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OLIY VA O‘RTAMAXSUS TA’LIM VAZIRLIGI

ISLOM KARIMOV NOMIDAGI

TOSHKENT DAVLAT TEXNIKA UNIVERSITETI

OLIY MATEMATIKA

Matematik fizika tenglamalaridan misol va masalalar yechish

Uslubiy qo‘llanma

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Ushbu uslubiy qoʻllanma ishlab chiqarish texnik sohasi bilim sohasi taʼlim yoʻnalishlari talabalari uchun moʻljallangan boʻlib, “Oliy matematika” fanining maxsus boʻlimlaridan biri boʻlgan “Matematik fizika tenglamalari” (Xususiy xosilali differensial tenglamalar),boʻlimi boʻyicha tayyorlangan. Uslubiy qoʻllanmada asosiy nazariy tushunchalar, ularga doir namunaviy mashqlar bajarib koʻrsatilgan, mustaqil bajarish uchun mashqlar berilgan.Talabalar oʻz bilimlarini tekshirib koʻrishlari uchun mavzuga oid oʻz-oʻzini tekshirish savollari keltirilgan.Uslubiy qoʻllanmadan talabalar, magistrantlar va yosh oʻqituvchilar ham foydalanishlari mumkin.

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Mazkur uslubiy qo‘llanma “Ishlab chiqarish texnik soha” bilim sohasi ta’lim yo‘nalishlari uchun 2018 yil 26 avgustda tasdiqlangan “Oliy matematika” fanining o‘quv dasturi asosida tayyorlangan. Xususiy hosilali differensial tenglamalar to‘g‘risida umumish tushunchalar, matematik fizikaning asosiy tenglamalari, ikkinchi tartibli ikki o‘zgaruvchili differensial tenglamalarni kanonik ko‘rinishga keltirish usullari misolar bilan batafsil keltirilgan. Koshi masalasi, tor tebranish tenglamasini keltirib chiqarish va uni yechish usullari bayon qilingan. Xususiy hosilali birinchi tartibli differensial tenglamalar va ularni yechish usullari hamda ikkinchi tartibli xususiy hosilali differensial tenglamalar va ularning turlari hamda kanonik ko‘rinishga keltirish usullari ustida to‘liq to‘xtalib o‘tilgan. Tor tebranish tenglamasini yechishning Dalamber va Furye usullari batafsil yoritib berilgan. Talabalarga Garmonik funksiya va Grin formulasi hamda uning tatbiqlari haqida ma’lumotlar berilgan . Chegaralangan va chegaralanmagan sterjen uchun issiqlik tarqalish tenglamalari va ularni yechish usullari keltirilgan. Har bir mavzuda dastlab mavzuning nazariy qismi yoritilib, so‘ngra bir neshta misol va masalalar yechib ko‘rsatilgan.Ushbu uslubiy qo‘llanmaga talabalarga yanada qulaylik yaratish maqsadida uslubiy qo‘llanmaning har bir bo‘limida mustaqil yechish uchun mashqlar va o‘z-o‘zini tekshirish maqsadida nazariy savollar keltirilgan. Qo‘llanmadan matematik fizika tenglamalari kursini mustaqil o‘rganuvchilar ham foydalanishlari mumkin.

1. Xususiy hosilali differensial tenglamalar

1.1. Umumiy tushunchalar

Biz faqat bitta erkli o'zgaruvchiga bog'liq bo'lgan noma'lum funksiya hamda uning hosilalariga bog'liq bo'lgan tenglamalar, ya'ni oddiy differensial tenglamalar nazariyasi bilan tanishgan edik.

Shuni ta'kidlash lozimki, fan va texnika masalalari, umuman tabiatda bo'ladigan barcha jarayonlar ko'p o'zgaruvchili noma'lum funksiya va uning xususiy hosilalariga bog'liq bo'lgan tenglama-xususiy hosilali differensial tenglamani yechishga keladi.

Xususiy hosilali tenglamada qatnashayotgan eng katta xususiy hosilaning tartibiga shu tenglamaning **tartibi** deyiladi.

Xususiy hosilali tenglamaning yechimi deb, shunday funksiyaga aytamizki, uni va barcha xususiy hosilalarini tenglamaga qo'yganda u ayniyatga aylansa.

1-misol. $x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = 0$ (1) tenglamani qaraymiz. Bunda $u = u(x, y)$ izlanayotgan funksiya. Butun $0xy$ tekislikda aniqlangan $u = x^2 + y^2$ funksiya shu tenglamaning yechimi ekanligini ko'rsatamiz.

Haqiqatan, bu funksiya $\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = 2y$ xususiy hosilalarga ega. (1) tenglamada $\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}$ xususiy hosilalarni uning qiymarlariga almashtirsak

$$x \cdot 2y - y \cdot 2x \equiv 0$$

ayniyatga ega bo'lmiz. Demak $u = x^2 + y^2$ funksiya (1) tenglamaning yechimi.

$x^2 + y^2$ ning istalgan differensiallanuvchi $u = F(x^2 + y^2)$ funksiyasi ham berilgan tenglamaning yechimi bo'lishini tekshirib ko'rish qiyin emas.

Haqiqatan, $x^2 + y^2 = u$ deb olsak murakkab funksiyani xususiy hosilasini topish qoidasiga ko'ra

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial u} \cdot 2x,\ \frac{\partial u}{\partial y} = \frac{\partial F}{\partial u} \cdot 2y$$

bo'ladi.

Bularni (1)ga qo'yib

$$x \cdot \frac{\partial F}{\partial u} \cdot 2y - y \cdot \frac{\partial F}{\partial u} \cdot 2x \equiv 0$$

ayniyatga ega bo'lamiz.

Shunday qilib, xususiy hosilali differensial tenglamalar oddiy differensial tenglamalardan farqli nafaqat ixtiyoriy o'zgarmaslarga bog'liq, (masalan, $y'' + y = 0$ oddiy differensial tenglama C_1, C_2 ixtiyoriy o'zgarmasga bog'liq $y = C_1\cos x + C_2\sin x$ yechimlarga ega) balki ixtiyoriy differensiallanuvchi funksiyalarga bog'liq bo'lgan yechimlarga ega bo'lishi mumkin ekan. Odatda bu funksiyalarning soni tenglamaning tartibiga teng bo'ladi.

Agar tenglama noma'lum funksiya va uning barcha xususiy hosilalariga nisbatan chiziqli bo'lsa, u **chiziqli xususiy hosilali tenglama** deyiladi.

Birinchi tartibli chiziqli xususiy hosilali differensial tenglamaning umumiy ko‘rinishi quyidagicha bo‘ladi:

$$A_1\frac{\partial u}{\partial x_1}+A_2\frac{\partial u}{\partial x_2}+...,+A_n\frac{\partial u}{\partial x_n}+B=0\,. \qquad (1.1)$$

Bu yerdagi $A_1, A_2,..., A_n, B$ koefitsiyentlar n ta erkli o‘zgaruvchilar $x_1, x_2,..., x_n$ ga bog‘liq ma’lum funksiyalar, $u=u(x_1, x_2,..., x_n)$-noma’lum funksiya. $B\equiv 0$ bo‘lganda bu tenglama bir jinsli xususiy hosilali differensial tenglama, $B\neq 0$ bo‘lganda u bir jinsli bo‘lmagan differensial tenglama deyiladi.

Bir jinsli bo‘lmagan tenglamani yechimini $v(x_1, x_2,..., x_n, u)$ ko‘rinishida izlab

$$\frac{\partial v}{\partial x_1}+\frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial x_1}=0 \quad \text{va bundan} \quad \frac{\partial u}{\partial x_1}=-\frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial u}},\quad i=\overline{1,n}$$

ekanini e’tiborga olib xususiy hosilalarning qiymatlarini (1.1) ga qo‘yib hosil bo‘lgan tenglamani-$\frac{\partial v}{\partial u}$ ga ko‘paytirsak

$$A_1\frac{\partial v}{\partial x_1}+A_2\frac{\partial v}{\partial x_2}+...,+A_n\frac{\partial v}{\partial x_n}-B\frac{\partial v}{\partial u}=0$$

bir jinsli chiziqli tenglama hosil bo‘ladi.

Demak berilgan birinchi tartibli bir jinsli bo‘lmagan chiziqli xususiy hosilali differensial tenglamani bir jinsli tenglamaga keltirish mumkin ekan. Shuning uchun bundan keyin faqat bir jinsli tenglamalar qaraladi.

Agar $u(x_1, x_2,..., x_n)$ birinchi tartibli bir jinsli tenglamaning yechimi bo‘lsa

$$\frac{dx_1}{A_1}=\frac{dx_2}{A_2}=....=\frac{dx_n}{A_n} \qquad (1.2)$$

oddiy differensial tenglamalar sistemasining umumiy integrali $u(x_1, x_2,..., x_n)=C$ bo‘ladi va aksincha $u=C$ (1.2) sistemaning integrali bo‘lsa, u holda $u(x_1, x_2,..., x_n)$ birinchi tartibli bir jinsli chziqli tenglamaning yechimi bo‘ladi. Bu da’voni noma’lum funksiya ikkita erkli-x, y o‘zgaruvchilarning funksiyasi bo‘lgan hol uchun isbotlaymiz, ya’ni $u(x, y)$ funksiya

$$A(x,y)\frac{\partial u}{\partial x}+B(x,y)\frac{\partial u}{\partial y}=0 \qquad (1.3)$$

tenglamaning yechimi bo‘lganda $u(x, y)=C$

$$\frac{dx}{A}=\frac{dy}{B} \qquad (1.4)$$

tenglamaning integrali bo‘lishini va aksincha $u(x, y)=C$ (1.4) ning integrali bo‘lganda $u(x, y)$ funksiya (1.3) ning yechimi bo‘lishini ko‘rsatamiz.

Faraz qilaylik $u(x, y)=C$ (1.4) sistemaning integrali bo‘lsin. U holda bu funksiyaning to‘liq differensiali

$$du(x,y)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$$

bo‘ladi. (1.4) ga asoslanib, bu yerdagi dx, dy larni ularga proporsional A va B ga almashtiramiz, ya’ni $dx=\lambda A,\ dy=\lambda B$ (λ -proporsionallik koeffitsiyenti) desak

$$du(x,y)=\lambda(A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y})=0 \text{ yoki } A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}=0$$

bo‘ladi. Bu o‘z navbatida $u(x,y)$ funksiya (1.3) tenglamaning yechimi bo‘lishini bildiradi. Da’voning ikkinchi qismi, ya’ni $u(x,y)$ (1.3) tenglamaning yechimi bo‘lganda $u(x,y)=C$ (1.4) sistemaning integrali bo‘lishi ham shunga o‘xshash isbotlanadi.

2-misol. a) $u=\frac{x^2}{y^2}+\frac{y^2}{z^2}$, b) $u=xyz$, c) $u=\frac{y}{x}e^{\frac{xz}{y^2}}$ funksiyalar

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

tenglamaning $x>0,\ y>0,\ z>0$ sohadagi yechimi bo‘ladimi?

Yechilishi. a) $u(x,y,z)$ funksiyaning xususiy hosilalarini topamiz.

$$\frac{\partial u}{\partial x}=\frac{2x}{y^2},\quad \frac{\partial u}{\partial y}=-\frac{2x^2}{y^3}+\frac{2y}{z^2},\quad \frac{\partial u}{\partial z}=-\frac{2y^2}{z^3}$$

xususiy hosilalarning qiymatlarini berilgan tenglamaga qo‘ysak

$$x\frac{\partial u}{\partial x}+\ y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2x^2}{y^2}-\frac{2x^2}{y^2}+\frac{2y^2}{z^2}-\frac{2y^2}{z^2}\equiv 0$$

ya’ni $u=\frac{x^2}{y^2}+\frac{y^2}{z^2}$ funksiyaning berilgan tenglamaning yechimi ekanligi kelib chiqadi.

b) $u=xyz,\ \frac{\partial u}{\partial x}=yz,\ \frac{\partial u}{\partial y}=xz,\ \frac{\partial u}{\partial z}=xy$ ga egamiz. Bu qiymatlarni berilgan tenglamaga qo‘yib qaralayotgan sohada $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz+xyz+xyz=3xyz\neq 0$ ga ega bo‘lamiz. Demak $u=xyz$ funksiya berilgan tenglamaning yechimi emas.

c) $u=\frac{y}{x}e^{\frac{xz}{y^2}}$ funksiyaning xususiy hosilalarini topamiz.

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}e^{\frac{xz}{y^2}}+\frac{y}{x}e^{\frac{xz}{y^2}}\cdot\frac{z}{y^2}=(-\frac{y}{x^2}+\frac{z}{xy})e^{\frac{xz}{y^2}},\quad \frac{\partial u}{\partial y}=(\frac{1}{x}-\frac{y}{x^2}\cdot\frac{2xz}{y^3})e^{\frac{xz}{y^2}}=(\frac{1}{x}-\frac{2z}{y^2})e^{\frac{xz}{y^2}},$$

$$\frac{\partial u}{\partial z}=\frac{y}{x}e^{\frac{xz}{y^2}}\cdot\frac{x}{y^2}=\frac{1}{y}e^{\frac{xz}{y^2}}.$$

Xususiy hosilalarning qiymatlarini berilgan tenglamaga qo‘ysak

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=e^{\frac{xz}{y^2}}(-\frac{y}{x}+\frac{z}{y}+\frac{y}{x}-\frac{27}{y}+\frac{z}{y})\equiv 0$$

bo‘ladi. Demak, $u=\frac{y}{x}e^{\frac{xz}{y^2}}$ funksiya berilgan tenglamaning yechimi.

3-misol. $f(x,y)$ uzluksiz differensiallanuvchi ixtiyoriy funksiya bo‘lganda $u(x,y)=x+y+f(xy)$ funksiya

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x-y$$

tenglamaning yechimi bo‘lishi isbotlansin.

Yechilishi. $u(x,y)$ ni $x$va y bo‘yicha differensiallab

$$\frac{\partial u}{\partial x}=1+y\cdot f'(xy),\quad \frac{\partial u}{\partial y}=1+x\cdot f'(xy)$$

larni topamiz. Bularni berilgan tenglamaga qo‘ysak

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=x+xy\cdot f'(xy)-y-yx\cdot f'(xy)=x-y$$

bo'ladi. Bu hosil qilingan funksiya berilgan tenglamaning yechimi ekanligini ko'rsatadi.

4-misol. $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$ Xususiy hosilali tenglamani yeching.

Yechilishi. (1.4) ga binoan $\frac{dx}{y} = -\frac{dy}{x}$ differensial tenglamaga ega bo'lamiz. $xdx = -ydy$ tenglamani integrallab $\frac{x^2}{2} + \frac{y^2}{2} = C_1$ yoki $x^2 + y^2 = C$ $(C = 2C_1)$ ga ega bo'lamiz.

Demak $u = x^2 + y^2$ funksiya berilgan tenglamaning yechimi.

Quyida asosiy e'tibor fizikaning muhim masalalari yuzaga keltirgan va "matematik fizika tenglamalari" nomini olgan ikkinchi tartibli chiziqli xususiy hosilali differensial tenglamalarni o'rganishga qaratilgan.

Fizik masalalarning ko'pchiligi ikkinchi tartibli chiziqli xususiy hosilali

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x, y) \qquad (1.5)$$

ko'rinishdagi differensial tenglamaga keltiriladi, bunda $A, B, C, D, E, F\ f$ koeffitsiyentlar x va y erkli o'zgaruvchilarning ma'lum uzluksiz funksiyalari, $u(x, y)$-noma'lum funksiya.

Agar tenglama koeffitsiyentlari erkli o'zgaruvchi x, y larga bog'liq bo'lmasa, u holda tenglama **o'zgarmas koeffitsiyentli tenglama** deyiladi. Agar (1.5) tenglamada $f(x, y) \equiv 0$ bo'lsa u **bir jinsli xususiy hosilali differensial tenglama** deyiladi.

(1.5) **xususiy hosilali differensial tenglamaning yechimi** deb, tenglamaga qo'yilganda uni ayniyatga aylantiradigan x va y ning ixtiyoriy $u(x, y)$ funksiyasiga aytiladi.

1.2. Matemarik fizikaning asosiy tenglamalari

I. To'lqin tenglamasi:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad (1.6)$$

Torning ko'ndalang tebranishi, sterjenning bo'ylama tebranishi, simdagi elektr tebranishlari, aylanuvchi silindrdagi aylanma tebranishlar, gazodinamika va akustikaning tebranish bilan bog'liq jarayonlarini tadqiq etish shunday tenglamaga olib keladi.

II. Issiqlikning tarqalish tenglamasi yoki Furye tenglamasi:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad (1.7)$$

Issiqlikning bir jinsli muhitda tarqalishi, diffuziya hodisalari, filtratsiya masalalari, ehtimolliklar nazariyasining ba'zi masalalari shunday tenglamaga keltirilib o'rganiladi.

III. Laplas tenglamasi:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad (1.8)$$

Elektr va magnit maydonlari haqidagi masalalarni, statsionar issiqlik holati haqidagi masalalarni, gidrodinamikaning siqilmaydigan suyuqlikning potensial harakati va statsionar issiqlik maydonlariga tegishli masalalarni yechish Laplas tenglamasiga keltirladi.

Agar izlanayotgan funksiya uchta erkli o'zgaruchilarga big'liq bo'lsa, to'lqin tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (1.6')$$

issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (1.7')$$

Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (1.8')$$

ko‘rinishda bo‘ladi.

Yuqoridagi tenglamalarning har biri cheksiz ko‘p xususiy yechimlarga ega. Biron bir aniq fizik masalalarni yechish vaqtida shu yechimlar ichidan aynan shu fizik mazmundan kelib chiqib qo‘yilgan qo‘shimcha talab (yoki shart)larni qanoatlantiruvchisini topish talab qilinadi. Ana shu qo‘shimcha shartlarga chegaraviy va boshlang‘ich shartlar deymiz. Har qanday tenglama uchun qo‘yilgan masala quyidagi uchta shartni bajarishi lozim:

1) yechim mavjud bo‘lishi kerak;
2) yechim yagona bo‘lishi kerak;
3) yechim turg‘un bo‘lishi, ya’ni masalada berilganlarning kichik o‘zgarishi yechimning ham kichik o‘zgarishini ta’minlashi kerak.

Mana shu uchta talabni bajargan masalaga **korrekt** qo‘yilgan masala deyiladi.

1.3. Ikkinchi tartibli ikki o‘zgaruvchili differensial tenglamalarning turlari va kanonik ko‘rinishlari

(1.5) tenglamani qaraymiz. Bu tenglamaning teskarisi $x = x(\xi,\eta),\ y = y(\xi,\eta)$ almashtirishga ega bo‘lgan

$$\xi = \xi(x,y),\ \eta = \eta(x,y) \quad (1.9)$$

almashtirish yordamida (1.5) tenglamani ya’ni ξ va η o‘zgaruvchilarga nisbatan soddaroq tenglamaga keltirish mumkin, bunda $\xi(x,y),\ \eta(x,y)$ funksiyalar almashtirish bajarilayotgan biror D sohada uzluksiz, ikki marta differensiallanuvchi.

Shu maqsadda ushbu xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x},\quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x^2} +$$

$$+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 u}{\partial \eta\partial \xi}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \xi}{\partial x} =$$

$$= \frac{\partial^2 u}{\partial \xi^2}\cdot\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \xi}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) = \frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y} +$$

$$+\frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial x}\cdot\frac{\partial \xi}{\partial y},$$

(1.10)

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}\cdot(\frac{\partial \xi}{\partial y})^2+2\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}(\frac{\partial \eta}{\partial y})^2+\frac{\partial u}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2}.$$

Xususiy hosilalarning topilgan qiymatlarini (1.5) tenglamaga qo‘yib, quyidagi tenglamani hosil qilamiz:

$$a_{11}\frac{\partial^2 u}{\partial \xi^2}+2a_{12}\frac{\partial^2 u}{\partial \xi \partial \eta}+a_{13}\frac{\partial^2 u}{\partial \eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right)=0 \tag{1.11}$$

bu yerda

$$a_{11}(\xi,\eta)=A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2, \tag{1.12}$$

$$a_{13}(\xi,\eta)=A(\cdot\frac{\partial \eta}{\partial x})^2+2B\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+C(\frac{\partial \eta}{\partial y})^2,$$

$$a_{12}(\xi,\eta)=A\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right)+C\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y}.$$

(1.12) da $\xi(x,y)$, $\eta(x,y)$ funksiyalarni shunday tanlaymizki, natijada a_{11}, a_{13} koeffitsiyentlar nolga aylansin, ya’ni

$$A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2=0, \qquad (\alpha)$$

$$A\left(\frac{\partial \eta}{\partial x}\right)^2+2B\frac{\partial \eta}{\partial x}\cdot\frac{\partial \eta}{\partial y}+C\left(\frac{\partial \eta}{\partial y}\right)^2=0 \tag{1.13}$$

bo‘lsin.

(α) tenglamani $(\frac{\partial \xi}{\partial y})^2$ ga bo‘lsak

$$A\left(\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}\right)^2+2B\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}+C=0$$

kvadrat tenglama hosil bo‘ladi. Bu tenglamani $\dfrac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}$ ga nisbatan yechsak,

$$\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}=\frac{-B\pm\sqrt{B^2-AC}}{A} \tag{1.14}$$

kelib chiqadi.

Bundan kvadrat uchhadni chiziqli ko‘paytuvchilarga ajratish qoidasiga asoslanib

$$A\left(\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}-\frac{-B+\sqrt{B^2-AC}}{A}\right)\cdot\left(\frac{\frac{\partial \xi}{\partial x}}{\frac{\partial \xi}{\partial y}}-\frac{-B-\sqrt{B^2-AC}}{A}\right)=0 \text{ yoki}$$

$$A\frac{\partial \xi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial \xi}{\partial y}=0,\quad A\frac{\partial \xi}{\partial x}-(B+\sqrt{B^2-AC})\frac{\partial \xi}{\partial y}=0$$

tengliklarni hosil qilamiz. Demak (1.13)ning har bir tenglamasi xususiy hosilali birinchi tartibli $A\frac{\partial\xi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial\xi}{\partial y}=0,\ A\frac{\partial\eta}{\partial x}-(B+\sqrt{B^2-AC})\frac{\partial\eta}{\partial y}=0$ chiziqli tenglamalarga ajraladi. Bu tenglamalar 1.1 mavzuda ko'rganimizdek

$$\frac{dx}{A}=\frac{dy}{B-\sqrt{B^2-AC}} \text{ va } \frac{dx}{A}=\frac{dy}{B+\sqrt{B^2-AC}}$$

yoki $Ady-(B-\sqrt{B^2-AC})dx=0,\ \ Ady-(B+\sqrt{B^2-AC})dx=0$ (1.15)

differensial tenglamalarga teng kuchli.

Bu yerdagi ikkita tenglamalarni

$$Ady^2-2Bdxdy+Cdx^2=0 \quad (1.16)$$

yoki buni dx^2 ga bo'lib uni $A(\frac{dy}{dx})^2-2B\frac{dy}{dx}+C=0$ (1.16')

ko'rinishdagi bitta tenglama shaklida yozish mumkin.

(1.15) ning umumiy integrallari $\varphi(x,y)=C_1,\ \psi(x,y)=C_2$ bo'lsin. Bu holda ular (1.5) tenglamaning ikkita egri chiziqlar oilasini tashkil qilib, **tenglamaning xarakteristikalari** deyiladi. (1.16) tenglama esa (1.5) tenglama **xarakteristikalarining differensial tenglamasi deyiladi.**

Shuningdek, (1.16) tenglama (1.5) tenglamaning **xarakteristik** tenglamasi, uning umumiy integrali esa **xarakteristika** deb ham yuritiladi.

Quyidagi lemmani isbotlaymiz.

74.1-lemma. Agar $z=\varphi(x,y)$ funksiya

$$A\left(\frac{\partial z}{\partial x}\right)^2+2B\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}+C\left(\frac{\partial z}{\partial y}\right)^2=0 \quad (1.17)$$

tenglamaning xususiy yechimlaridan biri bo'lsa, $\varphi(x,y)=C$ ifoda

$$A\left(\frac{dy}{dx}\right)^2-2B\frac{dy}{dx}+C=0$$

xarakteristik tenglamaning umumiy integrali bo'ladi va aksincha $\varphi(x,y)=C$ (1.16') xarakteristik tenglamaning umumiy integrali bo'lganda $z=\varphi(x,y)$ funksiya (1.17) tenglamaning xususiy yechimi bo'ladi.

Isboti. Ayniyatlik $z=\varphi(x,y)$ (1.17) tenglamaning yechimi bo'lsin. U holda (1.17) tenglamani $\left(\frac{\partial z}{\partial y}\right)^2$ ga bo'lib lemmaning shartiga ko'ra $\varphi(x,y)$ hosil bo'lgan tenglamaning yechimi ekanligini hisobga olsak

$$A\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2-2B\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)+C=0 \quad (1.18)$$

bo'ladi. $\varphi(x,y)=C$ tenglamani oshkormas shaklda berilgan funksiyaning tenglamasi deb qarasak

$$\frac{dy}{dx}=-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}$$

bo'lishi ravshan. Buni (1.16') ga qo'ysak (1.18) ga asosan

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = A\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2 - 2B\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right) + C = 0$$

bo‘ladi. Bu bilan lemmaning birinchi qismi ya’ni $\varphi(x, y)$ (74.17) tenglamaning yechimi bo‘lganda $\varphi(x, y) = C$ (1.16') tenglamaning umumiy integrali bo‘lishi isbotlandi.

Lemmaning ikkinchi qismini isbotlashni o‘quvchilarga qoldiramiz.

(1.16') tenglamani kvadrat tenglamani yechgandek yechib

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A} \quad \text{va} \quad \frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A}$$

tenglamalarni hosil qilamiz.

(1.15) tenglamalarning integrallarini qanday bo‘lishi va (1.5) tenglamaning sodda ko‘rinishga keltirilishi shu tenglamaning ikkinchi tartibli xususiy hosilalari oldidagi koeffitsiyentlardan tuzilgan $\Delta = B^2 - AC$ diskriminantning ishorasiga bog‘liq bo‘ladi.

Δ diskriminantning ishorasiga bog‘liq holda (1.5) tenglamani quyidagi turlarga ajratish mumkin:

1) agar D sohada $\Delta>0$ bo‘lsa, berilgan tenglama shu sohada **giperbolik** turdagi tenglama deyiladi;
2) agar D sohada $\Delta<0$ bo‘lsa, berilgan tenglama shu sohada **elliptik** turdagi tenglama deyiladi;
3) agar D sohada $\Delta=0$ bo‘lsa, berilgan tenglama shu sohada **parabolik** turdagi tenglama deyiladi;

Almashtirishning yakobiani $\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\cdot\frac{\partial\eta}{\partial x} \neq 0$ bo‘lganda $\xi = \xi(x, y),\ \eta = \eta(x, y)$ almashtirish natijasida tenglamaning turi o‘zgarmaydi. Bitta tenglama sohaning turli qismlarida turli turga mansub bo‘lishi mumkin, masalan.

$$\left(1 - x^2\right)\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\partial y} - \left(1 + y^2\right)\frac{\partial^2 u}{\partial y^2} - 2x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

tenglama uchun $B^2 - AC = (-xy)^2 + \left(1 - x^2\right)\left(1 + y^2\right) = 1 - x^2 + y^2$.

Shuning uchun bu tenglama $1 - x^2 + y^2 > 0$ sohada giperbolik turdagi, $x^2 - y^2 = 1$ chiziq ustida parabolik turdagi, $1 - x^2 + y^2 < 0$ sohada elliptik turdagi, ya‘ni $x^2 - y^2 = 1$ giperbola ustida parabolik, giperbola ichida giperbolik, giperboladan tashqarida elliptik bo‘ladi.

Barcha nuqtalarida (1.5) tenglama bir xil turga mansub bo‘lgan sohani qaraymiz. Bu sohaning har bir nuqtasi orqali giperbolik turdagi tenglamaning ikkita har xil haqiqiy xarakteristikalari, elliptik turdagi tenglamaning ikkita har xil o‘zaro qo‘shma kompleks xarakteristikalari va parabolik turdagi tenglamaning bitta haqiqiy xarakteristikasi o‘tadi.

Keltirilgan turlarga mansub har bir tenglamalarning o‘zlariga xos kanonik ko‘rinishlari mavjud. Ularni keltirib chiqarish uchun Δ diskriminantning ishorasiga bog‘liq holda (1.5) soddalashtiriladi.

1. Giperbolik turdagi tenglamalarning kanonik shakli

Giperbolik turdagi tenglamalar uchun $\Delta = B^2 - AC > 0$ bo‘lib (1.13) ga asosan (1.12) dan $a_{11} = 0,\ a_{13} = 0,\ a_{12} \neq 0$ ekani kelib chiqadi. Shuning uchun (1.11) ni $2a_{12}$ ga bo‘lib

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = -\frac{F}{2\overline{a}_{12}} \text{ yoki } \frac{\partial^2 u}{\partial\xi\partial\eta} = \overline{F}(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}) \qquad (1.19)$$

tenglamani hosil qilamiz. Bu yerda $\overline{F} = -\dfrac{F}{2a_{12}}$.

(1.19) giperbolik turdagi tenglamaning **kanonik ko‘rinishi** deyiladi.

Agar (1.19) da $\xi = \alpha + \beta,\ \eta = \alpha - \beta$ almashtirish bajarsak, giperbolik turdagi tenglamaning ikkinchi sodda ko‘rinishiga ega bo‘lamiz. Bu holda $\alpha = \dfrac{\xi + \eta}{2},\ \beta = \dfrac{\xi - \eta}{2}$ bo‘lib giperbolik turdagi tenglamaning kanonik shakli

$$\frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} = F_1(\alpha,\beta,u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\beta}) \qquad (1.20)$$

ko‘rinishni oladi.

5-misol. Ushbu $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} = 0$ tenglama kanonik ko‘rinishga keltirilsin.

Yechilishi. $A = x^2,\ B = 0,\ C = -y^2,\ \Delta = B^2 - AC = x^2y^2 > 0$. Demak, bu giperbolik tenglama ($0xy$ tekislikning koordinata o‘qlarida yotmagan barcha nuqtalarda).

Xarakteristik tenglamani tuzamiz:

$$x^2dy^2 - y^2dx^2 = 0 \text{ yoki } (xdy + ydx)(xdy - ydx) = 0.$$

Bu $\left.\begin{array}{l} xdy + ydx = 0, \\ xdy - ydx = 0 \end{array}\right\}$ differensial tenglamalarga ajraladi.

Bularni o‘zgaruvchilarini ajratib integrallaymiz:

$$\frac{dy}{y} + \frac{dx}{x} = 0,\ \frac{dy}{y} - \frac{dx}{x} = 0,\ \left.\begin{array}{l} \ln y + \ln x = \overline{C}_1, \\ \ln y - \ln x = \overline{C}_2 \end{array}\right\} \quad xy = C_1,\ \frac{y}{x} = C_2.$$

Berilgan tenglamani kanonik ko‘rinishga keltirish uchun $\xi = xy$ va $\eta = \dfrac{y}{x}$ yangi o‘zgaruvchilarni kiritamiz.

U holda $\dfrac{\partial\xi}{\partial x} = y,\ \dfrac{\partial\xi}{\partial y} = x,\ \dfrac{\partial\eta}{\partial x} = -\dfrac{y}{x^2},\ \dfrac{\partial\eta}{\partial y} = \dfrac{1}{x}$ bo‘lishi ravshan.

Murakkab funksiyaning xususiy hosilalarini topish qoidasiga asoslanib topamiz:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial u}{\partial\xi}y - \frac{\partial u}{\partial\eta}\frac{y}{x^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} = \frac{\partial u}{\partial\xi}x + \frac{\partial u}{\partial\eta}\cdot\frac{1}{x},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial\xi}\cdot y - \frac{\partial u}{\partial\eta}\cdot\frac{y}{x^2}) = \frac{\partial}{\partial x}(\frac{\partial u}{\partial\xi}\cdot y) - \frac{\partial}{\partial x}(\frac{\partial u}{\partial\eta}\cdot\frac{y}{x^2}) =$$

$$= (\frac{ä^2è}{ä\xi^2}\cdot\frac{ä\xi}{äx} + \frac{ä^2è}{ä\xiä\eta}\cdot\frac{ä\eta}{äx})y - \frac{ä}{äx}(\frac{äè}{ä\eta})\cdot\frac{y}{x^2} - \frac{äè}{ä\eta}\cdot\frac{ä}{äx}(\frac{y}{x^2}) =$$

$$= \frac{\partial^2 u}{\partial\xi^2}\cdot y^2 + \frac{\partial^2 u}{\partial\xi\partial\eta}\cdot y\cdot(-\frac{y}{x^2}) - (\frac{\partial^2 u}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x} + \frac{\partial^2 u}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x})\cdot\frac{y}{x^2} - \frac{\partial u}{\partial\eta}(-2\cdot\frac{y}{x^3}) =$$

$$= \frac{\partial^2 u}{\partial \xi^2} y^2 - \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{y^2}{x^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{y^2}{x^2} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{y^2}{x^4} + 2\frac{\partial u}{\partial \eta} \cdot \frac{y}{x^3} =$$

$$= \frac{\partial^2 u}{\partial \xi^2} y^2 - 2\frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{y^2}{x^2} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{y^2}{x^4} + 2\frac{\partial u}{\partial \eta} \cdot \frac{y}{x^3},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi} \cdot x + \frac{\partial u}{\partial \eta} \cdot \frac{1}{x}) = \frac{\partial}{\partial y}(\frac{\partial u}{\partial \xi})x + \frac{\partial}{\partial y}(\frac{\partial u}{\partial \eta}) \cdot \frac{1}{x} =$$

$$= (\frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y})x + (\frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial y})\frac{1}{x} =$$

$$= \frac{\partial^2 u}{\partial \xi^2} \cdot x^2 + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{1}{x^2} = \frac{\partial^2 u}{\partial \xi^2} x^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{1}{x^2}.$$

Xususiy hosilalarning ushbu qiymatlarini berilgan tenglamaga qo‘yib uni soddalashtiramiz.

$$x^2(\frac{\partial^2 u}{\partial \xi^2} \cdot y^2 - 2\frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{y^2}{x^2} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{y^2}{x^4} + 2\frac{\partial u}{\partial \eta} \cdot \frac{y}{x^3}) - y^2(\frac{\partial^2 u}{\partial \xi^2} \cdot x^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \cdot \frac{1}{x^2}) = 0,$$

$-4y^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + 2\frac{\partial u}{\partial \eta} \cdot \frac{y}{x} = 0$ yoki $-4y^2$ bo‘lsak

$$\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{1}{2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{1}{xy} = 0 \quad \text{yoki} \quad \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{1}{2\xi} \cdot \frac{\partial u}{\partial \eta} = 0$$

Bu berilgan tenglamaning kanonik ko‘rinishidir.

2. Parobolik turdagi tenglamaning kanonik shakli.

Parabolik turdagi tenglama uchun $B^2 - A \cdot C = 0$ bo‘lgani uchun (1.15) tenglamalar bitta tenglamani ifodalaydi va (1.16) xarakteristik tenglama yagona $\varphi(x, y) = C$ umumiy integralga ega bo‘ladi.

Bu holda $\xi = \varphi(x, y)$ va $\eta = \eta(x, y)$ deb olamiz, bu yerdagi $\eta(x, y)$ φ ga bog‘liq bo‘lmagan ikki marta differensiallanuvchi ixtiyoriy funksiya. Qaralayotgan holda $B = \sqrt{A} \cdot \sqrt{B}$ ekanini hisobga olsak

$$a_{11} = A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = \left(\sqrt{A}\frac{\partial \xi}{\partial x} + \sqrt{C}\frac{\partial \xi}{\partial y}\right)^2 = 0$$

va bunga asosan

$$a_{12} = A\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} =$$

$$= \left(\sqrt{A}\frac{\partial \xi}{\partial x} + \sqrt{C}\frac{\partial \xi}{\partial y}\right)\left(\sqrt{A}\frac{\partial \eta}{\partial x} + \sqrt{C}\frac{\partial \eta}{\partial y}\right) = 0$$

bo‘ladi. Shunday qilib, (1.11) tenglamada $\frac{\partial^2 u}{\partial \xi^2}$ va $\frac{\partial^2 u}{\partial \xi \partial \eta}$ ning oldidagi koeffitsiyentlar nolga teng bo‘lib, ikkinchi tartibli hosilalardan $\frac{\partial^2 u}{\partial \eta^2}$ qoladi, butun tenglamani uning oldidagi koeffitsiyentga a_{13} ga qisqartirish natijasida tenglama

$$\frac{\partial^2 u}{\partial \eta^2} = F_2(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})\left(F_2 = -\frac{F}{\overline{a}_{22}}\right)$$

ko‘rinishga ega bo‘ladi. Bu tenglama parabolik turdagi tenglamaning **kanonik** shaklidir.

6-misol. $\frac{\partial^2 u}{\partial x^2}\sin^2 x - 2y\sin x\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ tenglamani kanonik ko‘rinishga keltiring.

Yechilishi. $A = \sin^2 x,\ B = -y\sin x,\ C = y^2$ bo‘lgani uchun

$$\Delta = B^2 - A\cdot C = y^2\sin^2 x - y^2\sin^2 x = 0.$$

Demak, berilgan tenglama butun $0xy$ tekislikda parabolik turdagi tenglama.

$$\sin^2 xdy^2 + 2y\sin xdxdy + y^2dx^2 = 0 \text{ yoki } (\sin xdy + ydx)^2 = 0$$

berilgan tenglamaning xarakteristik tenglamasidir. $\sin xdy + ydx = 0$ tenglamani $\sin x\cdot y$ ga bo‘lsak $\frac{dy}{y} + \frac{dx}{\sin x} = 0$ o‘zgaruvchilari ajralgan tenglama hosil bo‘ladi. Uni integrallab

$$\ln y + \ln tg\frac{x}{2} = \ln C,\ \ln ytg\frac{x}{2} = \ln C,\ ytg\frac{x}{2} = C$$

xarakteristik tenglamaning umumiy integralini hosil qilamiz.

Berilgan tenglamani kanonik ko‘rinishga keltirish uchun $\xi = ytg\frac{x}{2},\ \eta = y$ almashtirish olib kerakli hosilalarni topamiz:

$$\frac{\partial\xi}{\partial x} = \frac{1}{2}\cdot y\cdot\frac{1}{\cos^2\frac{x}{2}},\quad \frac{\partial\xi}{\partial y} = tg\frac{x}{2},\quad \frac{\partial\eta}{\partial x} = 0,\quad \frac{\partial\eta}{\partial y} = 1,$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial u}{\partial\xi}\cdot y\cdot\frac{1}{\cos^2\frac{x}{2}} + \frac{\partial u}{\partial\eta}\cdot 0 = \frac{1}{2}\cdot\frac{\partial u}{\partial\xi}y\cdot\frac{1}{\cos^2\frac{x}{2}},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} = \frac{\partial u}{\partial\xi}\cdot tg\frac{x}{2} + \frac{\partial u}{\partial\eta},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\cdot y\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\xi}\cdot\frac{1}{\cos^2\frac{x}{2}}\right) = \frac{1}{2}\cdot y\cdot\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\xi}\right)\cdot\frac{1}{\cos^2\frac{x}{2}} + \frac{\partial u}{\partial\xi}\cdot\frac{\partial}{\partial x}\left(\frac{1}{\cos^2\frac{x}{2}}\right)\right) =$$

$$= \frac{1}{2}\cdot y\cdot\left(\frac{\partial^2 u}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}\cdot\frac{1}{\cos^2\frac{x}{2}} + \frac{\partial^2 u}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial x}\cdot\frac{1}{\cos^2\frac{x}{2}} + \frac{\partial u}{\partial\xi}\cdot(-2)\cdot\cos^{-3}\frac{x}{2}\cdot\left(-\sin\frac{x}{2}\right)\frac{1}{2}\right) =$$

$$= \frac{1}{2}y\left(\frac{\partial^2 u}{\partial\xi^2}\cdot\frac{1}{2}y\cdot\frac{1}{\cos^4\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}\frac{\partial u}{\partial\xi}\right) = \frac{1}{4}y^2\cdot\frac{1}{\cos^4\frac{x}{2}}\frac{\partial^2 u}{\partial\xi^2} + \frac{1}{2}y\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}\frac{\partial u}{\partial\xi}.$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial u}{\partial\xi}y\cdot\frac{1}{\cos^2\frac{x}{2}}\right) = \frac{1}{2\cos^2\frac{x}{2}}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial\xi}y\right) = \frac{1}{2\cos^2\frac{x}{2}}\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial\xi}\right)y + \frac{\partial u}{\partial\xi}\right] =$$

$$= \frac{1}{2\cos^2\frac{x}{2}}\left[\left(\frac{\partial^2 u}{\partial\xi^2}\frac{\partial\xi}{\partial y} + \frac{\partial^2 u}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial y}\right)y + \frac{\partial u}{\partial\xi}\right] = \frac{1}{2\cos^2\frac{x}{2}}\left[\left(\frac{\partial^2 u}{\partial\xi^2}\cdot tg\frac{x}{2} + \frac{\partial^2 u}{\partial\xi\partial\eta}\cdot 1\right)y + \frac{\partial u}{\partial\xi}\right] =$$

$$= \frac{y}{2\cos^2\frac{x}{2}}\cdot tg\frac{x}{2} + \frac{y}{2\cos^2\frac{x}{2}}\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{1}{2\cos^2\frac{x}{2}}\frac{\partial u}{\partial\xi}.$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}tg\frac{x}{2}+\frac{\partial u}{\partial \eta}\right)=tg\frac{x}{2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta}\right)=tg\frac{x}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial y}\right)+$$

$$+\frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y}=tg\frac{x}{2}\left(\frac{\partial^2 u}{\partial \xi^2}\cdot tg\frac{x}{2}+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot 1\right)+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot tg\frac{x}{2}+\frac{\partial^2 u}{\partial \eta^2}\cdot 1=$$

$$=tg^2\frac{x}{2}\frac{\partial^2 u}{\partial \xi^2}+2tg\frac{x}{2}\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}.$$

Topilgan xususiy hosilalarni berilgan tenglamaga qo‘yamiz.

$$\sin^2 x\cdot\left(\frac{1}{4}y^2\cdot\frac{1}{\cos^4\frac{x}{2}}\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{2}y\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}\frac{\partial u}{\partial \xi}\right)-2y\sin x\left(\frac{y}{2\cos^2\frac{x}{2}}tg\frac{x}{2}\frac{\partial^2 u}{\partial \xi^2}+\frac{y}{2\cos^2\frac{x}{2}}\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{2\cos^2\frac{x}{2}}\frac{\partial u}{\partial \xi}\right)+$$

$$y^2\left(tg^2\frac{x}{2}\frac{\partial^2 u}{\partial \xi^2}+2tg\frac{x}{2}\frac{\partial^2 u}{\partial \xi\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}\right)=0 \text{ yoki}$$

$$\left(\frac{1}{4}\sin^2\frac{x}{2}y^2\cdot\frac{1}{\cos^4\frac{x}{2}}-y^2\sin x\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot tg\frac{x}{2}+y^2\cdot tg^2\frac{x}{2}\right)\cdot\frac{\partial^2 u}{\partial \xi^2}-\left(y^2\frac{\sin x}{\cos^2\frac{x}{2}}-2y^2tg\frac{x}{2}\right)\frac{\partial^2 u}{\partial \xi\partial \eta}+y^2\frac{\partial^2 u}{\partial \eta^2}+$$

$$\left(\frac{\sin^2 x}{2}y\cdot\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}-y\sin x\cdot\frac{1}{\cos^2\frac{x}{2}}\right)\frac{\partial u}{\partial \xi}=0$$

$$\left(\frac{1}{4}\cdot 4\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}\cdot\frac{1}{\cos^4\frac{x}{2}}\cdot y^2-y^2\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\cdot tg\frac{x}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}+y^2\cdot tg\frac{x}{2}\right)\cdot\frac{\partial^2 u}{\partial \xi^2}-$$

$$-\left(y^2\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}-2y^2\cdot tg\frac{x}{2}\right)\cdot\frac{\partial^2 u}{\partial \xi\,\partial \eta}+y^2\cdot\frac{\partial^2 u}{\partial \eta^2}+$$

$$+\sin x\left(\frac{1}{2}y\cdot 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot tg\frac{x}{2}-y\cdot\frac{1}{\cos^2\frac{x}{2}}\right)\cdot\frac{\partial u}{\partial \xi}=0$$

yoki

$$\left(y^2\cdot tg^2\frac{x}{2}-2y^2\cdot tg^2\frac{x}{2}+y^2\cdot tg^2\frac{x}{2}\right)\cdot\frac{\partial^2 u}{\partial \xi^2}-\left(2y^2\cdot tg\frac{x}{2}-2y^2\cdot tg\frac{x}{2}\right)\cdot\frac{\partial^2 u}{\partial \xi\partial \eta}+$$

$$+y^2\cdot\frac{\partial^2 u}{\partial \eta^2}+\sin x\cdot y\left(tg^2\frac{x}{2}-\frac{1}{\cos^2\frac{x}{2}}\right)\cdot\frac{\partial u}{\partial \xi}=0.$$

Shunday qilib

$$y^2\cdot\frac{\partial^2 u}{\partial \eta^2}-\sin x\cdot y\cdot\frac{\partial u}{\partial \xi}=0 \text{ yoki } y\cdot\frac{\partial^2 u}{\partial \eta^2}=\sin x\cdot\frac{\partial u}{\partial \xi}$$

tenglikka ega bo‘lamiz.

$$\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}},\quad tg\frac{x}{2}=\frac{\xi}{y}=\frac{\xi}{\eta},\quad y=\eta$$

ekanini hisobga olsak

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\sin x}{y} \cdot \frac{\partial u}{\partial \xi}$$

tenglama

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{2\xi}{\xi^2 + \eta^2} \cdot \frac{\partial u}{\partial \xi}$$

ko‘rinishni oladi. Bu tenglama berilgan tenglamaning kanonik shakli.

3. Elliptik turdagi tenglamaning kanonik shakli

Elliptik turdagi tenglamalar uchun $B^2 - AC < 0$ va (1.17), (1.18) tenglamalarning o‘ng tomonlari qo‘shma kompleks kattaliklar bo‘ladi.

Shuning uchun elliptik turdagi tenglamalar haqiqiy xarakteristikalarga ega bo‘lmaydi.

Kompleks yechimlar o‘zaro qo‘shma bo‘lib

$$C_1 = \varphi + i\psi, \quad C_2 = \varphi - i\psi \tag{1.21}$$

ko‘rinishda bo‘ladi; $\varphi(x,y)$, $\psi(x,y)$ funksiyalar haqiqiy funksiyalardir.

(1.21) ni (1.13) ga qo‘yib, haqiqiy va mavhum qismlarini ajratamiz:

$$A(\frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x})^2 + 2B(\frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x})(\frac{\partial \varphi}{\partial y} + i\frac{\partial \psi}{\partial y}) + C(\frac{\partial \varphi}{\partial y} + i\frac{\partial \psi}{\partial y})^2 = 0,$$

$$A(\frac{\partial \varphi}{\partial x})^2 - A(\frac{\partial \psi}{\partial x})^2 + 2B\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} - 2B\frac{\partial \varphi}{\partial y} \cdot \frac{\partial \psi}{\partial x} + C(\frac{\partial \varphi}{\partial y})^2 - C(\frac{\partial \psi}{\partial y})^2 +$$

$$+ i\left[2A\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + 2B(\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial y} \cdot \frac{\partial \psi}{\partial x}) + 2C\frac{\partial \varphi}{\partial y} \cdot \frac{\partial \psi}{\partial y}\right] = 0.$$

Bundan kompleks son ham haqiqiy qismi, ham mavhum qismi nol bo‘lgandagina nolga teng bo‘lishini hisobga olsak

$$A(\frac{\partial \varphi}{\partial x})^2 + 2B\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} + C(\frac{\partial \varphi}{\partial y})^2 = A(\frac{\partial \psi}{\partial x})^2 + 2B\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} + C(\frac{\partial \psi}{\partial y})^2,$$

$$A\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + B(\frac{\partial \varphi}{\partial x} \cdot \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial y} \cdot \frac{\partial \psi}{\partial x}) + C\frac{\partial \varphi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0$$

tengliklar kelib chiqadi. Bundan (74.12)ga asosan

$$a_{11} = a_{13}, \; a_{12} = 0.$$

Bu holda (1.11)ni a_{11} ga bo‘lib

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_3(\xi, \eta, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \tag{1.22}$$

tenglamani hosil qilamiz, bunda $F_3 = -\dfrac{F}{a_{11}}$.

(1.22) elliptik turdagi tenglamaning kanonik shaklidir.

7-misol. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + 2\dfrac{\partial^2 u}{\partial y^2} = 0$ tenglamani kanonik ko‘rinishga keltiring.

Yechilishi. $A = 1, \; B = -1, \; C = 2, \; \Delta = B^2 - AC = 1 - 1 \cdot 2 = -1 < 0$ bo‘lgani uchun berilgan tenglama elliptik turdagi tenglamadir.

Endi xarakteristik tenglamani tuzamiz:

$dy^2 + 2dxdy + 2dx^2 = 0,$ bundan $y'^2 + 2y' + 2 = 0$ yoki $y' = -1 \pm i$ ni hosil qilamiz.

Demak $\dfrac{dy}{dx} = -1 \pm i, \; \dfrac{dy}{dx} + (1 \mp i) = 0, \; dy + (1 \mp i)dx = 0.$ Buni integrallab

$$y + x - ix = C_1, \quad y + x + ix = C_2$$

xarakterstik tenglamalarga ega bo‘lamiz.
$\xi = y + x,\ \eta = x$ yangi o‘zgaruvchilarni kiritamiz. Bulardan foydalanib xususiy hosilalarni topamiz.

$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 1,\ \frac{\partial \eta}{\partial y} = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi},,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial x} +$$

$$+\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial^2 u}{\partial \xi^2},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}.$$

Bularni barilgan tenglamaga qo‘yib soddalashtiramiz.

$$\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + 2\frac{\partial^2 u}{\partial \xi^2} = 0$$

yoki $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0.$

Bu berilgan tenglamaning kanonik ko‘rinishidir.

Shuni ta’kidlash lozimki matematik fizikaning asosiy tenglamalari to‘lqin tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

giperbolik turdagi tenglama, issiqlik tarqalish tenglamasi $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ parabolik turdagi tenglama, Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

elliptik turdagi tenglamadir.

Xulosa. Ikkinchi tartibli ikki o‘zgaruvchili chiziqli xususiy hosilali

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x, y)$$

differensial tenglamalarni kanonik ko‘rinishga keltirish uchun uning xarakteristikalarining differensial tenglamasi deb ataluvchi

$$Ady^2 - 2Bdxdy + Cdx^2 = 0$$

$\frac{dy}{dx}$ ga nisbatan kvadrat tenglama tuzilib uning umumiy integrallari (xarakteristikalari) topiladi:

a) $\Delta = B^2 - AC > 0$ bo‘lganda (1.16) tenglama ikkita har xil haqiqiy $\varphi(x, y) = C_1,\ \psi(x, y) = C_2$ xarakteristikalarga ega bo‘lib (1.5) tenglama $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ almashtirish yordamida kanonik ko‘rinishga keltiriladi;

b) $\Delta = B^2 - AC = 0$ bo‘lganda (1.16) tenglama bitta $\varphi(x,y)=C$ haqiqiy xarakteristikaga ega bo‘lib (1.5) tenglama $\xi=\varphi(x,y),\ \ \eta=\psi(x,y)$ almashtirish yordamida kanonik ko‘rinishga keltiriladi, bunda $\psi(x,y)$ φ ga bog‘liq bo‘lmagan ikki marta differensiallanuvchi ixtiyoriy funksiya;

c) $\Delta = B^2 - AC < 0$ bo‘lganda (1.16) tenglama o‘zaro qo‘shma $\varphi(x,y)+i\psi(x,y)=C_1,\ \ \varphi(x,y)-i\psi(x,y)=C_2$ kompleks xarakteristikalarga ega bo‘lib (1.5) tenglama $\xi=\varphi(x,y),\ \ \eta=\psi(x,y)$ almashtirish yordamida kanonik ko‘rinishga keltiriladi.

Mustaqil yechish uchun mashqlar

Quyidagi birinchi tartibli xususiy hosilali differensial tenglamalarning umumiy integrallari topilsin.

1. $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$. Javob: $\Phi(\dfrac{y}{x^2-y^2},\ z)=0$ yoki $z=\Psi(\dfrac{y}{x^2-y^2})$.

2. $yz\dfrac{\partial z}{\partial x}+xz\dfrac{\partial z}{\partial y}=-2xy.$ Javob: $x^2+\dfrac{z^2}{2}=\Psi(x^2-y^2)$.

3. $\dfrac{\partial z}{\partial x}\sin x+\dfrac{\partial z}{\partial y}\sin y=\sin z.$ Javob: $tg\dfrac{z}{2}=tg\dfrac{x}{2}\cdot\Psi(\dfrac{tg\frac{y}{2}}{tg\frac{x}{2}})$.

4. $yz\dfrac{\partial z}{\partial x}+xz\dfrac{\partial z}{\partial y}=xy.$ Javob: $z^2=x^2+\Psi(y^2-x^2)$.

5. $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=0.$ Javob: $z=\Psi(x^2+y^2)$.

Quyidagi tenglamalar kanonik shaklga keltirilsin.

6. $\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}+5\dfrac{\partial^2 u}{\partial y^2}-32u=0.$

7. $\dfrac{\partial^2 u}{\partial x^2}-2\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}+9(\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y})=0.$

8. $2\dfrac{\partial^2 u}{\partial x^2}+3\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}+7\dfrac{\partial u}{\partial x}+4\dfrac{\partial u}{\partial y}-2u=0.$

9. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial x\partial y}-2\dfrac{\partial^2 u}{\partial y^2}-3\dfrac{\partial u}{\partial x}-15\dfrac{\partial u}{\partial y}+27x=0.$

10. $9\dfrac{\partial^2 u}{\partial x^2}-6\dfrac{\partial^2 u}{\partial x\partial y}+\dfrac{\partial^2 u}{\partial y^2}+10\dfrac{\partial u}{\partial x}-15\dfrac{\partial u}{\partial y}-50u+x-2y=0.$

11. $\dfrac{\partial^2 u}{\partial x^2}-4\dfrac{\partial^2 u}{\partial x\partial y}+10\dfrac{\partial^2 u}{\partial y^2}-24\dfrac{\partial u}{\partial x}+42\dfrac{\partial u}{\partial y}+2x+2y=0.$

12. $\dfrac{\partial^2 u}{\partial x^2}+4\dfrac{\partial^2 u}{\partial x\partial y}+13\dfrac{\partial^2 u}{\partial y^2}+30\dfrac{\partial u}{\partial x}+24\dfrac{\partial u}{\partial y}-9u+9(x+y)=0.$

13. $(1+x^2)^2\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+2x(1+x^2)\dfrac{\partial u}{\partial x}=0.$

14. $y^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+x^2\dfrac{\partial^2 u}{\partial y^2}=0.$

15. $\frac{\partial^2 u}{\partial x^2}-(1+y^2)^2\frac{\partial^2 u}{\partial y^2}-2y(1+y^2)\frac{\partial u}{\partial y}=0.$

16. $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$ Javob: $\xi=\frac{y}{x},\ \eta=y,\ \frac{\partial^2 u}{\partial \eta^2}=0.$

17. $\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0.$ Javob: $\xi=x+y,\ \eta=3x+y,\ \frac{\partial^2 u}{\partial\xi\partial\eta}-\frac{\partial u}{\partial\eta}=0.$

18. $\frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{1}{y^2}\frac{\partial^2 u}{\partial y^2}=0.$ Javob: $\xi=y^2,\ \eta=x^2,\ \frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{2}(\frac{1}{\xi}\cdot\frac{\partial u}{\partial\xi}+\frac{1}{\eta}\cdot\frac{\partial u}{\partial\eta})=0.$

19. $(1+x^2)\frac{\partial^2 u}{\partial x^2}+(1+y^2)\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

Javob: $\xi=\ln(x+\sqrt{1+x^2}),\ \eta=\ln(y+\sqrt{1+y^2}),\ \frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=0.$

20. $\frac{\partial^2 u}{\partial x^2}-2\cos x\frac{\partial^2 u}{\partial x\partial y}-(3+\sin^2 x)\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}=0.$

Javob: $\xi=2x+\sin x+y,\ \eta=2x-\sin x-y,\ \frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\eta-\xi}{32}(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta})=0.$

21. $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$ Javob: $\xi=\frac{y}{x},\ \eta=y,\ \frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\cdot\frac{\partial u}{\partial\eta}=0$

22. $\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}=0.$

23. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}+3\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0.$

24. $\frac{\partial^2 u}{\partial x^2}-6\frac{\partial^2 u}{\partial x\partial y}+13\frac{\partial^2 u}{\partial y^2}=0.$

25. $y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=0.$

26. $y^2\frac{\partial^2 u}{\partial x^2}-x^2\frac{\partial^2 u}{\partial y^2}-2x\frac{\partial u}{\partial x}=0.$

27. $\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}=0\quad(x\geq 0\ \textit{sohada}).$

28. $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0.$

29. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0.$

30. $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+7\frac{\partial u}{\partial x}+9\frac{\partial u}{\partial y}+u=0.$

31. $\frac{\partial^2 u}{\partial x^2}+5\frac{\partial^2 u}{\partial x\partial y}+11\frac{\partial^2 u}{\partial y^2}+10u=0.$

32. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-5\frac{\partial u}{\partial x}=0.$

33. $\frac{\partial^2 u}{\partial x^2}+20\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}=0.$

34. $\frac{\partial^2 u}{\partial x^2} + 20\frac{\partial^2 u}{\partial x\partial y} - 10\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0.$

35. $\frac{\partial^2 u}{\partial x^2} + 15\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} - 5\frac{\partial u}{\partial y} = 0.$

36. $\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x\partial y} + 15\frac{\partial^2 u}{\partial y^2} + 8\frac{\partial u}{\partial x} = 0.$

37. $\frac{\partial^2 u}{\partial x^2} - 12\frac{\partial^2 u}{\partial x\partial y} + 30\frac{\partial^2 u}{\partial y^2} + 5\frac{\partial u}{\partial x} = 0.$

38. $8\frac{\partial^2 u}{\partial x^2} - 11\frac{\partial^2 u}{\partial x\partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0.$

39. $10\frac{\partial^2 u}{\partial x^2} + 9\frac{\partial^2 u}{\partial x\partial y} - 8\frac{\partial^2 u}{\partial y^2} = 0.$

40. $25\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

41. $8\frac{\partial^2 u}{\partial x^2} + 13\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

42. $\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x\partial y} + 14\frac{\partial^2 u}{\partial y^2} = 0.$

43. $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} - 7\frac{\partial^2 u}{\partial y^2} = 0.$

44. $\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} = 0.$

45. $\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

46. $\frac{\partial^2 u}{\partial x^2} + 9\frac{\partial^2 u}{\partial x\partial y} - 19\frac{\partial^2 u}{\partial y^2} = 0.$

47. $\frac{\partial^2 u}{\partial x^2} - 18\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$

48. $\frac{\partial^2 u}{\partial x^2} - 40\frac{\partial^2 u}{\partial x\partial y} + 11\frac{\partial^2 u}{\partial y^2} = 0.$

49. $\frac{\partial^2 u}{\partial x^2} + 20\frac{\partial^2 u}{\partial x\partial y} - 13\frac{\partial^2 u}{\partial y^2} = 0.$

50. $\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x\partial y} - 29\frac{\partial^2 u}{\partial y^2} = 0.$

51. $\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial^2 u}{\partial x\partial y} - 5\frac{\partial^2 u}{\partial y^2} = 0.$

Nazorat uchun savollar

1. Tenglama qachon xususiy hosilali deyiladi?
2. Xususiy hosilali tenglamaning yechimi deb nimaga aytiladi?
3. Xususiy hosilali tenglamaning tartibi deb nimaga aytiladi?
4. Chiziqli va nochiziqli xususiy hosilali tenglamalarga ta’rif bering.

5. Birinchi tartibli xususiy hosilali chiziqli tenglamaga te’rif bering.
6. Bir jinsli va bir jinsli bo‘lmagan xususiy hosilali differensial tenglamalarga ta’rif bering.
7. Birinchi tartibli xususiy hosilali bir jinsli bo‘lmagan chiziqli tenglama qanday qilib bir jinsliga keltiriladi?
8. Xarakterstik tenglama nima?
9. Xarakteristika nima?
10. Giperbolik, elliptik, parabolik turdagi tenglamalarning ta’rifini ayting.
11. Giperbolik, elliptik, parabolik turdagi tenglamalarning kanonik shakllarini yozing.

2. Masalaning qo‘yilishi. Tor tebranish tenglamasini keltirib chiqarish va uni yechish

2.1. Koshi masalasi, chegaraviy masalalar, aralash masalalarning qo‘yilishi.

Har qanday oddiy yoki xususiy hosilali differensial tenglamalar ham yechimga ega bo‘lgan taqdirda ham yechim cheksiz ko‘p bo‘ladi. Bu yechimlardan qo‘yilgan fizik masalaga to‘liq javob beruvchisi yagona yechimni ajratib olish kerak. Buning uchun tenglamaga shu fizik masalaning mohiyatiga ko‘ra qo‘shimcha shartlar berilishi kerak. Masalaning qo‘yilishiga qarab bu shartlar **boshlang‘ich, chegaraviy, aralash** deb nomlangan turlarga bo‘linadi.

Masalan, uzunligi ℓ ga teng, uchlari mahkamlangan bir jinsli torning erkin tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

ga quyidagi qo‘shimcha shartlarni qo‘yish mumkin. a) **boshlang‘ich shartlar**: $u(x,0) = u|_{t=0} = f(x)$, ya‘ni boshlang‘ich paytda torning har bir nuqtasini abssissa o‘qidan uzoqligi ma‘lum;

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x)$, ya‘ni torning har bir nuqtasini boshlang‘ich tezligi ma‘lum;

b) **chegaraviy shartlar**: $u(0,t) = 0$, $u(\ell,t) = 0$.

Bu shartlar torning ikki uchi mahkamlanganligini ko‘rsatadi. Faqat boshlang‘ich shartlarda yechish **Koshi masalasi** deyiladi.

Chegaraviy va boshlang‘ich shartlar to‘plami **chetki shartlar** deb ataladi.

Agar chegaraviy shartlardan bittasigina bajarilsa, torning bir uchigina mahkamlangan bo‘lib, ikkinchi uchi esa erkin tebranadi deymiz. Masalaning qo‘yilishida umuman chegaraviy shartlar berilmagan bo‘lishi ham mumkin, u holda tor chegaralanmagan (cheksiz tor) deb hisoblanadi. Bu fizika nuqtai nazardan tor shu qadar uzunki, qaralayotgan elementar bo‘lagiga ikki uchining ta‘siri yo‘q degan ma‘noni bildiradi.

σ sirt bilan chegaralangan ω soha(jism)da $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ ($\Delta u = 0$) Laplas tenglamasini qanoatlantiruvchi sohaning chegarasi σ sirtning har bir M nuqtasida berilgan.

$$u|_{\sigma} = f(M)$$

chegaraviy shartni qanoatlantiruvchi $u(x,y,z)$ funksiyani topish masalasi **Dirixlening ichki masalasi** yoki Laplas tenglamasi uchun **birinchi chetki masala** deyiladi.

$u|_{\sigma} = f(M)$ shart o‘rniga chegarada izlanuvchi $u(x,y,z)$ funksiyaning normalni musbat yo‘nalishi bo‘yicha hosilasi $\left.\frac{\partial u}{\partial n}\right|_{\sigma} = g(M)$ berilganda $\Delta u = 0$ tenglamaning yechimini topish masalasi **Neyman masalasi** yoki Laplas tenglamasi uchun **ikkinchi chetki masala** deyiladi.

Shuningdek σ sirtning bir qismida $u|_{\sigma} = f(M)$, qolgan qismida $u(x,y,z)$ funksiyaning normal musbat yo‘nalishi bo‘yicha hosilasi $\left.\frac{\partial u}{\partial n}\right|_{\sigma} = g(M)$ berilganda $\Delta u = 0$ tenglamaning yechimini topish masalasi Dirixle-Neyman (yoki aralash) masalasi deyiladi. Odatda bu masalalarni **ichki masalalar** deb yuritiladi. Bulardan tashqari **tashqi masalalar** deb ataluvchi masalalar ham berilishi mumkin. Tashqi masalalarda sohadan tashqarida $\Delta u = 0$ Laplas tenglamasini qanoatlantiruvchi va sohaning chegarasida berilgan chegaraviy shartni qanoatlantiruvchi hamda cheksizlikda nolga intiluvchi $u(x,y,z)$ funksiyani topish talab etiladi. Neyman masalasi uchun cheksizlikda $u(x,y,z)$ funksiyaning chegaralangan bo‘lishi kifoya.

Izoh. $\Delta u = 0$ Laplas tenglamasi o‘rnida matematik fizikaning boshqa biror tenglamasi bo‘lganda ham ta‘riflar o‘z kuchini saqlaydi.

Tenglamaning chetki shartlari shu tenglama ifodalaydigan jarayonning fizik mohiyatidan kelib chiqadi.

Chegaraviy shartlarni mohiyatining yanada yaxshiroq anglash maqsadida issiqlikning tarqalishi haqidagi masalani qarab chiqamiz.

Fizik jarayonlarning o‘zgarishi vaqtga bog‘liq bo‘lmaganda σ sirt bilan chegaralangan bir jinsli ω jismning turli nuqtalaridagi $u(x,y,z)$ harorat

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (2.2)$$

Laplas tenglamasini qanoatlantiradi. Haroratni to‘la aniqlash uchun bu tenglamaning o‘zi yetarli emas. (2.2) tenglamaning chegaraviy sharti-σ sirtdagi harorat berilishi kerak

$$u|_\sigma = f(M). \qquad (2.3)$$

Shunday qilib, ω jism ichida (2.2) tenglamani qanoatlantiruvchi va σ sirtning har bir M nuqtasida berilgan $f(M)$ qiymatni qabul qiluvchi $u(x,y,z)$ funksiyani topish kerak. Bu masala **Dirixle** masalasidir.

Agar sirtning har bir nuqtasida harorat emas, balki issiqlik oqimi berilgan bo‘lib, u $\frac{\partial u}{\partial n}$ (normal vektor yo‘nalishdagi hosila) ga proporsional bo‘lsa, sirtda (2.3) chegarviy shart o‘rniga

$$\left.\frac{\partial u}{\partial n}\right|_\sigma = g(M) \qquad (2.4)$$

shartga ega bo‘lamiz. (2.2) tenglamaning (2.4) chegaraviy shartni qanoatlantiruvchi yechimini topish masalasi **Neyman** masalasidir.

Agar sirtning bir qismida harorat, qolgan qismida issiqlik oqimi $\frac{\partial u}{\partial n}$ berilganda (2.2) tenglamaning ana shu chegaraviy shartni qanoatlantiruvchi yechimini topish masalasi **aralash** masaladir.

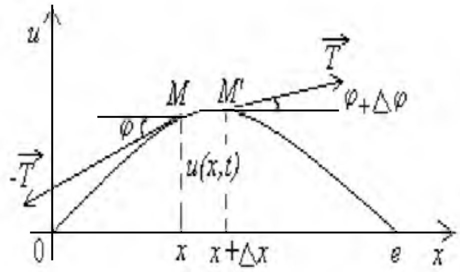
2.2. Tor tebranishlar tenglamasini keltirib chiqarish

Tor deganda ingichka egiluvchan va elastik ip tushuniladi.

Uzunligi ℓ ga teng bo‘lgan tor berilgan bo‘lib, uning uchlari to‘g‘ri burchakli dekart koordinatalar sistemasida nuqtalarga mahkamlangan deb faraz qilamiz. Agar tarang tortilgan torni dastlabki holatidan chetlashtirib, so‘ngra o‘z holatiga qo‘yib yuborilsa yoki uning nuqtalariga biror tezlik berilsa, u holda torning nuqtalari harakatga keladi, ya‘ni tor tebrana boshlaydi. Biz istalgan paytda tor shaklini aniqlash hamda torning har bir nuqtasi vaqtga bog‘liq ravishda qanday qonun bilan harakatlanishini aniqlash masalasi bilan shug‘ullanamiz.

Tor nuqtalarining boshlang‘ich (gorizontal) holatidan kichik chetlanishlarini qaraymiz. Shunga ko‘ra, tor nuqtalarining harakati $0x$ o‘qqa perpendikulyar holda va bir tekislikda vujudga keladi deb faraz qilish mumkin. Bunday farazda torning tebranish jarayoni bitta $u(x,t)$ funksiya bilan ifoda etiladi; bu funksiya abssissasi x bo‘lgan tor nuqtasining t paytda siljish (gorizontal holatidan uzoqlashish) miqdorini beradi (296-chizma).

Biz torning $0xu$ tekislikda kichik chetlanishlarini qarayotganimiz uchun, tor elementi uzunligi MM' uning $0x$ o‘qdagi proyeksiyasi Δx ga teng deb faraz qilamiz. Yana torning barcha nuqtalarida taranglik T bir xil deb faraz qilamiz. Torning MM' elementini qaraymiz (2.1-chizma).



2.1-chizma.

Bu elementning uchlarida T taranglik kuchlar torga urinma bo‘yicha ta‘sir etadi. Urinmalar $0x$ o‘q bilan φ va φ+Δφ burchaklar hosil qiladi. Bu holda MM' elementga ta‘sir etuvchi kuchlarning $0u$ o‘qdagi proyeksiyasi $T\sin(\varphi+\Delta\varphi)-T\sin\varphi$ ga teng bo‘ladi. φ burchak juda kichik bo‘lgani uchun $tg\varphi \approx \sin\varphi$ deb faraz qilish mumkin. U holda hosilaning geometrik ma‘nosini hamda $f(x+\Delta x)-f(x)=f'(x+\theta x)\Delta x$ Lagranj formulasini e‘tiborga olib quyidagiga ega bo‘lamiz:

$$T\sin(\varphi+\Delta\varphi)-T\sin\varphi \cong Tg(\varphi+\Delta\varphi)-Ttg\varphi = T\left[\frac{\partial u(x+\Delta x,t)}{\partial x}-\frac{\partial(x,t)}{\partial x}\right]=$$

$$=T\frac{\partial^2 u(x+\theta\Delta x,t)}{\partial x^2}\Delta x \approx T\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x \qquad 0<\theta<1.$$

Harakat tenglamasini hosil qilish uchun elementga qo‘yilgan tashqi kuchlarni inersiya kuchiga tenglash kerak. ρ torning chiziqli zichligi bo‘lsin. U holda tor elementining massasi $\rho\Delta x$ bo‘ladi.

Elementning tezlanishi $\frac{\partial^2 u}{\partial t^2}$ ga teng. Demak, Dalamber prinsipiga ko‘ra ushbu tenglikka ega bo‘lamiz:

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2}=T\frac{\partial^2 u}{\partial x^2}\Delta x,$$

Δx ga qisqartirib va $\frac{T}{\rho}=a^2$ deb belgilab, harakatning ushbu

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (2.5)$$

tenglamani hosil qilamiz.

Bu tenglama torning **erkin tebranish tenglamasi** yoki **bir o‘lchovli to‘lqin tenglamasi** deyiladi.

(2.5) tenglama tor harakatini to‘la aniqlashi uchun qo‘shimcha $u(x,0)=f(x),\ \frac{\partial u}{\partial t}\big|_{t=0}=F(x)$ -boshlang‘ich shartlar hamda $u(0,t)=0,\ u(\ell,t)=0$ - chegaraviy shartlar berilgan bo‘lishi kerak.

Shuningdek, gorizantal holda joylashgan membrana (yupqa tekis elastik plastinka) ni gorizantal holatdan chiqarilganda unga tashqi kuchlar ta‘sir etmasdan faqatgina ichki kuchlar ta‘sirida tebransa, u holda membrana erkin tebranishining tenglamasi

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \qquad (2.6)$$

ko‘rinishda bo‘lishi isbotlangan. Bu tenglama uchun chetki shartlar quyidagilardan iborat: *a*) boshlang‘ich shartlar $\grave{e}(\tilde{o},\acute{o},0)=f(x,y);\ \frac{\ddot{a}\grave{e}}{\ddot{a}t}\Big|_{t=0}=F(x,y)$ *b*) chegaraviy shartlar: membrananing konturi har xil bo‘lishi mumkin, masalan, kontur tenglamasi $x=\varphi(s),\ y=\psi(s)$ parametrik tenglamalari bilan berilgan egri chiziq shaklida bo‘lsa, membraning konturi mahkamlanganligi

$$u\Big|_{\substack{x=\varphi(s)\\ y=\psi(s)}}=0$$

shart bilan beriladi. Xususan membrana, tomonlari l va h bo‘lgan to‘g‘ri to‘rtburchak shaklida bo‘lsa, konturning mahkamlanganlik sharti

$$\grave{e}(\hat{\imath},\acute{o},t)=u(l,y,t)=0,$$

$$u(x,o,t)=u(x,h,t)=0$$

tengliklar bilan beriladi.

2.3. Tor tebranish tenglamasini Dalamber usuli bilan yechish.

Tor nihoyatda uzun bo‘lgan holni kuzatamiz. Bu holda uning o‘rtasidan unga biror tezlik berilsa, o‘ng va chap tomonga to‘lqinlar yo‘naladi. Natijada torning uchlariga to‘g‘ri to‘lqinlar borib, so‘ng teskari to‘lqinlar qaytadi. Biz akslangan teskari to‘lqinlarni hisobga olmaymiz, ya‘ni cheksiz uzunlikka ega torning tebranish masalasini qaraymiz. (2.5) tenglamani

$$u(x,0)=u\Big|_{t=0}=f(x),\quad u_t^1(x,0)=\frac{\partial u}{\partial t}\Big|_{t=0}=F(x) \tag{2.7}$$

boshlang‘ich sharlarni qanoatlantiruvchi yechimini izlaymiz. Bu yerda $f(x)$, $F(x)$ funksiyalar butun sonlar o‘qida berilgan. $u(x,t)$ funksiya uchun chegaraviy shartlar bo‘lmaydi. Shunday qilib qo‘yilgan Koshi masalasini Dalamber usuli bilan yechamiz.

Tenglamaning umumiy yechimini ikkita ixtiyoriy funksiyalar yig‘indisi sifatida qaraymiz:

$$u(x,t)=\varphi(x-at)+\psi(x+at)\,. \tag{2.8}$$

Bu yerda φ va ψ funksiyalarning ikkinchi tartibli xususiy hosilalari mavjud deb faraz qilamiz. U vaqtda ketma-ket hosilalar olsak,

$$\frac{\partial u}{\partial x}=\varphi'(x-at)+\psi'(x+at),\ \frac{\partial^2 u}{\partial x^2}=\varphi''(x-at)+\psi''(x+at)\,,$$

$$\frac{\partial u}{\partial t}=-a\varphi'(x-at)+a\psi'(x+at),\frac{\partial^2 u}{\partial t^2}=a^2\psi''(x-at)+a^2\psi''(x+at)$$

lar hosil bo‘lib, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t^2}$ larning ushbu qiymatlari (2.8) tenglamani qanoatlantiradi. Demak (2.8) funksiya umumiy yechim bo‘ladi. (2.7) boshlang‘ich shartlardan foydalanib, φ va ψ noma‘lum funksiyalarni topamiz:

$$t=0 \text{ da } \begin{cases}\varphi(x)+\psi(x)=f(x),\\ -a\varphi'(x)+a\psi'(x)=F(x)\end{cases} \tag{2.9}$$

sistemaga ega bo‘lamiz. Ikkinchi tenglamani 0 dan x gacha oraliqda integrallasak

$$-a[\varphi(x)-\varphi(0)]+a[\psi(x)-\psi(0)]=\int_0^x F(x)dx$$

yoki

$$-\varphi(x)+\psi(x)=\frac{1}{a}\int_0^x F(x)dx+C \tag{2.10}$$

ko‘rinishdagi ifodaga kelamiz. Bu yerda $C=-\varphi(0)+\psi(0)$ -o‘zgarmas son.

Shunday qilib (2.9) va (2.10) dan $\begin{cases} \varphi(x)+\psi(x)=f(x), \\ -\varphi(x)+\psi(x)=\frac{1}{a}\int_0^x F(x)dx+C \end{cases}$

sistemaga ega bo‘lamiz. Bu sistemani yechib $\varphi(x)$, $\psi(x)$ noma‘lum funksiyalarni aniqlaymiz:

$$\varphi(x)=\frac{1}{2}f(x)-\frac{1}{2a}\int_0^x F(x)dx-\frac{C}{2},$$

$$\psi(x)=\frac{1}{2}f(x)+\frac{1}{2a}\int_0^x F(x)dx+\frac{C}{2}.$$

Bu formulalarda argument x ni $x-at$ va $x+at$ ga almashtirib, ularni (2.8) formulaga qo‘ysak, $u(x,t)$ funksiya topiladi:

$$u(x,t)=\frac{1}{2}f(x-at)-\frac{1}{2a}\int_0^{x-at} F(x)dx+\frac{1}{2}f(x+at)+\frac{1}{2a}\int_0^{x+at} F(x)dx=$$

$$=\frac{f(x-at)+f(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at} F(x)dx. \qquad (2.11)$$

Bu formula tor tebranish tenglamasi uchun Koshi masalasini Dalamber usuli bilan yechgandagi yechimini ifodalaydi.

1-misol. $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ tenglamani $u|_{t=0}=x^2$, $\frac{\partial u}{\partial t}|_{t=0}=0$ boshlang‘ich shartlarda yeching.

Yechilishi. Bu yerda $a=1$, $f(x)=x^2$, $F(x)=0$ ekanini va (2.11) formulani hisobga olsak,

$$u(x,t)=\frac{f(x-t)+f(x+t)}{2}=\frac{(x-t)^2+(x+t)^2}{2}=x^2+t^2$$

yechimni hosil qilamiz.

2-misol. $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(x,0)=\sin x$ va $\frac{\partial u}{\partial t}|_{t=0}=0$ boshlang‘ich shartni qanoatlantiruvchi yechimini toping.

Yechilishi. $a=1$, $f(x)=\sin x$, $F(x)=0$. Bu qiymatlarni (2.11) formulaga qo‘yib

$$u(x,t)=\frac{\sin(x-t)+\sin(x+t)}{2}=\frac{2\sin x\cos t}{2}=\sin x\cos t$$

yechimni hosil qilamiz.

3-misol. $\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(x,0)=\sin 2x$, $\frac{\partial u}{\partial t}|_{t=0}=\cos x$ boshlang‘ich shartlarni qanoatlantiruvchi yechimining $t=\frac{\pi}{2}$ paytdagi qiymatini hisoblang.

Yechilishi. $a=3$, $f(x)=\sin 2x$, $F(x)=\cos x$ ekanligini e’tiborga olib, (2.11) Dalamber formulasidan foydalanamiz:

$$u(x,t)=\frac{\sin 2(x-3t)+\sin 2(x+3t)}{2}+\frac{1}{2\cdot 3}\int_{x-3t}^{x+3t}\cos x dx=\frac{2\sin 2x\cos 6t}{2}+\frac{1}{6}\sin x\Big|_{x-3t}^{x+3t}=\sin 2x\cos 6t+$$

$$+\frac{1}{6}(\sin(x+3t)-\sin(x-3t))=\sin 2x\cos 6t+\frac{1}{3}\sin 3t\cos x.$$

Topilgan $u(x,t)$ ga $t=\frac{\pi}{2}$ qiymatni qo‘yib yechimni hosil qilamiz, ya’ni

$$u(x,\frac{\pi}{2})=\sin 2x\cos 6\cdot\frac{\pi}{2}+\frac{1}{3}\sin 3\cdot\frac{\pi}{2}\cos x=-\sin 2x-\frac{1}{3}\cos x.$$

Mustaqil yechish uchun mashqlar

$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ tor tebranish tenglamasining $u|_{t=0}=f(x),\ \frac{\partial u}{\partial t}=F(x)$ boshlang'ich shartlarni qanoatlantiruvchi yechimini Dalamber formulasidan foydalanib toping:

1. $a=2,\ f(x)=0,\ F(x)=x$. Javob: $u(x,t)=xt$.
2. $a=a,\ f(x)=0,\ F(x)=\cos x$. Javob: $u(x,t)=\frac{1}{a}\cos x\sin at$.
3. $a=1,\ f(x)=\sin x,\ F(x)=0$. Javob: $u(x,t)=\sin x\cos t$.
4. $a=5,\ f(x)=0,\ F(x)=30\sin x$. Javob: $u(x,t)=6\sin x\sin 5t$.
5. $a=1,\ f(x)=\frac{x}{1+x^2},\ F(x)=\sin x$ bo'lsa yechimning $t=\pi$ paytdagi qiymatini toping.

   Javob: $\frac{1}{2}\left[\frac{x+\pi}{1+(x+\pi)^2}+\frac{x-\pi}{1+(x-\pi)^2}\right]$.
6. $a=4,\ f(x)=x^4,\ F(x)=\cos 2x$.
7. $a=4,\ f(x)=x^2,\ F(x)=\sin 2x$.
8. $a=4,\ f(x)=x^2,\ F(x)=\cos x$.
9. $a=9,\ f(x)=x,\ F(x)=\cos 3x$.
10. $a=9,\ f(x)=x^3,\ F(x)=\sin 4x$.
11. $a=16,\ f(x)=\sin 5x,\ F(x)=\cos 2x$.
12. $a=9,\ f(x)=x^4,\ F(x)=\sin 4x$.
13. $a=16,\ f(x)=\sin 5x,\ F(x)=\sin 2x$.
14. $a=16,\ f(x)=\sin 4x,\ F(x)=\cos x$.
15. $a=4,\ f(x)=\cos 5x,\ F(x)=\cos 2x$.
16. $a=4,\ f(x)=\cos 5x,\ F(x)=\sin 2x$.
17. $a=4,\ f(x)=\cos 4x,\ F(x)=\cos x$.
18. $a=4,\ f(x)=\cos 4x,\ F(x)=\sin x$.
19. $a=9,\ f(x)=\cos 5x,\ F(x)=e^{5x}$.
20. $a=9,\ f(x)=\sin 5x,\ F(x)=e^{6x}$.
21. $a=9,\ f(x)=\cos 4x,\ F(x)=e^{7x}$.
22. $a=9,\ f(x)=\sin 4x,\ F(x)=e^{8x}$.
23. $a=16,\ f(x)=\sin 3x,\ F(x)=e^{2x}$.
24. $a=16,\ f(x)=\sin 3x,\ F(x)=e^{9x}$
25. $a=16,\ f(x)=e^{5x},\ F(x)=\sin 2x$.
26. $a=4,\ f(x)=e^{-4x},\ F(x)=\sin 3x$.
27. $a=4,\ f(x)=e^{-5x},\ F(x)=\cos 2x$.
28. $a=4,\ f(x)=e^{-6x},\ F(x)=\cos 3x$.
29. $a=4,\ f(x)=e^{-7x},\ F(x)=\sin 4x$.

Nazorat uchun savollar

1. Matematik fizikaning asosiy tenglamasini yozing.
2. Bir jinsli tenglamaning ta‘rifini ayting.
3. Matematik fizikaning asosiy tenglamasi necha turga bo‘linadi?
4. Koshi masalasi deb nimaga aytiladi?
5. Tenglamaning chetki shartlari deb nimaga aytiladi?
6. Masalaning korrekt qo‘yilishi deganda nima tushuniladi?
7. Dirixle masalasini ayting.
8. Neyman masalasini ayting.
9. Dirixle-Neyman masalasini ayting.
10. Torning erkin tebranish tenglamasini keltirib chiqaring.
11. Membrananing erkin tebranish tenglamasini yozing.
12. Tor tebranish tenglamasi yechimini ifodalovchi Dalamber formulasini keltirib chiqaring.

3. Grin formulasi. Garmonik funksiyalarning asosiy xossalari

3.1. Grin formulasi

σ silliq sirt bilan chegaralangan uch o‘lchovli ω sohani qaraymiz. Agar $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar ω sohada differensiallanuvchi va σ sirtda uzluksiz bo‘lsa, u holda

$$\iiint\limits_{\omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz=\iint\limits_{\sigma}(P\cos\alpha+Q\cos\beta+R\cos\gamma)d\sigma \qquad (3.1)$$

Ostrogradskiy-Grin formulasi o‘rinli bo‘lar edi, bunda $cos\alpha,\ \cos\beta,\ \cos\gamma$ σ sirtga tashqi normal yo‘nalishidagi $\vec{n}$-birlik vektorning yo‘naltiruvchi kosinuslari. (3.1) formulani vektor shaklida qisqacha

$$\iiint\limits_{\omega} div\vec{a}d\omega=\iint\limits_{\sigma}\vec{a}\cdot\vec{n}d\sigma \qquad (3.2)$$

ko‘rinishda yozish ham mumkin, bunda $\vec{a}=P\vec{i}+Q\vec{j}+R\vec{k}$. $\omega+\sigma$ orqali σ sirt bilan chegaralangan yopiq ω sohani belgilaymiz.

Elliptik turdagi tenglamalarni yechishda Ostrogradskiy-Grin formulasining bevosita natijasi bo‘lgan Grin formulasidan foydalaniladi. Bu formulani keltirib chiqarishdan oldin garmonik funksiyani ta‘rifini eslaymiz.

Agar funksiya biror ω sohada ikkinchi tartibgacha uzluksiz hosilalarga ega bo‘lib, ω da

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad (\Delta u=0)$$

Laplas tenglamasini qanoatlantirsa bu funksiyani ω sohada garmonik funksiya deyilar edi.

Endi Grin formulasini keltirib chiqarishga kirishamiz.

Faraz qilaylik $u=u(x,y,z)$ va $v=v(x,y,z)$ funksiya $\omega+\sigma$ sohada uzluksiz va uzluksiz birinchi tartibli xususiy hosilalarga ega bo‘lib ω sohaning ichida uzluksiz ikkinchi tartibli xususiy hosilalarga ega bo‘lsin.

Agar

$$P=u\frac{\partial v}{\partial x},\ Q=u\frac{\partial v}{\partial y},\ R=u\frac{\partial v}{\partial z} \qquad (3.3)$$

desak, $div\vec{a}=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right)=u\cdot\Delta v+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+$

$$+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}=u\cdot\Delta v+\nabla u\cdot\nabla v$$

bo‘ladi.

Bunga va (3.3) ga asosan (3.2) ni quyidagicha yozish mumkin:

$$\iiint\limits_{\omega} u\Delta v d\omega=-\iiint\limits_{\omega}\nabla u\cdot\nabla v d\omega+\iint\limits_{\sigma}u\frac{\partial v}{\partial n}d\sigma\,. \qquad (3.4)$$

$u(x,y,z)$ va $v(x,y,z)$ funksiyalar teng huquqli bo‘lgani uchun oxirgi ifodada $u(x,y,z)$ bilan $v(x,y,z)$ ning o‘rinlarini almashtirsak,

$$\iiint\limits_{\omega} v\Delta u d\omega=-\iiint\limits_{\omega}\nabla u\cdot\nabla v d\omega+\iint\limits_{\sigma}v\frac{\partial u}{\partial n}d\sigma \qquad (3.5)$$

ga ega bo‘lamiz. (76.4) dan (76.5)ni hadma-had ayirsak

$$\iiint_{\omega} (u\Delta v - v\Delta u)d\omega = \iint_{\sigma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma \tag{3.6}$$

Grin formulasi hosil bo‘ladi.

Agar (3.6) formulada $u(x,y,z)$ funksiyani ω sohada garmonik hamda sohaning chegarasi σ da uzluksiz va birinchi tartibli hosilalarga ega va sohaning aniq $M_0(x_0,y_0,z_0)$ nuqtasi bilan uning ixtiyoriy $M\ (x,y,z)$ nuqtasi orasidagi masofani r deb belgilab, $v=\frac{1}{r}$ deb olsak, $M\ (x,y,z)$ nuqta M_0 nuqta ustiga tushganda v cheksizlikka aylanadi. Demak bu holda qaralayotgan ω sohaga Grin formulasini qo‘llab bo‘lmaydi. Grin formulasini qo‘llash uchun v funksiyani cheksizlikka aylantiruvchi M_0 nuqtani ρ radiusli va markazi M_0 nuqtada bo‘lgan S_ρ sfera bilan ajratib olamiz. Shundan so‘ng ω sohaning qolgan qismini ω_1 bilan belgilasak, ω_1 sohaning hech bir nuqtasida v funksiya cheksizlikka aylanmaydi va garmonik bo‘ladi. Shuning uchun unga Grin formulasini tatbiq etish mumkin. ω_1 soha ikkita sirt bilan chegaralanganligi uchun

$$\iiint_{\omega_1} \left(u\Delta\left(\frac{1}{r}\right) - \frac{1}{r}\Delta u \right) d\omega = \iint_{\sigma} \left(u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} \right) d\sigma + \iint_{S_\rho} \left(u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{n}\frac{\partial u}{\partial n} \right) d\sigma$$

bo‘ladi

$\Delta\left(\frac{1}{r}\right)=0$ ekanini tekshirib ko‘rish qiyin emas. S_ρ da $\frac{\partial}{\partial n}=-\frac{\partial}{\partial r}$, bularni hisobga olsak,

$$-\iiint_{\omega_1} \frac{\Delta u}{r} d\omega = \iint_{\sigma} \left(u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} \right) d\sigma + \iint_{S_\rho} \frac{1}{r^2} u\, d\sigma - \iint_{S_\rho} \frac{1}{r}\frac{\partial u}{\partial n} d\sigma\,. \tag{3.7}$$

Endi S_ρ sferaning radiusi ρ ni nolga intiltiramiz, u holda $\sigma_1 \to \sigma$ di (σ_1 – ω_1 ning chegarasi) va

$$\iint_{S_\delta} \frac{1}{r^2} u d\sigma = \frac{1}{\rho^2} \iint_{S_\rho} u d\sigma \qquad (r=\rho)$$

integralga o‘rta qiymat haqidagi teoremani qo‘llasak $\frac{1}{\rho^2}\cdot u(M^*)4\pi\rho^2 = 4\pi u(M^*)$ ga ega bo‘lamiz, bunda M^* sfera ustidagi qandaydir nuqta, $4\pi\rho^2 - S_\rho$ sferaning sirti, ρ nolga intilganda sferaning M^* nuqtasi ham sfera bilan birlikda M_0 nuqtaga (markazga) intiladi, shuning uchun

$$\iint_{S_\rho} \frac{1}{r^2} u d\sigma \xrightarrow[\rho\to 0]{} 4\pi u(M_0)\,.$$

(3.7) dagi oxirgi integralga o‘rta qiymat haqidagi teoremani qo‘llasak,

$$\iint_{S_\delta} \frac{1}{r}\frac{\partial u}{\partial n} d\sigma = \frac{1}{\rho}\left(\frac{\partial u}{\partial n}\right)_{M_\rho} 4\pi\rho^2 = 4\pi\rho\left(\frac{\partial u}{\partial n}\right)_{M_\rho} \xrightarrow[\rho\to 0]{} 0\,,$$

bunda $M_\rho - S_\rho$ sfera ustidagi biror nuqta. U holda (3.7) tenglikda $\rho \to 0$ da limitga o‘tib quyidagini hosil qilamiz:

$$-\iiint_{\omega} \frac{\Delta u}{r} d\omega = \iint_{\sigma} \left(u \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} \right) d\sigma + 4\pi u(M_0) \cdot$$

Bundan

$$u(M_0) = \frac{1}{4\pi}\iint_{\sigma} \left(\frac{1}{r}\frac{\partial u}{\partial n} - u \frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right) d\sigma - \frac{1}{4\pi}\iiint_{\omega} \frac{\Delta u}{r} d\omega \qquad (3.8)$$

Grin formulasini hosil qilamiz.

Tekislikda Grin formulasi:

$$\iint_S (u\Delta v - v\Delta u)\, dS = \int_L \left(u\frac{\text{äv}}{\text{än}} - v\frac{\text{äè}}{\text{än}} \right) dl \qquad (3.9)$$

ko'rinishida yoziladi, bunda S silliq L egri chiziq bilan chegaralangan tekis soha, $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$, $\frac{\partial}{\partial n} - L$ konturga normalning musbat yo'nalishi bo'yicha olingan hosila. Bu formulada $v = \ln r$ desak $\left(r = \overline{MM_0}\right)$ $\Delta(\ln r) = 0$ bo'lgani uchun yuqoridagidek mulohazalar yuritsak, (3.8) ga mos

$$u(M_0) = \frac{1}{2\pi}\int_L \left(u\frac{\partial(\ln r)}{\partial n} - \ln r\frac{\partial u}{\partial n} \right) dl - \frac{1}{2\pi}\iint_s \Delta u \ln r ds \qquad (3.10)$$

Grin formulasiga kelamiz. Bu formulalarni kelgusida Grin formulasining natijalari deb ishlatamiz.

3.2. Garmonik funksiyalarning asosiy xossalari

Garmonik funksiyalarning xossalarini o'rganish uchun Grin formulasi va undan chiqqan natijalardan foydalanamiz.

I. Yopiq σ sirt bilan chegaralangan uch o'lchovli ω sohada $u(x,y,z)$ garmonik funksiya berilgan bo'lsin. (3.6) formulada u ni garmonik funksiya, $v \equiv 1$ deb olsak, u holda $\Delta u = 0,\ \Delta v = 0$ va $\frac{\partial v}{\partial n} = 0$, demak

$$\iint_{\sigma} \frac{\partial u}{\partial n} ds = 0\,, \qquad (3.11)$$

ya'ni garmonik funksiyaning normal yo'nalishidagi hosilasidan sirt bo'yicha olingan integral nolga teng.

II. (3.8) formulalarni garomnik funksiyaga qo'llasak,

$$u(M_0) = \frac{1}{4\pi}\iint_{\sigma} \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right) d\sigma\,, \qquad (3.12)$$

ya'ni garmonik funksiyaning soha ichida olingan ixtiyoriy M_0 nuqtasidagi qiymati, funksiyaning o'zi va uning normal yo'nalishidagi hosilasining sirt ustidagi qiymatlari orqali ifodalangan (3.12) formula bilan aniqlanar ekan.

III. (3.12) formuladagi sirtning markazi M_0 nuqtada, radiusi R bo'lgan sfera deb faraz qilsak, u holda

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n}=\frac{\partial\left(\frac{1}{r}\right)}{\partial r}=-\frac{1}{r^2}$$

va sfera ustida $r=R$ bo‘lgani uchun (3.11) formulaga asosan

$$u(M_0)=\frac{\iint\limits_{\sigma} u d\sigma}{4\pi R^2} \tag{3.13}$$

bo‘ladi (Gauss formulasi). Garmonik funksiyaning sfera markazidagi qiymati shu funksiyaning sfera sirtidagi o‘rta qiymatiga tengdir. Bu xossa **o‘rta qiymat haqidagi teorema** deb yuritiladi.

IV. Sohaning ichida garmonik, sohaning chegarasida uzluksiz bo‘lgan funksiya o‘zining eng katta va eng kichik qiymatlarini soha chegarasida qabul qiladi (**maksimum prinsipi**).

Faraz qilaylik, sohaning ichki M_0 nuqtasida funksiya o‘zining eng katta $u_0=\max u$ qiymatiga ega bo‘lsin. M_0 nuqtani markaz qilib, qaralayotgan soha ichida to‘la yotgan, kichik R radiusli S_R sfera chizamiz. u_0 funksiyaning eng katta qiymati bo‘lgani uchun sfera ustidagi har qanday nuqtada $u<u_0$ bo‘ladi. Ikkinchi tomondan, Gauss formulasiga ko‘ra

$$u_0=\frac{\iint\limits_{S_R} u ds}{4\pi R^2}.$$

Suratdagi integraldan funksiyaning sirt ustidagi eng katta qiymatini chiqarsak,

$$u_0\le\frac{\max\limits_{S_R\ da} u \iint\limits_{S_R} ds}{4\pi R^2}=\max\limits_{S_R\ da} u.$$

Bu va yuqoridagi $u<u_0$ tengsizliklardan

$$u_0=\max\limits_{S_R\ da} u$$

degan xulosaga kelamiz. Bu soha ichida to‘la yotgan sferada funksiya o‘zgarmas, degan so‘z. Sfera ustida bitta ixtiyoriy nuqta olib uni markaz qilib sfera yasaymiz va bu sfera ustida ham yuqoridagi mulohazalarni yuritamiz va hokazo. Soha chegarasiga borguncha shunday mulohazalarni davom ettiramiz, natijada butun soha yuqorida aytilgandek sferalar bilan qoplanadi, bundan **funksiya sohada o‘zgarmas** degan xulosaga kelamiz. Shu bilan teorema isbot bo‘ldi, chunki funksiya soha ichida eng katta qiymatga ega deb, uning o‘zgarmasligiga erishdik. Eng kichik qiymatga erishish holi ham shunga o‘xshash isbot qilinadi. Ma‘lumki, chegaralangan yopiq sohada uzluksiz bo‘lgan funksiya shu sohada o‘zining eng katta va eng kichik qiymatlarini qabul qiladi. Demak garmonik funksiya o‘zining eng katta va eng kichik qiymatlarini sohaning chegarasida qabul qilar ekan.

Garmonik funksiyaning 4-xossasidan quyidagi natijalarga ega bo‘lamiz.

1-natija. Agar u va v funksiayalar $\omega+\sigma$ sohada uzluksiz, ω da garmonik bo‘lib σ da $u\le v$ bo‘lsa, u holda ω sohaning ichida ham $u\le v$ bo‘ladi.

Haqiqatan, $v-u$ funksiya $\omega+\sigma$ da uzluksiz, ω da garmonik va σ da $v-u\ge 0$. Shuning uchun maksimum prinsipiga asosan ω ning ichida ham $v-u\ge 0$ bo‘lib undan natijaning isboti kelib chiqadi.

2-natija. Agar u va v funksiyalar $\omega+\sigma$ sohada uzluksiz, ω da garmonik bo‘lib σ da $|u|\le v$ bo‘lsa, u holda ω sohaning ichida ham $|u|\le v$ bo‘ladi.

Shartga binoan uchta $-v$, u va v garmonik funksiyalar uchun σ da $-v \le u \le v$ tengsizliklar bajariladi. 1-natijani ikki marta qo‘llab ω sohaning ichida ham $-v \le u \le v$ yoki $|u| \le v$ ning bajarilishiga ishonch hosil qilamiz.

3-natija. ω sohada garmonik va $\omega+\sigma$ sohada uzluksiz $u(M)$ funksiya uchun $\omega+\sigma$ sohaning barcha nuqtalarida $|u| \le \max|u|\Big|_\sigma$ tengsizlik bajariladi. Isbotlash uchun $v = \max|u|\Big|_\sigma$ deb olib 2-natijadan foydalaniladi.

3.3 Grin funksiyasi va uni chegaraviy masalalarni yechishda qo‘llash

$$\Delta u = 0 \tag{3.14}$$

tenglamaning σ sirt bilan o‘ralgan ϖ soha **ichida**

$$u\big|_\sigma = f(P) \tag{3.15}$$

shartni qanoatlantiruvchi yechimini topish talab etiladi.

Masalani yechish uchun

$$\iiint\limits_{\varpi} (u\Delta v - v\Delta u)d\varpi = \iint\limits_{\sigma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma \tag{3.6}$$

Grin formulasidan va undan chiqqan natija

$$u(M_0) = \frac{1}{4\pi}\iint\limits_{\sigma} \left(\frac{1}{r_{PM_0}}\frac{\partial u}{\partial n} - u\frac{\partial(\frac{1}{r_{PM_0}})}{\partial n} \right) d\sigma - \frac{1}{4\pi}\iiint\limits_{\omega} \frac{\Delta u}{r_{\mu\mu_0}} d\omega \tag{3.8}$$

dan foydalanamiz, bunda M_0 sohaning aniq nuqtasi, M uning ixtiyoriy nuqtasi, P esa σ sirtning ixtiyoriy nuqtasi r_{MM_0} $-M(x,y,z)$ va $M_0(x_0,y_0,z_0)$ nuqtalar orasidagi masofa. (3.6) formulada $v(M)$ garmonik deb olsak, u

$$0 = \iint\limits_{\sigma} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})d\sigma - \iiint\limits_{\omega} v\Delta u d\omega \tag{3.16}$$

ko‘rinishni oladi. (3.8) va (3.16) tenglamani qo‘shamiz, u holda

$$u(M_0) = \iint\limits_{\sigma} \left[\frac{\partial u}{\partial n}\left(v + \frac{1}{4\pi r_{PM_0}} \right) - u\frac{\partial}{\partial n}\left(v + \frac{1}{4\pi r_{PM_0}} \right) \right] d\sigma - \iiint\limits_{\omega} \Delta u \left(v + \frac{1}{4\pi r_{MM_0}} \right) d\omega\,. \tag{3.17}$$

$$v + \frac{1}{4\pi r_{MM_0}} = G(M, M_0) \tag{3.18}$$

belgilashni kiritsak (3.17) tenglik

$$u(M_0) = \iint\limits_{\sigma} \left[\frac{\partial u}{\partial n}G(P,M_0) - u\frac{\partial}{\partial n}G(P,M_0) \right] d\sigma - \iiint\limits_{\omega} G(M,M_0)\Delta u d\omega \tag{3.19}$$

ko‘rinishni oladi.

Shu paytgacha $v(M)$ funksiyani ixtiyoriy garmonik funksiya deb qaradik. Endi bu funksiyaga sirt ustida $-\dfrac{1}{4\pi r_{PM_0}}$ gat eng bo‘lish talabini qo‘yamiz, ya’ni $v(M)$ funksiya $\Delta u = 0$ tenglamaning $v\big|_\sigma = -\dfrac{1}{4\pi r_{PM_0}}$ shartni qanoatlantiruvchi yechimi bo‘lishi kerak.

U holda (3.18) tenglik bilan aniqlangan $G(M,M_0)$ funksiya uchun quyidagilar o‘rinli:
1) $G(M,M_0)$ funksiya ω sohada M_0 nuqtadan boshqa barcha nuqtalarda garmonik funksiya,

M_0 nuqtada esa cheksizlikka aylanadi, shu bilan birga $G(M,M_0)-\frac{1}{4\pi r_{PM_0}}$ ayirma chekli bo‘lib qoladi; 2) sirt ustida $G(P,M_0)=0$. Aytilganlarga asoslanib (3.19) ni quyidagicha yozish mumkin:

$$u(M_0)=-\iint_\sigma u\frac{\partial G}{\partial n}d\sigma-\iiint_\omega G(M,M_0)\Delta u d\omega\,. \qquad (3.20)$$

Endi biz boshda qo‘yilgan ichki chegaraviy masalani yechishimiz mumkin. Haqiqatan, (3.20) dan $\Delta u=0$ va $u_\sigma=f$ ga asosan

$$u(M_0)=-\iint_\sigma f\frac{\partial G}{\partial n}d\sigma \qquad (3.21)$$

masalaning yechimi bo‘ladi. Demak, bu formulaga ko‘ra agar $G(M,M_0)$ funksiya aniq bo‘lsa, chegaraviy masalaning yechimi aniq bo‘lar ekan. Boshqacha aytganda Dirixlening ichki masalasini yechimi $G(M,M_0)$ funksiyaga bog‘liq bo‘lar ekan. $G(M,M_0)$ funksiya **Grin funksiyasi** deb ataladi. Grin funksiyasi quyidagi xossalarga ega: a) ω sohaning barcha ichki nuqtalarida $G(M,M_0)$ musbatdir.

Haqiqatan, σ sirt ustida $G(M,M_0)$ nolga teng, agar M_0 nuqtani markaz qilib kichik sfera chizsak, bu sfera ustida funksiya musbat, chunki $M\to M_0$ da $G(M,M_0)\to+\infty$; u holda garmonik funksiyalarning to‘rtinchi xossasiga asosan $G(M,M_0)$ funksiya butun soha ichida ham musbat bo‘lishi kerak.

$v(M)$ funksiya soha chegarasida $-\frac{1}{4\pi r_{PM_0}}$ manfiy qiymat qabul qilgani uchun butun ω sohada manfiy bo‘lishi kerak, endi (3.18) dan va $G(M,M_0)$ ning musbatligidan ω soha ichida quyidagi tengsizliklar o‘rinli bo‘ladi.

$$0<G(M,M_0)<\frac{1}{4\pi r_{MM_0}}\,.$$

$G(M,M_0)$ funksiyaning sirt ichi ω da musbat va sirt ustida 0 ga teng bo‘lishi (kamayuchiligi) dan $\frac{\partial G}{\partial n}|_\sigma\le 0$ ekanligi kelib chiqadi.

b) Grin funksiyasi simmetrik funksiyadir, ya‘ni $G(M,M_0)=G(M_0,M)$.

Izoh. Tekislikda Grin funksiyasi

$$G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}}+v(M)$$

ko‘rinishda bo‘ladi. Bunda Dirixlening ichki masalasini yechimi

$$u(M_0)=-\int_L f\frac{\partial G}{\partial n}dl \qquad (3.21')$$

bo‘ladi, bunda L yopiq silliq chiziq bilan chegaralangan D soha qaraladi, integral egri chiziqli integraldir.

3.4. Birinchi chetki masala yechimining yagonaligi

Garmonik funksiyalarning xossalarga asosanib Dirixle masalasi yechimining yagonaligini isbotlaymiz. Yopiq silliq σ sirt bilan chegaralangan uch o‘lchovli ω sohani qaraymiz. $f(P)$ funksiya σ sirtda berilgan uzluksiz funksiya bo‘lsin. Bu holda Laplas tenglamasi uchun Dirixle masalasini yechish quyidagi shartlarni qanoatlantiruvchi u funksiyani topish demakki u:

a) ω sohada va uning σ chegarasida aniqlangan va uzluksiz;

b) ω sohaning ichida $\Delta u = 0$ Laplas tenglamasini qanoatlantiradi;

c) σ chegarada berilgan $f(P)$ qiymatini qabul qiladi.

Yagonalik teoremasini isbotlaymiz.

Laplas tenglamasi uchun birinchi chetki masala ikkita har xil yechimlarga ega bo‘laolmaydi. Faraz qilaylik, masala ikkita har xil u_1 va u_2 yechimlarga ega bo‘lsin; ya‘ni u_1 va u_2 sohada va uning chegarasida uzluksiz, sohaning ichida garmonik funksiyalarning, har biri $\Delta u = 0$ Laplas tenglamasini qanoatlantiradi va σ chegarada bir xil $f(P)$ qiymatni qabul qiladi. U holda bu funksiyalarning ayirmasi $u = u_1 - u_2$ funksiya yopiq ($\omega + \sigma$) sohada uzluksiz, soha ichida garmonik, ya‘ni $\Delta u = 0$ va sohaning chegarasida

$$u\big|_{\sigma} = f(P) - f(P) = 0 .$$

Garmonik funksiyaning to‘rtinchi xossasiga ko‘ra bu funksiya soha ichida ham nolga teng bo‘ladi. Bundan butun $\omega + \sigma$ yopiq sohada $u_1 \equiv u_2$ degan xulosaga kelamiz. Yana shu xossaga asoslanib Laplas tenglamasi uchun birinchi chetki masalaning yechimi chegaraviy shartlarga uzluksiz bog‘liqligini ko‘rsatamiz.

Faraz qilaylik $\Delta u = 0$ tenglamaning u_1 va u_2 yechimi chegarada mos ravishda f_1 va f_2 funksiyalarga teng bo‘lsin va bu funksiyalar $|f_1 - f_2| < \varepsilon$ shartni qanoatlantirsin, ya‘ni chegarada berilgan funksiyalar bir-biriga yaqin deb faraz qilamiz. U holda bularga mos yechimlar ham bir-biriga yaqin bo‘lishini ko‘rsatamiz. Ikkala yechim ham $\omega + \sigma$ sohada uzluksiz, ω sohada garmonik bo‘lib σ chegarada $|u_1 - u_2| = |f_1 - f_2| < \varepsilon$ bo‘lganligi uchun 2-natijaga asosan $|u_1 - u_2| < \varepsilon$ tengsizlik ω sohaning ichida ham bajariladi. Bu tengsizlik u_1 va u_2 funksiyalarni butun yopiq sohada bir-biriga yaqinligini ko‘rsatadi.

Derixle tashqi masalasi yechimining yagonaligi ham shunga o‘xshash isbotlanadi.

Neyman ichki masalasining yechimlari ixtiyoriy o‘zgarmas aniqligida topilishini, ya‘ni u_1 va u_2 funksiyalar masalaning bir xil chegara shartlarni qanoatlantiruvchi yechimlari bo‘lganda $u_1 = u_2 + const$ bo‘lishini ta‘kidlab o‘tamiz.

Neyman tashqi masalasining yechimi chegarada uzluksiz hosilalarga ega bo‘lishlik sharti bilan yagona bo‘ladi.

Nazorat uchun savollar

1. Ostrogradskiy-Grin formulasini yozing.
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4. Matematik fizikaning ba'zi-bir tenglamalari va ularning yechimlari haqida

4.1. Fazoda issiqlikning tarqalishi

Issiqlikning tarqalish jarayonini uch o'lchovli fazoda qaraymiz. Fazoda notekis qizdirilgan jism berilgan bo'lsin.

$u(x, y, z, t)$- koordinatalari x, y, z bo'lgan nuqtaning t paytdagi harorati bo'lsin. ΔS yuzchadan o'tuvchi issiqlik tezligi, ya'ni bir birlik vaqt ichida oqib o'tuvchi issiqlik miqdori ushbu

$$\Delta Q = -k\frac{\partial u}{\partial n}\Delta S \qquad (4.1)$$

formula bilan aniqlanishi tajriba yo'li bilan tasdiqlangan; bunda k -qaralayotgan muhitning issiqlik o'tkazish koeffitsiyenti, $\vec{n}$ - issiqlik harakati yo'nalishida ΔS yuzchaga normal bo'yicha yo'nalgan birlik vektor. Maydon nazariyasidan ma'lumki, $u(x, y, z)$ funksiyaning normal vektor yo'nalishi bo'yicha hosilasi $grad\, u$ vektor bilan $\vec{n}$ vektorning skalyar ko'paytmasiga teng, ya'ni

$$\frac{\partial u}{\partial n} = gradu \cdot \vec{i}\,,$$

bunda $\vec{n}$- normal yo'nalishidagi birlik vektor. $\frac{\partial u}{\partial n}$ ning ifodasini (4.1) formulaga qo'yib

$$\Delta Q = -\kappa \cdot \vec{n} \cdot grad\, u \cdot \Delta S$$

tenglikni hosil qilamiz.

Δt vaqtda ΔS yuzchadan oqib o'tuvchi issiqlik miqdori

$$\Delta Q = -\kappa \cdot \vec{n} \cdot grad\, u \cdot \Delta t \cdot \Delta S$$

bo'ladi.

Endi yuqorida qo'yilgan masalaga qaytamiz. Qaralayotgan muhitda S sirt bilan chegeralangan kichik V jismni ajratamiz. S sirtdan oqib o'tuvchi issiqlik miqdori

$$Q = -\Delta t \iint_S k \cdot \vec{n} \cdot grad\, u \; dS \qquad (4.2)$$

bo'ladi, bunda $\vec{n}$ vektor S sirtga tashqi normal bo'yicha yo'nalgan birlik vektor. (4.2) formulaning Δt vaqtda V jismga kirib keluvchi (yoki V jismdan chiqib ketuvchi) issiqlik miqdorini berishi ravshan. V jismga kiruvchi issiqlik miqdori bu jismdagi moddaning haroratini ko'tarishga ketadi. Δv elementar jismni qaraymiz. Δt vaqtda uning harorati Δu ga ko'tarilgan bo'lsin. Δv element haroratini shu tariqa ko'tarishga sarf bo'lgan issiqlik miqdori quyidagiga teng bo'lishi ravshan.

$$c\Delta v\, \rho \Delta u \approx c\Delta v\, \rho \frac{\partial u}{\partial t}\Delta t,$$

bunda c- moddaning issiqlik sig'imi, ρ-zichligi, Δv-elementar jismning hajmi.

V -jismda harorat ko'tarilishiga sarf bo'lgan issiqlikning umumiy miqdori

$$\Delta t \iiint_V c\rho \frac{\partial u}{\partial t} dv$$

bo'ladi.

Ammo bu V jismga Δt vaqtda kirgan issiqlik miqdori bo'lib u yana (4.2) formula bilan ham aniqlanadi. Shunday qilib, ushbu tenglik hosil bo'ladi:

$$-\Delta t\iint_S k\vec{n}\cdot grad\,u\cdot dS=\Delta t\iiint_V c\rho\frac{\partial u}{\partial t}dv\,.$$

Δt ga qisqartirsak

$$-\iint_S (k\,grad\,u)\cdot\vec{n}\cdot dS=\iiint_V c\rho\frac{\partial u}{\partial t}dv$$

bo'ladi. Bu tenglikning chap tomonidagi sirt integraliga Ostrogradskiy-Grin formulasini qo'llab

$$-\iiint_v div(k\,grad\,u)dv=\iiint_V c\rho\frac{\partial u}{\partial t}dv$$

yoki

$$\iiint_v\left[div(k\,grad\,u)+c\rho\frac{\partial u}{\partial t}\right]dv=0 \tag{4.3}$$

tenglikni hosil qilamiz.

Bu tenglamaning chap tomonidagi uch o'lchovli integralga o'rta qiymat haqidagi teoremani tatbiq qilib

$$\left[div(kgrad\,u)+c\rho\frac{\partial u}{\partial t}\right]_{x=x_1,y=y_1,z=z_1}=0 \tag{4.4}$$

tenglikni hosil qilamiz, bunda $P(x_1,y_1,z_1)$ nuqta $-V$ jismdagi biror nuqta.

Issiqlik tarqalayotgan uch o'lchovli fazoda biz ixtiyoriy V jismni ajratishimiz mumkin bo'lgani uchun va (4.3) tenglamada integral ostidagi funksiyani uzluksiz funksiya deb faraz qilganimiz uchun (4.4) tenglik fazoning har bir nuqtasida bajariladi. Shunday qilib,

$$c\rho\frac{\partial u}{\partial t}=-div(k\,gradu)\,. \tag{4.5}$$

Ammo

$$k\,gradu=k\frac{\partial u}{\partial x}\vec{i}+k\frac{\partial u}{\partial y}\vec{j}+k\frac{\partial u}{\partial z}\vec{k}$$

va $div(k\,gradu)=\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right)$ edi. Bu qiymatni (4.5) tenglamaga qo'yib,

$$-c\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right) \tag{4.6}$$

tenglikni hosil qilamiz.

Agar k -o'zgarmas bo'lsa, u holda

$$div(k\,gadu)=kdiv(\,gradu)=k\left(\frac{\partial^2 è}{\partial x^2}+\frac{\partial^2 è}{\partial y^2}+\frac{\partial^2 è}{\partial z^2}\right)$$

bo'lib (4.6) tenglama

$$-ñ\rho\frac{\partial è}{\partial t}=k\left(\frac{\partial^2 è}{\partial x^2}+\frac{\partial^2 è}{\partial y^2}+\frac{\partial^2 è}{\partial z^2}\right)$$

yoki, $-\frac{\kappa}{c\rho}=a^2$ deb faraz qilsak

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{4.7}$$

ko'rinishga ega bo'ladi. Bu tenglama **Puasson tenglamasi** deb aytiladi.

(4.7) tenglamani qisqacha

$$\frac{\partial u}{\partial t} = a^2 \Delta u$$

ko'rinishda yozish ham mumkin, bunda $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplas operatori. (4.7) tenglama fazoda issiqlik o'tkazish tenglamasidir. Buning qo'yilgan masalaga to'liq javob beradigan birgina yechimini topish uchun chetki shartlarni berish zarur.

Sirti σ dan iborat bo'lgan Ω jismni olib unda issiqlikning tarqalish jarayonini qaraymiz. Boshlang'ich paytda jismning harorati berilgan. Bu $t=0$ **boshlang'ich shartda** yechimning qiymati ma'lum ekanini anglatadi:

$$u(x,y,z,0) = \varphi(x,y,z). \tag{4.8}$$

Bundan tashqari vaqtning ixtiyoriy t paytida jism sirtining har qanday M nuqtasidagi harorat ma'lum bo'lishi kerak-chegaraviy shart:

$$u(M,t) = \psi(M,t). \tag{4.9}$$

(Boshqa chegaraviy shartlar bo'lishi ham mumkin).

Izlanayotgan $u(x,y,z,t)$ funksiya z ga bog'liq bo'lmaganda

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.10}$$

-tekislikda issiqlik tarqalish tenglamasiga ega bo'lamiz. Agar chegarasi L bo'lgan tekis D sohada issiqlik tarqalishi qaralsa, u holda

(4.8) va (4.9) ga o'xshash chetki shartlar

$$u(x,y,0) = \varphi(x,y), \quad u(M,t) = \psi(M,t)$$

kabi ifodalanadi, bunda φ va ψ - berilgan funksiyalar, M esa L chegaradagi nuqta.

u funksiya faqat x va t ga bog'liq bo'lganda

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

-sterjenda issiqlik tarqalish tenglamasiga ega bo'lamiz. Bu tenglama uchun chetki shartlar

$$u(x,0) = \varphi(x), \tag{4.11}$$

$$u(0,t) = \psi_1(t), \tag{4.12}$$

$$u(l,t) = \psi_2(t) \tag{4.13}$$

kabi ifodalanishi mumkin, bunda l- sterjenning uzunligi. (4.11) shart (boshlang'ich shart) fizika nuqtai nazardan $t=0$ da (boshlang'ich paytda) sterjenning turli kesimlarida $\varphi(x)$ ga teng harorat berilganligini anglatadi. (4.12) va (4.13) shartlar (chegaraviy shartlar) $x=0$ va $x=l$ bo'lganda sterjen uchlarida mos tartibda $\psi_1(t)$ va $\psi_2(t)$ ga teng bo'lgan haroratning saqlanishiga to'g'ri keladi.

Endi $\Delta u = 0$ Laplas tenglamasiga keltiriladigan ba'zi-bir masalalarni qaraymiz.

4.2. Bir jinsli jismda issiqlikning statsionar taqsimoti masalasi

σ sirt bilan chegaralangan bir jinsli Ω jism berilgan bo'lsin. Jismning turli nuqtalaridagi harorati

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

tenglamani qanoatlantirishi ko‘rsatildi. Agar jarayon statsionar bo‘lsa, ya‘ni harorat vaqtga bog‘liq bo‘lmasdan, balki faqat jism nuqtalarning koordinatalari x, y, z larga bog‘liq bo‘lsa, u holda $\frac{\partial u}{\partial t} = 0$ va demak, harorat Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (4.14)$$

ni qanoatlantiradi. Jismning haroratini bu tenglamadan bir qiymatli aniqlash uchun σ sirtdagi haroratni bilish kerak. Shunday qilib, (4.14) tenglama uchun chetki masala quyidagicha ifodalanadi.

Ω jism ichida (4.14) tenglamani qanoatlantiruvchi va σ sirtning har bir M nuqtasida berilgan

$$u\big|_{\sigma} = \psi(M) \qquad (4.15)$$

qiymatni qabul qiluvchi $u(x, y, z)$ funksiyani topish talab etilsin. Bu masala Dirixle masalasi yoki (4.14) Laplas tenglamasi uchun birinchi chetki masala deb atalishi aytilgan edi.

Qaralayotgan holda tashqi masalada $u(M)$ ning cheksizlikda nolga aylanishi sharti M nuqta sirtdan uzoqlashgan sari sirt haroratining unga ta‘siri kamayib borishini anglatadi.

Agar harorat tarqalishi L kontur bilan chegaralangan D sohada qaralsa, u holda u funksiya ikkita x va y o‘zgaruvchilarga bog‘liq bo‘ladi, hamda

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (4.16)$$

tenglamani qanoatlantiradi; bu tenglama tekislikdagi Laplas tenglamasi deyiladi. Bu tenglama uchun (4.15) chegaraviy shart L konturda bajarilishi kerak.

4.3. Suyuqlik yoki gazning potensial oqimi

σ sirt bilan chegaralangan Ω jism ichida suyuqlik oqadigan bo‘lsin. ρ - suyuqlikning zichligi bo‘lsin. Suyuqlikning tezligini

$$\vec{v} = v_{\delta}\vec{i} + v_y\vec{j} + v_z\vec{k} \qquad (4.17)$$

bilan belgilaymiz, bunda $v_{\delta}, v_y, v_z - \vec{v}$ vektorning koordinata o‘qlaridagi proyeksiyalari. Ω jismda S sirt bilan chegaralangan kichik ω jism ajratamiz. Δt vaqtda S sirtning har bir ΔS elementi orqali

$$\Delta Q = \rho\,\vec{v}\cdot\vec{n}\,\Delta S\,\Delta t$$

miqdorda suyuqlik oqib o‘tadi, bunda $\vec{n}$ S sirtga tashqi normal bo‘yicha yo‘nalgan birlik vektor. ω jismga oqib kirgan (yoki ω jismdan oqib chiqqan) suyuqlikning umumiy miqdori Q ushbu integral bilan ifoda etiladi:

$$Q = \Delta t \iint\limits_{S} \rho\vec{v}\cdot\vec{n}\,ds\,. \qquad (4.18)$$

t paytda ω jismdagi suyuqlik miqdori

$$\iiint\limits_{\omega} \rho\, d\omega$$

bo‘lgan.

Δt vaqtda suyuqlik miqdori, zichlikning o‘zgarishiga binoan, quyidagi miqdorga o‘zgaradi:

$$Q = \iiint\limits_{\omega} \Delta\rho\, d\omega \approx \Delta t \iiint\limits_{\omega} \frac{\partial\rho}{\partial t}\, d\omega\,. \qquad (4.19)$$

ω jismda manbalar yo‘q deb faraz qilib, bu o‘zgarish miqdori (4.18) tenglik bilan aniqlangan suyuqlikning oqib kirishidan kelib chiqadi degan xulosaga kelamiz.

(4.18) va (4.19) tengliklarning o‘ng tomonlarini tenglashtirib va Δt ga qisqartirib, quyidagini hosil qilamiz:

$$\iint_S \rho\vec{v}\cdot\vec{n}ds = \iiint_\omega \frac{\partial\rho}{\partial t}d\omega . \qquad (4.20)$$

Chap tomondagi sirt integralini Ostrogradskiy-Grin formulasiga ko‘ra almashtirsak, (4.20) tenglik bunday ko‘rinishni oladi:

$$\iiint_\omega div(\rho\vec{v})d\omega = \iiint_\omega \frac{\partial\rho}{\partial t}d\omega$$

yoki

$$\iiint_\omega (\frac{\partial\rho}{\partial t} - div(\rho\vec{v}))d\omega = 0 .$$

ω jismning ixtiyoriy ekanligiga va integral ostidagi funksiyaning uzluksizligiga asosan ushbuni hosil qilamiz:

$$\frac{\partial\rho}{\partial t} - div(\rho\vec{v}) = 0 \qquad (4.21)$$

yoki

$$\frac{\partial\rho}{\partial t} - \frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z) = 0 . \qquad (4.21')$$

Ana shu tenglama **siqiladigan suyuqlik oqimining** uzluksizlik tenglamasidir.

Izoh. Ba‘zi masalalarda, masalan neft va gazning yer ostidagi g‘ovak muhitda quduqqa tomon harakati jarayonini qarashda

$$\vec{v} = -\frac{k}{\rho} grad\, p$$

deb qabul qilish mumkin, bunda p -bosim, k - o‘tkazuvchanlik koeffitsiyenti va

$$\frac{\partial\rho}{\partial t} \approx \lambda\frac{\partial p}{\partial t},$$

λ =const. Buni (4.21) uzluksizlik tenglamasiga qo‘yib, ushbuni hosil qilamiz:

$$\lambda\frac{\partial p}{\partial t} + div(k\, grad\, p) = 0$$

yoki

$$-\lambda\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial p}{\partial z}\right). \qquad (4.22)$$

Agar k o‘zgarmas son bo‘lsa, bu tenglama

$$\frac{\partial p}{\partial t} = -\frac{k}{\lambda}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) \qquad (4.23)$$

ko‘rinishni oladi va biz issiqlik o‘tkazish tenglamasiga kelamiz. (4.21) tenglamaga qaytamiz. Agar suyuqlik siqilmaydigan bo‘lsa, u holda $\rho = const$, $\frac{\partial\rho}{\partial t} = 0$ bo‘lib (4.21) tenglama bunday ko‘rinishni олади:

$$div\ \vec{v} = 0 . \qquad (4.24)$$

Agar harakat potensial bo‘lsa, ya‘ni $\vec{v}$ vektor biror φ funksiyaning gradiyenti, ya‘ni $\vec{v} = grad\,\varphi$ bo‘lsa, u holda (4.24) tenglik ushbu ko‘rinishni oladi: $div(grad\,\varphi) = 0$ yoki

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (4.25)$$

ya‘ni $\vec{v}$ tezlikning potensial funksiyasi φ Laplas tenglamasini qanoatlantiradi. Ko‘p masalalarda, masalan, filtrlanish (sizilish) masalalarida

$$\vec{v} = -k_1 \, grad \, p$$

deb qabul qilish mumkin, bunda p-bosim, k_1-o‘zgarmas son; u holda bosimni aniqlovchi

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \qquad (4.25')$$

Laplas tenglamasini hosil qilamiz. Bu tenglama uchun chetki shartlar quyidagicha berilishi mumkin:

1. σ sirtda izlanayotgan P funksiyaning qiymatlari-bosimlar beriladi:

$$p|_\sigma = f(M).$$

2. σ sirtda $\frac{\partial p}{\partial n}$ - normal bo‘yicha hosilaning qiymatlari beriladi–oqim sirt orqali beriladi:

$$\frac{\partial p}{\partial n}\Big|_\sigma = g(M).$$

3. σ sirtning bir qismida P-bosimlar, yana bir qismida $\frac{\partial p}{\partial n}$ hosila beriladi.

Matematik fizikaning barcha tenglamalari hamda ularning yechimlarini topish formulalarini keltirib chiqarishni lozim topmasdan matemarik fizikaning ko‘p uchraydigan ayrim tenglamalarini ko‘rinishlarini hamda ularning yechimlarini topish formlalarini keltirishni lozim topdik.

4.4. Chegaralangan torning erkin tebranishi tenglamasi

Ikki tomondan mahkamlangan torning erkin tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

ning $u|_{t=0} = \varphi(x)$, $\frac{\partial u}{\partial t}\Big|_{t=0} = F(x)$ boshlang‘ich shartlar va $u|_{x=0} = 0$, $u|_{x=l} = 0$ chegaraviy shartlarni qanoatlantiruchi xususiy yechimi

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\frac{an\pi}{l}t + D_n \sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

ko‘rinishda bo‘lishi, bu yerda

$$C_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}xdx, \quad D_n = \frac{2}{an\pi}\int_0^l F(x)\sin\frac{n\pi}{l}xdx$$

Furye tomonidan isbotlangan.

1-misol. Torning erkin tebranish tenglamasi

$$u|_{t=0} = \cos x, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 2\cos x$$

boshlang‘ich shartlar va $u|_{x=0} = 0$, $u|_{x=\pi} = 0$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini toping.

Yechilishi. C_n va D_n koeffiitsiyentlarni hisoblaymiz.

$$C_n = \frac{2}{l}\int_0^l \cos x \sin\frac{n\pi}{l}xdx = \frac{2}{\pi}\int_0^\pi \cos x \sin nxdx.$$

$n=1$ bo'lsin. U holda $C_1=\frac{2}{\pi}\int\limits_0^{\pi}\cos x\sin xdx=\frac{2}{\pi}\int\limits_0^{\pi}\sin xd(\sin x)=\frac{1}{\pi}\sin^2 x\Big|_0^{\pi}=0.$ $n\geq 2$ bo'lganda

$$C_n=\frac{2}{\pi}\int\limits_0^{\pi}\cos x\sin nxdx=\frac{2}{\pi}\cdot\frac{1}{2}\int\limits_0^{\pi}[\sin(nx+x)+\sin(nx-x)]dx=-\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]\Bigg|_0^{\pi}=$$

$$=-\frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}-\frac{1}{n+1}-\frac{1}{n-1}\right]=\frac{2n}{\pi(n^2-1)}-\frac{1}{\pi}\left(\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}\right).$$

Istalgan butun n son uchun $\cos 2n\pi=1,\quad \cos(2n+1)\pi=-1$ ekanligini hisobga olib

$$P=\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}$$

ifodani hisoblaymiz. n-juft son bo'lsin. U holda $n+1$ va $n-1$ lar toq son bo'lib, $P=-\frac{1}{n+1}-\frac{1}{n-1}=-\frac{2n}{n^2-1}$ bo'ladi.

n-toq son bo'lganda. $n+1$ va $n-1$ lar juft son bo'lib $P=\frac{1}{n+1}+\frac{1}{n-1}=\frac{2n}{n^2-1}$ bo'ladi.

Shunday qilib n juft son uchun

$$C_n=\frac{2n}{\pi(n^2-1)}-\frac{1}{\pi}\left(-\frac{2n}{n^2-1}\right)=\frac{4n}{\pi(n^2-1)},$$

n toq son uchun $C_n=\frac{2n}{\pi(n^2-1)}-\frac{2n}{\pi(n^2-1)}=0$ bo'ladi.

$$D_n=\frac{2}{an\pi}\int\limits_0^{\pi}2\cos x\sin nxdx=\frac{2}{an}\cdot\frac{2}{\pi}\int\limits_0^{\pi}\cos x\sin nxdx=$$

$$=\frac{2}{an}\cdot C_n=\begin{cases}\dfrac{8}{a\pi(n^2-1)}, & n\ \ juft\ \ son\ \ bo'lganda,\\ 0, & n\ \ toq\ \ son\ \ bo'lganda\end{cases}$$

Demak,

$$u(x,t)=\sum_{k=1}^{\infty}\left(\frac{8k}{\pi(4k^2-1)}\cdot\cos 2akt+\frac{8}{a\pi(4k^2-1)}\sin 2akt\right)\cdot\sin 2kx$$

yoki

$$u(x,t)=\frac{8}{\pi}\sum_{k=1}^{\infty}\left(\frac{k}{4k^2-1}\cdot\cos 2akt+\frac{1}{a(4k^2-1)}\sin 2akt\right)\cdot\sin 2kx$$

funksiya berilgan tenglamaning berilgan chetki shartlarini qanoatlartiruvchi xususiy yechimi bo'ladi.

4.5. Chegaralangan torning majburiy tebranish tenglamasi

Chegaralangan torning majburiy tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t)$$

ni $u\big|_{t=0}=\varphi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0}=F(x)$ boshlang'ich va $u\big|_{x=0}=0,\ u\big|_{x=l}=0$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini topamiz.

Yechimni $u(x,t)=v(x,t)+w(x,t)$ ko'rinishda izlaymiz:

Bu yerdagi $v(x,t)$ funksiyani shunday tanlaymizki, u bir jinsli $\frac{\partial^2 v}{\partial t^2}=a^2\frac{\partial^2 v}{\partial x^2}$ tenglamani $v|_{t=0}=\varphi(x),\ \frac{\partial v}{\partial t}|_{t=0}=F(x)$ boshlang‘ich va $v|_{x=0}=0,\ v|_{x=l}=0$ chegaraviy shartlarni qanoatlantirsin. $w(x,t)$ funksiya esa

$$\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2}+f(x,t)$$

tenglamani $w|_{t=0}=0,\ \frac{\partial w}{\partial t}|_{t=0}=0$ boshlangich va $w|_{x=0}=0,\ w|_{x=l}=0$ chegaraviy shartlarni qanoatlantirsin.

Chegaralangan torning majburiy tebranish tenglamasining yechimi $w(x,t)$ yig‘indi ko‘rinishida topiladi:

$$w(x,t)=\sum_{k=1}^{\infty}\gamma_k(t)\sin\frac{k\pi x}{l}, \tag{1}$$

bu yerda

$$\gamma_k(t)=\frac{1}{k\pi a}\int_0^t g_k(\tau)\sin\frac{k\pi a(t-\tau)}{l}d\tau, \tag{2}$$

$$g_k(t)=\frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi x}{l}dx. \tag{3}$$

2-misol. $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+x(x-1)t^2$ tenglamaning $u(x,0)=0,\ \frac{\partial u}{\partial t}|_{t=0}=0$ boshlang‘ich hamda $u(0,t)=0,\ \ u(1,t)=0$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini toping.

Yechilishi. (3) ga $f(x,t)=x(x-1)t^2,\ a=1,\ l=1$ qiymatlarni qo‘yib hisoblaymiz:

$$g_k(t)=2\int_0^1(x^2-x)t^2\sin k\pi x dx=2t^2\int_0^1(x^2-x)\sin k\pi x dx$$

$$\left|\begin{matrix}u=x^2-x, & du=(2x-1)dx\\ \sin k\pi x dx=dv, & v=-\frac{\cos k\pi x}{k\pi}\end{matrix}\right|=2t^2\left[-(x^2-x)\frac{\cos k\pi x}{k\pi}\Big|_0^1+\right.$$

$$\left.+\frac{1}{k\pi}\int_0^1\cos k\pi x(2x-1)dx\right]\left|\begin{matrix}2x-1=u, & du=2dx\\ \cos 2\pi x dx=dv, & v=\frac{\sin k\pi x}{k\pi}\end{matrix}\right|=$$

$$=\frac{2t^2}{k\pi}\left[(2x-1)\frac{\sin k\pi x}{k\pi}\Big|_0^1-\frac{1}{k\pi}\int_0^1\sin k\pi x 2dx\right]=-\frac{4t^2}{(k\pi)^2}\int_0^1\sin k\pi x dx=$$

$$=\frac{4t^2}{(k\pi)^3}\cos k\pi x\Big|_0^1=\frac{4t^2}{(k\pi)^3}(\cos k\pi-1)=\frac{4t^2}{(k\pi)^3}((-1)^k-1).$$

Shunday qilib $g_k(t)=-\frac{8t^2}{(k\pi)^3}$, k toq son bo‘lganda va $g_k(t)=0$ k juft son bo‘lganda.

$g_k(t)$ ning topilgan qiymatini (2) ga qo‘yib hisoblaymiz:

$$\gamma_k(t)=\frac{1}{(2k+1)\pi}\int_0^t\left(-\frac{8\tau^2}{[(2k+1)\pi]^3}\right)\cdot\sin(2k+1)\pi(t-\tau)d\tau=$$

$$= -\frac{8}{[(2k+1)\pi]^4}\int_0^1 t^2\sin(2k+1)\pi(t-\tau)d\tau \left| \begin{array}{l} \tau^2 = u,\ \ 2\tau d\tau = du, \\ dv = \sin(2k+1)\pi(t-\tau), \\ v = \dfrac{1}{(2k+1)\pi}\cdot\cos(2k+1)\pi(t-\tau) \end{array} \right| =$$

$$= -\frac{8}{[(2k+1)\pi]^4}\left[\frac{t^2\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi}\right|_0^t -$$

$$-\frac{1}{(2k+1)\pi}\int_0^t \cos(2k+1)\pi(t-\tau)d\tau\cdot 2\tau dt \left| \begin{array}{l} 2\tau = u,\ \ du = 2d\tau \\ \cos(2k+1)\pi(t-\tau) = dv \\ v = -\dfrac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi} \end{array} \right| =$$

$$= -\frac{8}{[(2k+1)\pi]^4}\left[\frac{t^2}{(2k+1)\pi} - \frac{1}{(2k+1)\pi}\left(-2\tau\frac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi}\right|_0^t + \right.$$

$$\left. + \frac{2}{(2k+1)\pi}\int_0^t \sin(2k+1)\pi(t-\tau)d\tau)\right] = -\frac{8}{[(2k+1)\pi]^4}\left[\frac{t^2}{(2k+1)\pi} - \right.$$

$$\left. - \frac{2}{[(2k+1)\pi]^2}\cdot\frac{\cos(2k+1)\pi(t-\tau)}{(2k+1)\pi}\right|_0^t \Bigg] = -\frac{8}{[(2k+1)\pi]^4}\left[\frac{t^2}{(2k+1)\pi} - \right.$$

$$\left. - \frac{2}{[(2k+1)\pi]^3}\cdot(1-\cos(2k+1)\pi t)\right].$$

Shunday qilib, $\gamma_k(t) = -\dfrac{8t^2}{(2k+1)^5\pi^5} - \dfrac{16}{(2k+1)^7\pi^7} + \dfrac{16\cos(2k+1)\pi t}{(2k+1)^7\pi^7}$. (1)ga asoslanib

$$u(x,t) = -\frac{8t^2}{\pi^5}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^5} - \frac{16}{\pi^7}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^7} + \frac{16}{\pi^7}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x\cdot\cos(2k+1)\pi t}{(2k+1)^7}$$

Izlanayotgan xususiy yechimni topamiz.

Shuni ta’kidlash joizki bir jinsli bo‘lmagan

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

to'lqin tenglamasining $-\infty < x < \infty,\ \ t > 0$ da aniqlangan va

$$u|_{t=0} = \varphi(x),\ \ \frac{\partial u}{\partial t}\Big|_{t=0} = F(x)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimi

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}F(x)dx + \frac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi\right]d\tau$$

formula yordamida topiladi. Bu formula Dyuamel formulasi deb ataladi.

3-misol. Bir jinsli bo‘lmagan

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + 2x$$

to‘lqin tenglamasining

$$u|_{t=0} = x^2,\ \ \frac{\partial u}{\partial t}\Big|_{t=0} = \cos x$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi topilsin.

Yechilishi. $a=2,\ \varphi(x)=x^2,\ F(x)=\cos x,\ f(x,t)=2x$ ekanini e'tiborga olib Dyuamel formulasini qo'llaymiz:

$$u(x,t)=\frac{(x-2t)^2+(x+2t)^2}{2}+\frac{1}{4}\int_{x-2t}^{x+2t}\cos x dx+\frac{1}{4}\left[\int_{x-2(t-\tau)}^{x+2(t-\tau)}2xdx\right]d\tau=$$

$$=\frac{x^2-4xt+4t^2+x^2+4xt+4t^2}{2}+\frac{1}{4}[\sin(x+2t)-\sin(x-2t)]+$$

$$+\frac{1}{4}\int_0^t\left[(x+2(t-\tau))^2-(x-2(t-\tau))^2\right]d\tau=x^2+4t^2+\frac{1}{2}\sin 2t\cos x+$$

$$+\frac{1}{4}\int_0^t\left[(x^2+4x(t-\tau))+4(t-\tau)^2-x^2+4x(t-\tau)-4(t-\tau)^2\right]d\tau=$$

$$=x^2+4t^2+\frac{1}{2}\sin 2t\cdot\cos x+\frac{1}{4}\cdot 8\int_0^t x(t-\tau)d\tau=x^2+4t^2+\frac{1}{2}\sin 2t\cos x+2x\int_0^t(t-\tau)d\tau=$$

$$=x^2+4t^2+\frac{1}{2}\sin 2t\cos x-2x\frac{(t-\tau)^2}{2}\Bigg|_0^t=x^2+4t^2+\frac{1}{2}\sin 2t\cos x+xt^2.$$

Demak, $u(x,t)=x^2+4t^2+xt^2+\frac{1}{2}\sin 2t\cos x$ qaralayotgan masalaning yechimi bo'lar ekan.

4.6.Chegaralangan sterjenda issiqlikning tarqalishi

Chegaralangan sterjenda issiqlikning tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\quad 0\le x\le l,\ t>0$$

ko'rinishga ega ekanligi eslatilgan edi.

Ana shu tenglamaning $u|_{t=0}=u(x,0)=\varphi(x)$ boshlang'ich shart hamda $u(0,t)=0,\ \ u(l,t)=0$ chegaraviy shartlarni qanoatlantiruchi yechimi

$$u(x,t)=\sum_{k=1}^{\infty}C_k e^{-(\frac{ak\pi}{l})^2 t}\sin\frac{k\pi}{l}x$$

formula yordamida topiladi, bu yerda

$$C_k=\frac{2}{l}\int_0^e\varphi(x)\sin\frac{k\pi x}{l}dx.$$

4-misol. $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(0,t)=0,\ u(l,t)=0,\ t>0$ chegaraviy shartlarni hamda

$$u(x,0)=\begin{cases}x, & 0\le x\le\frac{l}{2}\ \ bo'\lg anda,\\ l-x, & \frac{l}{2}<x\le l\ \ bo'\lg anda\end{cases}$$

boshlang'ich shartni qanoatlantiruvchi yechimini toping.

Yechilishi. Bo'laklab integrallash formulasiga asoslanib topamiz:

$$C_k=\frac{2}{l}\int_0^{\frac{l}{2}}x\sin\frac{k\pi x}{l}dx=\left|\begin{matrix}x=u,\ du=dx,\\ \sin\frac{k\pi}{l}xdx=dv,\ \ v=-\frac{l}{k\pi}\cos\frac{k\pi}{l}x\end{matrix}\right|+$$

$$+\frac{2}{l}\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{k\pi x}{l}dx\left|\begin{matrix}l-x=u,\ du=-dx,\\ v=-\frac{l}{k\pi}\cos\frac{k\pi x}{l}\end{matrix}\right|=\frac{2}{l}\left[-\frac{xl}{k\pi}\cos\frac{k\pi x}{l}\Bigg|_0^{l/2}+\frac{l}{k\pi}\int_0^{\frac{l}{2}}\cos\frac{k\pi x}{l}dx\right]+$$

$$+\frac{2}{l}\left[-(l-x)\frac{l}{k\pi}\cos\frac{k\pi x}{l}\Bigg|_{l/2}^{l}-\frac{l}{k\pi}\int_{\frac{l}{2}}^{l}\cos\frac{k\pi x}{l}dx\right]=\frac{2}{l}\left[-\frac{l^2}{2k\pi}\cos\frac{k\pi}{2}+(\frac{l}{k\pi})^2\sin\frac{k\pi x}{2}\Bigg|_{0}^{l/2}\right]+$$

$$+\frac{2}{l}\left[\frac{l^2}{2k\pi}\cos\frac{k\pi}{2}-(\frac{l}{k\pi})^2\sin\frac{k\pi x}{l}\Bigg|_{l/2}^{l}\right]=\frac{2}{l}\left[-\frac{l^2}{2k\pi}\cos\frac{k\pi}{2}+(\frac{l}{k\pi})^2\sin\frac{k\pi}{2}\right]+$$

$$+\frac{2}{l}\left[\frac{l^2}{2k\pi}\cos\frac{k\pi}{2}-(\frac{l}{k\pi})^2(\sin k\pi-\sin\frac{k\pi}{2})\right]=\frac{2}{l}\cdot 2\frac{l^2}{k^2\pi^2}\sin\frac{k\pi}{2}=\frac{4l}{\pi^2k^2}\sin\frac{k\pi}{2}.$$

Demak $C_k=\begin{cases}\frac{4l}{\pi^2k^2}\sin\frac{k\pi}{2}, & k-toq\ bo'lganda,\\ 0, & k-juft\ bo'lganda\end{cases}$ yoki $C_k=\frac{4l(-1)^k}{\pi^2(2k+1)^2}$.

Ushbu qiymatni yechimni topish formulasiga qo‘ysak

$$u(x,t)=\frac{4l}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2}e^{-\frac{(2k+1)^2a^2\pi^2}{l^2}t}\sin\frac{(2k+1)\pi x}{l}$$

berilgan tenglamaning xususiy yechimi hosil bo‘ladi.

4.7. Chegaralanmagan sterjenda issiqlikning tarqalishi

Chegaralanmagan sterjenda issiqlikning tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\quad -\infty<x<\infty$$

ko‘rinishga ega bo‘lishi isbotlangan.

Shu tenglamaning $u|_{t=0}=\varphi(x)$ boshlang‘ich shart qanoatlantiruvchi xususiy yechimi

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{(\xi-x)^2}{4a^2t}}d\xi$$

Puasson formulasi yordamida topiladi. Tenglikning o‘ng tomonidagi integral Puasson integralidir.

5-misol. $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$, $-\infty<x<\infty$ issiqlik tarqalish tenglamasining $u(x,0)=x$ boshlang‘ich shartni qanoatlantiruvchi yechimini toping.

Yechilishi. Puasson formulasiga binoan yechim

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\xi e^{-\frac{(\xi-x)^2}{4a^2t}}d\xi$$

bo‘ladi. Bu integralni hisoblash uchun $\frac{\xi-x}{2a\sqrt{t}}=\eta$ yangi o‘zgaruvchini kiritamiz. U holda

$\xi=2a\sqrt{t}\eta+x,\ d\xi=2a\sqrt{t}d\eta$ ekanini hisobga olsak

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}(2a\sqrt{t}\eta+x)e^{-\eta^2}\cdot 2a\sqrt{t}\eta=\frac{1}{\sqrt{\pi}}\cdot 2a\sqrt{t}\int_{-\infty}^{\infty}\eta e^{-\eta^2}d\eta+\frac{x}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\eta^2}d\eta$$

bo‘ladi. $\eta e^{-\eta^2}$ toq funksiya bo‘lgani uchun $\int_{-\infty}^{\infty}\eta e^{-\eta^2}d\eta=0$. Puasson integrali $\int_{-\infty}^{\infty}e^{-\eta^2}d\eta=\sqrt{\pi}$.

Demak

$$u(x,t)=\frac{x}{\sqrt{\pi}}\cdot\sqrt{\pi}=x$$

izlanayotgan yechim bo‘ladi.

4.8. Bir tomondan chegaralangan sterjenda issiqlikning tarqalishi

Bir tomondan chegaralangan sterjenda issiqlikning tarqalish tenglamasi

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<+\infty$$

ko‘rinishga ega bo‘ladi. Shu tenglamaning $u(x,t)=\varphi(x)$ boshlang‘ich shartni hamda $u(0,t)=F(t)$ chegaraviy shartni qanoatlantiruvchi yechimi

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_0^{\infty}\varphi(\xi)\left[e^{-\frac{(\xi-x)^2}{4a^2t}}-e^{-\frac{(\xi-x)^2}{4a^2t}}\right]d\xi+\frac{x}{2a\sqrt{\pi}}\int_0^{t}F(\tau)(t-\tau)^{-\frac{3}{2}}e^{-\frac{x^2}{(t-\tau)^2}}d\tau$$

formula yordamida topiladi.

4.9. Laplas tenglamasining ba’zi sodda yechimlari

6-misol. $x^2+y^2<a^2$ doirada

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Laplas tenglamasini qanoatlantiruvchi va doiraning chegarasi $x^2+y^2=a^2$ aylanada A qiymatni qabul qiluvchi $u(x,y)$ funksiyani toping.

Yechilishi. Izlanayotgan yechim $u(x,y)=A$ bo‘ladi, chunki $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}=0$ va $u(x,y)\big|_{x^2+y^2=a^2}=A\big|_{x^2+y^2=a^2}=A$.

7-misol. Laplas tenglamasining

$$u\big|_{x^2+y^2=a^2}=\frac{Ax}{a}$$

chegaraviy shartni qanoatlantiruvchi $x^2+y^2\le a^2$ doiradagi yechimini toping.

Yechilishi. $u(x,y)=\frac{Ax}{a}$ yechim, chunki $\frac{\partial u}{\partial x}=\frac{A}{a},\ \frac{\partial u}{\partial y}=0,\ \frac{\partial^2 u}{\partial x^2}=0,\ \frac{\partial^2 u}{\partial y^2}=0$ bo‘lgani uchun u Laplas tenglamasini qanoatlantiradi va

$$u\big|_{x^2+y^2=a^2}=\frac{Ax}{a}.$$

8-misol. Laplas tenglamasining

$$u\big|_{x^2+y^2=a^2}=\frac{Ay^2}{a^2}+\frac{Bx^2}{a^2}$$

shartni qanoatlantiruvchi, $x^2+y^2\le a^2$ doiradagi yechimini toping.

Yechilishi. $v(x,y)=\dfrac{A+B}{2}+\dfrac{B-A}{2a^2}(x^2-y^2)$ funksiyani olamiz. Shu funksiya qo‘yilgan masalaning yechimi ekanligini ko‘rsatamiz. $x=a\cos\varphi,\ y=a\sin\varphi$ desak, chegaraviy shart

$u\big|_{x^2+y^2=a^2}=A\sin^2\varphi+B\cos^2\varphi$ ko‘rinishni oladi. $v(x,y)$ funksiya esa ushbu

$$v(x,y)=\frac{A+B}{2}+\frac{B-A}{2a^2}(a^2\cos^2\varphi-a^2\sin^2\varphi)=\frac{A+B}{2}+\frac{(B-A)(\cos^2\varphi-\sin^2\varphi)}{2}=$$

$$=\frac{A(1-\cos^2\varphi+\sin^2\varphi)+B(1+\cos^2\varphi-\sin^2\varphi)}{2}=A\sin^2\varphi+B\cos^2\varphi$$

ko‘rinishga keladi.

Demak, $v(x,y)$ funksiya chegaraviy shartni qanoatlantiradi. Endi shu funksiyaning Laplas tenglamasini qanoatlantirishini ko‘rsatamiz

$$\frac{\partial v}{\partial x}=\frac{B-A}{a^2}x,\quad \frac{\partial^2 v}{\partial x^2}=\frac{B-A}{a^2},\quad \frac{\partial v}{\partial y}=\frac{A-B}{a^2}y,\quad \frac{\partial^2 v}{\partial y^2}=\frac{A-B}{a^2}$$

bo‘lgani uchun $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=\dfrac{B-A}{a^2}+\dfrac{A-B}{a^2}=0.$

Demak, $v(x,y)$ funksiya Laplas tenglamasining berilgan chegaraviy shartini qanoatlantiruvchi yechimi.

4.10. Dirixlening ichki masalasini shar uchun yechilishi

Markazi koordinatalar boshida bo‘lib, radiusi R ga teng sfera bilan chegaralangan sharni qaraymiz. Sharning ichki nuqtalarida garmonik va uning chegarasi $x^2+y^2+z^2=R^2$ sferada oldindan berilgan $f(P)=f(x,y,z)$ funksiyaga teng $u(x,y,z)$ funksiyani topish talab etilsin.

Masalani sferik koordinatalarga o‘tib hal qilgan ma’qul. Sharning qaralayotgan ichki M nuqtasi $M(\rho_0,\theta_0,\varphi_0)$ koordinatalarga sferaning P nuqtasi $P(R,\theta,\varphi)$ koordinatalarga ega deb hisoblansa masalaning yechimi

$$u(M)=\frac{1}{4\pi R}\int_0^{2\pi}\int_0^{\pi} f(P)\frac{(R^2-\rho_0^2)\sin\theta d\theta d\varphi}{\left[R^2+\rho_0^2-2R\rho_0\cos(\theta-\theta_0)\right]^{\frac{3}{2}}}$$

Puasson formulasi yordamida topiladi.

4.11. Dirixlening ichki masalasini doira uchun yechilishi

$x^2+y^2\le R^2$ doira hamda uning $x^2+y^2=R^2$ aylanasida biror $f(\varphi)$ funksiya berilgan bo‘lsin (φ-qutb burchagi).

Doiraning aylanasida $u\big|_{r=R}=f(\varphi)$ qiymatni qabul qiluvchi hamda doiraning ichida qutb koordinatalaridagi Laplas tenglamasi

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=0 \qquad (4.26)$$

ni qanoatlantiruvchi $u(r,\varphi)$ funksiyani topish talab etiladi.

Bu masalaning yechimi Puasson formulasi deb ataluvchi

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{r^2-R^2}{R^2-2Rr\cos(t-\varphi)+r^2}dt$$

formulasi yordamida topiladi.

Tenglikning o‘ng tomonidagi integral Puasson integrali deb yuritiladi. Shunday qilib Puasson integrali yordamida aniqlangan $u(r,\varphi)$ funksiya qutb koordinatalaridagi Laplas tenglamasini qanoatlantiradi hamda $r\to R$ da $u(r,\varphi)\to f(\varphi)$.

9-misol. (4.26) Laplas tenglamasining $u(r,\varphi)\big|_{r=R}=2\sin\varphi$ chegaraviy shartni qanoatlantiruvchi $r<1$ doira ichidagi yechimini toping.

Yechilishi. Puasson formulasiga ko‘ra:

$$u(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}2\sin t\frac{r^2-1}{1-2r\cos(t-\varphi)+r^2}dt=\frac{r^2-1}{\pi}\int_{-\pi}^{\pi}\frac{\sin[(t-\varphi)+\varphi]}{1-2r\cos(t-\varphi)+r^2}dt=$$

$$=\frac{r^2-1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(t-\varphi)\cos\varphi+\cos(t-\varphi)\sin\varphi}{1-2r\cos(t-\varphi)+r^2}dt=\frac{(r^2-1)\cos\varphi}{\pi}\int_{-\pi}^{\pi}\frac{\sin(t-\varphi)dt}{1-2r\cos(t-\varphi)+r^2}+$$

$$+\frac{(r^2-1)\sin\varphi}{\pi}\int_{-\pi}^{\pi}\frac{\cos(t-\varphi)dt}{1-2r\cos(t-\varphi)+r^2}=\frac{(r^2-1)\cos\varphi}{\pi}\cdot J_1+\frac{(r^2-1)\sin\varphi}{\pi}\cdot J_2.$$

Endi J_1 va J_2 integrallarni hisoblaymiz.

$$J_1=\frac{1}{2r}\int_{-\pi}^{\pi}\frac{2r\sin(t-\varphi)dt}{1-2r\cos(t-\varphi)+r^2}=\frac{1}{2r}\int_{-\pi}^{\pi}\frac{d\left[1-2r\cos(t-\varphi)+r^2\right]}{1+2r\cos(t-\varphi)+r^2}=\frac{1}{2r}\ln\left[1-2r\cos(t-\varphi)+r^2\right]\Big|_{-\pi}^{\pi}=$$

$$=\frac{1}{2r}\left[\ln(1-2r\cos(\pi-\varphi)+r^2)-\ln(1-2r\cos(-\pi-\varphi)+r^2)\right]=\frac{1}{2r}\left[\ln(1+2r\cos\varphi+\pi^2)-\right.$$

$$\left.-\ln(1+2r\cos\varphi+\pi^2)\right]=0.$$

J_2 integralni hisoblash jarayonida

$$A=\int_{-\pi}^{\pi}\frac{dx}{a^2\pm 2ab\cos x+b^2}$$

ko‘rinishdagi integralga duch kelamiz. Bu integralni $tg\frac{x}{2}=t$ almashtirish olib hisoblaymiz. U holda $x=2arctgt,\ dx=\frac{2dt}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$ bo‘lib

$$A=\int_{-\infty}^{+\infty}\frac{\frac{2dt}{1+t^2}}{a^2+b^2\pm 2ab\cdot\frac{1-t^2}{1+t^2}}=2\int_{-\infty}^{+\infty}\frac{dt}{(a^2+b^2)(1+t^2)\pm 2ab(1-t^2)}=2\int_{-\infty}^{+\infty}\frac{dt}{(a^2+b^2\pm 2ab)+(a^2+b^2\mp 2ab)t^2}=$$

$$=2\int_{-\infty}^{+\infty}\frac{dt}{(a\pm b)^2+(a\mp b)^2t^2}=\frac{2}{(a\mp b)^2}\int_{-\infty}^{+\infty}\frac{dt}{\frac{(a\pm b)^2}{(a\mp b)^2}+t^2}=\frac{2}{(a\mp b)^2}\cdot\frac{a\mp b}{a\pm b}\cdot arctg\frac{a\mp b}{a\pm b}\cdot t\Big|_{-\infty}^{+\infty}=$$

$$=\frac{2}{a^2-b^2}\left[\frac{\pi}{2}-(-\frac{\pi}{2})\right]=\frac{2\pi}{a^2-b^2}$$ yoki $A=\frac{2\pi}{a^2-b^2}$ bo‘ladi.

J_2 integralda integral ostidagi funksiyani $-2r$ ga ko‘paytirib bo‘lamiz, keyin suratga $1+r^2$ ni qo‘shib ayiramiz.

$$J_2=-\frac{1}{2r}\int_{-\pi}^{\pi}\frac{(1-2r\cos(t-\varphi)+r^2-(1+r^2))}{1-2r\cos(t-\varphi)+r^2}dt=-\frac{1}{2r}\int_{-\pi}^{\pi}dt+\frac{1+r^2}{2r}\int_{-\pi}^{\pi}\frac{dt}{1-2r\cos(t-\varphi)+r^2}=$$

$$=-\frac{1}{2r}\cdot 2\pi+\frac{1+r^2}{2r}\cdot\frac{2\pi}{1-r^2}=-\frac{\pi}{r}+\frac{\pi(1+r^2)}{r(1-r^2)}.$$

Demak,

$$u(r,\varphi)=\frac{(1-r^2)\sin\varphi}{\pi}\left(-\frac{\pi}{r}+\frac{\pi(1+r^2)}{r(1-r^2)}\right)=2r\sin\varphi.$$

Shunday qilib, masalaning yechimi $u(r,\varphi)=2r\sin\varphi$ bo‘ladi.

4.12. Dirixlening ichki masalasini halqa uchun yechilishi

$x^2+y^2=R_1^2$ va $x^2+y^2=R_2^2$ ($R_1<R_2$) aylanalar bilan chegaralangan $G=\{R_1^2<x^2+y^2<R_2^2\}$ sohada (halqada)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Laplas tegnlamasining

$u\big|_{x^2+y^2=R_1^2}=u_1$, $u\big|_{x^2+y^2=R_2^2}=u_2$ (bunda u_1 va u_2 o‘zgarmas sonlar) chegaraviy shartlarni qanoatlantiruvchi yechimi

$$u=\frac{u_2\ln\frac{r}{R_1}-u_1\ln\frac{r}{R_2}}{\ln\frac{R_2}{R_1}}$$

formula yordamida topiladi. Bunda r-nuqtaning qutb radiusi ($r=\sqrt{x^2+y^2}$).

10-misol. Laplas tenglamasining $x^2+y^2=\frac{1}{4}$ va $x^2+y^2=\frac{4}{9}$ aylanalar bilan chegaralangan $G=\left\{\frac{1}{4}<x^2+y^2<\frac{4}{9}\right\}$ sohadagi yechimi $u\Big|_{x^2+y^2=\frac{1}{4}}=2$, $u\Big|_{x^2+y^2=\frac{9}{4}}=4$ chegaraviy shartlarda topilsin.

Yechilishi. Masalaning shartiga ko‘ra $R_1=\frac{1}{2}$, $R_2=2$, $u_1=2$, $u_2=4$. Shuning uchun $u=\dfrac{4\ln\frac{r}{0{,}5}-2\ln\frac{r}{1{,}5}}{\ln\frac{1{,}5}{0{,}5}}=\dfrac{4\ln 2r-2\ln\frac{2r}{3}}{\ln 3}=\dfrac{2\ln 6r}{\ln 3}$.

Demak, $u(x,y)=\dfrac{2\ln 6\sqrt{x^2+y^2}}{\ln 3}$ funksiya qo‘yilgan Dirixle masalasining yechimi bo‘ladi.

4.13. Puasson tenglamasini doira uchun yechish

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

ko'rinishidagi tenglama Puasson tenglamasi deyiladi. Shu tenglamaning $u\big|_{x^2+y^2=R^2}=0$ shartni qanoatlantiruvchi yechimini topish talab etilsin.

Yechimni $u(x,y)=v(x,y)+w(x,y)$ ko‘rinishda izlanadi. Bu yerda $v(x,y)$ funksiya Puasson tenglamasining xususiy yechimi va $w(x,y)$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

Laplas tenglamasining $w\big|_{x^2+y^2=R^2}=-v\big|_{x^2+y^2=R^2}$ chegaaviy shartlarni qanoatlantiruvchi yechimidir.

11-misol. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-4$ Puasson tenglamasining $u\Big|_{x^2+y^2=9}=0$ chegaraviy shartni qanoatlantiruvchi yehimini toping.

Yechilishi. $v(x,y)=-x^2-y^2$ berilgan tenglamaning xususiy yechimi bo‘ladi, chunki

$$\frac{\partial v}{\partial x}=-2x,\ \frac{\partial^2 v}{\partial x^2}=-2,\ \frac{\partial v}{\partial y}=-2y,\ \frac{\partial^2 v}{\partial y^2}=-2.$$

Endi $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$ Laplas tenglamasining

$$w\Big|_{x^2+y^2=9}=-v\Big|_{x^2+y^2=9}=(x^2+y^2)_{x^2+y^2=9}=9$$

chegaraviy shartni qanoatlantiruvchi yechimini topamiz. Bu yechimni topishda $x=r\cos\varphi,\ y=r\sin\varphi$ deb olib, qutb koordinatalariga o‘tamiz, u holda $w(x,y)$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}=0$$

Laplas tenglamasining $u\big|_{r=9}=9$ shartni qanoatlantiruvchi yechimi bo‘ladi. $w(x,y)$ni Puasson formulasidan foydalanib topamiz:

$$w(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}9\cdot\frac{9-r^2}{9-6r\cos(t-\varphi)+r^2}dt=\frac{9(9-r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{dt}{3^2-2\cdot 3\cdot r\cos(t-\varphi)+r^2}=\frac{9(9-r^2)}{2\pi}\cdot\frac{2\pi}{9-r^2}=9.$$

Bu yerda $\int_{-\pi}^{\pi}\frac{dx}{a^2-2ab\cos x+b^2}=\frac{2\pi}{a^2-b^2}$ formuladan foydalandik. Demak, $u(x,y)=v(x,y)+\ +w(x,y)=9-x^2-y^2$.

Mustaqil yechish uchun mashqlar

1. Bir jinsli bo‘lmagan $\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}+4xt$ to‘lqin tenglamasining $u\big|_{t=0}=e^x,\ \frac{\partial u}{\partial t}\big|_{t=0}=x$ boshlang‘ich shartni qanoatlantiruvchi yechimini Dyuamel formulasidan foydalanib toping. Javob: $u(x,t)=\frac{e^{x+3t}+e^{x-3t}}{2}+xt+\frac{2xt^2}{3}$.

2. Bir jinsli bo‘lmagan $\frac{\partial^2 u}{\partial t^2}=16\frac{\partial^2 u}{\partial x^2}+xt$ to‘lqin tenglamasining $u\big|_{t=0}=e^{-x},\ \frac{\partial u}{\partial t}\big|_{t=0}=5x$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini Dyuamel formulasidan foydalanib toping. Javob: $u(x,t)=\frac{e^{-x+4t}+e^{-x-4t}}{2}+5xt+\frac{xt^3}{6}$.

3. Bir jinsli bo‘lmagan $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+2t$ to‘lqin tenglamasi uchun $u\big|_{t=0}=\sin x,\ \frac{\partial u}{\partial t}\big|_{t=0}=2$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini Dyuamel formulasidan foydalanib toping. Javob: $u(x,t)=\sin x\cos t+2t+\frac{t^3}{3}$.

4. Bir jinsli bo‘lmagan $\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}+xt$ to'lqin tenglamasi uchun $u\big|_{t=0}=2\cos 5x,\ \frac{\partial u}{\partial t}\big|_{t=0}=8x$ boshlang‘ich shartlarni qanoatlantiruvchi yechimini Dyuamen formulasidan foydalanib toping. Javob: $u(x,t)=2\cos 5x\cos 10t+8xt+\frac{xt^3}{2}$.

5. $\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial y^2}=0$ Laplas tenglamasining $u\big|_{x^2+y^2=a^2}=Axy$ shartni qanoatlantiruvchi $x^2+y^2\le a^2$ doiradagi yechimini toping. Javob: $u(x,y)=Axy$.

6. Laplas tenglamasining $u\big|_{x^2+y^2=a^2}=A+By$ shartni qanoatlantiruvchi $x^2+y^2\le a^2$ doiradagi yechimini toping. Javob: $u(x,y)=A+By$.

7. Chegaralangan torning erkin tebranish tenglamasi $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ning $u(x,0)=0,\ \frac{\partial u}{\partial t}\big|_{t=0}=0$ boshlang‘ich hamda $u(0,t)=0$ $u(l,t)=A\sin wt$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini Furye formulasidan foydalanib toping. Javob:
$$u(x,t)=\frac{A\sin\omega x\cdot\sin x\sin\omega t}{\sin\omega l}+\frac{2A\omega}{l}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\omega^2-\frac{n^2\pi^2}{l^2}}\sin\frac{n\pi t}{l}\cdot\sin\frac{n\pi x}{l}\cdot$$

8. $\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(x,0)=0,\ \frac{\partial u}{\partial t}\big|_{t=0}=0$ boshlang‘ich hamda $u(0,t)=0,\ u(\pi,t)=\sin\frac{t}{2}$ chegaravoy shartlarni qanoatlantiruvchi xususiy yechimini Furye formulasidan foydalanib toping. Javob:
$$u(x,t)=\sin\frac{x}{2}\cdot\sin\frac{t}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\frac{1}{4}-n^2}\sin nt\cdot\sin nx\cdot$$

9. Torning majburiy tebranish tenglamasi $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+5x(x-1)$ ning $u(x,0)=0,\ \frac{\partial u}{\partial t}\big|_{t=0}=0$ boshlang‘ich hamda $u(0,t)=0$, $u(l,t)=0$ chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini toping.

Javob: $u(x,t)=-\frac{5}{12}x(x^3-2x^2+1)+\frac{8}{\pi^5}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi t\cdot\sin(2n+1)\pi x}{(2n+1)^5}$.

10. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ -\infty<x<\infty$ issiqlikning tarqalish tenglamasini $u(x,0)=e^{-x^2}$ boshlang‘ich shartni qanoatlantiruvchi yechimini Puasson formulasidan foydalanib toping. Javob: $u(x,t)=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-(\tau^2+\frac{(x-\tau)^2}{4t})}d\tau.$

11. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ -\infty<x<\infty$ tenglamaning $u(x,0)=\begin{cases}1, & |x|\le 1 \text{ bo'lsa,}\\ 0, & x>1 \text{ bo'lsa}\end{cases}$ boshlang‘ich shartni qanoatlantiruvchi yechimi Puasson formulasidan foydalanib topilsin.

Javob: $u(x,t)=\frac{2}{\pi}\int_0^{+\infty}\frac{\sin\omega}{\omega}\cos\omega x e^{-\omega^2 t}d\omega.$

$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}=0$ Laplas tenglamasining berilgan chegaraviy shartlarni qanoatlantiruvchi doira ichidagi yechimini Puasson formulasidan foydalanib toping.

12. $u\big|_{r=a}=\sin^2\varphi$. Javob: $u(r,\varphi)=\frac{3}{a}r\sin\varphi-4(\frac{r}{a})^3\sin 3\varphi.$

13. $u\big|_{r=5}=2\sin^2\varphi+8$. Javob: $u(r,\varphi)=8+\frac{6}{5}r\sin\varphi-8(\frac{r}{5})^3\sin^3\varphi.$

14. $u\big|_{r=2}=5\sin\varphi$. Javob: $u(r,\varphi)=\frac{5}{2}r\sin\varphi.$

15. $u\big|_{r=1}=\sin^3\varphi$. Javob: $u(r,\varphi)=3\sin\varphi-4r^3\sin 3\varphi.$

16. $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ tenglamaning $u(0,t)=u(l,t)=0,\ u(x,0)=\frac{x(l-x)}{l^2}$ shartlarni qanoatlantiruvchi yechimini toping.

Javob: $u(x,t)=\frac{8}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\cdot e^{-\frac{(2n+1)^2a^2\pi^2t}{l^2}}\cdot\sin\frac{(2n+1)\pi x}{l}$.

Nazorat uchun savollar

1. Fazoda issiqlikning tarqalish tenglamasini keltirib chiqaring.
2. Laplas tenglamasini yozing.
3. Puasson tenglamasini yozing.
4. Laplas tenglamasiga keltiriladigan masalalardan ba‘zi-birlarini ayting.
5. Garmonik funksiya deb nimaga aytiladi.
6. Siqiladigan suyuqlik oqimining uzluksizlik tenglamasini yozing.
7. Bosimni aniqlovchi Laplas tenglamasi uchun chetki shartlar qanday bo‘lishi kerak?
8. Chegaralangan torning erkin tebranish tenglamsini hamda uning yechimini topish formulasini yozing.
9. Chegaralangan torning majburiy tebranish tenglamasini hamda uning yechimini topish formulasini yozing.
10. Chegaralangan sterjenda issiqlikning tarqalish tenglamasini hamda uning yechimini topish formulasini yozing.
11. Chegaralanmagan sterjenda issiqlikning tarqalish tenglamasini hamda uning yechimini topish formulasini yozing.
12. Bir tomondan chegaralangan sterjenda issiqlikning tarqalish tenglamasini hamda uning yechimini topish formulasini yozing.
13. Laplas tenglamasi uchun Direxlening ichki masalasini doira uchun yechimini yozing.
14. Puasson tenglamasi doira uchun qanday yechiladi?
15. Laplas tenglamasi uchun Direxlening ichki masalasini shar uchun yechimini yozing.
16. Laplas tenglamasi uchun Direxlening ichki masalasini halqa uchun yechimini yozing.

5. Matematik fizikaning chegaraviy masalalarini yechishning to‘r (katak)lar usuli

5.1. Matematik fizika tenglamalarini taqribiy yechishning to‘r (katak)lar usuli.

Ma’lumki, to‘lqin tenglamasi yoki tor tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

issiqlik tarqalish tenglamasi yoki Furye tenglamasi

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

ning yechimlarini bir qiymatni aniqlash uchun **boshlang‘ich** va **chegaraviy shartlar** deb ataluvchi qo‘shimcha shartlar ham berilishi lozim. Lekin, ko‘pgina tenglamalar yechimlarini analitik shaklda olishning iloji yo‘qligi sababli ularni yechishda taqribiy yoki sonli usullarga murojaat qilishga to‘g‘ri keladi.

Oddiy differensial tenglamalarni chekli ayirmalar usuli bilan taqribiy yechishda hosilalar chekli ayirmalarga almashtirilishi aytilgan edi. Xususiy hosilali differensial tenglamalarni ham bu usul bilan taqribiy yechishda xususiy hosilalar mos chekli ayirmalarga almashtiriladi:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}, \tag{5.2}$$

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+h) - u(x,t)}{h}. \tag{5.3}$$

Masalani yechishga bunday yondashish ayirma usuli, yoki chekli ayirma usuli yoki **kataklar** yoki **to‘rlar** usuli deb ataladi.

To‘rlar usulining mohiyati quyidagicha. Aytaylik, tenglamani x va y erkli o‘zgaruvchilarning biror G sohasidagi yechimini topish talab etilsin. Shu G sohani erkli o‘zgaruvchilarning diskret o‘zgarish sohasi G_h to‘rli soha bilan, ya’ni **to‘r tugunlari** deb ataladigan (x_i, y_i) nuqtalar to‘plami bilan almashtiriladi. To‘rli soha kvadrat, to‘g‘ri to‘rtburchak, uchburchak va hokazo kataklardan iborat bo‘lishi mumkin. To‘rli sohani tanlaganda, uning Γ_h chegarasi berilgan G sohaning Γ chegarasini iloji boricha yaxshi tasvirlaydigan bo‘lishi kerak. Γ_h koordinata o‘qlariga parallel kesmalardan tashkil topgan siniq yopiq chiziq bo‘ladi.

Misol sifatida h qadam bilan kvadrat to‘r tuzamiz:

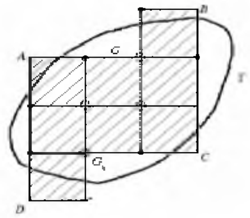
$$x_i = x_0 + ih, \quad y_j = y_0 + jh, \quad i, j = \pm 1, \pm 2, \ldots$$

bunda to‘rning (x_i, y_j) tugunlari yo G sohaga tegishli yoki uning chegarasidan h dan kichik bo‘lgan masofada yotadi. To‘r tugunlari-**qo‘shni**, **ichki** va **chegaraviy tugunlar** bo‘lishi mumkin.

Qo‘shni tugunlar bir-biridan koordinata o‘qlari yo‘nalishi bo‘yicha to‘r qadami h ga teng masofada yotadi.

Tugun G sohaga, unga qo‘shni bo‘lgan to‘rtta tugunlar to‘rga tegishli bo‘lsa, u holda bunday tugun **ichki tugun** deb ataladi.

Agar tugunning qo‘shni tugunlaridan aqalli birortasi to‘rga tegishli bo‘lmasa u **chegaraviy tugun** deb ataladi. Chegaraviy tugunlar birinchi va ikkinchi tur chegaraviy tugunlarga bo‘linadi.



5.1-chizma.

Chegaraviy tugun shu to‘rning qo‘shni ichki tuguniga ega bo‘lsa, u **birinchi tur chegaraviy tugun** deb ataladi. Qo‘shni ichki tugunga ega bo‘lmagan chegaraviy tugunlar **ikkinchi tur chegaraviy tugunlar** deb ataladi. 5.1-chizmada ichki tugunlar ochiq doiralar bilan, birinchi tur chegaraviy tugunlar qora doirachalar bilan belgilangan. A, B, C, D chegaraviy tugunlar ikkinchi tur chegaraviy tugunlardir. Ichki tugunlar va birinchi tur chegaraviy tugunlarni **hisoblash tugunlari** deb ataymiz. Endi izlanayotgan $u = u(x, y)$ funksiyaning (x_i, y_i) tugundagi qiymatini $u_{ij} = u(x_i, y_j)$ orqali belgilaymiz.

5.2 Issiqlik tarqalish tenglamasini kattaliklar usuli bilan taqribiy yechish

Issiqlik tarqalish tenglamasi

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (5.4)$$

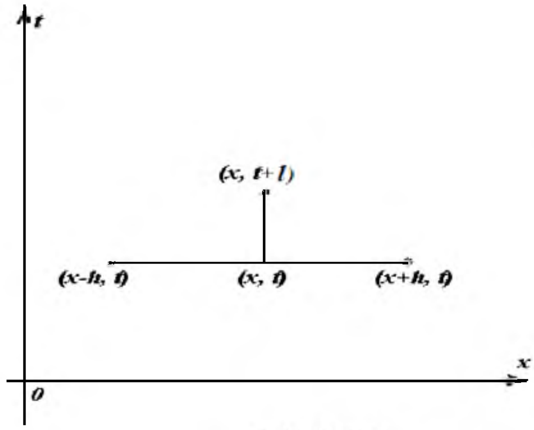
uchun Direxlining ichki masalasi quyidagicha ifodalanadi. (5.4) tenglamaning

$$u(x,0) = \varphi(x),\ 0 \le x \le L, \qquad (5.5)$$

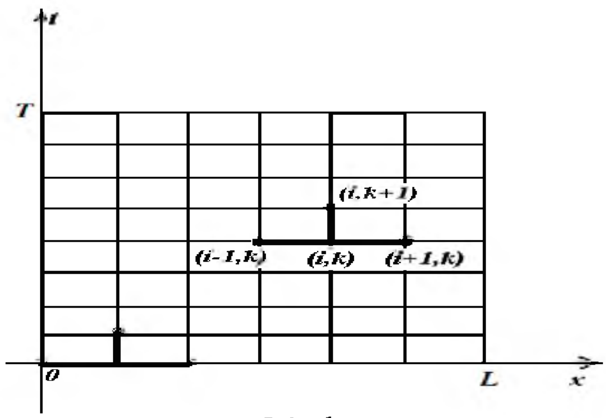
$$u(0,t) = \psi_1(t),\ 0 \le t \le T, \qquad (5.6)$$

$$u(L,t) = \psi_2(t),\ 0 \le t \le T \qquad (5.7)$$

chetki shartlarni qanoatlantiruvchi taqribiy yechimini topish talab etiladi, ya’ni agar izlanayotgan funksiyaning qiymatlari $t = 0$, $x = 0, x = L$, $t = T$ to‘g‘ri chiziqlar bilan chegaralangan to‘g‘ri to‘rtburchakning uchta $t = 0, x = 0, x = L$ tomonlarida berilgan bo‘lsa, shu to‘g‘ri to‘rtburchakning $t = T$ tomonida va ichki nuqtalarida $u(x,t)$ taqribiy yechimni topish talab etiladi.



5.2-chizma.



5.3-chizma.

$x = ih$, $(i = 1, 2, 3, \ldots)$ $t = kl$ $(k = 1, 2, 3, \ldots)$ to‘g‘ri chiziqlarni o‘tkazamiz. U holda qaralayotgan to‘g‘ri to‘rtburchak to‘r bilan qoplanadi, ya’ni kataklarga ajraladi.

To‘g‘ri chiziqlarning kesishishi nuqtalari to‘rning tugunlari bo‘ladi. To‘g‘ri to‘rtburchakning tugunlarida yechimning taqribiy qiymatlarini aniqlaymiz. $u(ih, kl) = U_{i,k}$ deb belgilaymiz. (5.4) tenglama o‘rniga unga mos chekli ayirmalar tenglamasini (ih, kl) tugun nuqtalar uchun yozamiz:

$$\frac{u_{i,k+1} - u_{i,k}}{\ell} = a^2 \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{h^2}. \quad (5.8)$$

Bundan $U_{i,k+1}$ ni topamiz:

$$u_{i,k+1} - u_{i,k} = \frac{la^2}{h^2}(u_{i+1,k} - 2u_{i,k} + u_{i-1,k})$$

yoki

$$u_{i,k+1} = (1 - \frac{2la^2}{h^2})u_{i,k} + \frac{la^2}{h^2}(u_{i+1,k} + u_{i-1,k}). \quad (5.9)$$

Bu formulaga asoslanib izlanuvchi funksiyaning k-qatordagi uchta qiymatlari $U_{i-1,k}, U_{i,k}, U_{i+1,k}$ ma'lum bo'lsa, u holda funksiyaning $(k+1)$-qatordagi $U_{i,k+1}$ qiymatini topish mumkin. (5.5) formulaga ko'ra funksiyaning $[0,L]$ kesmadagi qiymatlari ma'lum. (5.9) formuladan foydalanib $t=1$ to'g'ri chiziqning $[0,L]$ kesmasining barcha ichki tugun nuqtalaridagi qiymatlarini aniqlaymiz. Bu kesmaning chetlari $x=0$ va $x=L$ dagi qiymatlari $u(0,l)=\psi_1(l)$ va $u(L,l)=\psi_2(l)$ (5.6) va (5.7) qiymatlardan kelib chiqadi. Xuddi shuningdek $t=2l$ to'g'ri chiziqning barcha tugunlaridan funksiyaning qiymatlarini aniqlashimiz mumkin va hokazo. Shu jarayonni davom ettirib izlanayotgan yechimning qiymatlarini to'rning barcha tugunlarida aniqlaymiz. (5.9) formula bilan yechimning taqribiy qiymatini h va l qadamlar ixtiyoriy bo'lganda emas, balki $l \le \frac{h^2}{2a^2}$ bo'lgan holdagina hosil qilish mumkin.

Agar l qadam t o'q bo'yicha $1 - \frac{2a^2 l}{h^2} = 0$ yoki $l = \frac{h^2}{2a^2}$ bo'ladigan qilib tanlansa, (5.9) formula juda soddalashadi. Bu holda (5.9) tenglama

$$u_{i,k+1} = \frac{1}{2}(u_{i+1,k} + u_{i-1,k}) \quad (5.10)$$

ko'rinishni oladi.

Bu formula hisoblash uchun ancha qulay. Ko'rsatilgan usul bilan to'rning tugunlaridagi yechim aniqlanadi.

1-masala. Sterjenda issiqlik tarqalish tenglamasi

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (5.11)$$

ning

$$u(x,t)\big|_{t=0} = x(x - \frac{1}{4}) \quad (5.12)$$

boshlang'ich shartni hamda

$$u(x,t)\Big|_{x=0} = 0, \quad (5.13)$$

$$u(x,t)\Big|_{x=L} = \frac{3}{4} \quad (5.14)$$

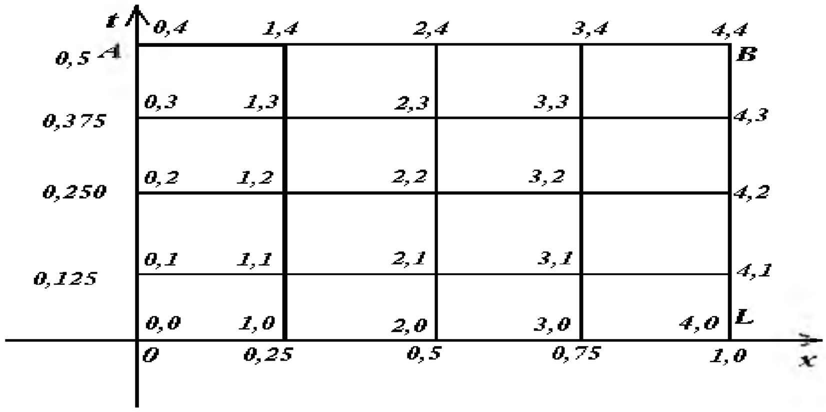
chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimi kataklar usuli yordamida topilsin. Bunda $h = 0{,}25$ deb olinsin va t ni esa $0 \le t \le 2l$ oraliqda qaralsin.

Yechilishi. Sterjenni gorizantal holatda joylashgan deb, uning kesimlaridagi haroratlarini o'zgarishini vertikal t o'q bo'yicha o'zgaradi deb qarymiz.

$0x$ o‘qni $h=0,25$ qadam bilan $0t$ o‘qni t bo‘yicha $l=\frac{h^2}{2a^2}=\frac{0,25^2}{2\cdot 0,5^2}=0,125m$ qadam bilan koordinata o‘qlariga parallel to‘g‘ri chiziqlar o‘tkazib $0xt$ tekislikni 5.4-chizmada ko‘rsatilgandek to‘r bilan qoplaymiz:

Masala shartiga ko‘ra, $ABLO$ to‘g‘ri to‘rtburchakning uchta tomoni: OA da (5.13) ga ko‘ra, OL da (5.12) ga ko‘ra, LB da (5.14)ga ko‘ra tugunlardagi haroratlarni aniqlash mumkin.

Ularni aniqlaymiz. (5.13)ga ko‘ra $U_{0,0}=U_{0,1}=U_{0,2}=U_{0,3}=U_{0,4}=0,$ (5.12) ga ko‘ra $h=0,25$ bo‘lgani uchun $U_{0,0}=0,$ $u_{1,0}=\frac{1}{4}(\frac{1}{4}-\frac{1}{4})=0,$ $u_{2,0}=0,5(0,5-0,25)=0,125,$ $u_{3,0}=0,75(0,75-0,4)=0,375,$ $u_{4,0}=0,75$ (5.14) ga ko‘ra $U_{4,0}=U_{4,1}=U_{4,2}=U_{4,3}=U_{4,4}=0,75.$



5.4-chizma.

Endi 2-qatorning ichki tugunlaridagi haroratlarni, 1-qatoning haroratlari yordamida

$$u_{i,k+1}=\frac{1}{2}(u_{i+1,k}+u_{i-1,k})$$

formula bilan hisoblaymiz:

$$u_{1,1}=\frac{1}{2}(u_{0,0}+u_{2,0})=\frac{1}{2}(0+0,125)=0,0625,$$

$$u_{2,1}=\frac{1}{2}(u_{1,0}+u_{3,0})=\frac{1}{2}(0+0,375)=0,1875,$$

$$u_{3,1}=\frac{1}{2}(u_{2,0}+u_{4,0})=\frac{1}{2}(0,125+0,75)=0,4375.$$

3-qatorning ichki tugunlaridagi haroratlarni 2-qatorning haroratlaridan foydalanib topamiz:

$$u_{1,2}=\frac{1}{2}(u_{0,1}+u_{2,1})=\frac{1}{2}(0+0,1875)=0,9275,$$

$$u_{2,2}=\frac{1}{2}(u_{1,1}+u_{3,1})=\frac{1}{2}(0,0625+0,4375)=0,25,$$

$$u_{3,2}=\frac{1}{2}(u_{2,1}+u_{4,1})=\frac{1}{2}(0,1875+0,75)=0,46875.$$

Xuddi shuningdek 4-va 5-qatorning ichki tugunlaridagi haroratlarni hisoblaymiz:

$$u_{1,3}=\frac{1}{2}(u_{0,2}+u_{2,2})=0,125,\quad u_{1,4}=0,140625,\quad u_{2,3}=0,28125,$$

$$u_{2,4}=0,3125,\quad u_{3,3}=0,5,\quad u_{3,4}=0,515625.$$

Berilgan va topilgan qiymatlarni quyidagi jadval ko‘rinishida yozamiz.

$U_{0,4}=0$	$U_{1,4}=0{,}140625$	$U_{2,4}=0{,}3125$	$U_{3,4}=0{,}51625$	$U_{4,4}=0{,}75$
$U_{0,3}=0$	$U_{1,3}=0{,}125$	$U_{2,3}=0{,}28125$	$U_{3,3}=0{,}5$	$U_{4,3}=0{,}75$
$U_{0,2}=0$	$U_{1,2}=0{,}09375$	$U_{2,2}=0{,}25$	$U_{3,2}=0{,}46875$	$U_{4,2}=0{,}75$
$U_{0,1}=0$	$U_{1,1}=0{,}0625$	$U_{2,1}=0{,}1825$	$U_{3,1}=0{,}4375$	$U_{4,1}=0{,}75$
$U_{0,0}=0$	$U_{1,0}=0$	$U_{2,0}=0{,}125$	$U_{3,0}=0{,}375$	$U_{4,0}=0{,}75$

5.3 Laplas tenglamasini kataklar usuli bilan taqribiy yechish

$0xy$ tekislikda yopiq Ã kontur bilan chegaralangan G soha, hamda Ã konturda uzluksiz $\varphi(x,y)$ funksiya berilgan bo‘lsin.

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0$$

(5.15)

Laplas tenglamasining

$$u(x,y)\big|_{\Gamma}=\varphi(x,y) \qquad (5.16)$$

chegaraviy shartni qanoatlantiruvchi taqribiy yechimni topish talab etilsin.

Xususan, yuqoridagi masalaga g‘ovak muhitlarda quduqqa tomon harakatlanayotgan (sizib o‘tayotgan) neft va gaz suyuqliklarini o‘rganish olib keladi.

Neft va gaz havzalarini gorizontal kesimlarini chegaralovchi chiziqlardagi bosim “razvetka” burg‘ilashlari orqali aniqlab olinadi. Demak (5.16) shartni oldindan hisoblash mumkin.

Havzaning gorizontal kesimlarining ichki nuqtalaridagi bosimni uning chegaralovchi chiziqdagi bosimni bilgan holda aniqlash mumkin. Boshqacha aytganda bosimni aniqlovchi Laplas tenglamasi uchun Dirixlening ichki masalasini yechish lozim.

Qo‘yilgan masalani kataklar usuli yordamida yechamiz.

Ikkita $x=ih$, va $y=jh$ (5.17) to‘g‘ri chiziqlar oilasini o‘tkazamiz, bunda h-berilgan son, i va j ketma-ket butun musbat qiymatlar qabul qiladi. Bu holda G soha to‘r bilan qoplanadi. To‘g‘ri chiziqlarning kesishish nuqtalari to‘rning tugunlari bo‘ladi. G sohani butunlay shu sohada yotuvchi hamma kvadratlardan va Ã ning chegaralari bilan kesishuvchi ba’zi kvadratlardan tashkil topuvchi to‘rli G_h soha bilan, G sohaning chegarasi Ã ni (5.17) ko‘rinishdagi to‘g‘ri chiziqlar kesmalaridan tashkil topgan $Ã_h$ egri (siniq) chiziq bilan taqribiy tasvirlaymiz.

Axtarilayotgan U funksiyaning qiymatlarini faqat to‘rning hisoblash tugunlaridagina qaraymiz. To‘rlar usulida Laplas tenglamasidagi xususiy hosilalar chekli ayirmalar bilan almashtiriladi:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=ih\\ y=jh}}=\frac{1}{h^2}(u_{i-1,k}-2u_{i,j}+u_{i+1,j}),\qquad \left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=ih\\ y=jh}}=\frac{1}{h^2}(u_{i,j-1}-2u_{i,j}+u_{i,j+1}).$$

(5.15) Laplas tenglamani quyidagi **ayirmali tenglama** bilan almashtiramiz:

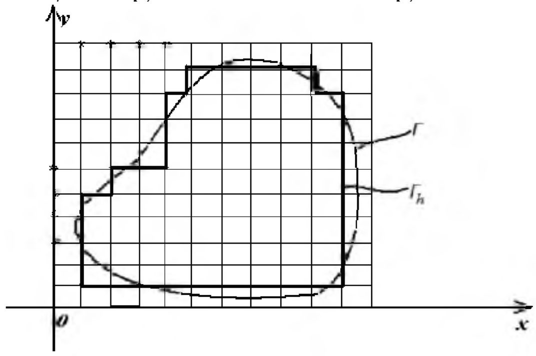
$$u_{i-1,j}-2u_{i,j-1}+u_{i+1,j}+u_{i,j+1}-4u_{i.j}=0$$

bundan

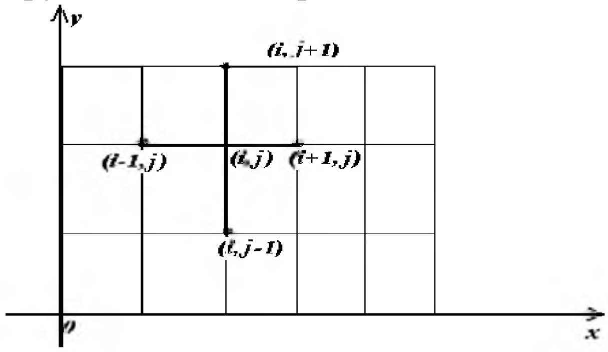
$$u_{i,j}=\frac{1}{4}(u_{i-1,j}+u_{i.j-1}+u_{i+1,j}+u_{i,j+1}) \qquad (5.18)$$

Agar berilgan masala uchun $\tilde{A}$ dagi (x,y) nuqtada $u(x,y)=\varphi(x,y)$ chegraviy shart berilgan bo'lsa, u holda to'rning birinchi tur chegaraviy tugun nuqtasida $u_{i,j}=u(x_i,y_j)=\varphi(x,y)$ deb olamiz, bunda (x,y) shu $\tilde{A}$ chegaraning (x_i,y_j) chegaraviy tugun nuqtaga eng yaqin nuqtasi.

To'rning har bir ichki tuguni uchun (5.18) tenglamani tuzamiz. Natijada, noma'lumlari va tenglamalari soni to'rning ichki tugunlari soniga teng bir jinsli bo'lmagan chiziqli tenglamalar sistemasiga ega bo'lamiz. Bunday sisitema har doim birgalikda va birgina yechimga ega. Uni yechib, izlanayotgan $u=u(x,y)$ funksiyaning G to'rli soha tugunlaridagi qiymatlarini hosil qilamiz.



5.5-chizma.



5.6-chizma.

2-masala. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ Laplas tenglamasi uchun $G=\{0<x<1,\ 0<y<1\}$ kvadratda Dirixle masalasini

$$u(x,y)=\begin{cases}0, \text{ agar } x=0 \text{ va } 0\le y\le 1 \text{ bo'lsa,}\\ 0, \text{ agar } y=0 \text{ va } 0\le x\le 1 \text{ bo'lsa,}\\ \frac{1}{2}y(y+1), \text{ agar } x=1 \text{ va } 0\le y\le 1 \text{ bo'lsa,}\\ \frac{1}{2}x(x+1), \text{ agar } y=1 \text{ va } 0\le x\le 1 \text{ bo'lsa.}\end{cases}$$

chegaraviy shartni qanoatlantiruvchi taqribiy yechimi to'rlar usuli bilan topilsin.

Yechilishi. $x=\frac{1}{3},\ x=\frac{2}{3},\ x=1$ va $y=\frac{1}{3},\ y=\frac{2}{3},\ y=1$ to'g'ri chiziqlarni o'tkazib $h=\frac{1}{3}$ qadam bilan to'rli G_h sohani hosil qilamiz. Qaralayotgan holda G va G_h sohalarning chegaralari bitta egri chiziq-kvadratdan iborat. To'r $(\frac{1}{3},\frac{1}{3}),\ (\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})$ va $(\frac{2}{3},\frac{2}{3})$ ichki tugun nuqtalariga hamda $(0,\frac{1}{3}),\ (0,\frac{2}{3}),(\frac{1}{3},1),\ (\frac{2}{3},1),\ (\frac{1}{3},0),(\frac{2}{3},0)$ birinchi tur tugunlarga ega. $0(0,0),\ C(0,1), B(1,1),\ A(1;0)$ nuqtalar to'rning ikkinchi tur tugunlaridir. **Hisoblash tugunlarida** funksiyaning qiymatlarini topamiz. Chegaraviy shartga asoslanib birinchi tur tugunlar uchun quyidagilarga ega bo'lamiz.

$u_{0,1}=u_{0,2}=u_{1,0}=u_{2,0}=0$, agar $x=0,\ 0\le y\le 1$ va $y=0,\ 0\le x\le 1$ bo'lsa.

$u_{3,1}=\frac{1}{2}\cdot\frac{1}{3}(\frac{1}{3}+1)=\frac{2}{9}=0{,}222,\ u_{3,2}=\frac{1}{2}\cdot\frac{2}{3}(\frac{2}{3}+1)=\frac{5}{9}=0{,}556$ agar $x=1,\ 0<y<1$ bo'lsa,

$u_{1,3}=\frac{1}{2}\cdot\frac{1}{3}(\frac{1}{3}+1)=\frac{2}{9}=0{,}222,\ u_{2,3}=\frac{1}{2}\cdot\frac{2}{3}(\frac{2}{3}+1)=\frac{5}{9}=0{,}556$ agar $y=1,\ 0<x<1$ bo'lsa.

Endi (5.18) formuladan foydalanib $u(x,y)$ funksiyaning ichki tugunlaridagi qiymatlarini yozib quyidagi to‘rt noma’lumli to‘rtta chiziqli tenglamalar sistemasini hosil qilamiz.

$$\begin{cases} u_{1,1} = \frac{1}{4}(u_{0,1} + u_{1,0} + u_{2,1} + u_{1,2}) = \frac{1}{4}(u_{2,1} + u_{1,2}), \\ u_{1,2} = \frac{1}{4}(u_{0,2} + u_{1,1} + u_{2,2} + u_{1,3}) = \frac{1}{4}(u_{1,1} + u_{2,2} + 0{,}222), \\ u_{2,1} = \frac{1}{4}(u_{1,1} + u_{2,0} + u_{3,1} + u_{2,2}) = \frac{1}{4}(u_{1,1} + 0{,}222 + u_{2,2}), \\ u_{2,2} = \frac{1}{4}(u_{1,2} + u_{2,1} + u_{3,2} + u_{2,3}) = \frac{1}{4}(u_{1,2} + u_{2,1} + \frac{5}{9} + \frac{5}{9}) \end{cases}$$

yoki

$$\begin{cases} 4u_{1,1} - u_{1,2} - u_{2,1} = 0, \\ u_{1,1} - 4u_{1,2} + u_{2,2} = -0{,}222, \\ u_{1,1} - u_{2,1} + u_{2,2} = -0{,}222, \\ u_{1,2} + u_{2,1} - 4u_{2,2} = -1{,}112. \end{cases}$$

$u_{1,1} = x,\ u_{1,2} = y,\ u_{2,1} = z,\ u_{2,2} = t$ belgilashni kiritamiz. U holda oxirgi sistema

$$\begin{cases} 4x - y - z = 0, \\ x - 4y + t = -0{,}222, \\ x - 4z + t = -0{,}222, \\ y + z - 4t = -1{,}112 \end{cases}$$

ko‘rinishga ega bo‘ladi.

Sistemaning 2-tenglamasidan 3-sini ayirsak $-4y + 4z = 0$ yoki $y = z$ kelib chiqadi. Sistemaga z o‘rniga y ni qo‘ysak

$$\begin{cases} 4x - 2y = 0, \\ x - 4y + t = 0, \\ 2y - 4t = -1{,}112 \end{cases}$$

bo‘ladi. Sistemaning birinchi tenglamasidan $x = \frac{y}{2}$. Buni sistemaning 2- va 3- tenglamalariga qo‘ysak

$$\begin{cases} x = \frac{y}{2}, \\ \frac{y}{2} - 4y + t = -0{,}222, \\ 2y - 4t = -1{,}112 \end{cases} \quad \text{yoki} \quad \begin{cases} x = \frac{y}{2}, \\ -\frac{7}{2}y + t = -0{,}222. \\ \frac{y}{2} - t = -0{,}278. \end{cases}$$

bo‘ladi. Sistemaning 2- va 3-tenglamalarini qo‘shsak $-3y = -0{,}5$ yoki bundan $y = 0{,}167$ hosil bo‘ladi. Demak, $x = \frac{y}{2} = 0{,}084,\ z = y = 0{,}167,\ t = \frac{y}{2} + 0{,}278 = 0{,}362.$ Shunday qilib, $u_{1,1} = 0{,}084,\ u_{1,2} = u_{2,1} = 0{,}167,\ u_{2,2} = 0{,}362$ va noma’lum funksiyaning hisoblash tugunlaridagi qiymatlarining quyidagi jadvaliga ega bo‘lamiz.

C	$u_{1,3} = 0{,}222$	$u_{2,3} = 0{,}556$	B
$u_{0,2} = 0$	$u_{1,2} = 0{,}167$	$u_{2,2} = 0{,}362$	$u_{3,2} = 0{,}556$
$u_{0,1} = 0$	$u_{1,1} = 0{,}084$	$u_{2,1} = 0{,}167$	$u_{3,1} = 0{,}222$
0	$u_{1,0} = 0$	$u_{2,0} = 0$	A

3-masala. 6.7-chizmada tasvirlangandek $\tilde{A} = AmBnA$ yopiq chiziq bilan chegaralangan G sohaning ichki nuqtalaridagi bosimning qiymati

$$\frac{\partial^2 P(x,y)}{\partial x^2} + \frac{\partial^2 P(x,y)}{\partial y^2} = 0 \qquad (5.19)$$

Laplas tenglamasi orqali berilgan. $\tilde{A}$ chiziqning AmB qismidagi bosimning qiymatlari

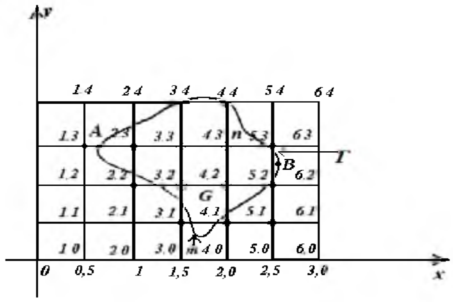
$$P(x,y)\big|_{AmB} = x(\frac{5}{2} - x) \qquad (5.20)$$

formula bilan, qolgan qismida esa

$$P(x,y)\big|_{BnA} = 0 \qquad (5.21)$$

formula bilan aniqlanadi. (6.19) tenglamaning (5.20) va (5.21) chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimni topish talab etiladi.

Yechilishi. Masalani to'rlar usuli bilan yechamiz. Dastlab G sohani $h = 0{,}5$ ga teng bo'lgan qadam orqali to'r bilan qoplaymiz.



5.7-chizma.

So'ngra $\tilde{A}$ chiziqni yopiq siniq chiziq bilan tasvirlaymiz. $\tilde{A}$ chiziqning AmB ga yaqin bo'lgan tugunlardagi bosimning qiymatlarini (5.20) formulaga ko'ra hisoblaymiz:

$$P_{1,3} = x(2{,}5 - x) = 0{,}5(2{,}5 - 0{,}5) = 1,\ P_{2,2} = 1\cdot(2{,}5 - 1) = 1{,}5,\quad P_{3,1} = 1{,}5(2{,}5 - 1{,}5) = 1{,}5,$$

$$P_{4,1} = 2(2{,}5 - 2) = 1,\ P_{5,1} = 2{,}5(2{,}5 - 2{,}5) = 0,\quad P_{5,2} = 2{,}5(2{,}5 - 2{,}5) = 0\ ,\quad P_{2,3} = 1(2{,}5 - 1) = 1{,}5.$$

$\tilde{A}$ egri chiziqning qolgan qismiga yaqin tugunlardagi bosim qiymatlari (5.21) chegaraviy shartga ko'ra nolga teng, ya'ni $P_{5,3} = P_{5,4} = P_{4,4} = P_{3,4} = P_{2,4} = 0$.

Ichki tugunlardagi $P_{3,3}$, $P_{4,3}$, $P_{3,2}$ va $P_{4,2}$ bosimlarni (5.18) formula yordamida hisoblaymiz.

$$\begin{cases} P_{3.3} = \frac{1}{4}(P_{2.3} + P_{3.4} + P_{4.3} + P_{3.2}), \\ P_{3.2} = \frac{1}{4}(P_{2.2} + P_{3.3} + P_{3.1} + P_{4.2}), \\ P_{4.2} = \frac{1}{4}(P_{4.3} + P_{3.2} + P_{5.2} + P_{4.1}), \\ P_{4.3} = \frac{1}{4}(P_{3..3} + P_{4.4} + P_{4.2} + P_{5.3}). \end{cases}$$

Bu tenglamalar sistemasining har birini 4 ga ko'paytirib, o'ng tomonlariga ma'lum qiymatlarni qo'yib quyidagiga ega bo'lamiz.

$$\begin{cases} 4P_{3\cdot3} = 1{,}5 + 0 + P_{4\cdot3} + P_{3\cdot2}, \\ 4P_{3\cdot2} = 1{,}5 + P_{3\cdot3} + 1{,}5 + P_{4\cdot2}, \\ 4P_{4\cdot2} = P_{4\cdot3} + P_{3\cdot2} + 0 + 1, \\ 4P_{4\cdot3} = P_{3\cdot3} + 0 + P_{4\cdot2} + 0. \end{cases}$$

$P_{3,3} = x,\quad P_{3,2} = y,\ P_{4,2} = z$ va $P_{4,3} = t$ deb belgilash kiritamiz. U holda

$$\left.\begin{array}{l}4x-y\ \ -t=1{,}5,\\ -x+4y-z=3,\\ y-4z+t=-1,\\ x+z-4t=0\end{array}\right\}$$

bo‘ladi. Hosil bo‘lgan to‘rt noma’lumli to‘rtta chiziqli tenglamalar sistemasini Gauss usuli bilan yechamiz.

Sistemaning 4-tenglamasini 4 ga ko‘paytirib birinchi tenglamaga hamda 4-tenglamani ikkinchi tengalmaga hadma-had qo‘shsak

$$\left.\begin{array}{l}-y-4z+15t=1{,}5,\\ 4y-4t=3,\\ y-4z+t=-1,\\ x+z-4t=0\end{array}\right\}\ \text{yoki}\ \left.\begin{array}{l}x+z-4t=0,\\ y-4z+t=-1,\\ -y-4z+1{,}5t=1{,}5,\\ 4y-4t=3\end{array}\right\}$$

hosil bo‘ladi. Endi oxirgi sistemaning ikkinchi tenglamasini uchinchi tenglamaga va ikkinchi tenglamani -4 ga ko‘paytirib sistemaning to‘rtinchi tenglamasiga hadma-had qo‘shamiz. U holda

$$\left.\begin{array}{l}x+z-4t=0,\\ y-4z+t=-1,\\ -8z+16t=0{,}5,\\ 16z-8t=7\end{array}\right\}$$

yoki bu sistemaning uchinchi tenglamasini 2 ga ko‘paytirib uning to‘rtinchi tenglamasiga hadma-had qo‘shsak $\left.\begin{array}{l}x+z-4t=0,\\ y-4z+t=-1,\\ -8z+16t=0{,}5,\\ 24t=8\end{array}\right\}$ hosil bo‘ladi. Bu sistemaning oxirgi tenglamasidan $t=\frac{1}{3}$ ni topamiz. Uchinchi tenglamaga $t=\frac{1}{3}$ ni qo‘yib $z=\frac{29}{48}$ ni topamiz. Sistemaning ikkinchi tenglamasiga $t=\frac{1}{3}$, $z=\frac{29}{48}$ qiymatlarni qo‘yib $y=\frac{13}{12}$ ni topamiz va nihoyat birinchi tenglamaga y,z,t o‘rniga ularning topilgan qiymatlarini qo‘yib $x=\frac{35}{48}$ ni topamiz.

Shunday qilib, $P_{3,3}=\frac{35}{48}$, $P_{3,2}=\frac{13}{12}$, $P_{4,2}=\frac{29}{48}$ va $P_{4,3}=\frac{1}{3}$.

$$P_{1,3}=1,\ P_{2,3}=1{,}5,\ P_{2,2}=1{,}5,\ P_{3,1}=1{,}5,\ P_{4,1}=1,\ \ P_{5,1}=P_{5,2}=P_{5,3}=P_{5,4}=P_{4,4}=P_{3,4}=P_{2,4}=0\,.$$

Mustaqil yechish uchun mashqlar

1.Uzunligi $1\,m$ bo‘lgan sterjenda issiqlik tarqalish tenglamasi $\frac{\partial u(x,t)}{\partial t}=\frac{1}{4}\frac{\partial^2 u(x,t)}{\partial x^2}$ ning $u(x,0)=x(x+\frac{1}{2})$ boshlang‘ich shartni hamda $u(0,t)=0,\ u(1,t)=0$ chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimi **to‘rlar** usuli yordamida topilsin.

Ko‘rsatma. $h=0{,}2\,m$ deb, $0\le t\le 3l$ shartlarda qaralsin.

2.Uzunligi $1\,m$ bo‘lgan sterjenda issiqlik tarqalish tenglamasi $\frac{\partial u(x,t)}{\partial t}=2\frac{\partial^2 u(x,t)}{\partial x^2}$ ning $u(x,0)=x(\frac{3}{2}-x),\ u(0,t)=0,\ u(1,t)=\frac{1}{2},\ 0<t<4l$ chetki shartlarni qanoatlantiruvchi taqribiy yechimi ($h=0{,}2\ m$ deb faraz qilib) topilsin.

3.Issiqlik o‘tkazish tenglamasi $\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}$ uchun $G=\{0<x<1,\ 0<t<0{,}1\}$ to‘g‘ri to‘rtburchakda $0\le x\le 1$ da $u(x,0)=x(x-\frac{1}{4})$ boshlang‘ich va $0\le t\le 0{,}1$ da $u(0,t)=u(1,t)=0$ chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimi toplsin.

4.Issiqlik o‘tkazuvchanlik tenglamasi $\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}$ ning $u(x,0)=(1{,}1x^2+1{,}1)\sin\pi x$ boshlang‘ich shartni hamda $u(0,t)=u(1,t)=0$ chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimini $0\le t\le 0{,}02$ uchun toping ($h=0{,}1$ deb olinsin).

5.Issiqlik o‘tkazish tenglamasi $\frac{\partial u(x,t)}{\partial t}=\frac{1}{5}\frac{\partial^2 u(x,t)}{\partial x^2}$ ning $u(x,0)=x(x+1)$ boshlang‘ich shartni hamda $u(0,t)=0,\ u(1,t)=2$ chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimini $0\le t\le 2$ uchun toping ($h=0{,}2$ deb olinsin).

6.$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ Laplas tenglamasining $G=\{0\le x\le 1{,}5;\ 0\le y<2{,}5\}$ to‘g‘ri to‘rtburchakdagi $u(x;0)=u(x;2{,}5)=0,\ \ u(0;y)=4y(2{,}5-y),\ u(1{,}5;y)=0$ chetki shartlarni qanoatlantiruvchi taqribiy yechimi $h=0{,}5$ qadam bilan topilsin.

7.$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ Laplas tenglamasining $G=\{0<x<1,\ 0<y<1\}$ birlik kvadratdagi

$$u(x,y)=\begin{cases}0, \text{ agar } 0\le x\le 1,\ y=0 \text{ bo'lsa},\\ \frac{8}{3}y(64y^2-60y+29), \text{ agar } x=0,\ 0\le y\le 1 \text{ bo'lsa},\\ \frac{8}{3}(1-x)(64x^2-68x+3), \text{ agar } 0\le x\le 1,\ y=1 \text{ bo'lsa},\\ 0, \text{ agar } x=1,\ 0\le y\le 1 \text{ bo'lsa}.\end{cases}$$

chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimi $h=0{,}25$ qadam bilan topilsin.

Javob:

	40	20	12	
$u_{0.3}=40$	28,5	17,0	8,6	0
$u_{0.2}=20$	17,0	11,3	5,6	0
$u_{0.1}=12$	8,6	5,6	2,8	0
	$u_{1.0}=0$	$u_{2.0}=0$	$u_{3.0}=0$	

8.$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ Laplas tenglamasining taqribiy yechimini $G=\{0<x<1,\ 0<y<1\}$ kvadrat uchun ko‘rsatilgan chegaraviy shartlarda yeching.

a) Javob:

0	16.18	38.63	50.0
0	0	0	30.10
0	0	0	12.38
0	26.15	29.34	4.31

0	16.18	38.63	50.0
0	14.12	26.09	30.10
0	15.20	20.53	12.38
0	26.15	29.34	4.31

b) Javob:

0	17.98	39.02	50
0	0	0	30.10
0	0	0	12.38
0	29.05	29.63	4.31

0	17.98	39.02	50
0	15.18	36.39	30.10
0	16.37	21.26	12.38
0	29.05	29.63	4.31

Nazorat uchun savollar

1. Chekli ayirmalar usuli nimaga asoslangan?
2. To‘r nima?
3. To‘rning tugunlari nima?
4. Ichki, chegaraviy va qo‘shni tugunlarga ta’rif bering.
5. Birinchi tur chegaraviy tugunga ta’rif bering.
6. Ikkinchi tur chegaraviy tugunga ta’rif bering.
7. Hisoblash tugunlari nima?
8. Issiqlik tarqalish tenglamasi uchun chetki shartlarni ayting.
9. Issiqlik tarqalish tenglamasini to‘rlar usuli bilan taqribiy yechishda tugunlarda funksiyaning qiymati qanday topiladi?
10. G‘ovak muhitlarda quduqqa tomon harakatlanayotgan neft va gaz suyuqliklarini o‘rganish qanaqa tenglamaga olib keladi.
11. Laplas tenglamasi uchun Dirixlening ichki masalasini ifodalang.
12. Laplas tenglamasini to‘rlar usuli bilan taqribiy yechganda tugunlarda funksiyaning qiymati qanday topiladi?

Foydalanilgan adabiyotlar

1. Соатов. Ё.У. Олий математика. II қисм,- Т.Ўқитувчи, 1993.
2. Xurramov SH.R. Oliy matematika. (masalalar to`plami, nazorat topshiriqlari) I , II , III qismlar .Toshkent: Fan va texnologiya , 2015.
3. Письменный Д. Конспект лекций по высшей математике. I , II, III части. М.Аирис –М:Пресс , 2006.
4. Xolmurodov E, Yusupov A.I., Aliqulov T. A.,Oliy matematika. O‘quv qollanma. III qism. -Toshkent,2018.

Mundarija

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