgaruvchili funksiya ta’rifi.
4. Kompleks o‘zgaruvchili funksiya limiti va o‘zluksizligi ta’rifi.
5. Kompleks o‘zgaruvchili funksiya differensiallanuvchi bo‘lish sharti.
6. Sohada analitik funksiya deb nimaga aytiladi.
7. Hosila argumenti va modulining geometrik ma’nosini tushuntirib bering.

III BOB. KOMPLEKS SOHADA INTEGRALLAR

Bu bobda $w = f(z)$ kompleks argumentli funksiyalarda tekislikdagi biror Γ chiziq bo‘ylab olingan egri chiziqli integral bilan tanishamiz.

3.1. Kompleks o‘zgaruvchi funksiya integralining ta’rifi

Kompleks (z) tekislikdagi biror G_1 sohada uzluksiz bir qiymatli

$$w = f(z) = u(x,y) + iv(x,y) \tag{3.1}$$

funksiya berilgan bo‘lsin. U holda $f(z)$ funksiya G soha ichidan olingan ixtiyoriy Γ – silliq chiziqda ham bir qiymatli va uzluksiz bo‘ladi.

Bu chiziqning tenglamasi $z = z(t)$, $(\alpha \le t \le \beta)$ bo‘lib, uning boshlang‘ich nuqtasi z_0 va ohirgi nuqtasi Z bo‘lsin, ya’ni $z_0 = z(\alpha)$, $Z = z(\beta)$.

Γ chiziqda ikki yo‘nalishni aniqlash mumkin: bulardan bittasi t parametrning o‘sishiga, ikkinchisi esa kamayishiga mos keladi. Odatda t ning o‘sishiga mos yo‘nalishni musbat yo‘nalish deb +Γ yoki Γ^+ orqali, bunga qarama – qarshi yo‘nalishni esa manfiy yo‘nalish deb – Γ yoki Γ^- bilan belgilaniladi (3.1 – chizma)

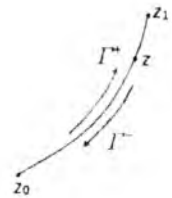
G soha ichida yotuvchi Γ silliq chiziqni ixtiyoriy ravishda

$$z_0, z_1, z_2, z_3, \ldots\ldots, z_n = z \tag{3.2}$$

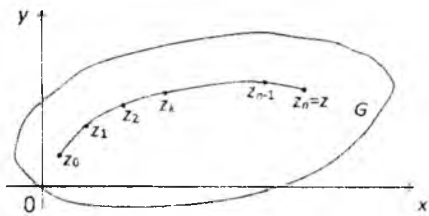
nuqtalar yordamida n – ta bo‘laklarga bo‘lamiz va shu bo‘laklardan har birining istalgan joyidan bittadan nuqta olib, bu nuqtalarni mos ravishda

$$\xi_0, \xi_1, \xi_2, \xi_3, \ldots\ldots, \xi_n \tag{3.3}$$

deb belgilaymiz (3.2 – chizma).



3.1 – chizma. Γ chiqda musbat va manfiy yo'nalishlar.



3.2 – chizma. G sohadagi Γ chiziqning bo'linishlari.

Yuqoridagi shartimizga muvofiq $w=f(z)$ funksiya ξ_k nuqtalarning har birida aniq chekli qiymatga ega.

Ushbu

$$f(\xi_1), f(\xi_2), ..., f(\xi_n),$$

sonlar bilan quyidagi

$$\Delta z_1 = z_1 - z_0 \qquad \Delta z_2 = z_2 - z_1 \qquad \Delta z_i = z_i - z_{i-1}$$

ayirmalarning mos ko'paytmalaridan tuzilgan ushbu

$$S_n = f(\xi_1)\Delta z_1 + f(\xi_2)\Delta z_2 + ... + f(\xi_k)\Delta z_k + ... + f(\xi_n)\Delta z_n = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \qquad (3.4)$$

yig'indini odatda integral yig'indi deyiladi. Bu yig'indining qiymati (3.2) va (3.3) nuqtalarga bog'liq. Agar o'sha nuqtalarni Γ bo'ylab bir marta siljitsak (z_0 va z_1 - larni qo'zg'atmasdan) yana bitta integral yig'indiga ega bo'lamiz. Umuman, mana shu usul bilan cheksiz ko'p

$$S_n^{(1)}, S_n^{(2)}, ..., S_n^{(m)}, ..$$

integral yig'indilar ketma-ketligini tuzish mumkin. Mana shulardan ixtiyoriy bittasi (3.4) dan S_n yig'indidan iborat.

Agar biz (3.2) nuqtalarni ketma – ket to‘g‘ri chiziqlar bilan tutashtirsak, Γ egri chiziq ichiga chizilgan siniq chiziq hosil bo‘ladi. Bu siniq chiziq bo‘laklarining uzunliklari

$$|\Delta z_1|, |\Delta z_2|, \ldots, |\Delta z_k|, \ldots, |\Delta z_n|$$

lardan iboratdir. Shartimizga muvofiq Γ silliq chiziq bo‘lgani uchun $|\Delta z_k|$ uzunlik (vatar) nolga yaqinlashganda uning qarshisidagi yoy uzunligi ham nolga intiladi. $\max|\Delta z_k| = \delta$ bo‘lsin. $|\Delta z_k| \le \delta \quad (k = 1,2\ldots n)$.

Ravshanki, δ – nolga intilganda n cheksizlikka intiladi.

Endi,

$$z_k = x_k + iy_k, \qquad \xi_k = \xi_k + ih_k,$$

$$\Delta z_k = z_k - z_{k-1} = (x_k - x_{k-1}) + i(y_k - y_{k-1}) = \Delta x_k + i\Delta y_k,$$

bo‘lgani uchun (3.1) muvofiq

$$f(\xi_k)\cdot \Delta z_k = [u(\xi_k, h_k) + iv(\xi_k, h_k)]\cdot(\Delta x_k + i\Delta y_k) =$$

$$= [u(\xi_k, h_k)\Delta x_k - v(\xi_k, h_k)\Delta y_k] + i[v(\xi_k, h_k)\Delta x_k + u(\xi_k, h_k)\Delta y_k]$$

Shularga asosan, S_n integral yig‘indini quyidagicha yozish mumkin:

$$S_n = \sum_{k=1}^{n} f(\xi_k)\Delta z_k = \sum_{k=1}^{n}[u(\xi_k, h_k)\Delta x_k - v(\xi_k, h_k)\Delta y_k] + i\sum_{k=1}^{n}[v(\xi_k, h_k)\Delta x_k + u(\xi_k, h_k)\Delta y_k] \quad (3.5)$$

3.1-ta’rif. Agar δ nolga intilganda (3.4) integral yig‘indi $z_1, z_2, \ldots, z_{n-1}$ va $\xi_1, \xi_2, \xi_3, \ldots \xi_k, \ldots, \xi_n$ nuqtalar Γ chiziqning qaysi joylaridan olinganiga bog‘liq bo‘lmay, aniq bir chekli limetga intilsa shu limit $f(z)$ funksiyadan Γ – chiziq bo‘ylab olingan integral deyiladi va quyidagicha yoziladi:

$$\int_{\Gamma} f(z)dz = \lim_{\delta\to 0}\sum_{k=1}^{n} f(\xi_k)\Delta z_k \quad (3.6)$$

Γ chiziq integrallash yo‘li yoki konturi deyiladi. Ta’rifda aytilgan limit haqiqatdan ham mavjud, chunki $f(z)$ funksiya Γ silliq chiziqda uzluksiz bo‘lgani uchun (3.1)– ga asosan $u(x,y)$ va $v(x,y)$ funksiyalar ham uzluksiz bo‘ladi.

U holda matematik analiz kursidagi egri chiziqli integrallar ta’rifiga asosan quyidagi tengliklarga ega bo‘lamiz:

$$\lim_{\delta\to 0}\sum_{k=1}^{n}[u(\xi_k, h_k)\Delta x_k - v(\xi_k, h_k)\Delta y_k] = \int_{\Gamma} u(x,y)dx - v(x,y)dy$$

$$\lim_{\delta\to 0}\sum_{k=1}^{n}[v(\xi_k, h_k)\Delta x_k + u(\xi_k, h_k)\Delta y_k] = \int_{\Gamma} v(x,y)dx + u(x,y)dy$$

Shularga va (3.5) asosan (3.6) ni quyidagicha yozish mumkin:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} u(x,y)dx - v(x,y)dy + i\int_{\Gamma} v(x,y)dx + u(x,y)dy = \int_{\Gamma}(u+iv)(dx+idy)$$
(3.7)

(3.7)– ni o'ng tomoni haqiqiy argumentli funksiyalardan olingan egri chiziqli integrallardan iborat.

3.2. Integrallarning asosiy xossalari

Kompleks o'zgaruvchining $f(z)$ funksiyasidan olingan integrallning xossalarini isbot qilishda (3.6) ta'rifiga asoslanamiz.

1–xossa. O'zgarmas ko'paytuvchini integral belgisidan tashqarisiga chiqarish mumkin:

$$\int_{\Gamma} cf(z)dz = c\int_{\Gamma} f(z)dz, \qquad c = const.$$

Haqiqatdan ham,

$$\sum_{k-1}^{n} cf(\xi_k)\Delta z_k = c\sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

tenglikning ikki tomonida $\delta \to 0$ faraz qilinib limitga o'tilsa, 1 – xossa isbot bo'ladi.

2–xossa. Integrallash konturining yo'nalishi qarama– qarshiga o'zgartirilsa, integral belgisi oldidagi ishora ham o'zgaradi:

$$\int_{\Gamma} f(z)dz = -\int_{\Gamma} f(z)dz$$

Haqiqatdan ham, agar integral yig'indi Г ning musbat yo'nalishida olinsa, u holda Δz_k orttirma $\Delta z_k = z_k - z_{k-1}$ ayirmani bildiradi, lekin qarama qarshi yo'nalishda olinsa $\Delta z_k = z_{k-1} - z_k = -(z_k - z_{k-1})$ ayirmani bildiradi.

Shu sababli $\sum_{k-1}^{n} f(\xi_k)(z_k - z_{k-1})$ va $\sum_{k-1}^{n} f(\xi_k)(z_{k-1} - z_k)$ yig'indilar faqat o'zaro ishoralari bilangina farq qiladi. Demak, ularning limitlari ham shu bilan farq qiladi.

3 – xossa. Chekli sondagi funksiyalar yig'indisidan olingan integral uning har bir hadidan olingan integrallar yig'indisiga teng:

$$\int_{\Gamma}[f_1(z) + f_2(z) + ... + f_k(z) + ...f_n(z)]dz = \int_{\Gamma} f_1(z)dz + \int_{\Gamma} f_2(z)dz + ...\int_{\Gamma} f_n(z)dz.$$

Haqiqatdan ham, chekli yi'gindi uchun

$$\sum_{k=1}^{n}[f_1(\xi_k) + f_2(\xi_k) + ... + f_n(\xi_k)]\Delta z_k = \sum_{k=1}^{n} f_1(\xi_k)\Delta z_k + \sum_{k=1}^{n} f_2(\xi_k)\Delta z_k + ...\sum_{k=1}^{n} f_n(\xi_k)\Delta z_k$$

bu tenglikni ikki tomonidan $\delta \to 0$deb limitga o'tish kifoya.

4 – xossa. Agar uzunligi l bo'lgan Γ chiziqning hamma nuqtalarida $M > 0$ son uchun $|f(z)| < M$ bo'lsa, u holda

$$\left|\int_{\Gamma} f(z)dz\right| \le Ml$$

bo'ladi.

Haqiqatdan ham

$$\left|\sum_{k=1}^{n}\int_{\Gamma} f(\xi_k)\Delta z_k\right| < \sum_{k=1}^{n}\left|\int_{\Gamma} f(\xi_k)\right||\Delta z_k| \le M\sum_{k=1}^{n}|\Delta z_k|.$$

Bunda

$$\sum_{k=1}^{n}|\Delta z_k| = |z_1| + |z_2| + \dots + |z_n|,$$

yig'indi Γ ichiga chizilgan siniq chiziq uzunligidan iborat bo'lgani uchun l dan katta bo'la olmaydi, demak

$$\left|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le M \cdot l.$$

Buning ikki tomonidan limit olinsa 4 – xossani isboti kelib chiqadi. Bu xossaga integrallni baholash teoremasi deyiladi.

5 – xossa.

$$\left|\int_{\Gamma} f(z)dz\right| \le \int_{\Gamma}|f(z)|dS.$$

Haqiqatdan ham,

$$\left|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le \sum_{k=1}^{n}|f(\xi_k)||\Delta z_k|$$

$\delta \to 0$ deb faraz qilib, ikkala tomonda limitga o'tish kifoya.

Bunda $|dz| = |dx + idy| = \sqrt{dx^2 + dy^2} = dS$.

6 – xossa.

$$\int_{\Gamma_1+\Gamma_2+\dots+\Gamma_n} f_1(z)dz = \int_{\Gamma_1} f_1(z)dz + \dots + \int_{\Gamma_n} f_n(z)dz.$$

3.3. Integralni hisoblash

(3.7)–ning o'ng tomonidagi har bir had egri chiziqli integrallardan iborat. Integrallni hisoblashning turli usullari mavjud. Agar Γ chiziqning tenglamasi dekart koordinatalari sistemasida berilgan bo'lsa, ya'ni

$$y = f(x) \quad a \le x \le b. \tag{3.8}$$

(3.7) dagi y va dy o'rniga (3.8) dan qiymatlar qo'yilib aniq integralga aylantiriladi. Agar Γ chiziqning tenglamasi

$$x = x(t),\ y = y(t) \quad t_0 \le t \le T, \tag{3.9}$$

parametrik ko'rinishda, ya'ni $z = z(t)$ berilgan bo'lsa, uni (3.6) – ga qo'yib aniq integral hosil qilinadi:

$$\int_{\Gamma} f(z)dz = \int_{t_0}^{T} f(z(t))\cdot z'(t)dt = \int_{t_0}^{T} R(t)dt + i\int_{t_0}^{T} Q(t)dt \tag{3.10}$$

Chap tomondagi integral belgisi ostidagi funksiyaning haqiqiy qismini $R(t)$ bilan, mavhum qismini esa $Q(t)$ bilan belgiladik. Integrallashga oid ba'zi tushunchalarni esga olib o'tamiz.

1)Agar $f(z)$ va $\varphi(z)$ funksiyalar bir bog'lamli G_1 sohada analitik, ya'ni hosilaga ega bo'lsa va z_0, z nuqtalar shu sohadan ixtiyoriy olingan bo'lsa, u holda quyidagi bo'laklab integrallash formulasi o'rinli bo'ladi:

$$\int_{z_0}^{z} f(z)\varphi'(z)dz = [f(z)\cdot\varphi(z)]\Big|_{z_0}^{z} - \int_{z_0}^{z}\varphi(z)f'(z)dz. \tag{3.11}$$

2)Integrallni soddalashtirish uchun ba'zan z o'zgaruvchi boshqa bir ω o'zgaruvchiga $z = \varphi(\omega)$ orqali aylantirishga to'g'ri keladi

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(\varphi(\omega))\cdot\varphi'(\omega)d\omega. \tag{3.12}$$

Γ_1 chiziq Γ ning ouv tekislikdagi aksidan iborat.

3)Agar Γ chiziq markazi $\alpha = a + ib$ nuqtaga joylashgan aylanadan iborat bo'lsa, integralni osonroq hisoblash uchun ushbu aylana tenglamasidan foydalanamiz.

$$z - \alpha = re^{i\varphi}. \tag{3.13}$$

4) Agar Γ chiziq α nuqtadan o'tuvchi to'g'ri chiziq – nurdan iborat bo'lsa ham (3.13) dan foydalanish tavsiya etiladi, bunda φ o'zgarmas bo'ladi $(0 \le r \le \infty)$.

3.1-misol. Ushbu integralni hisoblang:

$$J = \int (z\bar{z} + z^2)dz,$$

bu yerda Γ chiziq $|z| = 1$ aylananing yuqori yarimi, ya'ni $0 \le \arg z \le \pi$.

Yechilishi. $J-$ ni hisoblash uchun (3.13) dan foydalanamiz, bizning misolda $\alpha = 0$

$$r = 1;\ z = e^{i\varphi};\ dz = ie^{i\varphi}d\varphi;\quad 0 \le \varphi \le \pi\ ;\qquad z \cdot \bar{z} = e^{i\varphi} \cdot e^{-i\varphi} = 1.$$

Shuning uchun

$$J = i\int_0^{\pi}(1 + e^{2i\varphi}) \cdot e^{i\varphi}d\varphi = \int_0^{\pi}(e^{i\varphi} + e^{3i\varphi})d(i\varphi) = (e^{i\varphi} + \frac{1}{3}e^{3i\varphi})\Big|_0^{\pi} =$$

$$= (e^{\pi i} - e^0) + \frac{1}{3}(e^{3\pi i} - e^0) = -\frac{8}{3}.$$

Chunki $e^{\pi} = \cos\pi + i\sin\pi = -1$, $e^{3\pi} = \cos 3\pi + i\sin 3\pi = -1$.

Demak, $J = -\frac{8}{3}$.

3.2-misol. Ushbu integrallni hisoblang:

$$J = \int_{\Gamma}(1 + i - 2\bar{z})dz,$$

bu yerda Γ chiziq $z_0 = 0$, $z = 1 + i$ nuqtalardan o'tadigan $y = x^2$ paraboladan iborat.

Yechilishi. $y = x^2,\ x = 0,\ x = 1,\ dy = 2xdx,\ z = x + iy,\ \bar{z} = z - iy$

$$1 + i - 2\bar{z} = 1 + i - 2(x - iy) = (1 - 2x) + i(1 + 2y)$$

$$(1 + i - 2\bar{z})dz = [(1 - 2x) + i(1 + 2y)] \cdot (dx + idy) = [(1 - 2x) + i(1 + 2x^2)](1 + 2x)dx =$$

$$= (1 - 2x + i(1 + 2x^2) + (2x - 4x^2)i - 2x(1 + 2x^2))dx.$$

$$J = \int_0^1 (1 - 2x - 2x - 4x^3 + i(1 + 2x - 2x^2))dx = -2 + \frac{4}{3}i,$$

$$J = -2 + \frac{4}{3}i.$$

3.3-misol. Ushbu integrallni hisoblang:

$$J = \int_{\Gamma} e^{\bar{z}}dz$$

bu yerda Γ chiziq $z_0 = 0$ va $z = \pi - i\pi$ nuqtalarni tutashtiruvchi $y = -x$ to'g'ri chiziq kesmasidan iborat.

Yechilishi. Berilishiga ko'ra $x = 0,\ x = \pi;\ y = 0,\ y = -\pi$

Γ ning parametrik tenglamalarini tuzaylik

$$x = t,\ y = -x = -t,\ 0 \le t \le \pi$$

$$z = x + iy = t - it = (1-i)t;\ \ \bar{z} = (1+i)t$$

U holda

$$e^{\bar{z}} = e^{(1+i)t};\ dz = (1+i)dt$$

$$J = (1+i)\int_0^{\pi} e^{(1+i)t}dt = \frac{1-i}{1+i}\int_0^{\pi} e^{(1+i)t}d(1+i)dt = \frac{1-i}{1+i}e^{(1+i)t}\Big|_0^{\pi} = (1+e^{\pi})i.$$

3.4-misol. Ushbu integrallni hisoblang:

$$J = \int_0^i z\sin z dz$$

Yechilishi. J - ni bo‘laklab integrallaymiz:

$$\int_0^i z\sin z dz = -z\cos z\Big|_0^i + \int_0^i \cos z dz = -i\cos i + \sin i = \frac{-1}{2}i(e^{-1} + e^{1}) + \frac{1}{2i}(e^{-1} - e^{1}) =$$

$$= -\frac{1}{2}i(e^{-1} + e^{1}) - \frac{i}{2}(e^{-1} - e^{1}) = -\frac{i}{e}.$$

3.5-misol. Ushbu integrallni hisoblang:

$$J = \int_{\Gamma} \operatorname{Re} z dz,$$

bu yerda Γ chiziq $z = z(t) = (2+i)t$ tenglama bilan aniqlangan $0 \le t \le 1$.

Yechilishi. Ma’lumki ,

$$z = x + iy,\quad \operatorname{Re} z = x,\quad dz = dx + idy,\quad x = 2t,\ y = t$$

bundan

$$y = \frac{x}{2};\ dx = 2dt,\ dy = dt,\ \operatorname{Re} z dz = x(dx + idy) = xdx + ixdy = 4tdt + 2itdt$$

Demak,

$$J = \int_0^1 (4t + 2it)dt = (2t^2 + it^2)\Big|_0^1 = 2 + i.$$

3.6-misol. Ushbu integrallni hisoblang:

$$\oint_{\Gamma} z\bar{z}dz,$$

bu yerda Γ chiziq $|z| = 1$ aylanadan iborat.

Yechilishi. J ni (3.13) – ga asoslanib hisoblaymiz:

$$z = e^{i\varphi},\qquad z\bar{z} = e^{i\varphi}\cdot e^{-i\varphi} = 1,$$

$$dz = ie^{i\varphi}d\varphi,\quad 0 \le \varphi \le 2\pi,$$

bularga asoslanib

$$J = \int_0^{2\pi} e^{i\varphi}\cdot d(i\varphi) = e^{i\varphi}\Big|_0^{2\pi} = e^{2\pi i} - e^0 = \cos 2\pi + i\sin 2\pi - 1 = 1 + 0 - 1 = 0.$$

Demak, $J = 0$.

3.7-misol. Ushbu integrallni hisoblang:

$$J = \int_\Gamma z \operatorname{Im} z^2 dz,$$

bu yerda Γ chiziq $|z| = 1$ aylananing pastki yarmi $(-\pi \le \varphi \le 0)$.

Yechilishi. (3.13) formuladan foydalanamiz:

$$z = e^{i\varphi}, \qquad dz = ie^{i\varphi}d\varphi,\ z^2 = e^{2i\varphi} = \cos 2\varphi + i\sin 2\varphi,$$

$$\operatorname{Im} z^2 = \sin 2\varphi;\ zdz = ie^{2i\varphi}d\varphi\ ;$$

$$z\operatorname{Im} z^2 dz = i\sin 2\varphi e^{2i\varphi}d\varphi = i\sin 2\varphi(\cos 2\varphi + i\sin 2\varphi)d\varphi =$$

$$= i(\sin 2\varphi\cos 2\varphi + i\sin^2 2\varphi)d\varphi.$$

Demak,

$$J = i\int_{-\pi}^{0} \sin 2\varphi\cos 2d\varphi - \int_{-\pi}^{0} \sin^2 2\varphi d\varphi = \frac{i}{2}\int_{-\pi}^{0} \sin 2\varphi d(\sin 2\varphi) - \frac{1}{2}\int_{-\pi}^{0}(1 - \cos 4\varphi)d\varphi =$$

$$= \frac{i}{2}\frac{\sin^2 2\varphi}{2}\Big|_{-\pi}^{0} - \frac{1}{2}(\varphi - \frac{1}{4}\sin 4\varphi)\Big|_{-\pi}^{0} = -\frac{\pi}{2}.$$

3.4. Nyuton – Leybnits formulasi

Ba'zi integrallarni osonroq yo'l bilan hisoblash mumkin. Agar $f(z)$funksiya z_0 va z nuqtalarni o'z ichiga oluvchi bir bog'lamli G_1 sohada analitik, ya'ni hosilaga ega bo'lsa, u holda ushbu

$$\int_{z_0}^{z} f(z)dz = \phi(z)\Big|_{z_0}^{z} = \phi(z) - \phi(z_0), \tag{3.14}$$

Nyuton – Leybnits formulasi o'rinli bo'lib, bu yerda $\phi(z)$ funksiya $f(z)$ ning biror boshlang' ich funksiyadan iborat, ya'ni $\phi'(z) = f(z),\ z \in G_1$.

Misollar.

3.1. $\int_0^{1+i} z^3 dz = \frac{1}{4}z^4\Big|_0^{1+i} = \frac{1}{4}(1+i)^4 = -1$,bu yerda $f(z) = z^3$ analitik funksiya.

3.2. $\int_1^i (3z^4 - 2z^3)dz = (\frac{3}{5}z^5 - \frac{1}{2}z^4)\Big|_1^i = \frac{3}{5}(1-i)$.

3.3. $\int\limits_0^{1+i} \sin z \cos z dz = \frac{1}{2}\int\limits_0^{1+i} \sin 2z dz = \frac{1}{4}(-\cos 2z)\Big|_0^{1+i} = \frac{1}{4}[1-\cos(2+2i)]$.

3.4. $\int\limits_1^{i} \frac{\ln(1+z)}{1+z}dz$ integralning $|z|=1$ aylananing birinchi choragidagi qismi bo‘ylab olingan qismini hisoblang.

3.5. Ko‘p qiymatli funksiyalar.

Kompleks argumentli funksiyalar ham bir qiymatli va ko‘p qiymatli bo‘lib ikki gruppaga ajratiladi. Ma’lumki, agar $z = x + iy$ – ga bitta qiymat berilganda $w = f(z)$ funksiya ham birgina qiymat qabul qilsa, funksiya bir qiymatli, aks holda ko‘p qiymatli deyiladi. Bu holda funksiyaning har bir qiymati o‘sha funksiyaning tarmog‘ i deyiladi. Masalan,

a) $w = \sqrt{z^2 - 1}$ funksiya tarmog‘ i ikki qiymatli, chunki z – ga biror qiymat berganda w ikkita ildizga ega bo‘ladi. O‘sha tarmoqlarni quyidagicha yozish mumkin:

$$w_1 = \sqrt{z^2-1} \quad \text{va} \quad w_2 = -\sqrt{z^2-1}.$$

b) $w = f(z) = Lnz$ funksiya cheksiz ko‘p tarmoqlidir, chunki

$$w = Lnz = \ln|z| + i\arg z + 2k\pi i, \quad k = 0, \pm 1, \pm 2.$$

Buning uchun tarmoqlarni quyidagicha yozish mumkin:

$$w_0 = \ln|z| + i\varphi, \quad w_1 = \ln|z| + (2\pi + \varphi)i, \quad w_2 = \ln|z| + (4\pi + \varphi)i,$$

$$w_{-1} = \ln|z| + (\varphi - 2\pi)i, \quad w_{-2} = \ln|z| + (\varphi - 4\pi)i, \dots \ |z| = r.$$

Odatda teskari funksiyalar ko‘p qiymatli bo‘ladi.

3.8-misol. $\int\limits_\Gamma \frac{dz}{\sqrt{z}}$ ni hisoblang, bu yerda Γ-chiziq $|z| = 1$ aylananing yuqori qismi $w = \frac{1}{\sqrt{z}}$ funksiyaning shunday tarmog‘i olinsinki, natijada $\sqrt{1} = -1$ bo‘lsin.

Yechilishi. Misolni bir necha usullar bilan yechish mumkin.

a) $z = r(\cos\varphi + i\sin\varphi),\ r = |z|,\ \varphi = \arg z$

$$\alpha_k = \sqrt{z} = \sqrt{r}\left(\cos\frac{(\varphi + 2k\pi)}{2} + i\sin\frac{\varphi + 2k\pi}{2}\right),\ k = 0,1;\ r = 1;$$

$$\alpha_0 = \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2};\ \alpha_1 = \cos\frac{\varphi + 2\pi}{2} + i\sin\frac{\varphi + 2\pi}{2} = -(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}).$$

Berilgan $\sqrt{1}=-1$ shartga ko'ra α_1 – ildizni olishga to'g'ri keladi, chunki $z=1$ bo'lganda $\varphi=0$ bo'lib, $\alpha_1=-1$ bo'ladi. Nyuton – Leybnis formulasiga muvofiq

$$J=\int_1^{-1} z^{-\frac{1}{2}}dz=2\sqrt{z}\Big|_1^{-1}=2(\sqrt{-1}-1)=2(i-1).$$

Ma'lumki $z=-1$ bo'lganda $\varphi=\pi$ bo'lib, α_1 – ga muvofiq

$$\sqrt{-1}=-\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=-i.$$

Demak,

$$J=2(-i+1)=2(1-i).$$

b) $z=re^{i\varphi},\ r=|z|=1,\ z=e^{i\varphi},\ \sqrt{z}=\sqrt{e^{i\varphi}}=\sqrt{\cos\varphi+i\sin\varphi}=$

$$=\cos\frac{(\varphi+2\pi k)}{2}+i\sin\frac{\varphi+2\pi k}{2}=\alpha_k\quad k=0,1,2...$$

Endi $\sqrt{1}=-1$ shartga ko'ra α_1 ildizni olamiz, chunki $z=1$ bo'lganda $\varphi=0$ bo'lib,

$$\alpha_1=e^{\pi i}=\cos\pi+i\sin\pi=-1;\ dz=ie^{i\varphi}d\varphi$$

Aylananing ustki yarmida $0\le\varphi\le\pi$ Shu sababli berilgan integrallni aniq integrallga aylantira olamiz:

$$J=i\int_0^{\pi}e^{i\varphi}d\varphi=i\int_0^{\pi}e^{i(\frac{\varphi}{2}-\pi)}d\varphi=2\int_0^{\pi}e^{i(\frac{\varphi}{2}-\pi)}d\left[i\left(\frac{\varphi}{2}-\pi\right)\right]=2e^{i(\frac{\varphi}{2}-\pi)}\Big|_0^{\pi}=2(e^{-\frac{\pi}{2}i}-e^{\pi i})=2(1-i)$$

Mustaqil yechish uchun misollar

3.5. $\int_{\Gamma}(x^2-iy^2)dz$ Γ chiziq $z_0=1-i$ va $z=2-3i$ nuqtalarni tutashtiruvchi to'g'ri chiziq.

3.6. $\int_1^{1+i} zdz$ hisoblang.

3.7. $\oint_{\Gamma}\bar{z}dz$ bu yerda Γ chiziq $x=\cos t,\ y=\sin t\ \ (0\le t\le 2\pi)$.

3.8. $\oint_{\Gamma}\frac{dz}{z-4}$, bu yerda Γ chiziq $x=3\cos t,\ y=2\sin t$ Ellipsdan iborat.

3.9. $\int z^2dz$, bu yerda Γ chiziq $z_0=1$ va $z=i$ nuqtalarni tutashtiruvchi to'g'ri chiziq kesishmasi.

3.10. $\oint\limits_{\Gamma}\frac{dz}{z}$, bu yerda Γ chiziq $x=\cos t,\ y=\sin t$ aylanadan iborat.

3.11 $\int\limits_{1+i}^{1-i}(2z+1)dz$ hisoblang.

3.12. $\int\limits_{\Gamma}e^{z}dz$, bu yerda Γ chiziq $z_0=0$, $z=1+i$ nuqtalardan o‘tadigan $y=x^2$ parabolaning qismidir.

3.13. $\int\limits_{\Gamma}\cos zdz$, bu yerda Γ chiziq $z_0=\frac{\pi}{2}$ va $z=\pi+i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasi.

3.14. $\int\limits_{1+i}^{2i}(z^3-z)e^{\frac{z^2}{2}}dz$ hiisoblang.

3.15. $\int\limits_{0}^{i}z\cos zdz$ hisoblang.

3.16. $\int\limits_{1}^{i}z\sin zdz$ hisoblang.

3.17. $\int\limits_{\Gamma}\mathrm{Re}(\sin z)\cos zdz$, bu yerda Γ chiziq $|\mathrm{Im}\,z|\le 1$ $\mathrm{Re}\,z=\frac{\pi}{4}$.

3.18. $\int\limits_{\Gamma}\mathrm{Im}(z^2)dz$, bu yerda Γ chiziq: $|\mathrm{Im}\,z|\le 1$ $\mathrm{Re}\,z=1$.

3.19. $\int\limits_{-i}^{i}e^{z^2}dz$ hisoblang.

3.20. $\int\limits_{0}^{1}(1+it)^2dt$, t - haqiqiy o‘zgaruvchi.

3.21. $\int\limits_{0}^{1}\frac{dt}{1+it}$, t - haqiqiy o‘zgaruvchi.

3.22. $\int\limits_{0}^{1}\frac{1+it}{1-it}dt$, t - haqiqiy o‘zgaruvchi

3.23. $I=\int\limits_{\Gamma}(x^2+iy^2)dz$, bu yerda Γ chiziq: $z_0=1+i$ va $z=2+3i$ nuqtalarni tutashtiruvchi to‘g‘ri chiziq kesmasidan iborat.

3.24. $\oint\limits_{|z|=2}\frac{e^{z}dz}{z^2-1}$ hisoblang.

3.25. $I=\int\limits_{\Gamma}(1+i-z\bar{z})dz$, bu yerda Γ chiziq $z_0=1+i$, $z=1+2i$ nuqtalardan o‘tkazilgan $y=x^2$.

3.26. $I=\int\limits_{\Gamma}(z\bar{z}-z^2)dz$, bu yerda Γ chiziq: $|z|=1$ aylana $(-\pi\le\arg z\le 0)$.

3.27. $\oint\limits_{\Gamma}\frac{zdz}{(z-2)^2(z+1)}$, Γ chiziq: $|z|=1$ $|z-3|=6$ aylana.

3.28. $I=\int e^{|z|^2}\operatorname{Im}zdz$, Γ : $z_0=0$ $z=1+i$ nuqtalardan o‘tuvchi to‘g‘- ri chiziq.

3.29. $I=\oint\limits_{|z-1|}\frac{\sin\pi z}{(z^2-1)^2dz}$, $|z-1|<1$.

3.30. $I=\oint\limits_{|z|=2}\frac{\sin z\sin(z-1)}{z^2+z}dz$ hisoblang.

3.31. $I=\oint\limits_{|z|=3}\frac{z^2dz}{z-2i}$ hisoblang.

3.32. $\int\limits_{\Gamma}\frac{e^zdz}{(z^2+9)(z-\frac{1}{2})}$, bu yerda Γ : $x^2+y^2=2x+2y+4$.

3.33. $\oint\limits_{\Gamma}\frac{\sin(\cos 2z)}{(z+1)(z^2-1)}dz$, bu yerda L: $\left|z-\frac{1}{2}\right|=1$.

3.34. $\int\limits_{|z|=4}\frac{e^{2z}}{z^2+9}dz$ hisoblang.

3.35. $\int\limits_{|z+5i|=2}\frac{e^{z^2}+5z+6}{(z^2+16)(z+4i)}dz$ hisoblang.

3.36. $\int\limits_{|z-2|=2}\frac{\cos z}{(z-3)(z^2-9)}dz$ hisoblang.

3.37. $\int\limits_{|z-i|=2}\frac{e^{\sin(2z^2+5)}}{(z^2+4)(z-2i)}dz$ hisoblang.

3.38. $\int\limits_{|z-i|=2}\frac{\sin(e^{2z+3})}{(z-2)(z^2-4)}dz$ hisoblang.

3.39. $\int\limits_{|z-3|=3}\frac{e^{z^2}+4z}{(z-7)(z^2-49)}dz$ hisoblang.

3.40. $\int\limits_{|z+2|=2}\frac{3^{z^2}-4z+3}{(z+3)(z^2-9)}dz$ hisoblang.

3.41. $\oint\limits_{|z|=4}\frac{\sin iz}{z^2-4z+3}dz$ hisoblang.

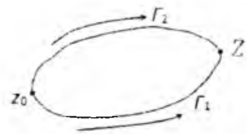
3.42. $\oint\limits_{|z|=3} \frac{\cos(z+\pi i)}{z(e^z+2)}dz$ hisoblang.

3.6. Yopiq kontur bo'ylab olingan integral

Ushbu

$$\oint\limits_{\Gamma} f(z)dz$$

integrallning qymati, umuman olganda, ikki narsaga, ya'ni berilgan funksiyaga va Γ chiziqning formasiga bog'liq. Agar z_0 va Z nuqtalarni tutashtiruvchi ikki xil Γ_1 va Γ_2 chiziqlarni olsak (3.3 – chizma),



***3.3 – chizma.** z_0 va Z nuqtalarni tutashtiruvchi ikki xil chiziqlar.*

$f(z)$ dan olingan integrallning qiymatlari ham umuman ikki turli bo'ladi:

$$\int\limits_{\Gamma_1} f(z)dz \neq \int\limits_{\Gamma_2} f(z)dz.$$

Ba'zi vaqtlarda bu ikkala integrallning qiymatlari bir–biriga teng bo'lib qolishi ham mumkin, ya'ni

$$\int\limits_{\Gamma_1} f(z)dz = \int\limits_{\Gamma_2} f(z)dz. \qquad (3.15)$$

Bunday holda integrallning qiymati integrallash yo'liga bog'liq emas deyiladi. Shu sababli (3.15) ni

$$\int\limits_{\Gamma_1} f(z)dz = \int\limits_{\Gamma_2} f(z)dz = \int\limits_{z_0}^{z} f(z)dz,$$

ko'rinishda yozish mumkin.

U holda (3.15) dan

$$0 = \int\limits_{\Gamma_1} f(z)dz - \int\limits_{\Gamma_2} f(z)dz = \int\limits_{\Gamma_1} f(z)dz + \int\limits_{\Gamma_2^-} f(z)dz = \int\limits_{\Gamma_1+\Gamma_2^-} f(z)dz = \oint\limits_{\Gamma} f(z)dz,$$

kelib chiqadi, ya'ni yopiq kontur bo'ylab olingan integral nolga teng:

$$\oint_{\Gamma} f(z)dz = 0.$$

Endi, $f(z)$ funksiyadan biror Γ chiziq bo'ylab olingan integral qiymatining integrallash yo'liga bog'liq bo'lmasdan, faqat uning boshlang'ich va ohirgi nuqtalariga bog'liq bo'lishi uchun $f(z)$ funksiya qanday shartlarga bo'ysunishi kerak, degan savol tug' iladi. Bu savolga Koshi teoremasi javob beradi.

3.7. Koshi teoremasi

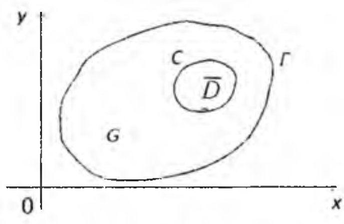
3.1-teorema. Agar bir bog'lamli G sohada $f(z)$ funksiya analitik bo'lsa, u holda G – da yotuvchi har qanday Γ yopiq kontur bo'ylab $f(z)$ funksiyadan olingan integral nolga teng bo'ladi:

$$\oint_{\Gamma} f(z)dz = 0.$$

Isbot. Agar qo'shimcha shart $f(z)$ funksiyaning G sohada uzluksizligini talab qilsa bu teoremaning o'rinli ekanligi Koshi – Riman shartlari va Grin formulasiga asosan

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} u(x;y)dx - v(x;y)dy + i\int_{\Gamma} v(x;y)dx + u(x;y)dy$$

formuladan bevosita kelib chiqadi. Haqiqatdan ham, matematik analiz kursidan ma'lumki, $P(x,y)$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$ lar yopiq $\overline{D}$ sohada uzluksiz bo'lsa (3.4- chizma),



3.4-chizma. Bir bog'lamli soha.

u holda ushbu

$$\int_{\Gamma} Pdx + Qdy = \iint_{D}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

Grin formulasi o‘rinlidur, bundagi D yopiq konturning ichki qismidan iborat.

Ravshanki, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ ning uzliksizligidan

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y},$$

hosilalarning, shuningdek, $u(x;y)$ va $v(x;y)$ funksiyalarning uzluksizligi kelib chiqadi. Grin formulasidan foydalanib, quyidagilarni hosil qilamiz:

$$\oint_{\Gamma} f(z)dz = \oint_{\Gamma}(udx - vdy) + i\oint_{\Gamma} vdx + udy = \iint_{D}\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_{D}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy.$$

Koshi – Riman shartlariga asosan

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

U holda

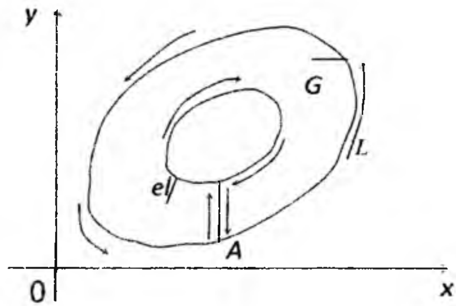
$$\oint_{\Gamma} f(z)dz = 0$$

o‘rinli ekanligi kelib chiqadi.

Ko‘p bog‘lamli soha uchun Koshi teoremasini ko‘ramiz.

3.2-teorema. Agar ko‘p bog‘lamli yopiq G sohada $f(z)$ funksiya analitik bo‘lsa, u holda shu sohaning butun konturi bo‘ylab musbat yo‘nalishda $f(z)$ funksiyadan olingan integral nolga teng bo‘ladi.

Isbot. Bu teoremani ikki bog‘lamli soha uchun isbotlaymiz (3.5- chizma).



3.5- chizma. Ikki bog'lamli soha.

G soha tashqi tomondan L chiziq bilan, ichki tomondan e chiziq bilan chegaralangan ikki bog'lamli soha bo'lsin. G sohaning hamma chegaralari birgalikda murakkab kontur deyiladi, bitta $c=L+e$ bilan belgilaymiz. U holda

$$\oint_c f(z)dz=0 \quad \text{yoki} \quad \oint_L f(z)dz+\oint_e f(z)dz=0 \qquad (3.16)$$

ekanligini ko'rsatishimiz keark.

G sohada A va B nuqtalarni qirqib, λ chiziq bilan tutashtiramiz. U holda ikki bog'lamli G soha bir bog'lamli C sohaga aylanadi. Shuning uchun bir bog'lamli Koshi teoremasiga asosan quyidagini hosil qilamiz.

$$\oint_L f(z)dx+\int_{AB} f(z)dz+\oint_e f(z)dz+\oint_{BA} f(z)dz=0,$$

bu yerda $\int_{AB} f(z)dz=\int_{BA} f(z)dz$ bo'lganligidan (3.16) formila kelib chiqadi.

Teoremani n bog'lamli soha uchun shunga o'xshash isbotlash mumkin.

(3.16) formuladan

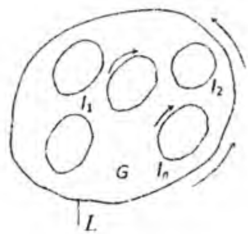
$$\int_L f(z)dz=\int_e f(z)dz \qquad (3.17)$$

kelib chiqadi. Bu (3.17) ni quyidagicha ta'riflash mumkin: Tashqi kontur bo'ylab olingan integral ichki kontur bo'ylab olingan integralga teng. Koshi teoremasining mana shu xususiy holi tajribada ko'p ishlatiladi.

n bog'lamli soha uchun (3.17) formulani o'rinli ekanligini ko'rsatish mumkin. G soha tashqi tomondan L chiziq bilan va ichki tomondan o'zaro kesishmaydigan oddiy $e_1, e_2, \dots e_n$ yopiq chziqlar bilan chegaralangan $n+1$ bog'lami soha bo'lsin (3.6-chizma).
U holda

$$\oint_L f(z)dz = \oint_{e_1} f(z)dz + \oint_{e_2} f(z)dz + \dots + \oint_{e_n} f(z)dz \qquad (3.18)$$

bo'ladi. Bunda L bo'ylab harakat soat strelkasi aylanishga teskari va $e_1, e_2, \dots e_n$ chiziqlar bo'ylab harakat soat strelkasi yo'nalishi bo'ycha harakat qiladi. $f(z)$ funksiya G sohada va uning barcha chegaralarida analitik bo'ladi. Ba'zan Koshining ikkinchi teoremasini boshqacha ta'riflash mumkin.



3.6- chizma. Ko'p bog'lamli soha.

Koshining ikkinchi teoremasi. Agar ko'p bog'lamli yopiq $\overline{G}$ sohada $f(z)$ funksiya analitik bo'lsa, u holda $f(z)$ funksiyadan tashqi kontur bo'ylab olingan integral ichki konturlar bo'ylab olingan integrallar yig'indisiga teng bo'lib, hamma konturlar bo'yicha yo'nalishi soat strelkasi harakati yo'nalishiga teskari olinadi.

Biz bu teoremani (3.18) tenglikka asoslanib yozdik. Bunda $e_1, e_2, \dots e_n$ konturlar bo'ylab olingan hamma integrallar musbat yo'nalish bo'yicha olingan.

Agar (3.18) tenglamani ikki tomonini (-1) ga ko'paytirib uni quyidagicha yozamiz:

$$-\oint_L f(z)dz = -\oint_{e_1} f(z)dz - \oint_{e_2} f(z)dz - \dots - \oint_{e_n} f(z)dz$$

yoki

$$\oint_{\bar{L}} f(z)dz = \oint_{\bar{e}_1} f(z)dz + \oint_{\bar{e}_2} f(z)dz + \ldots + \oint_{\bar{e}_n} f(z)dz$$

bunda konturlarning yo‘nalishi soat strelkasi yo‘nalishi bo‘yicha olingan bo‘ladi. Ikkala holda ham hamma konturlarning yo‘nalishlari bir hilda bo‘ladi. Shu sabali so‘ngi teoremada konturlar yo‘nalishi "soat strelkasi harakati yo‘nalishi bo‘yicha" deyish ham mumkin.

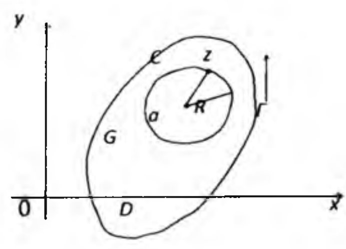
3.8. $\oint_{\Gamma} \frac{dz}{z-a}$ integralni hisoblash

Koshi teoremasidagi Γ konturni markazi a nuqtada radiusi R bo‘lgan aylana bilan almashtiramiz (3.7- chizma).

Kompleks sonning ko‘rsatkichli ko‘rinishi: $z = \mathrm{R}e^{i\varphi}$

Aylananing tenglamasi $|z-a| = R$ yoki $z - a = \mathrm{R}e^{i\varphi}$, $dz = i\mathrm{R}e^{i\varphi}d\varphi$,

$$\oint_{\Gamma} \frac{dz}{z-a} = \oint_{c} \frac{dz}{z-a} = \int_0^{2\pi} i\frac{\mathrm{R}e^{i\varphi}}{\mathrm{R}e^{i\varphi}}d\varphi = 2\pi i.$$



3.7-chizma. Γ konturni markazi a nuqtada radiusi R bo‘lgan aylana.

Agar a nuqta konturdan tashqarida yotsa, u holda Koshi teoremasiga muvofiq

$$\oint_{\Gamma} \frac{dz}{z-a} = 0.$$

Misol. $I = \oint_{c} \frac{dz}{(z-a)^n}$,c - aylana.

1. $n \le 0$ bo‘lsa $\frac{1}{(z-a)^n} = (z-a)^m$, $(n = -m)$

bu funksiya butun (z) tekislikda analitik bo‘lganligi uchun Koshi teoremasiga asosan $a \overline{\in} D$ da $I=0$ bo‘ladi.

2. $n>0; a \in D$. Bunda e ni kesmaydigan radiusi R, markazi a nuqtada bo‘lgan c aylana yasasak, ikki bog‘lamli soha hosil bo‘ladi. $\frac{1}{(z-a)^n}$ funksiya c va e orasidagi halqada analitik bo‘lganidan Koshi teoremasiga muvofiq:

$$I=\oint_e \frac{dz}{(z-a)^n}=\oint_c \frac{dz}{(z-a)^n}=\int_0^{2\pi} \frac{i\mathrm{Re}^{i\varphi}\,d\varphi}{R^n e^{in\varphi}}=\frac{1}{R^{n-1}}\int_0^{2\pi} e^{i(1-n)\varphi}d\varphi$$

Agar $n=1$ bo‘lsa $I=i\int_0^{2\pi} d\varphi=2\pi i$ bo‘ladi.

Agar $n>1$ bo‘lsa $I=\frac{i}{R^{n-1}}\cdot\frac{1}{i(1-n)}e^{i(1-n)\varphi}\Big|_0^{2\pi}=0$ bo‘ladi.

Demak,

$$\oint_e \frac{dz}{(z-a)^n}=\begin{cases}0, & agar \;\; n\neq 1\\ 2\pi i, & agar \;\; n=1\end{cases}$$ bo‘ladi.

3.9. Koshining integral formulasi

3.3-teorema. Agar bir bog‘lamli yopiq D sohada va uning chegaralovchi e konturda $f(z)$ funksiya analitik bo‘lib, a nuqta soha ichida, z nuqta esa e konturda yotsa, u holda

$$2\pi i f(a)=\oint_e \frac{f(z)}{z-a}dz \Rightarrow f(a)=\frac{1}{2\pi i}\oint_e \frac{f(z)}{z-a}dz \qquad (3.19)$$

o‘rinli bo‘ladi, (3.19) formulaga Koshining integral formulasi deyiladi.

Isbot. Teorema shartiga ko‘ra $\frac{f(z)}{z-a}$ funksiya e kontur va markaziy a nuqtada radiusi ρ bo‘lgan aylana bilan chegaralangan halqada analitikdir. ρ ni shunday tanlab olamizki, $|z-a|=\rho$ aylana o‘qining c chegarasi bilan e konturi ichida yotsin. Shu sababli Koshi teoremasiga muvofiq tashqi e kontur bo‘ylab olingan integral ichki c kontur bo‘ylab olingan integralga teng.

$$I=\frac{1}{2\pi i}\oint_e \frac{f(z)}{z-a}dz=\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz=$$

$$=\frac{1}{2\pi i}\oint_c \frac{f(z)-f(a)+f(a)}{z-a}dz = I_1+\frac{f(a)}{2\pi i}\oint_c \frac{dz}{z-a},$$

bu yerda $I_1=\frac{1}{2\pi i}\oint_c \frac{f(z)-f(a)}{z-a}dz;\ \frac{1}{2\pi i}\oint_c \frac{dz}{z-a}=1.$

$$I=\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}=I_1+f(a).$$

Teoremani isbotlash uchun $I_1=0$ ekanligini ko‘rsatish yetarlidir. $f(z)$ funksiya a nuqtada uzluksizligidan $\forall(\varepsilon>0)\exists\delta=\delta(\varepsilon)>0$ sonni topish mumkinki c aylananing ixtiyoriy z nuqtasi uchun $|z-a|=\rho<\delta$ tengsizlikni qanoatlantiradigan barcha qiymatlarda $|f(z)-f(a)|<\varepsilon$ tengsizlik o‘rinli bo‘ladi. Demak,

$$|I_1|<\frac{1}{2\pi}\oint_c \frac{|f(z)-f(a)|}{|z-a|}dz<\frac{1}{2\pi}\cdot\frac{\varepsilon}{\rho}\int_c dz=\frac{\varepsilon 2\pi\rho}{2\pi\rho}=\varepsilon.$$

I_1 integral ρ ga bog‘liq emas. Bundan $I_1=0$ kelib chiqadi. Demak

$$I=\frac{1}{2\pi i}\oint_e \frac{f(z)}{z-a}dz=f(a).$$

Teorema isbot bo‘ldi.

$\oint_e \frac{f(z)}{z-a}dz$ integralga Koshi integrali deyiladi. Koshi formulasining mohiyati shundaki, u D sohaning ichki nuqtasiga tegishli funksiya qiymatini o‘sha funksiyaning e konturdagi qiymati orqali aniqlanadi.Koshi formulasi ko‘p bog‘lamli soha uchun ham o‘z kuchini saqlaydi.

3.9-misol. $I=\oint_\Gamma \frac{dz}{z-a}$ ni hisoblang, bu yerda Γ chiziq markazi a nuqtada joylashgan ixtiyoriy aylana.

Yechilishi. Bu misolda $f(z)=1$bo‘lgani uchun $\oint_\Gamma \frac{f(z)dz}{z-a}=2\pi i f(a)$

(3.19) formulaga asoslanib $\oint_\Gamma \frac{dz}{z-a}=2\pi i$ agar $a=0$ bo‘lsa, aylana markazi 0 nuqtada bo‘ladi, u holda

$$\oint_\Gamma \frac{dz}{z}=2\pi i.$$

3.10-misol. $I=\oint\limits_{\Gamma}\frac{z^2dz}{z-2i}$ ni hisoblang, bu yerda Γ chiziq $|z|=3$ aylanadan iborat.

Yechilishi. $a=2i$ nuqta aylana ichida bo'lgani uchun (3.19) formulaga asosan:

$$f(z)=z^2=(2i)^2=-4 \qquad I=-4\cdot 2\pi i=-8\pi i.$$

3.11-misol. $I=\oint\limits_{\Gamma}\frac{\sin z dz}{z+i}$ ni hisoblang, bu yerda Γ chiziq $|z+i|=R$ aylanadan iborat bo'lib, $a=-i$ nuqta aylana ichida.

Yechilishi.

$$f(z)=\sin z=\frac{1}{2i}\left(e^{iz}-e^{-iz}\right), \qquad f(a)=f(-i)=\frac{1}{2i}\left(e-e^{-1}\right)=\frac{1}{i}sh1,$$

shu sababli (3.19) – formulaga muvofiq $I=2\pi i\frac{sh1}{i}=2\pi sh1$ iborat.

3.12-misol. $I=\oint\limits_{\Gamma}\frac{dz}{z^2+9}$ ni hisoblang. Bu yerda Γ chiziq $|z-2i|=2$ aylanadan iborat.

Yechilishi. $z^2+9=(z+3i)(z-3i)$ bu yerda ikkita nuqta bo'lib, shulardan $a=3i$ yuqoridagi aylana ichida. Shu sababli berilgan integralni quyidagicha yozib olamiz:

$$I=\oint\limits_{\Gamma}\frac{\frac{dz}{z+3i}}{z-3i} \qquad f(z)=\frac{1}{z+3i}; \quad f(a)=f(3i)=\frac{1}{3i+3i}=\frac{1}{6i}=-\frac{i}{6}.$$

Demak, (3.19) – ga muvofiq $I=2\pi i f(3i)=\frac{\pi}{3}$.

3.10. Analitik funksiyaning yuqori tartibli hosilasi

3.3-teorema. Bir bog'lamli yopiq D sohada analitik bo'lgan har qanday $f(z)$ funksiya shu sohada istalgan tartibli hosilaga ega bo'lib, ular

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint\limits_{e}\frac{f(\xi)}{(\xi-a)^{n+1}}d\xi, \quad (n=1,2,3.....) \qquad (3.20)$$

formula bilan ifodalanadi, bunda e kontur D sohaning chegarasidir.

Isbot. Biz $n=1$ uchun (3.20) formula to'g'riligini isbot qilamiz. Buning uchun Koshining integral formulasi (3.19) dan foydalanamiz.

U yerdagi $a=z$ va $z=\xi$ bilan almashtirib

$$f(z)=\frac{1}{2\pi i}\oint_e \frac{f(\xi)}{\xi - z}d\xi \qquad (3.21)$$

(3.21) hosil qilamiz. $z-$ ga shunday orttirma beramizki, $z+h\in D$ bo'lsin,

$$f(z+n)=\frac{1}{2\pi i}\oint_e \frac{f(\xi)}{\xi - z - h}d\xi. \qquad (3.22)$$

(3.22) dan (3.21) ni ayirib, h ga bo'lamiz, ya'ni

$$\frac{f(z+h)-f(z)}{h}=\frac{1}{h}\left[\frac{1}{2\pi i}\oint_e \frac{f(\xi)d\xi}{\xi - z - h}-\frac{1}{2\pi i}\oint_e \frac{f(\xi)d\xi}{\xi - z}\right]=\frac{1}{2\pi i}\oint_e \frac{f(\xi)d\xi}{(\xi - z - h)(\xi - z)}.$$

$h\to 0$ limitga o'tsak,

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\frac{1!}{2\pi i}\oint_e \frac{f(\xi)d\xi}{(\xi - z)^2},$$

bu yerda $f'(z)$ funksiya D da analitik. Yuqoridagi mulohazani $f'(z)$ uchun tatbiq qilsak

$$f''(z)=\lim_{h\to 0}\frac{f'(z+h)-f'(z)}{h}=\frac{2!}{2\pi i}\oint_e \frac{f(\xi)d\xi}{(\xi - z)^3}$$

formula kelib chiqadi.

Matematik induksiya metodidan foydalanib (3.20) formulani to'g'ri ekanligiga ishonch hosil qilamiz.

Natija. Agar $f(z)$ funksiya biror sohaning z nuqta atrofida analitik, ya'ni $f(z)$ mavjud bo'lsa, u holda shu nuqta atrofida istalgan tartibli hosilaga ega bo'lib, o'sha hosilaning har biri shu sohada uzluksiz bo'ladi. Bu xususiyat haqiqiy argumentli funksiya uchun bajarilmasligi mumkin.

3.13-misol. $I=\oint_{|z|=2}\frac{z^3+2}{(z-i)^2}dz$ hisoblansin.

Yechilishi. $f(z)=z^3+2$ funksiya $|z|=2$ doira ichida va chegarasida analitik.

$z=i$ nuqta $|z|=2$ doira ichida yotadi, shuning uchun (3.20) formulaga asosan $n=1$ bo'lganda $f'(z)=3z^2$, $f'(i)=3i^2=-3$

$$I=\frac{2\pi i}{1!}f'(i)=-6\pi i$$

3.14-misol. $I=\oint_{\Gamma}\frac{e^z dz}{(z+2)^4}$ ni hisoblang. Γ chiziq $a=-2$ nuqtani o‘z ichiga olgan har qanday yopiq kontur.

Yechilishi. (3.20) formulaga asosan:

$$f(z)=e^z;\ n=3 \quad f'''(z)=e^z;\ f'''(-2)=e^{-2}$$

$$I=\frac{2\pi i}{n!}f'''(a)=\frac{2\pi i}{3!}e^{-2}=\frac{\pi i}{3e^2}$$

3.15-misol. $I=\oint_{|z+1|=1}\frac{dz}{(z+1)(z-1)^3}$ ni hisoblang.

Yechilishi. $I=\int\frac{\frac{dz}{(z-1)^3}}{z+1}$ chunki $a=-1$ nuqta $|z+1|=1$ aylana ichida demak,

$$f(z)=\frac{1}{(z-1)^3},\ f(-1)=\frac{1}{(-2)^3}=-\frac{1}{8}.\ I=2\pi i\left(-\frac{1}{8}\right)=-\frac{\pi i}{4}.$$

3.16-misol. $I=\oint_{\Gamma}\frac{e^z dz}{z(1-z)^3}$ ni hisoblang. Γ chiziq $|z|<\frac{1}{2}$ doiraning chegarasidir.

Yechilishi. $a=0$ nuqta o‘sha chegara ichiga joylashgankigi uchun $f(z)$ deb ushbu funksiyani qabul qilamiz:

$$f(z)=\frac{e^z}{(1-z)^3},\qquad f(0)=\frac{e^0}{1}=1.$$

U holda (3.19) – ga ko‘ra $I=2\pi i$.

3.17-misol. 3.16 – misolda $|z|<\frac{3}{2}$ bo‘lsin.

Yechilishi. Bu doira ichida $a=0, a=1$ nuqtalar joylashadi. Shu sababli bu integralni bir necha integrallarga ajratamiz. Buning uchun matematik analiz kursidagi ushbu qoidadan foydalanamiz:

$$A(1-z)^3+Bz+Cz(1-z)+Dz(1-z)^2=1$$

bu yerda z ga $0;\pm1;\pm2$ qiymatlar berilsa $A=B=C=D=1$hosil bo‘ladi. Demak,

$$I = I_1 + I_2 + I_3 + I_4 = \oint_{\Gamma} \frac{e^z dz}{z} + \oint_{\Gamma} \frac{e^z dz}{(1-z)^3} - \oint_{\Gamma} \frac{e^z dz}{(1-z)^2} - \oint_{\Gamma} \frac{e^z dz}{(1-z)} = \oint_{\Gamma} \frac{e^z dz}{z} +$$

$$+ \oint_{\Gamma} \frac{e^z dz}{(z-1)^3} - \oint_{\Gamma} \frac{e^z dz}{(z-1)^2} - \oint_{\Gamma} \frac{e^z dz}{z-1},$$

bu yerda $f(z) = e^z$, $f'(z) = f''(z) = e^z$, $f(0) = e^0 = 1$, $f(1) = e$, $f'(1) = f''(1) = e$

U holda

$$I = 2\pi i \cdot 1 + \frac{2\pi i}{2!} e - 2\pi e i - 2\pi i e = (2 - e)\pi i.$$

Nazorat savollari.

1. Boshlang'ich va aniqmas integral ta'rifi.
2. Koshining integral formulasi yozib bering.
3. Garmonik funksiya ta'rifi.

IV BOB. QATORLAR

4.1. Teylor qatori

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, funksiya a nuqtaning biror atrofida analitik bo'lsa, uni $(z-a)$ ga nisbatan musbat darajali qatorga yig' ish mumkin:

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots \qquad (4.1)$$

Bu darajali qatorning yaqinlashish radiusi $R > 0$ bo'lsin. Abel teoremasiga muvofiq bu qator $R'(R' < R)$ radiusli har qanday doirada tekis yaqinlashuvchidir. Qatorning har bir hadi chekli tekislikda, jumladan, yuqorida aytilgan doirada analitik bo'lgani uchun Veyershtrass teoremasiga muvofiq, qator yig'indisi

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots \qquad (4.2)$$

shu doirada analitik bo'ladi hamda istalgancha (4.2) dan hosila olish mumkin bo'lib, hosilaviy qatorlar $|z-a| \le R$ doirada tekis yaqinlashadi:

$$f'(z) = c_1 + 2c_2(z-a) + \ldots + nc_n(z-a)^{n-1}$$

$$f''(z) = 1\cdot 2c_2 + 3\cdot 2\cdot c_3(z-a) + \ldots + n(n-1)c_n(z-a)^{n-2} + \ldots$$

$$f^{(k)}(z) = k(k-1)(k-2)(k-3)\ldots 2\cdot c_k + \ldots$$

$$f^{(n)}(z) = n(n-1)(n-2)(n-3)\ldots 2\cdot c_n + \ldots$$

Bulardan $z = x + iy$ yaqinlashish doirasi ichidagi ixtiyoriy nuqtadir.

Xususiy holda $z = a$ doira markazidan iborat bo'lib quyidagi

$$f(a) = c_0 \quad f'(a) = c_1, \quad f''(a) = 2!c_2, \quad f^{(n)}(a) = n!c_n \qquad (4.3)$$

yoki $c_0 = f(a)$ $c_1 = \dfrac{f'(a)}{1!}$, $c_2 = \dfrac{f''(a)}{2!}$, $\ldots, c_n = \dfrac{f^{(n)}(a)}{n!}, \ldots$

formulalar kelib chiqadi. Bu qiymatlarni (4.2) tanglikka qo'ysak

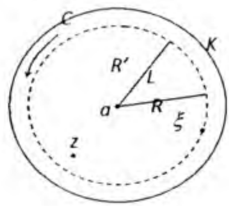
$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots \qquad (4.4)$$

hosil bo'lib, bu qator $f(z)$ funksiyaning Teylor qatori deyiladi. (4.3) ning o'ng tomondagi sonlarga Teylor koeffitsentlari deyiladi.

Demak yaqinlashish radiusga ($R>0$) musbat bo'lgan har qanday darajali qator bu Teylor qatoridir.

Har qanday funksiyani Teylor qatoriga yoyish mumkinmi degan savolga quyidagi teorema javob beradi.

4.1-teorema. Berilgan nuqtada analitik bo'lgan haq qanday funksiyani shu nuqtaning biror atrofida $(z-a)$ ning darajalari bo'yicha Teylor qatoriga yoyish mumkin. Bu qator $f(z)$ analitik bo'lgan $|z-a|<R$ doirada yaqinlashuvchidir (4.1- chizma).



4.1-chizma. Markazi a *nuqtada radiusi* R *ga teng bo'lgan aylana.*

Isbot. $f(z)$ funksiya, K aylana bilan chegaralangan $(z-a)=R$ doira ichida analitik bo'lsin. Mana shu K – aylana ichidagi ixtiyoriy nuqtani z bilan belgilab markazi "a" nuqtaga joylashgan R' radiusli $(R>R')$ C aylanani shunday olamizki, z nuqta shuning ichida qolsin. Berilgan $f(z)$ funksiya K aylanada va uning ichidagi C doirada analitik bo'lgani uchun Koshining formulasi o'rinlidir:

$$f(z)=\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{\xi-z},$$

bunda ξ nuqta C aylana ustidagi ixtiyoriy nuqta, z esa C ichidagi nuqtadir.

Qo'yiladigan masala $f(z)$ analitik funksiyani $(z-a)$ ni darajalari bo'yicha qatorga yoyishidan iborat. Ravshanki,

$$\frac{1}{\xi-z}=\frac{1}{(\xi-a)-(z-a)}=\frac{1}{(\xi-a)(1-\frac{z-a}{\xi-a})}. \tag{4.5}$$

Yuqoridagi chizmadan: $|z-a|<|\xi-a|$ demak, $q=\left|\dfrac{z-a}{\xi-a}\right|<1$ bo'lganligi uchun

$$\frac{1}{1-\dfrac{z-a}{\xi-a}}=1+\frac{z-a}{\xi-a}+\left(\frac{z-a}{\xi-a}\right)^2+....+\left(\frac{z-a}{\xi-a}\right)^4+... \qquad (4.6)$$

bo'lib, $(4.6)\to(4.5)$ qo'ysak, ushbu

$$\frac{1}{\xi-z}=\frac{1}{\xi-a}+\frac{z-a}{(\xi-a)^2}+\frac{(z-a)^2}{(\xi-a)^3}+....+\frac{(z-a)^n}{(\xi-a)^{n+1}}... \qquad (4.7)$$

C aylana ichidagi yopiq doirada absolyut va tekis yaqinlashuvchi qatorga ega bo'lamiz. Bu qatorning ikki tomonini $f(\xi)$ analitik funksiyaga ko'paytirilsa ham qator tekis yaqinlashuvchi bo'lib qolaveradi, chunki $|f(\xi)|$ cheklidir. Shu sababli uni hadlab integrallash mumkin:

$$\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(\xi-z)}=\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(\xi-a)}+(z-a)\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(\xi-a)^2}+(z-a)^2\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(\xi-a)^3}+$$

$$+.....+(z-a)^n\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(\xi-a)^{n+1}}+...$$

Tenglikning chap tomoni Koshi formulasiga asosan $f(z)$ ga teng va o'ng tomonidagi integrallarni esa $f(z)$ ning nuqtadagi mos hosilalari bilan almashtirilsa:

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+...+\frac{f^n(a)}{n!}(z-a)^n \qquad (4.8)$$

Teylor qatori hosil bo'ladi. Demak, (4.3) – ga asosan

$$c_n=\frac{f^n(a)}{n!}=\frac{1}{2\pi i}\oint_c\frac{f(\xi)d\xi}{(z-a)^{n+1}}\quad (n=1,2,....), \qquad (4.9)$$

Teylor koeffitsiyentlarini hisoblash uchun integral formulalarga ega bo'lamiz.

Xususiy holda $a=0$ bo'lib (4.8) dan:

$$f(z)=f(0)+\frac{f'(0)}{1!}(z)+\frac{f''(0)}{2!}(z)^2+...+\frac{f^n(0)}{n!}(z)^n \qquad (4.10)$$

qator hosil bo'ladi. (4.10) – ga Maklaren qatori deyiladi.

4.1-misol. $f(z)=z^4$ funksiyani $(z-i)$ ning darajalari bo'yicha Teylor qatoriga yoying.

Yechilishi. Funksiya hosilalarini topib, uning $z=i$ nuqtadagi qiymatini aniqlab (4.4) formulaga quyamiz:

$$f'(z)=4z^3;\quad f''(z)=12z^2;\quad f'''(z)=24z;\quad f^{IV}(z)=24,$$

$$f^{(5)}(z)=f^{(n)}(z)=0.\quad f(i)=1,\quad f'(i)=-4i,\quad f''(i)=-12.\ f'''(i)=24i,\ f^{IV}=24$$

$$f(z)=z^4=1-4i(z-i)-6(z-i)^2+8i(z-i)^3+\frac{24}{4!}(z-i)^4=1-4i(z-i)-$$

$$-6(z-i)^2+8i(z-i)^3+\frac{24}{4!}(z-i)^4=1-4i(z-i)-6(z-i)^2+i(z-1)^3+(z-1)^4.$$

4.2. Elementar funksiyalarni darajali qatorlarga yoyish

Bizga ma'lumki, e^z, $\sin z,\cos z$ funksiyalar tekislikning har qanday chekli qismida analitik bo'lgani uchun markazi nol nuqtada yotuvchi har qanday c aylana ichida ham analitik bo'lib, uni qatorga yoyish mumkin:

1) $f(z)=e^z,\ f'(z)=f''(z)=....=f^n(z)=e^z...,$

$f(0)=f'(0)=f''(0)=...=f^n(0)=1$

bo'lgani sababli (4.10) – ga asosan:

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+....+\frac{z^n}{n!}+...$$

2) $f(z)=\sin z\quad f'(z)=\cos z=\sin\left(z+\frac{\pi}{2}\right),$

$$f''(z)=-\sin z=\sin(z+\pi),.....,f^n(z)=\sin\left(z+\frac{\pi}{2}n\right).$$

bo'lgani uchun $f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=-1$ bo'lib, bularni (4.10) – ga asosan:

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5}-...+(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}+....$$

Huddi shu usulda $\cos z$ – ni qatorga yoyish mumkin:

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-...+(-1)^n\frac{z^{2n}}{(2n)!}+....$$

3) $f(z)=\ln z$ funksiyani $z-1$ ga nisbatan qatorga yoyilsin.

$$f'(z)=(\ln z)'=\frac{1}{z}=z^{-1};\qquad f''(z)=-z^{-2};\ f'''(z)=2z^{-3};$$

$$f^{IV}(z)=-2\cdot 3z^{-4};\ f^{V}(z)=1\cdot 2\cdot 3\cdot 4z^{-5}.$$

Umuman

$$f^n(z)=(-1)^{n-1}(n-1)!z^{-n} \quad (k=1,2,3....).$$

Bulardan

$$f(1)=\ln 1=0,\ f'(1)=1;\ f''(1)=-1;\ f'''(1)=2;\ f^{IV}(1)=-3!,...,f^n(1)=(-1)^{n-1}(n-1)!.$$

U holda (4.8) – ga asoslanib

$$\ln z=(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3+....+(-1)^{n-1}\frac{1}{n}(z-1)^n+....$$

Buning yaqinlashish doirasi $|z-1|<1$ dan iborat ekanligini tekshirib ko'rish mumkin.

4) $f(z)=arctgz$ ni $z=0$ ga nisbatan darajali qatorga yoyilsin.

$$f'(z)=\frac{1}{1+z^2};\quad f''(z)=\frac{-2z}{(1+z^2)^2};\quad f'''(z)=\frac{-2(1-3z)^2}{(1+z^2)^3}$$

Bulardan

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=-2!....,$$

hosil bo'lib, (4.10) ga qo'ygandan so'ng,

$$arctgz=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+...$$

yoyilmaga erishamiz.

Bu misolni 2 – chi usul bilan yechish qulayroqdir.Ma'lumki,

$$\frac{1}{1-\alpha}=1+\alpha+\alpha^2+\alpha^3+....+\alpha^n+..$$

Agar $\alpha=-z^2$ almashtirsak,

$$\frac{1}{1-\alpha}=\frac{1}{1+z^2}=1-z^2+z^4-z^6+.... \tag{4.11}$$

qatorning absolyut va tekis yaqinlashmog' i uchun $|z|=|-z^2|=r<1$ bo'lishi yetarli. Shu sababli bu qatorni hadlab integrallash mumkin, demak

$$arctgz=\int_0^z\frac{dz}{1+z^2}=\int_0^z dz-\int_0^z z^2dz+\int_0^z z^4dz-\int_0^z z^6dz+....=$$

$$=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+.....+(-1)^{n-1}\frac{z^{2n-1}}{2n-1}+1...$$

Bu usulning qulayligi shundaki, $f(z)=arctgz$ dan yuqori tartibli hosilalarning umumiy formulasini izlab o'tirishning hojati yo'q.

4.2-misol. $f(z)=\frac{1}{a^2+z^2}$ a -o'zgarmas son.

Yechilishi.

$$\frac{1}{a^2+z^2}=\frac{1}{a^2}\frac{1}{1+\left(\frac{z}{a}\right)^2};\quad \left(\frac{z}{a}\right)^2=t,$$

$$\frac{1}{a^2+z^2}=a^{-2}\frac{1}{1+t}=a^2\left(1-t+t^2-t^3+\ldots\ldots+(-1)^n t^n\ldots\right)=$$

$$=a^{-2}\left(1-\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^4+\ldots+(-1)^n\left(\frac{z}{a}\right)^n\right)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{a}\right)^{2n},$$

$$\left|\frac{z}{a}\right|<1\Rightarrow |z|<|a|\neq 0.$$

4.3-misol. $f(z)=\frac{1}{(z+1)(z-2)}$.

Yechilishi. $f(z)$ funksiyasini eng soda kasrlarga yoyamiz:

$$\frac{1}{(z+1)(z-2)}=\frac{A}{z+1}+\frac{B}{z-2},\qquad 1=A(z-2)+B(z+1),$$

$$z=-1 \text{ da } A=-\frac{1}{3},\qquad z=2 \text{ da } B=\frac{1}{3}.$$

$$\frac{1}{(z+1)(z-2)}=-\frac{1}{3(z+1)}+\frac{1}{3(z-2)}=-\frac{1}{3}\left(1-z+z^2-z^3+z^4-\ldots\right)-\frac{1}{6}\frac{1}{1-\frac{z}{2}}=$$

$$=-\frac{1}{3}\left(1-z+z^2-z^3+z^4-\ldots\right)-\frac{1}{6}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots+\left(\frac{z}{2}\right)^n+\ldots\right)=$$

$$=-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^n-\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k.$$

Quyidagi funksiyalarni $z=a$ nuqta atrofida Teylor qatoriga yoyilsin.

4.1. $f(z)=\ln z$ funksiyani $(z-1)$ ning darajalari bo‘yicha.

4.2. $f(z)=\frac{1}{1+3z}$ funksiyani $(z+2)$ ga nisbatan.

4.3. $f(z)=\sin(2z+1)$ funksiyani $(z+1)$ ga nisbatan.

4.4. $f(z)=\ln(z^2-3z+2)$ funksiyani $z=0$ nuqta atrofida.

4.5. $f(z)=\cos z$ funksiyani $\left(z+\frac{3\pi}{4}\right)$ ga nisbatan.

4.6. $f(z)=ch(1-z)$ funksiyani $z=\left(1-\frac{\pi i}{2}\right)$ ga nisbatan.

4.3. Manfiy darajali qatorlar

Yuqorida keltirilgan Teylor qatori $z-a$ ga nisbatan musbat darajali qator bo'lsin, uning yaqinlashish sohasi R radiusli biror doiradan iborat edi. $z-a$ bo'yicha manfiy darajali qatorlar ham ko'p uchrab turadi. Uni quyidagicha yozish mumkin:

$$\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+...+\frac{b_n}{(z-a)^n}+...=\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}. \tag{4.12}$$

Bu qatordagi a va $b_1, b_2, b_3, ... b_n, ...$ koeffitsiyentlar kompleks sonlardan iboratdir, xususiy holda haqiqiy sonlar bo'lishi ham mumkin. (4.12) qatorning yaqinlashish sohasini aniqlash maqsadida $Z=(z-a)^{-1}=\frac{1}{z-a}$ deb belgilaymiz. (4.12) qator $b_1Z+b_2Z^2+b_3Z^3+....b_nZ^n+.....$ musbat darajali qatorga aylanadi. Bu qatorning yaqinlashish sohasi $|z|<R$ doiradan iborat bo'lib,

$$r=\lim_{n\to\infty}\sqrt[n]{|b_n|}, \quad R=\frac{1}{r}. \tag{4.13}$$

$$|Z|=\left|\frac{1}{z-a}\right|<R \Rightarrow |z-a|>r, \tag{4.14}$$

tengsizlik hosil bo'ladi. Bu esa markazi a nuqtaga joylashgan r radiusli aylana tashqarisidagi z nuqtalar to'plamlaridir. Demak, manfiy darajali (4.12) qatorning yaqinlashish sohasi $|z-a|=r$ aylana tashqarisidan iborat ekan.

Xususiy holda a markazi koordinatalari boshiga joylashgan aylana bo'ladi.

Agar $r=0$ bo'lsa, yaqinlashish sohasi tekislikning a dan boshqa barcha nuqtalaridan iborat bo'ladi.

Agar $r=\infty$ bo'lsa, qator butun tekislikda uzoqlashuvchi bo'ladi. $r-$ chekli bo'lsin u holda (4.12) qator $|z-a|>r$ doira tashqarisidagi har qanday yopiq sohada yaqinlashuvchidir.

Veyershtrass teoremasiga muvofiq o'sha sohada qator yig'indisi biror $\phi(z)$ analitik funksiyaga teng bo'ladi:

$$\phi(z)=\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+...+\frac{b_n}{(z-a)^n}+.. \qquad (4.12')$$

$z=\infty$ nuqtada bu funksiya $\phi(z)=0$ qiymatga ega. Bu holda $\phi(z)$ funksiyani cheksiz uzoqlashgan nuqtada analitik deb ataymiz. Demak, funksiyaning cheksiz uzoqlashgan nuqtada analitik ekanligini bilmoqchi bo'lsak, uni (4.12) qatorga yoyib o'sha qatorni $z=\infty$ nuqta atrofida yaqinlashishni tekshirish kifoya bo'ladi.

Quyidagi manfiy darajali qatorlarning yaqinlashish sohalarini toping.

4.4-misol. $\sum\limits_{n=1}^{\infty} e^n (iz)^{-n}$.

Yechilishi.

$$C_{-n}=e^n i^{-n}=\frac{e^n}{i^n}, \quad C_{-(n-1)}=\frac{e^{n-1}}{i^{n-1}}$$

$$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{e}{|i|}=e, \qquad r=e.$$

Demak, qatorning yaqinlashish sohasi $|z|>e$ doira tashqarisidan iborat ekan.

4.5-misol. $\sum\limits_{n=1}^{\infty}\frac{z^{-n}}{\cos in}$.

Yechilishi.

$$C_{-n}=\frac{1}{\cos in}=\frac{1}{chn}; \quad C_{-(n-1)}=\frac{1}{ch(n-1)}$$

$$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{ch(n-1)}{chn}=\lim_{n\to\infty}\frac{e^{n-1}+e^{-(n-1)}}{e^n+e^{-n}}=\lim_{n\to\infty}\frac{e^{-1}+e\cdot e^{-2n}}{1+e^{-2n}}=e^{-1}.$$

Chunki $\lim\limits_{n\to\infty} e^{-2n}=0$, $r=e^{-1}$; $|z|>\frac{1}{e}$.

4.6-misol. $\sum\limits_{n=1}^{\infty}\frac{3^n+1}{(z+2i)^n}$.

Yechilishi. $a=-2i$ $C_{-n}=3^n+1$; $C_{-(n-1)}=3^{n-1}+1$,

$$r=\lim_{n\to\infty}\left|\frac{C_{-n}}{C_{-(n-1)}}\right|=\lim_{n\to\infty}\frac{3^n+1}{3^{n-1}}=\lim_{n\to\infty}\frac{3+\frac{1}{3^{n-1}}}{1+3^{\frac{1}{n-1}}}=3.$$

$r=3$ chunki, $\lim\limits_{n\to\infty}\frac{1}{3^{n-1}}=0$ demak, $|x+2i|>3$.

Mustaqil ishlash uchun misollar

4.7. $\sum\limits_{n=1}^{\infty}\frac{1}{(1-i)^n z^n}$.

4.8. $\sum\limits_{n=1}^{\infty}\frac{1}{4^n(1+z)^n}$.

4.9. $\sum\limits_{n=1}^{\infty}\frac{(z+1-i)^{-n}}{n+i}$.

4.10. $\sum\limits_{n=1}^{\infty}\frac{\left(\sqrt{2}+i\sqrt{2}\right)^n}{z^n}$.

4.11. $\sum\limits_{n=-\infty}^{-1}b^{-2(n+1)}z^{2n}$.

4.12. $\sum\limits_{n=-\infty}^{-1}(b-a)^{-(n-1)}(z-a)^n\ (a\neq b)$.

4.4. Loran qatori

Musbat hamda manfiy darajali hadlardan tuzilgan quyidagi umumiy qator berilgan bo‘lsin:

$$c_0+c_1(z-a)+c_2(z-a)^2+\ldots+c_n(z-a)^n+\ldots$$
$$+b_1(z-a)^{-1}+b_2(z-a)^{-2}+\ldots+b_n(z-a)^{-n}+$$

yoki bu qatorni

$$\begin{aligned}&\ldots+b_n(z-a)^{-n}+b_{n-1}(z-a)^{n-1}+\ldots+b_2(z-a)^{-2}+b_1(z-a)^{-1}+\ldots\\&+c_0+c_1(z-a)+c_2(z-a)^2+\ldots+c_n(z-a)^n+\ldots\end{aligned}\qquad(4.15)$$

ko‘rishda yozish mumkin.

Berildan qator ikki qismdan iborat bo‘lib

$$I_1=c_0+c_1(z-a)+c_2(z-a)^2+\ldots+c_n(z-a)^n+\ldots\qquad(4.16)$$

bu (4.15) ning to‘g‘ ri qismi

$$I_2=b_1(z-a)^{-1}+b_2(z-a)^{-2}+\ldots+b_n(z-a)^{-n}+\ldots\qquad(4.17)$$

(4.15) ning bosh qismi deyiladi.

(4.16) - Teylor qatori bo‘lib uning yaqinlashish radiusi

$$R=\frac{1}{L},\ L=\lim_{n\to\infty}\sqrt[n]{|C_n|}.$$

(4.16) qator $|z-a|<R$ doira ichida yaqinlashuvchi, tashqarisida uzoqlashuvchi (4.17) qatorning yaqinlashish radiusi $R=\frac{1}{r}$ bo‘lib $|z-a|>r$ doira tashqarisida yaqinlashadi, ichida uzoqlashadi. (4.16) va (4.17) qatorlarning yaqinlashish sohasi $r<R$ bo‘lgandagina umumiy nuqtalarga ega bo‘ladi, (4.15) qatorning yaqinlashish sohasi

$$r<|z-a|<R \qquad (4.18)$$

tengsizlikni qanoatlantiruvchi K halqadan iborat bo‘lib, uning ichida yaqinlashadi (4.2-chizma). Shu halqaning ichida yotuvchi har qanday sohada u tekis yaqinlashuvchi ekani shubhasiz. Jumladan, shu halqa ichida to‘la joylashgan torroq $r'<|z-a|<R'$ halqa ichida (4.15) qator tekis yaqinlashadi $(r<r'<R'<R)$. Shu sababli qator yig‘indisi biror $f(z)$ analitik funksiyani ifoda qiladi. Agar $b_n=c_{-n}$ deb belgilasak, u holda (4.15) qator

$$f(z)=....+c_{-n}(z-a)^{-n}+...+c_{-3}(z-a)^{-3}+c_{-2}(z-a)^{-2}+c_{-1}(z-a)^{-1}+$$
$$(4.19)$$
$$+c_0+c_1(z-a)+c_2(z-a)^2+...+c_n(z-a)^n+...=\sum_{-\infty}^{\infty}c_n(z-a)^n,$$

ko‘rinishga ega bo‘lamiz. Bu qatorni 1813 – 1854-yillarda yashagan fransuz matematigi Loran kashf etganligi uchun, uni nomiga qo‘yilgan.

Agar $c_{-1}=c_{-2}=...=c_{-n}=0$ bo‘lsa Teylor qatori kelib chiqadi. Agar $c_{-1}\neq 0;\ c_{-2}\neq 0;.....c_0=c_1=c_2=...=c_n=0$ bo‘lsa u holda manfiy darajali qator kelib chiqadi. Bundan ko‘rinib turibdiki, Teylor va manfiy darajali qatorlar Loran qatorining hususiy hollari ekan.

Agar $r<\rho<R$ bo‘lsa (4.19) qator $|z-a|=\rho$ bilan ifodalangan C aylanada ham tekis yaqinlashadi. Bu qatorni $\frac{1}{2\pi i}(z-a)^{-k-1}$ga ko‘paytirilsa ham tekis yaqinlashaveradi. Shu sababli u qatorni C bo‘ylab integrallash qonuniydir.

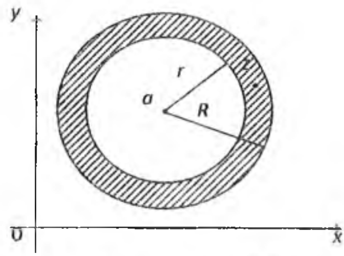
$$\frac{1}{2\pi i}\oint_C\frac{f(\xi)d\xi}{(\xi-a)^{k+1}}=\sum_{-\infty}^{\infty}C_n\oint_C(\xi-a)^{n-k-1}d\xi \qquad (4.20)$$

o‘ng tomonidagi integrallar

$$\oint_C (\xi - a)^m d\xi \qquad m = 0, \pm 1, \pm 2, \dots$$

ko'rinishga ega bo'lib, ular faqat $m = -1$ bo'lgandagina noldan farqli bo'ladi va

$$\oint_C \frac{d\xi}{(\xi - a)} = 2\pi i$$



4.2-chizma. $r < |z - a| < R$ halaqa.

teng bo'ladi. Ravshanki, buning uchun $k = n$ bo'lmog' i kerak. Demak, (4.20) dan

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi) d\xi}{(\xi - a)^{n+1}}, \qquad n = 0, \pm 1, \pm 2, \dots$$

Bu esa Loran qatoridagi koeffitsiyentlarni qatorning yi'g' indisi orqali ifoda qilib beradigan formuladir.

4.5. Maxsus nuqtalar. Funksiyaning nollari.

Kompleks argumentli funksiyalar nazariyasidan ma'lumki, $w = f(z)$ funksiya z_0 nuqtada hosilaga ega bo'lsa, bu nuqta to'g' ri nuqta deyiladi. Agar shu nuqtada analitik shartlari bajarilmasa, bu nuqta funksiyaning maxsus nuqtasi deyiladi. Masalan, $w = \dfrac{z-1}{z-i}$ funksiya uchun $z = i$ maxsus nuqtadir, chunki bu nuqtada hosila mavjud emas.

Faraz qilaylik, $w = f(z)$ funksiya biror a nuqtada analitik bo'lib, ya'ni hosilaga ega bo'lsin. Agar o'sha son $f(z)$ funksiyani nolga aylantirsa, ya'ni $f(a)=0$ bo'lsa, a soni $f(z)$ funksiyaning noli deyiladi. Agar

$$f(a)=0, f'(a)=0, \ \ldots \ f^{(n-1)}(a)=0, \ f^{n}(a)\neq 0 \qquad (4.21)$$

bo'lsa, u holda a soni $f(z)$ funksiyaning n – tartibli yoki n – karrali noli deyiladi.

Agar $n=1$ bo'lsa, a son oddiy nol deyiladi.

Demak, ta'rifga muvofiq a nuqtada analitik bo'lgan $f(z)$ funksiya uchun a nuqta n - karrali nol bo'lishi uchun shu nuqtaning biror atrofida ushbu

$$f(z)=(z-a)^n\varphi(z) \qquad (4.22)$$

tenglik bajarilib, $\varphi(z)$ funksiya a nuqtada analitik va $\varphi(a)\neq 0$ bo'lishi zarur va yetarlidir. Chunki $z=a$ nuqtada analitik funksiya bo'lganligi uchun u shu nuqta atrofida $(z-a)$ ga nisbatan darajali qatorga yoyiladi.

Quyidagi funksiyalarning nollarini va ularning karraliklarini aniqlang:

4.7-misol. $f(z)=z+i$.

Yechilishi. $f(-i)=-i+i=0$, $a=-i$, $f'(z)=1$, $f'(-i)=1\neq 0$

Demak, $z=a=-i$ son funksiyaning oddiy noli ekan.

4.8-Misol. $f(z)=z^2+1$.

Yechilishi. $f(\pm i)=(\pm i)^2+1=-1+1=0$; $z_1=i$; $z_2=-i$;

$$f'(z)=2z;\ f(\pm i)=2(\pm i)=\pm 2i\neq 0$$

Demak, ikkala ildiz ham oddiy noldir.

4.9-Misol. $f(z)=\sin z+1$.

Yechilishi. $\sin z+1=0$, $\sin z=-1$;

$$z=-\frac{\pi}{2}+2\pi n=(4n-1)\frac{\pi}{2};\ n=0,\pm1,\pm2.$$

Endi $f'(z)=\cos z=f'[(4n-1)\frac{\pi}{2}]=-\cos[(4n-1)\frac{\pi}{2}]=0$;

$$f''(z)=-\sin z;\ f''\left[(4n-1)\frac{\pi}{2}\right]=-\sin\left[(4n-1)\frac{\pi}{2}\right]=\pm1\neq 0.$$

Demak, $(4n-1)\frac{\pi}{2}$, $n=0,1,2$ funksiyaning ikki karrali nollaridir.

4.10-Misol. $f(z)=\dfrac{z^3}{\dfrac{z^2}{2}+\cos z}$.

Yechilishi. Ma'lumki,

$$\frac{z^2}{2}+\cos z=\frac{z^2}{2}+\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+....+(-1)^n\frac{z^{2n}}{2n!}+...\right)=1+\frac{z^4}{4!}-\frac{z^6}{6!}+...$$

$$f(z)=z^3\varphi(z);\quad \varphi(z)=\frac{1}{1+\frac{z^4}{4!}-\frac{z^6}{6!}+..},\quad f(0)=0;\quad \varphi(0)=1\neq 0.$$

Demak, $z=0$ son funksiyaning uch karrali nolidir.

Mustaqil yechish uchun misollar

4.13. $f(z)=1+\cos z$;

4.18. $f(z)=\dfrac{sh^2 z}{z}$;

4.14. $f(z)=1-e^z$;

4.19. $f(z)=\dfrac{(1-shz)^2}{z}$;

4.15. $f(z)=\dfrac{z^3}{z-\sin z}$;

4.20. $f(z)=\cos z^3$;

4.16. $f(z)=(z^2+1)^3 shz$;

4.21. $f(z)=\dfrac{z^3}{1+z-e^z}$;

4.17. $f(z)=(z+\pi i)shz$;

4.22. $f(z)=\dfrac{\sin z}{z}$.

4.6. To'g' ri va maxsus nuqtalar

Funksiyaning to'g'ri va maxsus nuqtalari haqidagi tushuncha katta ahamiyatga ega bo'lgani uchun yana bir marta eslatib o'tamiz.

4.1-ta'rif. Agar $f(z)$ funksiya a nuqtada anilitik bo'lsa, u holda a nuqta $f(z)$ ning to'g'ri nuqtasi deyiladi.

Demak, bu ta'rifga muvofiq $f(z)$ funksiyaning biror a nuqtasi to'g'ri yoki to'g'ri nuqta emasligini bilish uchun $f(z)$ ni va shu nuqta atrofidagi $f'(z)$ hosilasini tekshirib ko'rish kerak. Agar hosila mavjud bo'lsa, a nuqta to'g'ri nuqta deyiladi.

4.2-ta'rif. Berilgan $f(z)$ funksiyaning to'g'ri bo'lmagan nuqtasi maxsus nuqta deyiladi. Bu ta'rifga ko'ra, berilgan $f(z)$ funksiya maxsus nuqtada hosilaga ega emas.

4.7. Ajralgan maxsus nuqtalar

Maxsus nuqtalarning xillari juda ko'p bo'lib, ulardan amalda ko'p uchraydigani ajralgan maxsus nuqtalardir.

4.3-ta'rif. Agar $f(z)$ funksiya a nuqtaning biror $0<|z-a|<R$ atrofida analitik bo'lib, a nuqtaning o'zida analitik bo'lmasa, u holda a nuqta $f(z)$ funksiyaning ajralgan (yakkalangan) maxsus nuqtasi deyiladi.

Demak, bu ta'rifga ko'ra, a ajralgan maxsus nuqta bo'lsa, $f(z)$ funksiya $|z-a|<R$ doira ichidagi $z=a$ markazdan boshqa hamma nuqtalarda analitikdir. Boshqacha aytganda, $f(z)$ funksiya

$$r<|z-a|<R$$

doiraviy halqa ichida analitik bo'lib, bundagi r istalgancha kichik musbat sondan iborat. Bunday halqani chizish uchun $|z-a|<R$ doiraning $z=a$ markazini " o'yib " olib tashlash kerak.

Ajralgan maxsus nuqtalar uch xil bo'ladi:

1. Agar $f(z)$ funksiyaning maxsus nuqtasida

$$\lim_{z\to a} f(z)=A \tag{4.23}$$

mavjud bo'lib, A – aniq chekli son bo'lsa, u holda a nuqta $f(z)$ funksiyaning chetlashtiriladigan (tuzatiladigan) maxsus nuqtasi deyiladi.

2. Agar $f(z)$ funksiyaning maxsus nuqtasida

$$\lim_{z\to a} f(z)=\infty \tag{4.24}$$

bo'lsa, u holda a son $f(z)$ funksiyaning qutbi deyiladi.

4.8. Chetlashtiriladigan maxsus nuqtalar

Funksiyaning ajralgan maxsus nuqtasining birinchi tipiga kirishini quyidagi teorama aniqlab beradi.

4.2-teorema. $f(z)$ funksiyaning ajralgan a maxsus nuqtasi chetlashtiriladigan maxsus nuqta bo'lishining zarur va yetarli sharti shu funksiyaning a nuqta atrofidagi Loran qatoriga yoyilmasi bosh qismga ega bo'lmasligidan iboratdir:

$$f(z)=c_0+c_1(z-a)+c_2(z-a)^2+.....c_n(z-a)^n+..... \quad (4.25)$$

Isbot. Loran qatorining bosh qismi yoq, ya'ni $f(z)$ funksiya (4.25) qatorga yoyilgan deb faraz qilaylik. (4.25) ni ikki tomondan $z\to a$ faraz etib limitga o'tsak,

$$\lim_{z\to a} f(z)=c_0\neq\infty.$$

Bu esa ta'rifga ko'ra a ning tuzatiladigan nuqta ekanligidan darak beradi.

$\lim_{z\to 0} f(z)=A$ deb faraz etaylik, bunda A chekli son bo'lsin.

Bundan a nuqtaning shunday kichik radiusli atrofi mavjudki, unda $f(z)$ funksiya chegaralangan bo'ladi. $|f(z)|\le M$, $M>0$.

Endi, $\rho<|z-a|<R$ halqada $f(z)$ ni Loran qatoriga yoysak

$$C_n=\frac{1}{2\pi i}\oint\frac{f(\xi)d\xi}{(\xi-a)^{n+1}}$$

formulaga asosan:

$$|C_n|=\left|\frac{1}{2\pi i}\oint_\gamma\frac{f(\xi)d\xi}{(\xi-a)^{n+1}}\right|\le\frac{1}{2\pi}\frac{M}{\rho^{n+1}}\cdot 2\pi\rho=M\rho^{-n}.$$

Bunda γ radiusi ρ va markazi a nuqtada bo'lgan aylanadan iborat. Endi $n=-1,-2....$ faraz etib, $\rho\to 0$ da limitga o'tilsa, o'ng tomoni nolga teng bo'ladi: $|C_n|=0$. Demak,

$$C_{-1}=0,\ C_{-2}=0,\ C_{-3}=0.....\ C_{-n}=0,$$

bunda Loran qatorining bosh qismi yo'qoladi. Bu teoramada ko'rinadiki, bir tomondan a nuqta maxsus bo'lgani uchun $f(z)$ ning a dagi qiymati aniq emas. Lekin a nuqta chetlashtiriladigan nuqta bo'lsa,

$$A=\lim_{z\to 0} f(z)=c_0=f(a)$$

deb qabul qilsak a nuqta maxsuslikdan qutulgan bo'ladi. U holda Loran qatorining (4.25) qismi Teylor qatoriga aylanib, u $|z-a|<R$ doira ichida tekis va absolyut yaqinlashadi.

4.3-teorema. Agar $f(z)$ funksiya biror ajralgan maxsus a nuqta atrofida chegaralangan bo'lsa, u holda a nuqta $f(z)$ uchun qutulib bo'ladigan maxsus nuqta bo'ladi.

4.9. Qutblar

Agar ajralgan maxsus a nuqta qutb bo'lsa (4.24) tenglikni bajarilishi lozim ekanligini ta'rifdan ko'rgan edik. Endi qutbga ega bo'lgan ba'zi funksiyalar bilan tanishib o'tamiz. O'z-o'zidan ma'lumki, a nuqta $f(z)$ funksiyaning qutbi bo'lsa, u holda a nuqta $\varphi(z)=\dfrac{1}{f(z)}$ funksiyaning noli bo'ladi. Agar a son $\varphi(z)$ funksiyaning n karrali noli bo'lsa, u $f(z)$ funksiyaning n - karrali qutbi deyiladi.

Berilgan a nuqta $f(z)$ funksiyaning n karrali noli bo'lishi uchun $f(z)$ ni quyidagicha yozish mumkin bo'lishi zarur va yetarlidir:

$$f(z)=\frac{\varphi(z)}{(z-a)^n}, \qquad (4.26)$$

bu yerda $\varphi(z)$ funksiya a nuqtada analitik va $\varphi(z)\neq 0$.

4.4-ta'rif. $f(z)$ funksiyaning ajralgan maxsus nuqtasi qutb bo'lishi uchun $f(z)$ funksiyaning a nuqta atrofidagi Loran qatori bosh qismi hadlarining soni chekli bo'lishi zarur va yetarlidir.

$$f(z)=\frac{C_{-k}}{(z-a)^k}+\ldots+\frac{C_{-1}}{z-a}+C_0+C_1(z-a)+C_2(z-a)^2+\ldots+C_n(z-a)^n+\ldots \qquad (4.27)$$

bunda $C_{-k}\neq 0$ va k chekli natural son.

4.11-misol. $f(z)=\dfrac{1}{z-\sin z}$.

Yechilishi. $z=0$ ning ifodasidan ko'rinadiki. $z=0$ nuqta qutbdir. $z=0$ necha karrali ekanini tekshiraylik,

$$\varphi(z)=z-\sin z=z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\right)=\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}+\ldots$$

Bundan uch marta hosila olib $z=0$ nuqtada tekshirsa, $\varphi(0)=0,\ \varphi'(0)=0,\ \varphi''(0)=0,\ \varphi'''(0)\neq 0$ Demak, $z=0$ nuqta $\varphi(z)$ ning uch karrali noli, ya'ni $f(z)$ ning uch karrali qutbidir.

4.12-misol. $f(z)=\dfrac{1}{\cos z-1+\dfrac{z^2}{2}}$.

Yechilishi. Bu misolda $z=0$ nuqta (4.24) ni qanoatlantiradi, shu sababli u qutbdir. Endi uning necha karrali ekanligini aniqlaylik

$$\varphi(z)=\cos z-1+\frac{z^2}{2}=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots\right)-1+\frac{z^2}{2}=\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots$$

Bundan (4.26) ga binoan

$$f(z)=\frac{\varphi(z)}{z^4}=\frac{1}{\dfrac{z^4}{4!}-\dfrac{z^6}{6!}+\dfrac{z^8}{8!}-\ldots}=\frac{1}{z^4\left(\dfrac{1}{4!}-\dfrac{z^2}{6!}+\dfrac{z^4}{8!}\right)}$$

$$\varphi(z)=\frac{1}{\dfrac{1}{4!}-\dfrac{z^2}{6!}+\dfrac{z^4}{8!}-\ldots};\qquad \varphi(0)=4!\neq 0$$

Demak, $z=0$ nuqta $n=4$ karrali qutb ekan.

4.10. Muhim maxsus nuqtalar

Ta'rifdan ma'lumki, agar biror a nuqtada $f(z)$ funksiya hech qanday limitga ega bo'lmasa, ya'ni chekli ham, cheksiz ham limiti mavjud bo'lmasa, u holda $z=a$ nuqta $f(z)$ ning muhim maxsus nuqtasi deyiladi o'sha nuqtani aniqlash uchun $z=x+iy$ ni ikki turli yo'l bilan a ga intiltirib, $f(z)$ ning limit sonlarini aniqlash lozim. Agar bu limit sonlar har xil bo'lsa, u holda a muhim maxsus nuqta bo'ladi.

4.4-teorema. $f(z)$ funksiyaning ajralgan maxsus a nuqtasi muhim maxsus nuqta bo'lishi uchun Loran qatorining bosh qismi cheksiz ko'p hadlarga ega bo'lishi zarur va yetarlidir:

$$f(z)=\sum_{-\infty}^{+\infty}C_n(z-a)^n \qquad (4.28)$$

Misollar:

1. $f(z)=z\,sh\dfrac{1}{z}$ funksiya uchun $z=0$ maxsus nuqta

$$z\cdot sh\frac{1}{z}=z\left(\frac{1}{z}+\frac{1}{z^3 3!}+\frac{1}{5!z^5}+\dots\right)=1+\frac{1}{3!z^2}+\frac{1}{5!z^4}+\dots$$

Bu esa Loran qatorining to‘la bosh qismidan iborat bo‘lagani uchun $z=0$ muhim maxsus nuqtadir.

2. $f(z)=e^{\frac{1}{z+2}}$ funksiya uchun $z=-2$ maxsus nuqta. $f(z)$ ni Loran qatoriga yoyamiz.

Ma’lumki,

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$$

Shunga asosan

$$f(z)=e^{\frac{1}{z+2}}=1+\frac{1}{z+2}+\frac{1}{2!(z+2)^2}+\dots\dots+\frac{1}{n!(z+2)^2}+\dots$$

bu esa Loran qatorining bosh qismi bo‘lgani uchun $z=-2$ son berilgan funksiyaning muhim maxsus nuqtasidir.

3. $f(z)=\sin\frac{1}{1-z}$ funksiyaning maxsus nuqtasi $z=1$. $f(z)$ ni qatorga yoysak,

$$\sin\frac{1}{1-z}=\frac{1}{1-z}-\frac{1}{3!(1-z)^3}+\frac{1}{5!(1-z)^5}-\dots$$

bosh qismidan iborat bo‘lib, $z=1$ muhim maxsus nuqta ekan.

Mustaqil yechish uchun misollar

4.23. Quyidagi funksiyalar chetlashtiriladigan maxsus nuqtalarga ega ekanligini ko‘rsating.

1) $\frac{\sin 2z}{z}$,

2) $\frac{1-\cos z}{z^2}$,

3) $\frac{z^2-1}{z^3+1}$,

4) $\frac{z}{tgz}$,

5) $\frac{1}{\cos^2 z}-\frac{1}{\left(z-\frac{\pi}{2}\right)^2}$,

6) $ctgz - \frac{1}{z}$,

7) $\frac{1}{e^z - 1} - \frac{1}{\sin z}$.

4.24. Quyidagi funksiyalar qutblarini toping.

1) $\frac{1}{z}$,

2) $\frac{1}{(z^2+1)^2}$,

3) $\frac{z}{e^z+1}$,

4) $\frac{z}{(e^z-1)^2}$,

5) $ctg\frac{\pi}{z}$,

6) $tg\pi z$,

7) $\frac{\sin z}{z^2}$,

8) $\frac{z^4}{1+z^4}$.

4.25. Quyidagi funksiyalarning muhim maxsus nuqtalarini toping.

1) $z^2\cos\frac{\pi}{2}$,

2) e^{tgz},

3) $\sin\frac{\pi}{z^2}$,

4) $\cos\frac{z}{1+z}$,

5) $\sin\frac{\pi}{1+z^2}$,

6) $z^2\sin\frac{z}{z+1}$,

7) $\sin e^{\frac{1}{z}}$,

8) $e^{ctg\frac{\pi}{2}}$.

4.11. Qoldiqlar nazariyasi

Ma'lumki, agar $f(z)$ funksiya a nuqtani o'z ichiga olgan biror G sohada analitik bo'lsa, u holda a nuqtani o'rab olgan G soha ichida yotgan, yopiq Γ kontur bo'ylab olingan integral Koshi teoremasiga muvofiq nolga teng bo'ladi:

$$\oint_{\Gamma} f(z)dz = 0.$$

Agar a nuqta $f(z)$ ning ajralgan maxsus nuqtasi bo'lsa, u holda bu integral nolga teng bo'lmasligi ham mumkin. Shu integralning qiymatini topish talab qilinadi.

a nuqta $f(z)$ ning ajralgan maxsus nuqtasi bo'lsin deylik, u holda $f(z)$ ni $0<|z-a|<R$ halqada Loran qatoriga yoyish mumkin:

$$f(z)=c_0+c_1(z-a)+c_2(z-a)^2+....+c_n(z-a)^n+...+\frac{c_{-1}}{z-a}+\frac{c_{-2}}{(z-a)^2}+...+\frac{c_{-n}}{(z-a)^n}...$$

Endi $0<|z-a|<R$ halqa ichida bitta silliq yopiq Γ chiziq olaylik, u a nuqtani o'rab olsin. So'nggi qator Γ konturda tekis yaqinlashuvchi bo'lgani uchun o'sha qator bo'yicha integrallash mumkin. U holda

$$\oint_{\Gamma}(z-a)^n dz=\begin{cases}0, & n\neq -1\\ 2\pi i, & n=-1\end{cases}$$

bo'ladi. Demak,

$$\oint_{\Gamma} f(z)dz = 2\pi i\cdot c_{-1} \qquad (4.29)$$

Xususiy holda, Γ chiziq l aylanadan iborat bo'lib qolishi mumkin.

4.5-ta'rif. Agar $f(z)$ funksiya $0<|z-a|<R$ halqada analitik bo'lsa, shu funksiyaning ajralgan maxsus a nuqtaga nisbatan qoldig'i deb,

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz$$

integralning qiymatiga aytiladi va

$$\underset{z=a}{qold}\cdot f(z)=\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz \qquad (4.30)$$

ko'rinishida yoziladi.

Ba'zan qoldiqni quyidagicha yozadilar:

$$\operatorname{Res} f(a)=\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz \quad \text{yoki} \quad \operatorname{Res}[f(z),a]=\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz. \qquad (4.31)$$

Agar (4.29) tenglikni e'tiborga olsak,

$$c_{-1}=\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz=\underset{z=a}{qold}\cdot f(z) \qquad (4.32)$$

ekanligini ko'ramiz. Qoldiqni boshqacha ham ta'riflash mumkin: $f(z)$ funksiyaning ajralgan maxsus nuqtasiga nisbatan qoldig' i deb, shu funksiyaning a nuqta atrofidagi Loran qatori $(z-a)^{-1}$ hadining c_{-1} koeffitsiyentini qabul qilamiz.

Ma'lumki, agar a nuqta $f(z)$ funksiyaning to'g'ri nuqtasi yoki qutulib bo'ladigan maxsus nuqtasi bo'lsa, u holda Loran qatorining bosh qismi bo'lmaydi, ya'ni $c_{-1}=0$, $c_{-2}=0$,... Bundan (4.32) ga asosan

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz=\underset{z=a}{qold}\cdot f(z)=c_{-1}=0$$

bo'ladi. $f(z)$ funksiyani qoldiqlarga tegishli teoremani keltiramiz.

4.5-teorema. Agar $f(z)$ funksiya Γ chiziq bilan o'ralgan G yopiq sohaning ajralgan maxsus $a_1, a_2, ...a_k$ nuqtalaridan boshqa barcha nuqtalarida analitik bo'lsa, u holda $f(z)$ funksiyadan Γ bo'ylab olingan integralning qiymati Γ ichidagi barcha maxsus a_k nuqtalarga nisbatan funksiya qoldiqlari yig'indisining $2\pi i$ ga ko'paytirilganiga teng :

$$\oint_{\Gamma} f(z)dz=2\pi i\cdot\sum_{n=1}^{k}\underset{z=a}{qold}\cdot f(z).$$

Isbot. Funksiyaning maxsus nuqtalarini markaz qilib shunday $c_1, c_2, ...c_k$ aylanalar chizamizki, ular o'zaro kesishmasin va bir-birga urinib ham qolmasin. Undan tashqari, shu aylanalardan birortasi ham Γ dan tashqariga chiqib ketmasin. Shunday aylanalarni haqiqatdan ham chizish mumkin, chunki $a_1, a_2, ...a_k$ nuqtalar bir – biridan ajralgan holda turadi.

Yuqoridagi $\Gamma_1, \Gamma_2, ...\Gamma_k$ chiziqlar bilan chegaralangan ko'p bo'g' lamli yopiq sohada $f(z)$ funksiya analitik bo'lgani uchun Koshining murakkab konturga tegishli teoremasiga asosan tashqi kontur bo'ylab olingan integral ichki konturlar bo'ylab olingan integrallar yig'indisiga teng, demak,

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz = \frac{1}{2\pi i}\oint_{c_1} f(z)dz + \frac{1}{2\pi i}\oint_{c_2} f(z)dz + \frac{1}{2\pi i}\oint_{c_3} f(z)dz + \ldots \frac{1}{2\pi i}\oint_{c_k} f(z)dz .$$

Qoldiqlarning ta'rifiga muvofiq, o'ng tomondagi integralning har biri maxsus ajralgan nuqtaga nisbatan $f(z)$ funksiyaning qoldig' idir. Demak,

$$\frac{1}{2\pi i}\oint_{\Gamma} f(z)dz = \underset{z=a_1}{qold} f(z) + \underset{z=a_2}{qold} f(z) + \ldots + \underset{z=a_k}{qold} f(z) = c_{-1}^{(1)} + c_{-1}^{(2)} + \ldots + c_{-1}^{(k)},$$

yoki

$$\oint_{\Gamma} f(z)dz = 2\pi i\,[\underset{z=a_1}{qold}\, f(z) + \underset{z=a_2}{qold}\, f(z) + \ldots + \underset{z=a_k}{qold}\, f(z)]. \qquad (4.33)$$

Bu teoremadan ko'rinadiki, $f(z)$ funksiyadan yopiq Γ kontur bo'ylab olingan integralni hisoblash uchun Γ ichidagi ajralgan maxsus nuqtalarga nisbatan barcha qoldiqlarni hisoblab chiqish talab qilinadi. Mabodo ajralgan maxsus nuqtalar qutblardan iborat bo'lsa, ularga nisbatan qoldiqlarni Loran qatoridan foydalanmay, quyida berilgan oson yo'l bilan bevosita hisoblash mumkin.

4.12. Qutbga nisbatan funksiyaning qoldig' ini hisoblash

1. Agar a nuqta $f(z)$ funksiyaning oddiy qutbi bo'lsa, a nuqta atrofida $f(z)$ ning Loran qatori

$$f(z) = \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots$$

ko'rinishida bo'ladi. Buning ikki tomonini $z-a$ ga ko'paytirib limitga o'tilsa,

$$\lim_{z\to a}[(z-a)f(z)] = \lim_{z\to a}\left[c_{-1} + c_0(z-a) + c_1(z-a)^2 + \ldots\right] = c_{-1}$$

Demak, u holda

$$c_{-1} = \lim_{z\to a}[(z-a)f(z)] \qquad (4.34)$$

formula bilan aniqlanadi.

4.13-misol. $f(z) = \dfrac{z^3}{z+2}$ funksiya $z=-2$ oddiy qutbga ega bo'lgani uchun

$$c_{-1} = \underset{z=-2}{qold}\frac{z^3}{z+2} = \lim_{z\to -2}[(z+2)f(z)] = \lim_{z\to -2} z^3 = -8.$$

2. Agar a nuqta $f(z)$ ning k tartibli qutbi bo'lsa, uning bu nuqtaga nisbatan Loran qatori

$$f(z)=\frac{c_{-k}}{(z-a)^k}+\ldots+\frac{c_{-1}}{z-a}+c_0+c_1(z-a)+\ldots$$

ko'rinishida bo'lib, c_{-1} koeffitsiyentni topish uchun tenglikning ikki tomonini $(z-a)^k$ ga ko'paytirib, $(k-1)$ marta hosila olamiz:

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]=\frac{d^{k-1}}{dz^{k-1}}\left[c_{-k}+c_{-k+1}(z-a)+\ldots+c_{-1}(z-a)^{k-1}+c_0(z-a)^k+\ldots\right]=$$

$$=(k-1)!c_{-1}+k!c_0(z-a)+\ldots.$$

Endi $z\to a$ deb limitga o'tilsa,

$$c_{-1}=\frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\left[(z-a)^k f(z)\right]. \qquad (4.34')$$

4.14-misol. $f(z)=\dfrac{1}{(z^2+1)^3}$ funksiyaning qoldiqlarga nisbatan limitini toping.

Yechilishi. Bu funksiya qutblari $z=i$ va $z=-i$ dan iborat bo'lib, uchinchi tartiblidir. Shu sababli (4.34) formuladan foydalnamiz, buning uchun dastlab berilgan funksiyani

$$f(z)=\frac{1}{(z+i)^3(z-i)^3}$$

ko'rinishida yozib olamiz.

Avval funksiyaning $z=i$ ga nisbatan qoldig' ini topaylik.

$$\frac{d}{dz}\left[(z-i)^3 f(z)\right]=\frac{d}{dz}\left[(z+i)^{-3}\right]=-3(z+i)^{-4}$$

$$\frac{d^2}{dz^2}\left[(z-i)^3 f(z)\right]=\frac{d}{dz}\left[-3(z+i)^{-4}\right]=12(z+i)^{-5}.$$

Buning ikki tomonini $\frac{1}{2i}$ ga ko'paytirib limitga o'tilsa, $z=i$ ga nisbatan qoldiq

$$c_{-1}^{(1)}=\underset{z=i}{qold}\, f(z)=\frac{1}{2i}\lim_{z\to i}\left[(z+i)^{-5}\right]=\frac{6}{(2i)^5}=-\frac{3i}{16}.$$

Funksiyaning $z=-i$ ga nisbatan qoldig' i

$$(z+i)^3 f(z)=(z-i)^{-3},\quad \frac{d^2}{dz^2}\left[(z+i)^3 f(z)\right]=12(z-i)^{-5},$$

$$c_{-1}^{(2)}=\underset{z=-i}{qold}\, f(z)=\frac{1}{2i}12\lim_{z\to -i}\left[(z-i)^{-5}\right]=\frac{6}{-(2i)^5}=\frac{3i}{16}.$$

4.15-misol. Tenglamasi $x^2+y^2-2x-2y=0$ dan iborat bo'lgan c aylana bo'ylab olingan

$$\oint_c \frac{dz}{(z-1)^2(z^2+1)}$$

integral hisoblansin.

Yechilishi. c aylana tenglamasini $(x-1)^2+(y-1)^2=2$ ko'rinishga keltirsak, uning markazi $z=1+i$ nuqtada va radiusi $r=\sqrt{2}$ ga teng ekanligini ko'ramiz. Berilgan

$$\oint_c \frac{dz}{(z-1)^2(z^2+1)}$$

funksiya uchun $z=1$ ikkinchi tartibli va $z_2=i$ va $z_3=-i$ nuqta c ning tashqarisida bo'lgani uchun e'tiborga olmaymiz.

Bu massalani hal qilish uchun qoldiqlarga doir teoremani qo'llaymiz. Avval funksiyaning $z=1$ ga nisbatan qoldig' ini hisoblaymiz

$$\frac{d}{dz}(z-1)=\frac{d}{dz}\left[\left(z^2+1\right)^{-1}\right]=-\left(z^2+1\right)^{-2}\cdot 2z$$

Bundan limitga o'tilsa, $c_{-1}^{(1)}=-2\lim\limits_{z\to 1}\left(z^2+1\right)^{-2}=-2\cdot\frac{1}{2^2}=-\frac{1}{2}$

Endi $z=i$ ga nisbatan qoldig' ini topamiz: $z^2+1=(z-i)(z+i)$ dan va $f(z)=\frac{1}{(z-1)^2(z^2+1)}$ ni e'tiborga olib

$$(z-i)f(z)=\frac{1}{(z-1)^2(z+i)}$$

hosil qilamiz.Limitga o'tilsa $c_{-1}^{(2)}=\lim\limits_{z\to i}\frac{1}{(z-1)^2(z+i)}=\frac{1}{(i-1)^2 2i}=\frac{1}{4}$.

Demak,

$$\oint_c \frac{dz}{(z-1)^2(z^2+1)}=2\pi i\left(c_{-1}^{(1)}+c_{-1}^{(2)}\right)=2\pi i\left(-\frac{1}{2}+\frac{1}{4}\right)=\frac{-\pi i}{2}.$$

3. Agar $f(z)$ funksiya $f(z)=\frac{\varphi(z)}{\psi(z)}$ kasr ko'rinishida berilgan bo'lib, oddiy a qutbga ega bo'lsa, qoldiqni hisoblash ham osonlashadi. U holda, $\varphi(a)\neq 0$, $\psi(a)=0$, $\psi'(a)=0$ chunki $\psi(z)$ funksiya uchun a oddiy nol (4.34') ga muvofiq.

$$(z-a)f(z)=\frac{(z-a)\varphi(z)}{\psi(z)}=\frac{\varphi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}}$$

dan limitga o‘tilsa,

$$c_{-1}=\lim_{z\to 1}[(z-a)f(z)]=\frac{\lim\limits_{z\to a}\varphi(z)}{\lim\limits_{z\to a}\dfrac{\psi(z)-\psi(a)}{z-a}}=\frac{\varphi(a)}{\psi'(a)}.$$

Demak,

$$c_{-1}=\frac{\varphi(a)}{\psi'(a)}, \qquad (4.35)$$

yoki

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(z)}{\psi(z)}dz=\frac{\varphi(a)}{\psi'(a)}. \qquad (4.36)$$

4.16-misol. $f(z)=\dfrac{z^2}{z+2}$ funksiyaning qutbga nisbatan qoldig‘ini toping.

Yechilishi. Funksiya oddiy $z=-2$ qutbga ega bo‘lgani uchun

$$c_{-1}^{(1)}=\underset{z=-2}{qold}\,f(z)=\lim_{z\to -2}[f(z)(z+2)]=\lim_{z\to -2}z^2=4.$$

Demak, $c_{-1}=4$. Buni (4.36) ga asosan ham yechish mumkin, bunda $\varphi(z)=z^2,\ \psi(z)=z+2,\ \psi'(z)=1,\ z=-2,\ c_{-1}=\dfrac{\varphi(-2)}{\psi'(-2)}=\dfrac{(-2)^2}{1}=4.$

Nazorat savollari.

1. Sonli qator ta’rifi.Teylor qatorini yozib bering.
2. Elementlar funksiyalarni Teylor qatoriga yoyishga misollar ko‘rsating.
3. Loran qatorining bosh va to‘g‘ri qismini yozib bering.
4. Ajralgan maxsus nuqtalar va ularnmg turları.
5. Ajralgan maxsus nuqta atrofida funksiya chegirmasining ta’rifi.
6. Oddiy qutb nuqtasida chegirmani hisoblash formulasini yozib bering.
7. Chegirma yordamida integralni hisoblashga misol ko‘rsating.

V BOB. OPERATSION HISOB NAZARIYASI ELEMENTLARI

Operatsion hisob amaliy matematik analiz metodlaridan biri hisoblanadi. Uning yordamida ko‘p hollarda mexanika, elektronika, avtomatika va boshqa masalalar yechiladi.

5.1. Original va tasvirlar

1. Asosiy ta’riflar. Butun son o‘qida aniqlangan va quyidagi xossalarga ega bo‘lgan $f(t)$ funksyani qaraymiz:

a) $f(t)$ funksiya ot o‘qning istalgan chekli intervalida yo uzliksiz yoki chekli sondagi 1 to‘r uzilish nuqtalariga ega;

b) $t<0$ da $f(t)=0$;

c) shunday $M>0$ va $s_0\geq 0$ sonlar mavjudki barcha t-lar uchun: $|f(t)|\leq Me^{s_0 t}$.

b) - shart fizika va mexanikaning ko‘p masalalarida t argument vaqt sifatida qaralish munosabati bilan kiritiladi. Shuning uchun $f(t)$ funksya vaqtning biror boshlang‘ich paytigacha (uni har doim nolga teng deb olish mumkin) o‘zini qanday tutishi ahamiyatga ega emas.

c) – shart $t\to\infty$ da $f(t)$ funksyaning o‘sish xarakterini cheklaydi va bu bilan kelgusida uchraydugan ba’zi xosmas integrallarning mavjudligini ta’minlaydi. U $f(t)$ funksya $t\to\infty$ da ko‘rsatkichli funksiyadan sekinroq o‘sishini bildiradi. s_0 – soni o‘sish ko‘rsatkichi deyiladi.

Har bir $f(t)$ funksiyaga

$$F(p)=\int_0^\infty f(t)\exp(-pt)dt \qquad (5.1)$$

deb faraz qilib, $p=s+i\tau\left(s\geq s_0\right)$ kompleks o‘zgaruvchining $F(p)$ funksiyasini mos keltiramiz. $F(p)$ funksya $f(t)$ funksiyaning tasviri, yuqoridagi uchta shartni qanoatlantiruvchi $f(t)$ funksiya esa original deyiladi. M – barcha $f(t)$ originallar to‘plami, N – esa ularga mos tasvirlar to‘plami bo‘lsin. (5.1) formula M to‘plamni N to‘plamga akslantiradi. Bu akslantirish Laplas operatori yoki akslantirish deyiladi.

(5.1) formulani quyidagicha yozish mumkin:

$$F(p)=\int_0^\infty f(t)e^{-st-i\tau t}dt=\int_0^\infty f(t)e^{-st}e^{-i\tau t}dt=\int_0^\infty f(t)e^{-st}(Cos\,\tau t-iSin\,\tau t)dt=$$

$$=\int_0^\infty f(t)e^{-st}Cos\,\tau t\,dt-i\int_0^\infty f(t)e^{-st}Sin\,\tau t\,dt.$$

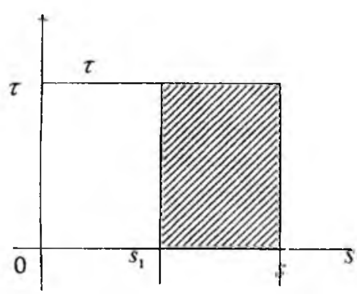
Agar $F(p)$ $f(t)$ funksyaning tasviri bo‘lsa, bu quyidagicha yoziladi:

$$f(t)\xrightarrow{L}F(p) \text{ yoki } L=\{f(t)\}=F(p).$$

Izoh. Agar $f(t)$ funksiya a)-c) shartlarni birontasi ham bajarilmasa u original bo‘lmaydi. Masalan, $tg\,t$ va $\frac{1}{t}$ funksiyalar original bo‘la olmaydi, chunki ular II tur uzilishga ega. e^{t^2} funksiya ham original bulolmaydi, chunki c) shartni qanoatlantirmydi. U $|f(t)|\le Me^{s_0 t}$ funksiyadan ko‘ra tezroq usadi. (M va s_0 lar har qanday bo‘lganda ham).

2. Mavjudlik va yagonalik teoremasi.

5.1-teorema (*tasvirning mavjudlik teoremasi*). Ixtiyoriy $f(t)$ original uchun $\mathrm{Re}\,p=s>s_0$ yarim tekislikda aniqlangan $F(p)$ tasvir mavjuddir. Yarim tekislikning har bir nuqtasida $F(p)$ funksya istalgan tartibli hosilaga ega. Bundan tashqari, agar $\mathrm{Re}\,p=s\to\infty$ bo‘lsa, u holda tasvir $F(p)\to 0$.



5.1- chizma.

5.2-teorema (***Originalning yagonalik teoremasi***). Agar $F(p)$ ikkita $f_1(t)$ va $f_2(t)$ originalning tasviri bo'lsa, u holda originallar ular uzluksiz bo'lgan barcha nuqtalarda o'zaro teng bo'ladi.

Tasvir chiziqlilik xossasiga ham ega:

1) Originalning songa ko'paytmasining tasviri, tasvirning bu songa ko'paytmasiga teng, ya'ni $f(t)\xrightarrow{L}F(p)$ bo'lsa, u holda $Cf(t)\xrightarrow{L}CF(p)$;

2) Bir nechta original algebraik yig'indisining tasviri, bu originallar tasvirlarining algebraik yig'indisiga teng, ya'ni agar masalan, $f_1(t)\xrightarrow{L}F_1(p)$, $f_2(t)\xrightarrow{L}F_2(p)$ bo'lsa, u holda $f_1(t)\pm f_2(t)\xrightarrow{L}F_1(p)\pm F_2(p)$.

Bu xossalar integralning chiziqliligidan kelib chiqadi.

Masalan,

$$f_1(t)\pm f_2(t)\xrightarrow{L}\int_0^\infty [f_1(t)\pm f_2(t)]e^{-pt}dt=\int_0^\infty f_1(t)e^{-pt}dt\pm\int_0^\infty f_2(t)e^{-pt}dt=F_1(p)\pm F_2(p).$$

5.2. Ba'zi funksiyalarning tasvilari

Lemma. Agar $z=x+iy$ kompleks son uchun uning haqiqiy qismi $\mathrm{Re}\,z=x>0$ va b haqiqiy uzgaruvchi bo'lsa, u holda $\lim\limits_{b\to\infty}e^{-zb}=0$ bo'ladi.

Isbot. Eyler formulasiga ko'ra

$$e^{-zb}=e^{-(x+iy)b}=e^{-xb}(Cosyb-iSinyb).$$

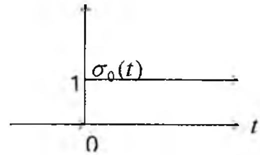
Demak,

$$\lim_{b\to\infty}e^{-zb}=\lim_{b\to\infty}e^{-xb}(Cosyb-iSinyb)=0,$$

chunki $x>0$ da $\lim\limits_{b\to+\infty}e^{-zb}=0$.

1. Birlik sakrash va uning tasviri. Quyidagicha aniqlangan funksiyani qaraymiz.

$$f(t)=\begin{cases}1,\ agar\ t\geq 0\ bo'lsa,\\ 0,\ agar\ t<0\ bo'lsa.\end{cases}$$



5.2-chizma. $f(t)$ ***funksiyaning grafigi.***

Bu funksiya Xevisaydning birlik funksiyasi yoki birlik sakrashi deyiladi (5.2-chizma). Uni $\sigma_0(t)$ orqali belgilashga kelishamiz. U holda uning tasviri

$$\sigma_0(t)\xrightarrow{L}\int_0^{\infty}1\cdot e^{-pt}dt=\lim_{b\to\infty}\int_0^{b}e^{-pt}dt=\lim_{b\to\infty}\left[-\frac{e^{-pt}}{p}\right]_0^b=-\frac{1}{p}\left(\lim_{b\to\infty}e^{-pb}-1\right)=\frac{1}{p}.$$

($\operatorname{Re}z=s>s_0$ bo'lgani uchun lemmaga ko'ra $\lim_{b\to\infty}e^{-zb}=0$). Shunday qilib,

$$\sigma_0(t)\xrightarrow{L}\frac{1}{p}. \qquad (5.2)$$

2. Ko'rsatkichli funksiyaning tasviri.

Ushbu funksiyaning tasvirini topamiz:

$$f(t)=\begin{cases}0, & agar\ t<0\,bo'lsa;\\ e^{\alpha t}, & agar\ t\geq 0\,bo'lsa,\end{cases}$$

bu yerda α kompleks son. (5.1) formulaga ko'ra quyidagiga egamiz:

$$f(t)\xrightarrow{L}F(p)=\int_0^{\infty}f(t)e^{-pt}dt=\int_0^{\infty}e^{\alpha t}e^{-pt}dt=\int_0^{\infty}e^{-(p-\alpha)t}dt=$$

$$=\lim_{b\to\infty}\int_0^{b}e^{-(p-\alpha)t}dt=\lim_{b\to\infty}\left[\frac{e^{-(p-\alpha)t}}{-(p-\alpha)}\right]_0^b=\lim_{b\to\infty}\left[\frac{e^{-(p-\alpha)b}}{-(p-\alpha)}+\frac{1}{p-\alpha}\right]=\frac{1}{p-\alpha},$$

bu yerda $\operatorname{Re}(p-\alpha)>0\Rightarrow\operatorname{Re}p>\operatorname{Re}\alpha$ deb faraz qilamiz, chunki lemmaga asosan $\lim_{b\to\infty}e^{-(p-\alpha)b}=0$. Shunday qilib,

$$f(t)\xrightarrow{L}\frac{1}{p-\alpha},\quad \operatorname{Re}p>\operatorname{Re}\alpha. \qquad (5.3)$$

Ya'ni

$$e^{\alpha t}\xrightarrow{L}\frac{1}{p-\alpha},\quad \operatorname{Re}p>\operatorname{Re}\alpha. \qquad (5.3')$$

Shunga o'xshash

$$e^{-\alpha t} \xrightarrow{L} \frac{1}{p+\alpha}, \quad \operatorname{Re} p > \operatorname{Re}(-\alpha). \tag{5.4}$$

3. $Sin\omega t$ va $Cos\omega t$ funksyalarning tasvirlari.

Ma'lumki,

$$Sin\omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right), \quad Cos\omega t = \frac{1}{2i}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

bo'lib (5.3) va (5.4) formulalar va tasvirning chiziqlilik xossasiga asosan

$$Sin\omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) \xrightarrow{L} \frac{1}{2i}\left(\frac{1}{p-i\omega} - \frac{1}{p+i\omega}\right) = \frac{\omega}{p^2+\omega^2}, \quad \operatorname{Re} p > 0$$

$$Cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \xrightarrow{L} \frac{1}{2}\left(\frac{1}{p-i\omega} + \frac{1}{p+i\omega}\right) = \frac{p}{p^2+\omega^2}, \operatorname{Re} p > 0.$$

Shunday qilib,

$$Sin\omega t \xrightarrow{L} \frac{1}{2i}\left(\frac{1}{p-i\omega} - \frac{1}{p+i\omega}\right) = \frac{\omega}{p^2+\omega^2}, \quad \operatorname{Re} p > 0, \tag{5.5}$$

$$Cos\omega t \xrightarrow{L} \frac{1}{2}\left(\frac{1}{p-i\omega} + \frac{1}{p+i\omega}\right) = \frac{p}{p^2+\omega^2}, \operatorname{Re} p > 0. \tag{5.6}$$

4. $sh\omega t$ va $ch\omega t$ funksiyalarning tasvirlari.

$$sh\omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right) \xrightarrow{L} \frac{1}{2}\left(\frac{1}{p-\omega} - \frac{1}{p+\omega}\right) = \frac{\omega}{p^2-\omega^2}, \quad \operatorname{Re} p > |\omega|.$$

$$ch\omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right) \xrightarrow{L} \frac{1}{2}\left(\frac{1}{p-\omega} + \frac{1}{p+\omega}\right) = \frac{p}{p^2-\omega^2}, \operatorname{Re} p > |\omega|.$$

Shunday qilib,

$$sh\,\omega t \xrightarrow{L} \frac{\omega}{p^2-\omega^2}, \quad \operatorname{Re} p > |\omega|, \tag{5.7}$$

$$ch\omega t \xrightarrow{L} \frac{p}{p^2-\omega^2}, \operatorname{Re} p > |\omega|. \tag{5.8}$$

5. Darajali funksiya t^n ning tasviri.

(5.1) formulaga ko'ra quyidagiga egamiz:

$$t^n \xrightarrow{L} \int_0^\infty t^n e^{-pt} dt.$$

n marta bo'laklab integrallab va $\operatorname{Re} p > 0$ bo'lganda $\lim_{t\to\infty} t^k e^{-pt} = 0 \ (k = 0,1,2,...,n)$ ekanini e'tiborga olib, topamiz:

$$t^n \xrightarrow{L} \int_0^\infty t^n e^{-pt} dt = \frac{n!}{p^{n+1}}, \ \operatorname{Re} p > 0.$$

Shunday qilib,

$$t^n \xrightarrow{L} \frac{n!}{p^{n+1}}, \ \operatorname{Re} p > 0. \quad (5.9)$$

5.1-misol. $f(t) = 5Sin4t + 3Cos2t$ funksiyaning tasvirini toping.

Yechilishi. Tasvirning chiziqlilik xossasi va (5.5) hamda (5.6) formulalar asosida topamiz:

$$f(t) \xrightarrow{L} 5\frac{4}{p^2+4^2} + 3\frac{p}{p^2+2^2} = \frac{20}{p^2+4^2} + \frac{3p}{p^2+2^2}.$$

5.22-misol. Tasviri $F(p) = \dfrac{2p+1}{p^2+4}$ ko'rinishga ega bo'lsa originalini toping.

Yechilishi. $F(p) = \dfrac{2p+1}{p^2+4} = 2\dfrac{p}{p^2+2^2} + \dfrac{1}{2}\dfrac{2}{p^2+2^2}$;

$$f(t) = 2Cos2t + \frac{1}{2}Sin2t.$$

5.3-misol. $f(t) = t^3 - 5t^2 + 2t + 6$ funksiyaning tasvirini toping.

Yechilishi. (5.2) va (5.9) formulalarga asosan:

$$f(t) \xrightarrow{L} \frac{3!}{p^{3+1}} - 5\frac{2!}{p^{2+1}} + 2\frac{1!}{p^{1+1}} + 6\frac{1}{p} = \frac{6}{p^4} - \frac{10}{p^3} + \frac{2}{p^2} + \frac{6}{p}.$$

5.3. Operatsion hisobning asosiy teoremalari

5.3-teorema (*O'xshashlik teoremasi*). $f(t) \xrightarrow{L} F(p)$ bo'lsin. U holda $f(\alpha t) \xrightarrow{L} \frac{1}{\alpha} F(\frac{p}{\alpha})$, bu yerda $\alpha > 0$ bo'ladi.

Isbot. Tasvirning ta'rifiga ko'ra quyidagiga egamiz:

$$f(t) \xrightarrow{L} \int_0^\infty f(\alpha t) e^{-pt} dt = \left| \begin{matrix} \alpha t = z \\ \alpha dt = dz \end{matrix} \right| = \frac{1}{\alpha}\int_0^\infty f(z) e^{-\frac{p}{\alpha}z} dz = \frac{1}{\alpha} F(\frac{p}{\alpha}).$$

Shunday qilib,

$$f(\alpha t) \xrightarrow{L} \frac{1}{\alpha} F(\frac{p}{\alpha}) \quad (5.10)$$

(5.10) munosabat originalning erkli o‘zgaruvchisini musbat songa ko‘paytirilsa, uning tasviri va tasvirning erkli o‘zgaruvchisi bu songa bo‘linishini ko‘rsatadi.

5.4-teorema (*Siljish teoremasi*). Agar $f(t)\xrightarrow{L}F(p)$ bo‘lsa, $e^{-\alpha t}f(t)\xrightarrow{L}F(p+\alpha)$ bo‘ladi. Bunda α haqiqiy son va $\mathrm{Re}(p+\alpha)>s_0$ deb faraz qilinadi.

Isbot. $e^{-\alpha t}f(t)$ funksiyaning tasvirini topamiz:

$$e^{-\alpha t}f(t)\xrightarrow{L}\int_0^\infty e^{-\alpha t}f(t)e^{-pt}dt=\int_0^\infty f(t)e^{-(p+\alpha)t}dt=F(p+\alpha).$$

Shunday qilib, $e^{-\alpha t}f(t)\xrightarrow{L}F(p+\alpha)$.

Bu teorema va (5.5) hamda (5.6) formulalar asosida topamiz:

$$\left.\begin{aligned} e^{-\alpha t}Sin\omega t &\xrightarrow{L}\frac{\omega}{(p+\alpha)^2+\omega^2}\\ e^{-\alpha t}Cos\omega t &\xrightarrow{L}\frac{p+\alpha}{(p+\alpha)^2+\omega^2}\end{aligned}\right| \qquad (5.11)$$

5.4-misol. $e^{-2t}(Sint+3Cost)$ funksiyaning tasvirini toping.

Yechilishi. Chiziqlilik xossasidan va (5.4) formulalardan foydalanib topamiz:

$$e^{-2t}(Sint+3Cost)\xrightarrow{L}\frac{1}{(p+2)^2+1}+3\frac{p+2}{(p+2)^2+1}=\frac{3p+7}{p^2+4p+5}.$$

5.5-misol. Ushbu $\frac{2p+1}{p^2+2p+2}$ tasvirga ko‘ra originalni toping.

Yechilishi. Chiziqlilik xossasidan va (5.11) formuladan foydalanib topamiz:

$$\frac{2p+1}{p^2+2p+2}=\frac{2(p+1)-1}{(p+1)^2+1^2}=2\frac{p+1}{(p+1)^2+1^2}-\frac{1}{(p+1)^2+1^2}\xleftarrow{L}2e^{-t}Cost-e^{-t}Sint=$$

$$=e^{-t}(2Cost-Sint).$$

$e^{-\alpha t}t^n$ funksiyaning tasvirini topamiz. (5.9) formulaga ko‘ra $f(t)=t^n\xrightarrow{L}F(p)=\frac{n!}{p^{n+1}}$. Siljish teoremasini qullab, topamiz:

$$e^{-\alpha t}t^n\xrightarrow{L}\frac{n!}{(p+\alpha)^{n+1}}. \qquad (5.12)$$

Bu formulaga $-\alpha$ ni α ga almashtiramiz:

$$e^{\alpha t}t^{n} \xrightarrow{L} \frac{n!}{(p-\alpha)^{n+1}}. \qquad (5.13)$$

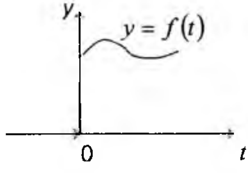
5.5-teorema (*Kechikish teoremasi*). $f(t)$ original bo'lsin.

Quyidagicha aniqlangan

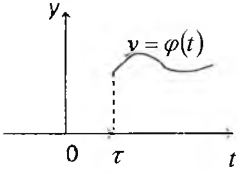
$\varphi(t)$ funksiyani qaraymiz:

$$\varphi(t)=\begin{cases} agar\ t<\tau\ bo'lsa,\ 0; \\ agar\ t\geq\tau\ bo'lsa,\ f(t-\tau) \end{cases}$$

bu yerda τ musbat son. $y=\varphi(t)$ funksiya grafigi original grafigi $y=f(t)$ ni ot o'q bo'ylab τ kattalikka surish orqali hosil qilinadi.



5.3- chizma. $f(t)$ originalning grafiki.



5.4- chizma. $\varphi(t)$ originalning grafiki.

Demak, agar $f(t)$ funksiya biror jarayonni tasvirlayotgan bo'lsa, u holda $\varphi(t)$ funksiya o'sha jarayonni τ ga kechikish bilan tavsiflaydi.

Originalning argumenti τ ga kechikkanda, bu originalning tasvirini topamiz.

5.6-teorema. $\tau>0$ va $f(t)\xrightarrow{L}F(p)$ bo'lsin. U holda $f(t-\tau)\xrightarrow{L}e^{-p\tau}F(p)$ bo'ladi.

Isbot. Tasvirning ta'rifiga ko'ra

$$f(t-\tau)\xrightarrow{L}\int_0^\infty f(t-\tau)e^{-pt}dt=\int_\tau^\infty f(t-\tau)e^{-pt}dt,$$

Chunki $t<\tau,\ \Rightarrow t-\tau<0$ uchun $f(t-\tau)=0,\ t-\tau=z\Rightarrow dt=dz,\ t=\tau+z$.

$$\int_\tau^\infty f(t-\tau)e^{-pt}dt=\int_0^\infty f(z)e^{-p(z+t)}dz=\int_0^\infty f(z)e^{-pz}e^{-p\tau}dz=e^{-p\tau}\int_0^\infty f(z)e^{-pz}dz=e^{-p\tau}F(p).$$

Shunday qilib, $f(t-\tau)\xrightarrow{L}e^{-p\tau}F(p)$.

5.6-misol. $(t-2)^3$ originalning tasvirini toping.

Yechilishi. $f(t)=t^3,\ \tau=2$.

$$t^3\xrightarrow{L}\frac{3!}{p^4};\ (t-2)^3=\frac{3!e^{-2p}}{p^4}=\frac{6e^{-2p}}{p^4}.$$

5.4. Originallarni differensiallash va integrallash

1. Originallarni differensiallash. $F(p)$ ushbu $f(t)$originalning tasviri bo'lsin. $f'(t)$ hosilaning tasvirini topish talab qilinadi.

5.7-teorema. Agar $f(t)\xrightarrow{L}F(p)$ va $f'(t)$ original bo'lsa, u holda

$$f'(t)\xrightarrow{L}pF(p)-f(0). \qquad (5.14)$$

Isbot. Tasvirni aniqlovchi formulaga ko'ra quyidagiga egamiz:

$$f'(t)\xrightarrow{L}\int_0^\infty f'(t)e^{-pt}dt=\left|\begin{matrix}u=e^{-pt}, du=-pe^{-pt}dt\\ dv=f'(t)dt, v=f(t)\end{matrix}\right|=\int_0^\infty e^{-pt}f'(t)dt=\lim_{b\to\infty}\int_0^b e^{-pt}f'(t)dt$$

$$=\lim_{b\to\infty}\left[e^{-pt}f(t)\right]_0^b+p\lim_{b\to\infty}\int_0^b e^{-pt}f(t)dt=\lim_{b\to\infty}\left[e^{-pb}f(b)-f(0)+\int_0^b f(t)e^{-pt}dt\right]=$$

$$=p\int_0^\infty f(t)e^{-pt}dt-f(o).$$

Chunki $\lim\limits_{b\to\infty}e^{-pb}f(b)=0$. Haqiqatan ham, $\left|e^{-pb}f(b)\right|=\left|f(b)\right|\left|e^{-pb}\right|=\left|f(b)\right|\left|e^{-(s+i\tau)b}\right|$.

Endi $e^{-(s+i\tau)b}=e^{-sb}e^{-i\tau b}=e^{-sb}(Cos\,\tau b-iSin\,\tau b)$ bo'lgani uchun $\left|e^{-(s+i\tau)b}\right|=e^{-sb}$ ga egamiz. Originalning c) xosasiga kura: $|f(b)|<Me^{s_0 b}$, demak, $\left|e^{-pt}f(b)\right|<Me^{s_0 b}\cdot e^{-sb}=Me^{(s_0-s)b}=Me^{-(s-s_0)b}$. Biroq tasvirning mavjudlik teoremasiga ko'ra $F(p)$ tasvir $\operatorname{Re}p=s>s_0$ uchun, ya'ni $s-s_0>0$ uchun mavjud. Shuning uchun $b\to\infty$ da $Me^{-(s-s_0)b}\to 0$. Demak, $\lim\limits_{b\to\infty}e^{-pb}f(b)=0$.

Biroq

$$\int_0^\infty f(t)e^{-pt}dt=F(p).$$

U holda

$$f'(t)\xrightarrow{L}pF(p)-f(0).$$

Xususan, agar $f(0)=0$ bo'lsa

$$f'(t)\xrightarrow{L}pF(p). \qquad (5.15)$$

(5.14) formulani $f''(t)$ga qullanib, quyidagiga ega bo'lamiz:

$$f''(t)\xrightarrow{L}pF_1(p)-f'(0),$$

bu yerda

$$f'(t)\xrightarrow{L}F_1(p)=pF(p)-f(0).$$

U holda

$$f''(t)\xrightarrow{L} p[pF(p)-f(0)]-f'(0)=p^2F(p)-pf(0)-f'(0). \tag{5.16}$$

Shunga o‘xshash

$$f'''(t)\xrightarrow{L} pF_2(p)-f''(0),\quad F_2(p)\xleftarrow{L} f''(t),$$

$$\begin{aligned} f'''(t)&\xrightarrow{L} p[p^2F(p)-pf(0)-f'(0)]-f''(0)=\\ &=p^3F(p)-p^2f(0)-pf'(0)-f''(0). \end{aligned} \tag{5.17}$$

$$f^{(n)}(t)\xrightarrow{L} p^nF(p)-p^{n-1}f(0)-p^{n-2}f'(0)-\cdots-pf^{(n-2)}(0)-f^{(n-1)}(0). \tag{5.18}$$

Xususan, agar $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$ bo‘lsa u holda

$$\left.\begin{aligned} f'(t)&\xrightarrow{L} pF(p)\\ f''(t)&\xrightarrow{L} p^2F(p)\\ f'''(t)&\xrightarrow{L} p^3F(p)\\ &\cdots\cdots\cdots\cdots\\ f^{(n)}(t)&\xrightarrow{L} p^nF(p) \end{aligned}\right| \tag{5.19}$$

Mavjudlik teoremasida ko‘rsatilishicha, $p\to\infty$ da ($\mathrm{Re}\,p=s\to\infty$ shartda). Har qanday originalning tasviri nolga intiladi. Shuning uchun agar $f'(t)\xrightarrow{L} F_1(p)$ bo‘lsa u holda $p\to\infty$da $F_1(p)\to 0$. Biroq (5.14) formulaga ko‘ra $F_1(p)=pF(p)-f(0)$. Demak,

$$\lim_{p\to\infty} F_1(p)=\lim_{p\to\infty}[pF(p)-f(0)]=0 \Rightarrow$$

$$\lim_{p\to\infty} pF(p)=f(0) \tag{5.20}$$

Bu formula $f(t)$ originalning boshlang‘ich qiymatini originalni hisoblab o‘tirmasdan uning $F(p)$ tasviri bo‘yicha topishga imkon beradi.

5.7-misol. $F(p)=\dfrac{p}{p^2+1}$ tasvir berilgan. Originalning boshlang‘ich $f(0)$ qiymatini toping.

Yechilishi. $f(0)=\lim\limits_{p\to\infty} pF(p)=\lim\limits_{p\to\infty}\dfrac{p^2}{p^2+1}=1$.

Originalni integrallash.

5.8-teorema. Agar $f(t)\xrightarrow{L} F(p)$ bo‘lsa, u holda $\int_0^t f(x)dx=\dfrac{F(p)}{p}$.

Isbot. $g(t)=\int_0^t f(x)dx$ funksya original ekanini ko‘rsatish mumkin.

Uning tasviri $\Phi(p)$ bo‘lsin:

$$g(t)\xrightarrow{L}\Phi(p) \tag{5.21}$$

$g(0)=\int_0^0 f(x)dx=0$ bo'lgani uchun (5.19) formulalarning birinchisiga ko'ra $g'(t)\xrightarrow{L} p\Phi(p)$. Biroq

$$g'(t)=\frac{d\int_0^t f(x)dx}{dt}=f(t)\xrightarrow{L} F(p).$$

Demak, $F(p)=p\Phi(p)\Rightarrow \Phi(p)=\frac{F(p)}{p}$, ya'ni $\int_0^t f(x)dx\xrightarrow{L}\frac{F(p)}{p}$.

5.8-misol. $F(p)=\frac{p^2+2p+2}{(p-2)^2(p+3)}$ tasvir berilgan. Originalni toping.

Yechilishi. $F(p)=\frac{p^2+2p+2}{(p-2)^2(p+3)}=\frac{4}{5(p-2)}+\frac{2}{(p-2)^2}+\frac{1}{p+3}$.

$$\frac{1}{p-2}\xleftarrow{L} e^{2t},\quad \frac{1}{(p-2)^2}\xleftarrow{L} te^{2t},\quad \frac{1}{p+3}\xleftarrow{L} e^{-3t}.$$

$$F(p)\xleftarrow{L}\frac{4}{35}e^{2t}+2te^{2t}+1e^{-3t}.$$

5.5. O'zgarmas koeffitsiyentli chiziqli differensial tenglamalarni integrallash

Ikkinchi tartibli chiziqli differensial tenglama berilgan bo'lsin.

$$y''(t)+a_1y'(t)+a_2y(t)=f(t), \qquad (5.22)$$

bu yerda a_1 va a_2– haqiqiy sonlar. Bu tenglamaning $y(0)=y_0$, $y'(0)=y_0'$ boshlang'ich shartlarni qanoatlantiruvchi yechimi $y(t)$ ni aniqlaymiz. Buning uchun $y(t), y'(t), y''(t)$ va $f(t)$ originallar bo'lsin.

$$y(t)\xrightarrow{L} Y(p),\ f(t)\xrightarrow{L} F(p),\ y'(t)\xrightarrow{L} pY(p)-y_0,$$

$$y''(t)\xrightarrow{L} p^2Y(p)-py_0-y_0'.$$

U holda

$$y''(t)+a_1y'(t)+a_2y(t)\xrightarrow{L} p^2Y(p)-py_0-y_0'+a_1(Y(p)p-y_0)+a_2Y(p)=F(p)$$

$$(p^2+a_1p+a_2)Y(p)=F(p)+-py_0+y_0'+a_1y_0 \qquad (5.23)$$

(5.23) tenglama yordamchi tenglama yoki (5.22) differensial tenglamag mos tasvirlardagi tenglama deyiladi. (5.23) dan

$$Y(p)=\frac{F(p)+py_0+y_0'+a_1y_0}{p^2+a_1p+a_2} \qquad (5.24)$$

(5.24) formula (5.23) tenglamaning operator yechimi deyiladi.

5.9-misol. $y'' - 3y' + 2y = 2e^{3t}$, $y(0) = 1$, $y'(0) = 3$ yeching.
Yechilishi.

$$Y(p) = \frac{\frac{2}{p-3} + p + 3 - 3}{p^2 - 3p + 2} = \frac{2 + p^2 - 3p}{(p-3)(p^2 - 3p + 2)} = \frac{1}{p-3} \xleftarrow{L} e^{3t}.$$

$$Y(t) = e^{3t}.$$

5.6. O'zgarmas koeffitsiyentli chiziqli differensial tenglamalar sistemalarini integrallash

Ushbu differensial tenglamalar sistemasi berilgan bo'lsin

$$\left.\begin{aligned} \frac{dx}{dt} + a_{11}x + a_{12}y = f_1(t) \\ \frac{dy}{dt} + a_{21}x + a_{22}y = f_2(t) \end{aligned}\right\} \qquad (5.25)$$

Bu sistemaning $x(0) = x_0$, $y(0) = y_0$ boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini topish talab qilinadi.

$$x(t) \xrightarrow{L} X(p),\ y(t) \xrightarrow{L} Y(p),\ f_1(t) \xrightarrow{L} F_1(p),\ f_2(t) \xrightarrow{L} F_2(p),$$

$$x'(t) \to pX(p) - x_0,\ y'(t) \to pY(p) - y_0$$

$$\left.\begin{aligned} pX(p) - x_0 + a_{11}x(p) + a_{12}y(p) = F_1(p) \\ pY(p) - y_0 + a_{11}x(p) + a_{12}y(p) = F_2(p) \end{aligned}\right\} \qquad (5.26)$$

Uni yechib $X(p)$ va $Y(p)$ tasvirlarni topamiz, biz izlanayotgan noma'lum $x(t)$ va $y(t)$ funksiyalar esa bularning originallaridan iborat bo'ladi.

5.9-teorema (*Kompozitsiyalash teoremalari*).

Agar $F_1(p) \xleftrightarrow{L} f_1(t)$, $F_2(p) \xleftrightarrow{L} f_2(t)$ bo'lsa

$$F_1(p)F_2(p) \to \int_0^t f_1(t) f_2(t-\tau)d\tau$$

bo'ladi.

Isbot.

$$L\left\{\int_0^\tau f_1(\tau)f_2(t-\tau)d\tau\right\} = \int_0^\infty e^{-pt}\left[\int_0^\tau f_1(\tau)f_2(t-\tau)d\tau\right]dt = \int_0^\tau\left[f_1(\tau)\int_0^\infty e^{-pt}f_2(t-\tau)dt\right]d\tau$$

$$\int_0^\infty e^{-pt} f_2(t-\tau)dt = \left| \begin{aligned} t - \tau = z \\ dt = dz,\ t = z + \tau \\ t \to \infty,\ \tau \to \infty \end{aligned} \right| = e^{-p\tau}F_2(p).$$

$$\int_0^{\infty} e^{-p\tau} f_2(\tau) dF_2(p) = F_1(p)F_2(p).$$

Tasvirni differensiallash. $f(t) \xrightarrow{L} F(p)$ bo'lsin. U holda

1. $-tf(t) \xrightarrow{L} F'(p)$;
2. $(-1)^n t^n f(t) \xrightarrow{L} F^{(n)}(p)$.

Tasvirni integrallash. $f(t) \xrightarrow{L} F(p)$ va $\frac{f(t)}{t}$ kasr mavjudligini shartlarini qanoatlantirsin. U holda

$$\frac{f(t)}{t} \xrightarrow{L} \int_p^{\infty} F(q) dq.$$

Urama haqida teorema (Tasvirlarni ko'paytirish teoremasi). Bu teoremani bayon qilishdan avval o'rash amalini (yoki o'ramaning) ta'rifini keltirishga to'g'ri keladi. U $(*)$ simvol bilan belgilanadi.

Biror $[\alpha, \beta]$ oraliqda aniqlangan $f_1(t)$ va $f_2(t)$ funksyalar berilgan bo'lsin. Ularning bu kesmadagi o'ramasi deb

$$f(t) = \int_{\alpha}^{\beta} f_1(\tau) f_2(t-\tau) d\tau = f_1(t) * f_2(t)$$

tenglik bilan aniqlanadigan yangi $f(t)$ funksyaga aytiladi. $[\alpha, \beta]$ kesma uchun $[0, t]$ kesmani olamiz. O'rama amali kommutativlik va assotsiativlik xossalariga (ya'ni o'rinalmashtirish va gruppalash qonuniyatlariga bo'ysinadi) va qo'shishga nisbatan distributivlik (taqsimot qonuni) xossasiga ega ekanligini ko'rish mumkin. Xususan, kommutativlik xossasi o'rama funksiyalardan qaysi biri integralga $t-\tau$ argument bilan kirishga bog'liq emasligini bildiradi.

Agar $f_1(t)$ va $f_2(t)$lar tasvir shartlarini qanoatlantirsa, ular o'ramasining tasviri ko'paytuvchilar tasvirlarining ko'paytmasidan iborat bo'ladi, ya'ni $f_1(t) \xrightarrow{L} F_1(p)$ va $f_2(t) \xrightarrow{L} F_2(p)$ dan $f_1(t) * f_2(t) \xrightarrow{L} F_1(p) \cdot F_2(p)$ kelib chiqadi.

Dyuamel teoremasi. Bu teorema oldingi teoremaning umumlashtirilgani sifatida qaralishi mumkin. U ikkita funksiya o'ramasini hosilasining tasviri uchun ifoda beradi.

Agar $f_1(t), f_2(t)$ funksiyalar $[0, \infty]$ da uzluksiz hosilalarga ega bo'lsa va $f_1(t) \xrightarrow{L} F_1(p)$, $f_2(t) \xrightarrow{L} F_2(p)$ bo'lsa, u holda

$$\frac{d}{dt}[f_1(t) * f_2(t)] \xrightarrow{L} pF_1(p) \cdot F_2(p).$$

Laplas almashtirishning qoidalari

$$F(p)=\int_0^\infty e^{-pt}f(t)dt$$

1-jadval.

№	nomi	original	tasvir
1.	Chiziqlilik xossasi	$\sum_{i=1}^{n} c_i f_i(t)$	$\sum_{i=1}^{n} l_i F_i(t)$
2.	Uxshashlik teoremasi.	$f(at)\ (a>0)$	$\frac{1}{a}F(\frac{p}{a})$
3.	Tasvirni siljish teoremasi	$e^{-\alpha t}f(t)$	$F(p+\alpha)$
4.	Originalni kechikish teoremasi	$f(t-\tau)\ (\tau>0)$	$e^{-p\tau}F(p)$
5.	Originalni o'zib ketish teoremasi	$f(t+\tau)\ \ (\tau>0)$	$e^{-p\tau}\left[F(p)-\int_0^\tau e^{-pt}f(t)dt\right]$
6.	Originalni differensial-lash	$f'(t)$ $f^{(m)}(t)$	$pF(p)-f(0)$ $p^nF(p)-[p^{n-1}f(0)+p^{n-2}f'(0)+....+pf^{(n-2)}($ $+f^{(n-1)}(0)]$
7.	Originalni integrallash	$\int_0^t f(\tau)d\tau$	$\frac{1}{p}F(p)$
8.	Tasvirni differensial-lash	$-tf(t)$ $(-1)^n t^n f(t)$	$F'(p)$ $F^{(n)}(p)$
9.	Tasvirni integrallash	$\frac{f(t)}{t}$	$\int_p^\infty F(q)dq$
10.	O'rama teoremasi	$f_1(t)*f_2(t)$	$F_1(p)\cdot F_2(p)$
11.	Dyua mel teoremasi	$\frac{d}{dt}[f_1(t)*f_2(t)]$	$pF_1(p)\cdot F_2(p)$

Quyida keltirilgan jadvalda tez-tez uchrab turadigan elementar funksiyalarning tasvirlari keltirilgan. Ularning ba'zilarini oldingi jadval qoidalari bo'yicha gruppalab, bir qator yangi formulalarni hosil qilish mumkin.

Ba'zi elementar funksiyalarning tasvirlari

2-jadval.

№	Original	tasvir
1	$\sigma_0(t)$	$\frac{1}{p}$
2	$e^{-\alpha t}$	$\frac{1}{p+\alpha}$
3	$\sin \omega t$	$\frac{\omega}{p^2+\omega^2}$
4	$\cos \omega t$	$\frac{p}{p^2+\omega^2}$
5	$sh\,\omega t$	$\frac{\omega}{p^2-\omega^2}$
6	$ch\,\omega t$	$\frac{p}{p^2-\omega^2}$
7	t	$\frac{1}{p^2}$
8	t^n	$\frac{n!}{p^{n+1}}$
9	$e^{-\alpha t}\sin \omega t$	$\frac{\omega}{(p+\alpha)^2+\omega^2}$
10	$e^{-\alpha t}\cos \omega t$	$\frac{p+\alpha}{(p+\alpha)^2+\omega^2}$
11	$e^{-\alpha t} sh\,\omega t$	$\frac{\omega}{(p+\alpha)^2-\omega^2}$
12	$e^{-\alpha t} ch\,\omega t$	$\frac{p+\alpha}{(p+\alpha)^2-\omega^2}$
13	$t^n e^{-\alpha t}$	$\frac{1}{(p+\alpha)^2}$
14	$t \sin wt$	$\frac{n}{(p+2)^{n+1}}$
15	$t \sin \omega t$	$\frac{2\omega p}{(p^2+\omega^2)^2}$
16	$t \cos \omega t$	$\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$

17	$\frac{1}{n_1} e^{nt} \sin n_1 t, n_1 = \sqrt{n^2 - k^2}$	$\frac{1}{p^2 + 2np + k^2}$
18	$\frac{1}{h} e^{-nt} shht, h = \sqrt{n^2 - k^2}$	$\frac{1}{p^2 + 2np + k^2}$
19	$\frac{1}{n_1^2}\left[1 - e^{nt}(\cos n_1 t + \frac{n}{n_1}\sin{}_1 t)\right]$ $n_1 = \sqrt{n^2 - k^2}$	$\frac{1}{p(p^2 + 2np + k^2)}$
20	$\frac{1}{n^2}\left[1 - e^{-nt}(chht + \frac{n}{h} shht)\right]$ $h = \sqrt{n^2 - k^2}$	$\frac{p}{p^2 - 2np + k^2}$
21	$e^{-nt}(chht - \frac{n}{h} shht),$ $h = \sqrt{n^2 - k^2}$	$\frac{p}{p^2 - 2np + k^2}$
22	$e^{-nt}(chht - \frac{n}{h} shht)$ $h = \sqrt{n^2 - k^2}$	$\frac{p}{p^2 + 2np + k^2}$
23	$e^{-nt}(\frac{t \cos n_1 t}{2n_1^2} + \frac{shht}{2h\sqrt{n}})$ $h = \sqrt{k^2 - n^2}$	$\frac{1}{(p^2 + 2np + k^2)^2}$
24	$e^{-nt}(\frac{tchht}{2h^2} + \frac{shht}{2h\sqrt{n}}),$ $h = \sqrt{k^2 - n^2}$	$\frac{1}{(p^2 + 2np + k^2)^2}$
25	$e^{-nt}\left(\frac{nt \cos n_1 t}{2n_1^2} + \frac{t \sin n_1 t}{2n_1} - \frac{n \sin n_1 t}{2n_1\sqrt{n_1}}\right),$ $n_1 = \sqrt{n^2 - k^2}$	$\frac{p}{(p^2 + 2np + k^2)^2}$
26	$e^{-nt}\left(\frac{ntch\,ht}{2h^2} + \frac{tsh\,ht}{2h} - \frac{nsh\,ht}{2h\sqrt{h}}\right),$ $h = \sqrt{n^2 - k^2}$	$\frac{p}{(p^2 + 2np + k^2)^2}$
27	$\frac{1}{(k_0^2 - k^2)^2 + 4n^2 k_0^2}\{e^{-nt}[(k_0^2 - k^2)\cdot$ $\cos k_1 t - \frac{n(k_0^2 + k^2)}{k_1}\sin k_1 t] -$ $(k_0^2 - k^2)\cos k_0 t + 2nk_0 \sin k_0 t\}\ k_1 = \sqrt{k^2 - n^2}$	$\frac{p}{(p^2 + 2np + k^2)(p^2 + k_0^2)}$

5.7. Differensial tenglamalar va ularning sistemalarini operatsion usullar yordamida yechish

Laplas almashtirishini differensial tenglamalarni yechishga tadbiq qilishning umumiy sxemasi oldingi paragraflarda keltirilgan misollarni tahlil qilishga to'xtalib o'tamiz.

O'zgarmas koeffitsiyentli chiziqli differensial tenglama berilgan bo'lsin.

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+a_2y^{(n-2)}(t)+....+a_ny(t)=f(t), \qquad (5.27)$$

bu tenglamaning

$$y(0)=y_0,\ y'(0)=y_0',\ .....,\ y^{(n-1)}(0)=y_0^{(n-1)} \qquad (5.28)$$

boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini toppish talab etiladi. Bunda $f(t)$ funksya kabi izlayotgan yechim ham Laplas bo'yicha tasvirning mavjudlik shartlariga bo'ysunadi deb faraz qilinadi. (5.27) tenglamaning ikkala qismiga Laplas almashtirishini tatbiq qilamiz. $y(t)$ va $f(t)$ larning tasvirlarini mos ravishda $Y(p)$va $F(p)$ orqali belgilaymiz:

$$y(t)\xrightarrow{L}Y(p),\quad f(t)\xrightarrow{L}F(p)$$

Originalni differensiallash qoidasini qo'llanib quyidagini topamiz:

$$y'(t)\xrightarrow{L}pY(p)-y_0,$$
$$y''(t)\xrightarrow{L}p^2Y(p)-[py_0+y'_0],$$
$$..$$
$$y^{(n-1)}(t)\xrightarrow{L}p^{n-1}Y(p)-[p^{n-2}y_0+p^{n-3}y'_0+...+y_0^{(n-2)}],$$
$$y^{(n)}(t)\xrightarrow{L}p^nY(p)-[p^{n-1}y_0+p^{n-2}y'_0+...+y_0^{(n-1)}].$$

Laplas almashtirishining chiziqligiga binoan chap tomonning tasvirini toppish uchun hosil qilingan ifodalarni tegishli a_i koeffitsiyentlarga ko'paytirish va qo'shish kifoya, o'ng tomonning tasviri esa $F(p)$ ga teng. Shunday qilib, quyidagiga egamiz:

$$\varphi(p)Y(p)-\psi(p)=F(p), \qquad (5.29)$$

bu yerda

$$\varphi(p)=p^n+a_1p^{n-1}+a_2p^{n-2}+....+a_{n-1}p+a_n$$

ifoda chiziqli tenglama uchun xarakteristik ko'phad. $\psi(p)$ esa

$$\psi(p)=[p^{n-1}y_0+p^{n-2}y'_0+...+y_0^{(n-1)}]+a_1[p^{n-2}y_0+p^{n-3}y'_0+...+y_0^{(n-2)}]+....$$
$$.+a_{n-2}[py_0+y_0']+a_{n-1}y_0.$$

Nol boshlang'ich shartlarda $\psi(p)=0$ bo'lib, (5.29) tenglamaning chap tomoni $\varphi(p)Y(p)$ ko'rinishni olishini qayd qilib o'tamiz, bunga (5.27) tenglamada differensiallash operatorini p ko'paytuvchi bilan almashtirib, $Y(p)$ ni qavs tashqarisiga chiqarish orqali kelish mumkin. (5.29) tenglama boshlang'ich shartlar sistemasi (5.28) bo'lgan (5.27) tenglama uchun yordamchi tenglamadir. Uni tasvirlovchi (operator) tenglama ham deb ataladi, (5.29) tenglamani $Y(p)$ ga nisbatan yechib,

$$Y(p)=\frac{F(p)+\psi(p)}{\varphi(p)} \tag{5.30}$$

ko'rinishga ega bo'lgan tasvirlovchi yoki operator yechimini hosil qilamiz. Originalga o'tish izlanayotgan $y(t)$ xususiy yechimni topishga imkon beradi. Bunda (5.30) operator yechimning o'ng tomoni odatda ratsional kasr bo'lib chiqadi va tasvirlar jadvalidan foydalanishni yengillatish uchun o'ng tomonni elementar kasrlarga yoyish kerak.

5.9-misol. Boshlang'ich shartlari $y(0)=3,\ y_0'(0)=2,\ y_0'(0)=1$ bo'lgan

$$y'''(t)-y'(t)=0$$

tenglamani yeching.

Yechilishi. Operator tenglama tuzamiz:

$$[p^3Y(p)-(3p^2+2p+1)]-[pY(p)-3]=0$$

yoki

$$(p^3-p)Y(p)=3p^2+2p-2.$$

Bundan

$$Y(p)=\frac{3p^2+2p-2}{p^3-p}.$$

O'ng tomonini eng soda kasrlarga yoyamiz:

$$\frac{3p^2+2p-2}{p^3-p}=\frac{A}{p}+\frac{B}{p-1}+\frac{C}{p+1}$$

A,B,C koeffitsiyentlarni toppish uchun o'ng tomonni umumiy maxrajga keltiramiz va ayniyatning ikkala tomonidagi suratlarni taqqoslaymiz:

$$3p^2+2p-2=A(p^2-1)+B(p^2+p)+C(p^2-p).$$

Bu ayniyatda $p=0$ deb $A=2$ ni, $p=1$ da $B=3/2$ ni, $p=-1$ da $C=-1/2$ hosil qilamiz.

Demak,

$$Y(p)=\frac{2}{p}+\frac{3}{2}\frac{1}{p-1}-\frac{1}{2}\frac{1}{p+1}$$

jadval yordamida originallarga o'tib

$$y(t)=2+\frac{3}{2}e^{t}-\frac{1}{2}e^{-t}$$

hosil qilamiz.

5.10-misol. Ushbu tenglamani yeching:

$$y'''(t)-2y''(t)+y'(t)=4;\ y_0=1, y_0'=2,\ y_0''=2.$$

Yechilishi. Bu yerda operator tenglama:

$$[p^3Y(p)-(p^2+2p+2)]-2[p^2Y(p)-(p+2)]+[pY(p)-1]=\frac{4}{p}$$

$$(p^3-2p^2+p)Y(p)=p^2-5+\frac{4}{p}$$

$$Y(p)=\frac{p^2-5+\frac{4}{p}}{p^3-2p^2+p}=\frac{p^3-5p+4}{p^2(p^2-2p+1)}=\frac{(p-1)(p^2+p-4)}{p^2(p-1)^2}=\frac{p^2+p-4}{p^2(p-1)}.$$

Bu kasrni elementar kasrlarga yoyib quyidagiga ega bo'lamiz:

$$Y(p)=\frac{3}{p}+\frac{4}{p^2}-\frac{2}{p-1}.$$

Originallarga o'tib quyidagini topamiz:

$$y(t)=3+4t-2e^{t}.$$

Operatsion usullar differensial tenglamaning faqat xususiy yechimlarigina topishga imkon beradi deb o'ylash kerak emas. Ulardan umumiy yechimni izlash ham mumkin. Buning uchun boshlang'ich shartlarni ixtiyoriy o'zgarmaslar uchun qabul qilish kerak, u holda ular hosil qilinadigan yechimning ifodasiga kiradi.

5.11-misol. Ushbu $y''+k^2y=aSinky$ tenglama umumiy yechimini toping.

Yechilishi. Umumiy yechimni topish uchun $y(0)=C_1,\ y'(0)=C_2$ deb olish kerak. Operator tenglamani hosil qilamiz:

$$[p^2Y(p)-(C_1p+C_2)]+k^2Y(p)=\frac{ak}{p^2+k^2}$$

$$Y(p)=\frac{ak}{(p^2+k^2)^2}+C_1\frac{p}{p^2+k^2}+C_2\frac{1}{p^2+k^2}.$$

Berilgan kasrni quyidagicha o'zgartiramiz:

$$\frac{ak}{(p^2+k^2)^2}=\frac{a[(p^2-k^2)+(p^2+k^2)]}{2k^2(p^2+k^2)^2}=\frac{a}{2k}\left[\frac{k^2-p^2}{(p^2+k^2)^2}+\frac{1}{p^2+k^2}\right].$$

Shunday qilib,

$$Y(p)=\frac{a}{2k}\left[\frac{k^2-p^2}{(p^2+k^2)^2}+\frac{1}{p^2+k^2}\right]+C_1\frac{p}{p^2+k^2}+C_2\frac{1}{p^2+k^2}.$$

Originallarga o‘tib, izlanayotgan umumiy yechimni hosil qilamiz:

$$Y(p)=\frac{a}{2k}(-tSinkt+\frac{1}{k}Sinkt)+C_1Coskt+\frac{C_2}{k}Sinkt.$$

O‘zgarmas koeffitsiyentli chiziqli differensial tenglamalarni yechishning operatsion usullarini bunday tenglamalar sistemalariga ham tatbiq qilish mumkin. Farq shundaki, bitta operator tenglama o‘rniga izlanayotgan funksiyalarning tasvirlariga nisbatan chiziqli algebraik operator tenglamalar sistemasi hosil bo‘ladi.

Birinchi tartibli differensial tenglamalar sistemasi berilgan bo‘lsin.

$$\left.\begin{array}{l} X_1'(t)+\sum\limits_{k=1}^{n}a_{1k}X_k(t)=f_1(t), \\ X_2'(t)+\sum\limits_{k=1}^{n}a_{2k}X_k(t)=f_2(t), \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ X_n'(t)+\sum\limits_{k=1}^{n}a_{nk}X_k(t)=f_n(t) \end{array}\right| \qquad (5.31)$$

Ushbu

$$X_k(0)=x_{k_0}\ (k=1,2,\dots,n) \qquad (5.32)$$

boshlang‘ich shartlarni qanoatlantiruvchi xususiy yechimni topish talab qilinadi, shu bilan birga

$$L\{f_k(t)\}=F_k(p),\ L\{X_k(t)\}=X_k(p).$$

U holda operator tenglamalar sistemasini quyidagicha yozish mumkin.

$$\begin{array}{l} pX_1(p)+\sum\limits_{k=1}^{n}a_{1k}X_k(p)=F_1(p)+x_{1_0}, \\ pX_2(p)+\sum\limits_{k=1}^{n}a_{1k}X_k(p)=F_2(p)+x_{2_0}, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ pX_n(p)+\sum\limits_{k=1}^{n}a_{1k}X_k(p)=F_n(p)+x_{n_0} \end{array}$$

Bu algebraik chiziqli tenglamalar sistemasini $X_k(p)$ tasvirlarga nisbatan yechish kerak, so'ngra topilgan tasvirlardan $x_k(t)$ originallarga o'tish kerak. Ana shular birgalikda (5.31) differensial tenglamalar Sistemasining (5.32) boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimni beradi.

5.12-misol. Ushbu $x(0)=y(0)=1$ boshlang'ich shartlarni qanoatlantiruvchi

$$\left.\begin{array}{l} x'(t)-2x(t)+y(t)=0, \\ y'(t)-x(t)-2y(t)=0 \end{array}\right\}$$

tenglamalar sistemasining xususiy yechimlari topilsin.

Yechilishi. Operator sistema:

$$\left.\begin{array}{l} [pX(p)-1]-2X(p)+Y(p)=0, \\ [pY(p)-1]-X(p)-2Y(p)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} (p-2)X(p)+Y(p)=1, \\ -X(p)+(p-2)Y(p)=1 \end{array}\right\}$$

Bu sistemaning yechimini Kramer qoidasi bo'yicha topamiz, buning uchun quyidagi determinantlarni hisoblaymiz:

$$\Delta=\begin{vmatrix} p-2 & 1 \\ -1 & p-2 \end{vmatrix}=(p-2)^2+1=p^2-4p+5,$$

$$\Delta_x=\begin{vmatrix} 1 & 1 \\ 1 & p-2 \end{vmatrix}=p-2-1=p-3,$$

$$\Delta_y=\begin{vmatrix} p-2 & 1 \\ 1 & 1 \end{vmatrix}=p-2+1=p-1.$$

Demak,

$$X(p)=\frac{p-3}{p^2-4p+5},\quad Y(p)=\frac{p-1}{p^2-4p+5},$$

bu yerda originallarga o'tib, izlanayotgan yechimni hosil qilamiz:

$$\left.\begin{array}{l} x(t)=e^{2t}(Cost+2S\text{int})-3e^{2t}S\text{int}=e^{2t}(Cost-S\text{int}), \\ y(t)=e^{2t}(Cost+2S\text{int})+e^{2t}S\text{int}=e^{2t}(Cost+3S\text{int}) \end{array}\right\}$$

5.13-misol. Differensial tenglamalar sistemasining umumiy yechimi topilsin:

$$\left.\begin{array}{l} x''(t)+y'(t)+2x=0, \\ y''(t)-3x'(t)-2y=0 \end{array}\right\}$$

$x(0)=c_1, x'(0)=c_2, y(0)=c_3, y'(0)=c_4$ bo'lsin.

Yechilishi. Operator tenglamalar sistemasini tuzamiz:

$$\left.\begin{array}{l} [p^2X(p)-(c_1p+c_2)]+[pY(p)-c_3]+2X(p)=0, \\ [p^2Y(p)-(c_3p+c_4)]-3[pX(p)-c_1]-2Y(p)=0 \end{array}\right\}$$

yoki

$$\left.\begin{aligned}(p^2+2)X(p)+pY(p)&=c_1p+c_2+c_3\\ -3pX(p)+(p^2-2)Y(p)&=c_3p-3c_1+c_4\end{aligned}\right\}$$

Determinantlarni hisoblaymiz:

$$\Delta=\begin{vmatrix}p^2+2 & p\\ -3p & p^2-2\end{vmatrix}=(p^2-1)(p^2+4),$$

$$\Delta_x=\begin{vmatrix}c_1p+c_2+c_3 & p\\ c_3p-3c_1+c_4 & p^2-2\end{vmatrix}=c_1(p^3+p)+c_2(p^2-2)-2c_3-c_4p,$$

$$\Delta_y=\begin{vmatrix}p^2+2 & c_1p+c_2+c_3\\ -3p & c_3p-3c_1+c_4\end{vmatrix}=-6c_1+3c_2p+c_3(p^2+5p)+c_4(p^2+2).$$

Demak, operator yechim quyidagi ko‘rinishga ega bo‘ladi:

$$X(p)=\frac{c_1(p^3+p)+c_2(p^2-2)-2c_3-c_4p}{(p^2-1)(p^2+4)},$$

$$Y(p)=\frac{-6c_1+3c_2p+c_3(p^3+5p)+c_4(p^2+2)}{(p^2-1)(p^2+4)}.$$

Tasvirlardan originallarga o‘tish uchun elementar kasrlarga yoyish o‘rniga originalni differensiallash teoremasiga asoslangan usulni qo‘llanamiz.

$$\frac{1}{(p^2-1)(p^2+4)}=\frac{1}{5}\left(\frac{1}{p^2-1}-\frac{1}{p^2+4}\right)\xrightarrow{L}\frac{1}{5}(sht-\frac{1}{2}Sin2t)$$

bo‘lgani uchun [$f(0)$ ko‘rinishdagi qo‘shimcha qushiluvchilar bo‘lmaydi, chunki har gal $f(0)=0$] quyidagilarni hosil qilamiz:

$$\frac{p}{(p^2-1)(p^2+4)}\xrightarrow{L}\frac{1}{5}(cht-Cos2t),$$

$$\frac{p^2}{(p^2-1)(p^2+4)}\xrightarrow{L}\frac{1}{5}(sht+2Sin2t),$$

$$\frac{p^3}{(p^2-1)(p^2+4)}\xrightarrow{L}\frac{1}{5}(cht+4Cos2t).$$

Bu formulalarni qo‘llanib, operator yechimdan izlanayotgan umumiy yechimni ushbu ko‘rinishda hosil qilish mumkin.

$$x(t)=\frac{2c_1-c_2}{5}cht-\frac{c_2+2c_3}{5}sht+\frac{3c_1+c_4}{5}Cos2t+\frac{3c_2+c_3}{5}Sin2t,$$

$$y(t)=\frac{3(c_2+2c_3)}{5}cht-\frac{3(2c_1-c_4)}{5}sht-\frac{3c_2+c_3}{5}Cos2t+\frac{3c_1+c_4}{5}Sin2t.$$

5.8. Operatsion hisobni fizikaviy va boshqa masalalarga tadbiqi

I. Mexanikaviy tebranishlar. Ba’zi mexanikaviy masalalarni operatsion usullar yordamida yechamiz.

1. Garmonik tebranishlar. Differensial tenglama va boshlang‘ich shartlar quyidagicha:

$$\ddot{x}(t) + k^2 x(t) = 0; \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 = v_0 .$$

Operator tenglama:

$$\left[p^2 x(p) - (x_0 p + v_0)\right] + k^2 x(p) = 0 .$$

Operator yechim:

$$x(p) = \frac{x_0 p + v_0}{p^2 + k^2} = x_0 \frac{p}{p^2 + k^2} + v_0 \frac{1}{p^2 + k^2} .$$

Izlanayotgan xususiy yechim:

$$x(t) = x_0 \cos kt + \frac{v_0}{k} \sin kt \text{ yoki } x(t) = A \sin(kt + \alpha), \text{ bu yerda}$$

$$A = \sqrt{x_0^2 + \frac{v_0^2}{k^2}}, \qquad \alpha = arctg \frac{x_0 k}{v_0}.$$

Xususan $v_0 = 0$ da $x(t) = x_0 \cos kt$ yoki $x(t) = x_0 \sin\left(kt + \frac{\pi}{2}\right)$.

2. So‘nuvchi tebranishlar. Tenglamani va boshlang‘ich shartlar sistemasi quyidagicha:

$$\ddot{x}(t) + 2n\,\dot{x}(t) + k^2 x(t) = 0; \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 = v_0 .$$

Operator tenglama:

$$\left[p^2 x(p) - (x_0 p + v_0)\right] + 2n\left[p x(p) - x_0\right] + k^2 x(p) = 0 ,$$

Operator yechim:

$$x(p) = \frac{x_0 p + v_0 + 2n x_0}{p^2 + 2np + k^2} = x_0 \frac{p}{p^2 + 2np + k^2} + \left(v_0 + 2n x_0 \cdot \frac{1}{p^2 + 2np + k^2}\right)$$

$k^2 - n^2 > 0$ da izlanayotgan xususiy yechim ($k_1 = \sqrt{n^2 - k^2}$ deb faraz qilamiz):

$$x(t) = x_0 e^{-nt}\left(\cos k_1 t - \frac{n}{k_1} \sin k_1 t\right) + \frac{v_0 + 2n x_0}{k_1} \cdot$$

$$\cdot\, e^{-nt} \sin k_1 t = e^{-nt}\left(x_0 \cos k_1 t + \frac{v_0 n x_0}{k_1} \sin k_1 t\right) = A e^{-nt}(\sin k_1 t + \alpha),$$

bu yerda

$$A=\sqrt{x_0^2+\left(\frac{v_0+nx_0}{k_1}\right)^2},\quad \alpha=arctg\frac{x_0k_1}{v_0+nx_0}.$$

Agar $k^2-n^2<0$ bo‘lsa, u holda $h=\sqrt{n^2-k^2}$ deb belgilab, yechimni quyidagi ko‘rinishda hosil qilamiz:

$$x(t)=e^{-nt}\left(x_0chht+\frac{v_0+nx_0}{h}shht\right).$$

Agar $n^2-k^2=0$ bo‘lsa, u holda operator yechim ushbu ko‘rinishni oladi:

$$x(p)=x_0\frac{p}{(p+n)^2}+(v_0+2nx_0)\frac{1}{(p+n)^2}=$$

$$=x_0\frac{p+n-n}{(p+n)^2}+\frac{v_0+2nx_0}{(p+n)^2}=\frac{x_0}{p+n}+\frac{v_0+nx_0}{(p+n)^2},$$

originalga o‘tib quyidagini topamiz:

$$x(t)=x_0e^{-nt}+(v_0+nx_0)te^{-nt}=e^{-nt}(x_0+(v_0+nx_0)t).$$

3.Muhitning qarshiligi hisobga olinmagandagi majburiy tebranishlar. Tenglama va boshlang‘ich shartlar sistemasi quyidagi ko‘rinishda:

$$\ddot{x}(t)+k^2x(t)=q\sin wt;\quad x(0)=x_0,\quad \dot{x}(0)=v_0.$$

Operator tenglama:

$$\left[p^2x(p)-(px_0+v_0)\right]+k^2x(p)=\frac{qw}{p^2+w^2}.$$

Operator yechim:

$$x(p)=\frac{qw}{(p^2+k^2)(p^2+w^2)}+\frac{x_0p+x_0}{p^2+k^2},$$

buni originalga o‘tkazish quyidagi ikki holga kelamiz:

1-hol. $w^2\neq k^2$. U holda

$$x(t)=\frac{qw}{k^2-w^2}\left(\frac{1}{w}\sin wt-\frac{1}{k}\sin kt\right)+x_0\cos kt+\frac{v_0}{k}\sin kt=$$

$$=\frac{q}{k^2-w^2}\sin wt+x_0\cos kt+\left(\frac{v_0}{k}-\frac{qw}{k(k^2+w^2)}\right)\sin kt=$$

$$=\frac{q}{k^2-w^2}\sin wt+x_0\cos kt+\frac{1}{k}\left(v_0-\frac{qw}{k^2-w^2}\right)\sin kt.$$

Bu yechimni quyidagi ko‘rinishga keltirish mumkin:

$$x(t)=\frac{q}{k^2-w^2}\sin wt+A\sin(kt+\alpha),$$

bu yerda

$$A=\sqrt{x_0^2+\frac{1}{k^2}\left(v_0-\frac{qw}{k^2-w^2}\right)^2},\quad \alpha=arctg\frac{x_0k}{v_0-\frac{qw}{(k^2-w)}}.$$

2-hol. $\omega^2=k^2$. Bu holda operator yechim ushbu ko‘rinishda bo‘ladi:

$$\chi(p)=\frac{q\omega}{\left(p^2+x^2\right)^2}+\frac{x_0p+v_0}{p^2+v^2}.$$

Bu yechimni orginalga o‘tkazib quyiladi

$$x(t)=-\frac{q}{2\kappa}t\cos\kappa t+x_0\cos\kappa t+\frac{1}{k}\left(v_0+\frac{q}{2k}\right)\sin kt=-\frac{q}{2k}t\cos kt+A\sin(kt+\alpha),$$

bu yerda

$$A=\sqrt{x_0^2+\frac{1}{k^2}\left(v_0+\frac{q}{2k}\right)^2},\quad \alpha=arctg\frac{x_0k}{v_0+q(2k)}.$$

II. Elektr tebranishlar. Elekrt yurituvchi kuch o‘zgarmas bo‘lgan holda RLG-zanjirdagi tokni topish masalasini operatsion usul bilan yechamiz.

Tok uchun deferensial tenglama mexanik tebranishlarining muhitning qarshilgni hisobga olingandagi tenglamalarga o‘xshash

$$i^k(t)+\frac{r}{l}i'(t)+\frac{1}{le}i(t)=0,$$

boshlang‘ich shartlariga esa quyidagicha:

$$i(0)=0\quad i'(0)=\frac{E}{L}.$$

Operator tenglama:

$$\left(p^2Jp\right)-\frac{r}{2}pJ(p)+\frac{1}{LC}J(p)=0$$

Operator yechim:

$$J(p)=\frac{E}{L}\frac{1}{p^2+\frac{R}{L}P+\frac{1}{LC}}.$$

$\frac{R}{L}=2\delta$ deb belgilaymiz va quydagi hollarni qarab chiqamiz.

1-hol. $\frac{1}{LC}-\frac{r^2}{4L^2}=\omega_1^2$. Tasvirdan orginalga o‘tib $i(t)$ tok uchun so‘nuvchi elektr tebranishlarni ifodalovchi ushbu i(t)=$\frac{E}{\omega_1L}e^{-\delta t}\sin\omega_1t$ funksiyani hosil qilamiz.

2- hol. $\frac{R^2}{4L^2}-\frac{1}{EC}=\beta^2>0$. U holda $i(t)=\frac{E}{BL}e^{-\delta t}\sin\beta t$ i - tok nodavriy – va zanjirda hech qanday tebranishlar bo‘lmaydi

3- hol. $\frac{R^2}{4L^2}-\frac{1}{LC}=0$. Bu holda operator yechim

$$J(p)=\frac{E}{L}\frac{1}{(p+v)^2},$$

demak.

$$i(t)=\frac{E}{L}te^{-\delta t},$$

ya'ni bu holga ham $i(t)$ tok nodavriy bo'lib, elektr tebranishlar bo'lmaydi.

Nazorat savollari.

1. Tasvir va original ta'rifi.
2. Laplas almashtirishini yozib bering.
3. Elementar funksiyalarning tasvirini hisoblashga misol ko'rsating.
4. Tasvir bo'yicha originalni tiklashga misol ko'rsating.
5. Originalni differensiallashning formulasi.
6. Originalni integrallash.
7. Operatsion hisobning differensial tenglamalarni yechishga qo'llanishiga misol keltiring.
8. Differensial tenglamalar sistemasini operatsioan usullar yordamida yechishga misol keltiring.

VI-BOB. MATEMATIK FIZIKA TENGLAMALARI

6.1. Matimatik fizika tenglamarining asosiy tiplari

Matimatik fizikaning ikkinchi tartibli asosiy ·differensial tenglamalari ikki o'zgaruvchili noma'lum $u(x,y)$ funksiya va uning hosilalariga nisbatan chiziqli bo'lib bunday tenglamalarning umumiy ko'rinishi quyidagicha bo'ladi:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=f(x,y), \qquad (6.1)$$

bu yerda A,B,C,D,E,F umumiy x va y larga bog'liq bo'lib, xususan o'zgarmaslardir, $f(x,y)$ esa berilgan funksiya. Agar $f(x,y)=0$ bo'lsa u holda bu tenglama ikkinchi tartibli bir jinsli chiziqli xususiy hosilali tenglama deyiladi:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0. \qquad (6.2)$$

Agar (6.2) tenglama berilgan sohaga:

$B^2-4AC>0$ bo'lsa (6.2) tenglama giperbolik

$B^2-4AC=0$ bo'lsa (6.2) tenglama parabolik

$B^2-4AC<0$ bo'lsa (6.2) tenglama elipik turga tegishli bo'ladi.

Torning ko'ndalang tebranishi, metal sterjenning uzunasiga tebranishi, simdagi elektr tebranishlar, aylanuvchi silindrdagi aylanma tebranishlar, gazning tebranishlari kabi masalalar giperbolik tipdagi eng sodda to'lqin tenglamasi

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (6.3)$$

ga olib keladi.

Issiqlikning tarqalish jarayoni, g'ovak muhitda suyuqlik va gazing oqish masalasi, ehtimollar nazariyasining bazi masalalari parabolik turdagi eng sodda issiqlik tarqalish tenglamasi (Fur'e tenglamasi)

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (6.4)$$

ga olib keladi.

Elektr va magnet maydonlari haqidagi masalalarni, statsionar issiqlik holati haqidagi masalalarni, gidrodinamik, diffuziya va shunga o'xshash masalalarni yechish elliptik turdagi Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (6.5)$$

ga olib keladi.

Agar izlanayotgan funksiya uchta o'zgaruvchiga bog'liq bo'lsa to'lqin tenglamasi:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (6.3')$$

issiqlik tarqalish tenglamasi:

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (6.4')$$

Laplas tenglamasi:

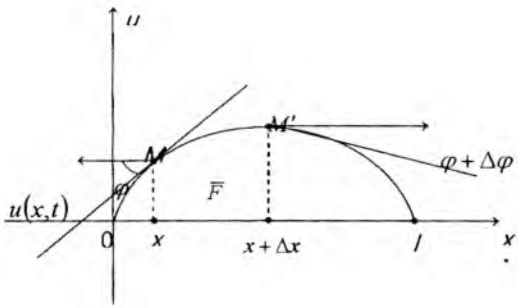
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2} = 0. \qquad (6.5')$$

6.2. Tor tebranish tenglamasi. Boshlang'ich va chetki shartlar

Uzunligi l ga teng bo'lgan egiluvchan va elastik ip (tor) berilgan bo'lib, uning uchlari to'g'ri burchakli dekart koordinatlarida $x=0$ va $x=l$ nuqtalarga biriktirilgan deb faraz qilamiz.

Agar tarang tortilgan torni dastlabki holatidan chetlashtirib, so'ngra o'z holatiga qo'yib yuborsak yoki uning nuqtalariga biror tezlik bersak, u holda torning nuqtalari harakatga keladi, ya'ni tor tebrana boshlaydi. Biz istagan momentga tor shaklini aniqlash hamda torning har bir nuqtasi vaqtga bog'liq ravishda qanday qonun bilan harakatlanishini aniqlash masalasini ko'ramiz.

Tor nuqtalari boshlang'ich holatidan kichik chetlanishlarga ega deb qarab, tor nuqtalarining harakati Ox uqqa perpendikulyar va bir tekislikda vujudga keladi, deb faraz qilamiz. U holda torning tebranish jarayoni bitta $u(x,t)$ funksiya orqali ifoda etiladi, bunga x tor nuqtasining t momentdagi siljish miqdorini bildiradi (6.1-chizma).



6.1-chizma. $x = 0$ ***va*** $x = l$ ***nuqtalarga tortilgan ip (tor).***

Torning barcha nuqtalarida taranglik T bir xil deb faraz qilamiz. Torning MM' elementiga ta'sir etuvchi kuchlarning Ou o'qdagi proyeksiyasi:

$$T\sin(\varphi + \Delta\varphi) - T\sin\varphi \approx Ttg(\varphi + \Delta\varphi) - Ttg\varphi =$$

$$= T\left[\frac{\partial u(x + \Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] = T\frac{\partial^2 u(x + \theta\Delta x, t)}{\partial x^2}\Delta x \approx \qquad (6.6)$$

$$\approx T\frac{\partial^2 u(x,t)}{\partial x^2}dx,\ 0 < \theta < 1$$

(bu yerda φ burchak kichik bo'lganligi uchun $tg\varphi \approx Sin\varphi$ va kvadrat qavsdagi ifodaga Lagranj teoremasini tatbiq etdik).

Harakat tenglamasini hosil qilish uchun $\breve{MM}$ elementga qo'yilgan tashqi kuchni energiya kuchiga tenglash kerak. Torning MM' elementiga t momentga ta'sir etuvchi kuch

$$F \approx g(x,t)\breve{MM'} \approx g(x,t)dx, \qquad (6.7)$$

bu yerda $\breve{MM'} = x_2 - x_1 = dx$, $g(x,t)$ - tor bo'ylab uzluksiz taqsimlangan Ou o'qiga parallel kuchlar zichligi. Torning chiziqli zichligi ρ bo'lsa MM' elementning massasi $\rho\breve{MM'} = \rho dx$ bo'ladi. Elementning tezlanishi $\frac{\partial^2 u}{\partial t^2}$ ga teng. Demak, Dalamber prinsipiga ko'ra (6.6) va (6.7) formulalarni hisobga olib, ushbu tenglikka ega bo'lamiz:

$$\rho dx\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}dx + g(x,t)dx \Rightarrow \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho}g(x,t), \qquad (6.8)$$

bu yerda $a^2 = \frac{T}{\rho}$.

Bu tenglama torning majburiy tebranish tenglamasi yoki bir o'lchovli to'lqin tenglamasi deyiladi.

Agar $g(x,t)=0$ bo'lsa, (6.8) tenglama tashqi kuch ta'sir etmagandagi bir jinsli erkin tebranish tenglamasi diyiladi.

Oddiy defferensial tenglamalarga umumiy yechimdan xususiy yechimlarni olish uchun ixtiyoriy o'zgarmaslarni aniqlash lozim. Harakatni to'la aniqlash uchun

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (6.9)$$

tenglamaning uzigina yetarl emas. Yana qo'shimcha

$$u(x,t)\big|_{x=0} = u(0,t) = 0, \quad u(x,t)\big|_{x=l} = u(l,t) = 0 \qquad (6.10)$$

chegaraviy shartlar

$$\left.\begin{aligned} &u(x,0) = u\big|_{t=0} = f(x), \\ &u_t'(x,0) = \frac{\partial u}{\partial x}\Big|_{t=0} = F(x) \end{aligned}\right| \qquad (6.11)$$

boshlang'ich shartlar berilishi kerak.

6.3. Torning tebranish tenglamasini o'zgaruvchilarni ajratish usuli (Fur'e usuli) bilan yechish

Ikkala uchidan mahkamlangan torning erkin tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (6.12)$$

ning boshlang'ich shartlar

$$u\big|_{t=0} = f(x), \quad \frac{\partial u}{\partial x}\Big|_{t=0} = F(x) \qquad (6.13)$$

va chetki shartlar

$$u(x,t)\big|_{x=0} = 0, \quad u(x,t)\big|_{x=l} = 0 \qquad (6.14)$$

berilgandagi xususiy yechimni topamiz.

Furye usulidan foydalanib nolga teng bulgan yechimni $X(x)$ va $T(t)$ funksiyalar ko'paytmasi shaklida qidiramiz:

$$u(x,t) = X(x)T(t) \qquad (6.15)$$

(6.15) dan x va t bo'yicha hosilalar olib (6.12) tenglamaga qo'yib ushbuni hosil qilamiz:

$$X(x)T''(t) = a^2 X''(x)T(t)$$

va bu tenglikning hadlarini $a^2 XT$ ga b o'lib

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda .$$

Bu tengliklardan

$$X'' + \lambda X = 0 \qquad (6.16)$$

$$T'' + \lambda a^2 T = 0 \qquad (6.17)$$

tenglamalar hosil bo'ladi.

Bu tenglamalarning umumiy yechimlarini topish uchun xarakteristik tenglamalarni tuzamiz. Xarakteristik tenglamalarni ildizlari kompleks bo'lganligi uchun

$$X(x) = ACos\sqrt{\lambda}x + BSin\sqrt{\lambda}x, \qquad (6.18)$$

$$T(t) = CCos\sqrt{\lambda}t + DSin\sqrt{\lambda}t \qquad (6.19)$$

yechimlarga ega bo'lamiz. Bunda A, B, C, D o'zgarmas sonlar bo'lib, $X(x)$ va $T(t)-$ lar uchun topilgan ifodalarni (6.15) tenglikka qo'yamiz:

$$u(x,t) = (ACos\sqrt{\lambda}x + BSin\sqrt{\lambda}x)(CCos\sqrt{\lambda}t + DSin\sqrt{\lambda}t) \qquad (6.20)$$

A va B o'zgarms sonlarni (6.14) shartdan foydalanib topamiz, (6.18) ga $x=0$, $x=l$ qiymatlarni qo'ysak,

$0 = A\cdot 1 + B\cdot 0$; $0 = ACos\sqrt{\lambda}l + BSin\sqrt{\lambda}l \Rightarrow A=0$, $BSin\sqrt{\lambda}l = 0$. $B \neq 0$ deb olamiz, aks holda $u=0$ bo'ladi. Shuning uchun

$$Sin\sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{l}, (n = 1,2,3,...) \qquad (6.21)$$

U holda

$$X = BSin\frac{n\pi}{l}x \qquad (6.22)$$

va

$$T(t) = CCos\frac{an\pi}{l}t + DSin\frac{an\pi}{l}t . \qquad (6.23)$$

(6.22) va (6.23) => (6.15) chegaraviy shartlarni qanoatlantiruvchi $u_n(x,t)$ yechimlarni hosil qilamiz:

$$u_n(x,t) = (C_n \cos\frac{an\pi}{l}t + D_n \sin\frac{an\pi}{l}t)\cdot \sin\frac{n\pi}{l}x$$

Tenglama chiziqli va bir jinsli bo'lgani uchun yechimlarining yig'indisi ham yechim bo'ladi va shuning uchun

$$u(x,t)=\sum_{n=1}^{\infty}(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t)\cdot\sin\frac{n\pi}{l}x \qquad (6.24)$$

qator bilan yozilgan funksiya ham (6.12) tenglamaning yechimi bo'ladi, C_n va D_n o'zgarmas miqdorlarini aniqlash uchun boshlang'ich (6.13) shrtdan foydalanamiz $t=0$ bo'lganda

$$f(x)=\sum_{n=1}^{\infty}C_n\sin\frac{\pi n}{l}x \qquad (6.25)$$

bo'lib, $f(x)$funksiya $(0,l)$ intervalda Fur'e qatoriga yoyilmasi mavjud deb faraz qilsak,

$$C_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{e}dx \qquad (6.26)$$

ga teng bo'ladi. (6.24) tenglikdan t bo'yicha hosila olib, $t=0$ da

$$F(x)=\sum_{n=1}^{\infty}D_n\frac{an\pi}{l}\sin\frac{n\pi}{l}x$$

tenglikni hosil qilamiz, bundan

$$D_n\frac{an\pi}{l}=\frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx$$

yoki

$$D_n=\frac{2}{an\pi}\int_0^l F(x)\sin\frac{n\pi x}{l}dx. \qquad (6.27)$$

Shunday qilib, biz C_n va D_n koeffitsiyentlarni aniqladik, demak, chegaraviy va boshlang'ich shartlarni qanoatlantiruvchi (6.12) tenglamaning yechimi bo'lgan $u(x,t)$ funksiyani aniqladik.

Izoh: Agar yuqorida $-\lambda$ urniga $\lambda=k^2$ ifodani olsak, tenglamaning umumiy yechimi (6.18)

$$X(x)=Ae^{kx}+Be^{-kx}$$

bo'lib chegaraviy (6.14) shartni qanoatlantirmaydi.

Xos funksiyani $u_k(x,t)=(C_kCos\frac{ak\pi}{l}t+D_kSin\frac{ak\pi}{l}t)Sin\frac{k\pi x}{l})$ ko'rinishda hosil qilgan edik. Uni quyidagi ko'rinishda ham yozish mumkin,

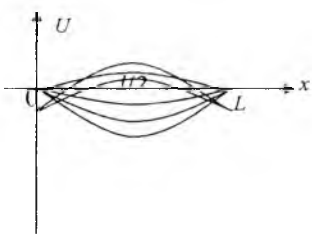
$$u_k(x,t)=F_kSin\frac{k\pi x}{l}\cdot Sin\left(\frac{k\pi a}{l}t+\varphi_k\right) \qquad (6.28)$$

bu yerda $F_k = \sqrt{C_k^2 + D_k^2}$ va $tg\varphi_k = \frac{C_k}{D_k}$.

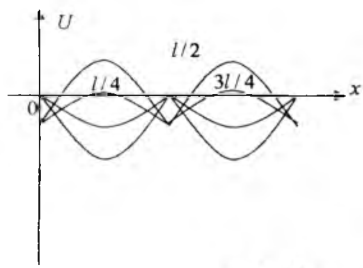
(6.28) formuladan ko‘rinadiki, torning barcha nuqtalari bir xil $\omega_k = \frac{k\pi a}{l}$ chastota va φ_k faza bilan garmonik tebranar ekan. Tebranish amplitudasi $F_k Sin\frac{k\pi x}{l}$ ga teng bo‘lib, u_x ga bog‘liq ekan. $k=1$ bo‘lganda (6.28) formuladan birinchi garmonik uchun·

$$u_1(x,t) = F_1 \sin\frac{\pi x}{l}\sin(\frac{\pi a}{l}t + \varphi_1)$$

formulani hosil qilamiz. $x=0$ va $x=l$ bo‘lganda qo‘zg‘almas nuqtalar torning chetlari bo‘lib, $x=\frac{l}{2}$ da torning chetlanishi eng katta bo‘lib, F_1 ga teng bo‘ladi (6.2- chizma).



6.2-chizma. $k=1$ ***bo‘lganda (6.28).***



6.3-chizma. $k=2$ ***bo‘lganda (6.28).***

$k=2$ ***bo‘lganda***

$$u_2(x,t) = F_2 Sin\frac{2\pi x}{l}Sin(\frac{2\pi a}{l}t + \varphi_2)$$

bo‘lib qo‘zg‘almas nuqta uchta bo‘ladi: $x=0$, $x=\frac{l}{2}$, $x=l$.

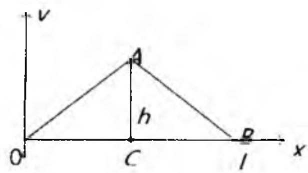
Amplitudaning eng katta qiymatiga $x=\frac{l}{4}$ va $x=\frac{3l}{4}$ nuqtada erishadi (6.3-chizma).

Umuman $Sin\frac{k\pi x}{l}=0$ tenglamaning ildizlari qancha bulsa $[0,l]$ kesmada shuncha qo‘zg‘almas nuqtalar bo‘ladi (ularga tugun nuqtalar deyiladi). Tugun nuqtalar orasida shunday bitta nuqta mavjud bo‘ladiki, bu nuqtada chetlanish eng katta- maksimumga erishadi, bunday nuqtalar tutamlik nuqtalari deyiladi. Torning eng kichik chastosi

$$\omega_1 = \frac{\pi a}{l} = \frac{\pi}{l}\sqrt{\frac{T}{\rho}} \qquad (6.29)$$

(6.29) formuladan ko'rinadiki, taranglik T qancha katta bo'lib, tor qancha yengil (l va ρ lar kichik) bo'lsa, ovoz shuncha yuqori bo'ladi. Qolgan ω_k chastotalarga mos kelgan ovozlar oberton yoki garmonikalar deyiladi.

6.1-misol. Chetlari $x=0,\ x=l$ nuqtalarda mahkamlangan tor berilgan bo'lsin. Uning boshlang'ich tezligi nolga teng. Boshlang'ich chetlanish uchi (c,h) nuqtada bo'lgan uchburchak shaklida bo'lsa (6.4-chizma) torning tebranishini toping (T_0 - taranglik, ρ - zichlik, $a=\sqrt{\frac{T_0}{\rho}}$ lar beilgan).



6.4-chizma. Chetlari $x=0,\ x=l$ nuqtalarda mahkamlangan tor.

Yechilishi. $f(x)=u|_{t=0}$ funksiyaning analitik ifodasi berilgan $O(0,0), A(c,h)$ va $B(l,0)$. OA va AB nuqtalardan o'tuvchi to'g'ri chiziq tenglamalarini tuzamiz.

OA: $\frac{x-o}{c-0}=\frac{y-0}{n-0}$ bundan $y=f(x)=\frac{hx}{c}(o\le x\le c)$

AB: $\frac{x-o}{l-c}=\frac{y-h}{-h}$ bundan $y=f(x)=\frac{h(l-x)}{l-c}(c\le x\le l)$

Masalaning shartiga ko'ra $\frac{\partial u}{\partial t}|_{t=0}=F(x)=0$ demak, (6.27) formulalaga asosan $D_k=0$ (6.26) furmulaga asosan:

$$C_k=\frac{2}{l}\int_0^l f(x)Sin\frac{k\pi x}{l}dx=\frac{2}{l}\left[\frac{h}{c}\int_0^l xSin\frac{k\pi x}{l}dx+\frac{h}{l-c}\int_c^l(l-x)Sin\frac{k\pi x}{l}dx\right]$$

Har bir integralni bo'laklab integrallaymiz va ushbu natijaga kelamiz:

$$C_k = \frac{2hl^2}{k^2\pi^2 c(l-c)} Sin\frac{k\pi c}{l}$$

C_k ni (6.24) formulaga quyamiz va ushbu yechimni olamiz:

$$u(x,t) = \frac{2hl^2}{\pi^2 c(l-c)} \sum_{k=1}^{\infty} \frac{1}{k^2} Sin\frac{k\pi c}{l} \sin\frac{k\pi x}{l} \cos\frac{ka\pi}{l} t$$

6.4. Torning majburiy tebranishi

Yuqorida ko'rilgan Fur'ye usuli tornig majburiy tebrranish tenglamasi (6.14) ni ham yechish uchun qulay ekanligini ko'ramiz. Torning tashqi kuch ta'sirida majburiy tebranishi masalasi bir jinsli bo'lmagan tebranma harakat tenglamasiga olib kelgan edi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + G(x,t), \tag{6.30}$$

bu yerda $G(x,t) = \frac{1}{\rho} g(x,t)$ belgilash kiritdik.

Boshlang'ich va chegaraviy shartlarni torning erkin tebranishidagi kabi qabul qilamiz:

$$u\big|_{t=0} = f(x), \quad \frac{\partial u}{dt}\Big|_{t=0} = F(x) \text{ va } u\big|_{x=0,l} = 0$$

(6.29) tenglamaning yechimi

$$u(x,t) = v(x,t) + w(x,t) \tag{6.31}$$

ko'rinishda qidiramiz.

(6.13) dagi $v(x,t)$ funksiyani shunday tanlab olamizki, u bir jinsli

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$$

tenglamani boshlangich $v\big|_{t=0} = f(x), v\big|_{t=l} = F(x)$ va $v\big|_{x=0,l} = 0$ shartlarni qanoatlantirsin. $w(x,t)$ funksiya esa bir jinsli emas

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + G(x,t) \tag{6.32}$$

tenglamani va quyidagi boshlang'ich va chegaraviy

$$w\big|_{t=0} = w'_t\big|_{t=0} = 0, \; w\big|_{x=0,l} = 0$$

shartlarini qanoatlantirsin. $v(x,t)$ torning erkin tebranishini ifodalagani uchun uning tenglamasini yechimini oldingi paragrafda ko'rib chiqilgan edik.

(6.32) tenglamani yechimini $Sin\frac{k\pi x}{l}$ xos funksiyalar bo‘yicha qator ko‘rinishda izlaymiz:

$$w(x,t)=\sum_{k=1}^{\infty}\gamma_k(t)\sin\frac{k\pi x}{l}, \qquad (6.33)$$

bu yerda $\gamma_k(t)$ hozircha noma’lum t – ga bog‘liq funksya $w(x,t)$ funksya chegaraviy shartlarini qanoatlantiradi. Haqiqatdan $x=0$ da $w(0,t)=0$, $x=l$ da $w(l,t)=0$. Barcha (6.33) dagi xos funksiyalar nolga teng bo‘ladi.

Agar (6.33) qatorga $\gamma_k|_{t=0}=0$, $\gamma_k'|_{t=0}=0$ bo‘lsin deb talab qilinsa, $w(x,t)$ funksiya uchun boshlang‘ich shartlar ham bajariladi. (6.33) qatordan x va t lar buyicha ikki marta xususiy hosilalar olib (6.32) tenglamaga qo‘yamiz. Natijada

$$\sum_{k=1}^{\infty}\left[\gamma_k''(t)+\frac{k^2\pi^2a^2}{l^2}\gamma_k(t)\right]\sin\frac{k\pi x}{l}=G(x,t) \qquad (6.34)$$

$G(x,t)$ funksiyani $(0,l)$ intervalga x argumentli sinuslar bo‘yicha Fur’ye qatoriga yoyamiz:

$$G(x,t)=\sum_{k=1}^{\infty}g_k(t)\sin\frac{k\pi x}{l}, \qquad (6.35)$$

bu yerda

$$g_k(t)=\frac{2}{l}\int_0^l G(x,t)\sin\frac{k\pi x}{l}dx \qquad (6.36)$$

(integralda t-o‘zgarmas).

Agar $G(x,t)=G(x)$ bo‘lsa, $g_k(t)$ funksiya o‘zgarmas bo‘ladi. Agar $G(x,t)=G(t)$ bo‘lsa

$$g_k(t)=\frac{2G(t)}{l}\int_0^l Sin\frac{k\pi x}{l}dx=\begin{cases}\frac{4}{k\pi}G(t), & k-toq\ \ bo'lsa\\ 0, & k-juft\ \ bo'lsa\end{cases} \qquad (6.37)$$

(6.34) va (6.35) yoyilmaning xos funksiyalri (6.37) oldidagi koeffitsiyentlarini tenglashtiramiz va noma’lum $\gamma_k(t)$funksiyalar uchun ushbu tenglamalarga ega bo‘lamiz:

$$\gamma_k''(t)+\frac{k^2\pi^2a^2}{e^2}\gamma_k(t)=g_k(t). \qquad (6.38)$$

Bu tenglamani

$$\gamma_k(0)=\gamma_k'(0)=0 \qquad (6.39)$$

boshlang'ich shartlarda yechamiz, (6.38) ga mos kelgan bir jinsli tenglamaning umumiy yechimi

$$\gamma_k(t) = A_k Cos\frac{k\pi at}{l} + B_k Sin\frac{k\pi at}{l}$$

ko'rinishda bo'ladi. Bir jinsli bo'lmagan (6.38) tenglamaning xususiy yechimini $g_k(t)$ funksiyaga qarab, tanlab olish usuli bilan aniqlash mumkin. Natijada boshlang'ich shartlardan foydalanib, ushbu yechimga ega bo'lamiz

$$\gamma_k(t) = \frac{1}{k\pi a}\int_0^t g_k(\tau) Sin\frac{k\pi a(t-\tau)}{l}d\tau. \qquad (6.40)$$

Topilgan $\gamma_k(t)$ larni (6.33) ga quyib, qidirilayotgan $w(x,t)$ funksiyalarni aniqlaymiz.

Nazorat savollari.

1. Ikkinchi tartibli xususiy hosilali differensial tenglama ta'rifi.
2. Giperbolik, parabolik va elliptik tipdagi tenglamalar ta'rifi.
3. Chegaraviy va boshlang'ich shartlar deganda nimani tushunasiz.
4. Torning erkin tebranish tenglamasi.
5. Torning erkin tebranish tenglamasining Dalamber yechimi ko'rinishi.
6. Torning erkin tebranish tenglamasining Fu're yechimi ko'rinishi.
7. Torning majburiy tebranish tenglamasi.

VII BOB. HODISA VA EHTIMOL TUSHUNCHASI

7.1. Elementlar hodisalar fazosi. Hodisalar ustida amallar

Ehtimollar nazariyasining asosiy tushunchalardan biri «tajriba» va tajriba natijasida kuzatilishi mumkin bo'lgan hodisa tushunchalaridir. Tajriba hodisani ro'yobga keltiruvchi shartlar majmui (shartlar kompleksi) S ning bajarilishini ta'minlashdan iboratdir. Tajribadan tajribagacha o'tganda ro'y berayotgan hollar hayotda keng miqyosda uchrab turadi, bu yerda albatta, tajribani vujudga keltiruvchi shartlar majmui (kompleks) S o'zgarmas bo'lgan hollar tushuniladi. Misollar:

1.O'tkazilayotgan tajriba simmetrik bir jinsi tangani muayyan sharoitda tashlashdan iborat bo'lsin. Albatta, bu yerda tajribadan tajribagacha o'tganda ro'y beruvchi hodisalar har xil bo'ladi, masalan, biror tajribada «gerb» (G) tushgan bo'lsa, boshqasida tanganing teskari tomoni «raqam» (R) tushishi mumkin (bunda tanga qirrasi bilan tushmaydi deb faraz qilinadi).

2.Kuzatilayotgan tajriba biror aloqa bo'lishidan bir kunda junatilayotgan telegrammalar soni bo'lsin, bu yerda ham tajribadan tajribaga o'tganda, ya'ni kundan-kunga o'tganda ro'y berishi mumkin bo'lgan hodisalar (telegrammalar sonining biror natural songa tengligi) har xil bo'lishi mumkin.

Tajriba natijasida ro'y berishi oldindan aniq bo'lmagan hodisa tasodifiy hodisa deyiladi. Tajribaning har qanday natijasi elementar hodisa deyiladi. Tajriba natijasida ro'y berishi mumkun Bo'lgan barcha elementar hodisalar to'plami elementar hodisalar fazosi deyiladi. Elementar hodisalar fazosini U orqali, har bir elementar hodisani esa e $(e \epsilon U)$ orqali belgilaymiz.

3. Tajriba simmetrik, bir jinsli tangani ikki marta tashlashdan iborat bo'lsin. Bunda elementar hodisalar quyidagicha bo'ladi:

$e_1 = (GG) -$ birinchi tashlashda gerb, ikkinchisida ham gerb tushish hodisasi,

$e_2 = (GR) -$ birinchi tashlashda gerb, ikkinchisida raqam tushish hodisasi,
$e_3 = (RG) -$ birinchi tashlashda raqam, ikkinchisida gerb tushish hodisasi,
$e_4 = (RR) -$ birinchi tashlashda raqam, ikkinchisida ham raqam tushish hodisasi.

Bu tajribada elementar hodisalar fazosi U to'rt elementdan iborat:

$$U = \{e_1,\ e_2,\ e_3,\ e_4\}$$

4. Agar o'sha tanga uch marta tashlansa ro'y berishi mumkin bo'lgan elementar hodisalar quyidagicha bo'ladi:

$$e_1 = (GGG),\quad e_2 = (GGR),\quad e_3 = (GRR),\quad e_4 = (RRR)$$
$$e_5 = (RGR),\quad e_6 = (RRG),\quad e_7 = (GRG),\quad e_8 = (RGG).$$

Bu holda elementar hodisalar fazosi sakkiz elementar hodisadan iborat:

$$U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}.$$

5. Tajriba yoqlari birdan oltigacha no'merlangan bir jinsli kubikni ikki marta tashlashdan iborat bo'lsin. Bu holda elementar hodisalar ushbu ko'rinishga ega:

$$e_{ij} = (i, j)$$

Bu hodisa kubikni birinchi tashlashda i raqamli yo'q, ikkinchi tashlashda j raqamli yo'q tushganligini bildiradi, bu yerda

$$U = \{e_{ij}\}, i, j = \overline{1,6}$$

va elementar hodisalar soni $N = 6 * 6 = 36$.

6. Tajriba nuqtani $[0, 1]$ segmentiga tasodifiy ravishda tashlashdan iborat bo'lsin, bu yerda elementar hodisalar fazosi $U = \{e\}\ [0, 1]$ to'plamdan iboratdur, ya'ni u kontinuum quvvatga ega.

Shunday qilib, har qanday tajriba natijasida ro'y berishi mumkin bo'lgan elementar hodisalar, hodisalar to'plamini vujudga keltiradi va bu hodisalar to'plami chekli, sanoqli va hatto kontinium quvvatiga ega bo'lishi mumkin.

Har qanday tasodifiy hodisa esa elementar hodisalar to'plamidan tashgil topgan bo'lib, uning «katta-kichikligi» unga kirgan elementar hodisalarning «soni» ga bog'liqdir. Tasodifiy hodisalarni, odatda, lotin alfavitining bosh harflari $A, B, C, ...$ lar bilan belgilanadi. «Eng katta» hodisa U bo‘lib, u barcha elementar hodisalar to‘plamidan iboratdur. Agar tajriba natijasida $A(A \subset U)$ ga kirgan e elementar hodisalarning birortasi ro‘y bersa, A hodisa ro‘y berdi deyiladi. Agar shu elementar hodisalardan birortasi ham ro‘y bermasa, A hodisa ro‘y bermaydi, unda A hodisaga teskari hodisa (uni $\bar{A}$ orqali belgilaymiz) ro‘y bergan deymiz. A va $\bar{A}$ o‘zaro qarama – qarshi hodisalar deyiladi.

Tajriba natijasida har gal ro‘y beradigan hodisa muqarrar hodisa va birorta ham elementar hodisani o‘z ichiga olmagan hodisa mumkin bo‘lmagan hodisa deyiladi va V orqali belgilanadi. Ro‘y bermaydigan hodisa V –ni to‘plam ma’nosida $\emptyset$ bo‘sh to‘plam bilan, muqarar hodisa U –ni Ω universal to‘plam bilan belgilaymiz, ya’ni

$$U = \Omega, \quad V = \emptyset.$$

Misollar.

7. A hodisa 4-misoldagi tajribada gerb ikki marta tushishidan iborat bo‘lsin. Bu holda

$$A = \{e_2, e_7, e_8\}$$

bo‘ladi, ya’ni tajriba natijasida e_2 ro‘y bersa, yoki e_7 ro‘y bersa, yoki e_8 ro‘y bersa, A hodisa ro‘y berdi deyiladi. Agar e_2, e_7, e_8 hodisalardan birortasi ham ro‘y bermasa A hodisa ro‘y bermadi deymiz, u holda A –ga qarama-qarshi hodisa $\bar{A}$ ro‘y bergan bo‘ladi.

8. B hodisa tangani uch marta tashlashda hech bo‘lmaganda (kamida, aqalli) ikki marta gerb tushishidan iborat bo‘lsin, u holda

$$B = \{e_1, e_2, e_7, e_8\}.$$

9. C hodisa tangani uch marta tashlashda hech bo‘lmaganda bir marta gerb tushishidan iborat bo‘lsin, unda

$$C = \{e_1, e_2, e_3, e_5, e_6, e_7, e_8\}.$$

Bu misollardan ko‘rinib turibdiki, C hodisaning ro‘y berish imkoniyati B dan ham, A dan ham ko‘proq, B –niki esa A dan ko‘proq.

Bu misollarda A, B, C hodisalarga qarama-qarshi hodisalar quyidagilardan iborat bo‘ladi:

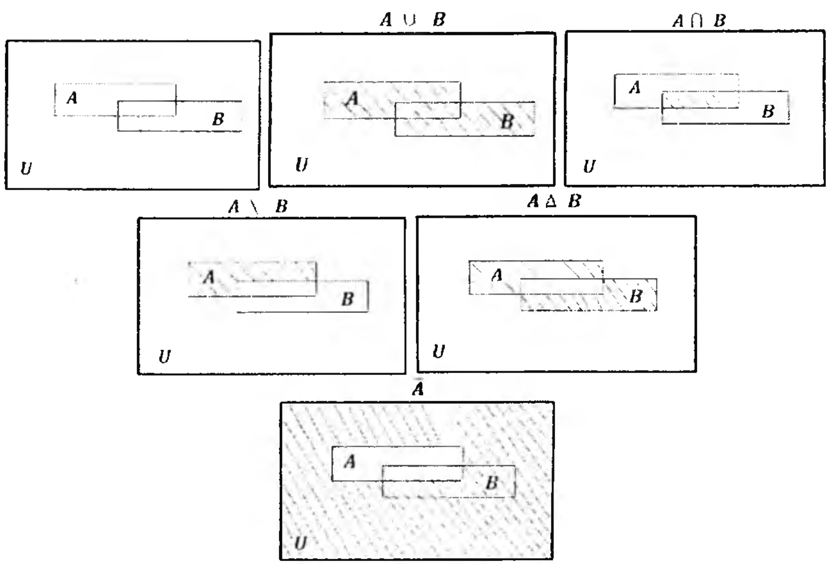
$$\bar{A} = \{e_1, e_3, e_4, e_5, e_6\}, \bar{B} = \{e_3, e_4, e_5, e_6\}, \quad \bar{C} = \{e_4\}.$$

Endi har qanday tasodifiy hodisalar orasidagi ayrim munosabatlarni ko‘rib chiqaylik.

1. Agar A hodisani tashkil etgan elementar hodisalar B hodisaga ham tegishli bo‘lsa A hodisa B hodisani ergashtiradi deyiladi va $A \subset B$ kabi belgilanadi. Ko‘rinib turibdiki, bu holda A ro‘y bersa, B albatta ro‘y beradi, lekin B ro‘y bersa, A ning ro‘y berishi shart emas.

2. A va B hodisalar bir xil elementar hodisalar to‘plamidan tashkil topgan bo‘lsa, ya’ni A –ni tashkil etgan barcha elementar hodisalar albatta B –ga ham tegishli va, aksincha, B –ni tashkil etgan barcha elementar hodisalar albatta A –ga ham tegishli bo‘lsa, A va B hodisalar teng deyiladi va $A = B$ kabi belgilanadi.

3. A va B hodisalarning yig‘indisi deb A yoki B ning, yoki ikkalasining ham ro‘y berishidan iborat C hodisaga aytamiz. A va B hodisalarning yig‘indisini $C = A \cup B$ (yoki $C = A + B$) orqali belgilanadi.



4. A va B hodisalarning bir vaqtda ro'y berishini ta'minlovchi barcha $\ell \in U$ lardan tashkil topgan C hodisa A va B hodisalarning ko'paytmasi deyiladi va $C = A \cap B$ (yoki $C = AB$) kabi belgilanadi.

5. A va B hodisalarning ayirmasi deb, A ro'y berib, B – ro'y bermasligidan iborat C hodisaga aytiladi. A va B hodisalarning ayirmasi $A \setminus B$ (yoki $A - B$) kabi belgilanadi.

6. Agar $A \cap B = \emptyset$ bo'lsa, A va B hodisalar birgalikda emas deyiladi.

7. A hodisaga qarama-qarshi $\bar{A}$ hodisa A –ga kirmagan barcha elementar hodisalar to'plamidan iboratdir, ya'ni $A \cap \bar{A} = \emptyset$ va $A \cup \bar{A} = U$.

8. Agar $A_1 \cup A_2 \cup \ldots \cup A_n = U$ bo'lsa, $A_1, A_2, \ldots, A_n$ hodisalar, hodisalarning to'liq gruppasini tashkil etadi deyiladi. Xususan, $A_i \cap A_j = \emptyset, i \neq j, i\, j = 1,2, \ldots, n$ va $A_1 + A_2 + \cdots + A_n = U$ bo'lsa, $A_1, \ldots, A_n$ hodisalar o'zaro birgalikda bo'lmagan hodisalarning to'liq gruppasini tashkil etadi beyiladi.

Tasodifiy hodisalarning ta'rifidan foydalanib, quyidagi munosabatlarning o'rinli ekanligini ko'rsatish mumkin:

a) $A \cup B = B \cup A$, $A \cap B = B \cap A$ kommutativlik qonuni;

b) $\left.\begin{matrix} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{matrix}\right\}$ assotsiativlik qonuni;

v) $A \cup A = A, A \cap A = A$ ayniylik qonuni;

g) $\left.\begin{matrix} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{matrix}\right\}$ distributlik qonuni;

7.2. Elementar hodisalar diskret fazosi. Ehtimollar fazosi.

Agar elementar hodisalar fazosi chekli yoki sanoqli miqdordagi elementar hodisalardan iborat bo'lsa, u elementar hodisalar diskret fazosi deyiladi. Agar Ω – da musbat qiymatli $P(e_i)$ funksiya berilgan bo'lsa va u

$$P(\Omega) = \sum_{\ell \in \Omega} P(e) = 1$$

shartni qanoatlantirsa, u holda Ω –da ehtimollar taqsimoti berilgan deyiladi.

Har qanday $A \in \Omega$ tasodifiy hodisalarning ehtimoli deb, ushbu

$$P(A) = \sum_{e \in A} P(e)$$

songa aytiladi

Misol. Bir jinsli kubni tanlashda i ochko ($i = \overline{1,6}$) tushish hodisasini e_i bilan belgilaylik. U holda elementar hodisalar fazosi $\Omega = \{e_i\}$, $i = \overline{1,6}$ bo'ladi. Kub bir jinsli bo'lgan uchun e_i hodisaning ro'y berish ehtimoli $P(e_i) = \frac{1}{6}$ deb hisoblash maqsadga muvofiqdir.

Bunday aniqlangan ehtimollik quyidagi xossalarga ega

1. $P(\emptyset) = 0,\ P(\Omega) = \sum_{e \in \Omega} P(e) = 1.$

2. $P(A \cup B) = \sum\limits_{e \in A \cup B} P(e) = \sum\limits_{e \in A} P(e) + \sum\limits_{e \in B} P(e) - \sum\limits_{e \in A \cap B} P(e) =$
$= P(A) + P(A) - P(AB)$

3. $P(\overline{A}) = \sum\limits_{e \in \overline{A}} P(e) =$

$$= \sum_{e \in \Omega \setminus A} P(e) = \sum_{e \in \Omega} P(e) - \sum_{e \in A} P(e) = 1 - P(A).$$

Ikkinchi xossadan xususiy holda $A \cap B = \emptyset$ bo'lganda $-P(A \cup B) = P(A) + +P(B)$ (qo'shish tenglamasi) kelib chiqadi. Buni ehtimollikning chekli additivlik xossasi deyiladi va u birgalikda ro'y bermaydigan ($A_i \cap A_j = \emptyset,\ i \neq j$) har qanday $\{A_i\}$ hodisalar uchun ham o'rinli, ya'ni

$$P\left\{\bigcup_{k=1}^{\infty} A_k\right\} = \sum_{k=1}^{\infty} P(A_k).$$

Bu xossa quyidagi

$$P\left\{\bigcup_{k=1}^{n} A_k\right\} = \sum_{k=1}^{\infty} P(A_k)$$

munosabatdan va $n \to \infty$ da

$$P\left(\bigcup_{k=n+1}^{\infty} A_k\right) \to 0$$

dan kelib chiqadi.

7.3. Kombinatorika elementlari. Ehtimollikning klassik, geometrik va statistik ta'riflari

1. Turli gruppadan bittadan tanlab olishlar kombinatsiyasi. $r-$ ta turli gruppa mavjud bo'lsin. Birinchi gruppa n_1 ta ($a_1^{(1)}$, $a_2^{(1)}, \ldots, a_{n_1}^{(1)}$) elementdan ikkinchi gruppa $n_2 -$ ta $a_1^{(2)}$, $a_2^{(2)}, \ldots, a_{n_2}^{(2)}$ elementdan va hokazo, r –gruppa $n_r -$ ta $(a_1^{(r)}, a_2^{(r)}, \ldots, a_{n_r}^{(r)})$ elementdan tuzilgan bo'lsin. Har bir gruppadan faqat bittadan element olib, nechta r elementli gruppa tuzish mumkin?

Shunday uchulda tuzish mumkin bo'lgan barcha gruppalar soni

$$N = n_1 n_2 \cdots n_r \tag{7.1}$$

dan iborat bo'ladi.

2. Qaytariladigan tanlashlar soni. Faraz qilaylik, $n -$ ta turli elementga ega bo'lgan, gruppa $(a_1, a_2, \ldots, a_n)$ berilgan bo'lsin. Bu gruppadan bittalab element olib uni fiksirlagach, o'rniga qaytarib qo'yamiz va bu protsessni yana takrorlaymiz. Bu usuldan r marta foydalanib, r elementli $(a_{i_1}, a_{i_2}, \ldots, a_{i_r})$ gruppani hosil qilamiz. Bu

3. Usulda tanlab olishlar soni $N = n^r$ ga teng.

4. O'rinlashtirishlar soni (qaytarilmaydigan tanlashlar). Kombinatorikada o'rinlashtirish deyilganda tartiblangan joylashtirishni tushiniladi. Agar $r -$ ta turli element $n -$ ta yacheykaga ko'pi bilan bittadan joylashtirilgan bo'lsa, u holda (7.1) formulaga ko'ra barcha o'rinlashtirishlar soni

$$N = A_n^r = n(n-1)(n-2)\cdots(n-r+1)$$

ga tengdir. Agar $n = r$ bo'lsa, o'rinlashtirishlar soni o'rinlashtirishlar soniga teng bo'ladi.

$$N = P_n = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

5. Gruppalashlar soni (kombinatsiyalar) $n-$ ta yacheykaga $r-$ ta turli element joylashgan bo'lsin. U holda gruppalashlar soni

$$N=C_n^r=\frac{n!}{r!\,(n-r)!}$$

Agar $n-$ ta elementli gruppa $k-$ ta gruppaga bo'lingan bo'lib, $i-$ gruppada $n_i(i=\overline{1,k})$ element $(n_1+n_2+\cdots+n_k=n)$ mavjud bo'lsa, u holda bu usulda gruppalashlar soni

$$N=\frac{n!}{n_1!\,n_2!\cdots n_k!} \qquad (7.2)$$

Ixtiyoriy $0\le m\le n$ lar uchun $C_n^m=C_n^{n-m}$ o'rinli.

Ehtimolning klassik ta'rifi. Agar Ω –chekli $n-$ ta elementar hodisadan tashkil topgan bo'lib, har bir elementar hodisa e_i teng ehtimoli $P(e_i)-$ ni $\frac{1}{n}-$ ga teng deb olinsa, e_i elementar hodisalar teng imkoniyatli deyiladi. Bunday fazoda har qanday A hodisaning ehtimolini quyidagicha aniqlash tabiiy:

$$P(A)=\sum_{e_i\in A}P(e_i)=\frac{A\ ga\ kirgan\ elementlar\ soni}{n}$$

Ehtimolning geometrik ta'rifi. Biror G soha berilgan bo'lib, bu soha G_1 sohani o'z ichiga olgan bo'lsin, $G_1\subset G$. G sohaga tavakaliga (tasodifan) tashlangan nuqtaning G_1 sohaga ham tushishi ehtimolini topish talab etilsin. Bu yerda Ω elementar hodisalar fazosi $G-$ ning barcha nuqtalaridan iborat va kontinuum quvvatga ega. Binobarin, bu holda klassik ta'rifdan foydalana olmaymiz. Tashlangan nuqta G –ga albatta tushsin va uning biror G_1 qismiga tushish ehtimoli shu G_1 qismining o'lchoviga (uzunligiga, yuziga, hajmiga) proporsional bo'lib, G_1 ning formulasiga va G_1 ni G –ning qayerida joylashganligiga bog'liq bo'lmasin. Bu shartlarda qaralayotgan hodisaning ehtimoli

$$P=\frac{mesG_1}{mesG}$$

formula yordamida aniqlanadi. Bu formula yordamida aniqlangan P funksiya ehtimolning barcha xossalarini qanoatlantirishini ko'rish qiyin emas.

7.1-misol. L uzunlikka ega bo'lgan kesmaga tavakaliga nuqta tashlansin. Tashlangan nuqtani kesmaning o'rtasidan ko'pi bilan ℓ masofada yotish hodisasining ehtimolini toping.

Yechilishi. Yuqoridagi shartlarni qanoatlantiradigan nuqtalar to'plami $-\ell \le x \le \ell$ dan iborat (umumiylikka halal keltirmasdan, kesmaning o'rtasini sanoq boshi deb qabul qilamiz). Bu kesmaning uzunligi 2ℓ –ga teng. Demak, qaralayotgan hodisaning ehtimoli

$$P = \frac{2\ell}{L}.$$

Ehtimolning statistik ta'rifi. Shartlar kompleksi o'zgarmas bo'lganda bir A hodisaning ro'y berishi yoki ro'y bermasligi ustida uzoq kuzatishlar o'tkazilganda, uning ro'y berishi yoki ro'y bermasligi ma'lum turg'unlik (barqarorlik) xarakteriga ega bo'ladi. A hodisaning n –ta tajribada ro'y berishlar sonini v deb olsak, u holda juda ko'p sondagi kuzatishlar seriyasi uchun $\frac{\text{v}}{n}$ nisbat deyarli o'zgarmas miqdor bo'lib qolaveradi. $\frac{\text{v}}{n}$ nisbat A hodisaning ro'y berish chastotasi deyiladi. Chastotaning turg'unlik xususiyati birinchi bor, demografik harakterdagi hodisalarda ochilgan. Bizning eramizdan 2238 yil burun qadimiy Xitoyda o'g'il bola tug'ilgan bolalar soniga nisbati deyarli $\frac{1}{2}$ ga tengligi hisoblangan.

Agar tajribalar soni yetarlicha ko'p bo'lsa, u holda shu tajribalarda qaralayotgan A hodisaning ro'y berish chastotasi biror o'zgarmas $p \in [0,1]$ son atrofida turg'un ravishda tebransa shu p sonni A hodisaning ro'y berish ehtimoli deb qabul qilamiz. Bunday usulda aniqlangan ehtimol hodisaning statistik ehtimoli deyiladi. Mizes hodisaning ehtimolini ushbu munosabat yordamida kiritgan:

$$p = \lim_{n\to\infty} \frac{n_r}{n}$$

7.4. Ehtimollar nazariyasini aksiomatik asosda qurish. Ehtimolning xossalari

Matematik S.N.Bernshteyn 1917-yilda birinchi bo'lib ehtimollar nazariyasini aksiomatik asosda qurishga harakat qildi. Akademik A.N.Kolmogorov tomonidan kiritilgan ehtimollar nazariyasining aksiomalari funksiyalarning metrik nazariyasiga asoslangandir. Ω –biror to'plam, $\mathcal{F}$ –uning qism to'plamlarining biror sistemasi bo'lsin. Agar

1. $\Omega\epsilon\mathcal{F}$; – qism to'plamlarining sistemasi;

2. $A_i\epsilon\mathcal{T}$; $i = \overline{1,n}$ dan $\bigcup_{i=1}^{n} A_i\epsilon\mathcal{F}$ kelib chiqsa ;

3. $A\epsilon\mathcal{F}$ dan $\bar{A}\epsilon\mathcal{F}$ kelib chiqsa, $\mathcal{F}$ sistema algebra tashkil etadi deyiladi.

Agar ikkinchi shart o'rniga $A_i\epsilon\mathcal{F}, i = 1,2,\dots,n,\dots$ dan

$$\bigcup_{i=1}^{\infty} A_i\epsilon\mathcal{F}$$

kelib chiqsin degan shartning bajarilishi talab qilinsa σ –algebra tashkil etadi deyiladi. Odatda, Ω –elementar hodisalar fazosi, $\Omega = \{\ell\}$ – fazoning elementlari, nuqtalari elementar hodisalar, σ – algebrasi deyiladi.

7.1-ta'rif. Agar Ω to'plam va bu to'plamning qism to'plamlaridan iborat $\mathcal{F}, \delta$ – algebra berilgan bo'lsa, u holda o'lchovli fazo berilgan deyiladi va uni $< \Omega, \mathcal{F} >$ kabi belgilanadi.

σ – algebraning ta'rifidan $\bar{\Omega} = \emptyset\epsilon\mathcal{F}$ ekani kelib chiqadi.

Ω –muqarrar hodisa, $\emptyset$ esa mumkin bo'lmagan hodisa deyiladi.

Endi A.N. Kolmogorov aksiomalarini keltiramiz.

1-aksioma. Ixtiyoriy $A\epsilon\mathcal{F}$ hodisaga uning ehtimoli deb ataluvchi $P(A) \geq 0$ son mos qo'yilsin.

2-aksioma. $P(\Omega) = 1$.

3-aksioma. Qo'shish aksiomasi. Agar $\{A_n\}$ hodisalar chekli ketma-ketligi juft-jufti bilan birgalikda bo'lmasa, u holda

$$P\left(\bigcup_{n=1}^{\infty} \mathrm{A}_n\right) = \sum_{n=1}^{\infty} P\left(\mathrm{A}_n\right).$$

Ehtimollar nazariyasining ko'pgina masalalarini hal qilishda 3 aksioma o'rniga undan kuchliroq bo'lgan aksiomaga zaruriyat tug'iladi.

3^I – aksiomani Qushishning kengaytirilgan aksiomasi. Agar $\{A_n\}$ hodisalar ketma-ketligi juft-jufti bilan birgalikda bo'lmasa, u holda unga ekvivalent quyidagi uzluksizlik aksiomasi bilan almashtirish mumkin.

3^{II} aksioma. Uzluksizlik aksiomasi. Agar $\mathcal{F}$ σ –algebraga tegishli bo'lgan

$$B_1 \supset B_2 \supset B_3 \supset \cdots \supset B_n \supset \cdots$$

bajarilsa va

$$\bigcap_{k=1}^{\infty} B_k = \emptyset$$

o'rinli bo'lsa, u holda

$$\lim_{n\to\infty} P\left(\bigcap_{k=1}^{n} B_k\right) = 0$$

bo'ladi.

$<\Omega, \mathcal{F}, P>$ uchlik, ehtimollik fazosi deyiladi. Shunday qilib, ehtimollik fazosi $<\Omega, \mathcal{F}, P>$ o'lchovli fazo va $\mathcal{F}$ da berilgan manfiy bo'lmagan, normalashtirilgan sanoqli additiv P o'lchovdan iborat bo'lar ekan P o'lchov ehtimollik o'lchovli deyiladi.

Ehtimolning xossalari.

1. $P(V) = 0$. Bu natija $V \cup \Omega = \Omega$ tenglikdan va 2, 3-aksiomalardan kelib chiqadi

$$P(V \cup \Omega) = \mathrm{P(V)} + \mathrm{P}(\Omega) = \mathrm{P}(\Omega) - \mathrm{P(V)} = 0.$$

2. $P(A) = 1 - P(A)$. Bu xossa $A \cup \overline{\mathrm{A}} = \Omega$ va $A \cap \overline{\mathrm{A}} = V$ dan kelib chiqadi.

$$P\left(A \bigcup \overline{\mathrm{A}}\right) = P(\mathrm{A}) +$$
$$+P(\overline{\mathrm{A}}) = P(\Omega) = 1 \quad P(A) = 1 - P(\overline{\mathrm{A}})$$

3. Agar $A \subset B$ bo'lsa, u holda $P(A) \leq P(B)$. $B = A \cup \bar{A} B$ dan bu munosibat kelib chiqadi.

4. $0 \leq P(A) \leq 1$. Bu xossaning isboti 3-xossadan va 1, 2-aksiomalardan kelib chiqadi.

5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, chunki $A \cup B = A \cup (B \setminus (A \cap B))$, $B = AB + (B \setminus AB)$.

6. $P(A \cup B) \leq P(A) + P(B)$. Bu xossaning isboti 5-xossadan kelib chiqadi.

7. Faraz qilaylik, $A_1, A_2, \cdots, A_n$ hodisalar berilgan bo'lsin, u holda

$$P\left(\bigcup_{i=1}^{n} A_i\right) =$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) + \cdots$$
$$+ (-1)^{n-1} P(A_1 A_2 \cdots A_n).$$

Bu munosibat Bul formulasi deyiladi.

Isbotni matematik induksiya metodiga asoslanib qilishni o'qituvchiga havola qilamiz.

8. $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$.

Haqiqatan ham, $B_n = \Omega \setminus \bigcup_{k=1}^{n-1} A_n$ deb belgilasak, u holda

$$A_j B_j \cap A_i B_i = V \quad (i \neq j),$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n A_n$$

tenglik o'rinli. Demak,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(B_n A_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

7.5. Shartli ehtimollar. Hodisalarning bog‘ liqsizligi

Hodisaning ehtimolini aniqlash asosida S shartlar kompleksi yotishini aytgan edik. Agar $P(A)$ ehtimolni hisoblashda S shartlar kompleksidan boshqa hech qanday shartlar talab qilinmasa, bunday ehtimol shartsiz ehtimol deyiladi. Ko‘p hollarda A hodisaning ehtimolini biror B hodisa ($P(B) > 0$ deb faraz qilinadi) ro‘y bergandan so‘ng hisoblashga to‘g‘ri keladi. Bunday ehtimol shartli ehtimol deyiladi va $P(A/B)$ kabi belgilanadi.

7.2-misol. Ikkita shashqoltosh tashlanayotgan bo‘lsin. Elementar hodisalar fazosi

$$\Omega = \{(1,1);(1,2),\dots,(1,6),(2,1),\dots,(2,6),\dots,(6,6)\}$$

bo‘ladi. Shashqoltosh tashlanganda uning yuqori yo‘qlaridagi raqamlar yig‘indisi 8-ga teng bo‘lish hodisasini A orqali, raqamlar yig‘indisining juft son bo‘lish hodisasini B orqali belgilaylik. U holda $\mathrm{P(A)} = \frac{5}{36}$, $P(B) = \frac{18}{36}$ bo‘ladi.

Endi B hodisa ro‘y berganda A hodisaning ehtimolini topaylik:

$$\mathrm{P}(A/B) = \frac{5}{18} = \frac{5}{36}\Big/\frac{18}{36} = \frac{P(A)}{P(B)}.$$

Lekin $\mathrm{A} \subset \mathrm{B}$ ekanidan $\mathrm{A}B = \mathrm{A}$ kelib chiqadi. Shuning uchun

$$\mathrm{P}(A/B) = \frac{P(AB)}{P(B)}$$

7.3-misol. Elementar hodisalar fazosi Ω $e_1, e_2, \dots, e_n$ lardan tuzilgan bo‘lsin. Shu elementar hodisalarning k –tasi A hodisaga, m –tasi B hodisaga va r –tasi AB hodisaga qulaylik tug‘dirsin. Klassik ta’rifga ko‘ra quyidagilar o‘rinli bo‘ladi:

$$\mathrm{P}(A) = \frac{k}{n}, \quad P(B) = \frac{m}{n}, \quad P(AB) = \frac{r}{n}, \quad \mathrm{P}(A/B) = \frac{r}{m} = \frac{r}{n}\Big/\frac{m}{n} = \frac{P(AB)}{P(B)}.$$

Umumiy holda shartli ehtimol ta’rifi quyidagicha kiritiladi.

7.2-Ta’rif. $(\Omega, \mathcal{F}, P)$ ehtimollik fazosi berilgan bo‘lib, $A, B \in$ va $P(B) > 0$ bo‘lsin. U holda A hodisaning B shartdagi ehtimoli deb, ushbu formula bilan aniqlanadigan ehtimolga aytiladi:

$$\mathrm{P}(\mathrm{A}/\mathrm{B}) = \frac{P(AB)}{P(B)}.$$

Shartli ehtimolning xossalari.

1. $\mathrm{P}(\mathrm{A}/\mathrm{B}) \geq 0$

2. $\mathrm{P}(\Omega/\mathrm{B}) \geq 1$

Isbot.

$$\mathrm{P}(\Omega/B) = \frac{P(\Omega B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3. $\mathrm{A}_1, \mathrm{A}_2 \epsilon \Omega$, $\mathrm{A}_1 \cap \mathrm{A}_2 = \mathrm{V}$ bo'lsa, u holda $\mathrm{P}(\mathrm{A}_1 + \mathrm{A}_2/B) =$

$$P(\mathrm{A}_1/B) + P(\mathrm{A}_2/B)$$

tenglik o'rinli bo'ladi.

Isbot.

$$P(A_1 + A_2/B) = \frac{P\big((A_1 + A_2)B\big)}{P(B)} = \frac{P(A_1B) + P(A_2B)}{P(B)} =$$

$$= \frac{P(A_1B)}{P(B)} + \frac{P(A_2B)}{P(B)} = P(A_1/B) + P(A_2/B),$$

chunki $\mathrm{A}_1 \cap \mathrm{A}_2 = V$ munosabatdan $\mathrm{A}_1 B \cap \mathrm{A}_2 B = V$ munosabat kelib chiqadi.

4. Agar A va $\bar{\mathrm{A}}$ hodisalar o'zaro qarama-qarshi hodisalar bo'lsa, u holda

$$\mathrm{P}(A/B) = 1 - \mathrm{P}(\bar{\mathrm{A}}/\mathrm{B})$$

tenglik o'rinli bo'ladi.

Isbot. Ta'rifga asosan

$$\mathrm{P}(A + \bar{A}/B) = \frac{\mathrm{P}((\mathrm{A} + \bar{\mathrm{A}})\mathrm{B})}{P(B)} = \frac{P(AB) + P(\bar{A}B)}{P(B)}$$

$$= P(A/B) + P(\bar{A}/B).$$

Shartga asosan $\mathrm{A} + \bar{\mathrm{A}} = \Omega$ bo'lgani uchun 2 xossaga asosan $P(\mathrm{A} + \bar{\mathrm{A}}/B) = 1$ bo'ladi. Hosil bo'lgan tengliklardan xossaning isboti kelib chiqadi.

Agar $\mathrm{P}(A) > 0$ bo'lsa, B hodisaning A shartdagi ehtimoli

$$P(B/A) = \frac{P(AB)}{P(A)}$$

formula yordamida topiladi.

Shartli ehtimolni topish formulasidan hodisalarning ko'paytmasi ehtimolini topish uchun ushbu formulani keltirib chiqarish mumkin:

$$P(AB) = P(B) \cdot P(A/B) = P(A)P(B/A) \qquad (7.3)$$

7.3-Ta'rif. Agar $P(A/B) = \mathrm{P}(A)$ tenglik bajarilsa, A hodisa B hodisa bog'liq emas deyiladi.

Agar A hodisa B hodisaga bog'liq bo'lmasa, u holda $\bar{A}$ hodisa B hodisada, B hodisa A hodisaga, $\bar{B}$ hodisa A hodisaga, $\bar{A}$ hodisa $\bar{B}$ hodisaga bog'liq bo'lmaydi. Bulardan A hodisa B hodisaga bog'liq bo'lmaganda B hodisa ham A hodisaga bog'liq bo'lmasligini isbotlaylik, boshqalari ham shu kabi isbotlanadi. Haqiqatan ham, A hodisa B –ga bog'liq bo'lmagani uchun 7.2-ta'rifga asosan $P(A/B) = \mathrm{P}(A)$ bo'ladi.

Agar (7.3) munosabatda bu tenglikdan foydalansak,

$$P(B/A) = \mathrm{P}(B)$$

munosabat kelib chiqadi. Shuningdek, agar A hodisa B_1 va B_2 hodisalarga bog'liq bo'lmasa hamda $B_1 \cap B_2 = V$ bo'lsa, u holda A hodisa $B_1 \cup B_2$ hodisaga bog'liq bo'lmaydi.

Isbot.

$$P(A(B_1 \cup B_2)) = P(AB_1 \cup AB_2) = P(A)P(B_1) + P(A) \cdot P(B_2)$$
$$= P(A) \cdot P(B_1 \cup B_2).$$

7.4-ta'rif. $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_n$ hodisalar berilgan bo'lsin. $1 \le i_1 \le i_2 \le \dots \le i_s \le n$ sonlarni olamiz. Agar

$$P\left(\bigcap_{k=1}^{s} A_{ik}\right) = \bigcap_{k=1}^{s} P(A_{ik}) \qquad (1 \le S \le n)$$

tenglik o'rinli bo'lsa, u holda $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_n$ hodisalar birgalikda bog'lik emas deyiladi. Shuni eslatib o'tish kerakki, hodisalarning juft-jufti bilan bog'liqmasligidan birgalikda bog'likmasligi kelib chiqmaydi.

7.6. To'la ehtimol formulasi. Beyes formulasi.

Faraz qilaylik, B hodisa n –ta juft-jufti bilan birgalikda bo'lmagan $\mathrm{A}_1, \mathrm{A}_2, \dots, \mathrm{A}_n$ hodisalarning (gipotezalarning) bittasi va faqat bittasi bilangina ro‘y berishi mumkin bo‘lsin, boshqacha qilib aytganda,

$$B = \bigcup_{i=1}^{n} BA_i,$$

bu yerda $(BA_i)\cap(BA_j) = V, i \neq j,$ u holda qo‘shish teoremasiga asosan

$$P(B) = \sum_{i=1}^{n} P(BA_i)$$

Agar $P(BA_i) = P(B/A_i) \cdot P(A_i)$ ligini e’tiborga olsak, u holda

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B/A_i).$$

Bu tenglik to‘la ehtimol formulasi deyiladi.

7.4-misol. Sostavi quyidagicha bo‘lgan beshta yashik bor. Ikkita (A_1 sostavli) yashikda 2-tadan oq, 1-tadan qora shar, bitta (A_2 sostavli) yashikda 10-ta qora shar, ikkita (A_3 sostavli) yashikda 3-tadan oq, 1-tadan qora shar bor. Tavakkaliga tanlangan yashikdan tavakkaliga olingan sharning oq shar (B hodisa) bo‘lishi ehtimolini toping.

Yechilishi. Yashikdan olingan shar yo A_1, yo A_2 yoki A_3 sostavli yashikdan olingan bo‘lish mumkin, u holda $B = A_1B\cup A_2B\cup A_3B$. To‘la ehtimol formulasiga muvofiq:

$$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + P(A_3)P(B/A_3)$$

Klassik ta’rifiga va shartli ehtimol ta’rifiga asosan quyidagilarni topamiz:

$$P(A_1) = \frac{2}{5}, \qquad P(A_2) = \frac{1}{5}, \qquad P(A_3) = \frac{2}{5},$$

$$P(B/A_1) = \frac{2}{3}, \qquad P(B/A_2) = 0, \qquad P(B/A_3) = \frac{3}{4},$$

Topilganlarni yuqoridagi formulaga quysak,

$$P(B) = \frac{2}{5} \cdot \frac{2}{3} + 0 + \frac{2}{5} \cdot \frac{3}{4} = \frac{17}{30}$$

hosil bo‘ladi.

Endi to‘la ehtimol formulasidan foydalanib, Beyes formulasini keltirib chiqaramiz. A_i va B hodisalarning qo‘paytmasi uchun ushbu

$$P(A_i/B) = P(B)\,P(A_i/B) = P(A_i)\,P(B/A_i)$$

formulasining o‘rinliligidan

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{P(B)}$$

munosabatga ega bo'lamiz, endi to'la ehtimollar formulasini qo'llasak, ushbu Beyes formulasini hosil qilamiz:

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{\sum_{j=1}^{n} P(A_j)P(B/A_j)}.$$

Nazorat savollari.

1. Hodisa ta'rifi.
2. Hodisalar ustida amallarga misollar keltiring.
3. Kambinatorika va uning elementlari ta'rifi.
4. Ehtimolning klassik ta'rifi.
5. Ehtimolning geometrik ta'rifi.
6. Shartli ehtimolni hisoblash formulasini yozib bering.
7. To'la ehtimol va Bayes formulasi.

VIII-BOB. SINOVLARNI TAKRORLASH.

8.1. Bog‘liq bo‘lmagan tajribalar ketma-ketligi. Bernulli formulasi.

Biror hodisani kuzatish uchun bir nechta tajriba o‘tkazilsa, u tajribalar bir-biriga bog‘liq bo‘lishi yoki bog‘liq bo‘lmasligi mumkin.

Faraz qilaylik, bog‘liq bo‘lmagan n –ta tajriba o‘tkazilayotgan bo‘lib, har bir tajribada kuzatiladigan A hodisaning ro‘y berish ehtimoli P va ro‘y bermaslik ehtimoli $q = 1 - p$ bo‘lsin. Kuzatilayotgan A hodisaning n –ta tajribada m –marta ro‘y berish ehtimoli $P_n(m)$ ni toppish kerak bo‘lsin. n –marta tajriba o‘tkazilganda kuzatilayotgan A hodisaning m –marta ro‘y berib, $n - m$ marta ro‘y bermaslik imkoniyatlarining soni C_n^m ga teng ekanini ko‘rish qiyin emas.

n –ta ketma-ket o‘tkazilgan tajribani bitta murakkab tajriba natijasida ro‘y beradigan hodisaning ko‘rinishi $A_1A_2\cdots A_n$ bo‘lib, $A_i(i = 1,\overline{n})$ A –ga yoki $\bar{A}$ –ga teng bo‘ladi. Bunday hodisalar soni 2^n –ga teng. Haqiqatan ham, $A_1A_2\cdots A_n$ hodisalar ichida:

1) $A_i = A(i = \overline{1,n})$ shartni qanoatlantiruvchi hodisalar bitta;

2) bittasi $\bar{A}$, qolganlari A dan iborat bo‘lgan hodisalar n –ta, chunki $\bar{A}$ –ni n –ta o‘ringa bir martadan qo‘yish bilan n –ta turli hodisa hosil qilish mumkin, $n - m$ tasi $\bar{A}$, qolganlari A dan iborat bo‘lmagan hodisalar soni n –ta o‘ringa $n - m$ –ta $\bar{A}$ –larni joylashtirishlar soni $C_n^{n-m} = C_n^m$ –ga teng va hokazo. Demak, biz ko‘rayotgan murakkab tajriba natijasida ro‘y berishi mumkin bo‘lgan barcha elementar hodisalar soni.

$$C_n^0 + C_n^1 + \cdots C_n^n = 2^n$$

ga teng ekan. Agar n –ta tajribada kuzatilayotgan A hodisaning m –marta ro‘y berish hodisani B desak.

$$B = (A \cdot A \cdots A \cdot \bar{A} \cdot \bar{A} \cdots \bar{A})\cup(A \cdot \bar{A} \cdot A \cdots A \cdot \bar{A}. A \cdots \bar{A})\cup \cdots \cup(\bar{A} \cdot \bar{A} \cdots \bar{A} \cdot A \cdot A \cdots A) \qquad (8.1)$$

bo'lib, u C_n^m ta qo'shiluvchidan iborat bo'ladi. Tajribalar ketma-ketligi bir-biriga bog'liq bo'lmagani uchun, masalan

$$P\left(\underbrace{A\,A \cdots A}_{m}\ \underbrace{\bar{A}\,\bar{A} \cdots \bar{A}}_{n-m}\right) = P(A)P(A) \cdots P(A) \cdot P(\bar{A}) \cdot P(\bar{A}) \cdots P(\bar{A}) = p^m q^{n-m}$$

bo'ladi. Bu yerda $A\,A \cdots A\bar{A} \cdots \bar{A}$ yozuv m –ta tajribada A –ning, $n-m$ tajribada esa $\bar{A}$ ning ro'y berganligini bildiradi. Bundan tashqari, (8.1) tenglikning o'ng tomonidagi C_n^m ta hodisaning ihtiyoriy ikkitasi bir vaqtda ro'y bermasligidan

$$P(B) = C_n^m p^m q^{n-m}$$

kelib chiqadi. Agar A hodisaning n –ta tajribada m –marta ro'y berish ehtimolini $P_n(m)$ deb belgilasak,

$$P_m(m) = C_n^m p^m q^{n-m} \qquad (8.2)$$

hosil bo'ladi. (8.2)-ni Bernulli formulasi deyiladi. $P_n(m)$ ehtimollar uchun

$$\sum_{m=0}^{n} P_n(m) = 1$$

munosabat o'rinli b o'lishini k o'rish qiyin emas. Haqiqatan ham

$$\sum_{m=0}^{n} C_n^m p^m q^{n-m} = (p+q)^m = 1. \qquad (8.2)$$

ifoda $(px+q)^m$ binom yoyilmasining x^m qatnashgan hadining koeffitsiyenti bo'lgani uchun $P_n(m)$ larni ehtimolning binomial taqsimot qonuni deyiladi.

Fiksirlangan n –da $P_n(m)$ ehtimol m –ning funksiyasi ekani ravshan. Bu funksiyani tekshiraylik. Quyidagi nisbatni ko'ramiz:

$$\frac{P_n(m+1)}{P_n(m)} = \frac{n-m}{m+1} \cdot \frac{p}{q}; \qquad (8.3)$$

a) agar $(n-m)p > (m+1)q$, ya'ni pn$-q > m$, bo'lsa, (8.3) tenglikdan $P_n(m+1) > P_n(m)$ natijaga ega bo'linadi;

b) agar pn$-q = m$ bo'lsa, (8.3) tenglikdan $P_n(m+1) < P_n(m)$ natijaga ega bo'linadi.

8.1-misol. Simmetrik tanga 8 marta tashlanganda 3 marta gerb tushish ehtimolini toping.

Yechilishi. $n = 8, m = 3, p = \frac{1}{2}, q = \frac{1}{2}$. U holda

$$P_n(m) = P_8(3) = C_8^3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 = \frac{7}{32}.$$

Demak,

$$P_8(3) = \frac{7}{32}.$$

8.2-misol. Har bir otilgan o'qning nishonga tegish ehtimoli $P = \frac{2}{3}$. Otilgan 20 ta o'qdan 4-tasining nishonga tegish ehtimolini toping.

Yechilishi. $p = \frac{2}{3}, q = \frac{1}{3}, n = 20, m = 4;$

$$P_{20}(4) = C_{20}^4 \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^{16}.$$

Yuqoridagi keltirilgan misollardan ko'rinadiki, ba'zi hollarda $P_n(m) = C_n^m p^m q^{n-m}$ ehtimolni hisoblash katta qiyinchiliklarga olib kelar ekan. Bunday hollarda hisoblashni osonlashtiruvchi formulalarga ehtiyoj tug'iladi. Bunday formulalarni navbatdagi paragraflarda keltirilib chiqariladi.

8.2. Muavr-Laplasning lokal teoremasi

Agar m va n katta sonlar bo'lsa, u holda $P_n(m)$ ehtimolni Bernulli formulasidan foydalanib hisoblash ma'lum qiyinchiliklarga olib keladi. Shu sababli $n \to \infty$ da $P_n(m)$ ehtimol uchun asimptotik formula topish masalasi tug'iladi.

8.1-teorema. Agar n –ta bog'lik bo'lmagan tajribaning har birida biror A hodisaning ro'y berish ehtimoli $P(0 < p < 1)$ bo'lsa, u holda m –ning ushbu

$$\frac{|m - np|}{\sqrt{npq}} < c \quad (c - \text{ o'zgarmas son})$$

shartni qanoatlantiruvchi barcha qiymatlari uchun tekis ravishda

$$P_n(m) = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{1}{2}\left(\frac{m-np}{\sqrt{npq}}\right)^2} \left(1 + 0\left(\frac{1}{\sqrt{n}}\right)\right)$$

tenglik bajariladi.

Bu teoremani Muavr 1730-yilda Bernulli sxemasining $p = q = \frac{1}{2}$ bo'lgan xususiy holi uchun, so'ngra Laplas ixtiyoriy $p\epsilon$] 0,1 [uchun isbotlagan.

Isbot. Teoremani analiz kursidan ma'lum bo'lgan ushbu

$$n! = \sqrt{2\pi n} \cdot n^n e^{-n} e^{\theta_n}, |\theta_n| \leq \frac{1}{12n}.$$

Stirling formulasidan foydalanib isbotlaymiz. Endi

$$x = x_{m,n,p} = \frac{m - np}{\sqrt{npq}}$$

belgilashni kiritsak, u holda

$$m = np + x\sqrt{npq} = np\left(1 + x\sqrt{\frac{q}{np}}\right) \qquad (8.4)$$

va

$$n - m = nq - x\sqrt{npq} = nq\left(1 - x\sqrt{\frac{p}{nq}}\right) \qquad (8.5)$$

tengliklar o'rinli bo'ladi (8.4) va (8.5) dan ko'rinadiki, $n \to \infty$ da va $|x| \leq c$ shart bajarilganda $m, n - m$ cheksizlikka intiladi. Shu sababli $(n - m)!$ va $m!$ sonlar uchun Stirling formulasini qo'llashimiz mumkin. U holda

$$P_n(m) = \frac{n!}{m!\,(n - m)!} p^m q^{n-m}$$

$$= \sqrt{\frac{n}{2\pi m(n - m)}} \cdot \frac{n^n p^m q^{n-m}}{m^m (n - m)^{n-m}} \cdot e^{\theta_{n,m}}$$

tenglikni hosil qilamiz, bu yerda

$$|\theta_{n,m}| \leq \frac{1}{12}\left(\frac{1}{n} + \frac{1}{m} + \frac{1}{n - m}\right) \qquad (8.6)$$

(8.4), (8.5), (8.6) munosibatlardan ushbu tengsizlik o'rinli bo'ladi:

$$|\theta_{n,m}| \leq \frac{1}{12n}\left(1 + \frac{1}{p + x\sqrt{\frac{pq}{n}}} + \frac{1}{q - x\sqrt{\frac{pq}{n}}}\right) \qquad (8.7)$$

Bundan ko‘rinadiki, $|x| \leq c$ bo‘lgani uchun $n \to \infty$ da $e^{\theta_{n,m}} \to 1$. Natijada (8.7) ga asosan katta n –lar uchun

$$e^{\theta_{n,m}} = 1 + 0\left(\frac{1}{n}\right) \qquad (8.8)$$

ifodani hosil qilamiz. Teorema shartiga asosan $x\sqrt{\frac{q}{np}}$ va $x\sqrt{\frac{p}{nq}}$ miqdorlar n –ning yetarlicha katta qiymatlarida istalgancha kichik bo‘ladi. Shu sababli

$$\ell n\left(1 + x\sqrt{\frac{q}{np}}\right) \quad va \quad \ell n\left(1 - x\sqrt{\frac{p}{np}}\right)$$

ifodalarni darajali qatorga yoyib,

$$\ell n\left(1 + x\sqrt{\frac{q}{np}}\right) = x\sqrt{\frac{q}{np}} - \frac{1}{2}\frac{qx^2}{np} + 0\left(\frac{1}{n^{3/2}}\right),$$

$$\ell n\left(1 - x\sqrt{\frac{p}{nq}}\right) = -x\sqrt{\frac{p}{nq}} - \frac{1}{2}\frac{px^2}{nq} + 0\left(\frac{1}{n^{3/2}}\right)$$

tengliklarni hosil qilamiz. Bu tengliklarga asosan

$$ln\frac{n^n p^m q^{n-m}}{m^m (n-m)^{n-m}} = ln\left(\frac{np}{m}\right)^m + ln\left(\frac{nq}{n-m}\right)^{n-m} = -m\, ln\frac{m}{np} \cdot$$

$$(n-m)ln\frac{n-m}{nq} =$$

$$= -(np + x\sqrt{npq}) \cdot \ell n\left(1 + x\sqrt{\frac{q}{np}}\right)$$

$$-(nq - x\sqrt{npq})\,\ell n\left(1 - x\sqrt{\frac{p}{nq}}\right) =$$

$$= -(np + x\sqrt{npq})\left[x\sqrt{\frac{q}{np}} - -\frac{1}{2}\frac{qx^2}{np} + 0\left(\frac{1}{n^{3/2}}\right)\right] -$$

$$-(np - x\sqrt{npq})\left[-x\sqrt{\frac{p}{nq}} - \frac{1}{2}\frac{px^2}{nq} + 0\left(\frac{1}{n^{3/2}}\right)\right] =$$

$$= -\frac{x^2}{2} + 0\left(\frac{1}{\sqrt{n}}\right). \qquad (8.9)$$

Natijada (8.9) dan $e^{\theta\left(\frac{1}{\sqrt{n}}\right)} = 1 + 0\left(\frac{1}{\sqrt{n}}\right)$ ni e'tiborga olgan holda

$$\frac{n^n p^m q^{n-m}}{m^m (n-m)^{n-m}} = e^{-\frac{x^2}{2}} + 0\left(\frac{1}{\sqrt{n}}\right) \qquad (8.10)$$

ni hosil qilamiz. Bevosita ishonch hosil qilish mumkinki,

$$\frac{1}{\sqrt{1 + 0\left(\frac{1}{\sqrt{n}}\right)}} = 1 + 0\left(\frac{1}{\sqrt{n}}\right).$$

Shuning uchun (8.4), (8.5) tengliklarga asosan

$$\sqrt{\frac{n}{2\pi m(n-m)}} = \sqrt{\frac{n}{2\pi\, npq\left(1 + 0\left(\frac{1}{\sqrt{n}}\right)\right)}} = \frac{1 + 0\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{2\pi\, npq}}$$

Shunday qilib, yetarlicha katta n – lar uchun (8.7), (8.8), (8.9), (8.10) ifodalardan teoremaning o'rinligiga ishonch hosil qilamiz. $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ funksiyaning x argument musbat qiymatlariga mos qiymatlaridan tuzilgan jadvallar mavjud, $\varphi(x)$ funksiyaning juftligidan bu jadvallardan argumentning manfiy qiymatlari uchun ham foydalaniladi.

8.3-misol. Har bir tajribada A hodisaning ro'y berish ehtimoli 0,2-ga teng bo'lsa, 400 ta tajribada bu hodisaning rosa 80 marta ro'y berish ehtimolini toping.

Yechilishi. Shartga ko'ra $n = 400, m = 80, p = 0{,}2, q = 0{,}8$. U holda

$$P_{400}(80) = \frac{\varphi(x)}{\sqrt{npq}} = \frac{\varphi(x)}{\sqrt{400 \cdot 0{,}2 \cdot 0{,}8}} = \frac{1}{8}\varphi(x),$$

bunda

$$x = \frac{m - np}{\sqrt{npq}} = \frac{80 - 400 \cdot 0{,}2}{8} = 0.$$

Jadvaldan $\varphi(0) = 0{,}3989$ ekanligini e'tiborga olsak,

$$P_{400}(80) = \frac{0{,}3989}{8} = 0{,}04986.$$

Asl qiymati 0,048813272.

8.3. Muavr – Laplasning integral teoremasi

Faraz qilaylik, n ta tajriba o'tkazilayotgan bo'lib, ularning har birida A hodisaning ro'y berish ehtimoli o'zgarmas va p $(0 < p < 1)$ ga teng bo'lsin. n –ta tajribada A hodisaning kamida k_1 marta va ko'pi bilan k_2 marta ro'y berish ehtimolini $P_n(k_1, k_2)$ ni qanday hisoblash mumkin. Bernulli sxemasiga ko'ra bu $P_n(k_1, k_2)$ ehtimol

$$P_n(k_1, k_2) = P_n(k_1 \le m \le k_2) = \sum_{m=k_1}^{k_2} C_n^m p^m q^{n-m} \qquad (8.11)$$

ga teng. Agar n, m, k_1, k_2 lar yetarlicha katta bo'lsa, (8.11) ifodani hisoblash ancha qiyindir. Bu qiyinchilikdan qutilish maqsadida yuqoridagi ifodaga asimptotik formula topamiz.

8.2-teorema. Agar har bir tajribada A hodisaning ro'y berish ehtimoli p o'zgarmas bo'lib, nol va birdan farqli bo'lsa, u holda $n \to \infty$ da

$$P_n(k_1, k_2) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx \qquad (8.12)$$

munosabat a va b $(-\infty \le a \le b \le +\infty)$ ga nisbatan tekis bajariladi, bu yerda

$$a = \frac{k_1 - np}{\sqrt{npq}} \qquad b = \frac{k_2 - np}{\sqrt{npq}}$$

Isbot. Aniqlik uchun a va b chekli bo'lsin. U holda Muavr – Laplasning lokal teoramasiga ko'ra (8.11) ni

$$P_n(k_1, k_2) = P_n\left(\frac{k_1 - np}{\sqrt{npq}} \le \frac{m - np}{\sqrt{npq}} \le \frac{k_2 - np}{\sqrt{npq}}\right) = P_n\left(a \le \frac{m - np}{\sqrt{npq}} \le b\right) =$$

$$= \sum_{a \le \frac{m-np}{\sqrt{npq}} \le b} \frac{1}{\sqrt{2\pi npq}} \cdot e^{-\frac{1}{2}\left(\frac{m-np}{\sqrt{npq}}\right)^2} \cdot (1 + \left(\frac{1}{\sqrt{n}}\right) =$$

$$= \sum_{a \le x_m \le b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_m^2}{2}} \Delta x_m (1 + 0\left(\frac{1}{\sqrt{n}}\right))$$

ko'rinishda yozamiz. Bu formulada

$$x_m = \frac{m - np}{\sqrt{npq}}, \quad \Delta x_m = x_{m+1} - x_m = \frac{1}{\sqrt{npq}}$$

Endi

$$J(x_m)=\int_{x_m}^{x_{m+1}} e^{-\frac{x^2}{2}}dx \qquad (8.13)$$

yordamchi funksiyani qaraylik. Bu integralda $x = x_m + u$ almashtirishni bajaramiz, natijada

$$J(x_m)=\int_0^{\Delta x_m} e^{-\frac{x_m^2}{2}} e^{-ux_m-\frac{u^2}{2}}du = e^{-\frac{x_m^2}{2}}\int_0^{\Delta x_m} e^{-ux_m-\frac{u^2}{2}}du$$

$n-$ni yetarlicha katta qiymatlarida $0 \le u \le \Delta x_m = \frac{1}{\sqrt{npq}}$ bo'lgani uchun u nolga yetarlicha yaqin bo'ladi, bu esa

$$e^{-ux_m-\frac{u^2}{2}} = 1 + 0\left(ux_m + \frac{u^2}{2}\right)$$

ekanini korsatadi. U holda $J(x_m) = e^{-\frac{x_m^2}{2}}\Delta x_m(1 + 0\left(\frac{1}{\sqrt{n}}\right))$

yoki

$$\frac{1}{\sqrt{2\pi}}\sum_{a\le x_m\le b}\int_{x_m}^{x_{m+1}} e^{-\frac{x^2}{2}}dx = \sum_{a\le x_m\le b}\frac{1}{\sqrt{2\pi}}e^{-\frac{x_m^2}{2}}\Delta x_m\left(1+0\left(\frac{1}{\sqrt{n}}\right)\right). \quad (8.14)$$

Natijada $n \to \infty$ da (8.12) va (8.14) larni solishtirib, ifodalarning o'ng tomonlari asimptotik tengligidan, chap tomonlarning ham asimptotik tengligiga ishonch hosil qilamiz. Shu bilan teorema isbot bo'ldi.

Muavr – Laplasning integral teoremasini qo'llash bilan yechiladigan masalalarda

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{n^2}{2}}du$$

Ifodani hisoblashga to'gri keladi. Bu integral uchun jadval mavjud bo'lib $\phi(x)$ funksiyaning musbat $x-$ larga mos qiymatlari keltirilgan. $\phi(x)$ funksiyaning toqligidan foydalanib, jadvaldan $x < 0$ bo'lganda ham foydalaniladi. Jadvalga $\phi(x)$ funksiyaning $x \in [0\,;5]$ segmentidagi qiymatlari berilgan. Agar $x > 5$ bo'lsa, u holda $\phi(x) \approx 0{,}5$ deb olinadi.

Jadvaldan foydalanish oson bo'lishi uchun quyidagi formuladan foydalanish qulaydir.

$$\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}dx = \phi(b)-\phi(a).$$

8.4. Puasson teoremasi

Muavr-Laplasning lokal teoremasi p va q ehtimollar yarimning atrofiga bo'lganda $P_n(m)$ ni hisoblash uchun yaxshi natija beradi, lekin p va q –lar bir yoki nol soniga yaqin bo'lgan hollarda bu formula ma'lum xatolikka olib keladi. Shuning uchun p va q lar 1 ga yoki 0 ga yaqin bo'lganda $P_n(m)$ uchun lokal teoremadan boshqa asimptotik formula topish zaruriyati paydo bo'ladi. Bu masalani hal qilish uchun quyidagi hodisalar seriyasini qaraymiz:

$$\varepsilon_{11}$$
$$\varepsilon_{12,}\varepsilon_{22,}$$
$$\varepsilon_{13,}\varepsilon_{23,}\varepsilon_{33}$$
$$- - - - - -$$
$$\varepsilon_{1n,}\varepsilon_{2n,}\cdots\cdots,\varepsilon_{nn}$$

Qaralayotgan har bir seriyadagi hodisalar o'zaro bog'liq bo'lmasdan, har birining ro'y berish ehtimoli P_n, ro'y bermaslik ehtimoli $q_n = 1 - p_n$ bo'lsin, bunda n –seriya nomeri n seriyadagi n –ta hodisadan m –tasining ro'y berish ehtimolini $P_n(m)$ deb belgilaymiz.

8.3-teorema. Agar $n \to \infty$ da $P_n \to 0$ munosabat bajarilsa, u holda

$$P_n(m) - \frac{(nP_n)^m}{m!}e^{-n}, \qquad P_n \longrightarrow 0$$

munosabat o'rinli bo'ladi.

Isbot. $a_n = nP_n$ deb belgilaymiz va $P_n(m) = C_n^m p^m q^{n-m}$ formuladan $P_n = \frac{a_n}{n}$ ekaligini e'tiborga olib, quyidagi ifodani hosil qilamiz

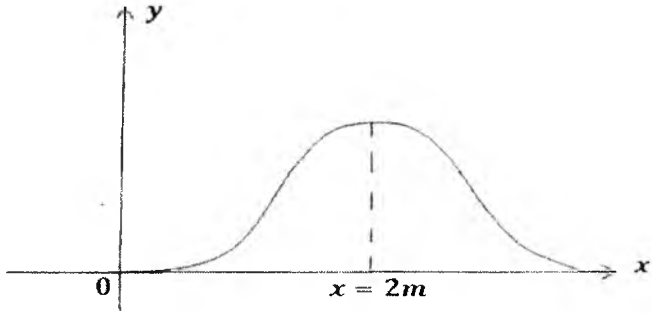
$$P_n(m) = C_n^m P_n^m (1 - P_n)^{n-m} = \frac{n!}{m!\,(n-m)!}\left(\frac{a_n}{n}\right)^m \left(1 - \frac{a_n}{n}\right)^{n-m} =$$
$$= \frac{a_n^m}{m!}\cdot\frac{n(n-1)(n-2)\cdots[n-(m-1)]}{n^m}\left(1 - \frac{a_n}{n}\right)^{n-m} =$$
$$= \frac{a_n^m}{m!}\cdot\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{a_n}{n}\right)^m}\left(1 - \frac{a_n}{n}\right)^n \qquad (8.15)$$

Aytaylik, m tayinlangan (fiksirlangan) bo'lsin. Quyidagi ikki holni ko'rib chiqamiz.

1-hol. a_n – chegaralanmagan, ya'ni $n \to \infty$ da $a_n \to \infty$ bo'lsin. U holda ixtiyoriy $0 \le x \le 1$ uchun $1 - x < e^{-x}$ ni va (8.15)ni e'tiborga olsak,

$$J = \left|P_n(m) - \frac{a_n^m}{m!}e^{-a_n}\right| \le P_n(m) + \frac{a_n^m}{m!}e^{-a_n} \le$$

$$\le \frac{a_n^m}{m!}e^{-\frac{n-m}{n}a_n} + \frac{a_n^m}{m!}e^{-a_n} \qquad (8.16)$$



8.1-chizma

hosil bo'ladi. Endi $y = \frac{x^m}{m!}e^{-\frac{x^2}{2}}$ funksiyani qaraymiz. Agar $x = 0$ bo'lsa, $y = 0$, $x \to \infty$ da $y \to 0$ va y eng katta qiymatni $x = 2m$ da oladi. Bu funksiyaning grafigi 8.1-chizmada keltirilgan. Natijada $\forall(\mathcal{E} > 0)$ uchun $\exists(A_\mathcal{E})$ –son topiladiki, yetarlicha katta $n(n > 2m)$ lar uchun $a_n > A_\mathcal{E}$ bo'lganda

$$\frac{a_n^m}{m!}e^{-\frac{n-m}{n}a_n} < \frac{\mathcal{E}}{2}; \qquad \frac{a_n^m}{m!}e^{-a_n} < \frac{\mathcal{E}}{2} \qquad (8.17)$$

bo'dadi. Demak, (8.16) va (8.17) dan $J < E$ kelib chiqadi.

2-hol. a_n – chegaralangan bo'lsin, u holda $\forall(\mathcal{E} > 0)$ uchun $\exists(n_o(\mathcal{E}))$ topiladiki, $n > n_o(\mathcal{E})$ bo'lganda ushbu tengsizliklar bajariladi $\left|\left(1 - \frac{a_n}{n}\right)^n - e^{-a_n}\right| < \frac{\mathcal{E}}{2}$,

$$\left|\frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right)}{\left(1 - \frac{a_n}{n}\right)^m}\right| < \frac{\mathcal{E}}{2}.$$

Bu tengsizliklardan va (8.15) dan quyidagiga ega bo'lamiz:

$$\left|P_n(m) - \frac{a_n^m}{m!}e^{-a_n}\right| =$$

$$= \left|\frac{a_n^m}{m!}\cdot\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{a_n}{n}\right)^m}\cdot\left(1-\frac{a_n}{n}\right)^n - \frac{a_n^m}{m!}e^{-a_n}\right| =$$

$$= \left|\frac{a_n^m}{m!}\cdot\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{a_n}{n}\right)^m}\cdot\left[\left(1-\frac{a_n}{n}\right)^n - e^{-a_n}\right] - \frac{a_n^m}{m!}\right.$$

$$\left.\cdot\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)e^{-a_n}}{\left(1-\frac{a_n}{n}\right)^m} - \frac{a_n^m}{m!}e^{-a_n}\right| \leq$$

$$\leq \frac{a_n^m}{m!}\cdot\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{a_n}{n}\right)^m}\cdot\left|\left(1-\frac{a_n}{n}\right)^n - e^{-a_n}\right|$$

$$+\frac{a_n^m}{m!}e^{-a_n}\left|\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{\left(1-\frac{a_n}{n}\right)^m} - 1\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

Bu esa teoremani isbotlaydi.

Endi

$$P(m) = \frac{a^m}{m!}e^{-a}, \qquad m = 0,1,2,\cdots$$

Ifodani kiritaylik. $P(m)$ miqdor

$$\sum_m P(m) = 1$$

tenglikni qanoatlantirishini ko'rish qiyin emas. Haqiqatan ham,

$$\sum_{m=0}^{\infty} P(m) = \sum_{m=0}^{\infty}\frac{a^m}{m!}e^{-a} = e^{-a}\sum_{m=0}^{\infty}\frac{a^m}{m!} = e^{-a}\cdot e^{a} = 1.$$

Hosil qilingan ehtimollar taqsimoti Puasson qonuni deyiladi.

Endi $P(m)$ ni m –ning funksiyani sifatida qarab, uni tekshiramiz. Shu maqsadda ushbu nisbatni tuzib olamiz:

$$\frac{P(m)}{P(m-1)} = \frac{a}{m}.$$

Bu nisbatdan $m > a$ bo'lganda $P(m) < P(m-1), m < a$ bo'lganda $P(m) > P(m-1)$ va, nihoyat, $m = a$ bo'lganda $P(m) = P(m-1)$ munosibatlarga ega bo'lamiz. Bularni e'tiborga olib, $P(m)$ miqdor m –ning 0 dan $m_0 = [a]$ gacha qiymatlarda o'sishi va keyingi qiymatlarda kamayishiga ishonch hosil qilamiz. Agar a butun son bo'lsa, $P(m)$ funksiya ikkita maksimum qiymatga ega bo'ladi va bu qiymatlar $m_0 = a, m_0^1 = a-1$ nuqtalarda topiladi.

8.4-misol. Har bir otilgan o'qning nishonga tegish ehtimoli 0,001 ga teng. Agar 5000 ta o'q otilgan nishonga tegish ehtimolini toping.

Yechish. Nishonga tekkan o'qlar sonini μ_n desak, izlanayotgan ehtimol $P(\mu_n \geq 2)$ dan iborat bo'lib, u quyidagicha teng bo'ladi:

$$P(\mu_n \geq 2) = \sum_{m=2}^{n} P(m) = 1 - P_n(0) - P_n(1).$$

$$a_n = np = 5000 \cdot 0{,}001 = 5.$$

$$P_{5000}(0) = \frac{5^0}{0!}e^{-5} = e^{-5}; \quad P_{5000}(1) = \frac{5}{1!}e^{-5} = 5e^{-5}.$$

U holda $P(\mu_{5000} \geq 2) = 1 - e^{-5} - 5e^{-5} \approx 0{,}9595.$

$P_{5000}(m)$ ehtimol $m = 4$ va $m = 5$ bo'lganda, ushbu maksimum qiymatga erishadi: $P_{5000}(4) = P_{5000}(5) \approx 0{,}1755.$

Nazorat savollar.

1. Bernulli formulasi.
2. Muav-Laplasning lokal teoremasi.
3. Muav-Laplasning integral teoremasi.
4. Puasson teoremasi.

IX-BOB. TASODIFIY MIQDORLAR

9.1. Tasodifiy miqdorlar va taqsimot funksiyalar

Oldingi boblardagi misollarga e'tibor bersak, ba'zi bir miqdorlarning u yoki bu tasodifiy ta'sir natijasida turli qiymatlarni qabul qilishini ko'ramiz.

Masalan, ya'ngi tug' ilgan chaqaloqlarni 100 tadan grupalarga ajratsak, har bir gruppadagi o'g'il (qiz) bolalar soni turlicha bo'lishi mumkin; olingan bir kilogramm paxtadan chiqqan tola uzunligi boshqa bir kilogramm paxtadan chiqqan tola uzunligidan farq qiladi, har yili yerga tushadigan meteorlar soni ham turlicha bo'ladi, ya'ni tasodifiy xarakterga ega.

Tasodifiy miqdor ta'rifini berishdan avval o'lchovli funksiya tushunchasini kiritamiz. Bizga $< \Omega, \mathcal{F} >, < R, G_1 >$ o'lchovli fazolar va $\tau: \Omega \to R$ funksiya berilgan bo'lib, bu funksiya uchun $A \in G_1$ ekanidan $\tau^{-1}(A) = \{\omega/\tau(\omega) \epsilon A\} \epsilon \mathcal{F}$ ekani kelib chiqsa, bunday funksiya o'lchovli funksiya deyiladi. Agar $< \Omega, \mathcal{F}, P >$ ixtiyoriy ehtimollik fazosi bo'lsa, har qanday $\tau: < \Omega, \mathcal{F} > \to < R, G_1 >$ o'lchovli funksiya tasodifiy miqdor deyiladi.

9.1-misol. Agar ikkita tanga tashlasak, ω elementar hodisalar GG; GR; RG; RR dan iborat bo'lib, «gerb» tushishlar sonini $\tau = \tau(\omega)$ deb belgilasak, bu tasodifiy miqdorni kuyidagi jadval bilan berish mumkin.

ω	RR	GR	RG	GG
$\tau(\omega)$	0	1	1	2

Tasodifiy miqdorning ta'rifiga ko'ra ixtiyoriy $x \epsilon R$ uchun

$$\{\omega/\tau(\omega) < x\} = \{\tau < x\} = \tau^{-1}(\,]-\infty, x[\,) \epsilon \mathcal{T},$$

chunki $]-\infty, x[\, \epsilon G_1$. Bundan $F_\tau(x) = P\{\omega: \tau < x\}$ funksiyani R da aniqlanganligi kelib chiqadi. Bu funksiya τ tasodifiy miqdorning taqsimot funksiyasi deyiladi.

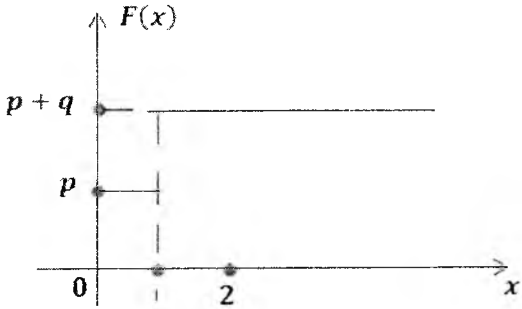
9.2-misol. τ tasodifiy miqdor 1 va 0 qiymatlarni mos ravishda p va $q(p+q=1)$ ehtimol bilan qabul qilsin, ya'ni

$$\tau: 0, 1, \qquad p = P\{\tau = 1\}$$
$$p: q, p, 1, \qquad q = P\{\tau = 0\}.$$

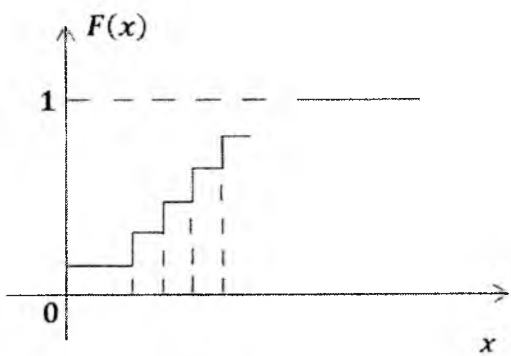
Bu holda

$$P\{\tau < x\} = F(x) = \begin{cases} 0, \text{agar } x \le 0 \text{ bo'lsa,} \\ p, \text{agar } 0 < x \le 1 \text{ bo'lsa,} \\ 1, \text{agar } 1 < x \text{ bo'lsa.} \end{cases}$$

Bu taqsimotning grafigi quyidagicha bo'ladi (9.1-chizma)



9.1-chizma



9.2-chizma

Agar tasodifiy miqdorlar 0,1,2,..., n qiymatlarni

$$P\{\tau = k\} = C_n^k p^k q^{n-k}, \quad k = \overline{0,n}$$

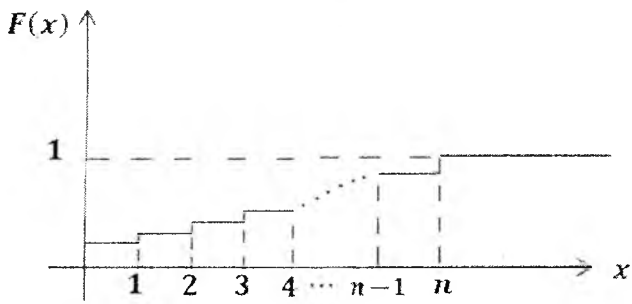
ehtimol bilan qabul qilsa, bu tasodifiy miqdor binomial qonun bo'yicha taqsimlangan tasodifiy funksiyasi quyidagicha bo'ladi:

$$F(x) = \begin{cases} 0, & \text{agar } x < 0 \text{ bo'lsa}, \\ \sum\limits_{k<x} C_n^k p^k q^{n-k}, & \text{agar } 0 \le x \le n \text{ bo'lsa}, \\ 1, & \text{agar } n < x \text{ bo'lsa}. \end{cases}$$

Grafigi quyidagicha bo'ladi (9.2-chizma).

9.3-misol. Agar τ tasodifiy miqdor 0,1,2,... qiymatlarni ehtimollar bilan qabul qilsa, uni Puasson qonun bo'yicha taqsimlangan tasodifiy miqdor deyiladi. Uning taqsimot funksiyasi quyidagicha aniqlanadi:

$$P\{\tau = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad \lambda > 0$$



9.3-chizma

$$F(x) = \begin{cases} 0, & \text{agar } k < 0 \text{ bo'lsa}, \\ \sum\limits_{k<x} \frac{\lambda^k}{k!} e^{-\lambda}, & \text{agar } 0 \le k \le x \text{ bo'lsa}. \end{cases}$$

Grafigi quyidagicha bo'ladi (9.3-chizma).

9.4-misol. τ tasodifiy miqdor $x_1, x_2, \dots, x_N$ qiymatlarni $P\{\tau = x_k\} = \frac{1}{N}$, $k = \overline{1,N}$ ehtimollar bilan qabul qilsin. Bu tasodifiy miqdor tekis taqsimlangan tasodifiy miqdor deyiladi. Uning taqsimot funksiyasi quyidagicha bo'ladi:

$$F(x)=\begin{cases} 0, & \text{agar } x \le x_1 \text{ bo'lsa,} \\ \frac{k}{N}, & \text{agar } \ x_k < x \le x_{k+1} \text{ bo'lsa,} \\ 1, & \text{agar } \ x_N < x \text{ bo'lsa.} \end{cases}$$

9.5-misol. Agar tasodifiy miqdorlarning taqsimot funksiyasi ko'rinishida bo'lsa, bunday tasodifiy miqdor normal taqsimlangan tasodifiy miqdor deyiladi. Bu yerda $c > 0, \delta > 0, -\infty < a < \infty$ –o'zgarmas sonlar.

$$F(x) = C \int_{-\infty}^{x} \ell^{-\frac{(u-a)^2}{2\delta^2}} du$$

Taqsimot funksiya quyidagi xossalarga ega:

1. Barcha haqiqiy x-lar uchun

$$0 \le F_\tau(x) \le 1.$$

2. $F_\tau(x)$ kamaymaydigan funksiya.

Haqiqatan ham, $x_1 < x_2$ bo'lsin.

Ushbu $A_1 = \{\tau < x_1\}$, $A_2 = \{\tau < x_2\}$, $B = \{x_1 < \tau < x_2\}$ hodisalarni kiritsak;

$$A_2 = A_1 \cup B, \quad A_1 \cap B = V$$

munosibatlar o'rinli bo'ladi. Natijada ehtimolning additivlik aksiomasiga ko'ra

$$P(A_2) = P(A_1) + P(B)$$

yoki

$$0 \le P(x_1 < \tau < x_2) = P(\tau < x_2) - P(\tau < x_1) = F(x_2) - F(x_1).$$

Bundan esa $x_1 \le x_2$ da $F(x_1) \le F(x_2)$ bo'lib, xossaning isboti kelib chiqadi.

3. Taqsimot funksiya chapdan uzluksiz, ya'ni

$$F_\tau(x) = F_\tau(x-0) = \lim_{x_m \to x} F(x_m).$$

Isbot. Faraz qilaylik, $a_1 < a_2 < \cdots < a_n < \infty$ va $n \to \infty$ da $a_n \to b$ bo'lsin, u holda

$$\bigcap_{n=1}^{\infty} \{\omega: \tau(\omega) \epsilon\ [a_n, b[\ \} = V.$$

Natijada uzluksizlik aksiomasiga ko'ra

$$F_\tau(b) - F_\tau(a_n) = P\{\omega: \tau(\omega)\epsilon[a_n, b[\}$$
ifoda $n \to \infty$ da nolga intiladi. Bu esa $F_\tau(x)$ ning chapdan uzluksizligini ko'rsatadi.

4. $\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to+\infty} F(x) = 1$

Agar $x = x_0$ nuqtada $F(x_0 + 0) - F(x_0 - 0) = C_0 > 0$ bo'lsa, funksiya $x = x_0$ nuqtada sakrashga ega bo'lib, uning kattaligi $C_0 -$ ga teng bo'ladi.

5. Taqsimot funksiyaning sakrashga ega bo'lgan nuqtalari to'plami ko'pi bilan sanoqli bo'lishi mumkin.

9.1-ta'rif. Agar τ tasodifiy miqdor chekli yoki sanoqli sondagi $\{x_n\}$ qiymatlarni $\{P_k\}$ $(\sum_k P_k = 1)$ ehtimol – lar bilan qabul qilsa, uni diskret tasodifiy miqdor deyiladi. Diskret tasodifiy miqdorning taqsimot funksiyasi

$$F(x) = \sum_{\{k:\, x_k < x\}} P_k$$

formula bilan aniqlanadi. Yuqorida keltirilgna 2-4 misollar diskret tasodifiy miqdorga misol bo'ladi.

9.2-ta'rif. τ tasodifiy miqdorning taqsimot funksiyasini

$$F(x) = \int_{-\infty}^{x} p(u)du.$$

ko'rinishda yozish mumkin bo'lsa, bu tasodifiy miqdorni absolyut uzluksiz taqsimlangan tasodifiy miqdor deyiladi. Bu yerdagi $p(u)$ funksiya τ tasodifiy miqdorning zichlik funksiyasi deyiladi.

9.2-ta'rifga ko'ra $F'(x) = p(x)$ bo'ladi. Zichlik funksiya quyidagi xossalarga ega.

1. Zichlik funksiya manfiy emas: $p(x) \geq 0$.

Haqiqatan ham, $F(x)$ taqsimot funksiyaning kamaymasligidan uning hosilasi deyarli hamma nuqtalarda doim musbat bo'ladi.

2. Agar $p(x)$ zichlik funksiya x_0 nuqtada uzluksiz bo'lsa, u holda $P(x_0 \leq \tau < x_0 + dx)$ ehtimol zichlik funksiyaning x_0 nuqtadagi qiymatiga nisbatan yuqori tartibli cheksiz kichik miqdor aniqligida ekvivalent bo'ladi:

$$P(x_0 \leq \tau < x_0 + dx) \approx P(x_0)dx$$

3. Zichlik funksiyadan $]-\infty; \infty[$ oraliq bo'yicha olingan integral 1-ga teng:

$$\int_{-\infty}^{\infty} p(x)dx \quad 1.$$

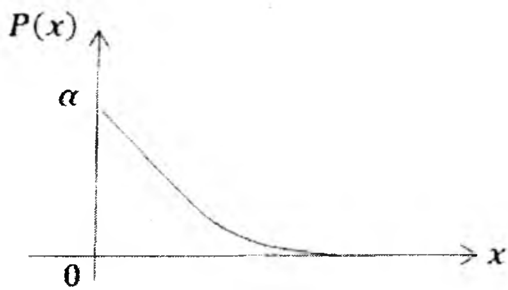
9.6-misol. $\alpha -$ parametrli eksponensial qonun bo'yicha taqsimlangan tasodifiy miqdorning zichlik funksiyasi

$$p(x) = \begin{cases} 0, x < 0, \\ \alpha e^{-\alpha} \ , x \geq 0 \end{cases}$$

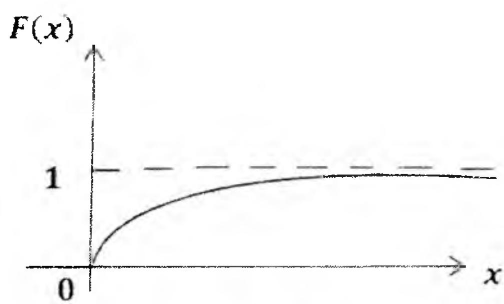
ko'rinishga ega, taqsimot funksiya esa

$$F(x) = \begin{cases} 0, \ \ x < 0, \\ 1 - e^{-\alpha} \ , \ \ x \geq 0 \end{cases}$$

ko'rinishga ega bo'ladi. Bu funksiyalarning grafiklari 9.4, 9.5-chizmalarda ko'rsatilgan.



9.4-chizma



9.5-chizma

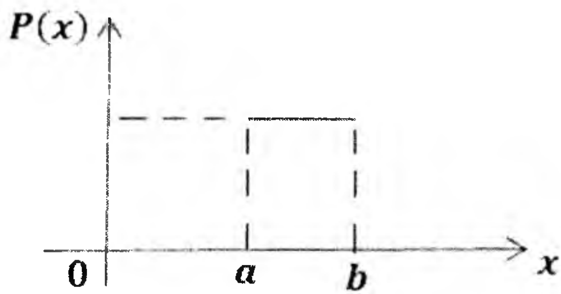
9.7-misol. Agar τ tasodifiy miqdorning zichlik funksiyasi

$$p(x)=\begin{cases}0, & agar\ x\overline{\epsilon}\ ]a,b[\quad bo'lsa,\\ \dfrac{1}{b-a}, & agar\ x\epsilon\,[a,b]\quad bo'lsa\end{cases}$$

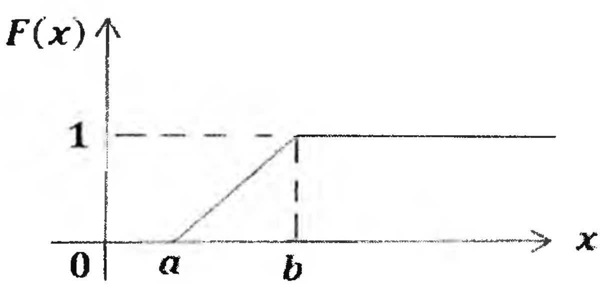
ko‘rinishda berilgan bo‘lsa, τ tasodifiy miqdor [a, b] da tekis taqsimlangan tasodifiy miqdor deyiladi. Bu tasodifiy miqdorning taqsimot funksiyasi quyidagi ko‘rinishga ega:

$$F(x)=\begin{cases}0, & agar\ x<a\ \ bo'lsa,\\ \dfrac{x-a}{b-a}, & agar\ a\le x<b\ bo'lsa,\\ 1, & agar\ x\ge b\ \ bo'lsa.\end{cases}$$

Bu funksiyalarning grafiklari 9.6, 9.7-chizmalarda ko‘rsatilgan.



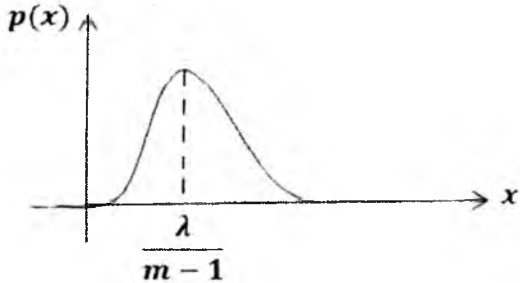
9.6-chizma



9.7-chizma

9.8-misol. Zichlik funksiyasi

$$p(x) = \begin{cases} \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{(m-1)!}, & x > 0, \\ 0, & x \le 0 \end{cases}$$



9.8-chizma.

ko'rinishda (9.8-chizma) bo'lgan tasodifiy miqdor (m, λ) parametrli Erlang qonuni bo'yicha taqsimlangan tasodifiy miqdor deyiladi. Uning taqsimot funksiyasi quyidagicha bo'ladi:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} \sum\limits_{k=1}^{m} \dfrac{\lambda^{k-1} x^{k-1}}{(k-1)!}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

9.2. Tasodifiy miqdorlar va ularning sonli xarakteristikalari

1. Matematik kutilma. Diskret va uzluksiz tasodifiy miqdorlarning matematik kutilmasi (o'rta qiymati) tushunchasini alohida-alohida ko'rib o'tamiz.

9.3-ta'rif. τ tasodifiy miqdor $\{x_k\}$ qiymatlarini $\{P_k\}$ ehtimollar bilan qabul qilsin.

$$\sum_{k=1}^{\infty} x_k P_k$$

qator yig'indisi (agar bu qator absolyut yaqinlashuvchi bo'lsa) τ tasodifiy miqdorning matematik qutilmasi deyiladi va

$$M\tau = \sum_{k=1}^{\infty} x_k P_k \qquad (9.1)$$

kabi belgilanadi.

9.9-misol. *A* hodisaning ro‘y berish ehtimoli p ga teng bo‘lsa, bitta tajribada *A* hodisa ro‘y berish sonining matematik kutilmasini toping.

Yechilishi. Bitta tajribada A hodisaning ro‘y berish sonini τ deb belgilaylik. U holda

$$\tau:\ 0 \qquad\qquad 1$$
$$p:\ q = 1 - p \quad p.$$

9.3-ta’rifga asosan

$$M\tau = 0 \cdot q + 1 \cdot p = p$$

9.10-misol. Binomial qonun bilan taqsimlangan tasodifiy miqdorning matematik kutilmasini toping.

Yechilishi. τ orqali A hodisaning n-ta o‘zaro bog‘liqmas tajribalarda ro‘y berish sonini belgilasak,

$$P(\tau = k) = C_n^k\, p^k\, q^{n-k}$$

tenglik o‘rinli ekani bizga ma’lum. Matematik kutilma (o‘rta qiymat) ta’rifga ko‘ra

$$M\tau = \sum_{k=1}^{n} k \cdot P(\tau = k) =$$

$$= \sum_{k=1}^{n} k \cdot C_n^k\, p^k\, q^{n-k} = np \sum_{k=1}^{n} C_{n-1}^{k-1}\, p^{k-1} q^{n-k} =$$
$$= np(p+q)^{n-1} = np.$$

9.11-misol. Puasson qonuni bilan taqsimlangan tasodifiy miqdorning matematik kutilmasini toping.

Yechilishi. Bu holda, ma’lumki,

$$P(\tau = k) = \frac{\lambda^k e^{-\lambda}}{k!},\ k = 0,1,2,\dots;\ \lambda > 0.$$

Ta’rifga asosan

$$M\tau = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda.$$

Demak, λ –parametrli Puasson qonuni bo'yicha taqsimlangan tasodifiy miqdorning matematik kutilmasi $\boldsymbol{\lambda}$ parametrga teng ekan.

9.4-ta'rif. Uzluksiz tasodifiy miqdorning matematik kutilmasi deb, ushbu

$$M\tau = \int_{-\infty}^{\infty} p(x)dx \tag{9.2}$$

integralga (agar bu integral absalyut yaqinlashuvchi bo'lsa) aytiladi.

9.12-misol. $(a,\ \sigma^2)$ parametrli normal qonun bilan taqsimlangan tasodifiy miqdorning matematik kutilmasini toping.

Yechilishi. ta'rifga asosan

$$M = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-a)e^{-\frac{(x-a)^2}{2\sigma^2}} dx + \frac{a}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}} dz + a = \quad .$$

Demak, $(a,\ \sigma^2)$ normal qonun bilan taqsimlangan tasodifiy miqdorning matematik kutilmasi $\boldsymbol{a}$ parametrga teng ekan.

9.13-misol. ν parametrli eksponensial qonun bo'yicha taqsimlangan τ tasodifiy miqdorning matematik kutilmasi

$$M\tau = \int_{0}^{\infty} x\nu e^{-\nu}\ dx \quad \frac{1}{\nu}.$$

9.14-misol. $[a, b]$ oraliqda tekis taqsimlangan τ tasodifiy miqdorning matematik kutilmasi quyidagicha topiladi:

$$M\tau = \int_{a}^{\ell} x\frac{1}{b-a}dx = \frac{b+a}{2}.$$

9.5-ta'rif. Taqsimot funksiyasi $F(x)$ bo'lgan tasodifiy miqdorning matematik kutilmasi

$$M\tau = \int_{-\infty}^{\infty} x dF(x) \qquad (9.3)$$

kabi aniqlanadi.

Umuman, $< \Omega, \mathcal{F}, P >$ ehtimollik fazosida berilgan τ tasodifiy miqdorning matematik kutilmasi deb,

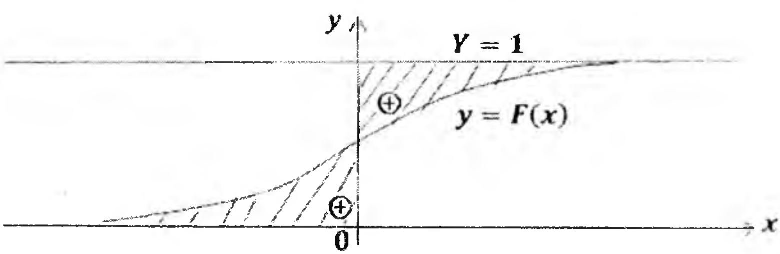
$$M\tau = \int_{\Omega} \tau(\omega)\, P(d\omega)$$

integralning son qiymatiga (agar bu integral mavjud bo'lsa) aytiladi.

Matematik kutilmaning geometrik tasvirini ko'z oldimizga keltirish maqsadida (9.3) integralni quyidagi ko'rinishda ifodalaymiz:

$$M\tau = -\int_{-\infty}^{0} F_\tau(x)dx + \int_{0}^{\infty} \left(1 - F_\tau(x)\right)d \ .$$

Bu ifodaning geometrik tavsiri 9.9-chizmada berilgan.



9.9-chizma.

Matematik kutilma quyidagi xossalarga ega:

1-xossa. O'zgarmas sonni matemetik kutilmasi shu sonning o'ziga teng.

Isbot. C o'zgarmas sonni faqat c qiymatni 1 ehtimol bilan qabul qiluvchi tasodifiy miqdor deb qarash mumkin. Shuning uchun

$$MC = C \cdot 1 = C$$

2-xossa. $|M\tau| \leq M|\tau|$ tengsizlik o'rinli.

Bu xossaning isboti matematik kutilmaning ta'rifidan bevosita kulib chikadi.

3-xossa. $M\xi, M\tau$ va $M(\xi+\tau)$ larning ixtiyoriy ikkitasi mavjud bo'lsa, u holda ushbu

$$M(\xi+\tau) = M\xi + M\tau$$

tenglik o'rinli bo'ladi.

Isbot. Isbotni diskret hol uchun keltiramiz. Faraz qilaylik, ξ tasodifiy miqdor $x_1, x_2, \dots, x_k, \dots$ qiymatlarni mos ravishda $P_1, P_2, \dots, P_k, \dots$ ehtimollar bilan, τ tasodifiy miqdor esa $y_1, y_2, \dots, y_l, \dots$ qiymatlarni mos ravishda $q_1, q_2, \dots, q_l, \dots$ ehtimollar bilan qabul qilsin, u holda $\xi+\tau$ yig'indining qabul qiladigan qiymatlari $\{x_k, y_l\}$ $(k, l = 1,2, \dots)$ qo'rinishdagi sonlardan iborat.

$P_{k,l}$ orqali ξ ning x_k va τ ning y_l qiymatlarni qabul qilish ehtimolning belgilaymiz. U holda to'la ehtimollik formulasiga asosan

$$M(\xi+\tau) = \sum_{k,e=1}^{\infty} (x_k + y_l)\, P_{k,l} =$$

$$= \sum_{k=1}^{\infty} x_k \left(\sum_{e=1}^{\infty} P_{k,l}\right) + \sum_{e=1}^{\infty} y_l \left(\sum_{k=1}^{\infty} P_{k,l}\right) =$$

$$= \sum_{k=1}^{\infty} x_k P_k + \sum_{e=1}^{\infty} y_l q_l = M\xi + M\tau.$$

Natija. $\tau_1, \tau_2, \dots, \tau_n$ tasodifiy miqdorlar yig'indisining matematik kutilmasi shu tasodifiy miqdorlar matematik kutilmalarining yig'indisiga teng, ya'ni

$$M\sum_{k=1}^{n} \tau_k = \sum_{k=1}^{n} M\tau_k.$$

4-xossa. O'zgarmas sonni matematik kutilma ishorasidan tashqarisiga chiqarib yozish mumkin, ya'ni

$$MC\tau = CM\tau, \qquad C - const.$$

5-xossa. Agar $\alpha \le \tau \le \beta$ bo'lsa, $\alpha \le M\tau \le \beta$ bo'ladi.

6-xossa. Agar $\tau \le 0$ va $M\tau = 0$ bo'lsa, u holda $\tau = 0$ tenglik 1 ehtimol bilan bajariladi.

7-xossa. Agar ξ va τ tasodifiy miqdorlar o'zaro bog'liq bo'lmasin. Agar $M\xi$ va $M\tau$ mavjud bo'lsa, u holda $M\xi = M\xi \cdot M\tau.$

Isbot. Faraz qilaylik, ξ tasodifiy miqdor $x_1, x_2, \ldots, x_k, \ldots$ qiymatlarni mos ravishda $P_1, P_2, \ldots, P_k, \ldots$ ehtimollar bilan, τ tasodifiy miqdor $y_1, y_2, \ldots, y_k, \ldots$ qiymatlarni $q_1, q_2, \ldots, q_k, \ldots$ ehtimollar bilan qabul qilsin.

ξ va τ tasodifiy miqdorlarning o'zaro bog'liq emasligidan $\xi \cdot \tau$ tasodifiy miqdor $x_i y_j$ ko'rinishdan qiymatlarni $p_i q_j$ ehtimol bilan qabul qiladi, natijada

$$M\xi\tau = \sum_{i,j} x_i y_j \, \mathrm{P}\left(\xi = x_i, \tau = y_j\right) = \sum_{i,j} x_i y_j p_i q_j = \sum_{i} x_i y_i \left(\sum_{j} y_j q_j\right) =$$
$$= M\xi M\tau$$

Teoremaning teskarisi har doim to'g'ri emas, ya'ni $M\xi M\tau = M\xi\tau$ dan ξ va τ ning o'zaro bog'liq bo'lmasligi kelib chiqmaydi.

2. Dispersiya.

9.6-Ta'rif. Tasodifiy miqdorning dispersiyasi deb,

$$M(\xi - M\xi)^2$$

Ifodaga aytiladi va $D\xi$ kabi belgilanadi.

Demak,

$$D\xi = M(\xi - M\xi)^2. \qquad (9.4)$$

Agar ξ tasodifiy miqdor $\{x_k\}$ qiymatlarni $\{P_k\}$ ehtimollar bilan qabul qilsa, $\tau = (\xi - M\xi)^2$ tasodifiy miqdor $\{(x_k - M\xi)^2\}$ qiymatlarni ham $\{P_k\}$ ehtimollar bilan qabul qiladi va shu tasodifiy miqdorning matematik kutilmasi uchun

$$M\tau = D\xi = \sum_{k=1}^{\infty} (x_k - M\xi)^2 P_k \qquad (9.5)$$

formula o'rinli bo'ladi.

ξ tasodifiy miqdorning dispersiyani ushbu formula bilan hisoblash qulaydir:

$$D\xi = M\xi^2 - (M\xi^2). \qquad (9.6)$$

Haqiqatan ham, matematik kutilmaning xossalaridan foydalanib, (9.6)ni isbotlash mumkin:

$$D\xi = M(\xi - M\xi)^2 = M(\xi^2 - 2\xi M\xi + (M\xi)^2)$$
$$= M\xi^2 - 2\, M\xi \cdot M\xi + (M\xi)^2 == M\xi^2 - (M\xi)^2.$$

9.15-misol. A hodisaning ro'y berish ehtimoli p -ga teng bo'lsa, bitta tajribada A hodisa ro'y berish sonining dispersiyasini toping.

Yechilishi. 9.1-§, 9.1-misoldagi masodifiy miqdor

$$\tau = \begin{cases} 0, & q = 1 - p, \\ 1, & p \end{cases}$$

ni kiritib, $M\tau = p$ ekanini e'tiborga olsak, (9.5) ga asosan

$$D\tau = (0 - M\tau)^2 q + (1 - M\tau)^2 \cdot p = p^2 q + (1 - p)^2 p$$
$$= p^2 q + p q^2 =$$
$$= pq(p + q) = pq.$$

9.16-misol. Binomial qonun bilan taqsilangan tasodifiy miqdorning dispersiyasini toping.

Yechilishi. 1-§, 2-misolga ko'ra $M\tau = np$ edi. $D\tau = M\tau^2 - (M\tau)^2$ tenglikka asosan.

$$D\tau = \sum_k k^2\, C_n^k\, p^k q^{n-k} - (np)^2 =$$

$$= np\left[(n-1)p\sum_k^n C_{n-2}^{k-2}\, p^{k-2} q^{n-k} + \sum_k^n C_{n-1}^{k-1} p^{k-1} q^{n-k}\right] -$$
$$-(np)^2 = np\left((n-1)p + 1\right) - (np)^2 = npq.$$

9.17-misol. Puasson qonuni bilan taqsimlangan tasodifiy miqdorning dispersiyasini toping.

Yechilishi. Shu bobdagi 9.1-§, 9.3-misolga asosan

$$M\tau = \lambda;$$

(9.6) tenglikka asosan

$$M\tau = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} - \lambda^2. \qquad (9.7)$$

(9.7)dagi qatorning yig'indisini hisoblaymiz:

$$\sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} =$$
$$= \lambda\left[\sum_{m=0}^{\infty} m \frac{\lambda^m e^{-\lambda}}{m!} + \sum_{m=0}^{\infty} m \frac{\lambda^m e^{-\lambda}}{m!}\right] =$$
$$= \lambda(\lambda + 1) = \lambda^2 + \lambda.$$

Buni (9.7)-ga qo'ysak,

$$D\tau = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Demak, Puasson qonuni bilan taqsimlangan tasodifiy miqdorning matematik qutilmasi va o'rta qiymati o'zaro teng ekan.

Endi uzluksiz tasodifiy miqdor dispersiyasining ta'rifini beramiz. τ tasodifiy miqdorning zichlik funksiyasi $p(x)$ bo'lsin.

9.7-ta'rif. Uzluksiz tasodifiy miqdorning dispersiyasi deb quyidagi

$$D\tau = \int_{-\infty}^{\infty}(x - M\tau)^2 p(x)dx \quad (9.8)$$

integralning qiymatiga aytiladi.

9.18-misol. (a, σ^2) –parametrli normal qonun bilan taqsimlangan tasodifiy miqdorning dispersiyasini toping.

Yechilishi. $M\tau = a$ ekanini e'tiborga olgan holda, (9.8) dan foydalanamiz:

$$D\tau = \int_{-\infty}^{\infty}(x - a)^2 \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx.$$

$\frac{x-a}{\sigma} = z$ almashtirishni kiritib, quyidagini hosil qilamiz:

$$D\tau = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dx.$$

Hosil bo'lgan integralni bo'laklab integrallaymiz:

$$D\tau = -\frac{\sigma^2}{\sqrt{2\pi}} \cdot z\, e^{-\frac{z^2}{2}}\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sigma^2.$$

Demak, $(a, \sigma^2)-$ parametrli normal qonun bilan taqsimlangan tasodifiy miqdorning dispersiyasi ikkinchi parametrga teng ekan.

9.19-misol. ν parametrli eksponensial qonun bilan taqsimlangan tasodifiy miqdorning dispersiyasini toping.

Yechilishi. $M\tau = \frac{1}{\nu}$ ekanini e'tiborga olib (9.6) formuladan foydalansak;

$$D\tau = \nu \int_{0}^{\infty} x^2 e^{-\nu x} dx - \frac{1}{\nu^2} = \frac{2}{\nu^2} - \frac{1}{\nu^2} = \frac{1}{\nu^2}.$$

9.20-misol. $[a, b]$ oraliqda tekis taqsimlangan τ tasodifiy miqdorning dispersiyasini toping.

Yechilishi. $M\tau = \frac{a+b}{2}$ ekanini hisobga olsak:

$$D\tau = \int_a^b x^2 \frac{dx}{b-a} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

9.8-ta'rif. Taqsimot funksiyani $F(x)$ bo'lgan tasodifiy miqdorning dispersiyasi quyidagicha aniqlanadi:

$$D\tau = \int_{-\infty}^{\infty} (x - M\tau)^2 dF(x). \qquad (9.9)$$

Tasodifiy miqdorning dispersiyasi tasodifiy miqdor bilan uning matematik kutilmasi orasidagi ayirmaning farqini kvadratiga bog'liq ekaniga e'tibor beraylik. Bu farq qanchalik katta bo'lsa, dispersiyaning qiymati ham katta va aksinchadir. Shuning uchun dispersiya qiymatini qaralayotgan tasodifiy miqdor qiymatlarining uni o'rta qiymatiga nisbatan tarqoqlik xarakteristikasi deb qarash mumkin.

Dispersiya quyidagi xossalarga ega:

1-xossa. O'zgarmas sonning dispersiyasi nolga teng.

Isbot. 1-ta'rifga asosan.

$$DC = M(C - MC)^2 = M(c - c) = M \cdot 0 = 0.$$

2-xossa. O'zgarmas sonni kvadratga oshirib, dispersiya ishorasidan tashqariga chiqarish mumkin, ya'ni

$$DC\tau = C^2 D\tau.$$

Isbot. Ta'rifga asosan

$$DC\tau = M(C\tau - MC\tau)^2 = M(C\tau - CM\tau\ ^2 = C^2 M(\tau - M\tau\ ^2 =$$
$$= C^2 D\tau.$$

3-xossa. O'zaro bog'liq bo'lmagan tasodifiy miqdorlar yig'indisining dispersiyasi bu tasodifiy miqdorlar dispersiyalarining yig' indisiga teng ya'ni

$$D(\xi + \tau) = D\xi + D\tau$$

Isbot. Ta'rifga asosan

$$D(\xi + \tau) = M[(\xi + \tau) - M(\xi + \tau)]^2$$

Matematik kutilmaning xossasidan foydalansak:

$$D(\xi + \tau) = M\{(\xi - M\xi) + (\tau - M\tau\ \}^2 =$$
$$= M(\xi - M\xi)^2 + M2(\xi - M\xi)(\tau - M\tau) +$$
$$+ M(\tau - M\tau)^2$$
$$= D\xi + 2M(\xi - M\xi)(\tau - M\tau) + D\ . \qquad (9.10)$$

$\xi - M\xi$ va $\tau - M\tau$ lar o'zaro bog' liq emas, u holda

$$M(\xi - M\xi)(\tau - M\tau) = M(\xi - M\xi) \cdot M(\tau - M\tau) = 0$$

bo'ladi. Buni e'tiborga olsak, (9.10) dan xossaning isboti kelib chiqadi.

τ tasodifiy miqdor berilgan bo'lsin. Odatda $\xi = \frac{\tau - M\tau}{\sqrt{D\tau}}$ tasodifiy miqdor normalashtirilgan va markazlashtirilgan tasodifiy miqdor beyiladi.

9.21-misol. $M\xi = 0, D\xi = 1$ ekanini isbotlang.

Yechilishi.

$$M\xi = M\frac{\tau - M\tau}{\sqrt{D\tau}} = \frac{1}{\sqrt{D\tau}}(M\tau - M\tau) = 0.$$

τ va $M\tau$ lar o'zaro bog'liq emasligidan dispersiyaning 2, 3 xossalariga ko'ra

$$D\left(\frac{\tau - M\tau}{\sqrt{D\tau}}\right) = \frac{D\tau + D(-M\tau)}{D\tau} = \frac{D\tau}{D\tau} = 1.$$

9.22-misol. ξ va τ tasodifiy miqdorlar o'zaro bog'liq bo'lmasa,

$$D(\xi - \tau) = D\xi + D\tau$$

bo'ladi.

Yechilishi. 2-3 xossalarga asosan

$$D(\xi - \tau) = D\xi + D(-\tau) = D\xi + (-1)^2 D\tau = D\xi + D\tau.$$

9.23-misol. O'zaro bog'liq bo'lmagan va har biri

$$\tau_k = \begin{cases} 0, & q \\ 1, & p \end{cases} \tag{9.11}$$

qonun bilan taqsimlangan $\tau_1, \tau_2, \dots, \tau_n$ tasodifiy miqdorlar berilgan, $\tau_1 + \tau_2 + \dots + \tau_n$ yig'indining dispersiyasini toping.

Yechilishi. Chekli sondagi o'zaro bog'liq bo'lmagan tasodifiy miqdorlar yig'indisining dispersiyasi, ular dispersiyalarining yig'Indisiga teng, ya'ni

$$D\sum_{k=1}^{n} \tau_k = \sum_{k=1}^{n} D\tau_k,$$

asosan

$$D\sum_{k=1}^{n} \tau_k = \sum_{k=1}^{n} D\tau_k = n \cdot D\tau_k,$$

Shu paragrafdagi 1-misolda $D\tau_k = pq$ ekani topilgan edi. U holda

$$D\sum_{k=1}^{n}\tau_k = npq.$$

9.3. Yuqori tartibli momentlar

9.9-ta'rif. τ tasodifiy miqdorning k –tartibli boshlang'ich momenti deb, diskret tasodifiy miqdorlar uchun

$$a_k = M\tau^k = \sum_{-\infty}^{\infty} x_i^k\, P(\tau = x_i)$$

ifodaga, uzluksiz tasodifiy miqdorlar uchun

$$a_k = M\tau^k = \int_{-\infty}^{\infty} x^k\, p_\tau(x)dx$$

ifodaga aytiladi. Bu yerda $p_\tau(x) - \tau$ ning zichlik funksiyasi.

9.10-ta'rif. τ tasodifiy miqdorning k tartibli absolyut momenti deb, diskret tasodifiy miqdorlar uchun

$$m_k = M|\tau|^k = \sum_{-\infty}^{\infty} |x_i|^k\, p(\tau = x_i)$$

ifodaga, uzluksiz tasodifiy miqdorlar uchun

$$m_k = M|\tau|^k = \int_{-\infty}^{\infty} |x|^k\, p_\tau(x)dx$$

ifodaga aytiladi.

9.11-ta'rif. $\tau -$ tasodifiy miqdorning k-tartibli markaziy momenti deb, diskret tasodifiy miqdorlar uchun

$$\nu_k = M(\tau - M\tau)^k = \sum_{-\infty}^{\infty} (x_i - M\tau)^k\, p(\tau = x_i)$$

ifodaga, uzluksiz tasodifiy miqdorlar uchun

$$\nu_k = M(\tau - M\tau)^k = \int_{-\infty}^{\infty} (x - M\tau)^k\, p_\tau(x)\, dx$$

ifodaga aytiladi.

Agar $M\tau = 0$ bo'lsa, u holda $\nu_k = a_n$ ya'ni markaziy moment boshlang'ich momentga teng bo'ladi.

Demak, a_1 moment τ-tasodifiy miqdorning matematik kutilmasi, ν_1 moment esa τ tasodifiy miqdorning dispersiyasi ekan.

9.12-ta'rif. τ tasodifiy miqdorning k tartibli markaziy absolyut momenti deb, diskret tasodifiy miqdorlar uchun

$$\mu_k = M|\tau - M\tau|^k = \sum_{i=-\infty}^{\infty} (i - M\tau)^k \, p(\tau = i)$$

ifodaga, uzluksiz tasodifiy miqdorlar uchun

$$\mu_k = M|\tau - M\tau|^k = \int_{-\infty}^{\infty} |x - M\tau|^k \, p_\tau(x)\, dx$$

ifodaga aytiladi.

Xususan, agar $M\tau = 0$ bo'lsa, k –tartibli markaziy absolyut moment k –tartibli boshlang' ich absolyut moment bilan usitma-ust tushadi.

Koshi-Bunyakovskiy tengsizlikligi

Ikkinchi tartibli momentiga ega bo'lgan ixtiyoriy τ va ξ tasodifiy miqdorlar uchun quyidagi tengsizlik o'rinli:

$$M|\xi \cdot \tau| \leq \sqrt{M\xi^2} \cdot \sqrt{M\tau^2} \qquad (9.12)$$

Isbot. Ma'lumki, $|\xi \cdot \tau| \leq \frac{1}{2}(\xi^2 + \tau^2)$ hamda $M\xi^2$ va $M\tau^2$ momentlarning chekliligidan $M|\xi \cdot \tau| < \infty$ kelib chiqadi. x va y o'zgaruvchilarga bog'liq bo'lgan musbat aniqlangan ushbu

$$M(x|\xi| + y|\tau|)^2 = x^2 M\xi^2 + 2xyM(|\xi| \cdot |\tau|) + y^2 M\tau^2$$

kvadratik formulaning diskriminatini

$$(2M(\xi\tau))^2 - 4M\xi^2 \cdot M\tau^2 \leq 0,$$

bundan esa (9.12) tengsizliklikning o'rinli ekanligi kelib chiqadi.

Agar $\tau = 1$ bo'lsa, (9.12) dan

$$M|\xi| \leq \sqrt{M\xi^2}.$$

Shuningdek, (9.12) munosabatdan ushbu muhim

$$\sqrt{M(\xi + \tau)^2} \leq \sqrt{M\xi^2} + \sqrt{M\tau^2}.$$

tengsizlik kelib chiqadi.

Darhaqiqat,

$$M(\xi + \tau)^2 = M\xi^2 + 2M\xi M\tau + M\tau^2 \leq \left(\sqrt{M\xi^2} + \sqrt{M\tau^2}\right)^2.$$

Yensen tengsizligi. Agar $M|\tau| < \infty$ va $g(x)$ funksiya botiq bo'lsa, u holda

$$g\big(M(\tau)\big) \le Mg(\tau).$$

Isbot. Agar $g(x)$ funksiya botiq bo'lsa, u holda har bir y uchun shunday $g_1(y)$ topiladiki,

$$g(x) \ge g(y) + (x - y)g_1(y)$$

bo'ladi. Agar $x = \tau, y = M\tau$ desak va bu tengsizlikning har ikki tomonidan matematik kutilma olsak,

$$Mg(\tau) \ge g(M\tau)$$

kelib chiqadi.

Lyapunov tengsizligi. Agar $0 < \varepsilon < t$ bo'lsa, u holda

$$(M|\tau|^{\varepsilon})^{\frac{1}{\varepsilon}} \le (M|\tau|^{t})^{\frac{1}{\varepsilon}}.$$

tengsizlik o'rinli bo'ladi.

Gyolder tengsizligi. Aytaylik, $\xi \ge 0$, $\tau \ge 0$ va p, q sonlar uchun $p > 1, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ munosabatlar o'rinli bo'lsin. Agar $M\tau^q < \infty, M\xi^p < \infty$ bo'lsa, u holda

$$M\xi\tau \le (M\xi^p)^{\frac{1}{p}}\, (M\tau^q)^{\frac{1}{q}}$$

tengsizlik o'rinli bo'ladi.

Ehtimollar nazariyasi va uning tatbiqlarida tasodifiy miqdorlarning quyidagi xarakteristikalari ham kerak bo'ladi.

9.13-ta'rif. Uzluksiz taqsimlangan τ tasodifiy miqdorning modasi deb, $p(x)$ zichlik funksiya maksimumga erishadigan nuqtalarga aytiladi va Mo kabi belgilanadi. Agar aytilayotgan nuqtalar bitta bo'lsa, $p(x)$ funksiyani uni modal, ikkita bo'lsa, bimodal, agar bir nechta bo'lsa, polimodal deyiladi. Agar $p(x)$ zichlik funksiya bitta ham maksimal qiymatga erishmasa, uni antimodal deyiladi.

9.14-ta'rif. $F(x) = p$ tenglamaning yechimi τ tasodifiy miqdorning p –tartibli kvantili deyiladi. Agar $p = \frac{1}{2}$ bo'lsa, bunday kvantil taqsimotning medianasi deyiladi va $Me = x$ kabi belgilanadi. Demak, $F(x)$ taqsimotning medianasi x argumentning shunday $x = Me$ qiymatiki, uning uchun

$$F(Me - 0) \le 0{,}5 \le F(Me + 0)$$

tengsizlik o'rinli bo'ladi.

Tasodifiy miqdorning matematik kutilmasi mavjud bo‘lmasa, ham uning medianasi doimo mavjud bo‘laveradi.

τ tasodifiy miqdorning variatsiya koeffitsiyenti deb $\nu = \frac{\delta}{m_1}$ ga aytiladi, bu yerda

$$m_1 = \int_{-\infty}^{\infty} x dF(x), \quad \sigma = \sqrt{D\tau}.$$

Variatsiya koeffitsiyenti tasodifiy miqdorning o‘zgaruvchanligini xarakterlaydi hamda protsentlarda ifodalandi. Nosimmetrik tasodifiy miqdorlarni xarakterlash uchun o‘lchamsiz miqdor-asimmetriya koeffitsiyenti tushunchasini kiritiladi. Odatda asimmetriya koeffitsiyenti $m_1 = M\tau$ –ga nisbatan, ya’ni o‘rta qiymatga nisbatan simmetriklikning buzilmaganligini bildiradi. τ tasodifiy miqdorning asimmetriya koeffitsiyenti deb $\gamma_1 = \frac{\nu_3}{\sigma^3}$ ga aytiladi, bunda $\nu_3 = M[(\tau - m_1)^3]$.

τ tasodifiy miqdorning ekssessiya koeffitsiyenti deb

$$\gamma_2 = \frac{\nu_4}{\sigma^4} - 3$$

ga aytiladi, bunda

$$\sigma_4 = M[(\tau - m_1)^4]$$

Normal taqsimot uchun $\frac{\nu_4}{\sigma^4} = 3$, shu sababli normal taqsimotning eksessiya koeffitsiyenti nolga teng.

Agar zichlik funksiya normal qonunning zichlik funksiyasiga nisbatan “tik” va “yuqori cho‘qili” bo‘lsa, ekssessiya koeffitsiyenti musbat, aks holda manfiy bo‘ladi.

9.4. Chebishev tengsizligi

Tasodifiy miqdorning matematik kutilma atrofida tarqoqlik darajasini xarakterlaydigan kattalik, xususan, tasodifiy miqdorning dispersiyasi ekanligi bizga ma’lum. Agar dispersiya kichik son bo‘lsa, u holda tasodifiy miqdor qiymatlari matematik kutilma atrofida zichroq joylashgan bo‘ladi. Bu faktni tasdiqlovchi quyidagi Chebishev tengsizligini keltiramiz.

Chebishev tengsizligi. Agar τ tasodifiy miqdor chekli dispersiyaga ega bo'lsa, u holda $\exists(\varepsilon > 0)$ ushbu tengsizlik o'rinli bo'ladi:

$$P\{|\tau - M\tau| \geq \varepsilon\} \leq \frac{D\tau}{\varepsilon^2}.$$

Isbot. Isbotni birinchi navbatda diskret tipdagi tasodifiy miqdorlar uchun keltiramiz. Ma'lumki, agar τ tasodifiy miqdor $x_1, x_2, \dots, x_k, \dots$ qiymatlarni mos ravishda $P_1, P_2, \dots, P_k, \dots$ ehtimollar bilan qabul qilsa, u holda

$$D\tau = \sum_{i=1}^{\infty} (x_i - M\tau)^2\, Pi \qquad (9.13)$$

bo'ladi. $\bar{A}_i$ bilan $|x_i - M\tau| < \varepsilon$ tengsizlikni qanoatlantiradigan i indekslar to'plamini, A_i bilan esa $|x_i - M\tau| \geq \varepsilon$ tengsizlikni qanoatlantiradigan i indekslar to'plamini belgilaymiz, u holda

$$D\tau \geq \sum_{i \in A_i} (x_i - M\tau)^2\, P_i \geq \varepsilon^{\,2} \sum_{i \in A_i} P_i = \varepsilon^{\,2} P\{|\tau - M\tau| \geq \varepsilon\},$$

bu esa Chebishev tengsizligining o'rinli ekanligini ko'rsatadi.

Quyidagi

$$P\{|\tau - M\tau| \geq \varepsilon\} = \int dF(x) \leq \frac{1}{\varepsilon^{\,2}} \int_{-\infty}^{\infty} (x - M\tau)^2\, dF(x = \frac{D\tau}{\varepsilon^2}$$

ifoda Chebishev tengsizligini $|X - M\tau| \geq \varepsilon$ umumiy holda ham o'rinli ekanligini ko'rsatadi.

9.24-misol. Tajriba Oyning suratini olish yordamida uning diametrini o'lchashdan iborat bo'lsin. Buning uchun Oyni turli vaqtda olingan suratlaridan foydalaniladi. Ma'lumki, turli vaqtda olingan surat atmosfera ta'siri natijasida turli kattalikka ega. Oy diskining haqiqiy qiymatini biror masshtabda a bilan belgilasak, u holda $\tau - a$ ayirma har bir o'lchash natijasi a dan qay darajada farqlanishini belgilaydi. Faraz qilaylik, bog'liqsiz ravishda n –ta o'lchash olib borilgan bo'lsin. Ushbu

$$S_n = \frac{1}{n}\,(\tau_1 + \tau_2 + \dots + \tau_n)$$

Ifodani tuzaylik. Bu yerda τ_i lar o'zaro bog'liq emas va $M\tau_i = a, D\tau_i = \sigma^2$ bo'lsin.

Demak, $MS_n = a, DS_n = \frac{\sigma^2}{n}$. n o'sishi bilan S_n va a orasidagi farq kichrayib berishini ko'ramiz. Shuning uchun a sonni S_n orqali baholash mumkin, ya'ni

$$P\{|S_n - a| \le \varepsilon\} \le \frac{DS_n}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \qquad (9.14)$$

Xususan, $P(|S_n - a| \ge 0{,}1) \le 0{,}02$ tengsizlik o'rinli bo'lish uchun kamida nechta o'lchash o'tkazish kerakligini aniqlaymiz. Buning uchun (9.14) dan foydalanamiz:

$$P(|S_n - a| \ge 0{,}1) \le \frac{\sigma^2}{0{,}01n} \le 0{,}02,$$

bunda $\sigma^2 = 1$ desak, $n \ge 5000$. $\forall (\varepsilon > 0)$ uchun

$$P(\tau \ge \varepsilon) \le \frac{M\tau}{\varepsilon}$$

tengsizlik o'rinlidir. Bu tengsizlikning to'g'riligi quyidagi hisoblashlardan kelib chiqadi.

$$M\tau \ge M(\tau;\ \tau > \varepsilon) \ge \varepsilon M(1; \tau \ge \varepsilon) = \varepsilon P(\tau \ge \varepsilon).$$

Nazorat savollari.

1. Tasodifiy miqdor ta'rifi.
2. Diskret tasodifiy miqdorning ta'rifi, misollar keltiring.
3. Uzluksiz tasodifiy miqdorning ta'rifi, misollar keltiring.
4. Tasodifiy miqdorlarning matematik ko'tilmasi ta'rifi, hisoblash formulasi va misollar keltiring.
5. Tasodifiy miqdorlarning dispersiyasi ta'rifi, hisoblash formulasi va misollar keltiring.
6. Tasodifiy miqdorlarning o'rta kvadratik chetlanishi formulasini yozib bering.

X-BOB. KATTA SONLAR QONUNI, YAQINLASHISH TURLARI

10.1. Katta sonlar qonuni

Biror o'zaro bog'liqmas tajribalar ketma-ketligi o'tkazilgan bo'lsin va har bir tajribada A hodisaning ro'y berish ehtimoli p ga ro'y bermaslik ehtimoli $1-p$ ga teng bo'lsin. A hodisaning k $-$tajribada ro'y berish sonini τ_k desak, u holda

$$\tau_k = \begin{cases} 0, & q = 1 - p, \\ 1, & p. \end{cases}$$

Natijada $\tau_1, \tau_2, \ldots, \tau_k, \ldots$ o'zaro bog'liq bo'lmagan bir xil taqsimlangan tasodifiy miqdorlar ketma-ketligi hosil bo'ladi. Bu tasodifiy miqdorlar uchun $M\tau_k = P, D\tau_k = pq$ ekani ma'lum. Bu tasodifiy miqdorlar $n-$ tasining o'rta arifmetigi A hodisa ro'y berishlarining nisbiy chastotasi $\frac{S_n}{n}$ bo'ladi, bu erda $S_n = \tau_1 + \tau_2 + \cdots + \tau_n$ $n-$ta o'zaro bog'liq bo'lmagan tajribada A hodisaning ro'Y berishlar soni. Ma'lumki, $MS_n = np,\ DS_n = npq$. Bunday sxema uchun quyidagi teorema (Bernulli teoremasi) o'rinli.

10.1-teorema. Ixtiyoriy $\varepsilon > 0$ son uchun:

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) = 0.$$

Isbot. Chebishev tengsizligiga asosan

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \varepsilon\right) \leq \frac{pq}{n\varepsilon^2}.$$

Bu tengsizlikdan $n \to \infty$ da limitga o'tilsa, teoremani isboti kelib chiqadi.

Demak, tajribalar soni etarlicha katta bo'lsa, A hodisa ro'y berishining nisbiy chastotasi A hodisaning ro'y berish ehtimoliga yaqin bo'lar ekan. 10.1-teoremadan shunday xulosa chiqarish mumkin: ayrim shartlar ostida qo'shuvchilar soni etarlicha katta bo'lganda tasodifiy miqdorlar yig'indisi o'zining tasodifiylik xarakterini ma'lum ma'noda "yo'qotar" ekan. Ana shu shartlarni bilish ehtimollar nazariyasida asosiy masalalardan biri hisoblanadi.

Faraz qilaylik, $\tau_1, \tau_2, \dots, \tau_k, \dots$ tasodifiy miqdorlar ketma-ketligi, va $\eta_n = f_n(\tau_1, \tau_2, \dots, \tau_n)$ funksiya berilgan bo'lsin.

10.1-ta'rif. Agar shunday o'zgarmas sonlar ketma – ketligi $\{a_n\}$, $n = 1,2,..$ mavjud bo'lsaki, $\forall\ (\varepsilon > 0)$ uchun $\lim_{n\to\infty} P\{|\xi_n - a_n| < \varepsilon\} = 1$ munosabat o'rinli bo'lsa, u holda $\tau_1, \tau_2, \dots, \tau_n, \dots$ tasodifiy miqdorlar ketma-ketmaligi katta sonlar qonuniga bo'ysunadi deyiladi.

10.2-teorema. (Chebishev teoremasi). Agar $\tau_1, \tau_2, \dots,$ o'zaro bog'liq bo'lmagan tasodifiy miqdorlar bo'lib, ularning dispersiyalari S son bilan tekis chegaralangan bo'lsa, u holda $\forall\ (\varepsilon > 0)$ uchun quyidagi tenglik o'rinli bo'ladi:

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\tau_j - \frac{1}{n}\sum_{j=1}^{n} M\tau_j\right| \geq \varepsilon\right\} = 0,$$

ya'ni $\tau_1, \tau_2, \dots, \tau_n, \dots$ tasodifiy miqdorlar katta sonlar qonuniga bo'ysunadi.

Isbot. 1-ta'rifdan

$$\xi_n = \frac{1}{n}\sum_{j=1}^{n}\tau_j, \quad a_n = \frac{1}{n}\sum_{j=1}^{n} M\tau_j$$

deb olish kerak. Chebishev tengsizligiga muvofiq $\{\tau_j\}$ larning juft-jufti bilan bog'liqsizligini hisobga olgan holda

$$P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\tau_j - \frac{1}{n}\sum_{j=1}^{n} M\tau_j\right| \geq \varepsilon\right\} \leq \frac{D\left(\frac{1}{n}\sum_{j=1}^{n}\tau_j\right)}{\varepsilon^2} =$$

$$= \frac{\sum_{j=1}^{n} D\tau_j}{n^2\varepsilon^2} \qquad (10.1)$$

tengsizlikni yoza olamiz. Agar $D\tau_j$ larning C bilan tekis chegaralanganligini e'tiborga olsak, (10.1) dan

$$P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\tau_j - \frac{1}{n}\sum_{j=1}^{n} M\tau_j\right| \geq \varepsilon\right\} \leq \frac{C}{n\varepsilon^2}$$

tengsizlikka ega bo'lamiz. Bunday esa

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\tau_j - \frac{1}{n}\sum_{j=1}^{n}M\tau_j\right| \geq \varepsilon\right\} = 0$$

Demak, Chebishev teoremasiga ko'ra $\tau_1, \tau_2, \ldots, \tau_n$ tasodifiy miqdorlar o'zaro bog'liqsiz va dispersiyalari tekis chegaralingan bo'lsa, u holda bu tasodifiy miqdorlarning o'rta arifmetigi n o'sishi bilan bu tasodifiy miqdorlar o'rta qiymatlarning o'rta arifmetrigiga istalgancha yaqin bo'lar ekan.

Markov teoremasi. Agar

$$\tau_1, \tau_2, \ldots, \tau_n, \ldots \tag{10.2}$$

Tasodifiy miqdorlar uchun $n \to \infty$ da

$$\frac{1}{n^2}D\left(\sum_{k=1}^{n}\tau_k\right) \to 0 \tag{10.3}$$

munosabat bajarilsa, (10.2) tasodifiy miqdorlar ketma-ketligi katta sonlar qonuniga bo'ysunadi.

Xususan, $\tau_1, \tau_2, \ldots, \tau_k, \ldots$ tasodifiy miqdorlar o'zaro bog'liq bo'lmasa va $D\tau_n < C$ bo'lsa, (10.3) shart bajariladi. Bu esa Chebishev teoremasi Markov teoremasining xususiy holi ekanligini ko'rsatadi.

10.2. Katta sonlar qonunining bajarilishi uchun zaruriy va yetarli shart

A.N.Kolmogorov 1926-yilda o'zaro bog'liq bo'lmagan $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligining katta sonlar qonuniga bo'ysunishi uchun zaruriy va yetarli shartni ko'rsatgan. 1928-yilda esa A.L.Xinchin o'zaro bog'liq bo'lmagan,bir xil taqsimlangan $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi katta sonlar qonuniga bo'ysunishi uchun $M\tau_n$ ning mavjudligi yetarli ekanini isbotladi. Keyinchalik, Chebishev metodidan foydalangan holda ixtiyoriy $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi uchun quyidagi teorema isbotlandi.

10.3-teorema. Ixtiyoriy $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi katta sonlar qonuniga bo'ysunishi uchun $n \to \infty$ da

$$M\frac{\sum_{k=1}^{n}(\tau_k - M\tau_k)^2}{n^2 + (\sum(\tau_k - M\tau_k))^2} \to 0 \tag{10.4}$$

munosabatning o'rinli bo'lishi zarur va yetarlidir.

Isbot. Faraz qilaylik, (10.4) shart bajarilsin.

$$F_n(x) = P\{\xi_n < x\} = P\left\{\frac{1}{n}\sum_{k=1}^{n}(\tau_k - M\tau_k) < x\right\}.$$

belgilashni kiritaylik. U holda

$$P\{|\xi_n| \geq \varepsilon\} = \int_{|x|\geq\varepsilon} dF_n(x) \leq \frac{1+\varepsilon^2}{\varepsilon^2}\int_{|x|\geq\varepsilon}\frac{x^2}{1+x^2}dF_n(x) \leq$$

$$\leq \frac{1+\varepsilon^2}{\varepsilon^2}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}dF_n(x) = \frac{1+\varepsilon^2}{\varepsilon^2}M\frac{\xi_n^2}{1+\xi_n^2}$$

tengsizlikka ega bo'lamiz va bu tengsizlikdan katta sonlar qonuning o'rinligi kelib chiqadi.

10.4-teorema (*Xinchin teoremasi*). Agar $\tau_1, \tau_2, \dots, \tau_k, \dots$ tasodifiy miqdorlar ketma-ketligi o'zaro bog'liq bo'lmagan, bir xil taqsimlangan bo'lib, matematik kutilmalari mavjud bo'lsa, bu tasodifiy miqdorlar ketma-ketligi uchun katta sonlar qonuni o'rinli bo'ladi, ya'ni $M\tau_i = a$ bo'lganda

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{j=1}^{n}\tau_j - a\right| < \varepsilon\right\} = 1.$$

10.3.Yaqinlashish turlari

Tasodifiy miqdorlar ketma-ketligining yaqinlashishi, masala talabiga qarab, turlicha bo'lishi mumkin.

10.2-ta'rif. Agar $\forall\,(\varepsilon > 0)$ uchun $n \to \infty$ da $P\{|\tau_n - \tau| \geq \varepsilon\} \to 0$ bo'lsa, u holda $\tau_1, \tau_2, \dots, \tau_k, \dots$ tasodifiy miqdorlar ketma-ketligi p ehtimol bo'yicha τ tasodifiy miqdorga yaqinlashadi deymiz va

$$\tau_n \xrightarrow{P} \tau$$

kabi belgilaymiz.

Aytaylik, g ixtiyoriy uzluksiz, chegaralangan funksiya bo'lsin. Agar $\tau_n \xrightarrow{P} \tau$ bo'lsa, u holda

$$Mg(\tau_n) \to Mg(\tau). \qquad (10.5')$$

Agar τ_n va τ larning taqsimot funksiyalarini mos ravishda $F_n(x)$ va $F(x)$ deb belgilasak, u holda (1) ni quyidagicha yozamiz:

$$\int_{-\infty}^{\infty} g(x)dF_n(x) \to \int_{-\infty}^{\infty} g(x)dF(x). \qquad (10.5)$$

10.3-ta'rif. Agar $\tau_1, \tau_2, \dots, \tau_n, \dots$ tasodifiy miqdorlar ketma-ketligi uchun

$$P\left\{\omega\colon \lim_{n\to\infty} \tau(\omega) = \tau(\omega)\right\} = 1$$

tenglik o'rinli bo'lsa, u holda $\tau_1, \tau_2, \dots, \tau_n, \dots$. tasodifiy miqdorlar ketma-ketligi τ tasodifiy miqdorga 1 ehtimol bilan yaqinlashadi deymiz, ya'ni bunday yaqinlashish uchun $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)$ munosabatni qanoatlantirmaydigan ω nuqtalarning o'lchovi nolga teng bo'ladi.

Biz bir ehtimol bilan yaqinlashishni $\tau_n \xrightarrow{P(1)} \tau$ kabi belgilaymiz. 1 ehtimol bo'yicha yaqinlashish

$$\lim_{n\to\infty} P\{\omega\colon sup(\ |\tau_n - \tau\ | > \varepsilon)\} = 0$$

ga teng kuchlidir.

10.4-ta'rif. Agar $n \to \infty$ da $M|\tau_n - \tau\ |^r \to 0$ shart bajarilsa, $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi τ –ga o'rtacha r –tartibda yaqinlashadi deymiz. Bu yaqinlashishni $\tau_n \xrightarrow{r} \tau$ kabi belgilaymiz.

Bizga $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi berilgan bo'lib, $F_n(x) = P\{\tau_n < x\}$ bo'lsin.

10.5-ta'rif. Agar $\{F_n(x)\}$ taqsimot funksiyalar ketma-ketligi $n \to \infty$ da $F(x) = P\{\tau < x\}$ ga $F(x)$ taqsimot funksiyaning har bir uzluksizlik nuqtasida yaqinlashsa, u holda $\{\tau_n\}$ tasodifiy miqdorlar ketma-ketligi τ –ga taqsimot bo'yicha yaqinlashadi deyiladi va $\tau_n \xrightarrow{D} \tau$ kabi belgilanadi (bu yerda D inglizcha "distribution"- taqsimot so'zining bosh harfidan olingan).

Nazorat savollari.

1. Bernulli teoremasi.
2. Chebishev teoremasi.
3. Yaqinlashish turlariga ta'rifi.

XI-BOB. XARAKTERISTIK FUNKSIYALAR. MARKAZIY LIMIT TEOREMA

11.1. Xarakteristik funksiya va uning xossalari

$\{\Omega, \mathcal{F}, P\}$ ehtimollik fazosi τ taosdifiy miqdor berilgan bo'lsin.

11.1-ta'rif. Tasodifiy miqdorning xarakteristik funksiyasi deb, haqiqiy o'zgaruvchining ushbu funksiyasiga aytiladi:

$$\varphi_\tau(t) = Me^{it\tau} = \int_{-\infty}^{\infty} e^{it\tau} dF(x) \qquad (11.1)$$

bu yerda t – haqiqiy son, $-\infty < t < \infty, F(x)$ esa τ ning taqsimot funksiyasi. Agar τ tasodifiy miqdorning zichlik funksiyasi $f(x)$ mavjud bo'Lsa, u holda

$$\varphi_\tau(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

bo'ladi, bu esa $f(x)$ funksiya Fure almashtirishning o'zidir. Umuman olganda, $\varphi(t)$ xarakteristik funksiya $F(x)$ taqsimot funksiyaning Fure-Stiltes almashtirishdir.

Ushbu

$$|\varphi_\tau(t)| = |Me^{it}| \leq 1$$

tengsizlikdan ixtiyoriy τ tasodifiy miыdorning xarakteristik funksiyasi mavjudligi kelib chiqadi.

Bog'liq bo'lmagan tasodifiy miqdorlar yig'indisining xossalarini o'rganishda xarakteristik funksiyalar metodi juda qulay metodlardan biri hisoblanadi.

Xarakteristik funksiyaning xossalari

1. Ixtiyoriy τ taosdifiy miqdor uchun $\varphi_\tau(0) = 1$ va barcha t lar uchun $|\varphi_\tau(t)| \leq 1$

2. $\varphi_{a\tau+b}(t) = e^{itb}\varphi_\tau(at)$.

Darhaqiqat,

$$\varphi_{a\tau+b}(t) = Me^{it(a\tau+b)} = e^{itb}Me^{iat\tau} = e^{i+b}\varphi_\tau(at)$$

3. Agar $\tau_1, \tau_2, \dots, \tau_n$ o'zaro bog'liq bo'lmagan taosdifiy miqdorlar bo'lsa, u holda

$S_n = \tau_1 + \dots + \tau_n$ yig'indining xarakteristik funksiyasi

$$\varphi_{S_n}(t) = \varphi_{\tau_1}(t) \cdots \varphi_{\tau_n}(t)$$

ga teng.

Isbot.

$$\varphi_{S_n}(t) = Me^{it(\tau_1,\tau_2+\dots+\tau_n)} = M\ ^{it_1} \cdot e^{it\tau_2} \dots e^{it\tau_n}$$
$$= Me^{ith_1} \cdot Me^{ith_2} \dots Me^{it\ \ n} =$$
$$= \varphi_{\tau_1}(t) \cdot \varphi_{\tau_2}(t) \cdots \varphi_{\tau_n}(t).$$

4. $\varphi_\tau(t)$ xarakteristik funksiya R^1 da tekis uzluksizdir.

Isbot.

$$|\varphi(t+h) - \varphi(t)|$$
$$\leq \int_{-\infty}^{\infty} |e^{ihx} - 1| dF_\tau(x) = \int_{|x|<N} |e^{ihx} - 1| dF_\tau(x) +$$
$$+ \int_{|x|\geq N} |e^{ihx} - 1| dF_\tau(x) = J_1 + J_2.$$

Bu yerda berilgan $\varepsilon > 0$ uchun N ni tanlash hisobiga $J_2 < \frac{\varepsilon}{2}$ qilish mumkin, sungra h – ni shunday tanlashimiz mumkinki, $J_1 < \frac{\varepsilon}{2}$ bo'ladi, natijada.

$$|\varphi(t+h) - \varphi(t)| \leq \frac{\varepsilon}{2} + {}^{\varepsilon}/_{2} = \varepsilon$$

5. $\overline{\varphi_\tau(t)} = \varphi_\tau(-t) = \varphi_{-\tau}(t)$, bu yerda funksiya ustidagi chiziqcha kompleks qo'shmani bildiradi.

Xossaning isboti

$$\overline{\varphi_\tau(t)} = M\overline{e^{it\tau}} = Me^{-i}$$

tenglikdan kelib chiqadi.

6. Poya teoremasi. Faraz qilaylik, $\varphi(t), t \in R$ funksiya quyidagi shartlarni qanoatlantirasin:

$a)$ $\varphi(t) \geq 0, \varphi(0) = 1, t \to \infty$ da $\varphi(t) \to 0$, *b*) $\varphi(t)$ uzluksiz, juft va botiq, u holda $\varphi(t)$ biror taqsimot funksiyaning xarakteristik funksiyasi bo'ladi.

7. Agar $M|\tau|^n < \infty$ bo'lsa, $\varphi_\tau(t)$ xarakteristik funksiya $n-$ tartibli uzluksiz hosilaga ega va quyidagi tengliklar o'rinli:

$$\varphi_\tau^{(k)}(t) = i^k \int_{-\infty}^{\infty} x^n e^{itx} dF_\tau(x), k \le n,$$

$$M\tau^k = \frac{\varphi_\tau^k(0)}{i^k}, \qquad k \le n,$$

$$\varphi_\tau(t) = \sum_{k=0}^{n-1} \frac{(it)^k}{k!} M\tau^k + \frac{(it)^n}{n!}[\varepsilon(t) + M\tau^n],$$

bu yerda $t \to \infty$ da $\varepsilon(t) \to 0$ va barcha $t-$ larda

$$|\varepsilon(t)| \le 3M|\tau|^n.$$

11.1-misol. Agar bir ehtimol bilan $\tau = c = const$ bo'lsa, $\varphi_\tau(t) = e^{itc}$ bo'ladi.

11.2-misol. Faraz qilaylik, τ tasodifiy miqdor uchun $P\{\tau = 0\} = q$, $P\{\tau = 1\} = p$ va $p + q = 1$ bo'lsin, u holda

$$\varphi_\tau(t) = Me^{it\tau} = pe^{it\tau} = pe^{it} + q$$

11.2. Markaziy limit teorema

Masalaning quyilishi

Ko'p hollarda tasodifiy miqdorlar yig'indisining taqsimot qonunlarini aniqlashga to'g'ri keladi. Faraz qilaylik, o'zaro bog'liq bo'lmagan $\tau_1, \tau_2, \dots, \tau_n$ tasodifiy miqdorlarning yig'indisi $S_n = \tau_1 + \tau_2 + \cdots \tau_n$ berilgan bo'lsin va har bir $\tau_i, i = \overline{1,n}$ tasodifiy miqdor "0" yoki "1" qiymatni mos ravishda q va p ehtimollik bilan qabul qilsin. U holda S_n tasodifiy miqdor binomial qonun bo'yicha taqsimlangan tasodifiy miqdor bo'lib, uning matematik kutilishi np dispersiyasi esa npq – ga teng bo'ladi.

S_n tasodifiy miqdor 0,1,2,..., n qiymatlarni qabul qila oladi va demak, n ning ortishi bilan S_n tasodifiy miqdorning qabul qiladigan qiymatlari istalgancha katta son bo'lishi mumkin shuning uchun S_n tasodifiy miqdor o'rniga $\tau_n = \frac{S_n - A_n}{B_n}$ tasodifiy miqdorni ko'rish maqsadga muvofiqdir. Bu ifodada A_n, B_n, lar $n-$ ga bog'liq bo'lgan sonlar. Xususan, A_n va B_n larni

$$A_n = MS_n = np, \ B_n = DS_n = npq$$

ko'rinishda tanlansa, u holda Muavr-Laplasning integral teoremasini quyidagicha bayon etish mumkin: agar $0 < p < 1$ bo'lsa, $n \to \infty$ da ixtiyoriy $a, b \in]-\infty, \infty[$ da

$$P\left\{a \leq \frac{S_n - np}{\sqrt{npq}} < b\right\} \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}} dz \qquad (11.2).$$

munosabat o'rinli bo'ladi.

11.2-ta'rif. $\xi_1, \xi_2, \dots, \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi berilgan bo'lsin. Agar shunday $\{A_n\}, \{B_n\}, B_n > 0$ sonlar ketma-ketligi mavjud bo'lsaki, $n \to \infty$ da

$$P\left\{\frac{\xi_1 + \xi_2 + \dots + \xi_n - A_n}{B_n} < x\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} du$$

munosabat $x \in]-\infty; \infty[$ da bajarilsa, $\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots$ tasodifiy miqdorlar ketma-ketligi uchun markaziy Limit teorema o'rinli deyiladi. Bu holda

$$\frac{\xi_1 + \xi_2 + \dots + \xi_n - A_n}{B_n}$$

tasodifiy miqdor $n \to \infty$ da asimtotik normal taqsimlangan deyiladi.

11.3. Bir xil taqsimlangan bog'liq bo'lmagan tasodifiy miqdorlar ketma-ketligi uchun markaziy Limit teorema

Matematik kutilmalishi a va dispersiyasi σ^2 bo'lgan va ularga bog'liq, bo'lmagan, bir-xil taqsimlangan $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi berilgan bo'lsin. Umumiylikka zarar keltirmasdan, $a = 0, \sigma^2 = 1$ deymiz. Quyidagi tasodifiy miqdorlarni keltiramiz:

$$S_n = \xi_1 + \xi_2 + \dots + \xi_n, \quad \tau_n = \frac{S_n}{\sqrt{n}}$$

11.1-teorema. Yuqori keltirilgan $\{\xi_n\}$ ketma-ketlik uchun $n \to \infty$ da

$$P\{\tau_n < x\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

munosabat ixtiyoriy x $(x \in R)$ da bajariladi.

Isbot. Teoremani isbotlash uchun $n \to \infty$ da τ_n tasodifiy miqdorning xarakteristik funksiyasi $\varphi_{\tau_n}(t)$ ning $e^{-\frac{t^2}{2}}$ –ga intilishini ko'rsatish kifoya. $\xi_1, \xi_2, \dots, \xi_n$ tasodifiy miqdorlar o'zaro bog'liq bo'lmaganligi uchun, xarakteristik funksiyaning xossaligi ko'ra

$$\varphi_{\tau_n}(t) = \varphi^n\left(\frac{1}{\sqrt{n}}\right), \qquad (11.3)$$

bu yerda $\varphi(t) = \varphi_{\xi_1}(t)$.

ξ_i tasodifiy miqdorlar chekli dispersiyaga ega bo'lganligi, xarakteristik funksiyaning xossasiga ko'ra

$$\varphi(t) = 1 - \frac{t^2}{2}(1 + \mathcal{E}(t)),$$

bu yerda $t \to 0$ da $\mathcal{E}(t) \to 0$.

SHuningdek,

$$\varphi(t/\sqrt{n}) = 1 - \frac{t^2}{2n} - \frac{t^2}{2n}\mathcal{E}\left(\frac{t}{\sqrt{n}}\right), \qquad (11.4)$$

$$\varphi^n(t/\sqrt{n}) = \left(1 - \frac{t^2}{2n} - \frac{t^2}{2n}\mathcal{E}\left(\frac{t}{\sqrt{n}}\right)\right)^n \sim \left(1 - \frac{t^2}{2n}\right)^n.$$

Shuning uchun ixtiyoriy fiksirlangan $t \epsilon R$ da (11.3) va (11.4) dan $n \to \infty$ da

$$\varphi_{\tau_n}(t) \to e^{-\frac{t^2}{2}}$$

ekani kelib chiqadi.

11.4. Bog'liq bo'lmagan tasodifiy miqdorlar ketma-ketligi uchun markaziy limit teorema

Bog'liq bo'lmagan $\{\xi_n\}$ tasodifiy miqdorlar ketma-ketligi uchun

$$M\xi_k = a_k, \qquad D\xi_k = \delta_k^2$$

bo'lsin quyidagi belgilarni kiritamiz:

$$A_n = \sum_{k=1}^{n} a_k, \quad B_n^2 = \sum_{k=1}^{n} \delta_k^2, \quad S_n = \xi_1 + \xi_2 + \dots + \xi_n$$

$$\tau_n = \frac{S_n - A_n}{B_n}$$

$$F_k(x) = P(\xi_k < x), \quad L_n(\tau) = = \frac{1}{B_n^2}\sum_{k=1}^{n} \int\limits_{|x-a_k|>\tau B_n} (x - a_k)^2 dF_k(x),$$

$$f_k(t) = Me^{it\xi_k}, \qquad \varphi_n(t) = Me^{it\,n}.$$

11.2-teorema. Ixtiyoriy $\tau > 0$ uchun $n \to \infty$ da

$$L_n(\tau) \to 0 \tag{11.5}$$

bo'Lsa, $\{\xi_n\}$ uchun markaziy Limit teorema o'rinli bo'ladi.

(11.5) shart Lindberg sharti deyiladi. Lindberg shartining bajarilishi ixtiyoriy k da $\frac{1}{B_n}(\xi_k - a_k)$ qo'shiluvchilarning tekis ravishda kichikligini ta'minlaydi. Xaqiqat ham,

$$P(|\xi_k - a_k| > \tau B_n) =$$

$$= \int_{|x-a_k|>\tau B_n} dF_k(x) \le$$

$$\le \frac{1}{(\tau B_n)^2} \int_{|x-a_k|>\tau B_n} (x-a_k)^2\, dF_k(x)$$

ekanligini e'tiborga olinsa,

$$P\left\{\max_{1\le k\le n} |\xi_k - a_k| > \tau B_n\right\} = P\left\{\bigcup_{k=1}^{n} |\xi_k - a_k| > \tau B_n\right\} \le$$

$$\le \sum_{k=1}^{n} P(|\xi_k - a_k| > \tau B_n) \le$$

$$\le \frac{1}{\tau^2 B_n^2} \sum_{k=1}^{n} \int_{|x-a_k|>\tau B_n} (x-a_k)^2 dF_k(x).$$

Agar Lindberg sharti bajarilsa, u holda oxirgi tengsizlikning o'ng tomoni, $\tau > 0$ son har qanday bo'lganda ham, $n \to \infty$ da nolga intiladi.

11.3-teorema. (A.M.Lyapunov). Agar $n \to \infty$ da $\frac{C_n}{B_n^{2+\delta}} \to 0$ shart bajarilsa, $n \to \infty$ da $P(\tau_n < x) \to \Phi(x)$ munosabat $x\epsilon]-\infty;\ \infty[$ da bajariladi.

Isbot. Lyapunov sharti bajarilganda Lindberg sharti o'rinli bo'lishini ko'rsatamiz. $|x - a_k| \ge \tau B_n$ tengsizlikdan ushbu

$$\frac{|x-a_k|}{\tau B_n} \ge 1$$

ni hosil qilamiz, u holda

$$\frac{1}{B_n^2}\sum_{k=1}^{n}\int\limits_{|x-a_k|>\tau B_n}(x-a_k)^2 dF_k(x) \leq$$

$$\leq \frac{1}{B_n^2}\sum_{k=1}^{n}\frac{1}{(\tau B_n)^2}\int\limits_{|x-a_k|>\tau B_n}|x-a_k|^{2+\delta} dF_x(x) \leq$$

$$\leq \frac{C_n}{\tau^\delta B_n^{2+\delta}}\xrightarrow[n\to\infty]{} 0$$

bu esa teoremani isbotlaydi.

Nazorat savollari.

1. Xarakteristik funksiyaga ta'rifi.
2. Xarakteristik funksiyalarning xossalarini misollar bilan tushuntiring.
3. Markaziy limit teoremasi.

XII-BOB. MATEMATIK STATISTIKA ELEMENTLARI

12.1. Matematik statistikaning asosiy masalalari

Statistika so'zi lotincha bo'lib, holat, vaziyat degan ma'noni anglatadi. Statistika tabiatda va jamiyatda bo'ladigan ommaviy hodisalarni o'rganadi. Matematik statistikaning vazifasi statistik ma'lumotlarni to'g' ilash, ularni tahlil qilish va shu asosda ba'zi bir xulosalar chiqarishdan iboratdir.

Endi matematik statistikaning asosiy masalalari bilan tanishib chiqamiz.

1. Faraz qilaylik, τ tasodifiy miqdor ustida $n-$ ta o'zaro bog'Liq bo'lmagan tajriba o'tkazib, $x_1, x_2, \dots, x_n$ qiymatlarni olgan bo'laylik. $x_1, x_2, \dots, x_n$ lar bo'yicha τ tasodifiy miqdorning noma'lum $F(x)$ taqsimot funksiyasini baholash matematik statistikaning vaziflaridan biridir.

2. τ tasodifiy miqdor k-ta noma'lum parametrga bog'liq ma'lum ko'rinishdagi taqsimot funksiyaga ega bo'lsin.

τ tasodifiy miqdor ustida kuzatishlarga asoslanib, bu no'malum parametrlarni baholash matematik statistikaning navbatdagi vazifasidir.

3. Kuzatilayotgan miqdorlarning taqsimot qonunlari ba'zi xarakteristikalarni haqidagi har qanday farazlarni "Statistik gipotezalar" deb ataladi. Faraz qilaylik, ba'zi mulohazalarga asoslanib, τ tasodifiy miqdorning taqsimot funksiyasini $F(x)$ deb hisoblash mumkin bo'Lsin, shu $F(x)$ funksiya haqiqatan ham τ ning taqsimot funksiyasini yoki yo'qmi degan savol statistik gipoteza hisoblanadi.

Umuman, kuzatish natijalari bilan nazariy jihatdan kutiladigan natija orasidagi farq turlicha bo'lishi mumkin. Shu farqni statistik baholash natijalasida u yoki bu gipotezani ma'lum ehtimol bilan qabul qilish mumkin, ya'ni shu farq katta bo'lsa, gipoteza qabul qilinmaydi, aks holda qabul qilinadi, albatta bu farq qanchalik bo'lganda gipotezani qabul qilish mumkinligini masalaning quyilishiga bog'liq bo'ladi.

Bosh to‘plam. Tanlanma to‘plam.

Faraz qilaylik, ma’lum paxta maydonidagi ko‘saklarning o‘rtacha og‘irligini aniqlash masalasini ko‘raylik. Buning uchun o‘sha maydonning to‘rli joylaridan teng miqdordagi ko‘saklarni yig‘ib olib, ularning og‘irliklarini o‘lchash va shu maydondagi ko‘saklarning o‘rtacha og‘irligi to‘g‘risida fikr yuritishi mumkin. Tekshirishning bunday usuli tanlanma uchul deyiladi, yig‘ib olingan kuzatish natijalari tanlanma to‘plam deyiladi. Keltirilgan misola kuzatilgan paxta maydonidagi barcha ko‘saklar to‘plami bosh to‘plam deyiladi. Shunday qilib, tanlanma to‘plam deb tasodifiy ravishda olingan obyektlar to‘plamiga, bosh to‘plam deb esa tanlanma operativ olinadigan obyektlar to‘plamiga aytiladi.

Bosh to‘plam yoki tanlanma to‘plamning hajmi deb bu to‘plamdagi obyektlar soniga aytiladi.

Misol. 1000 ta detaldan tekshirish uchun 100 ta detal olingan bo‘lsa, u holda bosh to‘plam hajmi N=1000, tanlanma hajmi esa n=100 bo‘ladi.

Takror va notakror tanlanma. Reprezentativ tanlanma.

Tanlanmani kuzatishda 2 xil yo‘l tutish mumkin: obyekt tanlanib, uni ustida kuzatish o‘tkazilgandan so‘ng, u bosh to‘plamga qaytarilishi yoki qaytarilmasligi mumkin. Shuning uchun tanlanma takror va notakror bo‘ladi. Takror tanlanma deb olingan obyekt bosh to‘plamga qaytariladi.

Notakror tanlanma deb tanlangan element yana bosh to‘plamga qaytarilmaydigan tanlanmaga aytiladi.

Tanlanmadan ma’lumotlarni etarlicha ishonch bilan fikr yuritish uchun tanlarmaning ob’ektlari bosh to‘plamini to‘g‘ ri tasvirlash zarur ya’ni reprezentativ bo‘lishi kerak (reprezentativ-tasvirlay oladigan).

Tanlash usullari

Praktikada tanlashning turli usullari mavjud bo‘lib, prinsipial jihatdan ikki turga bo‘lish mumkin:

1. Bosh to‘plamni qismlarga ajratishni talab qilmaydigan tanlash:

a) oddiy qaytarilmaydigan tasodifiy tanlash;
b) Odiy qaytariladigan;
2. Bosh to'plamni qismlarga ajratilgandan keyin tanlash:
a) tipik tanlash;
b) mexanik tanlash;
v) seriyali tanlash;

Bosh to'plamdan elementlar bittalab olinganidigan tanlash oddiy tasodifiy tanlash deyiladi. Bu turli usullar bilan amalga oshiriladi. M: N hajmli bosh to'plamdan n-ta obyekt tanlashda quyidagicha yo'l tutiladi. Kartochkalar olib ularni 1-dan N-gacha nomerlanadi. Yaxshilab aralashtirib bitta kartochka olinadi va olingan kartochka bilan bir xil nomerli obyekt tekshiriladi. Keyin kartochka dastaga qaytariladi va protsess n-mart takrorlanadi. Natijada n-ta hajmli oddiy takror tasodifiy tanlanma hosil qilinadi.

Agar olingan kartochkalar qaytarilmasa, u holda tanlanma oddiy notakror tanlanmaga tegishli bo'ladi.

Tipik tanlash deb obyetlar bosh to'plamdan emas, balki uning tipik qismlaridan olinadi.

Mexanik tanlash bosh to'plam tanlanmaga nechta obyekt kirishi lozim bo'lsa, shuncha gruppaga mexanik ravishda ajratiladi va har bir gruppadan bittadan obyekt tanlanadi.

Seriyali tanlash obyektlar bosh to'plamdan bitalab emas, balki seriyali olinadi va ular yalpisiga tekshiriladi.

Tanlashning statistik taqsimoti

Bosh to'plamdan tanlanma olingan. Bunda x_1 qiymat n_1 marta, x_2 qiymat n_2 –marta kuzatilgan va $\sum n_i = n$ bo'lsin. Kuzatilgan x_i qiymatlar, variantlar, variantlarning ortib borishi tartibida yozilgan ketma-ketligi esa variatsion qator deyiladi. Kuzatilar soni chastotalar, ularning tanlanma hajmiga nisbati $\frac{n_i}{n} = w_i$ esa nisbiy chastota deyiladi.

Tanlanmaning 208 variant 208c taqsimoti deb variantlar va ularga mos chastotalar yoki nisbiy chastotalar ro'yxatiga aytiladi.

Shunday qilib taqsimot deyilganda ehtimollar nazariyasida tasodifiy miqdorning mumkin bo'lgan qiymatlari va ularning ehtimolni orasida moslik, matematik statistikada esa kuzatilgan variantlar va ularning chastotalari yoki nisbiy chastotalari orasidagi moslik tushuniladi.

12.1-misol. Hajmi 20 bo'lgan tanlanmaning chastotalari taqsimligini berilgan:

x_i	2	6	12
n_i	3	10	7

Nisbiy chastotalar taqsimotini yozing.

Yechilishi. Nisbiy chastotalarni topamiz.

$$w_1 = \frac{3}{20} = 0{,}15, \qquad w_2 = \frac{10}{20} = 0{,}5, \qquad w_3 = \frac{7}{20} = 0{,}35$$

Nisbiy chastotalar taqsimotini yozamiz.

x_i	2	6	12
w_i	0,15	0,5	0,35

Kontrol qilish: $0{,}15 + 0{,}5 + 0{,}35 = 1$.

12.2. Taqsimotning empirik funksiyasi

$X -$ son belgi chastotalarning statistik taqsimoti ma'lum bo'lsin. $n_x - x -$dan kichik qiymati kuzatilgan kuzatishlar soni, $n -$ kuzatishlarning umumiy soni (tanlanma hajmi).

$X < x$ hodisaning nisbiy chastotasi $\frac{n_x}{n}$ – 209 variant. Agar x o'zgaradigan bo'lsa, u holda nisbiy chastota ham o'zgaradi, ya'ni $\frac{n_x}{n}$ nisbiy chastota x –ning funksiyasidir. Bu funksiya emperik (tajriba yo'li) yo'l bilan topiladigan bo'lgani uchun u empirik funksiyasi deyiladi.

Taqsimotning empirik funksiyasi (tanlanmaning taqsimot fuknsiyasi deb har bir x qiymati uchun X x hodisaning ehtimolini aniqlaydigan $F(x)$ funksiyasiga aytiladi va

$$F(x) = \frac{n_x}{n}$$

bu yerda $n_x - x -$ dan kichik variantlar soni; $n -$ tanlanma hajmi.

Bosh to'plam taqsimotning $F(x)$ integral funksiyasini, tanlanma taqsimotining emperik funksiyasidan farq qilib taqsimotning nazariy funksiyasi deyiladi. Emperik va nazariy funksiyalar orasidagi farq shunday $F(x)$ nazariy funksiya $X < x$

hodisa ehtimolini, $F(x)-$ emperik funksiya esa shu hodisaning o'zining nisbiy chastotasini aniqlaydi.

$F(x)$ va $F(x)$ sonlar bir-biridan ham farq qiladi. Shuning uchun $F(x)$ funksiya $F(x)$ning barcha xossalariga ega.

1) Emperik funksiyaning qiymatlari , $[0,\ 1]-$ ga tegishli;

2) $F(x)$ ning kamaymaydigan funksiya;

3) Agar x_1- eng kichik variantga bo'lsa, u holda $x \le x_1$ da $F(x)=0$, x_k- eng katta variant bo'lsa, u holda $x \le x_k$ da $F(x)=1$

12.2-misol. Tanlanmaning quyidagi berilgan taqsimoti emperik funksiyasini tuzing.

Variantlar	x_i	2	6	10
chastotalar	n_i	12	18	30

Yechilishi. Tanlanma hajmi: $n=12+18+30=60$ eng kichik variantga 2-ga teng, demak,

$$x \le 2 - \text{da} \quad F(x)=0$$

$x<6$ qiymat, xususan $x_1=2$ qiymat 12-marta kuzatilgan, demak,

$$2<x<6 \ \text{ da } F(x)=\frac{12}{60}=0{,}2.$$

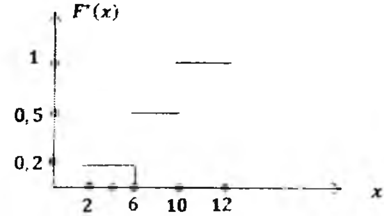
$x<10$ qiymatlar, jumladan, $x_1=2$ va $x_2=6$ qiymatlar 12+18=30 marta kuzatilgan; demak,

$$6<x<10 \ \text{ da } F(x)=\frac{30}{60}=0{,}5.$$

$x=10$ eng katta variantga bo'lgani uchun $x>10$ $F(x)=1$.

Izlanayotgan emperik funksiya:

$$F(x)=\begin{cases} x \le 2 \ da & 0 \\ 2<x\le 6 \ da & 0{,}2 \\ 6<x\le 10 \ da & 0{,}5 \\ x>0 \ da & 1 \end{cases}$$



12.3. Poligon va gistogramma

Chastotalar poligoni deb, kesmalari $(x_1, n_1), (x_2, n_2), \dots$ Nuqtalarni tutashtiruvchi siniq chiziqqa aytiladi.

Nisbiy chastotalar poligoni deb kesmalari $(x_1, w_1), (x_2, w_2), \dots$ nuqtalarni tutashtiruvchi siniq chiziqqa aytiladi.

Chastotalar gistogrammasi deb asoslari n uzunlikdagi intervallar, balandliklari esa $\frac{n_i}{n}$ nisbatlarga (chastota zichligi) teng bo‘lgan to‘g‘ri to‘rtburchaklardan iborat bo‘lgan pog‘onali figuraga aytiladi.

Nisbiy chastotalari gistogrammasi deb asoslari n uzunlikdagi intervallar, balandliklari $\frac{w_i}{n}$ nisbatga aytiladi.

Biror τ tasodifiy miqdor ustida n –marta kuzatish o‘tkazib

$$x_1, x_2, \dots, x_n \tag{12.1}$$

natijalar olingan bo‘lsin.

(12.1) tanlanmaning o‘rta arifmetik qiymati deb,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

ga aytamiz.

Agar τ tasodifiy miqdor ustida olib borilgan kuzatish natijalari $x_1, x_2, \dots x_n$ mos ravishda $n_1, \dots, n_n$ marta takrorlansa u holda o‘rta arifmetik

$$\bar{x} = \frac{\sum_{i=n}^{n} n_i x_i}{\sum_{i}^{n} n_i}$$

(12.1) tanlanmaning dispersiyasi deb,

$$\delta^2 = \frac{\sum_{i=1}^{k} n_i (x_i - \bar{x})^2}{\sum_{i=1}^{n} n_i}$$

ifodaga aytiladi.

$$\delta = \sqrt{\frac{\sum_{i=1}^{k} n_i (x_i - \bar{x})^2}{\sum_{i=1}^{k} n_i}}$$

o‘rtacha kvadratik xato (o‘rtacha kvadratik) deyiladi.

Bosh tanlanmaning dispersiyasini va o'rtacha kvadratik xatosini kiritish mumkin:

$$\delta_B^2 = \frac{\sum_{i=1}^{k} n_i (x_i - \bar{x})^2}{\sum_{i=1}^{k} N_i} = \frac{\sum_{i=1}^{n} n_i (x_i - \bar{x})^2}{N};$$

$$\delta_B = \sqrt{\frac{\sum_{i=1}^{n} n_i (x_i - \bar{x})^2}{N}}$$

Eng katta chastotaga ega bo'lgan variant moda deyiladi va μ_0 –belgilanganadi. Variatsion qatori variantlar soni teng bo'lgan ikki qismga ajratadigan 212 variant variatsion qatorning medianasi deyiladi va m_e ko'rinishda belgilanadi.

Agar variantlar soni toq $n = 2k + 1$ bo'lsa, u holda $m_e = x_{k+1}$, agar juft bo'lsa, ya'ni $n = 2k$

$$m_e = \frac{x_k + x_{k+1}}{2}.$$

Variatsiya koeffitsiyenti deb

$$v = \frac{\delta}{\bar{x}} \cdot 100\%$$

Taqsimotning asimmetriya (qiyshanganlik) koeffitsiyenti deb

$$A_s = \frac{\frac{1}{n}\sum_{i=1}^{n} n_i (x_i - \bar{x})^3}{\delta^3}.$$

Taqsimotning ekssessi deb

$$e_k = \frac{\frac{1}{n}\sum_{i=1}^{k} n_i (x_i - \bar{x})^4}{\delta^4} - 3.$$

Nazorat savollari.

1. Bosh va tanlanma to'plam ta'rifi.
2. Takror va notakror tanlanma deb nimaga aytiladi.
3. Tanlash usullari.
4. Tanlashning statistik taqsimoti deb nimaga aytiladi.
5. Taqsimotning emperik funksiyasi qanday tuziladi.
6. Polygon va gistagramma deb nimaga aytiladi.

12.4. Statistik baho. Statistik bahoga qo'yiladigan talablar.

Statistik baholash nazariyasi masalaning qo'yilishi nuqtai nazaridan parametrik va noparametrik hollarga bo'linadi.

Agar bosh to'plamning miqdoriy belgisini o'rganish talab etilgan bo'lsa, bu belgining taqsimotini aniqlaydigan parametrlarni baholash masalasi yuzaga keladi. Masalan, o'rganilayotgan belgi bosh to'plamda normal taqsimlanganligi oldindan ma'lum bo'lsa, u holda matematik kutilmani va o'rtacha kvadratik chetlanishni baholash (taqribiy hisoblash) zarur, chunki bu ikki parametr normal taqsimotni to'liq aniqlaydi.

Odatda tanlamadagi ma'lumotlargina, masalan, miqdoriy belgining o'zaro erkli deb faraz qilinuvchi n ta kuzatuv natijasida olingan $x_1, x_2, \dots, x_n$ qiymatlari ixtiyorda bo'ladi. Baholanayotgan belgi xuddi shu ma'lumotlar orqali ifodalanadi. $x_1, x_2, \dots, x_n$ larni erkli $X_1, X_2, \dots, X_n$ tasodifiy miqdorlar deb qarab, nazariy taqsimot noma'lum parametrining statistik bahosini topish kuzatilayotgan tasodifiy miqdorlarning baholanayotgan parametr taqribiy qiymatini beruvchi funksiyasini topishga teng kuchlidir deyish mumkin. Masalan, normal taqsimotning matematik kutilmasini baholash uchun belgining kuzatiladigan qiymatlarining o'rta arifmetik qiymati bo'ladigan $\overline{X} = (X_1 + X_2 + \ldots + X_n)/n$ funksiya xizmat qiladi.

SHunday qilib, nazariy taqsimot *noma'lum θ parametrining statistik bahosi* deb kuzatiladigan tasodifiy miqdorlarning ma'lum statistik ma'noda shu parametr haqiqiy qiymatiga yaqin $\overline{\theta} = \overline{\theta}(n) = \overline{\theta}(X_1, X_2, \ldots, X_n)$ funksiyasiga aytiladi.

Statistik bahoning baholanayotgan parametr haqiqiy qiymatiga yaqinligini aniqlaydigan eng muhim xossalari siljimaganlik, asoslilik va effektivlik xossalaridir.

$\overline{\theta}$ nazariy taqsimotning noma'lum θ parametrining statistik bahosi bo'lsin. Bosh to'plamdan ko'p marotalab n hajmli tanlanmalar olib, umuman olganda, bir-biridan farq qiluvchi $\overline{\theta}_1$, $\overline{\theta}_2$, ... , $\overline{\theta}_k$ baholarni olish mumkin. SHunday qilib, $\overline{\theta}$ bahoni tasodifiy miqdor sifatida, $\bar{\theta}_1$, $\bar{\theta}_2$, ... , $\bar{\theta}_k$ sonlarni esa uning mumkin bo'lgan qiymatlari sifatida qarash mumkin.

Agar $\bar{\theta}$ baho θ ning taqribiy qiymatini ortiri bilan bersa, u holda tanlanmadagi ma'lumotlar bo'yicha topilgan har bir $\bar{\theta}_i$ ($i=1,2,\ldots,k$) son θ ning haqiqiy qiymatidan katta bo'ladi. Bu holda $\bar{\theta}$ tasodifiy miqdorning matematik kutilmasi (o'rtacha qiymati) ham θ dan katta, ya'ni $M(\bar{\theta}) > \theta$ bo'lishi ravshan. Agar $\bar{\theta}$ bahoni kami bilan bersa, u holda $M(\bar{\theta}) < \theta$ bo'lishi muqarrar.

Bu erdan matematik kutilmasi baholanayotgan parametrga teng bo'lmagan statistik bahodan foydalanish o'lchashlar natijalarini tayinli bitta tomonga buzib ko'rsatuvchi tasodifiy bo'lmagan xatolar bo'lmish tizimli xatolarga olib kelishi ko'rinib turibdi. SHu sababga ko'ra, $\bar{\theta}$ baho matematik kutilmasining baholanayotgan parametrga tengligi $\bar{\theta}$ ning ba'zi qiymatlari θ dan katta, boshqalari esa kichik ekanligi tufayli xatolarni yo'qotmasa ham, lekin tizimli xatolarga yo'l qo'yilmasligini kafolatlaydi, chunki har xil ishorali xatolar deyarli teng miqdorda uchraydi.

Agar $\bar{\theta}$ statistik bahoning matematik kutilmasi baholanayotgan θ parametrga ixtiyoriy hajmdagi tanlanmada teng, ya'ni

$$M(\bar{\theta}) = \theta \qquad (12.2)$$

bo'lsa, bunday baho ***siljimagan baho*** deb ataladi.

Siljigan baho deb matematik kutilmasi baholanayotgan parametrga teng bo'lmagan bahoga aytiladi.

Biroq siljimagan baho baholanayotgan parametrga yaxshi yaqinlashishni har doim ham beravermaydi. Haqiqatan, $\bar{\theta}$ ning mumkin bo'lgan qiymatlari uning o'rta qiymati atrofida ancha tarqoq bo'lishi, ya'ni $D(\bar{\theta})$ dispersiya anchagina katta bo'lishi mumkin. Bunday holda bitta tanlanma ma'lumotlari bo'yicha topilgan baho $\bar{\theta}$ ning o'rta qiymatidan va, demak, baholanayotgan θ parametrning o'zidan ham ancha uzoqlashgan bo'lishi mumkin. Agar $D(\bar{\theta})$ dispersiyaning kichik bo'lishi talab etilsa, u holda katta xatoga yo'l qo'yishning imkoniyati yo'q bo'ladi.

Agar statistik baho tanlanmaning berilgan n hajmida eng kichik mumkin bo'lgan dispersiyaga ega bo'lsa, u holda bunday baho *effektiv baho* deb ataladi.

Agar $\bar{\theta}$ statistik baho baholanayotgan θ parametrga ehtimollik bo'yicha yaqinlashsa, ya'ni ixtiyoriy $\varepsilon > 0$ uchun

$$n \to \infty \text{ da } P\left(\left|\bar{\theta}(n) - \theta\right| \leq \varepsilon\right) \to 1 \qquad (12.3)$$

bo'lsa, u holda bunday baho *asosli baho* deb ataladi. Masalan, agar siljimagan bahoning dispersiyasi $n \to \infty$ da nolga intilsa, u holda bunday baho asosli baho ham bo'ladi.

Bosh to'plam X miqdoriy belgiga nisbatan o'rganilayotgan bo'lsin.

$\bar{x}_B$ *bosh o'rtacha qiymat* deb bosh to'plam belgisi qiymatlarining o'rta arifmetik qiymatiga aytiladi.

Agar N hajmli bosh to'plam belgisining barcha x_1, x_2, ... , x_N qiymatlari turlicha bo'lsa, u holda bosh o'rtacha qiymat

$$\bar{x}_B = (x_1 + x_2 + \ldots + x_N)/N \qquad (12.4)$$

ga teng bo'ladi.

Belgining x_1, x_2, ... , x_k qiymatlari mos ravishda N_1, N_2, ... , N_k chastotalarga ega va bunda $N_1 + N_2 + \ldots + N_k = N$ bo'lgan taqdirda esa bosh o'rtacha qiymat

$$\bar{x}_B = (x_1N_1 + x_2N_2 + \ldots + x_kN_k)/N \qquad (12.5)$$

ga teng bo'ladi.

Agar bosh to'plamning tekshirilayotgan X belgisi tasodifiy miqdor deb qaralsa hamda (12.5) formulani (9.1) formula bilan solishtirilsa, u holda belgining matematik kutilmasi shu belgining bosh o'rtacha qiymatiga teng degan xulosaga kelish mumkin:

$$\bar{x}_B = M(X). \qquad (12.6)$$

Endi bosh to'plamni X miqdoriy belgiga nisbatan o'rganish uchun n hajmli tanlanma olingan bo'lsin.

$\bar{x}_T$ *o'rtacha tanlanma qiymat* deb tanlanma to'plam belgisining kuzatilayotgan qiymatlarining o'rta arifmetik qiymatiga aytiladi.

Agar n hajmli tanlanma belgisining barcha $x_1, x_2, \ldots, x_n$ qiymatlari turlicha bo'lsa, u holda o'rtacha tanlanma qiymat

$$\bar{x}_T = (x_1 + x_2 + \ldots + x_n)/n \qquad (12.7)$$

ga teng bo'ladi.

Belgining $x_1, x_2, \ldots, x_k$ qiymatlari mos ravishda $n_1, n_2, \ldots,$ n_k chastotalarga ega va bunda $n_1 + n_2 + \ldots + n_k = n$ bo'lgan taqdirda esa o'rtacha tanlanma qiymat

$$\bar{x}_T = (x_1 n_1 + x_2 n_2 + \ldots + x_k n_k)/n \qquad (12.8)$$

ga yoki

$$\bar{x}_T = \left(\sum_{i=1}^{k} x_i n_i\right)\Big/ n \qquad (12.9)$$

teng bo'ladi.

O'rtacha tanlanma qiymat bosh o'rtacha qiymatning siljimagan bahosi ekan degan fikrga ishonch hosil qilaylik, ya'ni $\bar{x}_T$ ning matematik kutilmasi $\bar{x}_B$ ga teng ekanligini ko'rsatamiz. $\bar{x}_T$ ni tasodifiy miqdor va $x_1, x_2, \ldots, x_n$ larni erkli, bir xil taqsimlangan tasodifiy miqdorlar sifatida qaraymiz. Bu tasodifiy miqdorlar bir xil taqsimlangan bo'lgani uchun ular bir xil sonli tavsiflarga, xususan, bosh to'plam X belgisining matematik kutilmasiga teng bo'lgan bir xil matematik kutilmaga ega.

Tasodifiy miqdor matematik ko'tilmasining xossalaridan hamda (12.6) va (12.7) formulalardan foydalanib,

$$M(\bar{x}_T) = \bar{x}_B \qquad (12.10)$$

ni olamiz.

Chebishev teoremasidan foydalanib, o'rtacha tanlanma qiymat bosh o'rtacha qiymatning asosli bahosi ham ekanligini osongina ko'rsatish mumkin.

Bosh va tanlanma to'plamlar miqdoriy belgilari qiymatlarining o'zlarining o'rtacha qiymatlari atrofidagi tarqoqligini tavsiflash uchun jamlanma tavsiflar — mos ravishda bosh va tanlanma dispersiyalar hamda o‘rtacha kvadratik chetlanishlar kiritiladi.

D_B *bosh dispersiya* deb bosh to‘plam belgisi qiymatlarining ularning o‘rtacha qiymati $\bar{x}_B$ dan chetlanishlari kvadratlarining o‘rta arifmetik qiymatiga aytiladi.

Agar N hajmli bosh to‘plam belgisining barcha $x_1, x_2, \dots, x_N$ qiymatlari turlicha bo‘lsa, u holda bosh dispersiya

$$D_B = \left(\sum_{i=1}^{N} (x_i - \bar{x}_B)^2 \right) \Big/ N \qquad (12.11)$$

ga teng bo‘ladi.

Belgining $x_1, x_2, \dots, x_k$ qiymatlari mos ravishda $N_1, N_2, \dots, N_k$ chastotalarga ega va bunda $N_1 + N_2 + \dots + N_k = N$ bo‘lgan taqdirda esa bosh dispersiya

$$D_B = \left(\sum_{i=1}^{k} (x_i - \bar{x}_B)^2 N_i \right) \Big/ N \qquad (12.12)$$

ga teng bo‘ladi.

Bosh o‘rtacha kvadratik chetlanish deb bosh dispersiyadan olingan kvadrat ildizga aytiladi:

$$\sigma_B = \sqrt{D_B}. \qquad (12.13)$$

12.3-misol. Bosh to‘plam

x_i	2	4	5	6
N_i	8	9	10	3

taqsimot jadvali bilan berilgan. Bosh dispersiya va bosh o‘rtacha kvadratik chetlanish topilsin.

Yechilishi. Bosh o‘rtacha qiymatni topamiz:

$$\bar{x}_B = \frac{2 \cdot 8 + 4 \cdot 9 + 5 \cdot 10 + 6 \cdot 3}{8 + 9 + 10 + 3} = \frac{120}{30} = 4.$$

Bosh dispersiyani topamiz:

$$D_B = \frac{(2-4)^2 \cdot 8 + (4-4)^2 \cdot 9 + (5-4)^2 \cdot 10 + (6-4)^2 \cdot 3}{30} = \frac{54}{30} = 1{,}8.$$

Bosh o‘rtacha kvadratik chetlanishni topamiz:

$$\sigma_B = \sqrt{D_B} = \sqrt{1,8} \approx 1,34.$$

D_T *tanlanma dispersiya* deb tanlanma to‘plam belgisining kuzatiladigan qiymatlarining ularning o‘rtacha qiymati $\bar{x}_T$ dan chetlanishlari kvadratlarining o‘rta arifmetik qiymatiga aytiladi.

Agar n hajmli tanlanma belgisining barcha x_1, x_2, ... , x_n qiymatlari turlicha bo‘lsa, u holda tanlanma dispersiya

$$D_T = \left(\sum_{i=1}^{n} (x_i - \bar{x}_T)^2 \right) \Big/ n \qquad (12.14)$$

ga teng bo‘ladi.

Belgining x_1, x_2, ... , x_k qiymatlari mos ravishda n_1, n_2, ... , n_k chastotalarga ega va bunda $n_1 + n_2 + \ldots + n_k = n$ bo‘lgan taqdirda esa tanlanma dispersiya

$$D_T = \left(\sum_{i=1}^{k} (x_i - \bar{x}_T)^2 n_i \right) \Big/ n \qquad (12.15)$$

teng bo‘ladi.

Tanlanma o‘rtacha kvadratik chetlanish deb tanlanma dispersiyadan olingan kvadrat ildizga aytiladi:

$$\sigma_T = \sqrt{D_T}. \qquad (12.16)$$

12.4-misol. Tanlanma to‘plam

x_i	1	2	3	4
N_i	20	15	10	5

taqsimot jadvali bilan berilgan. Tanlanma dispersiya va tanlanma o‘rtacha kvadratik chetlanish topilsin.

Yechilishi. O‘rtacha tanlanma qiymatni topamiz:

$$\bar{x}_T = \frac{1 \cdot 20 + 2 \cdot 15 + 3 \cdot 10 + 4 \cdot 5}{20 + 15 + 10 + 5} = \frac{100}{50} = 2.$$

Tanlanma dispersiyani topamiz:

$$D_T = \frac{(1-2)^2 \cdot 20 + (2-2)^2 \cdot 15 + (3-2)^2 \cdot 10 + (4-2)^2 \cdot 5}{50} = \frac{50}{50} = 1.$$

Tanlanma o'rtacha kvadratik chetlanishni topamiz:

$$\sigma_T = \sqrt{D_T} = \sqrt{1} = 1.$$

Dispersiyalarni

$$D_Б = \left(\sum_{i=1}^{N} x_i^2\right) \Big/ N - (\bar{x}_Б)^2, \qquad (12.17)$$

$$D_Б = \left(\sum_{i=1}^{k} x_i^2 N_i\right) \Big/ N - (\bar{x}_Б)^2, \qquad (12.18)$$

$$D_T = \left(\sum_{i=1}^{n} x_i^2\right) \Big/ n - (\bar{x}_T)^2 \qquad (12.19)$$

va

$$D_T = \left(\sum_{i=1}^{k} x_i^2 n_i\right) \Big/ n - (\bar{x}_T)^2 \qquad (12.20)$$

formulalardan foydalanib hisoblash qulayroq bo'ladi.

Endi tanlanmadagi ma'lumotlar bo'yicha noma'lum D_B bosh dispersiyani baholash talab etilgan bo'lsin. D_T tanlanma dispersiya D_B ning siljigan bahosi bo'ladi, chunki

$$M(D_T) = \frac{n-1}{n} D_B. \qquad (12.21)$$

Bosh dispersiyaning bahosi sifatida D_T ni $n/(n-1)$ kasrga ko'paytirish natijasida hosil qilingan s^2 *tuzatilgan dispersiya* olingan taqdirda esa u bosh dispersiyaning siljimagan bahosi bo'ladi. Haqiqatan, (12.21) ni hisobga olgan holda

$$s^2 = \frac{n}{n-1} D_T = \frac{n}{n-1} \frac{\left(\sum_{i=1}^{k} (x_i - \bar{x}_T)^2 n_i\right)}{n} = \left(\sum_{i=1}^{k} (x_i - \bar{x}_T)^2 n_i\right) \Big/ (n-1)$$

va

$$M(s^2) = M\left(\frac{n}{n-1} D_B\right) = \frac{n}{n-1} M(D_B) = \frac{n}{n-1} \cdot \frac{n-1}{n} D_B = D_B$$

larni hosil qilamiz.

Nazorat savollari.

1. Noma'lum parametrning statistik bahosi deb nimaga aytiladi va u qanday muhim xossalarga ega bo'lishi mumkin.
2. Siljimagan baho nima va uning kiritilishi nima bilan asoslanadi.
3. Effektiv baho nima va uning kiritilishining zaruriyati nimada.
4. Siljigan baho va asosli baho deb nimaga aytiladi.
5. Bosh o'rtacha qiymat nima va u qaysi formulalar bo'yicha hisoblanadi.
6. O'rtacha tanlanma qiymat deb nimaga aytiladi va u qaysi formulalar bo'yicha hisoblanadi.
7. Bosh dispersiya nima va u qaysi formulalar bo'yicha hisoblanadi.
8. Bosh o'rtacha kvadratik chetlanish va tanlanma o'rtacha kvadratik chetlanish nima, ular hamda bosh va tanlanma dispersiyalar nima uchun kiritiladi.

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