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ODDIY DIFFERENSIAL TENGLAMALARDAN MISOL VA MASALALAR TO‘PLAMI

*Oliy texnika o‘quv yurtlari talabalari uchun
o‘quv qo‘llanma*

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Qo'llanmada oddiy differensial tenglamalar bo'yicha qisqacha nazariy ma'lumotlar va tipik masalalarning yechimlari keltirilgan. Bundan tashqari, mustaqil yechish uchun ham masalalar berilgan. Qo'llanma Oliy texnika o'quv yurtlari uchun oddiy differensial tenglamalar bo'limi bo'yicha dasturni to'la qamrab olgan.

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SO‘ZBOSHI

O‘zbek tiliga davlat tili maqomi berilishi munosabati bilan oliy o‘quv yurtlarida o‘zbek tilidagi o‘quv adabiyotlarining yetishmovchiligi sezilib qoldi. Shu munosabat bilan darslik va o‘quv qo‘llanmalar yaratishga ehtiyoj paydo bo‘ldi.

«Ta’lim to‘g‘risida»gi Qonunning va yangi davlat ta’lim standartlarining qabul qilinishi darslik va o‘quv qo‘llanmalarga yangi talablarni vujudga keltirdi.

Ushbu o‘quv qo‘llanma oddiy differensial tenglamalar mavzulari bo‘yicha amaliy mashg‘ulot darslari uchun mo‘ljallangan. Kitob uch bo‘limdan iborat bo‘lib, I.A. Gafarov tomonidan kitobning kirish qismi va birinchi tartibli differensial tenglamalarga bag‘ishlangan birinchi bo‘limi yozilgan. Ikkinchi bo‘lim Y.P. Oppoqov tomonidan yozilgan bo‘lib, yuqori tartibli tenglamalarni o‘z ichiga oladi. N.Turgunov tomonidan yozilgan uchinchi bo‘limda differensial tenglamalarning boshqa asosiy tushunchalari bayon etilgan.

Har bir mavzuda qisqa nazariy ma’lumotlar va foydalaniladigan asosiy formulalar hamda namuna uchun tipik misol va masalalar yechimlari bilan ko‘rsatilgan. Mustaqil yechish uchun tavsiya qilingan misollarning javoblari keltirilgan.

Kitobdagi masalalar, asosan, o‘zbek va rus tilidagi mavjud adabiyotlardan olingan, ayrim masalalar mualliflar tomonidan tuzilgan.

Texnika oliy o‘quv yurtlarida oliy matematikaning «Operatsion hisob elementlari» hamda «Matematik-fizika tenglamalari» bo‘limlariga ajratiladigan soatlarning kamligini e’tiborga olib, III bobga yuqoridagi ikki bo‘limni ham kiritishni lozim deb topildi.

O‘quv qo‘llanmadan universitetlar nomatematik mutaxassisligi hamda texnika oliy o‘quv yurtlari talabalari foydalanishlari mumkin.

Mualliflar qo‘lyozmani diqqat bilan ko‘rib chiqib, uni yaxshilash yuzasidan fikr-mulohaza bildirgan Namangan Muhandislik-pedagogika instituti va ushbu institut «Oliy matematika» kafedrasining a’zolariga minnatdorchilik bildiradilar.

Kitob to‘g‘risida bildirilgan fikrlarni mualliflar mamnuniyat bilan qabul qiladilar.

KIRISH

1- §. Differensial tenglamalar haqida umumiy tushunchalar

1- ta'rif. Differensial tenglama deb, erkli o'zgaruvchi x, noma'lum $y=f(x)$ funksiya va uning $y', y'', ..., y^{(n)}$ hosilalari orasidagi bog'lanishni ifodalaydigan tenglamaga aytiladi. Differensial tenglama umumiy holda quyidagicha yoziladi:

$$F(x, y, y', y'', ..., y^{(n)}) = 0$$

yoki

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dy^n}\right) = 0.$$

Agar izlanayotgan funksiya $y=f(x)$ bitta erkli o'zgaruvchining funksiyasi bo'lsa, u holda differensial tenglama *oddiy differensial tenglama* deyiladi.

Umuman, noma'lum funksiya ko'p argumentli bo'lgan hollar ham tez-tez uchraydi. Bunday holda differensial tenglama *xususiy hosilali differensial tenglama* deb ataladi. Biz faqat oddiy differensial tenglamalar bilan shug'ullanamiz.

2- ta'rif. Differensial tenglamaning tartibi deb, tenglamada qatnashgan hosilaning eng yuqori tartibiga aytiladi.

Masalan, $(y')^2 + 2y' + xy^3 = 0$ tenglama birinchi tartibli differensial tenglamadir.

Mana bu $(y'')^2 + ay' + by + \cos x = 0$ tenglama esa ikkinchi tartibli differensial tenglama.

3- ta'rif. Differensial tenglamaning yechimi yoki integrali deb, differensial tenglamaga qo'yganda uni ayniyatga aylantiradigan har qanday $y=f(x)$ funksiyaga aytiladi.

Masalan, ushbu tenglama berilgan bo'lsin:

$$\frac{d^2y}{dx^2} + y = 0.$$

$y = \sin x$, $y = 2\cos x$, $y = 3\sin x - \cos x$ funksiyalar, umuman, $y = C_1 \sin x$, $y = C_2 \cos x$ yoki $y = C_1 \sin x + C_2 \cos x$ ko'rinishdagi funksiyalar C_1 va C_2 o'zgarmas miqdorlarning har qanday qiymatlarida ham berilgan differensial tenglamaning yechimi bo'ladi. Buning to'g'riligiga ko'rsatilgan funksiyalarni berilgan tenglamaga qo'yib ko'rib, ishonish mumkin.

2- §. Differensial tenglamaga olib keladigan ba'zi bir masalalar

1- masala. Massasi m bo'lgan jism $v(0) = v_0$ boshlang'ich tezlik bilan biror balandlikdan tashlab yuborilgan. Jism tezligining o'zgarish qonunini toping.

Nyutonning ikkinchi qonuniga ko'ra: $m\frac{dv}{dt} = F$, bu yerda F — jismga ta'sir etayotgan kuchlarning yig'indisi. Jismga faqat ikkita kuch ta'sir etishi mumkin, deb hisoblaylik: havoning qarshilik kuchi $F_1 = -kv, k > 0$, yerning tortish kuchi $F_2 = mg$. U holda ushbu

$$m\frac{dv}{dt} = mg - kv \quad (k > 0)$$

differensial tenglamaga kelamiz. Bu differensial tenglamaning $v(0) = v_0$ shartni qanoatlantiruvchi yechimi

$$v(t) = \left(v_0 - \frac{mg}{k}\right) \cdot e^{-\frac{k}{m}t} + \frac{mg}{k}$$

ekanligini bevosita o'rniga qo'yish bilan tekshirish qiyin emas.

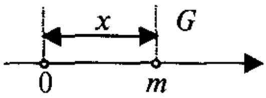
2- masala. Hayvonlarning biror turi o'zgarmas muhitda alohida yashasin deylik. Urchish va o'lishning davriyligini hisobga olmay, ko'rilayotgan tur individuumlari sonining o'zgarish qonunini toping.

Masalaning shartiga ko'ra vaqtning berilgan kichik intervalida urchish va o'lishlar soni berilgan individuumlar soniga proporsional bo'ladi. N individuumlar sonining o'sishi ko'rilayotgan intervalda N_0 soniga proporsional bo'lib, bu o'sish interval uzunligiga ham proporsional bo'ladi. Shunday qilib, $N(t)$ funksiyani uzluksiz va uzluksiz differensialanuvchi deb qarasak, ushbu

$$\frac{dN(t)}{dt} = \varepsilon \cdot N(t), \quad N(t_0) = N_0 > 0$$

differensial tenglamaga ega bo'lamiz, bu yerda ε — proporsionallik koeffitsiyenti («o'sish» koeffitsiyenti). Urchish qonuni differensial tenglama bilan berilgan funksiyaning ko'rinishi $N(t) = N_0 \cdot e^{\varepsilon(t-t_0)}$ ekaniga ishonch hosil qilish qiyin emas. Bundan kelib chiqadiki, vaqt arifmetik progressiya bo'yicha o'zgarsa, individuumlar soni geometrik progressiya bo'yicha o'zgaradi. Agar $\varepsilon > 0$ bo'lsa, $N(t)$ o'sadi; agar $\varepsilon < 0$ bo'lsa, $N(t)$ kamayadi. $\varepsilon = 0$ bo'lganda $N(t)$ o'zgarmas bo'lib, urchish o'lishni to'la qoplaydi.

3- masala. Massasi m bo'lgan moddiy nuqta to'g'ri chiziqli harakat qilmoqda. Uning harakat qonunini toping.



Har bir momentda G nuqtadan koordinata boshigacha bo'lgan masofa x bo'lsa, nuqta tezligi $\dot{x}$ $\left(\dot{x} = \frac{dx}{dt}\right)$ bo'ladi.

Moddiy nuqtaga ikki tashqi kuch: ishqalanish kuchi $-b\dot{x}$, $b > 0$ va taranglik kuchi $-kx$, $k > 0$ ta'sir etadi deylik.

Nyutonning ikkinchi qonuniga asosan G nuqtaning harakat qonuni

$$m\ddot{x} = -b\dot{x} - kx$$

bo'ladi. Bu ikkinchi tartibli differensial tenglamadir. Agar moddiy nuqta dvigatel bilan ta'minlangan bo'lib, dvigatelning G nuqtaga ta'sir kuchi F bo'lsa, u holda G ning harakat qonuni

$$m\ddot{x} = -b\dot{x} - kx + F$$

bo'ladi. Ko'pincha F miqdor $|F| \le F_0 = \text{const}$ munosabatga bo'ysunadi.

I BOB

BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR

1- §. Birinchi tartibli differensial tenglamalarga doir umumiy tushunchalar

Birinchi tartibli differensial tenglama

$$F(x,y,y') = 0 \quad (1.1)$$

ko'rinishda bo'ladi. Agar bu tenglamani y' ga nisbatan yechish mumkin bo'lsa, uni

$$y' = f(x,y) \quad (1.2)$$

ko'rinishda yozish mumkin.

Bu holda differensial tenglama hosilaga nisbatan yechilgan deyiladi. Bunday tenglama uchun quyidagi teorema o'rinli bo'lib, *bu teorema differensial tenglama yechimining mavjudligi va yagonaligi haqidagi teorema* deyiladi.

Teorema. Agar $y' = f(x,y)$ tenglamada $f(x, y)$ funksiya va undan y bo'yicha olingan $\frac{\partial f}{\partial y}$ xususiy hosila xOy tekislikdagi (x_0, y_0) nuqtani o'z ichiga oluvchi biror sohada uzluksiz funksiyalar bo'lsa, u holda berilgan tenglamaning $x=x_0$ bo'lganda $y=y_0$ shartni qanoatlantiruvchi birgina $y=\varphi(x)$ yechimi mavjuddir.

Bu teorema geometrik nuqtayi nazardan grafigi (x_0, y_0) nuqtadan o'tuvchi birgina $y=\varphi(x)$ funksiyaning mavjudligini ifodalaydi. Teoremadan (1.2) tenglama cheksiz ko'p turli yechimlarga ega ekanligi kelib chiqadi.

$x=x_0$ bo'lganda y funksiya berilgan y_0 songa teng bo'lishi kerak, degan shart *boshlang'ich shart* deyiladi. Bu shart ko'pincha

$$y\big|_{x=x_0} = y_0 \quad (1.3)$$

ko'rinishda yoziladi.

1- ta'rif. Birinchi tartibli differensial tenglamaning umumiy yechimi deb bitta ixtiyoriy C o'zgarmas miqdorga bog'liq bo'lgan hamda quyidagi shartlarni qanoatlantiruvchi $y=\varphi(x,C)$ funksiyaga aytiladi:

a) bu funksiya differensial tenglamani C o'zgarmas miqdorning har qanday aniq qiymatida qanoatlantiradi;

b) $x=x_0$ bo'lganda $y=y_0$, ya'ni $y\big|_{x=x_0}=y_0$ boshlang'ich shart har qanday bo'lganda ham C miqdorning shunday $C=C_0$ qiymatini topish mumkinki, bunda $y=\varphi(x,C_0)$ funksiya berilgan boshlang'ich shartni qanoatlantiradi. Ushbu holda x_0 va y_0 qiymatlar x va y o'zgaruvchilarning o'zgarish sohasining yechim mavjudligi va yagonaligi haqidagi teoremaning shartlari bajariladigan qismiga tegishli, deb faraz etiladi.

Biz differensial tenglamaning umumiy yechimini izlashda ko'pincha y ga nisbatan yechilmagan

$$\Phi(x,y,C)=0$$

ko'rinishdagi munosabatga kelib qolamiz. Bu munosabatni y ga nisbatan yechsak, umumiy yechimni hosil qilamiz. Ammo y ni $\Phi(x,y,C)=0$ munosabatdan foydalanib elementar funksiyalar bilan ifoda etish hamma vaqt ham mumkin bo'lavermaydi. Bunday hollarda umumiy yechim oshkormas ko'rinishda qoldiriladi.

Umumiy yechimni oshkormas holda ifodalovchi $\Phi(x,y,C)=0$ ko'rinishdagi tenglik *differensial tenglamaning umumiy integrali* deyiladi.

Misol. Birinchi tartibli

$$\frac{dy}{dx}=-\frac{y}{x}$$

tenglama uchun $y=\dfrac{C}{x}$ funksiyalar oilasi umumiy yechim bo'ladi: buning to'g'riligini y funksiyani tenglamaga qo'yib tekshirish mumkin.

2- §. O'zgaruvchilari ajralgan va ajraladigan tenglamalar

Ushbu $M(x)dx+N(y)dy=0$ ko'rinishdagi tenglamaga *o'zgaruvchilari ajralgan differensial tenglama* deyiladi. Uning o'ziga xos tomoni shundaki, dx oldida faqat x ga bog'liq ko'paytuvchi, dy oldida esa faqat y ga bog'liq ko'paytuvchi turadi. Bu tenglamaning yechimi uni hadma-had integrallash yo'li bilan aniqlanadi:

$$\int M(x)dx + \int N(y)dy = C.$$

Differensial tenglamaning oshkormas holda ifodalangan yechimi bu *tenglamaning integrali* deyiladi. Integrallash doimiysi C ni yechim uchun qulay ko'rinishda tanlash mumkin.

1- misol. $\mathrm{tg}x dx - \mathrm{ctg}y dy = 0$ tenglamaning umumiy yechimini toping.

Y e c h i s h . Bu yerda o'zgaruvchilari ajralgan tenglamaga egamiz. Uni hadma-had integrallaymiz:

$$\int \mathrm{tg}x\, dx - \int \mathrm{ctg}y\, dy = C \quad \text{yoki} \quad -\ln|\cos x| - \ln|\sin y| = -\ln \bar{C}.$$

Bu yerda integrallash doimiysi C ni $-\ln \bar{C}$, ya'ni $C = -\ln \bar{C}$ orqali belgilash qulaydir, bundan $\ln \sin y \cdot \cos x = \ln \bar{C}$ yoki $\sin y \cdot \cos x = \bar{C}$ umumiy integralni topamiz.

Ta'rif.

$$y' = f_1(x) f_2(y) \tag{1.4}$$

ko'rinishdagi tenglamalar *o'zgaruvchilari ajraladigan differensial tenglamalar* deb ataladi, bu yerda $f_1(x)$ va $f_2(y)$ — uzluksiz funksiyalar.

(1.4) tenglamani yechish uchun unda o'zgaruvchilarni ajratish kerak. Buning uchun (1.4) da y' ning o'rniga dy/dx ni yozib, tenglamaning ikki tomonini $f_2(y) \neq 0$ ga bo'lamiz va dx ga ko'paytiramiz. U holda (1.4) tenglama

$$\frac{dy}{f_2(y)} = f_1(x)dx \tag{1.5}$$

ko'rinishga keladi. Bu tenglamada x o'zgaruvchi faqat o'ng tomonda, y o'zgaruvchisi esa chap tomonda ishtirok etyapti, ya'ni o'zgaruvchilar ajratildi. (1.5) tenglikning har ikki tomonini integrallab,

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C$$

ekanligini hosil qilamiz, bu yerda C — ixtiyoriy o'zgarmas.

2- misol. $y' = y/x$ tenglamani yeching.

Y e c h i s h . Berilgan tenglama (1.4) ko'rinishdagi tenglama, bu yerda $f_1(x) = 1/x$ va $f_2(y) = y$. O'zgaruvchilarni ajratib, $\frac{dy}{y} = \frac{dx}{x}$

tenglamani hosil qilamiz. Uni integrallab $\int\frac{dy}{y}=\int\frac{dx}{x}+\ln C$, $C>0$ yoki $\ln y=\ln x+\ln C$ va bu tenglikni potensirlab, $y=Cx$ umumiy yechimni topamiz.

Faraz qilaylik, $y = Cx$ umumiy yechimdan $x_0=1$, $y_0=2$ boshlang'ich shartlarni qanoatlantiruvchi xususiy yechim topish talab qilinyapti. Bu qiymatlarni $y = C\cdot x$ ga x va y larning o'rniga qo'yib, $2=C\cdot 1$ yoki $C=2$ ni topamiz. Demak, xususiy yechim $y=2x$ ekan.

Quyidagi tenglamalarni yeching:

1. $x(y^2-4)dx+ydy=0$.
2. $y'\cos x=y/\ln y$, $y(0)=1$.
3. $y'=\text{tg}x\cdot\text{tg}y$.
4. $(1+x^2)dy+ydx=0$, $y(1)=1$.
5. $\ln\cos y\,dx+x\,\text{tg}y\,dy=0$.
6. $\frac{yy'}{x}+e^y=0$, $y(1)=0$.
7. $y/y'=\ln y$, $y(2)=1$.
8. $y'+\sin(x+y)=\sin(x-y)$.
9. $x\sqrt{1+y^2}dx+y\sqrt{1+x^2}dy=0$.
10. $y'=2^{x-y}$, $y(-3)=-5$.
11. $y'=\text{sh}(x+y)+\text{sh}(x-y)$.
12. $x(y^6+1)dx+y^2(x^4+1)dy$, $y(0)=1$.

3- §. Bir jinsli va bir jinsliga keltiriladigan differensial tenglamalar

Birinchi tartibli bir jinsli differensial tenglamalar

1- ta'rif. Agar ixtiyoriy λ uchun

$$f(\lambda x,\lambda y)=\lambda^n f(x,y)$$

ayniyat o'rinli bo'lsa, $f(x, y)$ funksiya x va y o'zgaruvchilarga nisbatan *n- o'lchovli bir jinsli funksiya* deb ataladi.

1- misol. $f(x,y)=\sqrt[3]{x^3+y^3}$ funksiya bir o'lchovli bir jinsli funksiya, chunki $f(\lambda x,\lambda y)=\sqrt[3]{(\lambda x)^3+(\lambda y)^3}=\lambda\sqrt[3]{x^3+y^3}=\lambda f(x,y)$.

2- misol. $f(x, y) = xy - y^2$ funksiya 2-o'lchovli bir jinsli funksiya, chunki $f(\lambda x, \lambda y) = (\lambda x) \cdot (\lambda x) - (\lambda y)^2 = \lambda^2(xy - y^2) = \lambda^2 f(x, y)$.

3- misol. $f(x,y) = \frac{x^2-y^2}{xy}$ funksiya 0- o'lchovli bir jinsli funksiya, chunki

$$f(\lambda x,\lambda y) = \frac{(\lambda x)^2-(\lambda y)^2}{\lambda x\cdot\lambda y} = \frac{\lambda^2(x^2-y^2)}{\lambda^2 xy} = \lambda^0\,\frac{x^2-y^2}{xy} = \lambda^0 f(x,y).$$

2- ta'rif. Birinchi tartibli

$$\frac{dy}{dx} = f(x,y) \tag{1.6}$$

differensial tenglama x va y *ga nisbatan bir jinsli differensial tenglama* deb ataladi (agar $f(x, y)$ funsiya x va y ga nisbatan 0- o'lchovli bir jinsli funksiya bo'lsa).

Bir jinsli differensial tenglamani yechish. Faraz qilaylik, (1.6) bir jinsli differensial tenglama berilgan bo'lsin, u holda shartga ko'ra $f(\lambda x,\lambda y) = \lambda^0 f(x,y)$. Bu ayniyatda $\lambda = \frac{1}{x}$ deb olsak, $f(x, y)= f\left(1, \frac{y}{x}\right)$ ni hosil qilamiz. Bu holda (1.6) tenglama quyidagi ko'rinishga keladi:

$$\frac{dy}{dx} = f\left(1, \frac{y}{x}\right). \tag{1.7}$$

(1.7) da $u = \frac{y}{x}$, ya'ni $y = u \cdot x$ almashtirish bajaramiz.

U holda $\frac{dy}{dx} = u + \frac{du}{dx}\cdot x$ ni hosil qilamiz. Hosilaning bu ifodasini (1.7) ga qo'yib, $u + \frac{du}{dx}\cdot x = f(1, u)$ yoki $\frac{du}{f(1,u)-u} = \frac{dx}{x}$ tenglikni hosil qilamiz. Bu esa o'zgaruvchilari ajralgan differensial tenglamadir. Integrallab quyidagini topamiz:

$$\int\frac{du}{f(1, u)-u} = \int\frac{dx}{x} + \ln C, \quad \int\frac{du}{f(1, u)-u} = \ln|Cx|.$$

Integrallarni topgandan so‘ng u o‘rniga $\frac{y}{x}$ ni qo‘yib, berilgan tenglamaning integralini $y = y(x, C)$ ko‘rinishida topamiz.

4- misol. $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$ tenglamani yeching.

Y e c h i s h . Tenglamaning o‘ng tomonidagi funksiya 0-o‘lchovli bir jinsli funksiya bo‘lgani uchun tenglama bir jinsli differensial tenglama, shuning uchun $\frac{y}{x} = u$ almashtirishni bajaramiz. U holda $y=ux$, $\frac{dy}{dx} = u + x \cdot \frac{du}{dx}$. Bularni tenglamaga qo‘yib $u + x \cdot \frac{du}{dx} = \frac{u}{1-u^2}$ yoki $x \cdot \frac{du}{dx} = \frac{u^3}{1-u^2}$ va o‘zgaruvchilarni ajratib, $\frac{(1-u^2)du}{u^3} = \frac{dx}{x}$, ya’ni $\left(\frac{1}{u^3} - \frac{1}{u}\right)du = \frac{dx}{x}$ tenglamaga kelamiz.

Integrallash natijasida $-\frac{1}{2u^2} - \ln|u| = \ln|x| + \ln|C|$ yoki $-\frac{1}{2u^2} = \ln|uxC|$ munosabatlarni hosil qilamiz. Oxirgi tenglikda u o‘rniga $\frac{y}{x}$ ni qo‘yib, $-\frac{x^2}{2y^2} = \ln|Cx|$ tenglamaning umumiy integralini topamiz. Ko‘rinib turibdiki, y ni x orqali elementar funksiyalar yordamida ifodalab bo‘lmaydi. Biroq x ni y orqali ifodalash mumkin: $x = y\sqrt{-2\ln|Cy|}$.

Bir jinsli tenglamalarga keltiriladigan differensial tenglamalar

$$\frac{dy}{dx} = \frac{ax+by+C}{a_1x+b_1y+C_1} \tag{1.8}$$

ko‘rinishdagi tenglamalarni bir jinsli tenglamalarga keltirish mumkin. Agar $C_1 = 0$, $C = 0$ bo‘lsa, tenglama bir jinsli bo‘lishini ko‘rish qiyin emas. Faraz qilaylik, C va C_1 larning birortasi noldan farqli bo'lsin. $x = x_1 + h$, $y = y_1 + k$ almashtirish bajaramiz. U holda

$$\frac{dy}{dx} = \frac{dy_1}{dx_1}. \tag{1.9}$$

x, y va $\frac{dy}{dx}$ ifodalarni (1.8) tenglamalarga qo'yib

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1 + ah + bk + C}{a_1x_1 + b_1y_1 + a_1h + b_1k + C_1} \tag{1.10}$$

tenglamaga ega bo'lamiz. h va k larni shunday tanlab olamizki,

$$\begin{cases} ah + bk + C = 0, \\ a_1h + b_1k + C_1 = 0 \end{cases} \tag{1.11}$$

tenglamalar o'rinli bo'lsin, ya'ni h va k larni (1.11) tenglamalar sistemasining yechimi sifatida olamiz. Bu holda (1.10) tenglamadan bir jinsli $\frac{dy_1}{dx_1} = \frac{ax_1 + by_1}{a_1x_1 + b_1y_1}$ tenglamani hosil qilamiz. Tenglamani yechib va x hamda y larga $x_1 = x - h$, $y_1 = y - h$ formulalar yordamida qaytib, berilgan (1.8) tenglamaning yechimini topamiz. Agar

$$\begin{vmatrix} ab \\ a_1b_1 \end{vmatrix} = 0$$

bo'lsa, ya'ni $ab_1 = a_1b$ bo'lganda, ma'lumki, (1.11) sistema yechimga ega bo'lmaydi. Ammo, bu holda $\frac{a_1}{a} = \frac{b_1}{b} = \lambda$, ya'ni $a_1 = \lambda a$, $b_1 = \lambda b$ bo'ladi.

Bundan kelib chiqadiki, (1.8) tenglamani

$$\frac{dy}{dx} = \frac{(ax+by)+C}{\lambda(ax+by)+C_1} \tag{1.12}$$

ko'rinishga keltirish mumkin ekan. Bu holda

$$z = ax + by \tag{1.13}$$

almashtirish yordamida tenglama o'zgaruvchilari ajraladigan differensial tenglamaga aylanadi, haqiqatdan, $\frac{dz}{dx} = a + b\frac{dy}{dx}$ tenglikdan

$$\frac{dy}{dx}=\frac{1}{b}\cdot\frac{dz}{dx}-\frac{a}{b} \qquad (1.14)$$

munosabatni hosil qilamiz hamda (1.13) va (1.14) ifodalarni (1.12) tenglamaga qo'yib, o'zgaruvchilari ajraladigan $\frac{1}{b}\cdot\frac{dz}{dx}-\frac{a}{b}=\frac{z+C}{\lambda z+C_1}$ tenglamani hosil qilamiz.

Yuqorida (1.8) tenglamaga qo'llanilgan usulni $\frac{dy}{dx}=f\left(\frac{ax+by+C}{a_1x+b_1y+C_1}\right)$ tenglamaga ham qo'llash mumkin, bu yerda f qandaydir uzluksiz funksiya.

5- misol. $\frac{dy}{dx}=\frac{x+y-3}{x-y-1}$ tenglamani yeching.

Y e c h i s h . Tenglamani bir jinsli tenglamaga aylantirish uchun $x=x_1+h$, $y=y_1+k$ almashtirishni bajaramiz. U holda tenglama $\frac{dy_1}{dx_1}=\frac{x_1+y_1+h+k-3}{x_1-y_1+h-k-1}$ ko'rinishni oladi. $h+k-3=0$, $h-k-1=0$ tenglamalar sistemasini yechib, $h=2$, $k=1$ ekanligini topamiz. Natijada bir jinsli $\frac{dy_1}{dx_1}=\frac{x_1+y_1}{x_1-y_1}$ tenglamani hosil qilamiz. $\frac{y_1}{x_1}=u$ almashtirishni bajarsak, u holda $y_1=ux_1$, $\frac{dy_1}{dx_1}=u+x_1\cdot\frac{du}{dx_1}$, $u+x_1\cdot\frac{du}{dx_1}=\frac{1+u}{1-u}$ bo'ladi va natijada $x_1\cdot\frac{du}{dx_1}=\frac{1+u^2}{1-u}$ o'zgaruvchilari ajraladigan tenglamaga ega bo'lamiz. O'zgaruvchilarni ajratamiz: $\frac{1-u}{1+u^2}du=\frac{dx_1}{x_1}$ ni integrallab

$$\operatorname{arctg}u-\frac{1}{2}\ln(1+u^2)=\ln|x_1|+\ln|C|,$$

$\operatorname{arctg}u=\ln\left|Cx_1\sqrt{(1+u^2)}\right|$ yoki $Cx_1\sqrt{1+u^2}=e^{\operatorname{arctg}u}$ ekanligini topamiz. u o'rniga $\frac{y_1}{x_1}$ ifodani qo'yib, $C\sqrt{x_1^2+y_1^2}=e^{\operatorname{arctg}\frac{y_1}{x_1}}$ ekanligini va nihoyat, x va y o'zgaruvchilarga o'tib, $C\sqrt{(x-2)^2+(y-1)^2}=e^{\operatorname{arctg}\frac{y-1}{x-2}}$ natijani hosil qilamiz.

6- misol. $y'=\frac{2x+y-1}{4x+2y+5}$ tenglamani yeching.

Y e c h i s h. Tenglamani $x=x_1+h$, $y=y_1+k$ almashtirish yordamida yechib bo'lmaydi, chunki bu holda h va k larni aniqlashga yordam beradigan sistema determinanti $\begin{vmatrix}2 & 1\\ 4 & 2\end{vmatrix}$ nolga teng.

Bu tenglamani $2x+y=z$ almashtirish yordamida o'zgaruvchilari ajraladigan differensial tenglamaga keltirish mumkin, haqiqatan, $y'=z'-2$ bo'lgani uchun tenglama

$$z'-2=\frac{z-1}{2z+5}$$

ko'rinishga yoki

$$z'=\frac{5z+9}{2z+5}$$

ko'rinishga keladi. Tenglamani yechib

$$\frac{2}{5}z+\frac{7}{25}\ln|5z+3|=x+C$$

munosabatni, z o'rniga $2x+y$ ni qo'yib esa

$$\frac{2}{5}(2x+y)+\frac{7}{25}\ln|10x+5y+9|=x+C \quad \text{yoki}$$

$$10y-5x+7\ln|10x+5y+9|=C_1 \quad \text{ni,}$$

ya'ni y ning x ga nisbatan oshkormas ko'rinishini hosil qilamiz.

Quyidagi tenglamalarni yeching:

13. $(x^2+2xy)dx+xydy=0$.

14. $y'=\frac{y}{x}+\sin\frac{y}{x},\ y(1)=\frac{\pi}{2}$.

15. $xy'\sin\left(\frac{y}{x}\right)+x=y\sin\left(\frac{y}{x}\right)$.

16. $xy+y^2=(2x^2+xy)\cdot y'$.

17. $xyy'=y^2+2x^2$.

18. $y'=\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)$.

19. $(x^2+y^2)dx-xydy=0$.

20. $(x+y+2)dx+(2x+2y-1)dy=0$.

21. $(2x+y+1)dx+(x+2y-1)dy=0$.

22. $2(x+y)dy+(3x+3y^{-1})dx=0, y(0)=2$.

23. $(x-2y+3)dy+(2x+y-1)dx=0$.

24. $(x-y+4)dy+(x+y-2)dx=0$.

4- §. Chiziqli differensial tenglamalar. Bernulli tenglamasi

1. Chiziqli differensial tenglamalar.

Ta'rif. Noma'lum funksiya va uning hosilasiga nisbatan chiziqli bo'lgan tenglama *chiziqli differensial tenglama* deyiladi. Bunday tenglama

$$\frac{dy}{dx}+P(x)\cdot y=Q(x) \qquad (1.15)$$

ko'rinishga ega bo'ladi, bu yerda $P(x)$ va $Q(x)$ – berilgan uzluksiz funksiyalar. (1.15) tenglama yechimini ikki funksiya ko'paytmasi ko'rinishida qidiramiz:

$$y=u(x)\cdot v(x) \qquad (1.16)$$

Bu funksiyalarning birini ixtiyoriy deb olish mumkin, ikkinchisi esa (1.15) tenglama orqali topiladi. (1.16) tenglikning ikki tomonini differensiallaymiz:

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Topilgan $\frac{dy}{dx}$ hosila ifodasini (1.15) tenglamaga qo'yib,

$$u\frac{dv}{dx}+\frac{du}{dx}v+Puv=Q \quad \text{yoki} \quad u\left(\frac{dv}{dx}+Pv\right)+\frac{du}{dx}v=Q \qquad (1.17)$$

bo'lishini topamiz. v funksiyani

$$\frac{dv}{dx}+Pv=0 \qquad (1.18)$$

shartni qanoatlantiradigan qilib olamiz. Bu differensial tenglamada v ga nisbatan o'zgaruvchini ajratib, quyidagini topamiz:

$\frac{dv}{v}=-Pdx$, integrallab $-\ln|C_1|+\ln|v|=-\int Pdx$ yoki $v=C_1e^{-\int Pdx}$

ni hosil qilamiz.

Bizga (1.18) tenglamaning noldan farqli biror yechimi yetarli bo'lgani uchun $v(x)$ sifatida

$$v=e^{-\int Pdx} \qquad (1.19)$$

funksiyani olamiz, bu yerda $\int Pdx$ – qandaydir boshlang'ich funksiya. Topilgan $v(x)$ ning qiymatini (1.17) tenglamaga qo'yib,

$v(x)\frac{du}{dx}=Q(x)$ yoki $\frac{du}{dx}=\frac{Q(x)}{v(x)}$ ekanligini topamiz, bu yerdan

$$u=\int\frac{Q(x)}{v(x)}dx+C$$

ni topamiz. u va v larni (1.16) formulaga qo'yib, nihoyat

$$y=v(x)\left[\int\frac{Q(x)}{v(x)}dx+C\right] \quad \text{yoki} \quad y=e^{-\int Pdx}\left[\int Q(x)e^{\int Pdx}dx+C\right] \qquad (1.20)$$

ifodani, ya'ni (1.15) ning umumiy yechimini topamiz.

1- misol. $\frac{dy}{dx}-\frac{2}{x+1}\cdot y=(x+1)^3$ tenglamani yeching.

Y e c h i s h . $y=uv$ deb olsak, u holda $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$.

$\frac{dy}{dx}$ ifodasini berilgan tenglamaga qo'ysak,

$$u\frac{dv}{dx}+\frac{du}{dx}v-\frac{2}{x+1}uv=(x+1)^3$$

yoki

$$u\left(\frac{dv}{dx}-\frac{2}{x+1}v\right)+\frac{du}{dx}v=(x+1)^3. \qquad (1.21)$$

v funksiyani aniqlash uchun $\frac{dv}{dx}-\frac{2}{x+1}v=0$ yoki $\frac{dv}{v}=\frac{2dx}{x+1}$ tenglamani hosil qilamiz. Bu yerdan $\ln|v|=2\ln|x+1|$ yoki $v=(x+1)^2$.

v ning ifodasini (1.21) tenglikka qo'yib, u ni aniqlash uchun $(x+1)^2\frac{du}{dx}=(x+1)^3$ yoki $\frac{du}{dx}=x+1$ tenglamani hosil qilamiz, bu yerdan $u=\frac{(x+1)^2}{2}+C$. Demak, berilgan tenglamaning umumiy yechimi $y=\frac{(x+1)^4}{2}+C(x+1)^2$ bo'lar ekan.

2. Bernulli tenglamasi.

Ta'rif.

$$\frac{dy}{dx}+P(x)\cdot y=Q(x)\cdot y^n,\quad n\geq 2 \qquad (1.22)$$

ko'rinishdagi tenglama *Bernulli tenglamasi* deb ataladi, bu yerda $P(x)$ va $Q(x)$ – berilgan uzluksiz funksiyalar, $n\neq 0;\ 1$.

Tenglamaning barcha hadlarini y^n ga bo'lamiz

$$y^{-n}\frac{dy}{dx}+P(x)\cdot y^{-n+1}=Q(x) \qquad (1.23)$$

va $z=y^{-n+1}$ almashtirishni bajaramiz, u holda

$$\frac{dz}{dx}=(-n+1)\cdot y^{-n}\frac{dy}{dx}.$$

Topilgan qiymatni (1.23) tenglamaga qo'yib, $\frac{dz}{dx}+(-n+1)P\cdot z=$ $=(-n+1)\cdot Q$ chiziqli tenglamani hosil qilamiz. Chiziqli tenglamaning umumiy integralini topgandan so'ng, z o'rniga y^{-n+1} ni qo'yib, Bernulli tenglamasining umumiy integralini hosil qilamiz.

2- misol. Ushbu

$$\frac{dy}{dx}+xy=x^3\cdot y^3 \qquad (1.24)$$

tenglamani yeching.

Y e c h i s h . Tenglamaning barcha hadlarini y^3 ga bo'lamiz

$$y^{-3}\frac{dy}{dx}+xy^{-2}=x^3 \qquad (1.25)$$

va $z=y^{-2}$ almashtirishni bajaramiz, u holda $\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx}$. Bu qiymatlarni (1.25) ga qo'yib

$$\frac{dz}{dx}-2xz=-2x^3 \qquad (1.26)$$

chiziqli tenglamani hosil qilamiz. Uning umumiy integralini topamiz:

$$z=uv\,,\ \frac{dz}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Bu ifodalarni (1.26) tenglamaga qo'yamiz:

$$u\frac{dv}{dx}+v\frac{du}{dx}-2xuv=-2x^3 \ \text{ yoki } \ u\left(\frac{dv}{dx}-2xv\right)+v\frac{du}{dx}=-2x^3.$$

Qavs ichidagi ifodani nolga tenglab,

$$\frac{dv}{dx}-2xv=0,\ \frac{dv}{v}=2xdx,\ \ln|v|=x^2,\ v=e^{x^2}$$

ekanligini topamiz. u ni aniqlash uchun

$$e^{x^2}\cdot\frac{du}{dx}=-2x^3$$

tenglamaga ega bo'lamiz. O'zgaruvchilarni ajratib

$$du=-2e^{-x^2}x^3dx,\ u=-2\int e^{-x^2}x^3dx+C$$

ekanligini topamiz. Oxirgi integralni bo'laklab

$$u=x^2e^{-x^2}+e^{-x^2}+C,\ z=u\cdot v=x^2+1+Ce^{x^2}$$

ifodalarni topamiz. Demak, berilgan tenglamaning umumiy integrali

$y^{-2} = x^2 + 1 + Ce^{x^2}$ yoki $y = \frac{1}{\sqrt{x^2+1+Ce^{x^2}}}$ bo‘lar ekan.

Quyidagi tenglamalarni yeching:

25. $y'\cos^2 x + y = \mathrm{tg}x$, $y(0)=0$.

26. $y' - y\mathrm{th}x = \mathrm{ch}^2 x$.

27. $y' + \frac{xy}{1-x^2} = \arcsin x + x$.

28. $xy' - y = x^2\cos x$.

29. $y' + 2xy = xe^{-x^2}$.

30. $y'\cos x + y = 1 - \sin x$.

31. $y' + \frac{y}{x} = x^2 y^4$.

32. $(x^2\ln y - x)y' = y$.

33. $y' + \frac{2y}{x} = 3x^2 y^{4/3}$.

34. $y' - \frac{y}{x-1} = \frac{y^2}{x-1}$.

35. $4xy' + 3y = -e^x \cdot x^4 y^5$.

36. $y' + \frac{3x^2 y}{x^3+1} = y^2(x^3+1)\sin x$, $y(0)=1$.

37. $ydx + (x + x^2y^2)dy = 0$.

5- §. To‘la differensialli tenglama. Integrallovchi ko‘paytuvchi

1. To‘la differensialli tenglama.

Ta’rif. Agar $M(x, y)dx + N(x, y)dy = 0$ ko‘rinishdagi tenglamaning chap qismi biror $u(x, y)$ funksiyaning to‘la differensiali, ya’ni

$$du = M(x, y)dx + N(x, y)dy \qquad (1.27)$$

bo‘lsa, u holda bunday tenglama *to‘la differensialli tenglama* deyiladi.

(1.27) tenglama to‘la differensialli tenglama bo‘lishi uchun

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

shart bajarilishi kerak.

To‘la differensialli tenglama ta’rifidan $du=0$, bundan $u(x, y)=C$ ekanligi kelib chiqadi (C – ixtiyoriy o‘zgarmas).

$u(x, y)$ ni topish uchun y ni o‘zgarmas deb hisoblaymiz, u holda $dy=0$ ekanidan $du=M(x, y)dx$ bo‘ladi. Bu tenglikni x bo‘yicha integrallasak,

$$u = \int M(x,y)dx + \varphi(y). \qquad (1.28)$$

(1.28) tenglikni y bo‘yicha differensiallaymiz va natijani $N(x, y)$ ga tenglaymiz, chunki $\frac{\partial u}{\partial y} = N(x,y)$,

$$\int \frac{\partial M}{\partial y} dx + \varphi'(y) = N(x,y)$$

yoki

$$\varphi'(y) = N(x,y) - \int \frac{\partial M}{\partial y} dx. \qquad (1.29)$$

(1.29) ifodani y bo‘yicha integrallab, $\varphi(y)$ ni topamiz:

$$\varphi(y) = \int\left(N(x,y) - \int \frac{\partial M}{\partial y} dx\right)dy + C.$$

Demak, $u(x, y)= \int M(x,y)dx + \int\left(N(x,y) - \int \frac{\partial M}{\partial y} dx\right)dy + C.$

Bu ifodani ixtiyoriy o‘zgarmasga tenglab, tenglamaning umumiy integralini hosil qilamiz.

1- misol. $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$ tenglamaning umumiy yechimini toping.

Yechish. Bu yerda $M(x, y)=3x^2+6xy^2$, $N(x, y)=6x^2y+4y^3$.

$\frac{\partial N}{\partial y} = 12xy$, $\frac{\partial N}{\partial x} = 12xy$, ya’ni $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\frac{\partial u}{\partial x} = M(x,y)$ bo‘lganligi sababli

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy^2.$$

Bu tenglikni x bo‘yicha integrallaymiz:

$$u = x^3+3x^2y^2+\varphi(y).$$

Bundan

$$\frac{\partial u}{\partial y} = 6x^2y + \varphi'(y).$$

$\frac{\partial u}{\partial y} = N(x, y)$ ekanligini hisobga olsak,

$$\varphi'(y) = 6x^2y + 4y^3 - 6x^2y \text{ yoki } \varphi'(y) = 4y^3.$$

Bundan

$$\varphi(y) = y^4 + C.$$

Demak,

$$u = x^3 + 3x^2y^2 + y^4 + C$$

yoki

$$x^3 + 3x^2y^2 + y^4 = C.$$

2. Integrallovchi ko'paytuvchi. Agar $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ bo'lsa, u holda ba'zi bir shartlar bajarilganda, shunday $\mu(x, y)$ funksiyani topish mumkinki, $\mu M dx + \mu N dy = du$ bo'ladi. Bu $\mu(x, y)$ funksiya *integrallovchi ko'paytuvchi* deyiladi.

Quyidagi hollarda integrallovchi ko'paytuvchini topish oson:

1) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \Phi(x)$ bo'lganda, $\ln\mu = \int \Phi(x)dx$ bo'ladi.

2) $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \Phi_1(y)$ bo'lganda, $\ln\mu = \int \Phi_1(y)dy$ bo'ladi.

2- misol. $(y + xy^2)dx - xdy = 0$ tenglamani yeching.

Y e c h i s h. Bu yerda $M = y + xy^2$, $N = -x$, $\frac{\partial M}{\partial y} = 1 + 2xy$,

$\frac{\partial N}{\partial x} = -1$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Demak, tenglamaning chap tomoni biror funksiyaning to'la differensiali emas. Bu tenglamaning faqat y ga bog'liq bo'lgan integrallovchi ko'paytuvchisi bormi, degan masalani qaraymiz.

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{-1-1-2xy}{y+xy^2}=-\frac{2}{y},$$

bundan

$$\ln\mu=-2\ln y,\text{ ya'ni }\mu=\frac{1}{y^2}.$$

Berilgan tenglamani μ ga ko'paytirganda

$$\left(\frac{1}{y}+x\right)dx-\frac{x}{y^2}dy=0$$

tenglama hosil bo'ladi. Bu to'la differensialli tenglamadir, chunki

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-\frac{1}{y^2}.$$

Tenglamani yechib

$$\frac{x}{y}+\frac{x^2}{2}+C=0$$

yoki

$$y=-\frac{2x}{x^2+2C}$$

umumiy integralni topamiz.

Quyidagi differensial tenglamalarning chap tomonlari to'liq differensialdan iborat ekanligi tekshirilsin va tenglamalar yechilsin:

38. $(e^x+y+\sin y)dx+(e^y+x+x\cos y)dy=0.$

39. $(x+y-1)dx+(e^y+x)dy=0.$

40. $(x\cos y-y\sin y)dy+(x\sin y+y\cos y)dx=0.$

41. $2xydx+(x^2-y^2)dy=0.$

42. $(2-9xy^2)xdx+(4y^2-6x^3)ydy=0.$

43. $\frac{y}{x}dx+(y^3+\ln x)dy=0.$

44. $(10xy-8y+1)dx+(5x^2-8x+3)dy=0.$

45. $3x^2(1+\ln y)dx=\left(2y-\frac{x^3}{y}\right)dy.$

46. $2x\cos^2 ydx+(2y-x^2\sin^2 y)dy=0.$

Quyidagi differensial tenglamalarning integrallovchi ko'paytuvchilari topilsin va tenglamalar yechilsin:

47. $(x^2 - y)dx + xdy = 0.$
48. $y^2dx + (yx - 1)dy = 0.$
49. $(x^2 + y^2 + x)dx + ydy = 0.$
50. $xy^2(xy' + y) = 1.$
51. $(x^2 + 3\ln y)ydx = xdy.$
52. $2x\mathrm{tg}ydx + (x^2 - 2\sin y)dy = 0.$
53. $(e^{2x} - y^2)dx + ydy = 0.$
54. $(1 + 3x^2\sin y)dx - x\,\mathrm{ctg}ydy = 0.$
55. $(\sin x + e^y)dx + \cos xdy = 0.$

6- §. Hosilaga nisbatan yechilmagan 1- tartibli differensial tenglamalar

Ta'rif.

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \qquad (1.30)$$

ko'rinishdagi tenglamalar *hosilaga nisbatan yechilmagan 1-tartibli tenglama* deb ataladi.

Bunday ko'rinishdagi tenglamani $\frac{dy}{dx}$ ga nisbatan yechib olish maqsadga muvofiq bo'ladi, ya'ni berilgan tenglamadan

$$\frac{dy}{dx} = f_i(x, y) \quad (i = 1, 2, ..., n) \qquad (1.31)$$

ko'rinishdagi bir yoki bir necha hosilaga nisbatan yechilgan tenglamalar hosil qilinadi. Ammo har doim ham (1.30) ko'rinishdagi tenglamani $\frac{dy}{dx}$ ga nisbatan yechib olish mumkin bo'lavermaydi, undan tashqari y' ga nisbatan yechilgandan hosil bo'lgan (1.31) ko'rinishdagi tenglamalar har doim ham oson integrallanavermaydi. Shuning uchun (1.31) ko'rinishdagi tenglamalarni ko'pincha parametr kiritish yo'li bilan yechiladi. Shu usulning eng oson variantlaridan biri bilan tanishib chiqamiz.

Faraz qilaylik, (1.30) tenglamani y yoki x ga nisbatan oson yechish mumkin bo'lsin. Masalan, uni $y = f(x, y')$ ko'rinishda yozib olish mumkin bo'lsin. $\frac{dy}{dx}=p$ parametr kiritib, $y=f(x, p)$ ni hosil qilamiz. Oxirgi tenglikning ikki tomonidan to'la differensial olib hamda dy ni pdx ga almashtirib

$$pdx=\frac{\partial f(x,p)}{\partial x}dx+\frac{\partial f(x,p)}{\partial p}dp,$$

ya'ni, $M(x,p)dx+N(x,p)dp=0$ ni hosil qilamiz. Agar bu tenglamaning $x=\Phi(p,C)$ yechimini topsak, u holda berilgan tenglamaning yechimi

$$\begin{cases} x=\Phi(p,C), \\ y=f(x,p) \end{cases}$$

parametrik ko'rinishda bo'ladi.

(1.30) tenglama uchun $y(x_0)=y_0$ Koshi masalasi (x_0, y_0) nuqtadan o'tuvchi va bu nuqtada umumiy urinmaga ega bo'lgan (1.30) tenglamaning ikki integral egri chizig'i mavjud bo'lmagandagina yagona yechimga ega bo'ladi. Aks holda Koshi masalasi yechimining yagonaligi buziladi, ya'ni (x_0, y_0) nuqta Koshi masalasi yechimining yagonaligi buziladigan nuqta bo'ladi.

(1.30) tenglama uchun Koshi masalasining yechimi mavjudligi va yagonaligining yetarlilik shartini quyidagi teorema aniqlab beradi.

Teorema. y_0, $F(x_0,y_0,y_0')=0$ tenglamaning yechimlaridan biri bo'lsin. Faraz qilaylik, $F(x,y,y')$ funksiya x bo'yicha uzluksiz, y va y' bo'yicha uzluksiz differensiallanuvchi hamda uning y' bo'yicha hosilasi noldan farqli bo'lsin:

$$\frac{\partial F}{\partial y'}(x_0,y_0,y_0')\neq 0.$$

U holda $F(x,y,y')=0$, $y(x_0)=y_0$ Koshi masalasining x_0 nuqtaning yetarlicha kichik atrofida $\varphi'(x_0)=y_0$ shartni qanoatlantiruvchi $y=\varphi(x)$ yagona yechimi mavjud bo'ladi.

Hosilaga nisbatan yechilgan tenglama kabi (1.30) ko'rinishdagi tenglamalar ham maxsus yechimlarga ega bo'lishi mumkin, ya'ni shunday yechimlarga ega bo'lishi mumkinki, bu integral chiziqlar faqat yagonalik sharti bajarilmaydigan nuqtalardan iborat bo'ladi.

Agar $F(x,y,y')$ funksiya x ga ko'ra uzluksiz hamda y va y' ga ko'ra uzluksiz differensiallanuvchi bo'lsa, (1.30) tenglamaning maxsus yechimi, agar u mavjud bo'lsa,

$$\begin{cases} F(x,y,y')=0, \\ \dfrac{\partial F}{\partial y'}(x,y,y')=0 \end{cases} \tag{1.32}$$

tenglamalar sistemasini qanoatlantiradi.

Shuning uchun, (1.30) tenglamaning maxsus yechimlarini topish uchun (1.32) tenglamalar sistemasidan y' ni yo'qotish kerak.

1- misol. $(y')^3-2x\cdot(y')^2+y'=2x$ tenglamani yeching.

Yechish. $(y')^3-2x\cdot(y')^2+y'-2x=(y'-2x)((y')^2+1)=0$ bo'lganligi uchun, berilgan tenglama $\frac{dy}{dx}-2x=0$ tenglamaga ekvivalent. Uning yechimlari $y=x^2+C$ ko'rinishga ega.

2- misol. $(y')^2+y\cdot(y-x)\cdot y'-xy^3=0$ tenglamani yeching.

Yechish. Berilgan tenglamani $(y'+y^2)\cdot(y'-xy)=0$ ko'rinishda yozib olish mumkin. Demak, berilgan tenglama $y'+y^2=0$ va $y'-xy=0$ tenglamalar yig'indisiga ekvivalent. Ulardan birinchisining yechimlari $y=0$ va $y=\frac{1}{x+C}$, ikkinchisiniki esa $y=C\cdot e^{\frac{x^2}{2}}$.

Demak, berilgan tenglama yechimlari $\left(y-\frac{1}{x+C}\right)\left(y-C\cdot e^{\frac{x^2}{2}}\right)=0$.

3- misol. $y=(y')^2\cdot e^{y'}$ tenglamani yeching.

Yechish. $p=y'=\frac{dy}{dx}$ parametr kiritamiz. U holda $y=p^2e^p$, $dy=(2pe^p+p^2e^p)dp$. Bu yerdan $p=0$ yoki $x=2e^p+e^p(p-1)+C=$ $=e^p(p+1)+C$.

Demak, berilgan tenglama yechimlari

$$y = 0 \text{ va } \begin{cases} x = (p+1)e^p + C, \\ y = p^2 e^p. \end{cases}$$

4- misol. $\ln y' + \sin y' - x = 0$ tenglamani yeching.

Y e c h i s h . $y'=p$ deb olsak, $x = \ln p + \sin p dy = pdx$ bo'lgani uchun $\frac{dy}{p} = \left(\frac{1}{p} + \cos p\right)dp$. Bu yerdan $y = \int(1 + p\cos p)dp =$ $= p + \cos p + p\sin p + C$. Demak, berilgan tenglama yechimlari

$$\begin{cases} x = \ln p + \sin p, \\ y = p + \cos p + p\sin p + C. \end{cases}$$

5- misol. $(y')^2 + (x+a)y' - y = 0$ tenglamani yeching.

Y e c h i s h . $p = y'$ parametr kiritamiz, u holda $y = p^2 + (x+a)p dy =$ $=pdx$ va $dy = 2pdp + (x+a)dp + pdx$ tenglamalardan $pdx = 2pdp +$ $+(x+a)dp + pdx$, $(2p + x + a)dp = 0$ tenglamalarni hosil qilamiz. Bu yerdan $p=C$ yoki $2p+x+a=0$ tenglamalar kelib chiqadi. Demak, berilgan tenglama yechimlari quyidagi ko'rinishga ega bo'ladi:

$$y = (x+a)\cdot C + C^2 \text{ va } \begin{cases} y = p^2 + (x+a)p, \\ 2p + x + a = 0. \end{cases}$$

Oxirigi ikki tenglikdan p parametrni yo'qotib, $y=C(x+a)+C^2$ va $y = -\frac{(x+a)^2}{4}$ ekanligini hosil qilamiz.

Quyidagi tenglamalarni yeching:

56. $(y')^2 = y^3 - y^2$.

57. $(y')^2 + y^2(\ln^2 y - 1) = 0$.

58. $(y')^3 + x(y')^2 - y = 0$.

59. $x(y')^3 - y(y')^2 + 1 = 0$.

60. $x(y')^2 + xy' - y = 0$.

61. $(3x+1)(y')^2 - 3(y+2)y' + 9 = 0$.

62. $x^2(y')^2 - 2xyy' - x^2 = 0$.

63. $x^4(y')^2 - xy' - y = 0$.

64. $y(y')^2 - 2xy' + y = 0$.

65. $\ln y' + 2(xy' - y) = 0$.

7- §. n- darajali 1- tartibli tenglama

Chap tomoni y' ga nisbatan butun ratsional funksiyadan iborat, ya'ni quyidagi

$$(y')^n + P_1(y')^{n-1} + P_2(y')^{n-2} + ... +P_{n-1}y' + P_ny = 0$$

ko'rinishga ega bo'lgan tenglama *n-darajali 1-tartibli tenglama* deyiladi. Bu yerda n– butun musbat son, P_1, P_2, P_3, ..., P_n lar x va y ning funksiyalari.

Bu tenglamani y' ga nisbatan echa olamiz, deb faraz qilaylik. Bundan y' uchun, umuman aytganda, n ta har xil ifoda hosil bo'ladi:

$$y' = f_1(x, y), \quad y' = f_2(x, y), ..., \quad y' = f_n(x, y). \qquad (1.33)$$

Bu holda

$$F(x, y, y') = 0 \qquad (1.34)$$

tenglamani integrallash birinchi tartibli n ta

$$y' = f(x, y) \qquad (1.35)$$

tenglamani integrallashga keltirildi. Ularni umumiy integrallari mos ravishda quyidagilar bo'lsin:

$$\Phi_1(x, y, C_1) = 0, \Phi_2(x, y, C_2) = 0, ..., \quad \Phi_n(x, y, C_n) = 0. \qquad (1.36)$$

Bu integrallarning chap tomonlarini o'zaro ko'paytirib, nolga tenglaymiz:

$$\Phi_1(x, y, C_1) \cdot \Phi_2(x, y, C_2) \cdot ... \cdot \Phi_n(x, y, C_n) = 0. \qquad (1.37)$$

Agar (1.37) tenglamani y ga nisbatan yechadigan bo'lsak, (1.34) tenglamaning yechimini hosil qilamiz, haqiqatan ham, (1.34) tenglamaning har qanday yechimi (1.37) tenglamalarning birini, binobarin, (1.35) tenglamalarning birortasini va shunday qilib, (1.34) tenglama (1.35) tenglamalarga yoyilgani uchun uni ham qanoatlantiradi. Umumiylikka ziyon keltirmasdan, (1.37) dagi barcha C_1, C_2, ..., C_n o'zgarmaslarni bitta C bilan almashtirish va tenglamani

$$\Phi_1(x, y, C) \cdot \Phi_2(x, y, C) \cdot ... \cdot \Phi_n(x, y, C) = 0 \qquad (1.38)$$

ko'rinishda yozish mumkin. Bu esa (1.34) tenglamaning yechimi bo'ladi. Bunga ishonch hosil qilish uchun (1.38) tenglamaning n ta tenglamaga ajralishini ko'rish mumkin:

$$\Phi_1(x, y, C) = 0, \Phi_2(x, y, C) = 0, ..., \quad \Phi_n(x, y, C) = 0, \qquad (1.39)$$

bu yerda C – istalgan qiymatlarni qabul qiluvchi ixtiyoriy o'zgarmas, shu sababli (1.36) tenglamadan hosil qilinadigan barcha yechimlar (1.39) tenglamadan hosil qilinadigan yechimlar orasida bo'ladi.

1- misol. $(y')^2 - \frac{xy}{a^2} = 0$ tenglamaning umumiy integralini toping.

Yechish. Tenglamaning chap tomonini ko'paytuvchilarga ajratib, quyidagini hosil qilamiz:

$$\left(y' - \frac{\sqrt{xy}}{a}\right)\cdot\left(y' + \frac{\sqrt{xy}}{a}\right) = 0, \text{ bu yerdan } y' - \frac{\sqrt{xy}}{a} = 0 \text{ va } y' + \frac{\sqrt{xy}}{a} = 0.$$

Bu ikkala tenglama o'zgaruvchilari ajraladigan tenglamadir. Ularning umumiy integrallari

$$\sqrt{y} - \frac{x\sqrt{x}}{3a} - C = 0, \sqrt{y} + \frac{x\sqrt{x}}{3a} - C = 0.$$

Shuning uchun berilgan tenglamaning umumiy integrali ushbu ko'rinishda bo'ladi:

$$(\sqrt{y} - C)^2 - \frac{x^3}{9a^2} = 0.$$

Quyidagi tenglamalar yechilsin:

66. $(y')^3 - 2x(y')^2 + y' = 2x.$

67. $(y')^2 + y(y - x)y' - xy^3 = 0.$

68. $(y')^2 + (\sin x - 2xy)y' - xy^3 = 0.$

69. $(y')^2 = 4.$

70. $(y')^2 + y^2 - 1 = 0.$

71. $x(y')^2 - 2yy' + 4x = 0.$

72. $(y')^2 - y^2 = 0.$

73. $8(y')^3 = 27y.$

74. $(y'+1)^3 = 27(x+y)^2.$

75. $y^2(y'^2+1) = 1.$

76. $(y')^2 - 4y^3 = 0.$

77. $x(y')^2 = y.$

78. $y(y')^3 + x = 1.$

79. $4(1 - y) = (3y - 2)^2(y')^2.$

8- §. $F(y, y') = 0$ va $F(x, y') = 0$ ko'rinishidagi tenglamalar

Bu tenglamalardan y ni (birinchi tenglamadan) yoki x ni (ikkinchi tenglamadan), shuningdek $p = y'$ ni t parametr orqali ifodalash mumkin, deb faraz qilamiz. Bu yerda tenglamaning umumiy yechimi parametrik shaklda hosil bo'ladi.

Masalan, $F(y, p)=0$ tenglama bo'lgan holni ko'raylik. $y=\varphi(t)$ deb tenglamadan $p=\psi(t)$ ni yoki, aksincha, $p=\varphi(t)$ deb tenglamadan $y=\varphi(t)$ ni topdik, deb faraz qilaylik. U holda bir tomondan, $dy=pdx=\psi(t)dx$, ikkinchi tomondan, $dy=\varphi'(t)dt$. dy uchun ikkala ifodani taqqoslab, $\psi(t)dx=\varphi'(t)dt$ ni hosil qilamiz, bundan:

$$dx = \frac{\varphi'(t)}{\psi(t)}dt \quad \text{va} \quad x = \int\frac{\varphi'(t)}{\psi(t)}dt+C.$$

Umumiy yechim parametrik shaklda quyidagicha yoziladi:

$$\begin{cases} x = \int\frac{\varphi'(t)}{\psi(t)}dt+C, \\ y = \varphi(t). \end{cases}$$

1- misol. $y = a\sqrt{1+(y')^2}$ tenglamaning umumiy yechimini toping.

Yechish. $p=y'=\text{sh}t$ deymiz, u holda $y = a\sqrt{1+\text{sh}^2t} = a\cdot\text{ch}t$, $\frac{dy}{dx} = p$ dan $dx = \frac{dy}{p}$ ni topamiz. $dy=a\text{sh}tdt$ bo'lganligidan $dx=adt$ va $x=at-C$.

Umumiy yechim parametrik shaklda quyidagicha yoziladi:

$$\begin{cases} x = at - C, \\ y = a\text{ch}t. \end{cases}$$

Bundan t parametrni yo'qotamiz. $t = \frac{x+C}{a}$ bo'lganligidan $y = a\text{ch}\frac{x+C}{a}$.

Quyidagi tenglamalar yechilsin:

80. $x(y'^2 - 1) = 2y'$.

81. $y'(x - \ln y') = 1$.

82. $x = y'^3 + y'$.

83. $x = y'\sqrt{y'^2+1}$.

84. $y = y'^2 + 2y'^3$.

85. $y = \ln(1+y'^2)$.

86. $(y'+1)^3 = (y' - y)^2$.

87. $y = (y' - 1)e^{y'}$.

88. $(y')^4 - (y')^2 = y^2$.

89. $(y')^2 - (y')^2 = y^2$.

9- §. Lagranj va Klero tenglamalari

1. Lagranj tenglamasi. Ushbu

$$y = x\varphi(y') + \psi(y') \qquad (1.40)$$

tenglama *Lagranj tenglamasi* deyiladi, bu yerda $\varphi(y')$, $\psi(y')$ lar y' ning ma'lum funksiyalari. Bunday tenglama ham p parametr kiritish usuli bilan yechiladi. $y'=p(x)$ deb belgilaymiz. U holda tenglama ushbu ko'rinishga keladi:

$$y = x\varphi(p) + \psi(p). \qquad (1.41)$$

Oxirgi tenglamani x bo'yicha differensiallab,

$$p = \varphi(p) + (x\varphi'(p) + \varphi'(p))\frac{dp}{dx}$$

yoki

$$p - \varphi(p) = (x\varphi'(p) + \varphi'(p))\frac{dp}{dx} \qquad (1.42)$$

tenglamani hosil qilamiz. $p - \varphi(p) \neq 0$ va $p - \psi(p) = 0$ bo'lgan hollarni qaraymiz:

a) $p - \varphi(p) \neq 0$ bo'lsin. (1.42) tenglamani $\frac{dp}{dx}$ ga nisbatan yechib, quyidagi ko'rinishda yozamiz: $\frac{dx}{dp} - x\frac{\varphi'(p)}{p-\varphi(p)} = \frac{\psi'(p)}{p-\varphi(p)}$.

Hosil qilingan tenglama x va $\frac{dx}{dp}$ ga nisbatan chiziqlidir, demak,

$$x = \Phi(p, C) \qquad (1.43)$$

umumiy yechimga ega. (1.43) ni (1.41) ga qo'yib, y ni p va C orqali ifodalaymiz:

$$y = \Phi(p, C)\cdot \varphi(p) + \psi(p) = f(p, C). \qquad (1.44)$$

(1.43) va (1.44) bizga Lagranj tenglamasining umumiy yechimini parametrik ko'rinishda beradi: $\begin{cases} x = \Phi(p,C), \\ y = f(p,C). \end{cases}$

Bu sistemada p parametrni yo'qotib, Lagranj tenglamasining umumiy yechimini quyidagi ko'rinishda hosil qilamiz:

$$F(x, y, C) = 0.$$

Tenglamaning umumiy yechimidan hosil bo‘lmaydigan maxsus yechimi ham bo‘lishi mumkin.

b) $p - \varphi(p) = 0$ bo‘lsin, ya’ni biror $p=p_0$ da $\varphi(p_0)=p_0$ bo‘lsin. Ushbu

$$\begin{cases} y = x\varphi(p) + \varphi(p), \\ p = p_0 \end{cases}$$

sistemada p ni yo‘qotib, $y = x\varphi(p_0) + \psi(p_0)$ yechimni hosil qilamiz. Bu esa Lagranj tenglamasining maxsus yechimidir.

1- misol. Ushbu $y = x + (y')^3$ Lagranj tenglamasining umumiy va maxsus yechimlarini toping.

Y e c h i s h . Bu tenglamada y' ni $p(x)$ ga almashtirib,

$$y = x + p^3 \tag{1.45}$$

tenglamani hosil qilamiz. Uni x bo‘yicha differensiallaymiz: $p = 1 + 3p^2 \frac{dp}{dx}$. Bundan $p - 1 = 3p^2 \frac{dp}{dx}$.

a) Agar $p - 1 \neq 0$ bo‘lsa, ushbu

$$dx = \frac{3P^2}{p-1} dp$$

tenglamani integrallab, quyidagini hosil qilamiz:

$$x = 3(\ln|p - 1| + p + \frac{p^2}{2}) + C, \tag{1.46}$$

x ning hosil qilingan ifodasini (1.45)ga qo‘yamiz:

$$y = 3(\ln|p - 1| + p + \frac{p^2}{2}) + C + p^3 .$$

(1.45) va (1.46) lar Lagranj tenglamasining umumiy yechimini parametr ko‘rinishida beradi.

b) Agar $p - 1 = 0$ bo‘lsa, $p = 1$ qiymatni (1.45) tenglamaga qo‘yib, $y = x+1$ maxsus yechimni hosil qilamiz.

2. Klero tenglamasi. Klero tenglamasi deb, Lagranj tenglamasining $\varphi(y') = y'$ bo‘lgan holiga aytiladi. Klero tenglamasining umumiy ko‘rinishi quyidagicha bo‘ladi:

$$y = xy' + \psi(y'). \tag{1.47}$$

$y' = p(x)$ deb olsak, (1.47) tenglama quyidagicha ko'rinishga keladi:

$$y = xp + \psi(p). \qquad (1.48)$$

x bo'yicha differensiallab, quyidagini topamiz:

$$y' = p + x\frac{dp}{dx} + \psi'(p)\frac{dp}{dx}, \text{ya'ni } \frac{dp}{dx}[x + \psi'(p)] = 0, \text{ bu yerdan } \frac{dp}{dx} = 0$$

yoki

$$x + \psi'(p) = 0. \qquad (1.49)$$

$\frac{dp}{dx} = 0$ tenglamadan $p=C$ kelib chiqadi,(1.48) da p o'rniga C ni qo'yib, Klero tenglamasining umumiy yechimini hosil qilamiz:

$$y = Cx + \psi(C). \qquad (1.50)$$

Bu geometrik nuqtai nazardan to'gri chiziqlar oilasini tasvirlaydi.

(1.49) tenglama (1.48) bilan birgalikda Klero tenglamasining parametrik shakldagi yechimini beradi:

$$\begin{cases} x = -\psi'(p), \\ y = -p\psi'(p) + \psi(p). \end{cases}$$

Haqiqatan ham, bu tenglamalardan: $dx = -\psi''(p)dp$.

$dy = [-p\psi''(p) - \psi'(p) + \psi'(p)]dp = -p\psi''(p)dp$, bu yerdan $\frac{dy}{dx} = p$.

Buni Klero tenglamasiga qo'yish $-p\psi'(p) + \psi(p) = -p\psi'(p) + \psi(p)$ ayniyatga olib keladi.

Sistemaning ikkala tenglamasidan p parametrni yo'qotib, (1.47) tenglamaning integrali $\Phi(x, y)=0$ ni hosil qilamiz. Bu integralda C ishtirok etmaydi, binobarin, u umumiy integral bo'la olmaydi. Uni, shuningdek, umumiy integraldan C ning hech qanday qiymatida hosil qilib bo'lmaydi, chunki chiziqli funksiya bo'lmagani uchun u *maxsus integral* deyiladi.

2- misol. Ushbu $y = xy' + y' - (y')^2$ Klero tenglamasining umumiy va maxsus yechimlarini toping.

Y e c h i s h. Klero tenglamasining umumiy yechimini y' ni C bilan almashtirib topamiz:

$$y = Cx + C - C^2$$

Bu tenglamani C bo'yicha differensiallaymiz:

$$0 = x + 1 - 2C.$$

Quyidagi

$$\begin{cases} y = Cx + C - C^2, \\ 0 = x + 1 - 2C \end{cases}$$

sistemadan C ni yo'qotib,

$$y = \frac{1}{4}(x+1)^2$$

maxsus yechimni hosil qilamiz. U parabola bo'lib, $y = Cx + C - C^2$ umumiy yechimlar oilasining o'ramasini tashkil qiladi.

Lagranj tenglamalarining umumiy va maxsus integrallarini toping:

90. $y = xy' - (y')^2$.

91. $y = 2xy' + \frac{1}{(y')^2}$.

92. $2y = \frac{x(y')^2}{y'+2}$.

93. $y = x(y')^2 + (y')^2$.

94. $y' + y = x(y')^2$.

95. $y + xy' = 4\sqrt{y'}$.

96. $y = x(y')^2 - 2(y')^3$.

97. $2xy' - y = \ln y'$.

98. $xy' - y = \ln y'$.

99. $y = xy' - (y')^2$.

100. $y = xy' - a\sqrt{1+(y')^2}$.

101. $y = xy' + \frac{1}{2y}$.

102. $\sqrt{(y')^2 + 1} + xy' - y = 0$.

103. $y = xy' - e^{y'}$.

104. $y = xy' - (2 + y')$.

105. $(y')^3 = 3(xy' - y)$.

106. $2(y')2(y - xy') = 1$.

107. $y = x\left(\frac{1}{x} + y'\right) + y'$.

10- §. Rikkati tenglamasi

Ushbu

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \qquad (1.50)$$

ko'rinishdagi tenglama *Rikkatining umumiy tenglamasi* deyiladi. Bu yerda $P(x)$, $Q(x)$, $R(x)$ – biror $a < x < b$ oraliqda o'zgaruvchi x ning uzluksiz funksiyalari ($-\infty < a, b < +\infty$).

Tenglamada $P(x) \equiv 0$ bo'lsa, chiziqli tenglama; $R(x) \equiv 0$ bo'lsa Bernulli tenglamasi hosil bo'ladi.

O'zgaruvchilarni quyidagicha almashtirish natijasida Rikkati tenglamasi o'z ko'rinishini saqlaydi:

1) x erkli o'zgaruvchini ixtiyoriy $x = \varphi(x_1)$ ko'rinishda (φ – differensiallanuvchi funksiya) o'zgartirish natijasida tenglamaning ko'rinishi o'zgarmaydi.

Haqiqatan ham, (1.50) tenglamada bu almashtirishni bajarib, yana Rikkati tenglamasini olamiz:

$$\frac{dy}{dx} = P[\varphi(x_1)]\varphi'(x_1)y^2 + Q[\varphi(x_1)]\varphi'(x_1)y + R[\varphi(x_1)]\varphi'(x_1);$$

2) y erksiz o'zgaruvchini kasr chiziqli $y = \frac{\alpha y_1+\beta}{\gamma y_1+\delta}$ ko'rinishda (α, β, γ, δ – qaralayotgan oraliqda $\alpha\delta - \beta\gamma \neq 0$ shartni qanoatlantiruvchi x ning ixtiyoriy differensiallanuvchi funksiyalari) almashtirish natijasida ham tenglama o'z ko'rinishini saqlaydi:

$$\frac{dy}{dx} = \frac{(\alpha\frac{dy_1}{dx}+\alpha' y_1+\beta')\cdot(\gamma y_1+\delta)-(\gamma\frac{dy_1}{dx}+\gamma' y_1+\delta')\cdot(\alpha y_1+\beta)}{(\gamma y_1+\delta)^2} =$$

$$= \frac{(\alpha\delta-\beta\gamma)\frac{dy_1}{dx}+(\alpha'\gamma-\gamma''\alpha)y_1^2+(\alpha'\delta+\beta'\gamma-\alpha\delta'-\beta\gamma')y'+(\beta'\delta-\delta'\beta)}{(\gamma y_1+\delta)^2}.$$

Natijani (1.50) tenglamaga qo'ysak, yana Rikkati tenglamasi hosil bo'lganiga ishonch hosil qilamiz.

Erkli o'zgaruvchi x yoki erksiz o'zgaruvchi y ning bunday shakl almashtirishlarini bajarib, Rikkati tenglamasi soddaroq (kanonik) ko'rinishga keltiriladi.

1) Tenglamada y^2 oldidagi koeffitsiyentni $y=w(x)z$ chiziqli almashtirish orqali ± 1ga tenglashtirish mumkin. Bu yerda $w(x)$ hozircha noma'lum funksiya, tegishli hosilalarni topib (1.50) tenglamaga qo'yamiz, u holda

$$w\frac{dz}{dx} + zw' = P(x)w^2z^2 + Q(x)wz + R(x)$$

yoki

$$\frac{dz}{dx} = P(x)wz^2 + \left(Q(x) - \frac{w'}{w}\right)z + \frac{R(x)}{w}.$$

Agar $w = \pm\frac{1}{P(x)}$ deb olinsa, tenglama ushbu ko'rinishga keladi:

$$\frac{dz}{dx} = \pm z^2 + \left(Q(x) - \frac{P'(x)}{P(x)}\right)z \pm P(x)\cdot R(x).$$

Bu almashtirish x ning $P(x) \neq 0$ bo'lgan o'zgarish oralig'i uchun o'rinlidir.

2) Tenglamada qidirilayotgan y funksiya oldidagi koeffitsiyentni $y=u+\alpha(x)$ almashtirish orqali nolga teng holga keltirish mumkin.

Tegishli hosilalarini topib, (1.50) tenglamaga qo'yamiz, u holda

$$\frac{du}{dx} = P(x)u^2 + [Q(x) + 2P(x)\alpha(x)]u + R(x) + P(x)\alpha^2.$$

u oldidagi koeffitsiyentning 0 ga teng bo'lishi uchun $\alpha(x) = -\frac{Q(x)}{2P(x)}$, ($P(x) \neq 0$) qilib tanlab olish kifoyadir.

Keltirilgan almashtirishlarni birgalikda qo'llab, Rikkati tenglamasini $\frac{dy}{dx} = \pm y^2 + R(x)$ ko'rinishda yozish mumkin.

1- misol. Ushbu $\frac{dy}{dx} = y^2 + \frac{1}{2x^2}$ tenglamani yeching.

Y e c h i s h. $y = \frac{1}{z}$ almashtirishni bajarib, tenglamani $\frac{dz}{dx} = -1 - \frac{1}{2}\left(\frac{z}{x}\right)^2$ shaklga keltiramiz. Bu bir jinsli tenglamani yechishda $\frac{z}{x} = u$ belgilashdan foydalanamiz. U holda $u + x\frac{dy}{dx} = -1 - \frac{1}{2}u^2$;

$\frac{du}{u^2+2u+2} = -\frac{dx}{2x}$; $\frac{du}{1+(u+1)^2} = -\frac{dx}{2x}$ tenglikni integrallab,

$$\text{arctg}(1 + u) = \frac{1}{2}\ln x + C$$

yoki

$$u+1=\operatorname{tg}\left(C-\frac{1}{2}\ln x\right),$$

$$z=x\left[-1+\operatorname{tg}\left(C-\frac{1}{2}\ln x\right)\right]$$

ifodaga ega bo'lamiz. Demak, izlangan yechim quyidagicha bo'ladi:

$$y=\frac{1}{x\left[-1+\operatorname{tg}\left(C-\frac{1}{2}\ln x\right)\right]}.$$

Quyidagi tenglamalrni yeching:

108. $y' + ay^2 - axy - 1 = 0.$

109. $y' + y^2 = 2/x^2.$

110. $xy^2 + xy + x^2y^2 = 4.$

111. $3y' + y^2 + 2/x^2 = 0.$

112. $xy' - (2x+1)y + y^2 = -x^2.$

113. $y' - 2xy + y^2 = 5 - x^2.$

114. $y' + 2ye^x - y^2 = e^{2x} + e^x.$

115. $3xy' - (2x+3)y + y^2 = -x^2.$

116. $2xy' - (3x+2)y + y^2 = -2x^2.$

117. $5xy' - (4x+5)y + y^2 = -3x.$

II BOB

YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR

1- §. Asosiy tushunchalar

n- tartibli oddiy differensial tenglama deb,

$$F(x, y, y', y'', \dots y^{(n)}) = 0 \qquad (2.1)$$

ko'rinishdagi tenglamaga aytiladi.

Bu tenglamaning yechimi deb, n marta differensiallanuvchi va (2.1) tenglamaga qo'yish natijasida uni ayniyatga aylantiruvchi $y = \varphi(x)$ funksiyaga aytiladi, ya'ni .

$$F\left[x, \varphi(x), \varphi'(x), \varphi''(x) \dots, \varphi^{(n)}(x)\right] = 0 .$$

Koshi masalasi. (2.1) tenglamaning

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad y''(x_0) = y_0'', \quad \dots, \quad y^{(n)}(x_0) = y_0^{(n)} \qquad (2.2)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

$$y = \varphi(x, C_1, C_2, \dots, C_n)$$

funksiya (2.1) tenglamaning umumiy yechimi bo'lsin. C_1, C_2, ..., C_n o'zgarmas sonlarni (2.2) Koshi shartlari orqali aniqlab, tegishli xususiy yechim hosil qilinadi.

Umumiy yechimdan xususiy yechimni hosil qilishda qaralayotgan oraliqning chetki nuqtalarida berilgan chegaraviy shartlardan ham foydalaniladi.

Koshi shartlari deb ataluvchi boshlang'ich shartlar soni tenglamaning tartibi bilan teng bo'lishini ta'kidlab o'tamiz.

n- tartibli differensial tenglamani faqat ayrim xususiy hollardagina bevosita integrallash mumkin.

Tartibini pasaytirish mumkin bo'lgan tenglamalar

2- §. $y^{(n)} = f(x)$ ko'rinishdagi tenglama

Bunday ko'rinishdagi tenglamani n marta ketma-ket integrallash natijasida umumiy yechimi topiladi:

$$y^{(n)} = f(x), \tag{2.3}$$

$$y^{(n-1)} = \int f(x)dx + C_1 = f_1(x) + C_1,$$

$$y^{(n-2)} = \int [f_1(x) + C_1]dx + C_2 = f_2(x) + C_1 x + C_2,$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,$$

$$y = f_n(x) + \frac{C_1}{(n-1)!}x^{n-1} + \frac{C_2}{(n-2)!}x^{n-2} + \ldots + C_{n-1}x + C_n, \tag{2.4}$$

bu yerda $f_n(x) = \int\int\ldots\int f(x)dx^n$. $\frac{C_1}{(n-1)!}, \frac{C_2}{(n-2)!}, \ldots, C_n$ lar o'zgarmas sonlar bo'lgani uchun (2.4) ni quyidagicha ham yozish mumkin:

$$y = f_n(x) + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_{n-1}x + C_n.$$

1- misol. $y''' = \sin x$ tenglamaning umumiy yechimi topilsin.

Y e c h i s h. $y''' = \frac{dy''}{dx}$ ekanligini e'tiborga olib, berilgan tenglamani $\frac{dy''}{dx} = \sin x$ yoki $dy'' = \sin x dx$ ko'rinishda yozish mumkin. Ketma-ket integrallab, quyidagiga ega bo'lamiz:

$$y'' = \int \sin x dx + C_1 = -\cos x + C_1,$$

$$y' = \int (-\cos x + C_1)dx + C_2 = -\sin x + C_1 x + C_2,$$

$$y = \int (-\sin x + C_1 x + C_2)dx + C_3 = \cos x + \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

Demak, $y = \cos x + Cx^2 + C_2 x + C_3$, $C = \frac{1}{2}C_1$.

Izlangan umumiy yechimga ega bo'ldik.

2- misol. $y'' = xe^{-x}$ tenglamaning $y(0) = 1, y'(0) = 0$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Berilgan tenglamani ketma-ket integrallash natijasida umumiy yechimni aniqlaymiz:

$$y' = \int xe^{-x}dx + C_1 = -xe^{-x} - e^{-x} + C_1,$$

$$y = \int(-xe^{-x} - e^{-x} + C_1)dx + C_2 = xe^{-x} + e^{-x} + e^{-x} + C_1x + C_2$$

yoki

$$y = e^{-x}(x + 2) + C_1x + C_2.$$

Boshlang'ich shartlarni e'tiborga olsak,

$$1 = e^{-0}(0 + 2) + C_1 \cdot 0 + C_2, C_2 = -1,$$

$$y' = -xe^{-x} - e^{-x} + C_1$$

dan

$$0 = -0e^{-0} - e^{-0} + C_1, \ C_1 = 1.$$

Demak, izlangan xususiy yechim quyidagi ko'rinishda bo'ladi:

$$y = e^{-x}(x + 2) + x - 1.$$

Quyidagi tenglamalarni yeching:

118. $y^{IV} = \cos^2 x, y(0) = \frac{1}{32}, y'(0) = 0, y''(0) = \frac{1}{8}, y'''(0) = 0.$

119. $y''' = x\sin x, y(0) = 0, y'(0) = 0, y''(0) = 2.$

120. $y'''\sin^4 x = \sin 2x.$

121. $y'' = 2\sin x\cos^2 x - \sin^3 x.$

122. $y''' = xe^{-x}, y(0) = 0, y'(0) = 2, y''(0) = 2.$

123. $y''' = \frac{6}{x^3}, y(1) = 2, y'(1) = 1, y''(1) = 1.$

124. $y'' = 4\cos 2x, y(0) = 0, y'(0) = 0.$

125. $y'' = \frac{1}{1+x^2}.$

126. $y'' = \frac{1}{\cos^2 x}, \; y\left(\frac{\pi}{4}\right) = \ln\sqrt{2}, \; y'\left(\frac{\pi}{4}\right) = 1$.

127. $y''' = x^{-2}$.

128. $y^{IV} = \cos x$.

129. $y'' = \frac{1}{\sin^2 x}$.

130. $y'' = xe^x, \; y(0) = 1, \; y'(0) = 2$.

131. $y'' = \sin 2x, \; y(0) = 6, \; y'(0) = 0$.

3- §. Noma'lum funksiya oshkor holda qatnashmagan tenglamalar

$$F(x, y^{(k)}, y^{(k+1)}, ..., y^{(n)}) = 0 \tag{2.5}$$

tenglamada y funksiya oshkor holda qatnashmagan. Bu tenglamada

$$y^{(k)} = p(x) \tag{2.6}$$

almashtirishni bajarib, uni

$$F(x, p, p', ..., p^{n-k}) = 0$$

ko'rinishga keltiriladi. Shunday qilib, (2.5) tenglamani tartibi k birlikka pasayadi.

1- misol. $xy'' = y' \ln\left(\frac{y'}{x}\right)$ tenglamaning umumiy yechimi topilsin.

Y e c h i s h . Bu tenglamada y funksiya oshkor holda qatnashmagani uchun $y' = p(x)$ almashtirishni bajaramiz. Bu holda $y'' = p'$ o'rinli bo'ladi. Bularni tenglamaga qo'ysak,

$$x \cdot p' = p \ln\frac{p}{x} \quad \text{yoki} \quad p' = \frac{p}{x}\ln\frac{p}{x}.$$

Hosil bo'lgan tenglama birinchi tartibli bir jinsli tenglama bo'lganidan $\frac{p}{x} = t$ yoki $p = x \cdot t$ almashtirishni bajarsak, $p' = t + xt'$ ga ega bo'lamiz. Buni e'tiborga olib, tenglamani $t + xt' = t\ln t$ yoki $xt' = t(\ln t - 1)$ ko'rinishda yozish mumkin. O'zgaruvchilarni ajratsak,

$$\frac{dt}{t(\ln t - 1)} = \frac{dx}{x}$$

tenglamaga ega bo'lamiz. Integrallash natijasida

$$\ln(\ln t - 1) = \ln x + \ln C_1 \quad \text{yoki} \quad \ln t - 1 = C_1 x,$$

bundan esa $t = e^{C_1 x + 1}$ kelib chiqadi. $t = \frac{p}{x}$ ekanini e'tiborga olsak,

$$p = xe^{C_1 x + 1}$$

hosil bo'ladi. $p(x) = y'$ dan $y' = xe^{C_1 x + 1}$ tenglik hosil bo'ladi. Bundan esa izlangan umumiy yechim

$$y = \int xe^{C_1 x + 1} dx = \frac{1}{C_1} xe^{C_1 x + 1} - \frac{1}{C_1^2} e^{C_1 x + 1} + C_2$$

ko'rinishda hosil bo'ladi.

2- misol. $y'''(x-1) - y'' = 0$ tenglamaning $y(2) = 2, y'(2) = 1, y''(2) = 1$ shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h. $y'' = p(x)$ va $y''' = p'$ almashtirish bajarsak, dastlabki tenglama $p'(x-1) = p$ yoki $\frac{dp}{p} = \frac{dx}{(x-1)}$ ko'rinishga keladi. Integrallash natijasida $\ln p = \ln(x-1) + \ln C_1$ yoki $p = C_1(x-1)$ yechim hosil bo'ladi. Dastlabki belgilashni e'tiborga olib, $y'' = C_1(x-1)$ natijaga ega bo'lamiz. Bu esa tartibi pasayadigan tenglamadan iborat. Ketma-ket integrallab:

$$y' = \int C_1(x-1)dx + C_2 = \frac{1}{2}C_1 x^2 - C_1 x + C_2,$$

$$y = \int\left(\frac{1}{2}C_1 x^2 - C_1 x + C_2\right)dx + C_3 = \frac{C_1}{6}x^3 - \frac{C_1}{2}x^2 + C_2 x + C_3$$

umumiy yechimni hosil qilamiz. Chetki shartlarni e'tiborga olib

$$y''(2)=1 \text{ dan } 1=C_1(2-1) \text{ yoki } C_1=1,$$

$$y'(2)=1 \text{ dan } 1=\frac{1}{2}\cdot 4-2+C_2 \text{ yoki } C_2=1,$$

$$y(2)=2 \text{ dan } 2=\frac{8}{6}-\frac{4}{2}+2+C_3 \text{ yoki } C_3=\frac{2}{3}$$

natijalarni hosil qilamiz. Bundan esa

$$y=\frac{1}{6}x^3-\frac{1}{2}x^2+x+\frac{2}{3}$$

xususiy yechimni topamiz.

3- masala. m massali jism samolyotdan boshlang'ich tezliksiz tashlandi. Unga o'z tezligining kvadratiga teng miqdorda havo qarshilik ko'rsatmoqda. Jismning harakat qonunini toping.

Y e c h i s h . Quyidagi belgilashlarni kiritamiz:

s — jism bosib o'tgan masofa;

$v=\frac{ds}{dt}$ — jism tezligi; $w=\frac{d^2s}{dt^2}$ — tezlanish.

Jismga quyidagi kuchlar ta'sir etadi:

$p=mg$ — harakati yo'nalishidagi og'irlik kuchi;

$F=mv^2=k\left(\frac{ds}{dt}\right)^2$ — qarama-qarshi yo'nalishdagi havo qarshiligi.

Nyutonning ikkinchi qonuniga asosan jismning harakat qonunini ifodalovchi quyidagi differensial tenglamani yozamiz:

$mw=p-kv^2$ yoki $m\frac{d^2s}{dt^2}=mg-k\left(\frac{ds}{dt}\right)^2\cdot\frac{ds}{dt}=v$ ekanini e'tiborga olsak, $m\frac{dv}{dt}=mg-kv^2$ yoki $\frac{dv}{dt}=\frac{k}{m}\left(\frac{gm}{k}-v^2\right)$ tenglama hosil bo'ladi.

$a^2=\frac{gm}{k}$ belgilash bajarsak, o'zgaruvchilari ajraladigan $\frac{dv}{dt}=\frac{k}{m}(a^2-v^2)$ tenglamani hosil qilamiz.

O‘zgaruvchilarini ajratib, $\frac{dv}{(a^2-v^2)}=\frac{k}{m}dt$ integrallash yordamida $\frac{1}{2a}\ln\left|\frac{a+v}{a-v}\right|=\frac{k}{m}t+C_1$ natijani hosil qilamiz.

Masala shartiga ko‘ra, t=0 da $v(0)$=0 ekanligini e'tiborga olsak, C_1=0 kelib chiqadi. Shunday qilib, $\ln\left|\frac{a+v}{a-v}\right|=\frac{2ak}{m}t$ tenglikdan v ni topsak, $v=a\left(e^{\frac{2akt}{m}}-1\right)\Big/\left(e^{\frac{2akt}{m}}+1\right)=a\frac{e^{\frac{akt}{m}}-e^{-\frac{akt}{m}}}{e^{\frac{akt}{m}}+e^{-\frac{akt}{m}}}=a\text{th}\frac{akt}{m}$ hosil bo‘ladi.

$\frac{ak}{m}=\sqrt{\frac{mg}{k}}\cdot\frac{k}{m}=\sqrt{\frac{kg}{m}}$ va $v=\frac{ds}{dt}$ ekanini hisobga olib, $\frac{ds}{dt}=a\text{th}\sqrt{\frac{kg}{m}t}$ tenglamani hosil qilamiz va uning yechimi $s=\sqrt{\frac{m}{kg}}a\ln\text{ch}\sqrt{\frac{kg}{m}}t+C_2=\frac{m}{k}\ln\text{ch}\sqrt{\frac{kg}{m}}t+C_2$, t=0 da $s(0)$=0 ekanligidan C_2=0 bo‘lib, jismni bosib o‘tgan yo‘li $s=\frac{m}{k}\ln\text{ch}\sqrt{\frac{kg}{m}}t$ formula bilan, tezligi esa $v=a th\sqrt{\frac{kg}{m}}t$ formula bilan ifodalanadi.

Bu formuladagi $a=\sqrt{\frac{mg}{k}}$, $\lim\limits_{t\to\infty}v=a\lim\limits_{t\to\infty}\text{th}\sqrt{\frac{kg}{m}}t=a=\sqrt{\frac{p}{k}}$ ekanligidan tushish tezligi cheksiz orta olmaydi hamda tezda $v=\sqrt{\frac{p}{k}}$ limit qiymatga erishadi va xuddi shu holat parashyut bilan sakrashda ham sodir bo‘ladi.

Quyidagi tenglamalarni yeching:

132. $x^3y'' + x^2y' = 1$.
133. $y'' + y'\mathrm{tg}x = \sin 2x$.
134. $y''x\ln x = y'$.
135. $xy'' - y' = e^x \cdot x^2$.
136. $y'' + 2xy'^2 = 0$.
137. $(1 - x^2)y'' - xy' = 2$.
138. $2xy''' \cdot y'' = y''^2 - a^2$.
139. $(1 + x^2)y'' + 1 + y'^2 = 0$.
140. $x^2y'' = y'^2$.
141. $y''(e^x + 1) + y' = 0$.
142. $(1 + x^2)y'' + 2xy' = x^3$.
143. $y''\mathrm{tg}x = y' + 1$.
144. $xy'' + y' + x = 0$
145. $y'' - \frac{1}{x-1}y' = x(x-1)$, $y(2) = 1, y'(2) = -1$.
146. $xy'' = y' + x\sin\frac{y'}{x}$.
147. $(1 - x^2)y'' + xy' = 2$.

4-§. Argument oshkor holda qatnashmagan tenglama

$$F\left(y, y', y'', \dots y^{(n)}\right) = 0 \qquad (2.7)$$

tenglamada erkli o'zgaruvchi x oshkor holda ishtirok etmaydi. Bu tenglama

$$y' = p(y) \qquad (2.8)$$

almashtirish bilan tartibini bittaga pasaytirib yechiladi.

(2.8) almashtirishda: $y'' = p'(y) \cdot y' = p \cdot p'$,

$$y''' = p\left[p \cdot p'' + p'^2\right], \dots$$

o'rniga qo'yishlar bajariladi.

1- misol. $1 + y'^2 = y \cdot y''$ tenglamaning umumiy yechimini toping.

Y e c h i s h. $y' = p(y)$ va $y'' = pp'$ almashtirishlarni bajarsak, dastlabki tenglama $1 + p^2 = y \cdot p \cdot p'$ ko'rinishga keladi, bu esa birinchi tartibli o'zgaruvchilari ajraladigan tenglamadir.

O‘zgaruvchilarni ajratib, $\frac{pdp}{1+p^2}=\frac{dy}{y}$ tenglamani hosil qilamiz. Tenglikni integrallab, quyidagiga ega bo‘lamiz:

$$\frac{1}{2}\ln\left|1+p^2\right|=\ln y+\ln C_1 \text{ yoki } 1+p^2=C_1^{\,2}y^2,\; p=\pm\sqrt{C_1^{\,2}y^2-1}.$$

Dastlabki o‘zgaruvchi y ga qaytib, $y'=\pm\sqrt{C_1^{\,2}y^2-1}$ yoki $\frac{dy}{\sqrt{C_1^{\,2}y^2-1}}=\pm dx$ natijaga ega bo‘lamiz. Tenglikni integrallab, $\frac{1}{C_1}\ln\left(C_1y+\sqrt{C_1^2y^2-1}\right)=\pm(x+C_2)$ yoki $y=\frac{1}{2C_1}\left(e^{\pm(x+C_2)C_1}+e^{\pm(x+C_2)C_1}\right)=$ $=\frac{1}{C_1}\operatorname{ch}C_1\left(x+C_2\right)$ izlangan umumiy yechimni hosil qilamiz.

2- misol. $M(0;\ 1)$ nuqtadagi urinmasi OX o‘q bilan $\alpha=45°$ burchak tashkil qiluvchi va egrilik radiusi normalning kubiga teng bo‘lgan chiziq tenglamasini tuzing.

Y e c h i s h . Egri chiziqning egrilik radiusi va normali tenglamalari quyidagicha edi:

$$R=(1+y'^2)^{3/2}/y'',\quad N=y\sqrt{1+y'^2}.$$

Masala shartiga asosan $R=N^3$ ekanligidan, quyidagi differensial tenglamaga ega bo‘lamiz:

$$(1+y'^2)^{3/2}/y''=y^3(\sqrt{1+y'^2})^3.$$

Tenglikning har ikki tomonini $(1+y'^2)^{3/2}$ ga bo‘lib, $1/y''=y^3$ yoki $y''y^3=1$ tenglamani hosil qilamiz. $y'=p(y)$ va $y''=pp'$ almashtirish bajarsak, $pp'y^3=1$ tenglik hosil bo‘ladi. O‘zgaruvchilarni ajratib va integrallab, quyidagi yechimni hosil qilamiz:

$$\frac{pdp}{dy}y^3=1,\; pdp=y^{-3}dy,\; \frac{1}{2}p^2=-\frac{1}{2}y^{-2}+\frac{1}{2}C_1$$

yoki

$$p^2=C_1-y^{-2}.$$

Dastlabki o‘zgaruvchiga qaytsak, $y'^2 = C_1 - y^{-2}$ tenglama hosil bo‘ladi. Masala shartiga asosan $y'(x_0) = \mathrm{tg}45^\circ = 1$ yoki $y(0)=1$, $y'(0)=1$, bundan $1=C_1-1$, ya’ni $C_1=2$. Shunday qilib, noma’lum funksiyani aniqlash uchun birinchi tartibli $y'^2 = 2 - y^{-2}$ yoki $y' = \frac{\sqrt{2y^2-1}}{y}$ tenglama kelib chiqadi. Bu tenglamaning o‘zgaruvchilarini ajratib, integrallaymiz:

$$\frac{ydy}{\sqrt{2y^2-1}} = dx, \quad \frac{1}{2}\sqrt{2y^2-1} = x + \frac{1}{2}C_2$$

yoki $y = \frac{1}{2}\left[(2x + C_2)^2 + 1\right]$. Izlangan chiziqning $M(0;\ 1)$ dan o‘tishini e’tiborga olsak, $1 = \frac{1}{2}\left[(2\cdot 0 + C_2)^2 + 1\right]$, $C_2 = 1$.

Demak, $y = 2x^2 + 2x + 1$ yechim hosil bo‘ladi.

Quyidagi tenglamalarni yeching:

148. $y \cdot y'' + y'^2 = 0$.

149. $y'' + 2y(y')^3 = 0$.

150. $y''\mathrm{tg}y = 2y'^2$.

151. $y''(2y+3) - 2y'^2 = 0$.

152. $y(1-\ln y)y'' + (1+\ln y)y'^2 = 0$.

153. $y''(1+y) = y'^2 + y'$.

154. $yy'' + y = y'^2$.

155. $y'^2 + 2yy'' = 0$.

156. $yy'' - y'^2 = 0$, $y(0) = 1,\ y'(0) = 2$.

157. Egrilik radiusining OY o‘qdagi proyeksiyasi o‘zgarmas a bo‘lib, OX o‘q bilan esa koordinata boshida kesishuvchi egri chiziq tenglamasini tuzing.

158. Suyuqlikka tashlangan m massali jism o‘z og‘irligi tufayli cho‘ka boshladi. Agar suyuqlik qarshiligi jism tezligiga proporsional bo‘lsa, harakat qonunini toping.

159. $2yy'' = (y')^2$.

160. $y''y^3 = 1$.

161. $2yy'' = 1 + y'^2$.

162. $y \cdot y'' = y'^2 + y^2 \ln y$.

163. $y'' = y'/\sqrt{y}$.

5- §. Noma'lum funksiya va hosilalarga nisbatan bir jinsli tenglamalar

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0 \qquad (2.9)$$

tenglama $x, y, y', y'', \ldots, y^{(n)}$ larga nisbatan bir jinsli bo'lsa,

$$\frac{y'}{y} = p(x) \qquad (2.10)$$

almashtirish yordamida (2.9) ni tartibini bittaga pasaytirib yechiladi.

1- misol. $3y'^2 = 4y \cdot y'' + y^2$ tenglamani yeching.

Y e c h i s h. Berilgan tenglama y, y', y'' larga nisbatan bir jinsli ekanligidan, tenglamaning har ikki tomonini y^2 ga bo'lib, $3 \cdot \left(\frac{y'}{y}\right)^2 - 4 \cdot \frac{y''}{y} = 1$ ko'rinishga keltiramiz. $\frac{y'}{y} = p(x)$, ya'ni $p'(x) = \frac{y''y - y'^2}{y^2}$, $\frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = p'$ yoki $\frac{y''}{y} = p' - p^2$ almashtirish bajarib, o'zgaruvchilari ajraladigan $3p^2 - 4p^2 - 4p' = 1$ yoki $4p' = -1 - p^2$ birinchi tartibli tenglamaga ega bo'lamiz.

O'zgaruvchilarni ajratib va integrallab, quyidagi natijaga kelamiz: $\frac{dp}{1+p^2} = -\frac{1}{4}dx$ yoki $\operatorname{arctg} p = C_1 - \frac{1}{4}x$, bundan esa $p = \operatorname{tg}\left(C - \frac{x}{4}\right)$ yoki $\frac{y'}{y} = \operatorname{tg}\left(C - \frac{x}{4}\right)$ hosil bo'lgan tenglamaning o'zgaruvchilarini ajratgandan so'ng, integrallab $\ln|y| = 4\ln\left|\cos\left(C_1 - \frac{x}{4}\right)\right| + \ln|C_2|$ yoki $y = C_2\cos^4\left(C_1 - \frac{x}{4}\right)$ yechimga ega bo'lamiz.

2- misol. $y'^2 + yy'' = yy'$ tenlamani yeching.

Y e c h i s h . Bu tenglama ham avvalgi tenglama kabi y, y', y'' larga nisbatan bir jinsli bo'lgani uchun yuqoridagi usulni qo'llash mumkin. Lekin tenglamaning chap tomonidagi ifoda $(yy')'$ ga tengligi, ya'ni $(yy')' = y'^2 + yy''$ ekanligidan $(yy')' = yy'$ tenglamaga ega bo'lamiz. $yy' = z$ almashtirish bajarsak, sodda $z' = z$ tenglamaga ega bo'lamiz va uning umumiy yechimi $z = C_1e^x$ ko'rinishda bo'ladi. Belgilashga asosan $yy' = C_1e^x$ yoki $ydy = C_1e^x dx$ ni integrallab, quyidagi umumiy yechimni hosil qilamiz:

$$y^2 = 2C_1e^x + C_2.$$

Quyidagi tenglamalarni yeching:

164. $yy'' - y'^2 = 0.$

165. $(y + y')y'' + y'^2 = 0.$

166. $2xy''' \cdot y'' = y'^2 - a^2.$

167. $y'' = y'e^y$, $y(0) = 0$, $y'(0) = 1.$

168. $xyy'' - xy'^2 = yy'.$

169. $yy'' = y'^2 + 15y^2\sqrt{x}.$

170. $(1 + x^2)(y'^2 - yy'') = xyy'.$

171. $xyy'' + xy'^2 = 2yy'.$

172. $x^2yy'' = (y - xy')^2.$

173. $y'' + \dfrac{y'}{x} + \dfrac{y}{x^2} = \dfrac{y'^2}{y}.$

174. $x^2yy'' + y'^2 = 0.$

175. $x^2(y'^2 - 2yy'') = y^2.$

176. $xyy'' = y'(y + y').$

177. $4x^2y^3y'' = x^2 - y^4.$

178. $x^3y'' = (y - xy')(y - xy' - x).$

6- §. Yuqori tartibli chiziqli tenglama

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x) \quad (2.11)$$

ko'rinishdagi tenglama *n- tartibli chiziqli bir jinsli bo'lmagan tenglama* deyiladi. Bu yerda $a_1(x), a_2(x), \ldots, a_n(x)$ va $f(x)$ — ma'lum va biror oraliqda uzluksiz bo'lgan funksiyalar.

Agar $f(x) = 0$ bo'lsa, bu tenglama *chiziqli bir jinsli tenglama* deyiladi.

Chiziqli bir jinsli tenglamaning birorta y_1 xususiy yechimini bilgan holda

$$y = y_1 \cdot \int z(x)dx \tag{2.12}$$

chiziqli almashtirish yordamida berilgan tenglamaning tartibini bittaga pasaytirish mumkin. U holda mos bir jinsli bo'lmagan tenglama ham $z(x)$ ga nisbatan $(n-1)$- tartibli chiziqli tenglamaga keladi.

1- misol. $y''' + \frac{2}{x}y'' - y' + \frac{1}{x\ln x}y = x$ tenglamani $y_1 = \ln x$ xususiy yechimini bilgan holda tartibini pasaytiring.

Yechish. (2.12) formulaga asosan $y = \ln x \int z(x)dx$ almashtirishni bajaramiz. Tegishli hosilalar

$$y' = \frac{1}{x}\int zdx + z\ln x,\ y'' = -\frac{1}{x^2}\int zdx + \frac{2z}{x} + z'\ln x,$$

$$y''' = \frac{2}{x^3}\int zdx - \frac{3z}{x^2} + \frac{3z'}{x} + z''\ln x$$

ni berilgan tengamaga qo'yib, $z(x)$ ga nisbatan quyidagi ikkinchi tartibli tenglamaga ega bo'lamiz:

$$z''\ln x + \left(\frac{3}{x} + \frac{2\ln x}{x}\right)\cdot z' + \left(\frac{1}{x^2} - \ln x\right)z = x.$$

2- misol. $y'' + \frac{2}{x}y' + y = 0$ tenglamaning xususiy yechimi $y_1 = \frac{\sin x}{x}$ ekanligini bilgan holda uning umumiy yechimini toping.

Yechish. (2.12) formulaga ko'ra $y = \frac{\sin x}{x}\int z(x)dx$ almashtirishni bajaramiz. Tegishli hosilalarni

$$y' = \frac{x\cos x - \sin x}{x^2}\int zdx + \frac{\sin x}{x}z,$$

$$y'' = \frac{\sin x}{x} z' + \frac{2(x\cos x - \sin x)}{x^2} \cdot z - \frac{(x^2-2)\sin x + 2x\cos x}{x^3} \int z dx$$

tenglamaga qo'ysak, quyidagi birinchi tartibli tenglama hosil bo'ladi:

$$\sin x \cdot z' + 2\cos x \cdot z = 0 \quad \text{yoki} \quad \frac{dz}{z} = -2\frac{\cos x}{\sin x} dx.$$

Tenglikni integrallab, $z = \frac{C_1}{\sin^2 x}$ yechimga ega bo'lamiz.

Natijani dastlabki almashtirishga qo'yib,

$$y = \frac{\sin x}{x} \int \frac{C_1}{\sin^2 x} dx = \frac{\sin x}{x}(C_2 - C_1 \text{ctg} x)$$

yoki

$$y = C_2 \frac{\sin x}{x} - C_1 \frac{\cos x}{x}$$

izlangan umumiy yechimni topamiz.

Misollarni yeching:

179. $y'' \sin^2 x = 2y$ tenglamaning $y = \text{ctg} x$ xususiy yechimini bilgan holda tartibini pasaytiring.

180. $y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$ tenglamaning $y=x$ xususiy yechimini bilgan holda tartibini pasaytirib integrallang.

181. $y''' + (\text{tg} x - 2\text{ctg} x)y'' + 2\text{ctg}^2 x y = 0$ tenglamaning $y=\sin x$ xususiy yechimini bilgan holda tartibini pasaytirib, uning umumiy yechimini toping.

7- §. Chiziqli bir jinsli tenglamalar

(2.11) tenglamada $f(x)=0$ bo'lsin, ya'ni

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_n(x)y = 0 \qquad (2.13)$$

ko'rinishdagi tenglama berilgan. y_1, y_2, ..., y_n funksiyalar (2.13) tenglamaning chiziqli erkli xususiy yechimlari bo'lsa, quyidagi teorema o'rinli.

Teorema. Agar (2.13) tenglamaning xususiy chiziqli erkli yechimlari y_1, y_2, ..., y_n funksiyalar bo'lsa,

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n \tag{2.14}$$

funksiya (2.13) tenglamaning umumiy yechimi bo'ladi (C_1, C_2, ..., C_n – ixtiyoriy o'zgarmas sonlar).

Izoh. y_1, y_2, ..., y_n funksiyalar $(a; b)$ oraliqda

$$\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n \neq 0 \tag{2.15}$$

shart noldan farqli $\alpha_1, \alpha_2, \ldots, \alpha_n$ sonlar uchun o'rinli bo'lsa, bu funksiyalar *chiziqli erkli funksiyalar*, aks holda *chiziqli bog'liq funksiyalar* deyiladi.

Ikkita funksiya uchun $\alpha_1 y_1 + \alpha_2 y_2 \neq 0$ (2.15) shart $\frac{y_1}{y_2} \neq -\frac{\alpha_1}{\alpha_2} = C$ shartga mos keladi, ya'ni ikkita funksiya chiziqli erkli bo'lishi uchun ularning nisbati o'zgarmas son bo'lmasligi kerak.

Masalan. 1. $y_1 = x, y_2 = x^2$ funksiyalar $\frac{y_1}{y_2} = \frac{x}{x^2} = \frac{1}{x} \neq C$ bo'lgani uchun chiziqli erkli.

2. $y_1 = e^x$, $y_2 = e^{-x}$ funksiyalar $\frac{y_1}{y_2} = e^{2x} \neq C$ bo'lganidan chiziqli erkli.

3. $y_1 = 2e^{3x}$, $y_2 = 5e^{3x}$ funksiyalar $\frac{y_1}{y_2} = \frac{2}{5} = 0{,}4$ bo'lgani uchun chiziqli bog'liq. (a, b) oraliqda berilgan $(n-1)$- tartibgacha uzluksiz hosilaga ega bo'lgan n ta funksiyaning chiziqli erkli bo'lishining yetarli sharti bo'lib, $W(y_1, y_2, \ldots, y_n)$ – Vronskiy determinantining noldan farqli bo'lishi xizmat qiladi, ya'ni

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} \neq 0. \tag{2.16}$$

Agar y_1, y_2, .., y_n funksiyalar (2.13) tenglamaning xususiy yechimlari bo'lsa, vronskianning noldan farqli bo'lishi zarur va yetarli.

(2.13) tenglamaning vronskiani (2.16) $a_1(x)$ koeffitsiyent bilan (a, b) oraliqning x_0 nuqtasida

$$W(y_1, y_2, \dots, y_n) = W(y_1, y_2, \dots, y_n)\Big|_{x=x_0} \cdot e^{-\int\limits_{x_0}^{x} a_1(x)dx} \qquad (2.17)$$

Liuvilli-Ostragradskiy formulasi bilan ifodalanadi.

(2.13) tenglamaning chiziqli erkli yechimlari to'plami *yechimlarning fundamental sistemasi* deyiladi.

Ikkinchi tartibli

$$y'' + a_1(x)y' + a_2(x)y = 0 \qquad (2.18)$$

chiziqli bir jinsli tenglamaning fundamental sistemasi $y_1(x)$ va $y_2(x)$ funksiyalardan iborat bo'lsa, uning umumiy yechimi

$$y = C_1 y_1(x) + C_2 y_2(x) \qquad (2.19)$$

ko'rinishda bo'ladi.

Agar (2.18) tenglamaning bitta xususiy yechimi $y_1(x)$ ma'lum bo'lsa, ikkinchi chiziqli erkli yechim Liuvilli-Ostragradskiy formulasi, ya'ni

$$y_2(x) = y_1(x) \int \frac{e^{-\int a_1(x)dx}}{y_1^2(x)} dx \qquad (2.20)$$

yordamida aniqlanadi. Bu usul ikkinchi tartibli bir jinsli tenglamaning bitta yechimi ma'lum bo'lganda, uning tartibini pasaytirmasdan birdaniga (2.20) formula yordamida $y_2(x)$ ni topib, (2.19) formula orqali umumiy yechimni yozishga imkon beradi.

1- misol. $y'' + \frac{2}{x} y' + y = 0$ tenglamaning xususiy yechimi $y_1 = \frac{\sin x}{x}$ bo'lgan holda uning umumiy yechimini toping.

Y e c h i s h . (2.20) formula yordamida $y_2(x)$ ni topamiz:

$$y_2(x) = \frac{\sin x}{x} \int \frac{e^{-2\int \frac{dx}{x}}}{\left(\frac{\sin x}{x}\right)^2} dx = \frac{\sin x}{x} \int \frac{dx}{\sin^2 x} = -\frac{\cos x}{x}.$$

Demak, (2.19) formulaga asosan tenglamaning umumiy yechimi quyidagi ko'rinishda bo'ladi:

$$y = C_1 \frac{\sin x}{x} - C_2 \frac{\cos x}{x}.$$

2- misol. $y = C_1 e^{3x} + C_2 e^{-3x}$ funksiya $y'' - 9y = 0$ tenglamaning umumiy yechimi ekanini ko'rsating.

Y e c h i s h . $y_1 = e^{3x}$ va $y_2 = e^{-3x}$ funksiyalarning har biri berilgan tenglamani qanoatlantiradi. Bu xususiy yechimlar o'zaro chiziqli erkli, chunki $\frac{y_1}{y_2} = \frac{e^{3x}}{e^{-3x}} = e^{6x} \neq C$. Shuning uchun bu ikki yechim fundamental sistemani tashkil etadi, demak,

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{3x} + C_2 e^{-3x}$$

umumiy yechim bo'ladi.

3- misol. $y''' - y' = 0$ tenglamaning $y_1 = e^x, y_2 = e^{-x}, y_3 = \text{ch}x$ xususiy yechimlari fundamental sistema tashkil etadimi?

Y e c h i s h . Buning uchun vronskianni hisoblaymiz:

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & \text{ch}x \\ e^x & -e^{-x} & \text{sh}x \\ e^x & e^{-x} & \text{ch}x \end{vmatrix} = 0,$$

chunki birinchi va uchinchi satr elementlari bir xil. Shunday qilib, bu funksiyalar chiziqli bog'liq, ya'ni ular fundamental sistemani tashkil etmaydi. Demak, ulardan umumiy yechim tuzib bo'lmaydi.

Misollarni yeching:

182. $y_1=\text{sh}x$ va $y_2=\text{ch}x$ funksiyalar $y''' - y = 0$ tenglamaning xususiy yechimlari bo'lsa, ular fundamental sistema tashkil etadimi?

183. $y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0, x \neq 0$ tenglamaning $y_1 = \frac{1}{\sqrt{x}} \sin x$, $y_2 = \frac{1}{\sqrt{x}} \cos x$ xususiy yechimlaridan umumiy yechim tuzib bo'ladimi?

Quyida berilgan funksiyalar o'zining aniqlanish sohasida chiziqli erkli bo'lishi yoki bo'lmasligini aniqlang:

184. $x+1,\ 2x+1,\ x+2.$

185. $2x^2+1,\ x^2-1,\ x+2.$

186. $\sqrt{x},\ \sqrt{x+a},\ \sqrt{x+2a}.$

187. $\ln(2x),\ \ln(3x),\ \ln(4x).$

188. $y_1=e^{-2x}$ va $y_2=e^x$ funksiyalari $y''+y'-2y=0$ tenglamaning xususiy yechimlari bo'lsa, umumiy yechim tuzilsin.

189. $y_1=1$ va $y_2=e^{2x}$ funksiyalar $y''-2y'=0$ tenglamaga xususiy yechim bo'lishini va fundamental sistema tashkil etishini ko'rsating.

190. $y''-4y'+5y=0$ tenglama uchun $y_1=e^{2x}\cos x$, $y_2=e^{2x}\sin x$ funksiyalar xususiy yechim bo'lsa, ularni fundamental sistema tashkil etishini ko'rsating va umumiy yechimni yozing.

191. $y''-y=0$ tenglamaga $y_1=e^{-x}$ xususiy yechim bo'lsa, y_2 – ikkinchi xususiy yechimni toping va umumiy yechimni yozing.

8- §. O'zgarmas koeffitsiyentli chiziqli bir jinsli tenglama

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\ldots+a_{n-1}y'+a_ny=0 \qquad (2.21)$$

tenglama o'zgarmas koeffitsiyentli chiziqli bir jinsli tenglama deyiladi, bu yerda $a_1, a_2, ..., a_n$ – o'zgarmas haqiqiy sonlar.

(2.21) tenglamaning yechimini

$$y=e^{kx} \qquad (2.22)$$

ko'rinishda qidirib, uni tenglamaga qo'yish orqali, (2.21) ning *xarakteristik tenglamasi* deb ataluvchi

$$k^n+a_1k^{n-1}+a_2k^{n-2}+\ldots+a_{n-1}k+a_n=0 \qquad (2.23)$$

algebraik tenglamani hosil qilamiz.

(2.21) tenglamaning yechimi (2.23) xarakteristik tenglamaning yechimiga mos ravishda:

1) har bir oddiy haqiqiy k yechimga Ce_{kx} qo'shiluvchi mos keladi, bu holda umumiy yechim quyidagicha bo'ladi:

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} + \ldots + C_n e^{k_n x}; \qquad (2.24)$$

2) har bir karrali yechimga

$$y = \left(C_1 + C_2 x + \ldots + C_m x^{m-1}\right) e^{kx} \qquad (2.25)$$

ko'rinishdagi yechim mos keladi;

3) har bir $k_{1,2} = \alpha \pm i\beta$ oddiy kompleks yechimga esa

$$e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right) \qquad (2.26)$$

qo'shiluvchi mos keladi;

4) har bir $k_{1,2} = \alpha \pm i\beta$ m- karrali yechimga

$$e^{\alpha x}\left[\left(C_1 + C_2 x + \ldots + C_{m-1} x^{m-1}\right)\cos\beta x + \left(c_1 + c_2 x + \ldots + c_{m-1} x^{m-1}\right)\cdot \sin\beta x\right]$$

qo'shiluvchi mos keladi.

1- misol. $y'' - 7y' + 6y = 0$ tenglamaning umumiy yechimi topilsin.

Y e c h i s h. $k^2 - 7k + 6 = 0$ xarakteristik tenglamani tuzib, $k_1=1$ va $k_2=6$ ildizlarga ega bo'lamiz, bularga esa e^x va e^{6x} xususiy yechimlar mos keladi. Bu yechimlar chiziqli erkli bo'lganidan, umumiy yechim (2.29) formulaga asosan quyidagi ko'rinishda yoziladi:

$$y = C_1 e^x + C_2 e^{6x}.$$

2- misol. $y^{IV} - 13y'' + 36y = 0$ tenglamaning umumiy yechimi topilsin.

Y e c h i s h. Xarakteristik tenglama $k^4 - 13k^2 + 36 = 0$ ko'rinishda bo'lib, uning ildizlari $k_{1,2} = \pm 3$, $k_{3,4} = \pm 2$. Bunga mos e^{-3x}, e^{3x}, e^{-2x}, e^{2x} funksiyalar chiziqli erkli bo'lganligidan, umumiy yechim (2.24) formulaga asosan

$$y = C_1 e^{-3x} + C_2 e^{3x} + C_3 e^{-2x} + C_4 e^{2k}.$$

3- misol. $y'' - y' - 2y = 0$ tenglamaning $y(0)=0$ va $y'(0) = 3$ boshlang‘ich shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

Yechish. Mos xarakteristik tenglama $k^2 - k - 2 = 0$ ko‘rinishda bo‘ladi va uning yechimlari $k_1 = -1$, $k_2 = 2$. Umumiy yechim esa (2.24) formuladan

$$y = C_1 e^{-x} + C_2 e^{2x}$$

ko‘rinishda bo‘ladi.

Boshlang‘ich shartlardan C_1 va C_2 larga nisbatan

$$\begin{cases} C_1 + C_2 = 0, \\ -C_1 + 2C_2 = 3 \end{cases}$$

sistema hosil bo‘ladi va $C_1 = -1$, $C_2 = 1$ ekanligini topamiz. Demak, xususiy yechim $y = -e^{-x} + e^{2x}$.

4- misol. $y'' - 2y' = 0$ tenglamaning $y(0)=0$ va $y(\ln 2)=3$ chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Yechish. Xarakteristik tenglama $k^2 - 2k = 0$ ko‘rinishda bo‘ladi va $k_1=0$, $k_2=2$ uning yechimlari bo‘ladi. Demak, umumiy yechim (2.24) formuladan $y(x) = C_1 + C_2 e^{2x}$ ko‘rinishda bo‘ladi.

Chegaraviy shartlarga ko‘ra quyidagi sistemaga ega bo‘lamiz:

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 + C_2 e^{2\ln 2} = 3 \end{cases} \quad \text{yoki} \quad \begin{cases} C_1 + C_2 = 0, \\ C_1 + 4C_2 = 3. \end{cases}$$

Bundan esa $C_1=-1$, $C_2=1$. Izlangan xususiy yechim $y(x) = e^{2x} - 1$ ko‘rinishda bo‘ladi.

5- misol. $y''' - 2y'' + y' = 0$ tenglamaning umumiy yechimi topilsin.

Yechish. Xarakteristik tenglama $k^3 - 2k^2 + k = 0$ ko‘rinishda bo‘lib, $k_1=0$, $k_2=k_3=1$. Bu yerda 1 ikki karrali yechim bo‘lgani uchun e^{ox}, e^x, $x \cdot e^x$ funksiyalar xususiy yechimlar bo‘lib xizmat qiladi va umumiy yechim (2.25) formuladan $y = C_1 + C_2 e^x + C_2 x e^x$ ko‘rinishda bo‘ladi.

6- misol. $y'' - 4y' + 13y = 0$ tenglamaning umumiy yechimi topilsin.

Y e c h i s h . Xarakteristik tenglama $k^2 - 4k + 13 = 0$ ko'rinishda bo'lib, $k_{1,2} = 2 \pm 3i$. Bularga mos xususiy yechimlar $e^{2x}\cos 3x$ va $e^{2x}\sin 3x$ ko'rinishda bo'lgani uchun umumiy yechim, (2.26) formulaga asosan, $y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$.

Quyidagi tenglamalarning umumiy yechimlari topilsin:

192. $y'' - 4y' + 3y = 0.$

193. $y'' - 4y' + 4y = 0.$

194. $y'' - 4y' + 13y = 0.$

195. $y'' - 4y = 0.$

196. $y'' + 4y = 0.$

197. $y'' + 4y' = 0.$

198. $y'' - y' - 2y = 0.$

199. $y'' + 25y = 0.$

200. $y'' - y' = 0.$

201. $y'' + 4y' + 4y = 0.$

202. $y^{IV} - 2y''' + y'' = 0.$

203. $y^{IV} + a^4 y = 0.$

204. $y^{IV} + 5y'' + 4y = 0.$

205. $y'' - 3y' + 2y = 0.$

206. $y'' + 2ay' + a^2 = 0.$

207. $y'' + 2y' + 5y = 0.$

208. $x''(t) - 2x'(t) - 3x(t) = 0.$

209. $x''(t) + w^2 x(t) = 0\ (w = \text{const}).$

210. $s''(t) + as'(t) = 0\ \ (a = \text{const}).$

Quyidagi tenglamalarning boshlang'ich yoki chetki shartlarni qanoatlantiruvchi yechimi topilsin:

211. $y'' + 5y' + 6y = 0,\ \ y(0) = 1,\ \ y'(0) = -6.$

212. $y'' - 10y' + 25y = 0,\ \ y(0) = 0,\ \ y'(0) = 1.$

213. $y'' - 2y' + 10y = 0,\ \ y\left(\frac{\pi}{6}\right) = 0,\ \ y'\left(\frac{\pi}{6}\right) = e^{\frac{\pi}{6}}.$

214. $y'' + 3y' = 0,\ \ y(0) = 1,\ \ y'(0) = 2.$

215. $y'' + 9y = 0,\ \ y(0) = 0,\ \ y\left(\frac{\pi}{4}\right) = 1.$

216. $y'' + y = 0, \ y(0) = 1, \ y'\left(\frac{\pi}{3}\right) = 0.$

217. $9y'' + y = 0, \ y\left(\frac{3\pi}{2}\right) = 2, \ y'\left(\frac{3\pi}{2}\right) = 0.$

218. $y'' - y = 0, \ y(0) = 2, \ y'(0) = 4.$

219. $y'' + 2y' + 2y = 0, \ y(0) = 1, \ y'(0) = 1.$

9- §. Chiziqli bir jinsli bo'lmagan tenglama

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad (2.28)$$

tenglama *chiziqli bir jinsli bo'lmagan*, ya'ni *o'ng tomoni 0 dan farqli tenglama* deyiladi. (2.28) tenglamaning umumiy yechimi quyidagi teorema bilan aniqlanadi.

Teorema. Agar $U = U(x)$ funksiya (2.28) tenglamaning birorta xususiy yechimi bo'lib, $y_1, y_2, \ldots, y_n$ funksiyalar esa mos bir jinsli tenglamaning fundamental yechimlar sistemasini tashkil etsa, bir jinsli bo'lmagan tenglamaning umumiy yechimi.

$$y = U + C_1y_1 + C_2y_2 + \ldots + C_ny_n \qquad (2.29)$$

ko'rinishda bo'ladi.

Boshqacha aytganda, bir jinsli bo'lmagan tenglamaning umumiy yechimi uning biror xususiy yechimi bilan unga mos bir jinsli tenglamaning umumiy yechimlari yig'indisiga teng.

Masalaning muhim jihati shundaki, bir jinsli tenglamaning umumiy yechimini xarakteristik tenglama orqali topishni bilamiz, ammo bir jinsli bo'lmagan tenglamaning birorta xususiy yechimini topish masalasi ancha murakkab.

Chiziqli bir jinsli bo'lmagan tenglamaning birorta xususiy yechimini topishning ikki usuli bilan tanishib o'tamiz. (Mos bir jinsli tenglamaning umumiy yechimi ma'lum deb olamiz)

I. O'zgarmasni variatsiyalash usuli

Bu usul bir jinsli bo'lmagan tenglamaning birorta xususiy yechimini topish uchun qo'llaniladi va koeffitsiyentlar o'zgarmas bo'lgan hol uchun ham yaroqlidir.

Mos bir jinsli tenglamaning fundamental yechimlari $y_1, y_2, ..., y_n$ ma'lum bo'lsa, (2.28) ning birorta xususiy yechimini

$$U(x) = C_1(x)y_1 + C_2(x)y_2 + \ldots + C_n(x)y_n \qquad (2.30)$$

ko'rinishda qidiramiz.

(2.30) ni (2.28) ga qo'yib, $C_1(x), C_2(x), \ldots, C_n(x)$ funksiyalarni aniqlash uchun quyidagi sistemani hosil qilamiz:

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 + \ldots + C_n'(x)y_n = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' + \ldots + C_n'(x)y_n' = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ C_1'(x)y_1^{(n-2)} + C_2'(x)y_2^{(n-2)} + \ldots + C_n'(x)y_n^{(n-2)} = 0, \\ C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \ldots + C_n'(x)y_n^{(n-1)} = f(x). \end{cases} \qquad (2.31)$$

Bu sistemadan $C_1(x), \; C_2(x), \; \ldots, \; C_n(x)$ larni aniqlab (2.30) ga qo'ysak, qidirilgan xususiy yechimga ega bo'lamiz.

Yuqoridagi sistema

$$y'' + a_1(x)y' + a_2(x)y = f(x)$$

ikkinchi tartibli tenglama uchun

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = f(x) \end{cases} \qquad (2.32)$$

ko'rinishni oladi va bu sistemaning yechimi quyidagi ko'rinishda bo'ladi:

$$C_1(x) = -\int \frac{y_2 f(x)dx}{W(y_1, y_2)}; \quad C_2(x) = \int \frac{y_1 f(x)dx}{W(y_1, y_2)}.$$

U holda (2.30) formulaga asosan xususiy yechim

$$U(x) = -y_1 \int \frac{y_2 f(x)dx}{W(y_1, y_2)} + y_2 \int \frac{y_1 f(x)dx}{W(y_1, y_2)} \qquad (2.33)$$

ko'rinishda bo'lib, bu yerda $W(y_1, y_2) - y_1$ va y_2 yechimlar vronskianidir.

1- misol. $y'' + \frac{2}{x}y' + y = \frac{\mathrm{ctg}x}{x}$ tenglamaning umumiy yechimini toping.

Y e c h i s h . $y'' + \frac{2}{x}y' + y = 0$ bir jinsli tenglama uchun 7- § dagi 1- misolda $y_1 = \frac{\sin x}{x}$ ekanini bilgan holda $y_2 = -\frac{1}{x}\cos x$ ni aniqlagan edik va $W(y_1, y_2) = \begin{vmatrix} \frac{\sin x}{x} & -\frac{\cos x}{x} \\ \frac{x\cos x - \sin x}{x^2} & \frac{x\sin x + \cos x}{x^2} \end{vmatrix} = \frac{1}{x^2}$.

Demak, y_1 va y_2 yechimlar chiziqli erkli, ya'ni fundamental sistemani tashkil etadi. U holda bir jinsli tenglamaning umumiy yechimi $y = C_1\frac{\sin x}{x} - C_2\frac{\cos x}{x}$ ko'rinishda bo'ladi. Bundan esa xususiy yechimni (2.33) formulaga asosan aniqlash mumkin:

$$U(x) = -\frac{\sin x}{x}\int \frac{-\frac{\cos x}{x}\cdot\frac{\mathrm{ctg}x}{x}}{\frac{1}{x^2}}dx - \frac{\cos x}{x}\int \frac{\frac{\sin x}{x}\cdot\frac{\mathrm{ctg}x}{x}}{\frac{1}{x^2}}dx = \frac{\sin x}{x}\int \frac{\cos^2 x}{\sin x}dx -$$

$$-\frac{\cos x}{x}\int \cos x dx = \frac{\sin x}{x}\left[\ln\left|\mathrm{tg}\frac{x}{2}\right| + \cos x\right] - \frac{\cos x}{x}\sin x = \frac{\sin x}{x}\ln\left|\mathrm{tg}\frac{x}{2}\right|.$$

Natijada (2.29) formulaga asosan

$$y = C_1\frac{\sin x}{x} - C_2\frac{\cos x}{x} + \frac{\sin x}{x}\ln\left|\mathrm{tg}\frac{x}{2}\right|$$

umumiy yechimni hosil qilamiz.

Yuqoridagi misoldan ko'rinadiki, (2.28) tenglamaning bir jinsli tenglamasining $y_1(x)$ birorta xususiy yechimi ma'lum bo'lsa, uning umumiy yechimi

$$y = C_1y_1 + C_2y_2 + U(x)$$

ko'rinishda aniqlanib, bu yerda

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

formula orqali, $U(x)$ esa (2.30) formuladan topilar ekan.

II. Noma'lum koeffitsiyentlar usuli

Bu usuldan faqat (2.28) tenglamada koeffitsiyentlar o'zgarmas bo'lgan holdagina foydalanish mumkin.

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = f(x) \qquad (2.34)$$

tenglama berilgan bo'lib,

$$f(x) = e^{\alpha x}\left[P_n(x)\cos\beta x + Q_m(x)\sin\beta x\right] \qquad (2.35)$$

ko'rinishda bo'lsa (bu yerda $P_n(x)$ va $Q_m(x)$ – mos ravishda n va m darajali ko'phadlar), u holda birorta xususiy yechim

$$U(x) = x^r e^{\alpha x}\left[P_l(x)\cos\beta x + Q_l(x)\sin\beta x\right]$$

ko'rinishda qidiriladi, bu yerda r daraja – $k^n + a_1 k^{n-1} + \ldots + a_n = 0$ xarakteristik tenglamaning $\alpha+\beta i$ ildizi tartibiga teng bo'lgan sondir. Agar xarakteristik tenglama $\alpha+\beta i$ kompleks ildizga ega bo'lmasa, $r=0$ olinadi. $P_l(x)$ va $Q_l(x)$ lar esa l tartibli ko'phadlar bo'lib, $l = \max(n,m)$ va $P_l(x) = A_0 x^l + A_1 x^{l-1} + \ldots + A_l$, $Q_l(x) = B_0 x^l +$ $+ B_1 x^{l-1} + \ldots + B_l$.

$$y'' + a_1 y' + a_2 y = f(x) \qquad (2.36)$$

tenglama uchun yuqorida aytilganlarni tartiblab, quyidagicha yozish mumkin.

1. $f(x) = P_n(x)e^{\alpha x}$ bo'lgan holda:

a) α son $k^2 + a_1 k + a_2 = 0$ xarakteristik tenglamaning ildizi bo'lmasa, xususiy yechim

$$U(x) = Q_n(x)e^{\alpha x} \qquad (2.37)$$

ko'rinishda qidiriladi;

b) α son xarakteristik tenglamaning bir karrali ildizi bo'lsa, xususiy yechim

$$U(x) = xQ_n(x)e^{\alpha x} \tag{2.38}$$

ko'rinishda qidiriladi;

d) α son xarakteristik tenglamaning ikki karrali ildizi bo'lsa, xususiy yechim

$$U(x) = x^2 Q_n(x)e^{\alpha x} \tag{2.39}$$

ko'rinishda qidiriladi.

2. $f(x) = e^{\alpha x}\left[P_n(x)\cos\beta x + Q_m(x)\sin\beta x\right]$ bo'lgan holda:

a) $\alpha + \beta i$ xarakteristik tenglamaning ildizi bo'lmasa, u holda xususiy yechim

$$U(x) = e^{\alpha x}\left[P_l(x)\cos\beta x + Q_l(x)\sin\beta x\right] \tag{2.40}$$

ko'rinishda qidiriladi, bu yerda $l = \max(n,m)$;

b) $\alpha + \beta i$ son xarakteristik tenglamaning ildizi bo'lsa, xususiy yechim

$$U(x) = x \cdot e^{\alpha x}\left[P_l(x)\cos\beta x + Q_l(x)\sin\beta x\right] \tag{2.41}$$

ko'rinishda qidiriladi, bu yerda $l = \max(n,m)$.

2- misol. $y'' - 2y' - 3y = e^{4x}$ tenglamaning $y(\ln 2) = 1$, $y(2\ln 2) = 1$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimi topilsin.

Y e c h i s h. Xarakteristik tenglamaning $k^2 - 2k - 3 = 0$ yechimlari $k_1=-1$, $k_2=3$. Demak, bir jinsli tenglamaning umumiy yechimi

$$y = C_1e^{-x} + C_2e^{3x}$$

ko'rinishda bo'ladi. $\alpha = 4$, $P_0(x) = 1$ bo'lgani uchun xususiy yechimni (2.37) formulaga asosan

$$U(x) = Ae^{4x}$$

ko'rinishda izlaymiz. Bu yechimni tenglamaga qo'ysak:

$$16Ae^{4x} - 8Ae^{4x} - 3Ae^{4x} = e^{4x} \text{ yoki } 5A = 1,\ A = \frac{1}{5}.$$

Demak, umumiy yechim (2.29) formulaga asosan

$$y = C_1 e^{-x} + C_2 e^{3x} + \frac{1}{5}e^{4x}$$

ko'rinishda bo'ladi. C_1 va C_2 larni aniqlash uchun chegaraviy shartlardan foydalanamiz:

$$\begin{cases} C_1 e^{-\ln 2} + C_2 e^{3\ln 2} + \frac{1}{5}e^{4\ln 2} = 1, \\ C_1 e^{-2\ln 2} + C_2 e^{6\ln 2} + \frac{1}{5}e^{8\ln 2} = 1. \end{cases}$$

$$\begin{cases} \frac{1}{2}C_1 + 8C_2 + \frac{16}{5} = 1, \\ \frac{1}{4}C_1 + 64C_2 + \frac{256}{5} = 1 \end{cases} \quad \text{yoki} \quad C_1 = \frac{652}{75},\ C_2 = -\frac{491}{600}.$$

Demak, izlanayotgan xususiy yechim:

$$y = \frac{652}{75}e^{-x} - \frac{491}{600}e^{3x} + \frac{1}{5}e^{4x}.$$

3- misol. $y'' + y' - 2y = \cos x - 3\sin x$ tenglamaning $y(0)=1$, $y'(0)=2$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h. Xarakteristik tenglama $k^2 + k - 2 = 0$, uning yechimlari esa $k_1=-2$, $k_2=1$ bo'lgani uchun bir jinsli tenglamaning umumiy yechimi

$$y = C_1 e^{-2x} + C_2 e^{x}$$

ko'rinishda bo'ladi. $f(x) = e^{0x}(\cos x - 3\sin x)$, ya'ni $\alpha=0$, $\beta=1$ bo'lgani uchun xususiy yechimni (2.40) formulaga asosan

$$U(x) = A\cos x + B\sin x$$

ko'rinishda izlaymiz. $U(x)$ ni tenglamaga qo'ysak:

$$-A\cos x - B\sin x - A\sin x + B\cos x - 2A\cos x - 2B\sin x = \cos x - 3\sin x$$

yoki

$$(B - 3A)\cos x - (3B + A)\sin x = \cos x - 3\sin x.$$

Mos koeffitsiyentlarni tenglab, quyidagi sistemaga ega bo'lamiz:

$$\begin{cases} B-3A=1, \\ 3B+A=3 \end{cases} \quad \text{yoki} \quad A=0,\ B=1.$$

Bundan esa umumiy yechim $y=C_1e^{-2x}+C_2e^x+\sin x$ ko'rinishda ekanligini topamiz. C_1 va C_2 koeffitsiyentlarni topish uchun boshlang'ich shartlardan foydalanib, quyidagi sistemani hosil qilamiz: $\begin{cases} C_1e^0+C_2e^0+\sin 0=1, \\ -2C_1e^0+C_2e^0+\cos 0=2 \end{cases}$ yoki $C_1=0$, $C_2=1$. Demak, $y=e^x+\sin x$ izlangan yechim bo'ladi.

4- misol. $y''-y'=\operatorname{ch}2x$ tenglamaning $y(0)=y'(0)=0$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Xarakteristik tenglama $k^2-k=0$ va uning yechimlari $k_1=0$, $k_2=1$ bo'lgani uchun bir jinsli tenglamaning umumiy yechimi

$$y=C_1+C_2e^x$$

ko'rinishda bo'ladi. $f(x)=e^{0x}(\operatorname{ch}2x+0\cdot\operatorname{sh}2x)$ bo'lgani uchun (2.40) formulaga asosan xususiy yechimni

$$U(x)=A\operatorname{ch}2x+B\operatorname{sh}2x$$

ko'rinishda izlanadi. $U(x)$ ni tenglamaga qo'ysak:

$$4A\operatorname{ch}2x+4B\operatorname{sh}2x-2A\operatorname{sh}2x-2B\operatorname{ch}2x=\operatorname{ch}2x$$

yoki

$$(4A-2B)\operatorname{ch}2x+(4B-2A)\operatorname{sh}2x=\operatorname{ch}2x+0\cdot\operatorname{sh}2x.$$

Shunday qilib, mos koeffitsiyentlarni tenglab, quyidagi sistemaga ega bo'lamiz:

$$\begin{cases} 4A-2B=1, \\ -2A+4B=0. \end{cases} \quad \text{Buning yechimi } A=\frac{1}{3},\quad B=\frac{1}{6}.$$

Demak, umumiy yechim quyidagi ko'rinishda bo'ladi:

$$y=C_1+C_2e^x+\frac{1}{3}\operatorname{ch}2x+\frac{1}{6}\operatorname{sh}2x.$$

Noma'lum koeffitsiyentlarni aniqlash uchun boshlang'ich shartlardan foydalanamiz:

$$\begin{cases} C_1 + C_2 e^0 + \frac{1}{3}\mathrm{ch}\,0 + \frac{1}{6}\mathrm{sh}\,0 = 0, \\ C_2 e^0 + \frac{2}{3}\mathrm{sh}\,0 + \frac{1}{3}\mathrm{ch}\,0 = 0 \end{cases}$$

yoki

$$\begin{cases} C_1 + C_2 = -\frac{1}{3}, \\ C_2 + \frac{1}{3} = 0. \end{cases}$$

Bundan, $C_1 = 0,\ C_2 = -\frac{1}{3}$. Demak, boshlang'ich shartlarni bajaruvchi xususiy yechim $y = -\frac{1}{3}e^x + \frac{1}{3}\mathrm{ch}\,2x + \frac{1}{6}\mathrm{sh}\,2x$ ko'rinishda bo'ladi.

Izoh: $f(x) = \mathrm{ch}\,2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{1}{2}(e^{2x} + e^{-2x})$ ekanligidan xususiy yechimni $U = U_1 + U_2 = A_1 e^{2x} + B_1 e^{-2x}$ ko'rinishda qidirsak ham aynan yuqoridagi yechim hosil bo'ladi.

5- misol. $y'' - 2y' + 2y = x^2$ tenglamaning umumiy yechimi topilsin.

Yechish. Xarakteristik tenglama $k^2 - 2k + 2 = 0$ va uning ildizlari $k_{1,2} = 1 \pm i$ bo'lgani uchun bir jinsli tenglamaning umumiy yechimi

$$y = e^x (C_1 \cos x + C_2 \sin x)$$

ko'rinishda bo'ladi.

$f(x) = x^2 = e^{0x} P_2(x)$ bo'lgani uchun xususiy yechimni $U(x) = Ax^2 + Bx + C$ ko'rinishda qidiramiz. Tenglamaga qo'yish natijasida

$$2A - 4Ax - 2B + 2Ax^2 + 2Bx + 2C = x^2 \text{ yoki}$$

$$2Ax^2 + (-4A + 2B)x + 2A - 2B + 2C = x^2 + 0x + 0$$

ekanligidan quyidagilarni hosil qilamiz:

$$\begin{cases} 2A = 1, \\ -4A + 2B = 0, \\ 2A - 2B + 2C = 0 \end{cases} \quad \text{yoki} \quad A = \frac{1}{2},\ B = 1,\ C = \frac{1}{2}.$$

Bundan esa dastlabki tenglamaning umumiy yechimi

$$y = e^x (C_1 \cos x + C_2 \sin x) + \frac{1}{2}(x + 1)^2.$$

6- misol. $y'' + y = xe^x + 2e^{-x}$ tenglamaning umumiy yechimini toping.

Y e c h i s h. Xarakteristik tenglama $k^2 + 1 = 0$, uning ildizlari esa $k_{1,2} = \pm i$ bo'ladi. Shuning uchun bir jinsli tenglamaning umumiy yechimi

$$y = C_1 \cos x + C_2 \sin x$$

ko'rinishda bo'ladi. $f(x) = f_1(x) + f_2(x) = xe^x + 2e^{-x}$ bo'lgani uchun $\alpha_1 = 1,\ \alpha_2 = -1,\ \beta_1 = \beta_2 = 0,\ p_1(x) = x$, demak, xususiy yechimni $U(x) = U_1(x) + U_2(x) = (Ax + B)e^x + Ce^{-x}$ ko'rinishda izlaymiz. Tegishli hosilalarni hisoblab tenglamaga qo'ysak:

$$2Ae^x + (Ax + B)e^x + Ce^{-x} + (Ax + B)e^x + Ce^{-x} = xe^x + 2e^{-x},$$

$$(2Ax + 2A + 2B)e^x + 2Ce^{-x} = (1x + 0)e^x + 2e^{-x}.$$

Noma'lum koeffitsiyentlarni aniqlash uchun quyidagi sistemani hosil qilamiz:

$$\begin{cases} 2A = 1, \\ 2A + 2B = 0, \\ C = 1 \end{cases} \quad \text{yoki} \quad A = \frac{1}{2},\ B = -\frac{1}{2},\ C = 1.$$

Demak, dastlabki tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2}(x - 1)e^x + e^{-x}.$$

7- misol. $y''' + y'' - 2y' = x - e^x$ tenglamaning umumiy yechimini toping.

Y e c h i s h . Xarakteristik tenglamasi $k^3 + k^2 - 2k = 0$, uning ildizlari esa $k_1 = -2, \quad k_2 = 0, \quad k_3 = 1$ bo'ladi. Demak, bir jinsli tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$y = C_1 + C_2 e^x + C_3 e^{-2x},$$

$$f(x) = f_1(x) + f_2(x) = x - e^x.$$

$\alpha_1 = 0,\ P_1(x) = x,\ \alpha_2 = 1,\ P_0(x) = -1,\ \beta_1 = \beta_2 = 0$ bo'lgani uchun xususiy yechimni (2.38) formulaga asosan:

$$U(x) = U_1(x) + U_2(x) = x \cdot (Ax + B) + x \cdot Ce^x$$

ko'rinishga qidiramiz. Buni asosiy tenglamaga qo'yib,

$$3Ce^x + Cxe^x + 2A + 2Ce^x + Cxe^x - 4Ax - 2B - 2Ce^x - 2Cxe^x = x - e^x$$

yoki

$$-4Ax + (2A - 2B) + 3Ce^x = x - e^x \cdot$$

ifodani hosil qilamiz. Noma'lum koeffitsiyentlarni aniqlash uchun quyidagi sistema hosil bo'ladi:

$$\begin{cases} -4A = 1, \\ 2A - 2B = 0, \\ 3C = -1 \end{cases} \quad \text{yoki} \quad A = -\frac{1}{4},\ B = -\frac{1}{4},\ C = -\frac{1}{3}.$$

Natijada dastlabki tenglamaning izlangan umumiy yechimiga ega bo'lamiz:

$$y = C_1 + C_2 e^x + C_3 e^{-2x} - \frac{1}{4}x(x+1) - \frac{1}{3}xe^x.$$

8- misol. $y'' + y = 3\sin x$ tenglamaning $y(0) + y'(0) = 0$, $y\left(\frac{\pi}{2}\right) + y'\left(\frac{\pi}{2}\right) = 0$ chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Xarakteristik tenglama $k^2+1=0$ va uning ildizlari $k_{1,2}=\pm i=0\pm i$ bo'lgani uchun mos bir jinsli tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$y=C_1\cos x+C_2\sin x.$$

$f(x)=e^{0x}(3\sin x+0\cos x)$, ya'ni $\alpha+\beta i=0+i$, $\alpha=0$, $\beta=1$ bo'lgani hamda bu xarakteristik tenglamaning ildizi bilan aynan bir xil bo'lganligi uchun xususiy yechimni (2.41) formulaga asosan

$$U(x)=x(A\cos x+B\sin x)$$

ko'rinishda izlaymiz.

$$U'=(-A\sin x+B\cos x)x+(A\cos x+B\sin x),$$

$$U''=2(-A\sin x+B\cos x)+(-A\cos x-B\sin x)x$$

ifodalarni tenglamaga qo'ysak,

$$-2A\sin x+2B\cos x-Ax\cos x-Bx\sin x+Ax\cos x+Bx\sin x=3\sin x$$

$$\text{yoki}\quad -2A\sin x+2B\cos x=3\sin x+0\cos x$$

hosil bo'ladi.

Noma'lum koeffitsiyentlarni aniqlash uchun quyidagi sistemaga ega bo'lamiz:

$$\begin{cases}-2A=3,\\ 2B=0\end{cases}\quad \text{yoki}\quad A=-\frac{3}{2},\quad B=0.$$

Natijada, dastlabki tenglamaning umumiy yechimi

$$y=C_1\cos x+C_2\sin x-\frac{3}{2}x\cos x$$

ko'rinishda bo'ladi. Noma'lum C_1 va C_2 koeffitsiyentlarni aniqlash uchun chegaraviy shartlarni qanoatlantiramiz:

$$y'=-C_1\sin x+C_2\cos x-\frac{3}{2}\cos x+\frac{3}{2}x\sin x,$$

$$y(0)=C_1\cos 0+C_2\sin 0-\frac{3}{2}\cdot 0\cdot\cos 0=C_1,$$

$$y'(0)=-C_1\sin 0+C_2\cos 0-\frac{3}{2}\cdot\cos 0+\frac{3}{2}\cdot 0\cdot\sin 0=C_2-\frac{3}{2},$$

$$y\left(\frac{\pi}{2}\right)=C_1\cos\frac{\pi}{2}+C_2\sin\frac{\pi}{2}-\frac{3}{2}\cdot\frac{\pi}{2}\cdot\cos\frac{\pi}{2}=C_2,$$

$$y'\left(\frac{\pi}{2}\right)=-C_1\sin\frac{\pi}{2}+C_2\cos\frac{\pi}{2}-\frac{3}{2}\cdot\cos\frac{\pi}{2}+\frac{3}{2}\cdot\frac{\pi}{2}\sin\frac{\pi}{2}=-C_1+\frac{3\pi}{4}.$$

Shunday qilib,

$$\begin{cases} C_1+C_2-\frac{3}{2}=0, \\ C_2-C_1+\frac{3\pi}{4}=0 \end{cases} \quad \text{yoki} \quad \begin{cases} C_1+C_2=\frac{3}{2}, \\ -C_1+C_2=-\frac{3\pi}{4} \end{cases}$$

sistema hosil bo'ladi va uning yechimi $C_1=\frac{3(2+\pi)}{8}$, $C_2=\frac{2-\pi}{8}$ bo'ladi. Demak, berilgan tenglamaning chegaraviy shartlarni qanoatlantiruvchi xususiy yechimi:

$$y=\frac{3}{8}[(\pi+2)\cos x-(\pi-2)\sin x]-\frac{3}{2}x\cos x.$$

9- misol. $y''+6y'+10y=80e^x\cos x$ tenglamaning $y(0)=4$, $y'(0)=10$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h. Xarakteristik tenglama $k^2+6k+10=0$ va uning ildizlari $k_{1,2}=-3\pm i$ bo'lgani uchun mos bir jinsli tenglamaning umumiy yechimi $y=e^{-3x}(C_1\cos x+C_2\sin x)$ ko'rinishda bo'ladi. $f(x)=e^x(80\cos x+0\cdot\sin x)$ bo'lgani hamda $\alpha+\beta i=1+i$ ekanligidan xususiy yechimni $U(x)=e^x(A\cos x+B\sin x)$ ko'rinishda izlaymiz. Tegishli hosilalarni hisoblab tenglamaga qo'ysak:

$$e^x(-2A\sin x+2B\cos x)+6e^x(A\cos x+B\sin x-A\sin x+B\cos x)+$$
$$+10e^x(A\cos x+B\sin x)=80e^x\cos x.$$

Noma'lum koeffitsiyentlarni aniqlash uchun quyidagi sistemani hosil qilamiz:

$$\begin{cases} 16A+8B=80, \\ -8A+16B=0 \end{cases} \quad \text{yoki} \quad A=4, \quad B=2.$$

Demak, dastlabki tenglamaning umumiy yechimi

$$y = e^{-3x}\left(C_1 \cos x + C_2 \sin x\right) + 2e^x\left(2\cos x + \sin x\right).$$

Boshlang'ich shartlarni qanoatlantirib, C_1 va C_2 larni aniqlaymiz:

$y' = e^{-3x}(-3C_1 \cos x - 3C_2 \sin x - C_1 \sin x + C_2 \cos x) + 2e^x(3\cos x - \sin x)$,

$y(0) = C_1 + 4 = 4$, $y'(0) = -3C_1 + C_2 + 6 = 10$, bundan C_1=0, C_2=4.

Shunday qilib, boshlang'ich shartlarni qanoatlantiruvchi xususiy yechim:

$$y = 4e^{-3x}\sin x + 2e^x\left(2\cos x + \sin x\right).$$

10- misol. $y'' + y = \mathrm{tg}x$ tenglamaning $y(0) = y\left(\frac{\pi}{6}\right) = 0$ chegaraviy shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h. Xarakteristik tenglama $k^2 + 1 = 0$, uning ildizlari esa $k_{1,2} = \pm i$. Shuning uchun mos bir jinsli tenglamaning umumiy yechimi:

$$y = C_1 \cos x + C_2 \sin x.$$

$f(x) = \mathrm{tg}x = e^{0x} \cdot \mathrm{tg}x$ bo'lgani uchun xususiy yechimni noma'lum koeffitsiyentlar usuli bilan izlab bo'lmaydi.

Shuning uchun, o'zgarmasni variatsiyalash usulidan foydalanamiz.

$U(x) = C_1(x)\cos x + C_2(x)\sin x$ deb olsak, $C_1(x)$ va $C_2(x)$ funksiyalarni aniqlash uchun (2.32) formulaga asosan, quyidagi sistemaga ega bo'lamiz:

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = f(x) \end{cases} \quad \text{yoki} \quad \begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0, \\ -C_1'(x)\sin x + C_2'(x)\cos x = \mathrm{tg}x. \end{cases}$$

Bu sistemani yechib,

$$C_1(x) = -\int \frac{\sin^2 x}{\cos x}dx + A = \sin x - \ln\left|\mathrm{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + A,$$

$$C_2(x) = -\cos x + B$$

ekanligini topamiz.

Shunday qilib, dastlabki tenglamaning umumiy yechimi:

$$y = A\cos x + B\sin x - \cos x \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$$

Chegaraviy shartlarni qanoatlantirib, A va B ni aniqlash uchun, quyidagi sistemaga ega bo'lamiz:

$$\begin{cases} A\cos 0 + B\sin 0 - \cos 0 \cdot \ln\left|\operatorname{tg}\left(\frac{\pi}{4}\right)\right| = 0, \\ A\cos\frac{\pi}{6} + B\sin\frac{\pi}{6} - \cos\frac{\pi}{6} \cdot \ln\left|\operatorname{tg}\left(\frac{\pi}{3}\right)\right| = 0. \end{cases}$$

Bundan $A = 0$, $B = \frac{\sqrt{3}}{2}\ln 3$. Demak, chegaraviy shartlarni qanoatlantiruvchi yechim, quyidagicha bo'ladi:

$$y = \frac{\sqrt{3}}{2} \cdot \ln 3 \cdot \sin x - \cos x \cdot \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$$

Quyidagi tenglamalarni yeching:

220. $y'' - 2y' + y = e^{2x}$.

221. $y'' - 4y = 8x^3$.

222. $y'' + 3y' + 2y = \sin 2x + 2\cos 2x$.

223. $y'' + y = x + 2e^x$.

224. $y'' + 3y' = 9x$.

225. $y'' + 4y' + 5y = 5x^2 - 32x + 5$.

226. $y'' - 3y' + 2y = e^x$.

227. $y'' + 5y' + 6y = e^{-x} + e^{-2x}$.

228. $y''' + y'' = 6x + e^{-x}$.

229. $y'' + y' - 2y = 6x^2$.

230. $y'' - 5y' + 6y = 13\sin 3x$.

231. $y'' + 2y' + y = e^x$.

232. $y'' + y' + 2{,}5y = 25\cos 2x$.

233. $4y'' - y = x^3 - 24x$.

234. $y'' - 4y' + 3y = e^{5x}$, $y(0) = 3$, $y'(0) = 9$.

235. $y'' - 8y' + 16y = e^{4x}$, $y(0) = 0$, $y'(0) = 1$.

236. $y'' + y = \cos 3x, \quad y\left(\frac{\pi}{2}\right) = 4, \quad y'\left(\frac{\pi}{2}\right) = 1.$

237. $2y'' - y' = 1, \quad y(0) = 0, \; y'(0) = 1.$

238. $y'' + 4y = \sin 2x + 1, \; y(0) = \frac{1}{4}, \; y'(0) = 0.$

239. $y'' + 4y = \cos 2x, \quad y(0) = y\left(\frac{\pi}{4}\right) = 0.$

240. $y'' - y = 2\mathrm{sh}x, \quad y(0) = 0, \; y'(0) = 1.$

241. $y'' - 4y' + 8y = 61e^{2x}\sin x, \quad y(0) = 0, \quad y'(0) = 4.$

10- §. Eyler tenglamasi

O'zgaruvchi koeffitsiyentli chiziqli

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + ... + a_{n-1} x y^1 + a_n y = f(x) \qquad (2.42)$$

yoki

$$(ax+b)^n y^{(n)} + a_1 (ax+b)^{n-1} y^{(n-1)} + ... + a_{n-1}(ax+b)y' + a_n y = f(x) \quad (2.43)$$

tenglama *Eyler tenglamasi* deb ataladi, a_i — bu tenglamalar uchun o'zgarmas koeffitsiyentlar.

(2.42) tenglamani $x=e^t$ va (2.43) tenglamani esa $ax + b = e^t$ almashtirish orqali o'zgarmas koeffitsiyentli chiziqli tenglama holiga keltiriladi.

1- misol. $x^2 y'' - xy' + y = 0$ tenglamani yeching.

Yechish. $x = e^t$ yoki $t = \ln x$, $\frac{dt}{dx} = \frac{1}{x} = \frac{1}{e^t} = e^{-t}$ almashtirish bajarib, $y = y(x) = y[x(t)]$ funksiyaning murakkab funksiya sifatida hosilalarini topamiz:

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y}e^{-t},$$

$$y'' = \frac{d}{dt}(e^{-t}\dot{y})\frac{dt}{dx} = (\ddot{y}e^{-t} - e^{-t}\dot{y})e^{-t} = e^{-2t}(\ddot{y} - \dot{y}).$$

Bu yerda $\dot{y}$ va $\ddot{y}$ ko'rinishda t bo'yicha hosilalar belgilandi.

Bularni e'tiborga olsak, dastlabki tenglama quyidagi holga keladi:

$$e^{2t}\cdot e^{-2t}(\ddot{y}-\dot{y})-e^{t}\cdot e^{-t}\dot{y}+y=0$$

yoki

$$\ddot{y}-2\dot{y}+y=0.$$

Bu tenglamaning xarakteristik tenglamasi:

$$k^2-2k+1=0,\ \left(k_{1,2}=1\right),$$

umumiy yechimi esa

$$y=(C_1+C_2t)e^{t}=(C_1+C_2\ln x)\cdot x$$

ko'rinishda bo'ladi.

2- misol. $(4x-1)^2y''-2(4x-1)y'+8y=0$ tenglama yechilsin.

Y e c h i s h. $4x-1=e^{t}$ yoki $x=\frac{1}{4}\left(e^{t}+1\right)$, $\frac{dx}{dt}=\frac{1}{4}e^{t}$ yoki $\frac{dt}{dx}=4e^{-t}$ almashtirishlarni bajarsak,

$$y'=\frac{dy}{dt}\frac{dt}{dx}=4e^{-t}\dot{y},$$

$$y''=\frac{d}{dt}\left(4\cdot e^{-t}y'\right)\frac{dt}{dx}=\left(-4e^{-t}\cdot\dot{y}+4e^{-t}\ddot{y}\right)4e^{-t}=16e^{-2t}(\ddot{y}-\dot{y}).$$

Bularni e'tiborga olsak, dastlabki tenglama

$$16e^{2t}e^{-2t}(\ddot{y}-\dot{y})-4\cdot 2e^{t}\cdot e^{-t}\dot{y}+8y=0$$

yoki

$$2\ddot{y}-3\dot{y}+y=0$$

ko'rinishdagi o'zgarmas koeffitsiyentli chiziqli bir jinsli tenglamaga aylanadi. Xarakteristik tenglamasi:

$$2k^2-3k+1=0,\ \left(k_1=1,\ \ k_2=\frac{1}{2}\right).$$

Natijada umumiy yechim

$$y=C_1e^{t}+C_2e^{\frac{1}{2}t}$$

yoki

$$y = C_1(4x-1) + C_2\sqrt{4x-1}$$

ko'rinishda bo'ladi.

3- misol. $y'' - xy' + y = \cos(\ln x)$ tenglamani yeching.

Yechish. $x = e^t$ yoki $t = \ln x$, $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$ almashtirishlarni bajarib, tegishli hosilalarni hisoblaymiz:

$$y' = e^{-t}\dot{y}, \quad y'' = e^{-2t}(\ddot{y} - \dot{y}).$$

Topilganlarni tenglamaga qo'ysak, quyidagi o'zgarmas koeffitsiyentli tenglama hosil bo'ladi:

$$\ddot{y} - 2\dot{y} + y = \cos t.$$

Xarakteristik tenglama $k^2 - 2k + 1 = 0$, $(k_{1,2} = 1)$ bo'lganidan, bir jinsli tenglamaning umumiy yechimi

$$y = (C_1 + C_2 t)e^t$$

ko'rinishda bo'ladi.

$f(x) = (1 \cdot \cos t + 0 \cdot \sin t)e^{0t}$ bo'lgani uchun xususiy yechimni

$$U(t) = A\cos t + B\sin t$$

ko'rinishda qidiramiz. Hosilalarni hisoblab:

$$U' = -A\sin t + B\cos t, \quad U'' = -A\cos t - B\sin t,$$

tenglamaga qo'ysak,

$$-A\cos t - B\sin t + 2A\sin t - 2B\cos t + A\cos t + B\sin t = \cos t$$

yoki

$$-2B\cos t + 2A\sin t = \cos t.$$

Noma'lum koeffitsiyentlarni aniqlaymiz:

$$\begin{cases} -2B = 1, \\ A = 0 \end{cases} \text{ yoki } B = -\frac{1}{2}, \ A = 0.$$

Demak, $U(t) = -\frac{1}{2}\sin t$ hamda umumiy yechim $y = (C_1 + C_2 t)e^t - \frac{1}{2}\sin t$ ko'rinishda bo'ladi.

Dastlabki o'zgaruvchiga qaytsak,

$$y = (C_1 + C_2 \ln x)x - \frac{1}{2}\sin \ln x$$

umumiy yechimni hosil qilamiz.

Quyidagi Eyler tenglamalarini yeching:

242. $x^2 y'' - 2y = 0.$
243. $x^2 y'' + 2xy' - n(n+1)y = 0.$
244. $x^2 y'' + 5xy' + 4y = 0.$
245. $x^2 y'' + xy' + y = 0.$
246. $xy'' + 2y' = 10x.$
247. $x^2 y'' - 6y = 12\ln x.$
248. $x^2 y'' - xy' + 2y = 0.$
249. $x^2 y'' - 3xy' + 3y = 3\ln^2 x.$
250. $x^2 y'' + xy' + y = \sin(2\ln x)$
251. $x^2 y'' - 2xy' + 2y = 4x.$
252. $x^3 y'' + 3x^2 y' + xy = 6\ln x.$
253. $x^2 y'' - 4xy' + 6y = x^5.$
254. $x^2 y'' + xy' + y = x.$
255. $x^3 y''' - 3xy' + 3y = 0.$
256. $x^2 y'' + 3xy' + y = \frac{1}{x}, \quad y(1) = 1, \quad y'(1) = 0.$
257. $x^2 y'' - 3xy' + 4y = \frac{1}{2}x^3, \quad y(1) = \frac{1}{2}, \quad y(4) = 0.$

11- §. Differensial tenglamalarni qator yordamida yechish

Ba'zi bir differensial tenglamalarni elementar funksiyalar yordamida integrallash mumkin bo'lmaydi, bunday tenglamalarning yechimini

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n \qquad (2.44)$$

darajali qator ko‘rinishida izlanadi.

Noma'lum C_n koeffitsiyentlarni (2.44) ni tenglamaga qo‘yib, tenglikning har ikki tomonidagi bir xil darajali hadlar oldidagi koeffitsiyentlarni tenglab topiladi, ya’ni

$$y' = f(x;y) \qquad (2.45)$$

tenglamaga qo‘yilgan $y(x_0) = y_0$ boshlang‘ich shartni qanoatlantiruvchi yechimni topish haqidagi Koshi masalasining yechimini

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n \qquad (2.46)$$

Teylor qatori yordamida topish qulay, bu yerda

$$y(x_0) = y_0, \quad y'(x_0) = f(x_0;y_0), \ldots$$

1- misol. $y'' - x^2 y = 0$ tenglamani yeching.

Y e c h i s h . Bu tenglamaning yechimini

$$y = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n + \ldots$$

darajali qator ko‘rinishda qidiramiz.

Tegishli hosilalarni hisoblab,

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1} + \ldots,$$

$$y'' = 2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot C_3 x + \ldots + n(n-1) C_n x^{n-2} + \ldots,$$

natijalarni tenglamaga qo‘yamiz:

$$2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot C_3 x + \ldots + n(n-1) C_n x^{n-2} -$$
$$-x^2 \left(C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n + \ldots\right) = 0.$$

x ni bir xil darajalari bo‘yicha guruhlasak:

$$2 \cdot 1 \cdot C_2 + 3 \cdot 2 \cdot C_3 x + (4 \cdot 3C_4 - C_0)x^2 + (5 \cdot 4 \cdot C_5 - C_1)x^3 + \ldots$$
$$+ [(n+4)(n+3)C_{n+4} - C_n] x^{n+2} \ldots, = 0$$

yoki

$$2\cdot 1\cdot C_2 + 3\cdot 2\cdot C_3 x + \sum_{n=0}^{\infty}[(n+4)(n+3)C_{n+4} - C_n]x^{n+2} = 0.$$

Bundan

$$C_2 = 0,\ C_3 = 0, ...,\ (n+4)(4+3)C_{n+4} - C_n = 0$$

yoki

$$C_{n+4} = \frac{C_n}{(n+3)(n+4)} \quad (n = 0,1,2...).$$

Bu tenglik barcha noma'lum koeffitsiyentlarni aniqlashga yordam beradi:

$$C_{4n} = \frac{C_0}{3\cdot 4\cdot 7\cdot 8 ... (4n-1)4n}, \quad C_{4n+1} = \frac{C_1}{4\cdot 5\cdot 8\cdot 9 ... 4n(4n+1)},$$

$$C_{4n+2} = C_{4n+3} = 0 \quad (n = 0,1,2,...).$$

Shunday qilib, quyidagi umumiy yechimga ega bo'ldik:

$$y = C_0 \sum_{n=0}^{\infty} \frac{x^{4n}}{3\cdot 4\cdot 7\cdot 8 .. (4n-1)4n} + C_1 \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4\cdot 5\cdot 8\cdot 9 ... 4n(4n+1)}.$$

Hosil bo'lgan qator son o'qidagi barcha nuqtalarda yaqinlashuvchi bo'lib, u ikkita chiziqli erkli yechimlar yig'indisidan iborat:

2- misol. $y' = x^2 + y^2$ tenglamaning, $y(0)=1$ shartni bajaruvchi yechimini Teylor qatori yordamida birinchi oltita hadlari yig'indisi shaklida toping.

Y e c h i s h . $y(0)=1$ boshlang'ich shartga asosan $y'(0) = 0^2 + 1^2 = 1$,

ikkinchi tartibli hosila $y'' = 2x + 2y\cdot y'$ va uning qiymati

$$y''(0) = 2\cdot 0 + 2\cdot 1\cdot 1^2 = 2;$$

uchinchi tartibli hosila $y''' = 2 + 2y'^2 + 2yy''$ va uning qiymati

$$y'''(0) = 2 + 2\cdot 1^2 + 2\cdot 1\cdot 2 = 8;$$

to'rtinchi tartibli hosila $y^{IV} = 6y'y'' + 2yy'''$ va uning qiymati

$$y^{IV}(0) = 6\cdot 1\cdot 2 + 2\cdot 1\cdot 8 = 28;$$

beshinchi tartibli hosila $y^{V}=6y''^{2}+8y'y'''+2yy^{IV}$ va uning qiymati

$$y^{V}(0)=6\cdot 2^{2}+8\cdot 1\cdot 8+2\cdot 1\cdot 28=144.$$

Izlangan yechim formulasi

$$y=1+\frac{x}{1!}y'(0)+\frac{x^{2}}{2!}y''(0)+\frac{x^{3}}{3!}y'''(0)+\frac{x^{4}}{4!}y^{IV}(0)+\frac{x^{5}}{5!}y^{V}(0).$$

Demak, $y=1+\frac{x}{1!}+\frac{2x^{2}}{2!}+\frac{8x^{3}}{3!}+\frac{28x^{4}}{4!}+\frac{144x^{5}}{5!}.$

3- misol. $y''=x+y^{2}$ tenglamaning $y(0)=0$, $y'(0)=1$ shartlarni qanoatlantiruvchi yechimini Teylor qatori ko‘rinishida to‘rtta noldan farqli had yig‘indisi ko‘rinishida toping.

Yechish. Teylor formulasiga asosan yechim ko‘rinishi $y=y(0)+\frac{x}{1!}y'(0)+\frac{x^{2}}{2!}y''(0)+\frac{x^{3}}{3!}y'''(0)+\frac{x^{4}}{4!}y^{IV}(0)+...$ bo‘lgani uchun boshlang‘ich shartlardan foydalanib:

$y''(0)=0+0^{2}=0$,

$y'''=1+2yy'$ va uning qiymati $y'''(0)=1+2\cdot 0\cdot 1=1$,

$y^{IV}=2y'^{2}+2yy''$ va uning qiymati $y^{IV}(0)=2\cdot 1^{2}+2\cdot 0\cdot 0=2$,

$y^{V}=6y'y''+2yy'''$ va uning qiymati $y^{V}(0)=6\cdot 1\cdot +2\cdot 0\cdot 1=0$,

$y^{VI}=6y''^{2}+8y'y'''+2y\cdot y^{IV}$ va uning qiymati

$y^{VI}(0)=6\cdot 0+8\cdot 1\cdot 1+2\cdot 0\cdot 2=8$.

Demak, $y=\frac{x}{1!}+\frac{x^{3}}{3!}+\frac{2x^{4}}{4!}+\frac{8x^{6}}{6!}+...=x+\frac{x^{3}}{6}+\frac{x^{4}}{12}+\frac{x^{6}}{90}+...$

Quyidagi differensial tenglamalarning yechimlarini darajali qatorlar ko‘rinishida toping:

258. $y'+xy=0.$

259. $y'=x-2y$, $y(0)=0.$

260. $y''+xy'+y=0.$

261. $y''-xy'-2y=0.$

262. $y''+x^{2}y=0$, $y(0)=0$, $y'(0)=1.$

Quyidagi tenglamalarning yechimlarini ko'rsatilgan aniqlikda noldan farqli Teylor qatori yig'indisi shaklida toping:

263. $y' = x^2 y + y^3$, $y(0) = 1$, to'rtta noldan farqli hadlar yig'indisi shaklida.

264. $y' = x + 2y^2$, $y(0) = 0$, ikkita noldan farqli had yig'indisi shaklida.

265. $y'' - xy^2 = 0$, $y(0) = 1$, $y'(0) = 1$, to'rtta noldan farqli had yig'indisi shaklida.

266. $y' = 2x - y$, $y(0) = 2$, aniq yechimi topilsin.

267. $y' = y^2 + x$, $y(0) = 1$, birinchi beshta hadi yig'indisi ko'rinishidagi yechimi topilsin.

268. $y'' = (2x - 1)y - 1$, $y(0) = 0$, $y'(0) = 1$, birinchi beshta hadi yig'indisi ko'rinishidagi yechimi topilsin.

III BOB

DIFFERENSIAL TENGLAMALAR SISTEMASI

1- §. Normal sistema

Ushbu

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n), \quad (i = \overline{1,n}) \qquad (3.1)$$

ko'rinishdagi sistema *birinchi tartibli n ta differensial tenglamalarning normal sistemasi* yoki $x_i = x_i(t)$ noma'lum funksiyaning hosilasiga nisbatan yechilgan differensial tenglamalar sistemasi deyiladi. Bunda tenglamalar soni noma'lum funksiyalar soniga teng, deb faraz qilinadi.

Agar (3.1) sistemaning o'ng tomonidagi f_i $(i = \overline{1,n})$ funksiyalar $x_1, x_2, \ldots, x_n$ larga nisbatan chiziqli bo'lsa, u vaqtda (3.1) sistema *chiziqli differensial tenglamalar sistemasi* deyiladi.

(3.1) sistemaning $(a; b)$ intervaldagi yechimi deb, $(a; b)$ intervalda uzluksiz differensiallanuvchi va sistemaning hamma tenglamasini qanoatlantiradigan n ta $x_1(t)$, $x_2(t)$, $x_3(t)$, ..., $x_n(t)$ funksiya to'plamiga aytiladi.

Differensial tenglamalarning normal sistemasi uchun Koshi masalasi shunday yechimni topishdan iboratki, u $t = t_0$ da berilgan quyidagi qiymatlarni qabul qilsin:

$$x_1\big|_{t=t_0} = x_{10}, \quad x_2\big|_{t=t_0} = x_{20}, \ldots, \quad x_n\big|_{t=t_0} = x_{n0}. \qquad (3.2)$$

Bu qiymatlar (3.1) *normal sistemaning boshlang'ich shartlari* deyiladi. Ularning soni noma'lum funksiyalar soni bilan bir xil.

(3.1) sistemaning umumiy yechimi deb, n ta ixtiyoriy C_1, C_2, ..., C_n o'zgarmaslarga bog'liq bo'lgan ushbu $x_i = \varphi_i(t, C_1, C_2, \ldots, C_n)$ funksiyalar sistemasiga aytiladi. Ixtiyoriy o'zgarmaslarning mumkin bo'lgan ba'zi qiymatlarida hosil bo'ladigan yechimlar xususiy yechimlar deyiladi.

n- tartibli bitta differensial tenglamani tenglamalarning normal sistemasiga keltirish mumkin. Umuman aytganda, buning aksi ham o'rinli, ya'ni birinchi tartibli n ta differensial tenglamaning normal sistemasi n- tartibli bitta differensial tenglamaga ekvivalentdir.

1- misol.

$$\begin{cases} \dfrac{dx}{dt} = ax + by + f(t), & (3.3) \\ \dfrac{dy}{dt} = cx + dy + g(t) & (3.4) \end{cases}$$

sistema berilgan bo'lsin. Bu yerda a, b, c, d — o'zgarmas koeffitsiyentlar, $f(t)$ va $g(t)$ — berilgan funksiyalar, $x(t)$ va $y(t)$ —noma'lum funksiyalar.

(3.3) tenglamadan

$$y = \frac{1}{b}\left(\frac{dx}{dt} - ax - f(t)\right) \qquad (3.5)$$

ni topamiz va uning ikkala qismini differensiallaymiz:

$$\frac{dy}{dt} = \frac{1}{b}\left(\frac{d^2x}{dt^2} - a\frac{dx}{dt} - \frac{df}{dt}\right). \qquad (3.6)$$

(3.5) va (3.6) ni (3.4) ga keltirib qo'yamiz. Natijada $x(t)$ ga nisbatan ikkinchi tartibli differensial tenglamani hosil qilamiz:

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx + P(t) = 0\,, \qquad (3.7)$$

bu yerda A, B, C — o'zgarmaslar .

2- misol. Quyidagi tenglamalar sistemasining yechimini toping:

$$\begin{cases} \dfrac{dx}{dt} = y + 1, \\ \dfrac{dy}{dt} = x + 1. \end{cases}$$

Birinchi tenglamadan

$$y = \frac{dx}{dt} - 1 \qquad (3.8)$$

ni topib, uning ikkala tomonini t bo'yicha differensiallaymiz:

$$\frac{dy}{dt}=\frac{d^2x}{dt^2}. \tag{3.9}$$

(3.8) va (3.9) ifodalarni sistemaning ikkinchi tenglamasiga keltirib qo'yib, $x(t)$ ga nisbatan o'zgarmas koeffitsiyentli ikkinchi tartibli differensial tenglamani hosil qilamiz:

$$\frac{d^2x}{dt^2}-x-1=0.$$

Bu tenglamaning umumiy yechimi:

$$x=C_1e^t+C_2e^{-t}-1. \tag{3.10}$$

(3.10) funksiyani t bo'yicha differensiallab, (3.8) ifodaga keltirib qo'ysak,

$$y=C_1e^t-C_2e^{-t}-1$$

ni topamiz. Demak, sistemaning umumiy yechimi:

$$\begin{cases} x=C_1e^t+C_2e^{-t}-1, \\ y=C_1e^t-C_2e^{-t}-1. \end{cases}$$

2- §. O'zgarmas koeffitsiyentli chiziqli bir jinsli differensial tenglamalar sistemasini Eyler usulida integrallash

Quyidagi bir jinsli chiziqli

$$\begin{cases} \frac{dx}{dt}=ax+by+cz, \\ \frac{dy}{dt}=a_1x+b_1y+c_1z, \\ \frac{dz}{dt}=a_2x+b_2y+c_2z \end{cases} \tag{3.11}$$

sistemani qaraymiz va undagi koeffitsiyentlarni o'zgarmas deb hisoblaymiz. (3.11) sistemaning yechimini ko'rsatkichli funksiyalar ko'rinishida izlaymiz:

$$x=\lambda e^{rt},\ y=\mu e^{rt},\ z=\nu e^{rt}, \tag{3.12}$$

bu yerda r, λ, μ, ν o'zgarmas bo'lib, ularni (3.12) ifodalar (3.11) sistemani qanoatlantiradigan qilib aniqlash lozim. (3.11) sistemaga (3.12) qiymatlarni qo'yib, e^{rt} ga qisqartirib va λ, μ, ν oldidagi koeffitsiyentlarni tanlab, quyidagi algebraik tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} (a-r)\lambda + b\mu + c\nu = 0, \\ a_1\lambda + (b_1 - r)\mu + c_1\nu = 0, \\ a_2\lambda + b_2\mu + (c_2 - r)\nu = 0. \end{cases} \tag{3.13}$$

(3.13) sistema — λ, μ, ν ga nisbatan chiziqli bir jinsli tenglamalar sistemasidir. Demak, sistema noldan farqli yechimlarga ega bo'lishi uchun sistemaning determinanti nolga teng bo'lishi zarur va yetarlidir. Shunday qilib,

$$\Delta = \begin{vmatrix} a-r & b & c \\ a_1 & b_1 - r & c_1 \\ a_2 & b_2 & c_2 - r \end{vmatrix} = 0 \tag{3.14}$$

tenglik bajarilishi kerak.

(3.14) tenglama r ga nisbatan uchinchi darajali tenglamadir, u (3.11) sistemaning xarakteristik tenglamasi deyiladi.

a) Xarakteristik tenglamaning r_1, r_2, r_3 ildizlari haqiqiy va har xil bo'lsin. Bu ildizlarning har biri uchun mos (3.13) tenglamalar sistemasini yozamiz va λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3 koeffitsiyentlarni aniqlaymiz. Agar (3.14) tenglamaning r_1, r_2, r_3 ildizlariga mos (3.11) sistemaning xususiy yechimlarini x_1, y_1, z_1; x_2, y_2, z_2; x_2, y_3, z_3 orqali belgilasak, (3.11) differensial tenglamalar sistemasining umumiy yechimi

$$\begin{cases} x(t) = C_1x_1 + C_2x_2 + C_3x_3, \\ y(t) = C_1y_1 + C_2y_2 + C_3y_3, \\ z(t) = C_1z_1 + C_2z_2 + C_3z_3 \end{cases} \tag{3.15}$$

ko'rinishda bo'ladi.

1- misol. Ushbu sistemaning umumiy yechimini toping:

$$\begin{cases} \dfrac{dx}{dt} = 3x - y + z, \\ \dfrac{dy}{dt} = -x + 5y - z, \\ \dfrac{dz}{dt} = x - y + 3z. \end{cases} \qquad (3.16)$$

Y e c h i s h . Sistemaning xarakteristik tenglamasini tuzamiz:

$$\begin{vmatrix} 3-r & -1 & 1 \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix} = 0$$

yoki $r^3 - 11r^2 + 36r - 36 = 0$. Uning ildizlari: $r_1 = 2$, $r_2 = 3$, $r_3 = 6$.
Demak, (3.16) sistemaning xususiy yechimlarini

$$x_1 = \lambda_1 e^{2t},\ y_1 = \mu_1 e^{2t},\ z_1 = \nu_1 e^{2t},$$
$$x_2 = \lambda_2 e^{3t},\ y_2 = \mu_2 e^{3t},\ z_2 = \nu_2 e^{3t},$$
$$x_3 = \lambda_3 e^{6t},\ y_3 = \mu_3 e^{6t},\ z_3 = \nu_3 e^{6t}$$

ko'rinishda izlaymiz.

r_1=2 da λ, μ, ν ni aniqlash uchun (3.13) tenglamalar sistemasi quyidagicha yoziladi:

$$\begin{cases} (3-2)\lambda_1 - \mu_1 + \nu_1 = 0, \\ -\lambda_1 + (5-2)\mu_1 - \nu_1 = 0, \\ \lambda_1 - \mu + (3-2)\nu = 0 \end{cases} \quad \text{yoki} \quad \begin{cases} \lambda_1 - \mu_1 + \nu_1 = 0, \\ -\lambda_1 + 3\mu_1 - \nu_1 = 0, \\ \lambda_1 - \mu_1 + \nu_1 = 0. \end{cases}$$

Bu sistema yechimlari: $\lambda_1 = 1$, $\mu_1 = 0$, $\nu_1 = -1$.

r_2=3 uchun

$$\begin{cases} \lambda_2 - \mu_2 + \nu_2 = 0, \\ -\lambda_2 + 2\mu_2 - \nu_2 = 0, \\ \lambda_2 - \mu_2 = 0 \end{cases}$$

sistemani hosil qilamiz. Bu sistemaning yechimlari sifatida λ_2=1, μ_2=1, ν_2=1 ni olish mumkin.

$r_3=6$ da (3.13) tenglamalar sistemasi quyidagicha bo'ladi:

$$\begin{cases} -3\lambda_3 - \mu_3 + \nu_3 = 0, \\ -\lambda_3 - \mu_3 - \nu_3 = 0, \\ \lambda_3 - \mu_3 - 3\nu_3 = 0. \end{cases}$$

$\lambda_3=1$ deb, $\mu_3=-2$, $\nu_3=1$ larni topamiz.

Shunday qilib, (3.16) sistemaning xususiy yechimlari:

$$\begin{aligned} &x_1 = e^{2t}, && y_1 = 0, && z_1 = -e^{2t}; \\ &x_2 = e^{3t}, && y_2 = e^{3t}, && z_2 = e^{3t}; \\ &x_3 = e^{6t}, && y_3 = -2e^{6t}, && z_3 = e^{6t}. \end{aligned}$$

Bu xususiy yechimlar (3.16) sistemaning fundamental yechimlar sistemasidir. Demak, (3.16) sistemaning umumiy yechimi (3.15) formulaga ko'ra quyidagicha bo'ladi:

$$\begin{aligned} x(t) &= C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t}, \\ y(t) &= C_2 e^{3t} - 2C_3 e^{6t}, \\ z(t) &= -C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t}. \end{aligned} \tag{3.17}$$

b) Xarakteristik tenglamaning ildizlari kompleks sonlar bo'lgan holni qaraymiz.

2- misol. Sistemaning yechimini toping:

$$\begin{cases} \dfrac{dx}{dt} = x - 5y; \\ \dfrac{dy}{dt} = 2x - y. \end{cases} \tag{3.18}$$

Y e c h i s h . Berilgan sistemaning xarakteristik tenglamasi

$$\begin{vmatrix} 1-r & -5 \\ 2 & -1-r \end{vmatrix} = r^2 + 9 = 0$$

ko'rinishda bo'lib, u ildizlarga ega. (3.13) formulaga asosan

$$\begin{cases} (1-r)\lambda - 5\mu = 0, \\ 2\lambda - (1+r)\mu = 0 \end{cases} \tag{3.19}$$

sistemaga ega bo'lamiz. $r_1=3i$ uchun

$$\begin{cases}(1-3i)\lambda_1 - 5\mu_1 = 0,\\ 2\lambda_1 - (1+3i)\mu_1 = 0.\end{cases}$$

$\lambda_1=5$ deb, $\mu_1=1-3i$ ni topamiz. U holda

$$x_1 = 5e^{3it}, \quad y_1 = (1-3i)e^{3it} \tag{3.20}$$

xususiy yechimlarni topamiz. $r_2=-3i$ ni (3.19) ga qo'yib, $\lambda_2=5$, $\mu_2=1+3i$ larni topamiz.

U holda xususiy yechimlar

$$x_1 = 5e^{-3it}, \quad y_1 = (1+3i)e^{-3it} \tag{3.21}$$

ko'rinishda bo'ladi.

Yangi fundamental yechimlar sistemasiga o'tamiz:

$$\left.\begin{aligned}\overline{x_1} &= \frac{x_1+x_2}{2}, & \overline{x_2} &= \frac{x_1-x_2}{2},\\ \overline{y_1} &= \frac{y_1+y_2}{2}, & \overline{y_1} &= \frac{y_1-y_2}{2}.\end{aligned}\right\} \tag{3.22}$$

Bundan Eyler formulasi $e^{+\alpha it} = \cos\alpha t \pm i\sin\alpha t$ dan foydalanib

$$\overline{x_1} = 5\cos 3t, \qquad \overline{x_2} = 5\sin 3t,$$
$$\overline{y_1} = \cos 3t + 3\sin 3t, \quad \overline{y_2} = \sin 3t - 3\cos 3t$$

larni topamiz. U holda berilgan sistemaning umumiy yechimi quyidagi ko'rinishda bo'ladi:

$$x(t) = 5C_1\cos 3t + 5C_2\sin 3t;$$
$$y(t) = C_1(\cos 3t + 3\sin 3t) + C_2(\sin 3t - 3\cos 3t).$$

d) Xarakteristik tenglamaning ildizlari karrali bo'lsin.

3- misol. Sistemaning yechimini toping:

$$\begin{cases}\dfrac{dx}{dt} = 2x + y,\\ \dfrac{dy}{dt} = 4y - x.\end{cases} \tag{3.23}$$

Y e c h i s h . Sistemaning xarakteristik tenglamasi

$$\begin{vmatrix} 2-r & 1 \\ -1 & 4-r \end{vmatrix} = r^2 - 6r + 9 = 0$$

$r_1 = r_2 = 3$ ildizga ega. Sistemaning yechimini

$$\begin{cases} x = (\lambda_1 + \mu_1 t)e^{3t}, \\ y = (\lambda_2 + \mu_2 t)e^{3t} \end{cases} \tag{3.24}$$

ko'rinishda izlash kerak. (3.24) ifodani (3.23) sistemaning birinchi tenglamasiga qo'yib

$$3(\lambda_1 + \mu_1 t) + \mu_1 = 2(\lambda_1 + \mu_1 t) + (\lambda_2 + \mu_2 t) \tag{3.25}$$

tenglikka ega bo'lamiz. Chap va o'ng tomondagi bir xil darajali t ning koeffitsiyentlarini tenglashtirib

$$\begin{cases} 3\lambda_1 + \mu_1 = 2\lambda_1 + \lambda_2, \\ 3\mu_1 = 2\mu_1 + \mu_2 \end{cases}$$

sistemani hosil qilamiz. Bundan

$$\begin{cases} \lambda_2 = \lambda_1 + \mu_1, \\ \mu_2 = \mu_1 \end{cases}$$

ni topamiz. λ_1 va μ_1 sonlarni ixtiyoriy parametr deb olishimiz mumkin. $\lambda_1 = C_1$ va $\mu_2 = C_2$ deb belgilasak, (3.13) sistemaning umumiy yechimi

$$\begin{cases} x = (C_1 + C_2 t)e^{3t}, \\ y = (C_1 + C_2 + C_2 t)e^{3t} \end{cases}$$

ko'rinishda bo'ladi.

3- §. Differensial tenglamalar sistemasining birinchi integrali

Differensial tenglamalar sistemasi

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n), \quad (i = \overline{1, n}) \tag{3.26}$$

ni integrallashning bu usuli quyidagidan iborat: arifmetik amallar (qo'shish, ayirish, ko'paytirish, bo'lish) yordamida (3.26) tenglamalar sistemasi osongina integrallanadigan

$$F\left(t,\ U,\ \frac{dU}{dt}\right) = 0 \tag{3.27}$$

tenglamaga keltiriladi, bu yerda $U = U(t,\ x_1,\ x_2,\ \ldots,\ x_n)$.

1- misol. Sistemaning yechimini toping:

$$\begin{cases} \dfrac{dx}{dt} = 2(x^2 + y^2)t, \\ \dfrac{dy}{dt} = 4xyt. \end{cases} \tag{3.28}$$

Y e c h i s h. Tenglamalarni hadma-had qo'shib,

$$\frac{d(x+y)}{dt} = 2(x+y)^2 t$$

tenglamani hosil qilamiz. Uni integrallab

$$\frac{1}{x+y} + t^2 = C_1$$

ni topamiz. Tenglamalarni hadma-had ayirib, quyidagini topamiz:

$$\frac{d(x-y)}{dt} = 2t(x-y)^2,$$

bundan

$$\frac{1}{x-y} + t^2 = C_2$$

ni hosil qilamiz. Shunday qilib, sistemaning ikkita birinchi integralini topdik:

$$t^2 + \frac{1}{x+y} = C_1,\quad t^2 + \frac{1}{x-y} = C_2. \tag{3.29}$$

(3.29) ifoda – (3.28) sistemaning umumiy integrali. (3.29) sistemani x va y noma'lum funksiyalarga nisbatan yechib, (3.28) differensial tenglamalarning umumiy yechimini topamiz:

$$x(t) = \frac{C_1 + C_2 - 2t^2}{2(C_1 - t^2)(C_2 - t^2)},\quad y(t) = \frac{C_2 - C_1}{2(C_1 - t^2)(C_2 - t^2)}.$$

2- misol. Quyidagi sistemaning yechimini toping:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x-y}{z-t}, \\ \dfrac{dy}{dt} = \dfrac{x-y}{z-t}, \\ \dfrac{dz}{dt} = x - y + 1. \end{cases} \qquad (3.30)$$

Y e c h i s h . Sistemaning birinchi tenglamasidan ikkinchi tenglamasini hadma-had ayirib,

$$\frac{d(x-y)}{dt} = 0$$

tenglamani hosil qilamiz. Uni integrallab, (3.30) sistemaning birinchi integralini topamiz:

$$x - y = C_1. \qquad (3.31)$$

(3.31) ifodani (3.30) sistemaning ikkinchi va uchinchi tenglamalariga qo'yib, ikki noma'lumli tenglamalar sistemasiga kelamiz:

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{C_1}{z-t}, \\ \dfrac{dz}{dt} = C_1 + 1. \end{cases} \qquad (3.32)$$

(3.32) sistemaning ikkinchi tenglamasidan

$$z = (C_1 + 1)t + C_2 \qquad (3.33)$$

ni topamiz. (3.33) ni (3.32) sistemaning birinchi tenglamasiga keltirib qo'yamiz va

$$y = \ln|C_1 t + C_2| + C_3 \qquad (3.34)$$

ni topamiz. Shunday qilib, (3.30) sistemaning umumiy yechimi:

$$x(t) = \ln|C_1 t + C_2| + C_1 + C_3,$$
$$y(t) = \ln|C_1 t + C_2| + C_3,$$
$$z(t) = (C_1 + 1)t + C_2.$$

3- misol. Quyidagi

$$\begin{cases} \dfrac{dx}{dt} = 3x + 5y, \\ \dfrac{dy}{dt} = -2x - 8y \end{cases} \qquad (3.35)$$

sistemaning $x|_{t=0} = 2$, $y|_{t=0} = 5$ boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini toping.

Y e c h i s h . Sistemaning birinchi tenglamasini 2 ga ko'paytirib, ikkinchi tenglamaga hadma-had qo'shib,

$$\frac{d(2x+y)}{dt} = 2(2x + y)$$

tenglamani hosil qilamiz. Bundan

$$2x + y = C_1 e^{2t}$$

yoki

$$y = C_1 e^{2t} - 2x \qquad (3.36)$$

birinchi integralni topamiz. (3.36) ni (3.35) sistemaning birinchi tenglamasiga keltirib qo'yib, x ga nisbatan chiziqli tenglamaga kelamiz:

$$\frac{dx}{dt} + 7x = 5C_1 e^{2t}. \qquad (3.37)$$

Bundan

$$x(t) = C_2 e^{-7t} + \frac{5}{9} C_1 e^{2t} \qquad (3.38)$$

yechimni topamiz. Shunday qilib, (3.35) sistemaning umumiy yechimi

$$\begin{cases} x(t) = C_2 e^{-7t} + \dfrac{5}{9} C_1 e^{2t}, \\ y(t) = -\dfrac{1}{9} C_1 e^{2t} - 2C_2 e^{-7t}. \end{cases} \qquad (3.39)$$

Sistemaning boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini topish uchun (3.39) ga t, x va y larning o'rniga mos ravishda 0, 2 va 5 sonlarni qo'yib, C_1 va C_2 larga nisbatan quyidagi sistemani hosil qilamiz:

$$\begin{cases} C_2 + \frac{5}{9}C_1 = 2, \\ -\frac{1}{9}C_1 - 2C_2 = 5. \end{cases} \qquad (3.40)$$

Bundan $C_1=9$, $C_2=-3$ ni topamiz, demak, (3.35) sistemaning xususiy yechimi:

$$x(t) = 5e^{2t} - 3e^{-7t},$$
$$y(t) = -e^{2t} + 6e^{-7t}.$$

4- §. O'zgarmas koeffitsiyentli chiziqli bir jinsli bo'lmagan differensial tenglamalar sistemasini integrallash usullari

1. O'zgarmaslarni variatsiyalash usuli. Ushbu sistema berilgan bo'lsin:

$$\begin{cases} x' + a_1 x + b_1 y + c_1 z = f_1(t), & (1) \\ y' + a_2 x + b_2 y + c_2 z = f_2(t), & (2) \\ z' + a_3 x + b_3 y + c_3 z = f_3(t). & (3) \end{cases} \qquad (3.41)$$

Bunda $f_i(t)$ ($i=1, 2, 3$) o'zgaruvchining berilgan uzluksiz funksiyasi.

Faraz qilaylik:

$$\begin{cases} x = C_1 x_1 + C_2 x_2 + C_3 x_3, \\ y = C_1 y_1 + C_2 y_2 + C_3 y_3, \\ z = C_1 z_1 + C_2 z_2 + C_3 z_3 \end{cases} \qquad (3.42)$$

funksiyalar (3.41) sistemaga mos bir jinsli sistemaning umumiy yechimi bo'lsin. U holda (3.41) sistemaning yechimini

$$x = C_1(t)x_1 + C_2(t)x_2 + C_3(t)x_3,$$
$$y = C_1(t)y_1 + C_2(t)y_2 + C_3(t)y_3, \qquad (3.43)$$
$$z = C_1(t)z_1 + C_2(t)z_2 + C_3(t)z_3$$

ko‘rinishda izlaymiz, bu yerda $C_1(t)$, $C_2(t)$, $C_3(t)$ – noma’lum funksiyalar.

(3.43) ifodalarni (3.41) sistemaga keltirib qo‘ysak, (3.41) sistemaning (1) tenglamasi quyidagi ko‘rinishga keladi:

$$\begin{aligned}&C_1'x_1+C_2'x_2+C_3'x_3+C_1(x_1'+a_1x_1+b_1y_1+c_1z_1)+\\&+C_2(x_2'+a_1x_2+b_1y_2+c_1z_2)+C_3(x_3'+a_1x_3+b_1y_3+c_1z_3)=f_1(t).\end{aligned}\qquad(3.44)$$

Bunda (3.42) ga asosan barcha qavslar nolga teng, demak,

$$C_1'x_1+C_2'x_2+C_3'x_3=f_1(t).\qquad(3.45)$$

Xuddi shuningdek, (3.41) sistemaning (2) va (3) tenglamalaridan

$$\begin{cases}C_1'y_1+C_2'y_2+C_3'y_3=f_2(t)\\C_1'z_1+C_2'z_2+C_3'z_3=f_3(t)\end{cases}\qquad(3.46)$$

tenglamalar sistemasini hosil qilamiz.

C_1', C_2', C_3' larga nisbatan chiziqli bo‘lgan (3.45), (3.46) sistema yechimga ega, chunki uning determinanti Vronskiy determinanti bo‘lib, u noldan farqli, ya’ni:

$$\Delta=\begin{vmatrix}x_1&x_2&x_3\\y_1&y_2&y_3\\z_1&z_2&z_3\end{vmatrix}\neq 0.$$

(3.45), (3.46) sistemadan C_1', C_2', C_3' larni topib, so‘ng integrallab, C_1, C_2, C_3 larni topamiz, shu bilan birga (3.41) sistemaning (3.43) yechimini topamiz.

1- misol. Ushbu sistemaning umumiy yechimini toping:

$$\begin{cases}\dfrac{dx}{dt}+2x+4y=1+4t,\\\dfrac{dy}{dt}+x-y=\dfrac{3}{2}t^2.\end{cases}\qquad(3.47)$$

Y e c h i s h . Avvalo

$$\begin{cases}\dfrac{dx}{dt}+2x+4y=0,\\\dfrac{dy}{dt}+x-y=0\end{cases}\qquad(3.48)$$

sistemaning umumiy yechimini topamiz. (3.48) sistemaning ikkinchi tenglamasini

$$x = y - \frac{dy}{dt} \tag{3.49}$$

ko'rinishda yozib, uni t bo'yicha differensiallab,

$$\frac{dx}{dt} = \frac{dy}{dt} - \frac{d^2y}{dt^2} \tag{3.50}$$

tenglikni hosil qilamiz. (3.49) va (3.50) ifodalarni (3.48) sistemaning birinchi tenglamasiga keltirib qo'yib, y ga nisbatan ikkinchi tartibli tenglamaga kelamiz:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0\,.$$

Bu tenglamaning umumiy yechimi:

$$y = C_1e^{2t} + C_2e^{-3t}.$$

(3.49) dan

$$x = -C_1e^{2t} + 4C_2e^{-3t}$$

ni topamiz. Demak, (3.48) sistemaning umumiy yechimi:

$$x = -C_1e^{2t} + 4C_2e^{-3t},$$
$$y = C_1e^{2t} + C_2e^{-3t}.$$

Endi (3.47) sistemaning yechimini

$$\begin{cases} x = -C_1(t)e^{2t} + 4C_2(t)e^{-3t}, \\ y = C_1(t)e^{2t} + C_2(t)e^{-3t} \end{cases} \tag{3.51}$$

ko'rinishda izlaymiz.

(3.51) ni (3.47) ga keltirib qo'yib va ba'zi bir elementar amallarni bajarib, $C_1'(t)$, $C_2'(t)$ larga nisbatan

$$\begin{cases} -C_1'(t)e^{2t} + 4C_2'(t)e^{-3t} = 1 + 4t, \\ C_1'(t)e^{2t} + C_2'(t)e^{-3t} = \frac{3}{2}t^2 \end{cases}$$

chiziqli tenglamalar sistemasiga kelamiz. Bundan

$$C_1'(t) = \frac{(6t^2-4t-1)e^{-2t}}{5}, \quad C_2'(t) = \frac{(3t^2+8t+2)e^{3t}}{10}$$

larni topib, so'ngra integrallab,

$$C_1(t) = -\frac{1}{5}(t+3t^2)e^{-2t} + C_3, \quad C_2(t) = \frac{1}{10}(2t+t^2)e^{3t} + C_4 \quad (3.52)$$

ni topamiz, bu yerda C_3 va C_4 – ixtiyoriy o'zgarmaslar. (3.52) ni (3.51) ga keltirib qo'yib, (3.47) sistemaning umumiy yechimini hosil qilamiz:

$$\begin{aligned} x(t) &= -C_1e^{2t} + 4C_2e^{-3t} + t + t^2, \\ y(t) &= C_1e^{2t} + C_2e^{-3t} - \frac{1}{2}t^2. \end{aligned} \quad (3.53)$$

2. Aniqmas koeffitsiyentlar usuli. Agar o'zgarmas koeffitsiyentli chiziqli bir jinsli bo'lmagan differensial tenglamalar sistemasining o'ng tomonidagi ifoda $f_i(t)$ – funksiya, $P_k(t)$ – ko'phad, $e^{\alpha t}$ – ko'rsatkichli funksiya, $\sin\beta t$, $\cos\beta t$ – sinus va kosinus yoki ularning ko'paytmasi ko'rinishida bo'lsa, sistemaning xususiy yechimini aniqmas koeffitsiyentlar usuli bilan topish maqsadga muvofiqdir.

2- misol. Ushbu

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = x - 5\sin t \end{cases} \quad (3.54)$$

sistemaning umumiy yechimini toping.

Y e c h i s h. (3.54) sistemaga mos bo'lgan bir jinsli sistema:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = x. \end{cases} \quad (3.55)$$

Birinchi tenglamani t bo'yicha differensiallaymiz:

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2\frac{dy}{dt}.$$

Bundan

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0 \qquad (3.56)$$

tenglamani hosil qilamiz. Bu tenglama $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 10\sin t$ tenglamaga mos bir jinsli tenglama. (3.56) ning $k^2 - k - 2 = 0$ xarakteristik tenglamasi $k_1 = -1$, $k_2 = 2$ ildizlarga ega.

Mos bir jinsli tenglamaning umumiy yechimi

$$\overline{x} = C_1 e^{-t} + C_2 e^{2t} \qquad (3.57)$$

bo'ladi. Bir jinsli bo'lmagan tenglamaning o'ng tomoni $f(t) = 10\sin t$ ko'rinishga ega. Bunda α=0, β=1, shuning uchun xususiy yechimni

$$x^* = A\cos t + B\sin t \qquad (3.58)$$

ko'rinishda izlaymiz. Bundan $x^{*'}$, $x^{*''}$ larni topamiz:

$$x^{*'} = -A\sin t + B\cos t,$$
$$x^{*''} = -A\cos t - B\sin t. \qquad (3.59)$$

(3.58) va (3.59) larni differensial tenglamaga qo'yib, quyidagiga ega bo'lamiz:

$$(3A + B)\cos t + (A + B)\sin t = 10\sin t.$$

cos*t* va sin*t* larning koeffitsiyentlarini tenglashtirib,

$$\begin{cases} 3A + B = 0, \\ A + B = 10 \end{cases}$$

sistemani hosil qilamiz. Sistemani yechib

$$A = -5, \quad B = 15$$

larni topamiz. Demak, xususiy yechim

$$x^* = -5\cos t + 10\sin t$$

ko'rinishda, umumiy yechim esa

$$x = \overline{x} + x^* = C_1 e^{-t} + C_2 e^{2t} - 5\cos t + 10\sin t \qquad (3.60)$$

ko'rinishda bo'ladi. (3.60) ni *t* bo'yicha differensiallab

$$x' = -C_1e^{-t} + 2C_2e^{2t} + 5\sin t + 10\cos t \qquad (3.61)$$

ni topamiz. (3.60) va (3.61) larni (3.54) sistemaning birinchi tenglamasiga keltirib qo'yamiz.

U holda

$$y = -C_1e^{-t} + \frac{1}{2}C_2e^{2t} + \frac{15}{2}\cos t - \frac{5}{2}\sin t$$

ni topamiz. Demak, (3.54) sistemaning umumiy yechimi

$$x = C_1e^{-t} + C_2e^{2t} - 5\cos t + 10\sin t,$$
$$y = -C_1e^{-t} + \frac{1}{2}C_2e^{2t} + \frac{15}{2}\cos t - \frac{5}{2}\sin t$$

ko'rinishda bo'ladi.

3. Birinchi integrallarini topish usuli (Dalamber usuli).

Ushbu

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + f_1(t), \\ \dfrac{dy}{dt} = a_2x + b_2y + f_2(t) \end{cases} \qquad (3.62)$$

sistemani qaraymiz. Ikkinchi tenglamani biror λ songa ko'paytirib, birinchi tenglamaga hadma-had qo'shamiz:

$$\frac{d(x+\lambda y)}{dt} = (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + f_1(t) + \lambda f_2(t). \qquad (3.63)$$

(3.63) tenglamani quyidagi ko'rinishda yozib olamiz:

$$\frac{d(x+\lambda y)}{dt} = (a_1 + \lambda a_2)\left(x + \frac{b_1 + \lambda b_2}{a_1 + \lambda a_2}y\right) + f_1(t) + \lambda f_2(t). \qquad (3.64)$$

Endi λ sonni shunday tanlaymizki, u

$$\frac{b_1 + \lambda b_2}{a_1 + \lambda a_2} = \lambda \qquad (3.65)$$

bo'lsin. U holda (3.64) tenglama $(x+\lambda y)$ ga nisbatan chiziqli tenglama ko'rinishiga keladi:

$$\frac{d(x+\lambda y)}{dt} = (a_1 + \lambda a_2)(x + \lambda y) + f_1(t) + \lambda f_2(t). \qquad (3.66)$$

(3.66) ni integrallab

$$x+\lambda y=e^{(a_1+\lambda a_2)t}\left\{C+\int[f_1(t)+\lambda f_2(t)]e^{-(a_1+a_2\lambda)t}dt\right\} \quad (3.67)$$

ni topamiz.

Agar (3.65) tenglama ikkita haqiqiy har xil $\lambda_1 \neq \lambda_2$ ildizga ega bo'lsa, u holda (3.67) dan (3.62) sistemaning ikkita birinchi integrali topiladi. Demak, sistemani integrallash tugallangan bo'ladi.

3- misol. Ushbu sistemani Dalamber usuli bilan yeching:

$$\begin{cases}\dfrac{dx}{dt}=5x+4y+e^t,\\ \dfrac{dy}{dx}=4x+5y+1.\end{cases}$$

Y e c h i s h. Bu yerda $a_1=5$, $b_1=4$, $a_2=4$, $b_2=5$, $f_1(t)=e^t$, $f_2(t)=1$. λ sonni (3.65) formuladan topamiz:

$$\frac{4+5\lambda}{5+4\lambda}=\lambda \Rightarrow 4+5\lambda=\lambda(5+4\lambda). \quad (3.67')$$

Bu tenglama $\lambda_1=-1$, $\lambda_2=1$ ildizlarga ega. U holda (3.67′) formuladan λ=1 uchun

$$x+y=e^{9t}(C_1+\int(e^{-8t}+e^{-9t})dt)=C_1e^{9t}-\frac{1}{8}e^t-\frac{1}{9};$$

$\lambda=-1$ uchun

$$x-y=e^t(C_2+\int(1-e^{-t})dt)=C_2e^t+te^t+1$$

larni topamiz. Shunday qilib, berilgan sistemaning bog'liqmas ikkita birinchi integrali

$$\left(x+y+\frac{1}{8}e^t+\frac{1}{9}\right)e^{-9t}=C_1,$$

$$(x-y-te^t-1)e^{-t}=C_2$$

ko'rinishda bo'ladi.

Quyidagi differensial tenglamalar sistemasining yechimini bitta tenglamaga keltirish usuli bilan toping:

269. $\begin{cases} \dfrac{dx}{dt} = -9y, \\ \dfrac{dy}{dt} = x. \end{cases}$

270. $\begin{cases} \dfrac{dx}{dt} = y + t, \\ \dfrac{dy}{dt} = x - t. \end{cases}$

271. $\begin{cases} \dfrac{dx}{dt} = 3 - 2y, \\ \dfrac{dy}{dt} = 2x - 2t. \end{cases}$

272. $\begin{cases} \dfrac{dx}{dt} + 3x + y = 0, \\ \dfrac{dy}{dt} - x + y = 0. \end{cases}$

273. $\begin{cases} \dfrac{dx}{dt} = 3x - \dfrac{1}{2}y - 3t^2 - \dfrac{1}{2}t + \dfrac{3}{2}, \\ \dfrac{dy}{dt} = 2y - 2t - 1. \end{cases}$

274. $\begin{cases} \dfrac{dx}{dt} = x + 5y, \\ \dfrac{dy}{dt} = -x - 3y, \end{cases}$
$x(0) = -2, \quad y(0) = 1.$

275. $\begin{cases} \dfrac{d^2x}{dt^2} = x^2 + y, \\ \dfrac{dy}{dt} = -2\dfrac{dx}{dt} + x, \end{cases}$
$x(0) = x'(0) = 1, \quad y(0) = 0.$

276. $\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + x = 0, \\ \dfrac{dx}{dt} + \dfrac{d^2y}{dt^2} = 0. \end{cases}$

277. $\begin{cases} \dfrac{dx}{dt} = x - 4y, \\ \dfrac{dy}{dt} = x + y. \end{cases}$

278. $\begin{cases} 4\dfrac{dx}{dt} - \dfrac{dy}{dt} + 3x = \sin t, \\ \dfrac{dx}{dt} + y = \cos t. \end{cases}$

Quyidagi differensial tenglamalar sistemasining yechimini sistemaning birinchi integrallarini topish usuli bilan toping:

279. $\begin{cases} \dfrac{dx}{dt} = x^2 + y^2, \\ \dfrac{dy}{dt} = 2xy. \end{cases}$

280. $\begin{cases} \dfrac{dx}{dt} = \dfrac{y}{x-y}, \\ \dfrac{dy}{dt} = \dfrac{x}{x-y}. \end{cases}$

281. $\begin{cases} \dfrac{dx}{dt} = \sin x \cos y, \\ \dfrac{dy}{dt} = \cos x \sin y. \end{cases}$

282. $\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = \dfrac{y^2 - t}{x}. \end{cases}$ $\quad z = t^2 + 2xy,\ z = x - ty^2$ funksiyalar sistemaning birinchi integrali bo‘la oladimi?

283. $\begin{cases} \dfrac{dx}{dt} = y^2 - \cos x, \\ \dfrac{dy}{dt} = -y \sin x. \end{cases}$ $\quad z = 2t\cos x - \ln y,\ z = 3y\cos x - y^3$ funksiyalar sistemaning birinchi integrali bo‘la oladimi?

284. $\begin{cases} \dfrac{dx}{dt} = \cos^2 x \cos^2 y + \sin^2 x \cos^2 y, \\ \dfrac{dy}{dt} = -\dfrac{1}{2}\sin 2x \sin 2y, \end{cases}$ $x(0) = 0, \quad y(0) = 0.$

Quyidagi differensial tenglamalar sistemasining yechimini Eyler usuli bilan toping:

285. $\begin{cases} \dfrac{dx}{dt} = 8y - x, \\ \dfrac{dy}{dt} = x + y. \end{cases}$

286. $\begin{cases} \dfrac{dx}{dt} = 2x + y, \\ \dfrac{dy}{dt} = x - 3y. \end{cases}$

$x(0) = 0, \ y(0) = 0.$

287. $\begin{cases} \dfrac{dx}{dt} = x + y, \\ \dfrac{dy}{dt} = 4y - 2x, \end{cases}$

$x(0) = 0, \ y(0) = -1.$

288. $\begin{cases} \dfrac{dx}{dt} = 4x - 3y, \\ \dfrac{dy}{dt} = 3x + 4y. \end{cases}$

289. $\begin{cases} \dfrac{dx}{dt} = 5x - y, \\ \dfrac{dx}{dt} = x + 3y. \end{cases}$

290. $\begin{cases} \dfrac{dx}{dt} = x + 5y, \\ \dfrac{dy}{dt} = -3y - x, \end{cases}$

$x(0) = -2, \quad y(0) = 1.$

291. $\begin{cases} \dfrac{dx}{dt} + 2\dfrac{dy}{dt} = 17x + 8y, \\ 13\dfrac{dx}{dt} = 53x + 2y, \end{cases}$
$x(0) = 2, \quad y(0) = -1.$

293. $\begin{cases} \dfrac{dx}{dt} = x - z, \\ \dfrac{dy}{dt} = x, \\ \dfrac{dz}{dt} = x - y. \end{cases}$

292. $\begin{cases} \dfrac{dx}{dt} = 6x - 12y - z, \\ \dfrac{dy}{dt} = x - 3y - z, \\ \dfrac{dz}{dt} = -4x + 12y + 3z. \end{cases}$

Quyidagi differensial tenglamalar sistemasining yechimini o'zgarmaslarni variatsiyalash usuli bilan toping:

294. $\begin{cases} \dfrac{dx}{dt} + 2x - y = -e^{2t}, \\ \dfrac{dy}{dt} + 3x - 2y = 6e^{2t}. \end{cases}$

297. $\begin{cases} \dfrac{dx}{dt} + y = \cos t, \\ \dfrac{dy}{dt} + x = \sin t. \end{cases}$

295. $\begin{cases} \dfrac{dx}{dt} = x + y - \cos t, \\ \dfrac{dy}{dt} = -y - 2x + \cos t + \sin t. \end{cases}$

298. $\begin{cases} \dfrac{dx}{dt} = 2x - y, \\ \dfrac{dy}{dt} = 2y - x - 5e^{t} \sin t. \end{cases}$

296. $\begin{cases} \dfrac{dx}{dt} - y = \cos t, \\ \dfrac{dy}{dt} = 1 - x. \end{cases}$

299. $\begin{cases} \dfrac{dx}{dt} = 2x + y - 2z - t + 2, \\ \dfrac{dy}{dt} = -x + t, \\ \dfrac{dz}{dt} = x + y - z - t + 1. \end{cases}$

Quyidagi differensial tenglamalar sistemasining yechimini aniqmas koeffitsiyentlar usulida toping:

300. $\begin{cases} \dfrac{dx}{dt} = 3 - 2y, \\ \dfrac{dy}{dt} = 2x - 2t. \end{cases}$

301. $\begin{cases} \dfrac{dx}{dt} = -y + \sin t, \\ \dfrac{dy}{dt} = x + \cos t. \end{cases}$

302. $\begin{cases} \dfrac{dx}{dt} = 4x - 5y + 4t - 1, \\ \dfrac{dy}{dt} = x - 2y + t, \end{cases}$
$x(0) = 0, \ \ y(0) = 0.$

304. $\begin{cases} \dfrac{dx}{dt} = x + y + t, \\ \dfrac{dy}{dt} = x - 2y + 2t, \end{cases}$
$x(0) = -\dfrac{7}{9}, \ \ y(0) = -\dfrac{5}{9}.$

303. $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + y = e^{-t}, \\ 2\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = \sin t. \end{cases}$

305. $\begin{cases} \dfrac{dx}{dt} = x\cos t, \\ 2\dfrac{dy}{dt} = (e^{t} + e^{-t})y. \end{cases}$

Quyidagi differensial tenglamalar sistemasining yechimini Dalamber usulida toping:

306. $\begin{cases} \dfrac{dx}{dt} = 5x + 4y, \\ \dfrac{dy}{dt} = x + 2y. \end{cases}$

309. $\begin{cases} \dfrac{dx}{dt} = 2x + 4y + \cos t, \\ \dfrac{dy}{dt} = x - 2y + \sin t. \end{cases}$

307. $\begin{cases} \dfrac{dx}{dt} = 6x + y, \\ \dfrac{dy}{dt} = 4x + 3y. \end{cases}$

310. $\begin{cases} \dfrac{dx}{dt} = 3x + y + e^{t}, \\ \dfrac{dy}{dt} = x + 3y - e^{t}. \end{cases}$

308. $\begin{cases} \dfrac{dx}{dt} = 2x - 4y + 1, \\ \dfrac{dy}{dt} = -x + 5y. \end{cases}$

311. $\begin{cases} \dfrac{dx}{dt} = x + 5y, \\ \dfrac{dy}{dt} = -3y - x, \end{cases}$
$x(0) = -2, \ \ y(0) = 1.$

Mustaqil ish topshiriqlari

312. $\begin{cases} \dfrac{dy}{dt} = y + z, \\ \dfrac{dz}{dt} = y + z + t. \end{cases}$

313. $\begin{cases} \dfrac{dy}{dt} = \dfrac{y^2}{z}, \\ \dfrac{dz}{dt} = \dfrac{1}{2}y. \end{cases}$

314. $\begin{cases} \dfrac{dy}{dt} = 1 - \dfrac{1}{z}, \\ \dfrac{dz}{dt} = \dfrac{1}{y-t}. \end{cases}$

318. $\begin{cases} \dfrac{dy}{dt} = x - y, \\ \dfrac{dz}{dt} = x + 3y. \end{cases}$

315. $\begin{cases} \dfrac{dy}{dt} = \dfrac{z^2}{y}, \\ \dfrac{dz}{dt} = \dfrac{y^2}{z}. \end{cases}$

319. $\begin{cases} \dfrac{dx}{dt} = -y + z, \\ \dfrac{dy}{dt} = z + x, \\ \dfrac{dz}{dt} = x + y. \end{cases}$

316. $\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{yz}, \\ \dfrac{dz}{dx} = \dfrac{x}{y^2}. \end{cases}$

320. $\begin{cases} \dfrac{dx}{dt} = -x + y + z + e^t, \\ \dfrac{dy}{dt} = x - y + z + e^{3t}, \\ \dfrac{dz}{dt} = x + y + z + 4. \end{cases}$

317. $\begin{cases} \dfrac{dy}{dt} = -7x + y, \\ \dfrac{dz}{dt} = -2x - 5y. \end{cases}$

5- §. Operatsion hisob

1. Boshlang'ich funksiya va uning tasviri.

Boshlang'ich funksiya (original) deb quyidagi shartlarni qanoatlantiruvchi $f(t)$ funksiya qabul qilinadi:

1°. Istalgan chekli intervalda $f(t)$ va $f'(t)$ chekli sondan ko'p bo'lmagan birinchi tur uzilish nuqtalariga (chekli sakrashlarga) ega.

2°. $t < 0$ uchun $f(t)=0$.

3°. $f(t)$ funksiya ko'rsatkichli funksiyadan tez o'smaydi, ya'ni shunday t ga bog'liq bo'lmagan musbat haqiqiy o'zgarmas M va S_0 sonlari mavjudki, bunda yetarlicha katta t lar uchun

$$|f(t)| \le Me^{S_0 t} \qquad (3.68)$$

tengsizlik bajariladi. Bunda S_0 – originalning o'sish tartibini ko'rsatuvchi son. Original o'zgarmas bo'lsa, $S_0=0$ deb qabul qilish mumkin.

$f(t)$ funksiyaning haqiqiy o'zgaruvchi t ning kompleks funksiyasi e^{-pt} ga ko'paytmasini, ya'ni

$$e^{-pt}\cdot f(t) \quad (p=a+ib,\ a\succ 0) \tag{3.69}$$

ni qaraymiz. (3.69) funksiya ham haqiqiy o'zgaruvchi t ning kompleks funksiyasidir:

$$e^{-pt}\cdot f(t)=e^{-at}f(t)\cos bt-ie^{-at}f(t)\sin bt. \tag{3.70}$$

So'ngra ushbu xosmas integralni qaraymiz:

$$\int_0^\infty e^{-pt}f(t)dt=\int_0^\infty e^{-at}f(t)\cos btdt-i\int_0^\infty e^{-at}f(t)\sin btdt. \tag{3.71}$$

Agar $f(t)$ funksiya (3.68) tengsizlikni qanoatlantirsa va $a>S_0$ bo'lsa, (3.71) tenglikning o'ng qismida turgan xosmas integrallar mavjud va ular absolyut yaqinlashuvchi.

(3.71) integral p ning bironta funksiyasini aniqlaydi, u funksiya $F(p)$ ni bilan belgilaymiz:

$$F(p)=\int_0^\infty e^{-pt}f(t)dt. \tag{3.72}$$

Ta'rif. Kompleks $p=a+ib$ o'zgaruvchiga bog'liq bo'lgan

$$F(p)=\int_0^\infty e^{-pt}f(t)dt=L\{f(t)\}$$

tenglik bilan aniqlangan $F(p)$ funksiyaga $f(t)$ *funksiyaning tasviri* yoki *Laplas almashtirishi* deyiladi, $f(t)$ funksiyaning o'zi esa $F(p)$ *ning originali* deyiladi va quyidagicha yoziladi:

$F(p)\xrightarrow{\cdot} f(t)$ tasvir-original yoki $f(t)\xrightarrow{\cdot} F(p)$ original-tasvir, yoki

$$L\{f(t)\}=F(p).$$

2. Laplas almashtirishining asosiy xossalari

1°. Ixtiyoriy α va β kompleks o'zgarmaslar uchun

$$\alpha f(t)+\beta g(t)\xleftarrow{\cdot}\alpha F(p)+\beta G(p). \tag{3.73}$$

Asosiy originallar va tasvirlar jadvali

№	$f(t)$ original	$F(p)$ tasvir	№	$f(t)$ original	$F(p)$ tasvir
1.	1	$\frac{1}{p}$	9.	$e^{\alpha t}\sin at$	$\frac{\alpha}{(p-\alpha)^2+a^2}$
2.	t^n	$\frac{n!}{p^{n+1}}$	10.	$t^n e^{\alpha t}$	$\frac{n!}{(p-\alpha)^{n+1}}$
3.	$e^{\alpha t}$	$\frac{1}{p-\alpha}$	11.	$t\cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$
4.	$\sin at$	$\frac{a}{p^2+a^2}$	12.	$t\sin at$	$\frac{2pa}{(p^2+a^2)^2}$
5.	$\cos at$	$\frac{p}{p^2+a^2}$	13.	$\sin(t-\alpha)$	$e^{-\alpha p}\frac{1}{p^2+1}$
6.	$\mathrm{sh}at$	$\frac{a}{p^2-a^2}$	14.	$\cos(t-\alpha)$	$e^{-\alpha p}\frac{p}{p^2+1}$
7.	$\mathrm{ch}at$	$\frac{p}{p^2-a^2}$	15.	$\frac{\sin t}{t}$	$\mathrm{arcctg}p$
8.	$e^{\alpha t}\cos at$	$\frac{p-\alpha}{(p-\alpha)^2+a^2}$	16.	$\int_0^t \frac{\sin t}{t}dt$	$\frac{\mathrm{arcctg}p}{p}$

2°. Ixtiyoriy o‘zgarmas $\alpha > 0$ uchun

$$f(at) \leftarrow\!\!\!\!\!\cdot\!\!\!-\!\!- \frac{1}{a}F\left(\frac{p}{a}\right). \tag{3.74}$$

3°. Agar $f(t) \leftarrow\!\!\!\!\!\cdot\!\!\!-\!\!- F(p)$ bo‘lib, $f'(t)$ original bo‘lsa, u holda

$$f'(t) \leftarrow\!\!\!\!\!\cdot\!\!\!-\!\!- pF(p) - f(0). \tag{3.75}$$

4°. Agar $f(t) \leftarrow\!\!\!\!\!\cdot\!\!\!-\!\!- F(p)$ bo‘lsa, u holda istalgan α da

$$e^{\alpha t} f(t) \leftarrow\!\!\!\!\!\cdot\!\!\!-\!\!- F(p-\alpha). \tag{3.76}$$

5°. Agar $f(t) \leftarrow\!\!\!\!\!\cdot\!\!\!\!\cdot\; F(p)$ bo'lsa, u holda $\tau > 0$ bo'lganda

$$f(t-\tau) \leftarrow e^{-p\tau} F(p). \quad (3.77)$$

6°. Agar $f(t) \leftarrow F(p)$ bo'lsa, u holda

$$\int_0^t f(t)dt \leftarrow \frac{F(p)}{p}. \quad (3.78)$$

7°. Agar $f(t,x) \leftarrow F(p,x)$ bo'lsa, u holda

$$\frac{\partial f(t,x)}{\partial x} \leftarrow \frac{\partial F(p,x)}{\partial x}. \quad (3.79)$$

8. Agar $f(t) \leftarrow F(p)$ bo'lsa, u holda

$$-tf(t) \leftarrow F'(p). \quad (3.80)$$

9°. Agar $\int^{\infty} F(z)dz$ integral yaqinlashuvchi va $f(t) \leftarrow F(p)$ bo'lsa, u holda

$$\frac{f(t)}{t} \leftarrow \int_p^{\infty} F(z)dz. \quad (3.81)$$

3. Funksiyaning tasvirini topishga doir misollar.

Asosiy xossalardan va tasvirlar jadvalidan foydalanib, haqiqiy o'zgaruvchining bir qator elementar funksiyalarining tasvirini topamiz.

Ba'zi funksiyalarning tasvirini topishda to'g'ridan-to'g'ri jadvaldan foydalanib bo'lmaydi. Bunday hollarda shakl almashtirishlar yordamida funksiya ko'rinishini jadvalga moslab olamiz.

1- misol. $f(t) = a^t$ funksiyaning tasvirini toping.

Y e c h i s h . Logarifmning asosiy ayniyatidan:

$$a^t = e^{\ln a^t} = e^{t\ln a}.$$

U holda (3) formuladan

$$e^{t\ln a} \leftarrow \frac{1}{p-\ln a}$$

ifodani topamiz. Demak, a^t funksiyaning tasviri $F(p)=\frac{1}{p-\ln a}$, ya'ni

$$a^t \leftarrow \frac{1}{p-\ln a}.$$

2- misol. $f(t)=t\cdot\cos at$ originalning tasvirini toping.

Y e c h i s h. 8° ga asosan

$$t\cdot\cos at \leftarrow -\left(\frac{p}{p^2+a^2}\right)' = -\frac{a^2-p^2}{(a^2+p^2)^2} = \frac{p^2-a^2}{(a^2+p^2)^2}.$$

Demak, $L\{t\cos at\}=\frac{p^2-a^2}{(p^2+a^2)^2}$.

3- misol. $t^n e^{\alpha t}$ originalning tasvirini toping.

Y e c h i s h. $e^{\alpha t} \leftarrow \frac{1}{p-\alpha}$ moslikning ikkala tomonini α parametr bo'yicha n marta differensiallab, quyidagilarni hosil qilamiz:

$$te^{\alpha t} \leftarrow \frac{1!}{(p-\alpha)^2};\quad t^2e^{\alpha t} \leftarrow \frac{2!}{(p-\alpha)^3};$$

$$t^3e^{\alpha t} \leftarrow \frac{3!}{(p-\alpha)^4};\ \ldots;\ t^n e^{\alpha t} \leftarrow \frac{n!}{(p-\alpha)^{n+1}}.$$

Demak, $L\{t^n e^{\alpha t}\}=\frac{n!}{(p-\alpha)^{n+1}}$.

4- misol. $f(t)=(t-1)^2\,e^{t-1}$ funksiyaning tasvirini toping.

Y e c h i s h. $t-1=z$ deb, funksiyani z^2e^z ko'rinishga keltiramiz. Endi jadvalning 10- formulasidan

$$z^2e^z \leftarrow \frac{2}{(p-1)^3}$$

ni topamiz. U holda 5°-xossaga asosan

$$(t-1)^2\,e^{t-1} \leftarrow \frac{2}{(p+1)^3}$$

ga egamiz.

4. Originalni tasviri bo'yicha topish usullari

Operatsion hisobda originalni ma'lum tasviri bo'yicha izlash uchun *yoyish teoremalari* deb ataladigan teoremalardan hamda tasvirlar jadvalidan foydalaniladi.

Yoyish teoremasi. Agar izlanayotgan $f(t)$ funksiyaning $F(p)$ tasvirini $\frac{1}{p}$ ning darajalari bo'yicha darajali qatorga yoyish mumkin bo'lsa, ya'ni

$$F(p)=\frac{a_0}{p}+\frac{a_1}{p^2}+\frac{a_2}{p^3}+\ldots+\frac{a_n}{p^{n+1}}+\ldots. \qquad (3.82)$$

bo'lib, u $\frac{1}{|p|}<R$ da $F(p)$ ga yaqinlashsa, u holda original quyidagi formula bo'yicha topiladi:

$$f(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}. \qquad (3.83)$$

Bu qator $t>0$ qiymatlar uchun yaqinlashadi va $t<0$ da $f(t)=0$ deb olinadi.

Endi $F(p)$ funksiya p ning kasr ratsional funksiyasi, ya'ni

$$F(p)=\frac{A(p)}{B(p)} \qquad (3.84)$$

bo'lsin, bu yerda $A(p)$ va $B(p)$ — mos ravishda m va n darajali ($m \prec n$) ko'phadlar. U holda $F(p)$ ga mos originallar quyidagicha topiladi.

Agar $B(p)$ maxrajning barcha ildizlari ma'lum bo'lsa, u holda uni eng sodda ko'paytuvchilarga yoyish mumkin:

$$B(p)=(p-p_1)^{k_1}(p-p_2)^{k_2}\ldots(p-p_r)^{k_r}, \qquad (3.85)$$

bu yerda $k_1+k_2+\ldots+k_r=n$ Ma'lumki, bu holda $F(p)$ funksiyani eng sodda kasrlar yig'indisiga yoyish mumkin:

$$F(p)=\sum_{j=1}^{r}\sum_{s=1}^{k_j}\frac{A_{js}}{(p-p_j)^{k_j-s+1}}. \qquad (3.86)$$

Bu yoyilmaning barcha koeffitsiyentlarini

$$A_{js} = \frac{1}{(s-1)!} \lim_{p \to p_j} \frac{d^{s-1}}{dp^{s-1}} [(p - p_j)^{k_j} \cdot F(p)] \tag{3.87}$$

formula bo‘yicha aniqlash mumkin.

A_{js} koeffitsiyentlarni aniqlash uchun (3.87) formulaning o‘rniga integral hisobda ratsional kasrlarni integrallashda qo‘llaniladigan elementar usullardan foydalanish mumkin. Xususan, bu usulni qo‘llash $B(p)$ maxrajning barcha ildizlari tub, ya’ni sodda va juft-jufti bilan qo‘shma bo‘lganda maqsadga muvofiqdir.

Agar $B(p)$ ning barcha ildizlari sodda, ya’ni

$$B(p) = (p - p_1)(p - p_2)(p - p_3)...(p - p_n),$$

bu yerda $j \neq k,\ p_j \neq p_k$ bo‘lsa, yoyilma soddalashadi:

$$F(p) = \sum_{j=1}^{n} \frac{A_j}{p - p_j}, \text{ bu yerda } A_j = \frac{A(p_j)}{B(p_j)}. \tag{3.88}$$

$F(p)$ ning u yoki bu usul bilan sodda kasrlarga yoyilmasini tuzishda $f(p)$ original quyidagi formulalar bo‘yicha izlanadi:

a) $B(p)$ maxrajning ildizlari sodda bo‘lgan holda:

$$f(t) = \sum_{j=1}^{n} \frac{A(p_j)}{B'(p_j)} e^{p_j t}. \tag{3.89}$$

b) $B(p)$ maxrajning ildizlari karrali bo‘lgan holda:

$$f(t) = \sum_{j=1}^{r} \sum_{s=1}^{k_j} A_{js} \frac{t^{k_j - s}}{(k_j - s)!} e^{p_j t}. \tag{3.90}$$

5. Originalni tasvir bo‘yicha topishga misollar

1- misol. $F(p) = \frac{1}{p} e^{-\frac{1}{p^2}}$ tasvir uchun originalni toping.

Y e c h i s h . $F(p)$ funksiyani $p(p \neq 0)$ kompleks o‘zgaruvchining butun tekisligida ushbu Loran qatoriga yoyamiz:

$$F(p)=\frac{1}{p}e^{-\frac{1}{p^2}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,p^{2n+1}}=\frac{1}{p}-\frac{1}{1!}\cdot\frac{1}{p^3}+\frac{1}{2!}\cdot\frac{1}{p^5}-\frac{1}{3!}\cdot\frac{1}{p^7}+\ldots=$$

$$=\frac{1}{p}\left(1-\frac{1}{1!\,p^2}+\frac{1}{2!\,p^4}-\frac{1}{3!\,p^6}+\frac{1}{4!\,p^8}-\ldots\right).$$

Yoyilma birinchi teoremaning shartlarini qanoatlantirganligi sababli bu funksiyaning originali quyidagicha bo‘ladi:

$$f(t)=1-\frac{t^2}{2!}+\frac{t^4}{2!\,4!}-\frac{t^6}{3!\,6!}-\frac{t^8}{4!\,8!}\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!\cdot(2n)!}.$$

Demak,

$$\frac{1}{p}e^{-\frac{1}{p^2}}\overset{\cdot}{\underset{\cdot}{\longrightarrow}}\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!(2n)!}.$$

2- misol. $F(p)=\dfrac{p^2+p+1}{p(p^4-1)}$ tasvirning originalini toping.

Y e c h i s h . Tasvirning maxraji $p_1=0$, $p_2=1$, $p_3=1$, $p_4=i$, $p_5=-i$ tub ildizlarga ega. Bu holda $F(p)$ funksiyaning yoyilmasi (3.88) ko‘rinishda bo‘ladi:

$$F(p)=\frac{A_1}{p}+\frac{A_2}{p-1}+\frac{A_3}{p+1}+\frac{A_4}{p-i}+\frac{A_5}{p+i}.$$

A_1, A_2, A_3, A_4, A_5 koeffitsiyentlar

$$A_j=\frac{A(p_i)}{B'(p_j)}$$

formula bilan aniqlanadi, bu yerda $A(p)=p^2+p+1$, $B'(p)=5p^4-1$.

$$A_1=\frac{A(0)}{B'(0)}=-1;\quad A_2=\frac{A(1)}{B'(1)}=\frac{3}{4};\quad A_3=\frac{A(-1)}{B'(-1)}=\frac{1}{4};$$

$$A_4=\frac{A(i)}{B'(i)}=\frac{i}{4};\quad A_5=\frac{A(-i)}{B'(-i)}=-\frac{i}{4}.$$

Endi (3.89) formula bo'yicha originalni topamiz:

$$f(t) = -1 \cdot e^{0 \cdot t} + \frac{3}{4}e^{1 \cdot t} + \frac{1}{4}e^{-1 \cdot t} + \frac{i}{4}e^{i \cdot t} - \frac{i}{4}e^{-i \cdot t} =$$

$$= -1 + \frac{1}{4}(3e^t + e^{-t}) - \frac{1}{2}\frac{e^{it} - e^{-it}}{2i} = -1 + \frac{1}{4}(3e^t + e^{-t}) - \frac{1}{2}\sin t.$$

Mustaqil ish topshiriqlari

Quyidagi funksiyalarning tasvirlarini toping:

321. $f(t) = \sin^2 t.$

322. $f(t) = e^t \cos^2 t.$

323. $f(t) = \operatorname{sh} at \cos bt.$

324. $f(t) = \operatorname{ch} at \cos bt.$

325. $f(t) = t \operatorname{sh} bt$

326. $f(t) = \operatorname{ch} at \sin bt.$

327. $f(t) = e^{-7t} \operatorname{ch} 7t.$

328. $f(t) = \int_0^t \cos^2 \alpha t dt.$

329. $f(t) = \frac{e^t - 1}{t}.$

330. $f(t) = \frac{\sin t}{t}.$

Quyidagi tasvirlarning originallarini toping:

331. $F(p) = p - \sin\frac{1}{p}.$

332. $F(p) = p \ln\left(1 + \frac{1}{p^2}\right).$

333. $F(p) = \frac{p+3}{p(p^2 - 4p + 3)}.$

334. $F(p) = \frac{p+1}{p(p-1)(p-2)(p-3)}.$

335. $F(p) = \frac{1}{p(1+p^4)}.$

336. $F(p) = \frac{1}{p(p^2+1)(p^2+4)}.$

337. $F(p) = \frac{p}{p^2 - 2p + 5}.$

338. $F(p) = \frac{p}{p^2 - 2p + 5}.$

339. $F(p) = \frac{p+2}{(p+1)(p-2)(p^2+4)}.$

340. $F(p) = \frac{p+2}{p^3 (p-1)^2}.$

341. $F(p)=\dfrac{p}{p^4-1}$.

342. $F(p)=\dfrac{1}{p(p^4-1)}$.

343. $F(p)=\dfrac{1}{p^4-1}$.

344. $F(p)=\dfrac{1}{(p-1)(p^2+1)}$.

345. $F(p)=\dfrac{p}{(p-1)(p^2+1)}$.

346. $F(p)=\dfrac{1}{p(p-1)(p^2+1)}$.

347. $F(p)=\dfrac{p^2}{(p^2+1)^2}$.

348. $F(p)=\dfrac{1}{(p+1)(p^2+2p+2)}$.

349. $F(p)=\dfrac{p}{(p+1)(p^2+2p+2)}$.

350. $F(p)=\dfrac{1}{p(p+1)(p^2+2p+2)}$.

6. Differensial tenglamalar va ularning sistemalarini operatsion hisob usuli bilan yechish.

Ushbu

$$x''(t)+a_1x'(t)+a_2x(t)=f(t) \qquad (3.91)$$

chiziqli differensial tenglamaning $x(0)=x_0$, $x'(0)=x_1$ boshlang'ich shartlarni qanoatlantiruvchi yechimini topish talab qilinsin. Bu yerda a_1, a_2 – berilgan haqiqiy sonlar, $f(t)$ – ma'lum funksiya. Izlanayotgan $x(t)$ funksiya, uning qaralayotgan barcha hosilalari va $f(t)$ funksiya originallar bo'lsin deb faraz qilaylik.

$$x(t)\leftarrow\!\!\!\!\cdot\!\!-\bar{x}(p) \text{ va } x(t)=t^2-3t+4$$

bo'lsin. Originalni differensiallash qoidasiga asosan quyidagilarga ega bo'lamiz:

$$x'(t)\leftarrow\!\!\!\!\cdot\!\!-p\bar{x}(p)-x(0),$$

$$x''(t)\leftarrow\!\!\!\!\cdot\!\!-p^2\bar{x}(p)-px(0)-x'(0).$$

Tasvirlarning chiziqliligidan foydalanib, (3.91) tenglamada tasvirlarga o'tamiz:

$$p^2\bar{x}(p) - px(0) - x'(0) + a_1[p\bar{x}(p) - x(0)] + a_2\bar{x}(p) = F(p)$$

yoki

$$(p^2 + a_1 p + a_2)\bar{x}(p) - (p + a_1)x_0 - x_1 = F(p). \quad (3.92)$$

(3.92) tenglamani $\bar{x}(p)$ ga nisbatan yechib

$$\bar{x}(p) = \frac{(p+a_1)x_0 + x_1}{p^2 + a_1 p + a_2} + \frac{F(p)}{p^2 + a_1 p + a_2} \quad (3.93)$$

ni topamiz. $\bar{x}(p)$ ning originali (3.91) tenglamaning boshlang'ich shartlarni qanoatlantiruvchi yechimi bo'ladi.

Shu kabi istalgan n-tartibli o'zgarmas koeffitsiyentli chiziqli differensial tenglamaning yechimini boshlang'ich shartlarda topish mumkin.

1-misol. $x'' - 5x' + 4x = 4$ tenglamani $x(0) = 0,\ \ x'(0) = 2$ boshlang'ich shartlarda integrallang.

Yechish. $x(t) \leftarrow\!\!\!\cdot\!\!\!- \bar{x}(p)$ deymiz, u holda berilgan boshlang'ich shartlarga asosan

$$x' \leftarrow\!\!\!\cdot\!\!\!- p\bar{x}(p) - x(0) = p\bar{x}(p),$$

$$x'' \leftarrow\!\!\!\cdot\!\!\!- p^2\bar{x}(p) - px(0) - x'(0) = p^2\bar{x}(p) - 2,$$

$$4 \leftarrow\!\!\!\cdot\!\!\!- \frac{4}{p}.$$

Berilgan tenglamada barcha funksiyalarni ularning tasvirlari bilan almashtirib, quyidagi operatorli tenglamani hosil qilamiz:

$$(p^2 - 5p + 4)\bar{x}(p) = \frac{4}{p} + 2.$$

Bu tenglamadan $\bar{x}(p)$ ni aniqlaymiz:

$$\bar{x}(p) = \frac{4+2p}{p(p^2 - 5p + 4)}.$$

Tenglikning o'ng tomonini elementar kasrlarga ajratamiz, u holda

$$\bar{x}(p) = \frac{1}{p} - \frac{2}{p-1} + \frac{1}{p-4}$$

ni hosil qilamiz. Bunda originalga o'tib, tenglamaning yechimini topamiz:

$$x(t) = 1 - 2e^{t} + e^{4t}.$$

Endi quyidagi o'zgarmas koeffitsiyentli differensial tenglamalar sistemasini qaraymiz:

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y + f_1(t), \\ \dfrac{dy}{dt} = a_2 x + b_2 y + f_2(t). \end{cases} \tag{3.94}$$

Bu sistemaning

$$x(0) = x_0, \quad y(0) = y_0 \tag{3.95}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topamiz. Bunda biz $f_1(t)$, $f_2(t)$, $x(t)$, $y(t)$ funksiyalarni $x'(t)$ va $y'(t)$ larning originallari deb faraz qilamiz:

$$x(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- \bar{x}(p), \quad y(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- \bar{y}(p), \quad f_1(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- F_1(p), \quad f_2(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- F_2(p)$$

bo'lsin.

(3.95) boshlang'ich shartlarni e'tiborga olib, originallarni differensiallash qoidasidan foydalanib

$$x'(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- p\bar{x}(p) - x_0, \quad y'(t) \leftarrow\!\!\!\!\cdot\!\!\!\!- p\bar{y}(p) - y_0$$

larni topamiz.

Endi (3.94) sistema har bir tenglamasining ikkala tomoniga Laplas almashtirishlarini qo'llab, $\bar{x}(p)$ va $\bar{y}(p)$ larga nisbatan quyidagi sistemani hosil qilamiz:

$$\begin{cases} p\bar{x}(p) = a_1\bar{x}(p) + b_1\bar{y}(p) + F_1(p) + x_0, \\ p\bar{y}(p) = a_2\bar{x}(p) + b_2\bar{y}(p) + F_2(p) + y_0. \end{cases} \tag{3.96}$$

(3.96) sistemaning yechimi:

$$\bar{x}(p) = \frac{b_1[F_2(p)+y_0]+(p-b_2)[F_1(p)+x_0]}{(p-a_1)(p-b_2)-a_2 b_1}, \tag{3.97}$$

$$\bar{y}(p) = \frac{a_2[F_1(p)+x_0]+(p-a_1)[F_2(p)+y_0]}{(p-a_1)(p-b_2)-a_2 b_1}. \tag{3.98}$$

(3.97) va (3.98)da originalga o'tib, (3.94) sistemaning (3.95) boshlang'ich shartlarni qanoatlantiruvchi yechimini hosil qilamiz.

Mustaqil ish topshiriqlari

Quyidagi tenglamalarning yechimini toping:

351. $x' + 3x = e^{-2t}$, $x(0) = 0$.

352. $x' - x = \cos t - \sin t$, $x(0) = 0$.

353. $2x' + 6x = te^{-3t}$, $x(0) = -\frac{1}{2}$.

354. $x'' + 6x' = 12t + 2$, $x(0) = 0$, $x'(0) = 0$.

355. $x'' + 4x' + 4 = 4$, $x(0) = 1$, $x'(0) = -4$.

356. $x'' + x = \cos t$, $x(0) = -1$, $x'(0) = 1$.

357. $x'' + 3x' + 2x = 2t^2 + 1$, $x(0) = 4$, $x'(0) = -3$.

358. $x'' - x' = 2\sin t$, $x(0) = 2$, $x'(0) = 0$.

359. $x'' - 4x' + 5x = 2e^{2t}(\sin t + \cos t)$, $x(0) = 1$, $x'(0) = 2$.

360. $x'' - 4x' = 1$, $x(0) = 0$, $x'(0) = -\frac{1}{4}$, $x''(0) = 0$.

Quyidagi tenglamalar sistemasining yechimini toping:

361. $\begin{cases} \frac{dx}{dt} + x - 2y = 0, \\ \frac{dy}{dt} + x + 4y = 0. \end{cases}$ $x(0)=1$, $y(0)=1$.

362. $\begin{cases} \frac{dx}{dt} + 2y = 3t, \\ \frac{dy}{dt} - 2x = 4. \end{cases}$ $x(0)=2$, $y(0)=3$.

363. $\begin{cases} \frac{dx}{dt} - \frac{dy}{dt} = -\sin t, \\ \frac{dx}{dt} + \frac{dy}{dt} = \cos t. \end{cases}$ $x(0) = \frac{1}{2}$, $y(0) = -\frac{1}{2}$.

364. $\begin{cases} \dfrac{dx}{dt} = 4y + z, \\ \dfrac{dy}{dt} = z, \\ \dfrac{dz}{dt} = 4y. \end{cases}$ $x(0)=5,\ y(0)=0,\ z(0)=4.$

365. $\begin{cases} \dfrac{dx}{dt} - \dfrac{dy}{dt} - 2x + 2y = 1 - 2t, \\ \dfrac{d^2x}{dt^2} + 2\dfrac{dy}{dt} + x = 0. \end{cases}$ $x(0)=y(0)=x'(0)=0.$

366. $\begin{cases} \dfrac{dx}{dt} + 4y + 2x = 4t + 1, \\ \dfrac{dy}{dt} + x - y = \dfrac{3}{2}t^2. \end{cases}$ $x(0)=y(0)=0.$

367. $\begin{cases} \dfrac{dx}{dt} + y - 2x = 0, \\ \dfrac{dy}{dt} + x - 2y = -5e^t \sin t. \end{cases}$ $x(0)=2,\ y(0)=3.$

368. $\begin{cases} \dfrac{d^2x}{dt^2} + x + y = 5, \\ \dfrac{d^2y}{dt^2} - 4x - 3y = -3. \end{cases}$ $x(0)=y(0)=x'(0)=y'(0)=0.$

369. $\begin{cases} \dfrac{dx}{dt} + 2\dfrac{dy}{dt} + x + y + z = 0, \\ \dfrac{dx}{dt} + \dfrac{dy}{dt} + x + z = 0. \\ \dfrac{dz}{dt} - 2\dfrac{dy}{dt} - y = 0, \end{cases}$ $x(0)=y(0)=1,\ z(0)=-2.$

370. $\begin{cases} \dfrac{d^2x}{dt^2} = x - 4y, \\ \dfrac{d^2y}{dt^2} = -x + y. \end{cases}$ $x(0) = 2,\quad y(0) = 0,$ $x'(0) = -\sqrt{3},\quad y'(0) = \dfrac{\sqrt{3}}{2}.$

6- §. Matematik fizika tenglamalarining tiplari

Ushbu ko'rinishdagi

$$F\left(x,y,U,\ \frac{\partial U}{\partial x},\ \frac{\partial U}{\partial y},\ \frac{\partial^2 U}{\partial x^2},\ \frac{\partial^2 U}{\partial x\cdot\partial y},\ \frac{\partial^2 U}{\partial y^2}\right)=0 \qquad (3.99)$$

differensial tenglamaga *ikkinchi tartibli ikki o'zgaruvchili xususiy hosilali differensial tenglama* deyiladi.

$$a_{11}(x,y)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x,y)\frac{\partial^2 u}{\partial x\,\partial y}+a_{22}(x,y)\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \quad (3.100)$$

ko'rinishdagi tenglama *ikkinchi tartibli xususiy hosilalarga nisbatan chiziqli tenglama* deyiladi.

Agar (3.100) tenglama ushbu

$$a_{11}(x,y)\frac{\partial^2 u}{\partial x^2}+2a_{12}(x,y)\frac{\partial^2 u}{\partial x\partial y}+a_{22}(x,y)\frac{\partial^2 u}{\partial y^2}+$$
$$+a_{13}(x,y)\frac{\partial u}{\partial x}+a_{23}(x,y)\frac{\partial u}{\partial y}+a_{33}(x,y)u+f=0 \qquad (3.101)$$

ko'rinishda bo'lsa, bunday tenglama *chiziqli* deyiladi. Agar (3.101) tenglamaning koeffitsiyentlari x va y o'zgaruvchilarga bog'liq bo'lmasa, tenglama *o'zgarmas koeffitsiyentli* deyiladi. (3.101) tenglamada $f(x, y)=0$ bo'lsa, unga *bir jinsli* deyiladi.

$$a_{11}(dy)^2-2a_{12}dxdy+a_{22}(dx)^2=0 \qquad (3.102)$$

tenglama (3.101) tenglamaning *xarakteristik tenglamasi* deyiladi.

(3.102) tenglama quyidagi ikkita birinchi tartibli oddiy differensial tenglamalarga ajraladi:

$$\frac{dy}{dx}=\frac{a_{12}+\sqrt{a_{12}^2-a_{11}\cdot a_{22}}}{a_{11}},\quad \frac{dy}{dx}=\frac{a_{12}-\sqrt{a_{12}^2-a_{11}\cdot a_{22}}}{a_{11}}. \qquad (3.103)$$

Bu tenglamalardagi ildiz ostidagi ifodaning ishorasi (3.101) tenglamani tiplarga (turlarga) ajratadi.

Agar M nuqtada $a_{12}^2-a_{11}\cdot a_{22}>0$ bo'lsa, (3.101) tenglama M nuqtada *giperbolik tipdagi tenglama* deyiladi. Giperbolik tipdagi (3.101) tenglamada x va y o'zgaruvchilarni

$$\xi = \varphi(x,y), \quad \eta = \psi(x,y)$$

tengliklarga asosan ξ va η larga almashtirsak, (3.101) tenglama

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + a_{13}\frac{\partial u}{\partial\xi} + a_{13}\frac{\partial u}{\partial\eta} + a_{33}u + f = 0 \qquad (3.104)$$

yoki

$$\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial^2 u}{\partial\eta^2} + a_{13}\frac{\partial u}{\partial\xi} + a_{23}\frac{\partial u}{\partial\eta} + a_{33}u + f = 0 \qquad (3.105)$$

ko'rinishdagi giperbolik tipdagi tenglamaga keladi.

Torning ko'ndalang tebranishi, sterjenning uzunasiga tebranishi, o'tkazgichdagi elektr tebranishlar, aylanuvchi silindrdagi (valdagi) aylanma tebranishlar, gazning tebranishlari va shunga o'xshash tebranish jarayonlarini o'rganish giperbolik tipdagi tenglamalarga olib keladi.

1- misol. $x^2 \cdot \frac{\partial^2 u}{\partial x^2} - y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0$ tenglamani kanonik ko'rinishga keltiring.

Y e c h i s h. Bunda $a_{11} = x^2$, $a_{12} = 0$, $a_{22} = -y^2$, $a_{12}^2 - 4a_{11}a_{22} = x^2y^2 \succ 0$. Demak, tenglama giperbolik tipda ekan. Xarakteristik tenglamasini tuzamiz:

$$x^2(dy)^2 - y^2(dx)^2 = 0 \text{ yoki } (xdy - ydx)(xdy + ydx) = 0.$$

Bu tenglik ikkita differensial tenglamaga ajratiladi:

$$xdy - ydx = 0 \text{ va } xdy + ydx = 0.$$

Bundan $\frac{dy}{y} + \frac{dx}{x} = 0$ yoki $xy = C_1$, $\frac{dy}{y} - \frac{dx}{x} = 0$ yoki $\frac{y}{x} = C_2$.

Endi $\xi = xy$, $\eta = \frac{y}{x}$ almashtirishlarni bajaramiz. x va y o'zgaruvchilarning xususiy hosilalarini yangi ξ va η o'zgaruvchilar orqali ifodalaymiz:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial u}{\partial\xi}\cdot y - \frac{\partial u}{\partial\eta}\cdot\frac{y}{x^2};$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} = \frac{\partial u}{\partial\xi}\cdot x + \frac{\partial u}{\partial\eta}\cdot\frac{1}{x};$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\cdot y\right)-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\cdot\frac{y}{x^2}\right)=\left(\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial^2 u}{\partial \xi\,\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)\cdot y-$$

$$-\left(\frac{\partial^2 u}{\partial \eta\,\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\right)\cdot\frac{y}{x^2}+\frac{\partial u}{\partial \eta}\cdot\frac{2y}{x^3}=\left(\frac{\partial^2 u}{\partial \xi^2}\cdot y-\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{y}{x^2}\right)\cdot y-$$

$$-\left(\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot y-\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{y}{x^2}\right)\cdot\frac{y}{x^2}+\frac{\partial u}{\partial \eta}\cdot\frac{2y}{x^3}=\frac{\partial^2 u}{\partial \xi^2}\cdot y^2-2\cdot\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{y^2}{x^2}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{y^2}{x^4}+2\cdot\frac{\partial u}{\partial \eta}\cdot\frac{y}{x^3};$$

$$\frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)\cdot x+\left(\frac{\partial^2 u}{\partial \eta\cdot\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y}\right)=$$

$$=x\cdot\left(\frac{\partial^2 u}{\partial \xi^2}\cdot x+\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{1}{x}\right)+\left(\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot x+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{1}{x}\right)\cdot\frac{1}{x}=\frac{\partial^2 u}{\partial \xi^2}\cdot x^2+2\cdot\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{1}{x^2}.$$

Hosil bo‘lgan tengliklarni berilgan differensial tenglamaga qo‘yamiz:

$$x^2\left(\frac{\partial^2 u}{\partial \xi^2}\cdot y^2-2\cdot\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{y^2}{x^2}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{y^2}{x^4}+2\cdot\frac{\partial u}{\partial \eta}\cdot\frac{y}{x^3}\right)-$$

$$-y^2\left(\frac{\partial^2 u}{\partial \xi^2}\cdot x^2+2\cdot\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot\frac{y^2}{x^2}+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{1}{x^2}\right)=0.$$

Bundan

$$-4\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}\cdot y^2+2\cdot\frac{\partial u}{\partial \eta}\cdot\frac{y}{x}=0 \quad \text{yoki} \quad \frac{\partial^2 u}{\partial \xi\cdot\partial \eta}-\frac{1}{2}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{1}{xy}=0$$

$$\text{yoki} \quad \frac{\partial^2 u}{\partial \xi\cdot\partial \eta}-\frac{1}{2\xi}\cdot\frac{\partial u}{\partial \eta}=0.$$

Demak, tenglamaning kanonik ko‘rinishi:

$$\frac{\partial^2 u}{\partial \xi\cdot\partial \eta}-\frac{1}{2\xi}\cdot\frac{\partial u}{\partial \eta}=0.$$

Agar M nuqtada $a_{12}^2-a_{11}a_{22}<0$ bo‘lsa, (3.101) tenglama M nuqtada *elliptik tipdagi tenglama* deyiladi. Elliptik tipdagi (3.101) tenglama

$$\xi=\varphi(x,y),\quad \eta=\overline{\varphi(x,y)}$$

($\overline{\varphi(x,y)}$ funksiya φ funksiyaga qo‘shma kompleks funksiya) almashtirishga asosan ushbu

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+a_{13}\frac{\partial u}{\partial\xi}+a_{23}\frac{\partial u}{\partial\eta}+a_{33}u+f=0 \qquad (3.106)$$

kanonik ko'rinishga keladi.

Elektr va magnit maydonlar haqidagi masalalarni, statsionar issiqlik holat haqidagi masalalarni, gidrodinamika, diffuziya va shunga o'xshash masalalarni o'rganish elliptik tipdagi tenglamalarga olib keladi.

2- misol.

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+2\cdot\frac{\partial^2 z}{\partial y^2}=0$$

tenglamani kanonik ko'rinishga keltiring.

Yechish. Tenglamada $a_{11}=1$, $a_{12}=-1$, $a_{22}=2$, $a_{12}^2-a_{11}a_{22}=$ $=-1<0$, bu esa tenglamaning elliptik tipda ekanini bildiradi.

Xarakteristik tenglamasi:

$$(dy)^2+2dxdy+2(dx)^2=0 \quad \text{yoki} \quad y'^2+2y'+2=0.$$

Bundan $y'=-1\pm i$; $y+x-ix=C_1$, $y+x+ix=C_2$. Quyidagicha almashtirishni bajaramiz: $\xi=y+x$, $\eta=x$. U holda:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial z}{\partial\eta}\frac{\partial\eta}{\partial x}=\frac{\partial z}{\partial\xi}+\frac{\partial z}{\partial\eta};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial z}{\partial\eta}\frac{\partial\eta}{\partial y}=\frac{\partial z}{\partial\xi};$$

$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial^2 z}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)+\left(\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2 z}{\partial^2\eta}\cdot\frac{\partial\eta}{\partial x}\right)=\frac{\partial^2 z}{\partial\xi^2}+2\cdot\frac{\partial^2 z}{\partial\xi\partial\eta}+\frac{\partial^2 z}{\partial\eta^2};$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial y}=\frac{\partial^2 z}{\partial\xi^2}+\frac{\partial^2 z}{\partial\xi\partial\eta};$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial\xi^2}\frac{\partial\xi}{\partial y}+\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial y}=\frac{\partial^2 z}{\partial\xi^2}.$$

Hosil bo'lgan tengliklarni tenglamaga qo'yib, kanonik tenglama ko'rinishini hosil qilamiz:

$$\frac{\partial^2 z}{\partial\xi^2}+2\frac{\partial^2 z}{\partial\xi\ \partial\eta}+\frac{\partial^2 z}{\partial\eta^2}+\frac{\partial^2 z}{\partial\xi^2}-2\frac{\partial^2 z}{\partial\xi\ \partial\eta}+\frac{\partial^2 z}{\partial\xi^2}=0$$

yoki

$$\frac{\partial^2 z}{\partial\xi^2}+\frac{\partial^2 z}{\partial\eta^2}=0.$$

Agar M nuqtada $a_{12}^2-a_{11}a_{22}=0$ bo'lsa, (3.119) tenglama M nuqtada *parabolik tipdagi tenglama* deyiladi. Parabolik tipdagi (3.101) tenglamada o'zgaruvchilarni

$$\xi=\varphi(x,y),\ \eta=\eta(x,y)$$

shaklda almashtirsak, u ushbu

$$\frac{\partial^2 u}{\partial\xi^2}+a_{13}\frac{\partial u}{\partial\xi}+a_{23}\frac{\partial u}{\partial\eta}+a_{33}u+f=0 \qquad (3.107)$$

kanonik ko'rinishga keladi.

Issiqlikning tarqalish jarayoni, g'ovak muhitda suyuqlik va gazning filtrlanish masalasi va shunga o'xshash masalalarni o'rganish parabolik tipdagi tenglamaga olib keladi.

3- misol.

$$\frac{\partial^2 z}{\partial x^2}\cdot\sin^2 x-2y\sin x\cdot\frac{\partial^2 z}{\partial x\partial y}+y^2\cdot\frac{\partial^2 z}{\partial y^2}=0$$

tenglamani kanonik ko'rinishga keltiring.

Y e c h i s h. Tenglamada $a_{11}=\sin^2 x$, $a_{12}=-y\sin x$, $a_{22}=y^2$. Bundan esa $a_{12}^2-a_{11}a_{22}=y^2\sin^2 x-y^2\sin^2 x=0$. Demak, tenglama parabolik tipda ekan.

Xarakteristik tenglamasini tuzamiz:

$\sin^2 x(dy)^2+2y\sin x dxdy+y^2(dx)^2=0$ yoki $(\sin x dy+y dx)^2=0$.

$\xi=y\cdot\mathrm{tg}\frac{x}{2}$, $\eta=y$ almashtirish yordamida x va y o'zgaruvchilardan ξ va η o'zgaruvchilarga o'tamiz:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial z}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}=\frac{1}{2}\frac{\partial z}{\partial\xi}\cdot y\sec^2\frac{x}{2};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial z}{\partial \xi}\cdot \mathrm{tg}\frac{x}{2}+\frac{\partial z}{\partial \eta};$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial^2 z}{\partial \xi\ \partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)\cdot y\sec^2\frac{x}{2}+\frac{1}{2}\frac{\partial z}{\partial \xi}\cdot y\sec^2\frac{x}{2}\mathrm{tg}\frac{x}{2}=$$
$$=\frac{1}{4}\frac{\partial^2 z}{\partial \xi^2}\cdot y^2\sec^4\frac{x}{2}+\frac{1}{2}y\frac{\partial z}{\partial \xi}\cdot\sec^2\frac{x}{2}\mathrm{tg}\frac{x}{2};$$

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \xi\ \partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)\cdot \mathrm{tg}\frac{x}{2}+\frac{\partial^2 z}{\partial \eta\ \partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \eta^2}\frac{\partial \eta}{\partial y}=$$
$$=\frac{1}{4}\frac{\partial^2 z}{\partial \xi^2}\cdot \mathrm{tg}^2\frac{x}{2}+2\frac{\partial^2 z}{\partial \xi\ \partial \eta}\mathrm{tg}\frac{x}{2}+\frac{\partial^2 z}{\partial \eta^2};$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 z}{\partial \xi\ \partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)\cdot y\sec^2\frac{x}{2}+\frac{1}{2}\frac{\partial z}{\partial \xi}\cdot\sec^2\frac{x}{2}=$$
$$=\frac{1}{2}\left(\frac{\partial^2 z}{\partial \xi^2}\mathrm{tg}\frac{x}{2}+\frac{\partial^2 z}{\partial \xi\ \partial \eta}\right)\cdot y\sec^2\frac{x}{2}+\frac{1}{2}\frac{\partial z}{\partial \xi}\cdot\sec^2\frac{x}{2}.$$

Hosil bo‘lgan tengliklarni berilgan tenglamaga qo‘yamiz:

$$\frac{1}{4}\frac{\partial^2 z}{\partial \xi^2}\cdot y^2\sec^4\frac{x}{2}\sin^2 x+\frac{1}{2}y\frac{\partial z}{\partial \xi}\cdot\sec^2\frac{x}{2}\cdot \mathrm{tg}\frac{x}{2}\sin^2 x-$$
$$-\left(\frac{\partial^2 z}{\partial \xi^2}\mathrm{tg}\frac{x}{2}+\frac{\partial^2 z}{\partial \xi\ \partial \eta}\right)\cdot y^2\sec^2\frac{x}{2}\sin x-\frac{\partial z}{\partial \xi}y\cdot\sec^2\frac{x}{2}\cdot\sin x+$$
$$+y^2\left(\frac{\partial^2 z}{\partial \xi^2}\cdot \mathrm{tg}^2\frac{x}{2}+2\frac{\partial^2 z}{\partial \xi\ \partial \eta}\mathrm{tg}\frac{x}{2}+\frac{\partial^2 z}{\partial \eta^2}\right)=0.$$

Qavslarni ochib chiqib elementar amallarni bajarsak, tenglama quyidagi ko‘rinishga keladi:

$$\frac{1}{2}\cdot y\frac{\partial z}{\partial \xi}\cdot\sec^2\frac{x}{2}\mathrm{tg}\frac{x}{2}\sin^2 x+\frac{\partial^2 z}{\partial \eta^2}y^2-\frac{\partial z}{\partial \xi}\cdot y\sec^2\frac{x}{2}\sin x=0$$

yoki

$$y\frac{\partial^2 z}{\partial\eta^2}=\frac{\partial z}{\partial\xi}\sin x\,.$$

Lekin $\sin x=\dfrac{2\,\mathrm{tg}(x/2)}{1+\mathrm{tg}^2(x/2)}$, $\mathrm{tg}\dfrac{x}{2}=\dfrac{\xi}{\eta}$ ekanidan $\sin x=\dfrac{2\xi\eta}{\xi^2+\eta^2}$ ni topamiz. Demak, berilgan tenglamaning kanonik ko'rinishi :

$$\frac{\partial^2 z}{\partial\eta^2}=\frac{2\xi}{\xi^2+\eta^2}\cdot\frac{\partial z}{\partial\xi}.$$

Agar (3.104)–(3.107) tenglamalarda $U=U(\xi,\eta)$ funksiyani $U=e^{\lambda\xi+\mu\eta}\cdot V$ tenglikka asosan yangi $V=V(\xi,\eta)$ funksiyaga almashtirsak, ular quyidagi sodda ko'rinishga keladi:

$$\begin{cases}\dfrac{\partial^2 V}{\partial\xi\cdot\partial\eta}+\gamma V+f_1=0,\\[2mm] \dfrac{\partial^2 V}{\partial\xi^2}-\dfrac{\partial^2 V}{\partial\eta^2}+\gamma V+f_1=0,\end{cases}\qquad \text{(giperbolik tip)}$$

$$\frac{\partial^2 V}{\partial\xi^2}+\frac{\partial^2 V}{\partial\eta^2}+\gamma V+f_1=0\qquad \text{(elliptik tip)}$$

$$\frac{\partial^2 V}{\partial\xi^2}+a_{23}\frac{\partial V}{\partial\eta}+f_1=0\,.\qquad \text{(parabolik tip)}$$

Bunda γ, a_{13}, a_{23}, a_{33} – parametrlarga bog'liq bo'lgan o'zgarmas kattalik: $f_1=f\cdot e^{-(\lambda\xi+\mu\eta)}$; $\lambda=-a_{13}/2$; $\mu=-a_{23}/2$.

(3.101) tenglama yechimlarining analitik ifodasini xususiy hollarda Dalamber, Furye, Riman, Grin, potensiallar va h.k. usullar bilan topish mumkin. Agar yechimlarning son qiymatlarini topish talab qilinsa, u vaqtda chekli ayirmalar, setkalar, variatsion va h.k. usullar qo'llaniladi.

Har bir usulning o'ziga xos qulayligi bor. Shuning uchun masalaning qo'yilishiga qarab, mos usullaridan birini tanlab olish maqsadga muvofiqdir.

Mustaqil yechish uchun misollar

Quyidagi tenglamalarni kanonik ko'rinishga keltiring:

371. $x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$

372. $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial y} = 0.$

373. $\frac{1}{x^2} \cdot \frac{\partial^2 z}{\partial x^2} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} = 0.$

374. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial y} = 0.$

375. $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + 5\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 0.$

376. $y^2 \frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + 2x^2 \frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial y} = 0.$

7- §. Tor tebranish tenglamasini Dalamber usuli bilan yechish

Dalamber usulida (3.101) tenglama (3.103) xarakteristikalar yordamida kanonik ko'rinishga keltiriladi. Kanonik ko'rinishdagi tenglamani integrallab, avvalgi o'zgaruvchilarga o'tilsa, (3.101) tenglamaning izlangan yechimi hosil bo'ladi.

Bu usulni chegaralanmagan tor tebranishi masalasida ko'raylik:

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2}, \qquad (a\text{=const}) \qquad (3.108)$$

$$\left.\begin{aligned} U(x,t)\big|_{t=0} &= f_1(x), \\ \frac{\partial U}{\partial t}\bigg|_{t=0} &= f_2(x). \end{aligned}\right\} \qquad (3.109)$$

Ushbu (3.109) ifoda boshlang'ich shartlar bo'lib, $f_1(x)$ funksiya torning boshlang'ich holatini, $f_2(x)$ funksiya esa boshlang'ich tezligini ifodalaydi.

(3.108) tenglamaning xarakteristik tenglamasi

$$dx^2 - a^2 dt^2 = 0 \tag{3.110}$$

ko‘rinishda bo‘lib, unda $a_{12}^2 - a_{11}a_{22} = a^2 > 0$, demak, tenglama giperbolik tipdagi tenglama. Uning xarakteristikalari

$$x - at = C_1, \quad x + at = C_2, \tag{3.111}$$

u holda

$$\xi = x - at, \quad \eta = x + at \tag{3.112}$$

almashtirish yordamida (3.108) tenglama

$$\frac{\partial^2 U}{\partial\xi\partial\eta} = 0 \tag{3.113}$$

ko‘rinishdagi kanonik tenglamaga keladi. (3.113) tenglamani fiksirlangan η da ξ o‘zgaruvchi bo‘yicha integrallab, birinchi tartibli

$$\frac{\partial U}{\partial\eta} = Q(\eta) \tag{3.114}$$

xususiy hosilali tenglamani hosil qilamiz. Bunda $Q(\eta)$ – ixtiyoriy funksiyadir. So‘ng (3.114) tenglamani fiksirlangan ξ da η o‘zgaruvchi bo‘yicha integrallab,

$$U = \varphi(\xi) + \psi(\eta) \tag{3.115}$$

ifodani, ya’ni (3.113) tenglamaning yechimini topamiz. Bu yerda $\varphi(\xi)$ ham ixtiyoriy funksiya, $\psi(\eta) = \int Q(\eta)d\eta$. (3.115) ifodada ξ va η o‘zgaruvchilardan x va t o‘zgaruvchilarga o‘tsak,

$$U(x,t) = \varphi(x - at) + \psi(x + at). \tag{3.116}$$

Oxirgi ifoda (3.108) tenglamaning umumiy yechimi bo‘lib, *Dalamber integrali* deyiladi. Qo‘yilgan masalaning yechimini topish uchun φ va ψ funksiyalarni shunday tanlash kerakki, bunda $U(x, t)$ funksiya (3.109) ni qanoatlantirsin. Buning uchun (3.116) da $t = 0$ desak, (3.109) ning birinchisiga asosan

$$U(x,0) = \varphi(x) + \psi(x) = f_1(x). \tag{3.117}$$

(3.116) ning t o‘zgaruvchi bo‘yicha xususiy hosilasini topib, unda t=0 desak, (3.117) ning ikkinchisiga asosan

$$\frac{\partial U(x,0)}{\partial t} = -a\varphi'(x) + a\psi'(x) = f_2(x)$$

yoki

$$-\varphi'(x) + \psi'(x) = \frac{1}{a} f_2(x).$$

Bundan

$$-\varphi(x) + \psi(x) = \frac{1}{a}\int_{x_0}^{x} f_2(z)dz + C \qquad (3.118)$$

ni topamiz. Bu yerda x_0, C = const.

(3.117) va (3.118) tenglamalarni birgalikda yechib,

$$\begin{aligned} \varphi(x) &= \frac{1}{2} f_1(x) - \frac{1}{2a}\int_{x_0}^{x} f_2(z)dz - \frac{C}{2}, \\ \psi(x) &= \frac{1}{2} f_1(x) + \frac{1}{2a}\int_{x_0}^{x} f_2(z)dz + \frac{C}{2} \end{aligned} \qquad (3.119)$$

ni topamiz. Demak, (3.116) dagi ixtiyoriy φ va ψ funksiyalarni (3.119) ko'rinishda olsak, $U(x, t)$ funksiya (3.109) shartlarni qanoatlantiradi. (3.119) ni (3.116) ga qo'yib, qo'yilgan masalaning yechimini topamiz:

$$U(x,t) = \frac{f_1(x-at)+f_1(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} f_2(z)dz. \qquad (3.120)$$

Bu formula *Dalamber formulasi* deyiladi.

1- misol.

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}$$

tenglamaning

$$U|_{t=0} = x, \quad \left.\frac{\partial U}{\partial t}\right|_{t=0} = -x$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Bunda $f_1(x)=x$, $f_2(x)=-x$ va $a^2=1$ ekanligini e'tiborga olib, (3.120) formuladan

$$U(x,t)=\frac{x-t+x+t}{2}-\frac{1}{2}\int_{x-t}^{x+t} z dz=$$

$$=x-\frac{1}{4}z^2\Big|_{x-t}^{x+t}=x-\frac{1}{4}\left[(x+t)^2-(x-t)^2\right]=x-xt=x(1-t)$$

ni topamiz. Demak, masalaning yechimi

$$U(x,\ t)=x(1-t).$$

2- masala. Dalamber usuli bilan $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $\frac{\partial u}{\partial t}\Big|_{t=0}=x^2,\ \frac{\partial u}{\partial t}\Big|_{t=0}=0$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Masala shartiga ko'ra, $a=1$, $\varphi(x)=x^2$, $\psi(x)=0$. Dalamber formulasiga asosan, masalaning yechimi

$$u=\frac{(x-t)^2+(x+t)^2}{2} \text{ yoki } u=x^2+t^2$$

ko'rinishda bo'ladi.

3- masala. Dalamber usuli bilan $\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0$ tenglamaning $u\big|_{t=0}=\cos x,\ \frac{\partial u}{\partial t}\Big|_{t=0}=\sin x$ boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Y e c h i s h . Bunda $\varphi(x)=\cos x$ va $\psi(x)=\sin x$ bo'lganligi uchun Dalamber formulasiga asosan

$$u(x,y)=\frac{\cos(x-at)+\cos(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\sin z dz=$$

$$=\frac{1}{2}\cdot 2\cdot\cos\frac{x-at+x+at}{2}\cdot\cos\frac{x-at-x-at}{2}-\frac{1}{2a}\cos z\Big|_{x-at}^{x+at}=$$

$$=\cos x\cdot\cos at-\frac{1}{2a}\cdot 2\sin\frac{x+at+x-at}{2}\cdot\sin\frac{x-at-x-at}{2}=$$

$$=\cos x\cdot\cos at+\frac{1}{a}\sin x\cdot\sin at$$

yechimga ega bo'lamiz.

4- masala. $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$ tenglamaning $u|_{t=0} = x(a-x)$ va $\left.\frac{\partial u}{\partial t}\right|_{t=0} = e^{-3x}$ boshlang'ich shartlarni qanoatlantiruvchi yechimini Dalamber usuli bilan topilsin.

Y e c h i s h . Bu masalada $\varphi(x) = x(a-x) = ax - x^2$ va $\psi(x) = e^{-3x}$. Yuqoridagi formuladan foydalansak,

$$u(x,y) = \frac{1}{2}\left[a(x-at) - (x-at)^2 + a(x+at) - (x+at)^2\right] + \frac{1}{2a}\int_{x-at}^{x+at} e^{-3z}dz =$$

$$= ax - x^2 - a^2t^2 - \frac{1}{6a}\left[e^{-3(x-at)} - e^{-3(x+at)}\right]$$

izlanayotgan yechimni topamiz.

Mustaqil yechish uchun misollar

Dalamber usuli bilan $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ **tenglamaning** $u|_{t=0} = f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x)$ **boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin:**

377. $f(x) = x(2-x)$, $F(x) = e^{-x}$.

378. $f(x) = \cos x$, $F(x) = \sin x$.

379. $f(x) = e^{-x}$, $F(x) = \sin^2 x$.

380. $f(x) = x(2-x)$, $F(x) = e^{x}$.

381. $f(x) = e^{x}$, $F(x) = 4x$.

382. $f(x) = \cos x$, $F(x) = \cos^2 x$.

383. $f(x) = \sin x$, $F(x) = 8x^3$.

384. $f(x) = \sin^2 x$, $F(x) = \cos x$.

385. $f(x) = e^{2x}$, $F(x) = x^3$.

386. Dalamber usuli bilan $u|_{t=0} = 0$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = x$ boshlang'ich shartlarni qanoatlantiruvchi $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ tenglamaning yechimi topilsin.

387. $u|_{t=0}=0$, $\frac{\partial u}{\partial t}|_{t=0}=-x$ bo'lsa, $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning yechimi topilsin.

388. $u|_{t=0}=\sin x$, $\frac{\partial u}{\partial t}|_{t=0}=1$ boshlang'ich shartlarni qanoatlantiruvchi $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $t=\frac{\pi}{2a}$ vaqtdagi yechimi topilsin.

389. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u|_{t=0}=0$, $\frac{\partial u}{\partial t}|_{t=0}=\cos x$ shartlarni qanoatlantiruvchi yechimi topilsin.

390. $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $t=\pi$ momentda $u|_{t=0}=\sin x$, $\frac{\partial u}{\partial t}|_{t=0}=\cos x$ shartlarni qanoatlantiruvchi yechimi topilsin.

391. $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u|_{t=0}=x^2$, $\frac{\partial u}{\partial t}|_{t=0}=\sin x$ shartlarni qanoatlantiruvchi yechimi topilsin.

392. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u|_{t=0}=\cos x$, $\left.\frac{\partial u}{\partial t}\right|_{t=0}=\sin x$ shartlarni qanoatlantiruvchi yechimi topilsin.

393. $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u|_{t=0}=\sin x$, $\left.\frac{\partial u}{\partial t}\right|_{t=0}=0$ shartlarni qanoatlantiruvchi yechimi topilsin.

8- §. Furye usuli

Matematik fizika tenglamalariga qo'yilgan masalalarni yechishda keng qo'llaniladigan usullardan yana biri *o'zgaruvchilarni ajratish* yoki *Furye usulidir.* Bu usul boshlang'ich va nolga teng bo'lgan chegaraviy shartlar bilan berilgan masalalarni yechishda samarali natija beradi.

Furye usulini uzunligi l ga teng bo'lgan va ikki uchi mahkamlangan torning erkin tebranish masalasida ko'raylik.

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \tag{3.121}$$

tenglamaning

$$u\big|_{t=0}=f_1(x),\ \frac{\partial u}{\partial t}\bigg|_{t=0}=f_2(x) \qquad (3.122)$$

boshlang'ich va

$$U(0,t)=U(l,t)=0 \qquad (3.123)$$

chegaraviy shartlarni qanoatlantiruvchi yechimini topish talab qilingan bo'lsin. (3.121) tenglamaning yechimini Furye usuliga ko'ra

$$U(x,t)=X(x)T(t) \qquad (3.124)$$

ko'rinishda izlaymiz.

(3.124) ni (3.121) ga qo'yib, izlanayotgan $X(x)$, $T(t)$ funksiyalarning har biriga nisbatan oddiy differensial tenglamalarni hosil qilamiz:

$$\frac{d^2T}{dt^2}+\lambda^2T=0,\quad \frac{d^2X}{dx^2}+\frac{\lambda^2}{a^2}X=0, \qquad (3.125)$$

bu yerda λ – hozircha no'ma'lum bo'lgan tebranish chastotasi, bu tenglamalarning umumiy yechimlari quyidagicha bo'ladi:

$$T(t)=C_1\cos\lambda t+C_2\sin\lambda t, \qquad (3.126)$$

$$X(x)=C_3\cos\frac{\lambda}{a}x+C_4\sin\frac{\lambda}{a}x, \qquad (3.127)$$

bunda C_1, C_2, C_3, C_4 – ixtiyoriy o'zgarmas sonlar.

$U(x,t)=X(x)T(t)$ funksiya (3.123) chegaraviy shartlarni qanoatlantirishi uchun $X(x)$ funksiya shu shartlarga bo'ysunadigan, ya'ni $X(0)=X(l)=0$ bo'lishi kerak. $x=0$ va $x=l$ qiymatlarni (3.127) tenglikka qo'yib, (3.123) shartlarga asosan quyidagilarni topamiz:

$$C_3=0,\ C_4\sin\frac{\lambda}{a}l=0.$$

Ixtiyoriy o'zgarmas $C_4\neq0$ bo'lgani uchun

$$\sin\frac{\lambda}{a}l=0$$

bo'lishi kerak, bundan $n\in N$ uchun

$$\frac{\lambda}{a}l=n\pi.$$

Shunday qilib, tebranish chastotasi λ ushbu

$$\lambda_1=\frac{a\pi}{l},\ \lambda_2=\frac{2a\pi}{l},\ \ldots,\ \lambda_n=\frac{an\pi}{l},\ \ldots$$

qiymatlardan birini qabul qiladi xolos. n ning har bir qiymati uchun, demak, har bir λ uchun (3.126) va (3.127) ifodalarni (3.124) ga qo'yib va $C_1 \cdot C_4$, $C_2 \cdot C_4$ larning $\lambda = \lambda_n$ ga mos qiymatlarini a_n va b_n lar bilan belgilab, (3.121) tenglamaning (3.123) chegaraviy shartlarni qanoatlantiruvchi xususiy yechimlari ketma-ketligini hosil qilamiz:

$$U_n(x,t) = X_n(x)T_n(t) = \sum_{n=1}^{\infty}(a_n \cos\frac{an\pi}{l}t + b_n \sin\frac{an\pi}{l}t)\sin\frac{n\pi}{l}x\,. \quad (3.128)$$

(3.121) tenglama chiziqli va bir jinsli bo'lgani uchun (3.128) yechimlarning yig'indisi

$$U(x,t) = \sum_{n=1}^{\infty}U_n(x,t) = \sum_{n=1}^{\infty}(a_n \cos\frac{an\pi}{l}t + b_n \sin\frac{an\pi}{l}t)\sin\frac{n\pi}{l}x \quad (3.129)$$

ham (3.121) tenglamaning (3.123) chegaraviy shartlarni qanoatlantiruvchi yechimi bo'ladi.

(3.129) yechim (3.122) boshlang'ich shartlarni ham qanoatlantirishi kerak. Bunga biz a_n va b_n koeffitsiyentlarni tanlab olish yo'li bilan erishamiz.

(3.129) yechimda va uning t bo'yicha xususiy hosilasida $t = 0$ desak, (3.122) shartlarga asosan ushbu

$$U(x,0) = \sum_{n=1}^{\infty}a_n \sin\frac{n\pi}{l}x = f_1(x),$$

$$\frac{\partial U(x,0)}{\partial t} = \sum_{n=1}^{\infty}\frac{an\pi}{l}t \cdot b_n \sin\frac{n\pi}{l}x = f_2(x) \quad (3.130)$$

tengliklarni hosil qilamiz. Bundan a_n va b_n koeffitsiyentlarni (Furye koeffitsiyentlari kabi) quyidagi formulalar orqali topamiz:

$$a_n = \frac{2}{l}\int_0^l f_1(x)\sin\frac{n\pi}{l}xdx, \qquad b_n = \frac{2}{an\pi}\int_0^l f_2(x)\sin\frac{n\pi}{l}xdx. \quad (3.131)$$

Bularni (3.129) ga qo'ysak, masalaning ushbu

$$U(x,l) = 2\sum_{n=1}^{\infty}\sin\frac{n\pi}{l}x(\frac{1}{l}\cos\frac{an\pi}{l}t\int_0^l f_1(\xi)\cdot\sin\frac{n\pi}{l}\xi d\xi +$$

$$+\frac{1}{an\pi}\sin\frac{an\pi}{l}t\int_0^l f_2(\xi)\sin\frac{n\pi}{l}\xi d\xi) \quad (3.132)$$

yechimi hosil bo'ladi. Bunday ko'rinishdagi yechim *Bernulli integrali* deyiladi.

1- masala. Uchlari $x=0$ va $x=l$ da mahkamlangan torning boshlang'ich holati $u=\left(\frac{4h}{l^2}\right)\cdot x(l-x)$ parabolani ifodalasa hamda boshlang'ich tezligi $\frac{\partial u(x,0)}{\partial t}=0$ bo'lsa, uning OX o'qidan og'ishi aniqlansin.

Yechish. Masala shartiga ko'ra, $\varphi(x)=\frac{4h}{l^2}\cdot x(l-x),\quad \psi(x)=0.$

Tor tenglamasining yechimini (3.147) qator ko'rinishida izlaymiz. Qatorning koeffitsiyentlari quyidagicha aniqlanadi:

$$a_k=\frac{2}{l}\int_0^l f(x)\cdot\sin\frac{k\pi x}{l}dx=\frac{8h}{l^3}\int_0^l(lx-x^2)\cdot\sin\frac{k\pi x}{l}dx,\ b_k=0.$$

Integralni bo'laklab integrallaymiz:

$$u_1=lx-x^2,\ dv_1=\sin\frac{k\pi x}{l}dx,$$

$$du_1=(l-2x)dx,\ v=-\frac{l}{k\pi}\cdot\cos\frac{k\pi x}{l};$$

$$a_k=-\frac{8h}{l^3}(lx-x^2)\frac{l}{k\pi}\cdot\cos\frac{k\pi x}{l}\Bigg|_0^l+\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx,$$

bundan,

$$a_k=\frac{8h}{k\pi l^2}\int_0^l(l-2x)\cos\frac{k\pi x}{l}dx,$$

$$u_2=l-2x,\ du_2=-2dx,$$

$$dv_2=\cos\frac{k\pi x}{l}dx,\ v_2=\frac{l}{k\pi}\sin\frac{k\pi x}{l},$$

$$a_k=\frac{8h}{k^2\pi^2 l}(l-2x)\sin\frac{k\pi x}{l}\Bigg|_0^l+\frac{16h}{k^2\pi^2 l}\int_0^l\sin\frac{k\pi x}{l}dx=-\frac{16h}{k^3\pi^3}\cos\frac{k\pi x}{l}\Bigg|_0^l=$$

$$=\frac{16h}{k^3\pi^3}(\cos k\pi-1)=\frac{16h}{k^3\pi^3}(1-(-1)^k).$$

Topilgan a_k va b_k larni (3.129) tenglikka qo'yamiz:

$$u(x,t)=\sum_{k=1}^{\infty}\frac{16h}{k^3\pi^3}(1-(-1)^k)\cos\frac{k\pi at}{l}\cdot\sin\frac{k\pi x}{l}.$$

Agar $k=2n$ bo'lsa, $1-(-1)^k=0$, agar $k=2n+1$ bo'lsa, $1-(-1)^k=2$. U holda

$$u(x,t)=\frac{32h}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\cos\frac{(2n+1)\pi at}{l}\sin\frac{(2n+1)\pi x}{l}$$

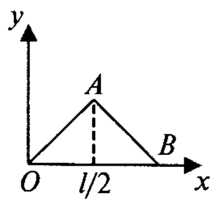
yechimga ega bo'lamiz.

2- masala. Uchlari $x=0$, $x=l$ nuqtalarga mahkamlangan tor berilgan bo'lib, boshlang'ich holati OAB siniq chiziqdan iborat.

Agar boshlang'ich tezlik

$$f_2(x)=\begin{cases}2\alpha x, & 0\le x\le l/2\\ 2\alpha(l-x), & l/2\le x\le l\end{cases}$$

bo'lsa, ixtiyoriy t momentdagi tor holati topilsin.



Y e c h i s h . Chizmaga asosan OB va AB to'g'ri chiziqlarning tenglamasi:

$$OA:\ \frac{2x}{l}=\frac{y}{h}\Rightarrow y=\frac{2h}{l}x, \qquad \text{agar } 0\le x\le l/2;$$

$$AB:\ \frac{x-l/2}{l-l/2}=\frac{y-h}{-h}\Rightarrow y=\frac{2h(l-x)}{l}, \qquad \text{agar } l/2\le x\le l.$$

Demak, torning boshlang'ich holati

$$f_1(x)=\begin{cases}\dfrac{2hx}{l}, & 0\le x\le l/2,\\ \dfrac{2h(l-x)}{l}, & l/2\le x\le l.\end{cases}$$

Furye usuliga asosan qo'yilgan masala yechimini (3.129) tenglik ko'rinishida izlaymiz, ya'ni

$$U(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{an\pi}{l}\cdot t+b_n\sin\frac{an\pi}{l}\cdot t\right)\cdot\sin\frac{n\pi}{l}\cdot x.$$

Bu tenglikdan a_n va b_n koeffitsiyentlarni quyidagi formulalar yordamida topamiz:

$$a_n = \frac{2}{l}\int_0^l f_1(x)\sin\frac{n\pi}{l}\cdot xdx = \frac{4h}{l^2}\int_0^{l/2} x\cdot\sin\frac{n\pi}{l}\cdot xdx + \frac{4h}{l^2}\int_{l/2}^{l/2}(l-x)\cdot\sin\frac{n\pi}{l}\cdot xdx.$$

Bo'laklab integrallash formulasiga asosan:

$$u = x,\ dv = \sin\frac{n\pi}{l}xdx$$

desak, bundan

$$du = dx,\ v = -\frac{l}{n\pi}\cdot\cos\frac{n\pi}{l}x.$$

U holda

$$\int x\cdot\sin\frac{n\pi}{l}\cdot xdx = -\frac{lx}{n\pi}\cos\frac{n\pi}{l}\cdot x + \frac{l}{n\pi}\int\cos\frac{n\pi}{l}\cdot xdx =$$

$$= -\frac{lx}{n\pi}\cos\frac{n\pi}{l}\cdot x + \frac{l^2}{n^2\pi^2}\cdot\sin\frac{n\pi}{l}\cdot x.$$

Demak,

$$a_n = \frac{4h}{l^2}\int_0^{l/2} x\cdot\sin\frac{n\pi}{l}\cdot xdx + \frac{4h}{l}\int_{l/2}^{l}\sin\frac{n\pi}{l}\cdot xdx - \frac{4h}{l^2}\int_{l/2}^{l} x\cdot\sin\frac{n\pi}{l}\cdot xdx =$$

$$= -\frac{4h}{l\pi n}\cdot x\cdot\cos\frac{n\pi}{l}\cdot x\Big|_0^{l/2} + \frac{4h}{n^2\pi^2}\cdot\sin\frac{n\pi}{l}\cdot x\Big|_0^{l/2} - \frac{4h}{n\pi}\cdot\cos\frac{n\pi}{l}\cdot x\Big|_{l/2}^{l} +$$

$$+\frac{4h}{l\pi n}\cdot x\cdot\cos\frac{n\pi}{l}\cdot x\Big|_{l/2}^{l} - \frac{4h}{n^2\pi^2}\cdot\sin\frac{n\pi}{l}\cdot x\Big|_{l/2}^{l} = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2},$$

$$b_n = \frac{2}{n\pi a}\int_0^l f_2(x)\sin\frac{n\pi}{l}x\cdot dx = \frac{4\alpha}{n\pi a}\int_0^{l/2} x\sin\frac{n\pi}{l}x\cdot dx + \frac{4\alpha}{n\pi a}\int_{l/2}^{l}(l-x)\sin\frac{n\pi}{l}x\cdot dx.$$

Yuqoridagi hisoblashlarni aynan takrorlab,

$$b_n = \frac{8\alpha l^2}{n^3\pi^3 a}\cdot\sin\frac{n\pi}{2}$$

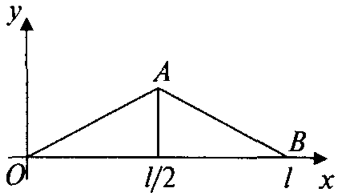
ni topamiz.

Demak, tor tebranishining ixtiyoriy t momentdagi holati

$$U(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(h\cos\frac{n\pi at}{l}-\frac{\alpha l^2}{n\pi a}\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}.$$

Mustaqil yechish uchun misollar

394. Uchlari x=0 va x=l da mahkamlangan, boshlang'ich holati OAB siniq chiziqni ifodalovchi torning ixtiyoriy t vaqtdagi holatini boshlang'ich tezligi 0 bo'lgan holda aniqlang.



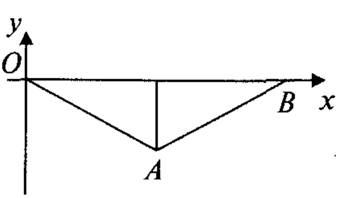
395. Uchlari x=0 va x=l da mahkamlangan torning boshlang'ich og'ishi nolga teng bo'lib, boshlang'ich tezligi esa

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\begin{cases}\cos\dfrac{\pi\left(x-\frac{l}{2}\right)}{h}, & \text{agar } \left|x-\dfrac{l}{2}\right|<\dfrac{h}{2},\\ 0, & \text{agar } \left|x-\dfrac{l}{2}\right|>\dfrac{h}{2}\end{cases}$$

formula bilan aniqlansa, torning ixtiyoriy t vaqtdagi holatini aniqlang.

396. Uchlari x=0 va x=1 da mahkamlangan, boshlang'ich holati $u=h(x^4-2x^3+x)$ ni ifodalovchi boshlang'ich tezligi 0 bo'lgan torning ixtiyoriy t vaqtdagi holatini aniqlang.

397. Uchlari x=0 va x=3 da mahkamlangan, boshlang'ich holati OAB siniq chiziqni ifodalovchi torning ixtiyoriy t vaqtdagi holatini aniqlang. Bunda $O(0, 0)$, $A(2, 1)$, $B(3, 0)$ koordinatalarga ega.



398. Uchlari x=0 va x=l da mahkamlangan torning dastlabki og'ishi 0 bo'lib, boshlang'ich tezligi esa

$$\frac{\partial u}{\partial t}=\begin{cases}u_0, & \text{agar } \left|x-\dfrac{l}{2}\right|<\dfrac{h}{2},\\ 0, & \text{agar } \left|x-\dfrac{l}{2}\right|>\dfrac{h}{2}\end{cases}$$

formula bilan ifodalansa, torning ixtiyoriy t vaqtdagi holatini aniqlang.

9- §. Sterjenda issiqlik tarqalish tenglamasi. Chegaraviy masalaning qo'yilishi

I. Issiqlikning chegaralanmagan sterjenda tarqalishi.

$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $t>0$, $-\infty<x<+\infty$ sohada $u(x,0)=f(x)$, $-\infty<x<+\infty$ boshlang'ich shartni qanoatlantiruvchi yechimi

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\cdot\int_{-\infty}^{+\infty}f(\xi)\cdot e^{-(\xi-x)^2/(4a^2t)}d\xi \qquad (3.133)$$

Puasson integrali orqali aniqlanadi.

II. Issiqlikning bir tomondan chegaralangan sterjenda tarqalishi.

$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamani $\{x>0,\ t>0\}$ sohada $u(x,0)=f(x)$ boshlang'ich va $u(0,t)=\varphi(t)$ chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\cdot\int_{0}^{+\infty}f(\xi)\cdot\left[e^{-(\xi-x)^2/(4a^2t)}-e^{-(\xi+x)^2/(4a^2t)}\right]d\xi+$$

$$+\frac{x}{2a\sqrt{\pi}}\cdot\int_0^t\varphi(\eta)\cdot e^{-x^2/(4a^2(t-\eta))}(t-\eta)^{-\frac{3}{2}}d\eta \qquad (3.134)$$

ko'rinishda topiladi.

III. Issiqlikning chegaralangan sterjenda tarqalishi.

$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(x,t)\big|_{t=0}=f(x)$ boshlang'ich va $u(0,t)=u(l,t)=0$ chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u(x,t)=\sum_{n=1}^{\infty}b_n e^{-\frac{a^2\pi^2n^2t}{l^2}}\cdot\sin\frac{\pi nx}{l} \qquad (3.135)$$

ko'rinishda aniqlanadi. Bunda $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{\pi nx}{l}dx$.

1- masala. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ tenglamaning

$$u(x,t)\big|_{t=0} = f(x) = \begin{cases} u_0, & \text{agar } x_1 < x < x_0, \\ 0, & \text{agar } x < x_1 \text{ yoki } x > x_2 \end{cases}$$

boshlang'ich shartni qanoatlantiruvchi yechimi topilsin.

Yechish. Sterjen chegaralanmagan bo'lgani uchun yechimni Puasson integrali ko'rinishida izlaymiz:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(\xi) \cdot e^{-(\xi - x)^2/(4a^2 t)} d\xi .$$

Shartga ko'ra $f(x)$ funksiya $[x_1, x_2]$ oraliqda o'zgarmas u_0 temperaturaga, qolgan oraliqda esa 0 ga teng bo'lgani uchun:

$$u(x,t) = \frac{u_0}{2a\sqrt{\pi t}} \cdot \int_{x_1}^{x_2} e^{-(\xi - x)^2/(4a^2 t)} \cdot d\xi .$$

Bunda quyidagi almashtirishni bajaramiz:

$$\frac{x - \xi}{2a\sqrt{t}} = \mu, \quad d\xi = -2a\sqrt{t} \cdot d\mu .$$

U holda

$$u(x,t) = -\frac{u_0}{\sqrt{\pi}} \int_{\frac{x-x_1}{2a\sqrt{t}}}^{\frac{x-x_2}{2a\sqrt{t}}} e^{-\mu^2} d\mu = \frac{u_0}{\sqrt{\pi}} \int_0^{\frac{x-x_1}{2a\sqrt{t}}} e^{-\mu^2} d\mu - \frac{u_0}{\sqrt{\pi}} \int_0^{\frac{x-x_2}{2a\sqrt{t}}} e^{-\mu} d\mu$$

yoki

$$u(x,t) = \frac{u_0}{2}\left[\Phi\left(\frac{x - x_1}{2a\sqrt{t}}\right) - \Phi\left(\frac{x - x_2}{2a\sqrt{t}}\right)\right]$$

izlangan yechim bo'ladi.

Bu yerda $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ integral *Puasson integrali* deb ataladi.

2- masala. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $x>0$, $t>0$ da $u|_{t=0}=f(x)=u_0$ boshlang'ich va $u|_{x=0}=0$ chegaraviy shartlarni qanoatlantiruvchi yechimini toping.

Y e c h i s h . Sterjen bir tomondan chegaralangani uchun berilgan shartlarni qanoatlantiruvchi yechim ushbu ko'rinishga ega bo'ladi:

$$u(x,t)=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}u_0\left[e^{-\frac{(\xi-x)^2}{4t}}-e^{-\frac{(\xi+x)^2}{4t}}\right]d\xi$$

yoki

$$u(x,t)=\frac{u_0}{2\sqrt{\pi t}}\int_0^{\infty}\left[e^{-\frac{(\xi-x)^2}{4t}}-e^{-\frac{(\xi+x)^2}{4t}}\right]d\xi.$$

Birinchi integralda $\frac{x-\xi}{2\sqrt{t}}=\mu$, $d\xi=-2\sqrt{t}d\mu$ almashtirishni bajarib,

$$\frac{u_0}{2\sqrt{\pi t}}\int_0^{\infty}e^{-\frac{(\xi-x)^2}{4t}}d\xi=\frac{u_0}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2\sqrt{t}}}e^{-\mu^2}d\mu=\frac{u_0}{2}\left[1+\Phi\left(\frac{x}{2\sqrt{t}}\right)\right],$$

ikkinchi integralda esa $\frac{x+\xi}{2\sqrt{t}}=\mu$, $d\xi=2\sqrt{t}d\mu$ deb

$$\frac{u_0}{2\sqrt{\pi t}}\int_0^{+\infty}e^{\frac{(x+\xi)^2}{4t}}d\xi=\frac{u_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{+\infty}e^{-\mu^2}d\mu=\frac{u_0}{2}\left[1-\Phi\left(\frac{x}{2\sqrt{t}}\right)\right]$$

ga ega bo'lamiz.

Shunday qilib, yechim ushbu ko'rinishni oladi:

$$u(x,t)=u_0\Phi\left(\frac{x}{2\sqrt{t}}\right).$$

3- masala. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ $(0<x<l,\ t>0)$ tenglamaning

$$u\big|_{t=0}=f(x)=\begin{cases} x, & \text{agar } 0<x\le \frac{l}{2},\\ l-x, & \text{agar } \frac{l}{2}<x<l\end{cases}$$

bo'lsa, boshlang'ich va $u\big|_{x=0}=u\big|_{x=l}=0$ chegaraviy shartlarni qanoatlantiruvchi yechimini toping.

Y e c h i s h . Sterjen chegaralangan bo'lganidan, berilgan chegaraviy shartlarni qanoatlantiruvchi yechimni ushbu ko'rinishda izlaymiz:

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-\frac{\pi^2 n^2 t}{l^2}}\cdot \sin\frac{\pi n x}{l},$$

bu yerda

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}dx=\frac{2}{l}\int_0^{\frac{l}{2}} x\sin\frac{\pi n x}{l}dx+\frac{2}{l}\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{\pi n x}{l}dx=$$

$$=\left\{\begin{matrix} u=x, & du=dx\\ dv=\sin\frac{\pi n x}{l}dx, & v=-\frac{l}{\pi n}\cos\frac{\pi n x}{l}\end{matrix}\right\}=\frac{2}{l}\left(-\frac{lx}{\pi n}\cos\frac{\pi n x}{l}+\frac{l^2}{\pi^2 n^2}\sin\frac{\pi n x}{l}\right)\Bigg|_0^{\frac{l}{2}}+$$

$$+\frac{2}{l}\left(-\frac{l^2}{\pi n}\cos\frac{\pi n x}{l}+\frac{lx}{\pi n}\cos\frac{\pi n x}{l}-\frac{l^2}{\pi^2 n^2}\sin\frac{\pi n x}{l}\right)\Bigg|_{\frac{l}{2}}^{l}=\frac{4l}{\pi^2 n^2}\sin\frac{\pi n}{2}.$$

Demak, izlanayotgan yechim ushbu ko'rinishga ega:

$$u(x,t)=\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi n}{2}e^{-\frac{\pi^2 n^2 t}{l^2}}\cdot\sin\frac{\pi n x}{l}$$

yoki $$u(x,t)=\frac{4l}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)}e^{-\frac{\pi^2(2n+1)^2 t}{l^2}}\cdot\sin\frac{\pi(2n+1)x}{l}.$$

Mustaqil yechish uchun masalalar

399. Uzunligi l ga teng, tashqi muhit ta'siridan muhofazalangan va $u|_{t=0} = f(x) = \frac{cx(l-x)}{l^2}$ boshlang'ich temperaturaga ega bo'lgan bir jinsli sterjen berilgan. Sterjenning uchlari nolga teng temperaturada tutib turiladi. Sterjenning $t > 0$ vaqtdagi temperaturasi topilsin.

400. Agar sterjenning $u|_{t=0} = f(x) = \frac{2\pi}{l}x - \sin\frac{2\pi x}{l}$ boshlang'ich temperaturasi berilgan va uchlari issiqlikdan muhofazalangan, ya'ni $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0$ bo'lsa, uzunligi l ga teng va sirti ham issiqlikdan muhofazalangan sterjenda temperatura taqsimotini toping.

401. Agar uzunligi l ga teng, sirti issiqlikdan muhofazalangan sterjenning boshlang'ich temperaturasi

$$f(x) = \begin{cases} \frac{2u_0}{l}, & \text{agar } 0 \le x \le \frac{l}{2}, \\ \frac{2u_0}{l}(l-x), & \text{agar } \frac{l}{2} < x < l \end{cases}$$

bo'lib, sterjenning uchlari ham issiqlikdan muhofazalangan bo'lsa, shu sterjenda issiqlik taqsimotini toping.

Quyidagi masalalarni Puasson formulasi yordamida hal qiling:

402. $4u_t = u_{xx},\ u\big|_{t=0} = e^{2x-x^2}$. **403.** $u_t = u_{xx},\ u\big|_{t=0} = x \cdot e^{-x^2}$.

404. $4u_t = u_{xx},\ u\big|_{t=0} = \sin x e^{-x^2}$.

10- §. Laplas masalasining yechimlarini tekshirishga keltiriladigan masalalar

Markazi $O(0,0)$ nuqtada bo'lgan doiraning chegarasida biror $f(\varphi)$ funksiya berilgan bo'lsin. Doirada va uning chegarasida uzluksiz bo'lib, doira ichida $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Laplas tenglamasini va

$u_{r=R} = f(\varphi)$ chegaraviy shartni qanoatlantiradigan $u(r, \varphi)$ funksiyani topish Dirixle masalasi bo‘lib, uning yechimi

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\frac{R^2-r^2}{R^2-2rR\cos(\tau-\varphi)+r^2}d\tau$$

ko‘rinishda bo‘ladi.

1- masala. Bir jinsli yupqa doiraviy plastinkada temperaturaning statsionar taqsimtini toping. Plastinka radiusi R ga teng bo‘lib, uning yuqori qismi 1° C da, pastki qismi 0° C da tutib turiladi.

Y e c h i s h. Masala shartiga ko‘ra $f(\tau)=\begin{cases}0, & \text{agar } -\pi<\tau<0,\\ 1, & \text{agar } 0<\tau<\pi\end{cases}$

bo‘lsa, temperatura taqsimoti $u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2R\cos(\tau-\varphi)+r^2}d\tau$ integral bilan aniqlanadi.

a) yuqori yarim doira $(0<\varphi<\pi)$ nuqtalar uchun $\mathrm{tg}\frac{\tau-\varphi}{2}=t$ almashtirishni kiritamiz, bundan $\cos(r-\varphi)=\frac{1-t^2}{1+t^2}$; $d\tau=\frac{2dt}{1+t^2}$, ya’ni t integrallash o‘zgaruvchisi $\left(-\mathrm{tg}\frac{\varphi}{2}\right)$ dan $\mathrm{ctg}\frac{\varphi}{2}$ gacha o‘zgaradi.

Shunday qilib,

$$u(r,\varphi)=\frac{1}{\pi}\int_{-\mathrm{tg}\frac{\varphi}{2}}^{\mathrm{ctg}\frac{\varphi}{2}}\frac{R^2-r^2}{(R-r)^2+(R+r)^2t^2}dt=\frac{1}{\pi}\mathrm{arctg}\left(\frac{R+r}{R-r}t\right)\Bigg|_{-\mathrm{tg}\frac{\varphi}{2}}^{\mathrm{ctg}\frac{\varphi}{2}}=$$

$$=\frac{1}{\pi}\left[\mathrm{arctg}\left(\frac{R+r}{R-r}\mathrm{ctg}\frac{\varphi}{2}\right)+\mathrm{arctg}\left(\frac{R+r}{R-r}tg\frac{\varphi}{2}\right)\right]=$$

$$=\frac{1}{\pi}\mathrm{arctg}\frac{\frac{R+r}{R-r}\left(\mathrm{ctg}\frac{\varphi}{2}+\mathrm{tg}\frac{\varphi}{2}\right)}{1-\left(\frac{R+r}{R-r}\right)^2}=-\frac{1}{\pi}\mathrm{arctg}\frac{R^2-r^2}{2Rr\sin\varphi}$$

yoki

$$\operatorname{tg}(u\pi) = -\frac{R^2 - r^2}{2Rr\sin\varphi}, \quad 0 < \varphi < \pi .$$

Bu tenglikning o'ng tomoni manfiy, demak, $0 < \varphi < \pi$ da u funksiya $\frac{1}{2} < u < 1$ tengsizliklarni qanoatlantiradi. Bu hol uchun, ya'ni $0 < \varphi < \pi$ da ushbu yechimga ega bo'lamiz:

$$\operatorname{tg}(\pi - u\pi) = \frac{R^2 - r^2}{2Rr\sin\varphi}$$

yoki

$$u = 1 - \frac{1}{\pi}\operatorname{arctg}\frac{R^2 - r^2}{2Rr\sin\varphi} .$$

b) Pastki yarim doirada joylashgan nuqtalar uchun $(\pi < \varphi < 2\pi)$ $\operatorname{ctg}\frac{\tau-\varphi}{2} = t$ o'rniga qo'yishdan foydalanamiz, bundan $\cos(\tau - \varphi) = \frac{t^2-1}{t^2+1}$, $d\tau = -\frac{2dt}{t^2+1}$, yangi integrallash o'zgaruvchisi t esa $\left(-\operatorname{ctg}\frac{\varphi}{2}\right)$ dan $\operatorname{tg}\frac{\varphi}{2}$ gacha o'zgaradi. U holda φ ning bu qiymatlari uchun ushbuga egamiz:

$$u(r,\varphi) = -\frac{1}{\pi}\int_{-\operatorname{ctg}\frac{\varphi}{2}}^{\operatorname{tg}\frac{\varphi}{2}} \frac{R^2 - r^2}{(R+r)^2 + (R-r)^2 t^2} dt =$$

$$= -\frac{1}{\pi}\left[\operatorname{arctg}\left(\frac{R-r}{R+r}\operatorname{tg}\frac{\varphi}{2}\right) + \operatorname{arctg}\left(\frac{R-r}{R+r}\operatorname{ctg}\frac{\varphi}{2}\right)\right]$$

yoki

$$u = -\frac{1}{\pi}\operatorname{arctg}\frac{(R^2 - r^2)}{2Rr\sin\varphi}, \quad \pi < \varphi < 2\pi.$$

O‘ng tomon musbat (chunki $\sin\varphi < 0$), shuning uchun $0 < u < \frac{1}{2}$.

Mustaqil yechish uchun masalalar

Doira ichida Laplas tenglamasini qanoatlantiruvchi va doira chegarasida $u\big|_{r=1} = f(\varphi)$ funksiyaga teng bo‘lgan garmonik funksiya topilsin.

405. $f(\varphi) = \cos^2\varphi$.

406. $f(\varphi) = \sin^3\varphi$

407. $f(\varphi) = \cos^4\varphi$.

408. $f(\varphi) = \sin^6\varphi + \cos^6\varphi$.

JAVOBLAR

1. $y^2 - 4 = Ce^{-x^2}$.
2. $\frac{1}{2}\ln 2y \ln \operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)$.
3. $\sin y \cos x = C$.
4. $y = e^{\frac{\pi}{4}\operatorname{arctg}x}$.
5. $y = \arccos e^{cx}$.
6. $2e^{-y}(y+1) = x^2 + 1$.
7. $2(x-2) = \ln^2 y$.
8. $2\sin x + \ln\left|\operatorname{tg}\frac{x}{2}\right| = C$.
9. $\sqrt{1+x^2} + \sqrt{+y^2} = C$.
10. $2^x - 2^y = \frac{3}{32}$.
11. $y = \ln \operatorname{tg}(\operatorname{ch}x + C)$.
12. $\operatorname{arctg}x^2 + 2\operatorname{arctg}y^3 = \frac{\pi}{2}$.
13. $\ln|x+y| + \frac{x}{x+y} = C$.
14. $y = 2x\operatorname{arctg}x$.
15. $Cx = e^{\cos\frac{y}{x}}$.
16. $y^2 = Cxe^{\frac{-y}{x}}$.
17. $y^2 = 4x^2 \ln Cx$.
18. $1+\sin(y/x) = Cx\cos(y/x)$.
19. $y^2 = x^2 \ln Cx^2$.
20. $x+2y+5\ln|x+y-3| = C$.
21. $x^2+y^2+xy+x-y = C_1, C_1 = C^2-1$.
22. $3x+2y-4+2\ln|x+y-1| = 0$.
23. $x^2+xy-y^2-x+3y = C$.
24. $x^2+2xy-y^2-4x+8y = C$.
25. $y = \operatorname{tg}x - 1 + e^{-\operatorname{tg}x}$.
26. $y = \operatorname{ch}x(\operatorname{sh}x + C)$.
27. $y = \sqrt{1-x^2}\left[\frac{1}{2}(\arcsin x)^2 - \sqrt{1-x^2} + C\right]$.
28. $y = x(\sin x + C)$.
29. $y = e^{-x^2}(x^2/2 + C)$.
30. $\cos x(x+C)/(1+\sin x)$.
31. $y = \frac{1}{x\sqrt[3]{3\ln(C/x)}}$.
32. $x = \frac{1}{\ln y + 1 - Cy}$.
33. $y^{-1/3} = Cx^{2/3} - (3/7)x^3$.
34. $y = (x-1)(C-x)$.
35. $y^{-4} = x^3(e^x + C)$.
36. $y = \sec x/(x^3+1)$.
37. $x = 1/[y(y+C)]$.
38. $e^x + xy + x\sin y + e^y = C$.
39. $e^y + \frac{1}{2}x^2 + xy - x = C,\ C = C_1 + 1$.
40. $e^x(x\sin y + y\cos y - \sin y) = C$.
41. $3x^2y - y^3 = C$.
42. $x^2 - 3x^3y^2 + y^4 = C$.
43. $4y\ln x + y^4 = C$.
44. $5x^2y - 8xy + x + 3y = C$.
45. $x^3 + x^3\ln y - y^2 = C$.
46. $x^2\cos^2 y + y^2 = C$.
47. $\mu = 1/x^2;\ x + y/x = C$.
48. $\mu = 1/y;\ xy - \ln y = 0$.
49. $2x + \ln(x^2+y^2) = C$.
50. $2x^3y^3 - 3x^2 = C$.
51. $x^2 + \ln y = Cx^3;\ x = 0$.
52. $\mu = \cos y;\ x^2\sin y + \frac{1}{2}\cos 2y = C$.
53. $\mu = e^{-2x};\ y^2 = (C-2x)e^{2x}$.
54. $\mu = 1/\sin y;\ x/\sin y + x^3 = C$.
55. $\mu = e^{-y};\ e^{-y}\cos x = C + x$.
56. $y = \frac{1}{\cos^2 x + \frac{C}{2}};\ y = 0,\ y = 1$.
57. $y = e^{\sin(x+C)},\ y = e,\ y = \frac{1}{e}$.

58. $x=-\frac{1}{2}-p+\frac{c}{(p-1)^2},\ y=-\frac{p^p}{2}+\frac{cp^2}{(p-1)^2};$
$y=0;\ y=x+1.$

59. $y=Cx+\frac{1}{C^2}$, $4y^3=27x^2$.

60. $x=Cp^2e^p$, $y=C(p+1)e^p$; $y=0$.

61. $3Cy=3C^2x+(C-3)^2$; $y^2+4y=12x$.

62. $2Cy+x^2=C^2$.

63. $xy=C^2x+C$; $4x^2y=-1$.

64. $y^2=2Cx-C^2$; $y=\pm x$.

65. $y=Cx+\frac{1}{2}\ln C$, $2y+1+\ln(-2x)=0$.

66. $y=x^2+C$.

67. $\left(y-\frac{1}{x+C}\right)(y-Ce^{x^2/2})=0$.

68. $(y-\cos x-C)(ye^{-x^2}-C)=0$.

69. $y=(C\pm x)^2$.

70. $y=\sin(C\pm x)$.

71. $y=Cx^2+1/C$.

72. $y=e^{C\pm x}$.

73. $y^2=(x+C)^3$.

74. $y+x=(x+C)^3$; $y=-x$.

75. $(x+C)^2+y^2=1$; $y=\pm 1$.

76. $y(x+C)^2=1$; $y=0$.

77. $(y-x)^2=2C(x+y)-C^2$; $y=0$.

78. $(x-1)^{4/3}+y^{4/3}=C$.

79. $y^2(1-y)=(x+C)^2$; $y=1$.

80. $x=\frac{2p}{p^2-1},\ y=\frac{2p}{p^2-1}-\ln\left|p^2-1\right|+C.$

81. $x=\ln p+\frac{1}{p},\ y=p-\ln p+C.$

82. $x=p^3+p,\ 4y=3p^4+2p^2+C.$

83. $x=p\sqrt{p^2+1},\ 3y=(2p^2-1)\sqrt{p^2+1}+C.$

84. $x=3p^2+2p+C,\ y=2p^3+p^2,\ y=0.$

85. $x=2\operatorname{arctg}p+C,\ y=\ln(1+p^2),\ y=0.$

86. $x=\ln|p|\pm\frac{3}{2}\ln\left|\frac{\sqrt{p+1}-1}{\sqrt{p+1}+1}\right|+3\sqrt{p+1}+C,$
$y=p\pm(1+p)^{\frac{3}{2}},\ y=\pm 1.$

87. $x=e^p+C,\ y=(p-1)e^p,\ y=-1.$

88. $x=\pm\left(2\sqrt{p^2-1}+\arcsin\frac{1}{|p|}\right)+C,$
$y=\pm p\sqrt{p^2-1},\ y=0.$

89. $x=\pm(\ln\left|\frac{1-\sqrt{p-1}}{1+\sqrt{1-p}}\right|\pm 3\sqrt{1-p})+C,$
$y=\pm\sqrt{1-p}),\ y=0.$

90. $y=(C+\sqrt{x+1})^2$; maxsus interal $y=0$.

91. $x=Ct^2-2t^3$; $y=2Ct-3t^2$, bunda $t=1/p$.

92. $Cy=(x-C)^2$, maxsus intervallar $y=0$ va $y=-4x$.

93. $(\sqrt{y}+\sqrt{x+1})^2=C$, $y=0$.

94. $x=\frac{p-\ln p+C}{(p-1)^2}$.

95. $x\sqrt{p}=\ln p+C$, $y=\sqrt{p}\,(4-\ln p-C)$; $y=0$.

96. $x=C(p-1)-2+2p+1$,
$y=Cp^2(p-1)-2+p^2$; $y=0$; $y=x-2$.

97. $xp^2=p+C$, $y=2+2Cp-1-\ln p$.

98. $y=Cx-\ln C$; $y=\ln x+1$.

99. $Cx-C^2$; maxsus integral $y=\frac{x^2}{4}$.

100. $y=Cx-a^{\sqrt{1+C^2}}$; maxsus integral $x^2+y^2=a^2$.

101. $y=Cx+\frac{1}{2c^2}$; maxsus integral
$y=1{,}5x^{\frac{2}{3}}$.

102. $y=\sqrt{1-x^2}$.

103. $y=Cx-eC$.

104. $y=Cx-C^2$.

105. $C^3=3(Cx-y)$; $9y^2=4x^3$.

106. $2C^2(y-Cx)=1$; $8y^3=27x^2$.

107. $y=Cx+C^2+1$; $y=1-\frac{x^2}{4}$.

108. $y=+x\dfrac{e^{-\frac{ax^2}{2}}}{C+a\int e^{-\frac{ax^2}{2}}dx}$; $y=x$.

109. $y=\dfrac{2Cx^3+1}{(Cx^3-1)x}$; $y=\dfrac{2}{x}$.

110. $y=\dfrac{2}{x}+\dfrac{4}{Cx^5-x}$; $y=\dfrac{2}{x}$.

111. $y=\dfrac{1}{x}+\dfrac{1}{Cx^{\frac{2}{3}}+x}$; $y=\dfrac{1}{x}$.

112. $y=x+\dfrac{x}{x+C}$; $y=x$.

113. $y=x+2+\dfrac{4}{Ce^{4x}-1}$; $y=x+2$.

114. $y=e^x-\dfrac{1}{x+C}$; $y=e^x$.

115. $y=\dfrac{x}{3C+x}+x$, $y=x$.

116. $y=\dfrac{x}{3Ce^{-\frac{x}{2}}+1}+x$, $y=x$.

117. $y=\dfrac{2x}{2Ce^{-\frac{2x}{5}}+1}+x$, $y=x$.

118. $y=\frac{1}{48}x^4+\frac{1}{8}x^2+\frac{1}{32}\cos 2x$.

119. $y=x\cos x-3\sin x+x^2+2x$.

120. $y=\ln|\sin x|+c_1x^2+c_2x+c_3$.

121. $y=\frac{1}{3}\sin^3 x+c_1x+c_2$.

122. $y=-(x+3)e^{-x}+\frac{3}{2}x^2+3$.

123. $y=3\ln x+2x^2-6x+6$.

124. $y=1-\cos 2x$.

125. $y=C_1x+x\operatorname{arctg}x-\ln\sqrt{1+x^2}+C_2$.

126. $y=c_1x+c_1-\ln|\cos x|$ – umumiy yechim, xususiy yechim esa $y=-\ln|\cos x|$.

127. $y=x(1-\ln|x|)+\frac{1}{2}c_1x^2+c_2x+c_3$.

128. $y=\cos x+\frac{1}{6}c_1x^3+\frac{1}{2}c_2x^2+c_3x+c_4$.

129. $y=-\ln|\sin x|+c_1x+c_2$.

130. $y=e^x(x-2)+c_1x+c_2$.

131. $y=-\frac{1}{4}\sin 2x+\frac{1}{2}x+6$.

132. $y=\frac{1}{x}+c_1\ln x+c_2$.

133. $y=c_1\sin x-x-\frac{1}{2}\sin 2x+c_2$.

134. $y=c_1x(\ln x-1)+c_2$.

135. $y=e^x(x-1)+c_1x^2+c_2$.

136. $y=c_2+\frac{1}{\sqrt{c_1}}\operatorname{arctg}\frac{x}{\sqrt{c_1}}$.

137. $y=(\arcsin x)^2+c_1\arcsin x+c_2$.

138. $y=\pm 4\left[(c_1x+a^2)^{\frac{5}{2}}+c_2x+c_3\right]\cdot\frac{1}{15c_1^2}$.

139. $y=(1+c_1^{-2})\ln|1+c_1x|-c_1^{-1}x+c_2$.

140. $y=\frac{x}{c_1}-\frac{1}{c_1^2}\ln|1+c_1x|+c_2$.

141. $y = c_1\left(x - e^{-x}\right) + c_2.$

142. $y = \frac{x^3}{12} - \frac{x}{4} + c_1 \text{arctg} x + c_2.$

143. $y = c_2 - c_1 \cos x - x.$

144. $y = -\frac{x^2}{4} + c_1 \ln|x| + c_2.$

145. $y = (3x^4 - 4x^3 - 36x^2 + 72x + 8)/24.$

146. $y = (x^2 + c_1^3)\text{arctg}\frac{x}{c_1} + c_1 x + c_2.$

147. $y = x^2 + \frac{c_1}{2}(x\sqrt{1-x^2} + \arcsin x) + c_2.$

148. $y = c_1 x + c_2.$

149. $y^3 + c_1 y + c_2 = 3x.$

150. $\text{ctg} y - c_1 x = c_2.$

151. $\frac{1}{2}\ln|2y + 3| = c_1 x + c_2.$

152. $y = e^{\frac{x+c_2}{x+c_1}}.$

153. $\ln[c_1(y + 1) - 1] = c_1(x + c_2).$

154. $c_1^2 y + 1 = \pm\text{ch}(c_1 x + c_2).$

155. $y^3 = c_1(x + c_2)^2, \quad y = c.$

156. $y = e^{2x}.$

157. $y = -a \ln\left|\cos\frac{x}{a}\right|.$

158. $s = \frac{m^2 g}{k^2}\left(e^{-\frac{kt}{m}} - 1\right) + \frac{mgt}{k}.$

159. $y = (c_1 x + c_2)^2.$

160. $c_1 y^2 = 1 + (c_1 x + c_2)^2.$

161. $4(c_1 y - 1) = (c_1 x + c_2)^2.$

162. $\ln|y| = c_1 e^x + c_2 e^{-x}.$

163. $x = \sqrt{y} - \frac{1}{2}c_1 \ln(2\sqrt{y} + c_1) + c_2.$

164. $y = c_2 e^{c_1 x}.$

165. $y\sqrt{y^2 + c_1^2} + c_2^2 \ln\left|y + \sqrt{y^2 + c_1^2}\right| =$
$= \pm(-y^2 + 2c_1^2 x + 3c_2).$

166. $y = c_2 x + c_3 \pm \frac{4}{15c_1^2}(c_1 x + a^2)^{5/2}.$

167. $y = -\ln|1 - x|.$

168. $y = c_2 e^{c_1 x^2}.$

169. $\ln c_2 y = 4x^{5/2} + c_1 x, \quad y = 0.$

170. $y = c_2(x + \sqrt{x^2 + 1}).$

171. $y^2 = c_1 x^3 + c_2.$

172. $y = c_2 x e^{-\frac{c_1}{x}}.$

173. $y = C_2 |x|^{C_1 - \frac{1}{2}\ln|x|}.$

174. $|y|^{c_1^2+1} = c_2\left(x - \frac{1}{c_1}\right)(x - c_1)^{c_1^2}.$

175. $y = c_2 x(\ln c_1 x)^2.$

176. $\ln|y| = \ln\left|x^2 - 2x + c_1\right| + \int\frac{2dx}{(x-1)^2 + c_2 - 1}.$

177. $4c_1 y^2 = 4x + x(c_1 \ln c_2 x)^2.$

178. $y = -x\ln(c_2 \ln c_1 x), \quad y = cx.$

179. $y = c_2 + (c_1 - c_2 x)\mathrm{ctg}x.$

180. $y = \frac{1}{2}x\ln^2 x + c_1 x\ln x + c_2 x.$

181. $y = c_1 \sin x + c_2 \sin^2 x.$

182. Tashkil etadi.

183. Tuzib bo'ladi.

184. Chiziqli erkli emas.

185. Chiziqli erkli.

186. Chiziqli erkli.

187. Chiziqli erkli emas.

188. $y = c_1 e^{-2x} + c_2 e^x.$

189. Chiziqli erkli. $y = c_1 + c_2 e^{2x}$.

190. Tashkil etadi. $y = e^{2x}(c_1 \cos x + c_2 \sin x).$

191. $y_2 = e^x$ va $y = c_1 e^{-x} + c_2 e^x.$

192. $y = c_1 e^x + c_2 e^{3x}.$

193. $y = (c_1 + c_2 x)e^{2x}.$

194. $y = e^{2x}(A\cos 3x + B\sin 3x).$

195. $y = c_1 e^{2x} + c_2 e^{-2x} = A\mathrm{ch}2x + B\mathrm{sh}2x.$

196. $y = A\cos 2x + B\sin 2x = a\sin(2x + \varphi).$

197. $y = c_1 + c_2 e^{-4x}.$

198. $y = c_1 e^{2x} + c_2 e^{-x}.$

199. $y = c_1 \cos 5x + c_2 \sin 5x.$

200. $y = c_1 + c_2 e^x.$

201. $y = (c_1 + c_2 x)e^{2x}.$

202. $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x.$

203. $y = (c_1 e^{xa\sqrt{2}/2} + c_2 e^{-xa\sqrt{2}/2})\cos(xa\sqrt{2}/2) +$
$+ (c_3 e^{xa\sqrt{2}/2} + c_4 e^{-xa\sqrt{2}/2})\sin(xa\sqrt{2}/2).$

204. $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x.$

205. $y = c_1 e^{-2x} + c_2 e^{-x}.$

206. $y = (c_1 x + c_2)e^{ax}.$

207. $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$

208. $x(t) = c_1 e^{3t} + c_2 e^{-t}.$

209. $x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$

210. $s(t) = c_1 + c_2 e^{-at}.$

211. $y = 4e^{-3x} - 3e^{-2x}.$

212. $y = xe^{5x}.$

213. $y = -\frac{1}{3}e^x \cos 3x.$

214. $y = \frac{1}{3}(5 - 2e^{-3x}).$

215. $y = \sqrt{2}\sin 3x.$

216. $y = \sin x + \frac{1}{\sqrt{3}}\cos x.$

217. $y = 2\sin\frac{x}{3}.$

218. $y = 3e^x - e^{-x}.$

219. $y = e^{-t}(\cos t + 2\sin t).$

220. $y = (c_1 x + c_2)e^x + e^{2x}.$

221. $y = c_1 e^{2x} + c_2 e^{-2x} - 2x^3 - 3x.$

222. $y=c_1e^{-x}+c_2e^{-2x}+0{,}25\sqrt{2}\cos\left(\frac{\pi}{4}-2x\right)$.

223. $y = c_1\cos x + c_2\sin x + x + e^x$.

224. $y = c_1 + c_2e^{-3x} + \frac{3}{2}x^2 - x$.

225. $y=e^{-2x}(c_1\cos x+c_2\sin x)+x^2-8x+7$.

226. $y = c_1e^{2x} + (c_2 - x)e^x$.

227. $y=\frac{1}{2}e^{-x}+xe^{-3x}+c_1e^{-2x}+c_2e^{-3x}$.

228. $y=c_1+c_2x+(c_3+x)e^{-x}+x^3-3x^2$.

229. $y=c_1e^x+c_2e^{-2x}-3(x^2+x+1{,}5)$.

230. $y=c_1e^{2x}+c_2e^{3x}+\frac{1}{6}(5\cos 3x-\sin 3x)$.

231. $y = (c_1x + c_2)e^{-x} + \frac{1}{4}e^x$.

232. $y=e^{-\frac{x}{2}}(c_1\cos\frac{3x}{2}+c_2\sin\frac{3x}{2})-$
$-6\cos 2x+8\sin 2x$.

233. $y = c_1e^{\frac{x}{2}} + c_2e^{-\frac{x}{2}} - x^3$.

234. $y = \frac{1}{8}(e^{5x} + 22e^{3x} + e^x)$.

235. $y = \frac{1}{2}x(x+2)e^{4x}$.

236. $y = -\frac{11}{8}\cos x + 4\sin x - \frac{1}{8}\cos 3x$.

237. $y = 4e^{\frac{x}{2}} - x - 4$.

238. $y = \frac{1}{8}\sin 2x - \frac{1}{4}(x\cos 2x - 1)$.

239. $y = \frac{1}{16}(4x - \pi)\sin 2x$.

240. $y = x\operatorname{ch}x$.

241. $y=e^{2x}(5\cos 2x-\sin 2x+6\sin x-5\cos x)$.

242. $y = \frac{c_1}{x} + c_2x^2$.

243. $y = c_1x^n + c_2x^{-(n+1)}$.

244. $y = x^{-2}(c_1 + c_2\ln x)$.

245. $y = c_1\cos(\ln x) + c_2\sin(\ln x)$.

246. $y = \frac{5}{3}x^2 + c_1x^{-1} + c_2$.

247. $y = c_1x^3 + c_2x^{-2} - \ln x + \frac{1}{3}$.

248. $y = x(c_1\cos(\ln x) + c_2\sin(\ln x))$.

249. $y=c_1x+c_2x^3+\frac{1}{9}(9\ln^2 x+24\ln x+26)$.

250. $y=c_1\cos(\ln x)+c_2\sin(\ln x)-$
$-\frac{1}{3}\sin(2\ln x)$.

251. $y = c_1x + c_2x^2 - 4x\ln x$.

252. $y = \frac{1}{x}(c_1 + c_2\ln x + \ln^3 x)$.

253. $y = x^2(\frac{1}{6}x^3 + c_1x + c_2)$.

254. $y = \frac{1}{2}x + c_1\cos(\ln x) + c_2\sin(\ln x)$.

255. $y = c_1x + c_2x^{-1} + c_3x^3$.

256. $y = \frac{1}{2x}(\ln^2 x + 2\ln x + 2)$.

257. $y = \frac{1}{2}x^3 - \frac{1}{\ln 2}x^2\ln x$.

258. $y = c_0(1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4} - \frac{x^6}{2\cdot 4\cdot 6} + \ldots) =$
$= c_0e^{-\frac{x^2}{2}}$.

259. $y=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{4\cdot n!}=\frac{1}{4}e^{-2x}-\frac{1}{4}+\frac{x}{2}.$

260. $y=c_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2\cdot 4\cdot 6\ldots 2n}+c_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{1\cdot 3\cdot 5\ldots(2n+1)}.$

261. $y=c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{1\cdot 3\ldots(2n-1)}+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2\cdot 4\ldots 2n}.$

262. $y=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{4\cdot 5\cdot 8\cdot 9\ldots 4n(4n+1)}.$

263. $y=1+\frac{x}{1!}+\frac{3x^2}{2!}+\frac{17x^3}{3!}+\ldots$

264. $y=\frac{x^2}{2!}+\frac{12x^5}{5!}+\ldots$

265. $y=1+\frac{x}{1!}+\frac{x^3}{3!}+\frac{4x^4}{4!}+\ldots$

266. $y=4(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots)$

267. $y=1+x+\frac{3x^2}{2!}+\frac{8x^3}{3!}+\frac{34x^4}{4!}+\ldots$

268. $y=x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{5}{24}x^4-\frac{1}{24}x^5-\ldots$

269. $\begin{cases} x=3c_1\cos 3t-3c_2\sin 3t,\\ y=c_2\cos 3t+c_1\sin 3t.\end{cases}$

270. $\begin{cases} x(t)=c_1e^t-c_2e^{-t}+t-1,\\ y(t)=c_1e^t+c_2e^{-t}-t+1,\end{cases}$

271. $\begin{cases} x(t)=t+c_1\cos 2t+c_2\sin 2t,\\ y(t)=1+c_1\sin 2t-c_2\cos 2t.\end{cases}$

272. $\begin{cases} x(t)=e^{-2t}(1-2t),\\ y(t)=e^{-2t}(1+2t).\end{cases}$

273. $\begin{cases} x(t)=t^2+t+c_1e^{2t}+c_2e^{3t},\\ y(t)=t+1+2c_1e^{2t}.\end{cases}$

274. $\begin{cases} x(x)=(\sin t-2\cos t)e^{-t},\\ y(t)=e^{-t}\cos t.\end{cases}$

275. $\begin{cases} x(t)=e^t.\\ y(t)=e^t-e^{2t}.\end{cases}$

276. $\begin{cases} x(t)=c_1+c_2t+c_3t^2,\\ y(t)=-(c_1+2c_3)t-\frac{c_2}{2}t^2-c_3\frac{t^3}{3}+c_4.\end{cases}$

277. $\begin{cases} x(t)=\left(\frac{\sqrt{2}}{2}+1\right)e^{t\sqrt{2}}+\left(1-\frac{\sqrt{2}}{2}\right)e^{-t\sqrt{2}},\\ y(t)=\frac{\sqrt{2}}{2}e^{t\sqrt{2}}-\frac{\sqrt{2}}{2}e^{-t\sqrt{2}}.\end{cases}$

278. $\begin{cases} x(t)=c_1e^{-t}+c_2e^{-3t},\\ y(t)=c_1e^{-t}+3c_2e^{-3t}+\cos t.\end{cases}$

279. $\begin{cases} \frac{1}{x+y}+t=c_1,\\ \frac{1}{x-y}+t=c_2.\end{cases}$

280. $\begin{cases} x^2-y^2=c_1,\\ x-y+t=c_2.\end{cases}$

281. $\begin{cases} \operatorname{tg}\frac{x+y}{2}=c_1e^t,\\ \operatorname{tg}\frac{x-y}{2}=c_2e^t.\end{cases}$

284. $\begin{cases} \operatorname{tg}(x+y)=t,\\ \operatorname{tg}(x-y)=t.\end{cases}$

285. $\begin{cases} x(t)=2c_1e^{3t}-4c_2e^{-3t},\\ y(t)=c_1e^{3t}+c_2e^{-3t}.\end{cases}$

286. $\begin{cases} x(t)\equiv 0,\\ y(t)\equiv 0.\end{cases}$

287. $\begin{cases} x(t)=e^{2t}-e^{3t},\\ y(t)=e^{2t}-2e^{3t}.\end{cases}$

288. $\begin{cases} x(t) = e^{4t}(c_1 \cos 3t + c_2 \sin 3t). \\ y(t) = e^{4t}(-c_1 \sin 3t + c_2 \cos 3t). \end{cases}$

289. $\begin{cases} x(t) = e^{4t}(c_1 t + c_2), \\ (t) = e^{4t}(c_1 t + c_2 - c_1). \end{cases}$

290. $\begin{cases} x(t) = (\sin t - 5\cos t)e^{-t}, \\ y(t) = e^{-t}\cos t. \end{cases}$

291. $\begin{cases} x(t) = e^{5t} + e^{3t}, \\ y(t) = 6e^{5t} - 7e^{3t}. \end{cases}$

292. $\begin{cases} x(t) = 2c_1 e^t + 7c_2 e^{2t} + 3c_3 e^{3t}, \\ y(t) = c_1 e^t + 3c_2 e^{2t} + c_3 e^{3t}, \\ z(t) = -2c_1 e^t - 8c_2 e^{2t} - 3c_3 e^{3t}. \end{cases}$

293. $\begin{cases} x(t) = c_1 e^t + c_2 \cos t + c_3 \sin t, \\ y(t) = c_1 e^t + c_2 \sin t + c_3 \cos t, \\ z(t) = c_2(\cos t + \sin t) + c_3(\sin t - \cos t). \end{cases}$

294. $\begin{cases} x(t) = \frac{8}{3}e^{2t} + 2c_1 e^t + c_2 e^{-t}, \\ y(t) = \frac{29}{3}e^{2t} + 3c_1 e^t + c_2 e^{-t}. \end{cases}$

295. $\begin{cases} x(t) = (1 - t)\cos t - \sin t, \\ y(t) = (t - 2)\cos t + t\sin t. \end{cases}$

296. $\begin{cases} x(t) = c_1 \cos t + c_2 \sin t + \frac{t}{2}\cos t + 1, \\ y(t) = -c_1 \sin t + c_2 \cos t - \frac{t}{2}\sin t - \frac{1}{2}\cos t. \end{cases}$

297. $\begin{cases} x(t) = c_1 e^t + c_2 e^{-t} + \sin t, \\ y(t) = -c_1 e^t + c_2 e^{-t}. \end{cases}$

298. $\begin{cases} x(t) = c_1 e^t + c_2 e^{3t} + e^t(2\cos t - \sin t), \\ y(t) = c_1 e^t - c_2 e^{3t} + e^t(3\cos t + \sin t). \end{cases}$

299. $\begin{cases} x(t) = c_1 e^t + c_2 \sin t + c_3 \cos t, \\ y(t) = -c_1 e^t + c_2 \cos t - c_3 \sin t + t, \\ z(t) = c_2 \sin t + c_3 \cos t + 1. \end{cases}$

300. $\begin{cases} x(t) = c_1 \cos 2t + c_2 \sin 2t + t, \\ y(t) = c_1 \sin 2t - c_2 \cos 2t + 1. \end{cases}$

301. $\begin{cases} x(t) = -c_1 \sin t + (c_2 - 1)\cos t, \\ y(t) = c_1 \cos t + c_2 \sin t. \end{cases}$

302. $\begin{cases} x(t) = -t, \\ y(t) = 0. \end{cases}$

303. $\begin{cases} x(t) = -c_1 t + c_2 - 2e^{-t} - \cos t - \sin t, \\ y(t) = c_1 - 2e^{-t} + \cos t. \end{cases}$

304. $\begin{cases} x(t) = -\frac{4}{3}t - \frac{7}{9}, \\ y(t) = \frac{1}{3}t - \frac{5}{9}. \end{cases}$

305. $\begin{cases} x(t) = c_1 e^{\sin t}, \\ y(t) = c_2 e^{\operatorname{sh} t}. \end{cases}$

306. $\begin{cases} x(t) = 4c_1 e^{6t} + c_2 e^t, \\ y(t) = c_1 e^{6t} + c_2 e^t. \end{cases}$

307. $\begin{cases} x(t) = c_1 e^{2t} + 4c_2 e^{7t}, \\ y(t) = -4c_1 e^{2t} + 4c_2 e^{7t}. \end{cases}$

308. $\begin{cases} x(t) = 4c_1 e^t + c_2 e^{6t} - \frac{5}{6}, \\ y(t) = c_1 e^t - c_2 e^{6t} - \frac{1}{6}. \end{cases}$

309. $\begin{cases} x(t) = c_1(1 + 2t) - 2c_2 - 2\cos t - 3\sin t, \\ y(t) = -c_1 t + c_2 + 2\sin t. \end{cases}$

310. $\begin{cases} x(t) = c_1 e^{4t} + c_2 e^{2t} - e^t, \\ y(t) = c_1 e^{4t} - c_2 e^{2t} + e^t. \end{cases}$

311. $\begin{cases} x(t) = (\sin t - 2\cos t)e^{-t}, \\ y(t) = e^{-t}\cos t. \end{cases}$

312. $\begin{cases} y(t) = c_1 + c_2 e^{2t} - \frac{1}{4}(t^2 + t), \\ z(t) = c_2 e^{2t} - c_1 + \frac{1}{4}(t^2 - t - 1). \end{cases}$

313. $\begin{cases} y(t) = \dfrac{2c_1}{(c_2 - t)^2}, \\ z(t) = \dfrac{c_1}{c_2 - t}. \end{cases}$

314. $\begin{cases} y(t) = t + \dfrac{c_1}{c_2} e^{-\frac{t}{c_1}}, \\ z(t) = c_2 e^{\frac{t}{c_1}}. \end{cases}$

315. $\begin{cases} x(t) = \sqrt{\dfrac{c_1}{2} e^{2t} + \dfrac{c_2}{2} e^{-2t}}, \\ y(t) = \sqrt{\dfrac{c_1}{2} e^{2t} - \dfrac{c_2}{2} e^{-2t}}. \end{cases}$

316. $\begin{cases} z = c_1 y, \\ zy^2 - \frac{3}{2} x^2 = c_2. \end{cases}$

317. $\begin{cases} x(t) = e^{-6t}(c_1 \cos t + c_2 \sin t), \\ y(t) = e^{-6t}[(c_1 + c_2)\cos t - (c_1 - c_2)\sin t]. \end{cases}$

318. $\begin{cases} x(t) = (c_1 + c_2 t)e^{2t}, \\ y(t) = -(c_1 + c_2(1 + t))e^{2t}. \end{cases}$

319. $\begin{cases} x(t) = c_1 e^{-t} + c_2 e^{2t}, \\ y(t) = c_3 e^{-t} + c_2 e^{2t}, \\ z(t) = -(c_1 + c_2)e^{-t} + c_2 e^{2t}. \end{cases}$

320. $\begin{cases} x + y - z = c_1 e^{-t} + \frac{1}{2} e^{t} + \frac{1}{4} e^{3t} - 4, \\ x + y + 2z = c_2 e^{2t} + \frac{1}{2} e^{t} + \frac{1}{4} e^{3t} + 8, \\ x - y = c_3 e^{-2t} + \frac{1}{2} e^{t} - \frac{1}{4} e^{3t}. \end{cases}$

321. $f(t) = \sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n+1}}{[(2n+1)!]^2}$.

322. $f(t) = \sum_{n=0}^{\infty} (-1)^n \dfrac{t^{2n}}{(n+1)(2n!)}$.

323. $f(t) = 1 - e^{2t} + e^{3t}$.

324. $f(t) = -\frac{1}{6} + e^{t} - \frac{3}{2} e^{2t} + \frac{2}{3} e^{3t}$.

325. $f(t) = \sum_{n=1}^{\infty} \dfrac{t^{4n}}{(4n!)} (-1)^{n+1}$.

326. $f(t) = \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos 2t$.

327. $f(t) = -\frac{1}{3} e^{t} + \frac{1}{4} e^{2t} + \frac{1}{12} e^{-2t}$.

328. $f(t) = e^{t}\{\cos 2t + \frac{1}{2}\sin 2t\}$.

329. $f(t) = -\frac{1}{15} e^{-t} + \frac{1}{6} e^{2t} - \frac{1}{10}\cos 2t - \frac{1}{5}\sin 2t$.

330. $f(t) = 8 + 5t + t^2 + (3t - 8)e^{t}$.

331. $F(p) = \dfrac{2}{p(p^2+4)}$.

332. $F(p) = \dfrac{p(p^2+2p+3)}{(p-1)(p^2-2p+5)}$.

333. $F(p) = \dfrac{a(p^2-a^2-b^2)}{p[(p-a)^2+b^2][(p+a)^2+b^2]}$.

334. $F(p) = \dfrac{p(p^2-a^2+b^2)}{[(p-a)^2+b^2][(p+a)^2+b^2]}$.

335. $F(p) = \dfrac{2pb}{(p^2+b^2)^2}$.

336. $F(p) = \dfrac{b(p^2+a^2-b^2)}{[(p-a)^2+b^2][(p+a)^2]}$.

337. $F(p)=\frac{p-7}{(p+7)^2-49}$.

338. $F(p)=\frac{p^2+2\alpha^2}{p^2(p^2+4\alpha^2)}$.

339. $F(p)=\ln\frac{p}{p-1}$.

340. $F(p)=\operatorname{arctg}\frac{1}{p}$.

341. $f(t)=\frac{1}{2}(\operatorname{ch}t-\cos t)$.

342. $f(t)=\frac{1}{2}(\operatorname{ch}t+\cos t-2)$.

343. $f(t)=\frac{1}{2}(\operatorname{sh}t-\sin t)$.

344. $f(t)=\frac{1}{2}(e^t-\sin t-\cos t)$.

345. $f(t)=\frac{1}{2}(e^t+\sin t-\cos t)$.

346. $f(t)=\frac{1}{2}(e^t+\cos t-\sin t-2)$.

347. $f(t)=\frac{1}{2}(t\cos t+\sin t)$.

348. $f(t)=e^{-t}(1-\cos t)$.

349. $f(t)=e^{-t}(\sin t+\cos t-1)$.

350. $f(t)=\frac{1}{2}e^{-t}(\cos t-\sin t-2)+\frac{1}{2}$.

351. $x(t)=e^{-2t}-e^{-3t}$.

352. $x(t)=\sin t$.

353. $x(t)=\frac{t^2-2}{4}\cdot e^{-3t}$.

354. $x(t)=t^2$.

355. $x(t)=1-4te^{-2t}$.

356. $x(t)=(t+1)\sin t-\cos t$.

357. $x(t)=t^2-3t+4$.

358. $x(t)=e^t+\cos t-\sin t$.

359. $x(t)=e^{2t}[(1-t)\cos t+(1+t)\sin t]$.

360. $x(t)=-\frac{t}{4}$.

361. $\begin{cases}x(t)=4e^{-2t}-3e^{-3t},\\ y(t)=3e^{-3t}-2e^{-2t}.\end{cases}$

362. $\begin{cases}x(t)=-\frac{5}{4}+\frac{13}{4}\cos 2t-3\sin 2t,\\ y(t)=\frac{3}{2}t+3\cos 2t+\frac{13}{4}\sin 2t.\end{cases}$

363. $\begin{cases}x(t)=\frac{1}{2}(\sin t+\cos t),\\ y(t)=\frac{1}{2}(\sin t-\cos t).\end{cases}$

364. $\begin{cases}x(t)=1+3e^{2t}+e^{-2t},\\ y(t)=e^{2t}-e^{-2t},\\ z(t)=2e^{2t}+2e^{-2t}.\end{cases}$

365. $\begin{cases}x(t)=2(1-e^{-t}-te^{-t}),\\ y(t)=2t-2e^{-t}-2te^{-t}.\end{cases}$

366. $\begin{cases}x(t)=t-\frac{t^3}{6}+e^t,\\ y(t)=1+\frac{1}{24}t^4-e^t.\end{cases}$

367. $\begin{cases}x(t)=e^t(2\cos t-\sin t),\\ y(t)=e^t(3\cos t+\sin t).\end{cases}$

368. $\begin{cases}x(t)=12(\operatorname{ch}t-1)-\frac{7}{2}t\cdot\operatorname{sh}t,\\ y(t)=7t\cdot\operatorname{sh}t-17(\operatorname{ch}t-1).\end{cases}$

369. $\begin{cases} x(t) = 3 - 2e^{-t}, \\ y(t) = e^{-t}, \\ z(t) = e^{-t} - 3. \end{cases}$

370. $\begin{cases} x(t) = \cos t + e^{-\sqrt{3}t}, \\ y(t) = \frac{1}{2}(\cos t - e^{-\sqrt{3}t}). \end{cases}$

371. $\frac{\partial^2 z}{\partial \eta^2} = 0$, $\xi = \frac{y}{x}$, $\eta = y$.

372. $\frac{\partial^2 z}{\partial \xi \cdot \partial \eta} - \frac{\partial z}{\partial \xi} = 0$, $\xi = x + y$, $\eta = 3x + y$.

373. $\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{1}{2}\left(\frac{1}{\xi} \cdot \frac{\partial z}{\partial \xi} + \frac{1}{\eta} \cdot \frac{\partial z}{\partial \eta}\right) = 0$,
$\xi = y^2$, $\eta = x^2$.

374. $\frac{\partial^2 z}{\partial \xi \cdot \partial \eta} + \frac{1}{2}\frac{\partial z}{\partial \xi} = 0$, $\xi = x + y$, $\eta = 3x - y$.

375. $\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \eta} = 0$, $\xi = 2x - y$, $\eta = x$.

376. $\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{1}{\xi - \eta} \cdot \frac{\partial z}{\partial \xi} + \frac{1}{2\eta} \cdot \frac{\partial z}{\partial \eta} = 0$,
$\xi = x^2$, $\eta = x^2$.

377. $U(x,t) = 2x - x^2 - a^2t^2 + \frac{1}{2a}e^{-x}\text{sh}at$.

378. $U(x,t) = \cos x \cos at - \frac{1}{2}\sin x \sin at$.

379. $U(x,t) = e^{-x}chat + \frac{t}{2} - \frac{1}{4a}\cos 2x \sin 2at$.

380. $U(x,t) = 2x - x^2 - a^2t^2 + \frac{1}{2}e^{x}\text{sh}at$.

381. $U(x,t) = e^{x}\text{ch}at + 4xt$.

382. $U(x,t) = \cos x \cos at + \frac{t}{2} +$
$+ \frac{1}{4a}\cos 2x \sin 2at$.

383. $U(x,t) = \cos x \sin at + 8xt(x^2 + a^2t^2)$.

384. $U(x,t) = \frac{1}{2} - \frac{1}{2}\cos 2x \cos 2at +$
$+ \frac{1}{a}\cos x \sin at$.

385. $U(x,t) = e^{2x}\text{ch}2at + xt(x^2 + a^2t^2)$.

386. $x \cdot t$.

387. $x(1 - t)$.

388. $u = \frac{\pi}{2a}$.

389. $\frac{1}{a}\cos x \sin at$.

390. $u = -\sin x$.

391. $x^2 + t^2 + \sin x \sin t$.

392. $\cos x \cos at + \frac{1}{a}\sin x \sin at$.

393. $\sin x \cos at$.

394. $u(x,t) = \frac{8h}{\pi^2} \cdot \sum_{k=1}^{\infty}\frac{1}{k^2} \cdot \sin\frac{k\pi}{2} \times$
$\times \sin\frac{k\pi x}{l} \cdot \cos\frac{k\pi at}{l}$.

395. $u(x,t) = \frac{4l^2h}{\pi^2 a} \cdot \sum_{k=1}^{\infty}\frac{1}{k^2} \cdot \frac{\sin\frac{k\pi}{2}\cos\frac{k\pi h}{l}}{l^2 - k^2h^2} \times$
$\times \sin\frac{k\pi x}{l} \cdot \sin\frac{k\pi at}{l}$.

396. $u(x,t) = \frac{96h}{\pi^5} \cdot \sum_{k=0}^{\infty}\frac{1}{(2k+1)^5} \times$
$\times \cos(2k+1)\pi at \cdot \sin(2k+1)\pi x$.

397. $u(x,t) = -\frac{0{,}9}{\pi^2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin\frac{2k\pi}{3} \times$
$\times \sin\frac{k\pi x}{3} \cdot \cos\frac{k\pi a t}{3}.$

398. $u(x,t) = \frac{4v_0 l}{\pi^2 u} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin\frac{k\pi}{2} \times$
$\times \sin\frac{k\pi h}{2l} \cdot \sin\frac{k\pi a t}{l} \cdot \sin\frac{k\pi k}{l}.$

399. $u(x,t) = \frac{8c}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} e^{-\frac{(2n+1)^2\pi^2 a^2 t}{l^2}} \times$
$\times \sin\frac{(2n+1)\pi x}{l}.$

400. $u(x,t) = \pi + \sum_{k=0}^{\infty} \frac{32}{\pi(2k+1)^2(2k-1)(2k+3)} \times$
$\times e^{-\frac{a^2(2k+1)^2\pi^2 t}{l^2}} \cdot \cos\frac{(2n+1)\pi x}{l}.$

401. $u(x,t) = \frac{u_0}{2} - \frac{4u_0}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\frac{2(2n+1)}{l}\pi x}{(2n+1)^2} \times$
$\times e^{-\frac{2(2n+1)^2\pi^2 a^2 t}{l^2}}.$

402. $u(x,t) = (1+t)^{-\frac{1}{2}} e^{\frac{2x - x^2 + t}{1+t}}.$

403. $u(x,t) = x \cdot (1+4t)^{-\frac{3}{2}} e^{-\frac{x^2}{1+4t}}.$

404. $u(x,t) = (1+t)^{-\frac{1}{2}} \sin\frac{x}{1+t} e^{-\frac{4x^2+t}{4(1+t)}}.$

405. $u(r,\varphi) = \frac{1}{2}\left(1 + r^2\cos^2\varphi\right)$

406. $u(r,\varphi) = \frac{r}{4}\left(3\sin\varphi - r^2\sin 3\varphi\right)$

407. $u(r,\varphi) = \frac{3}{8} + \frac{r^2}{2}\cos^2\varphi + \frac{r^4}{8}\cos 4\varphi.$

408. $u(r,\varphi) = \frac{5}{8} + \frac{3}{8} r^4 \cos 4\varphi.$

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MUNDARIJA

III BOB. DIFFERENSIAL TENGLAMALAR SISTEMASI

YUSUPJON PULATOVICH OPPOQOV,
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ILGOR AXMEDJONOVICH GAFAROV

ODDIY DIFFERENSIAL TENGLAMALARDAN MISOL VA MASALALAR TO'PLAMI

Oliy texnika o'quv yurtlari talabalari uchun o'quv qo'llanma

Toshkent – «Voris-nashriyot» – 2009

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