O. S. ZIKIROV

# MATEMATIK FIZIKA TENGLAMALARI

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MIRZO ULUG‘BEK NOMIDAGI
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O. S. ZIKIROV

# MATEMATIK FIZIKA TENGLAMALARI

*Uzbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi tomonidan o‘quv qo‘llanma sifatida tavsiya etilgan*

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O'quv qo'llanma matematik fizika tenglamalari faniga bag'ishlangan. Unda matematik fizika tenglamalari haqida umumiy tushunchalar, matematik fizikaning asosiy tenglamalari va ularni keltirib chiqarish hamda bu tenglamalar uchun asosiy boshlang'ich–chegaraviy masalalarning qo'yilishi va ularni yechishning ayrim usullari bayon qilingan. Ayrim mavzularga oid tipik misol va masalalarning yechilishi ko'rsatilgan hamda mustaqil yechish uchun masalalar berilgan. Qo'llanmada nazorat uchun testlardan namunalar keltirilgan.

Ushbu o'quv qo'llanma universitetlarning matematika, mexanika, tadbiqiy matematika va informatika yo'nalishlari bo'yicha bakalavrlar tayyorlaydigan fakultetlar talabalari uchun mo'ljallangan.

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**Mas'ul muharrir:**
**J. O. Toxirov** – fizika–matematika fanlari doktori, professor.

**Taqrizchilar:**
**Q. S. Fayazov** – fizika–matematika fanlari doktori, professor;
**A. Q. O'rinov** – fizika–matematika fanlari doktori, professor.

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# SO'Z BOSHI

Matematik fizika tenglamalari fani klassik mexanika, fizika, gidrodinamika, akustika va boshqa sohalarda sodir bo'ladigan jarayonlarning matematik modellarini yaratish va bu masalalarni yechish usullarini qurish bilan uzviy bog'liq. Bu modellashtirish muayyan jarayonlarni ifodalovchi fizikaviy kattaliklar asosida tenglamalarni keltirib chiqarish bilan xarakterlanadi. Kvant mexanikasi, atom va yadro fizikasi, qattiq jismlar nazariyasi, elementar zarralar fizikasi kabi sohalarning rivojlanishi matematik tadqiqotlarning asosini tashkil etadi. Mexanika va fizikaning ko'plab masalalari xususiy hosilali differensial tenglamalarni tadqiq etishga keladi. Shuning uchun xususiy hosilali differensial tenglamalar fani matematik fizikaning zamonaviy holatini o'rganish va tushunish uchun zarur bo'lgan boshlang'ich bilimlarni beradi.

Matematik fizika tenglamalari faniga bag'ishlangan darsliklar, o'quv qo'llanmalar ingliz, rus va boshqa tillarda ko'plab nashr qilingan. Bu adabiyotlarning nazariy qismi bilan misol va masalalarga oid bo'limlari orasida biroz tafovut borligi seziladi. Hozirgi davr talabiga javob beradigan yuqori malakali mutaxassislar tayyorlash, ularning nazariy va amaliy masalalarni chuqur o'zlashtirishiga ko'maklashuvchi o'zbek tilida yozilgan darsliklar, o'quv qo'llanmalar yaratish muhim ahamiyatga ega.

Ushbu o'quv qo'llanma universitetlarning matematika, mexanika, amaliy matematika va informatika yo'nalishlari o'quv rejasidagi asosiy fanlardan biri bo'lgan matematik fizika tenglamalari faniga bag'ishlangan. Fizikaviy jarayonlarning matematik modelini tadqiq etish xususiy hosilali differensial tenglamalar kursining asosiy qismini tashkil qiladi.

Talabalar bu fanni mukammal o'zlashtirishlari uchun ularga ma'lum bilimlar majmuasi zarur bo'ladi. Masalan, matematik fizika tenglamalari kursi oddiy differensial tenglamalar fanining bevosita davomi hisoblanadi. Uni mukammal tushunib, o'zlashtirishlari uchun oddiy differensial tenglamalar fanidan ma'lum bilimlar talab qilinadi. Bu borada talabalar kuzatuvchi bo'lib qolmasdan, balki o'zlari misol va masalalar yechish, mavzularni mustaqil o'zlashtirishlari lozim.

Qo'llanmaning asosiy maqsadi, talabalarni matematik fizika tenglamalari va ular uchun qo'yiladigan asosiy masalalar bilan tanishtirish, ular uchun zarur bo'lgan boshlang'ich bilimlarni berishdan iborat. Unda matematik fizikaning xususiy hosilali differensial tenglamalar bilan ifodalanadigan ayrim masalalari o'rganilgan. Asosiy e'tibor xususiy hosilali differensial tenglamalarning klassifikatsiyasi, ularni kanonik ko'rinishga keltirish, Koshi masalasining qo'yilishi va uni yechish, giperbolik, parabolik va elliptik tipdagi tenglamalar uchun asosiy boshlang'ich–chegaraviy masalalarning qo'yilishi va ularni yechishning ayrim usullarini bayon qilishga qaratilgan.

Ushbu qo'llanmani yozishda va mustaqil yechish uchun masalalarni tanlashda xususiy hosilali differensial tenglamalar sohasida o'zbek va rus tillarida chop etilgan adabiyotlardan keng foydalanildi hamda muallifning Mirzo Ulug'bek nomidagi O'zbekiston Milliy universiteti talabalariga "Matematik fizika tenglamalari" fanidan olib borgan nazariy va amaliy mashg'ulotlari katta yordam bo'ldi.

Qo'lyozmani ko'rib chiqib, o'z fikr–mulohazalarini bildirgan qo'llanmaning mas'ul muharriri professor J.O.Toxirovga hamda professor A.Q.O'rinovga va professor Q.S.Fayazovga muallif samimiy minnatdorchilik bildiradi.

Qo'llanmadagi kamchiliklarni bartaraf etish, uning sifatini yaxshilashga qaratilgan tanqidiy fikr va mulohazalarini bildirgan hamkasblarga va kitobxonlarga muallif avvaldan minnatdorchilik bildiradi.

*Muallif*

# I BOB

# MATEMATIK FIZIKA TENGLAMALARI. ASOSIY MASALALARNING QO'YILISHI

Matematik fizikaning ayrim masalalari ikkinchi tartibli xususiy hosilali differensial tenglamalar orqali ifodalanadi. Qo'llanmaning ushbu dastlabki qismida xususiy hosilali tenglamalar haqida tushunchalar va ta'riflar qisqacha bayon qilingan.

Matematik fizikaning asosiy tenglamalarini keltirib chiqarish, bu tenglamalar uchun boshlang'ich–chegaraviy shartlar va korrekt qo'yiladigan masalalar berilgan.

Nokorrekt qo'yilgan masalalarga misollar keltirilgan.

## 1–§. Xususiy hosilali differensial tenglamalar. Asosiy tushunchalar va ta'riflar

Noma'lum $u(x) = u(x_1, x_2, \ldots, x_n)$ funksiya uning xususiy hosilalarini va $x_1, x_2, \ldots, x_n$ erkli o'zgaruvchilarni bog'lovchi quyidagi ifoda

$$F\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \ldots\right) = 0 \qquad (1)$$

*xususiy hosilali differensial tenglama* deyiladi.

Bu yerda $F(\cdot)$ — o'z argumentlarining berilgan funksiyasi, $x \in D \subset R^n$, $n \geq 2$; $k_1 + k_2 + \ldots + k_n = k$, $k = \overline{0, m}$; $m \geq 1$; $D$ esa (1) tenglamaning *berilish sohasi* deyiladi.

Misollar.

$$1)\ \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \ldots + \frac{\partial u}{\partial x_n} + u^2 = 0;$$

$$2)\ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} + \sin u = 0;$$

$$3)\ \frac{\partial u}{\partial x_1} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_3 \partial x_3} = f(x_1, x_2, x_3, u);$$

Yuqorida keltirilgan tenglamalar mos ravishda *birinchi tartibli*, *ikkinchi tartibli* va *uchinchi tartibli* xususiy hosilali differensial tenglamalardir.

Demak, differensial tenglamada noma'lum funksiya xususiy hosilasining eng yuqori tartibiga shu *tenglamaning tartibi* deyiladi.

Agar $u(x) = u(x_1, x_2, \dots, x_n)$ funksiya biror $D$ sohada aniqlangan, uzluksiz va tenglamada qatnashgan uzluksiz hosilalarga ega bo'lib, shu sohada tenglamani qanoatlantirsa, u holda bu funksiya *tenglamaning yechimi* deb ataladi.

Agar $F$ barcha

$$\frac{\partial^k u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \qquad 0 \le k \le m,$$

hosilalarga nisbatan chiziqli bo'lsa, u holda (1) tenglama *chiziqli xususiy hosilali differensial tenglama* deyiladi.

Chiziqli tenglamani quyidagi ko'rinishda yozish mumkin:

$$\sum_{k=1}^{m} \sum_{k_1, \cdots, k_n} a_{k_1, \cdots, k_n}(x) \frac{\partial^k u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} = f(x), \qquad \sum_{j=1}^{n} k_j = k$$

yoki

$$\mathcal{L}u = f(x)$$

bu yerda

$$\mathcal{L} \equiv \sum_{k=0}^{n} \sum_{k_1 \dots k_n} a_{k_1 \dots k_n}(x) \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \qquad \sum_{j=1}^{n} k_j = k,$$

$m$ – *tartibli chiziqli differensial operator* deb ataladi.

Odatda xususiy hosilali chiziqli ikkinchi tartibli differensial tenglama quyidagicha ifodalanadi

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x), \qquad (2)$$

bu yerda $a_{ij}(x)$, $b_i(x)$, $c(x)$ biror chekli yoki cheksiz $D$ sohada berilgan funksiyalar, ular tenglamaning *koeffitsiyentlari*, $f(x)$ esa tenglamaning *ozod hadi* deyiladi.

Agar chiziqli tenglamada $f(x) = 0$ bo'lsa, u *bir jinsli*, aksincha, ya'ni $f(x) \neq 0$ bo'lsa, *bir jinsli bo'lmagan tenglama* deyiladi.

Agar (2) tenglamada uning chap tomonini $L(u)$ orqali belgilasak, u holda (2) tenglamani quyidagi ko'rinishda yozib olish mumkin:

$$L(u) = f(x), \tag{3}$$

bu yerda

$$L(u) \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \tag{4}$$

*ikkinchi tartibli xususiy hosilali differensial operator* deyiladi.

Masalan, ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali chiziqli differensial tenglama ushbu

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} +$$

$$+a_1(x,y)\frac{\partial u}{\partial x} + b_1(x,y)\frac{\partial u}{\partial y} + c_1(x,y)u = f(x,y)$$

ko'rinishga ega bo'ladi.

Bu yerda $a(x,y)$, $b(x,y)$, $c(x,y)$, $a_1(x,y)$, $b_1(x,y)$ va $c_1(x,y)$ – tenglamaning koeffitsiyentlari, $f(x,y)$ esa uning ozod hadi bo'lib, ular $x, y$ argumentlarning berilgan funksiyalari.

Agar (1) tenglamada $F$ faqat yuqori tartibli

$$\frac{\partial^m u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \qquad \sum_{j=1}^{n} k_j = m$$

hosilalariga nisbatan chiziqli bo'lsa, u holda (1) *kvazichiziqli tenglama* deyiladi.

Masalan, quyidagi tenglama

$$a\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x^2}+2b\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x\partial y}+$$

$$+c\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2}+F_1\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0,$$

ikki o'zgaruvchili ikkinchi tartibli kvazichiziqli tenglamadir.

Xususiy hosilalar uchun quyidagi belgilashlardan foydalanamiz:

$$u_x=\frac{\partial u}{\partial x},\quad u_y=\frac{\partial u}{\partial y},\quad u_{xx}=\frac{\partial^2 u}{\partial x^2},\quad u_{xy}=\frac{\partial^2 u}{\partial x\partial y},\quad u_{yy}=\frac{\partial^2 u}{\partial y^2},$$

va h. k.

Agar (4) ifodada $a_{ij}(x)\equiv 0$, $i,j=\overline{1,n}$, va $b_i(x)$ koeffitsiyentlardan kamida bittasi noldan farqli bo'lsa, u holda (4) ifoda *birinchi tartibli chiziqli operator* deb ataladi.

Birinchi va ikkinchi tartibli chiziqli differensial operatorlar quyidagi xossalarga ega:

$$1)\quad L(cu)=cL(u),\qquad c=const;$$

$$2)\quad L(u_1+u_2)=L(u_1)+L(u_2).$$

Bu xossalarni tekshirib ko'rish qiyinchilik keltirib chiqarmaydi.

Yuqoridagi xossalardan chiziqli bir jinsli differensial tenglamalar uchun muxim bo'lgan ushbu xulosalar kelib chiqadi:

1–XULOSA. Agar $u(x)$ funksiya biror $D$ sohada bir jinsli xususiy hosilali differensial $L(u)=0$ tenglamaning yechimi bo'lsa, u holda $cu(x)$, $c=const$ funksiya ham $D$ sohada shu tenglamaning yechimi bo'ladi.

2–XULOSA. Agar $u_1(x)$ va $u_2(x)$ funksiyalar biror $D$ sohada bir jinsli xususiy hosilali differensial $L(u)=0$ tenglamaning yechimlari bo'lsa, u holda $u_1(x)+u_2(x)$ funksiya ham $D$ sohada shu $L(u)=0$ tenglamaning yechimi bo'ladi.

Bu xulosalardan quyidagi natijani olamiz:

NATIJA. Agar $u_1(x)$, $u_2(x), \ldots, u_n(x)$ funksiyalar biror $D$ sohada bir jinsli xususiy hosilali differensial $L(u) = 0$ tenglamaning yechimlari bo'lsa, u holda bu funksiyalarning chiziqli kombinatsiyasi

$$c_1u_1(x) + c_2u_2(x) + \ldots + c_nu_n(x)$$

ham $D$ sohada shu $L(u) = 0$ tenglamaning yechimi bo'ladi.

Bu erda $c_i$, $i = \overline{1, n}$ ixtiyoriy o'zgarmaslar.

Agar $R^n$ fazoning o'lchami ikkiga teng, ya'ni $n = 2$ bo'lsa, u holda $x_1 = x$, $x_2 = y$ deb, agar $n = 3$ bo'lsa, u holda $x_1 = x$, $x_2 = y$, $x_3 = z$ deb olamiz.

1–MISOL. Quyidagi berilgan

$$a) \quad u(x, y, z) = \frac{x^2}{y^2} + \frac{y^2}{z^2}, \qquad b) \quad u(x, y, z) = xyz,$$

funksiyalar $x > 0$, $y > 0$, $z > 0$ sohada ushbu

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial x} = 0$$

tenglamaning yechimi bo'ladimi?

YECHISH. a) Berilgan funksiyaning $u_x$, $u_y$ va $u_z$ xususiy hosilalarini topamiz:

$$\frac{\partial u}{\partial x} = \frac{2x}{y^2}, \qquad \frac{\partial u}{\partial y} = -\frac{2x^2}{y^3} + \frac{2y}{z^2}, \qquad \frac{\partial u}{\partial z} = -\frac{2y^2}{z^3}$$

va ularni berilgan tenglamaga ko'yib,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial x} = \frac{2x^2}{y^2} - \frac{2x^2}{y^2} + \frac{2y^2}{z^2} - \frac{2y^2}{z^2} = 0$$

tenglikni olamiz.

Demak, $u(x, y, z) = \dfrac{x^2}{y^2} + \dfrac{y^2}{z^2}$ funksiya qaralayotgan sohada berilgan tenglamaning yechimi ekan.

b) Xuddi yuqoridagi kabi berilgan funksiyaning xususiy hosilalarini topamiz: $u_x = yz$, $u_y = xz$, $u_z = xy$ va bularni tenglamaga qo'ysak,

$$xu_x + yu_y + zu_z = xyz \neq 0$$

bo'ladi. Shunday qilib, $u = xyz$ funksiya $x > 0$, $y > 0$, $z > 0$ sohada berilgan tenglamani qanoatlantirmas ekan.

2–MISOL. Ushbu

$$u(x,y) = \ln\frac{1}{r}, \quad r^2 = (x - x_0)^2 + (y - y_0)^2,$$

funksiya $R^2$ tekislikda

$$\Delta u \equiv u_{xx} + u_{yy} = 0$$

Laplas tenglamasining yechimi bo'ladimi?

Bu erda $(x_0, y_0) \in R^2$ tekislikdagi biror fiksirlangan nuqta.

YECHISH. Berilgan funksiyaning hosilasini hisoblash uchun uni qulay ko'rinishda yozib olamiz

$$u(x,y) = \ln\frac{1}{r} = -\frac{1}{2}\ln r^2.$$

Bu funksiyaning xususiy hosilalarini olaylik

$$u_x = -\frac{1}{2}\frac{1}{r^2}(r^2)_x = -\frac{x - x_0}{r^2};$$

$$u_{xx} = (u_x)_x = -\frac{1}{r^2} + \frac{2(x - x_0)^2}{r^4}.$$

Xuddi shu kabi berilgan funksiyaning $y$ bo'yicha ikkinchi tartibli hosilasini hisoblaymiz:

$$u_{yy} = -\frac{1}{r^2} + \frac{2(y - y_0)^2}{r^4}.$$

Endi olingan $u_{xx}$ va $u_{yy}$ hosilalarni Laplas tenglamasiga qo'yamiz. Natijada barcha $(x,y) \neq (x_0, y_0)$ nuqtalarda

$$u_{xx} + u_{yy} = -\frac{2}{r^2} + \frac{2(x - x_0)^2 + 2(y - y_0)^2}{r^4} = -\frac{2}{r^2} + \frac{2}{r^2} = 0$$

ayniyatga ega bo'lamiz.

Demak, $u(x,y) = \ln\frac{1}{r}$ funksiya $R^2$ tekislikning barcha $(x,y) \neq (x_0, y_0)$ nuqtalarida Laplas tenglamasining yechimi bo'lar ekan.

Xususiy hosilali differensial tenglamalar xuddi oddiy differensial tenglamalar kabi aksariyat hollarda berilgan tenglamani qanoatlantiruvchi cheksiz ko'p xususiy yechimlarga ega. Bularning yig'indisi qaralayotgan tenglamaning umumiy yechimini tashkil qiladi. Oddiy differensial tenglamaning umumiy yechimi bilan xususiy hosilali differensial tenglamaning umumiy yechimi o'rtasida keskin farq bor.

Oddiy differensial tenglamalar kursidan ma'lumki, ushbu

$$y'' = f(x, y, y'), \tag{5}$$

ikkinchi tartibli tenglamaning umumiy yechimi 2 ta ixtiyoriy o'zgarmas songa bog'liq bo'lib, u

$$y(x) = g(x, c_1, c_2), \tag{6}$$

ko'rinishdagi egri chiziqlar oilasidan iborat.

Berilgan tenglamaning ixtiyoriy xususiy yechimi $c_1$, $c_2$ parametrlarga qiymatlar berish natijasida hosil bo'ladi.

Masalan, ushbu

$$y'' + 4y = 0, \tag{7}$$

ikkinchi tartibli bir jinsli chiziqli differensial tenglamaning umumiy yechimi

$$y(x) = c_1 \cos 2x + c_2 \sin 2x,$$

bu erda $c_1$, $c_2$ ixtiyoriy o'zgarmaslar.

Agar berilgan tenglamaning $y(0) = 1$, $y'(0) = 1$ boshlang'ich shartlarni qanoatlantiruvchi yechimini topish talab qilinsa, u holda umumiy yechimda qatnashgan $c_1$, $c_2$ o'zgarmaslarni $c_1 = 0$, $c_2 = 1/2$ ekanligi ko'rinadi. Bundan berilgan (7) tenglamaning xususiy yechimi

$$y(x) = \frac{1}{2} \sin 2x$$

ekanligi kelib chiqadi.

Endi ikki o'zgaruvchili birinchi tartibli xususiy hosilali quyidagi

$$F\left(x, y, u, \frac{\partial u}{\partial x}\right) = 0, \tag{8}$$

tenglamani qaraylik.

Bu tenglamada faqat $u_x$ hosila qatnashyapti, $y$ o'zgaruvchini fiksirlangan, deb qaraymiz. Barcha fiksirlangan $y$ larda (8) tenglamani oddiy differensial tenglama deb qarash mumkin, bunda $x$ erkli o'zgaruvchili, $u$ noma'lum funksiya.

Faraz qilaylik, bu tenglamaning umumiy yechimi

$$u = \varphi(x, y, C), \tag{9}$$

formula bilan aniqlansin. Bu yechimda $y$ parametr va ixtiyoriy $C$ o'zgarmaslar uchun u (8) tenglamani qanoatlantiradi. (9) funksiya xususiy hosilali (8) tenglamaning yechimi bo'lishi uchun $C$ parametr $x$ ga nisbatan o'zgarmas bo'lishi zarur, ya'ni u $y$ o'zgaruvchining ixtiyoriy funksiyasi bo'lishi kerak. Demak, $C$ ning o'rniga $y$ ga bog'liq bo'lgan ixtiyoriy $\psi(y)$ funksiya qo'ysak, natijada (8) tenglamaning (9) ga qaraganda umumiyroq bo'lgan

$$u = \varphi(x, y, \psi(y))$$

ko'rinishdagi yechimiga ega bo'lamiz.

Shunday qilib, birinchi tartibli xususiy hosilali (8) tenglamaning umumiy yechimi $C(R)$ sinfiga tegishli bo'lgan ixtiyoriy $\psi(y)$ funksiyaga bog'liq bo'lar ekan.

Buni ayrim misollarda ko'raylik.

3–MISOL. Ushbu

$$u - x\frac{\partial u}{\partial x} - x^2y^2 = 0, \tag{10}$$

birinchi tartibli xususiy hosilali differensial tenglamaning umumiy yechimini toping.

YECHISH. Berilgan tenglamani quyidagi

$$\frac{\partial u}{\partial x} - \frac{1}{x}u = -xy^2$$

ko'rinishda ifodalaymiz.

Bu tenglamada $y$ o'zgaruvchini parametr deb qaraymiz. Oxirgi tenglama birinchi tartibli chiziqli differensial tenglama bo'lgani uchun uning umumiy yechimi $u = cx - x^2y^2$ bo'ladi. U holda xususiy hosilali (10) differensial tenglamaning umumiy yechimi

$$u(x, y) = x\psi(y) - x^2y^2, \qquad \psi(y) \in C(R),$$

ko'rinishda topiladi.

Xususiy hosilali differensial tenglamaning umumiy yechimi oddiy differensial tenglamadan farqli o'laroq berilgan tenglamaning tartibiga teng bo'lgan sondagi ixtiyoriy funksiyalarga bog'liq bo'lar ekan.

4-MISOL. Ushbu $u_{xy} = 0$ yoki

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 0 \tag{11}$$

tenglamaning umumiy yechimini toping.

YECHISH. Berilgan tenglamani $x$ bo'yicha integrallab,

$$u_y = \psi(y)$$

tenglamani hosil qilamiz. Bunda $\psi(y)$ – ixtiyoriy funksiya.

Oxirgi tenglamani $y$ bo'yicha integrallab,

$$u(x, y) = \int_0^y \psi(z)dz + \psi_1(x), \tag{12}$$

tenglikni olamiz. Bu erda $\psi_1(x)$ – ixtiyoriy funksiya.

Endi (12) tenglikda

$$\int_0^y \psi(z)dz = \psi_2(y)$$

deb belgilab,

$$u(x, y) = \psi_1(x) + \psi_2(y) \tag{13}$$

formulaga ega bo'lamiz.

Bunda $\psi(y)$ ixtiyoriy funksiya bo'lganligi uchun $\psi_2(y)$ ham $y$ o'zgaruvchining ixtiyoriy funksiyasi bo'ladi.

Agar $\psi_1(x)$ va $\psi_2(y)$ funksiyalar $R$ sohada bir marta uzluksiz differensiallanuvchi, ya'ni $\psi_1(x)$, $\psi_2(y) \in C^1(R)$ bo'lsa, u holda (13) formula orqali aniqlangan $u(x,y)$ funksiya $R^2$ tekislikda (11) tenglamaning umumiy yechimi bo'ladi.

Xususiy hosilali (11) differensial tenglamaning umumiy yechimi yordamida uning xususiy yechimini ham topish mumkin. Buning uchun qaralayotgan masalaning berilgan shartlari asosida $\psi_1(x)$ va $\psi_2(y)$ funksiyalarning aniq ko'rnishlari topiladi.

Shuni ta'kidlash muximki, ayrim xususiy hosilali differensial tenglamalarning xususiy yechimlarini aniq ko'rinishlarini topish mumkin. Ko'p hollarda xususiy hosilali differensial tenglamalarining xususiy yechimlarini topish usullari yaratilgan. Qaralayotgan tenglamani, ma'lum boshlang'ich va chegaraviy shartlarni qanoatlantiradigan yechimlari topiladi.

Fizikaviy jarayonlarning matematik modelini qurish va uni tadqiq etish matematik fizikaning asosiy vazifasi hisoblanadi.

Mexanika va fizikaning juda ko'p masalalari ikkinchi tartibli xususiy hosilali differensial tenglamalar orqali ifodalanadi.

MASALAN:

1. Bir jinsli torning ko'ndalang tebranishi, sterjenning bo'ylama tebranishi, o'tkazgichdagi elektr tebranishlar, turli muhitlarda tovush tarqalishi va shu kabi jarayonlar

$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) + f(x,y,z,t), \quad a = const, \tag{14}$$

*to'lqin tenglamasi* bilan ifodalanadi, va bu tenglama *uch o'lchovli to'lqin tarqalish tenglamasi* deyiladi.

Ushbu tenglama

$$u_{tt} = a^2(u_{xx} + u_{yy}) + f(x,y,t),$$

*ikki o'lchovli to'lqin tenglamasi* deb atalidi. (14) tenglama bir o'lchovli

$$u_{tt} = a^2 u_{xx} + f(x,t),$$

bo'lgan holda bir jinsli torning majburiy tebranishini ifodalaydi.

2. Bir jinsli izotrop jismlarda issiqlikning tarqalishi, diffuziya jarayoni va g'ovak muhitlarda suyuqlik va gazlarning filtrlanishi kabi jarayonlar

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) + f(x, y, z, t), \quad a = const, \qquad (15)$$

*issiqlik tarqalish tenglamasi* orqali ifodalanadi.

Yuqoridagi (15) tenglama *uch o'lchovli issiqlik tarqalish tenglamasi* deyiladi. Ushbu ko'rinishdagi

$$u_t = a^2(u_{xx} + u_{yy}) + f(x, y, t),$$

tenglama *ikki o'lchovli issiqlik tarqalish tenglamasi* deb ataladi. Bir o'lchovli bo'lgan

$$u_t = a^2 u_{xx} + f(x, t),$$

hol esa, *bir o'lchovli issiqlik tarqalish tenglamasi* deyiladi.

3. Statsionar issiqlik holati, o'tkazgich sirtida zaryadlarning muvozanatlashuvi, siqilmaydigan suyuqliklarning harakati va shu kabi jarayonlar

$$u_{xx} + u_{yy} + u_{zz} = -f(x, y, z), \qquad (16)$$

*Puasson tenglamasi* orqali ifodalanadi.

Agar (16) tenglamada $f(x, y, z) \equiv 0$ bo'lsa, u holda bu tenglama

$$u_{xx} + u_{yy} + u_{zz} = 0, \qquad (17)$$

*Laplas tenglamasi* deb ataladi.

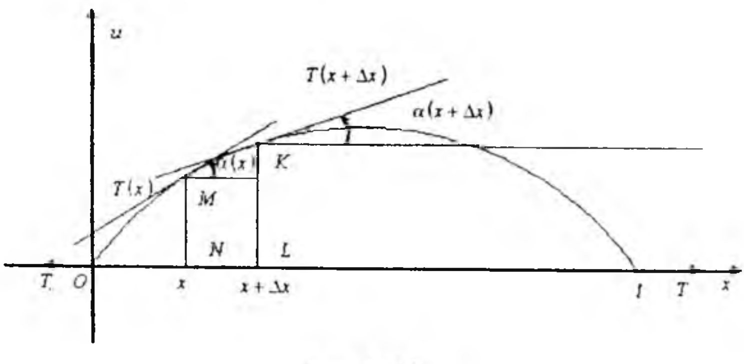
Elektr zaryadi va massasi hisobga olinmagan tortishish maydoni potensiallari va statsionar elektr maydoni masalalari Laplas tenglamasi orqali ifodalanadi.

Yuqorida keltirilgan (14)–(17) tenglamalar *matematik fizikaning asosiy tenglamalari* deb yuritiladi. Bu tenglamalarni mukammal o'rganish turli fizikaviy masalalarni yechish va umumiyroq bo'lgan xususiy hosilali differensial tenglamalar nazariyasini yaratishga imkoniyat beradi.

## 2-§. Tor tebranish tenglamasini keltirib chiqarish. Asosiy boshlang'ich–chegaraviy masalalarning qo'yilishi

Tekislikda, $Ox$ o'qi bo'yicha uchlari mahkamlangan uzunligi $l$ ga teng bo'lgan torni (ingichka, elastik ipni) qaraylik.
*Ingichka* — bu torning ko'ndalang kesimi uning uzunligiga nisbatan cheksiz kichik miqdor, *egiluvchan* deganda tor uzunligining o'zarishiga bog'liq bo'lmagan holda shaklini o'zgarishiga torning hech qanday qarshilik qilmasligi tushuniladi. Bu tushunchalarning matematik ma'nosi — torda sodir bo'ladigan $T(x)$ taranglik kuchi doimo uning oniy uzunligiga o'tkazilgan normal bo'yicha yo'nalgan bo'ladi.

Faraz qilaylik, $Ox$ o'q bo'yicha torning uchlariga qarama-qarshi tomonlarga yo'nalgan $T_0$ taranglik kuchi qo'yilgan bo'lsin. Agar tor tashqi kuchlar ta'sirida muvozanat holatidan chiqarilsa, u holda tor tebranma harakat qiladi. Bunda torning muvozanat holatidagi $N(x)$ nuqtasi $t$ vaqtda $M$ holatga o'tadi (1–shakl).




1 — shakl.

Tor tebranish tenglamasini keltirib chiqarish uchun quyidagilarni talab qilamiz:

1) torning barcha nuqtalari bir tekislikda $Ox$ o'qiga perpendikulyar tebransin, ya'ni tor ko'ndalang tebransin;

2) torning kichik tebranishlari hisobga olinsin;

3) og'irlik kuchining ta'siri inobatga olinmasin, ya'ni taranglik kuchi shunchalik kattaki, buning natijasida og'irlik kuchining ta'siri sezilmaydi.

Torning tebranishi bir tekislikda sodir bo'layotgani uchun torning tebranish qonuni, ya'ni muvozanat holatidan og'ishi $NM$ ikki o'zgaruvchili bitta $u(x,t)$ funksiya orqali ifodalanadi. Bunda $u$ – torning $t$ vaqtdagi abssissasi $x$ bo'lgan $N$ nuqtasining $M$ nuqtagacha muvozanat holatidan $NM$ og'ishi.

Agar torning kichik tebranishini inobatga olsak, u holda $u(x,t)$ funksiya ham kichik va etarlicha silliq torning $x$ nuqtasiga $t$ vaqtda o'tkazilgan urinmaning burchak koeffitsienti $u_x(x,t)$ ham kichik bo'ladi.

Torning tebranishi shunchalik kichik–ki, bunda

$$\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \ll 1,$$

bo'lsin. Bu torning kichik tebranishlarida uning uzunligini o'zgarmasligini bildiradi.

Haqiqatdan ham, $t$ vaqtda torning $MK$ yoyining uzunligi

$$l_{MK} = \int_x^{x+\Delta x} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx \approx \Delta x$$

formula bilan aniqlanadi.

Demak, torning kichik tebranishlarida uning uzunligi o'zgarmaydi. U holda Guk qonuniga ko'ra taranglik koeffitsiyenti $T$ vaqtga ham $x$ ga ham bog'liq emas va u torning barcha nuqtalarida bir xil $T_0$ ga teng.

Endi tor tebranish tenglamasini keltirib chiqaraylik. Buning uchun torning $MK$ bo'lakchasini ajratib olamiz va bunga ta'sir qilayotgan kuchlarni koordinata o'qlariga proeksiyasini tushiramiz. Dalamber prinsipiga asosan, barcha kuchlar proeksiyalarining yig'indisi, inersiya kuchini hisobga olganda nolga teng bo'ladi. Taranglik kuchining gorizantal o'qdagi proeksiyalarining yig'indisi

$$F_{gor} = -T(x)\cos\alpha(x) + T(x+\Delta x)\cos\alpha(x+\Delta x) =$$

$$= -T_0 \cos\alpha(x) + T_0 \cos\alpha(x + \Delta x) \equiv 0,$$

bu yerda

$$\cos\alpha(x) = \frac{1}{\sqrt{1 + \operatorname{tg}^2\alpha(x)}} = \frac{1}{\sqrt{1 + u_x^2(x,t)}} \cong 1.$$

Endi taranglik kuchini vertikal o‘qqa proeksiyasini qaraylik:

$$F_{ver} = T_0 \sin\alpha(x) - T_0 \sin\alpha(x + \Delta x) =$$

$$= T_0\left(\frac{\operatorname{tg}\alpha(x + \Delta x)}{\sqrt{1 + \operatorname{tg}^2\alpha(x + \Delta x)}} - \frac{\operatorname{tg}\alpha(x)}{\sqrt{1 + \operatorname{tg}^2\alpha(x)}}\right) =$$

$$= T_0\left(\frac{u_x(x + \Delta x, t)}{\sqrt{1 + u_x^2(x + \Delta x, t)}} - \frac{u_x(x,t)}{\sqrt{1 + u_x^2(x,t)}}\right) =$$

$$\cong T_0[u_x(x + \Delta x, t) - u_x(x,t)].$$

Oxirgi formuladan Lagranj teoremasiga asosan

$$F_{ver} \cong T_0 u_{xx}(x', t), \qquad x' \in (x, x + \Delta x)$$

kelib chiqadi.

Torning ko‘ndalang tebranishlari qaralayotgani uchun inersiya kuchi va tashqi kuchlar $Ou$ o‘qiga parallel yo‘nalgan. Shuning uchun ularning $Ou$ o‘qdagi proeksiyasini topamiz.

Faraz qilaylik, $p(x,t)$ – torning $MK$ bo‘lagiga ta’sir qilayotgan uzluksiz tashqi kuch, $\rho(x)$ esa torning uzluksiz chiziqli zichligi bo‘lsin. U holda tashqi kuchlarning $Ou$ o‘qiga proeksiyasi

$$F_{tashqi} \cong p(x,t)\Delta x$$

bo‘ladi. Torning zichligi $\rho(x)$ bo‘lgani uchun uning $MK$ bo‘lagining massasi

$$m \cong \rho(x)\Delta x$$

ga teng.

Nyuton qonuniga ko'ra inersiya kuchi

$$F_{in} = ma \cong \rho(x)\Delta x u_{tt}(x'', t), \qquad x'' \in [x, x + \Delta x]$$

formula bilan aniqlanadi.

U holda barcha kuchlarning $Ou$ o'qdagi proeksiyasi

$$T_0 u_{xx}(x', t)\Delta x - \rho(x)u_{tt}(x'', t)\Delta x + p(x, t)\Delta x = 0$$

formula bilan ifodalanadi.

Bu tenglikni $\Delta x \neq 0$ ga qisqartirib, so'ngra $\Delta \to 0$ limitga o'tsak,

$$T_0 u_{xx}(x, t) - \rho(x)u_{tt}(x, t) + p(x, t) = 0 \tag{1}$$

torning majburiy tebranish tenglamasiga ega bo'lamiz.

Agar tor bir jinsli bo'lsa, ya'ni $\rho(x) = const$, u holda (1) tenglama quyidagi

$$u_{tt} = a^2 u_{xx} + f(x, t), \tag{2}$$

ko'rinishga keladi. Bu erda $a^2 = T_0/\rho$, $f(x, t) = p(x, t)/\rho$.

Agar (1) yoki (2) tenglamada tashqi kuchlar qatnashmasa, ya'ni $p(x, t) \equiv 0$, bo'lsa, u holda (2) tenglama ushbu

$$u_{tt} = a^2 u_{xx}, \tag{3}$$

ko'rinishga keladi.

Oxirgi (3) tenglama bir jinsli *torning erkin tebranish tenglamasi* deyiladi. Bu tenglama *bir o'lchovli to'lqin tarqalish tenglamasi* deb ham yuritiladi.

Sterjenning bo'ylama tebranishlari, trubkadagi gazning tebranishlari va boshqa tebranma harakatlar (1) ko'rinishdagi tenglama orqali ifodalanadi.

### Asosiy boshlang'ich-chegaraviy masalalarning qo'yilishi

Xususiy hosilali (1) tenglamaning koeffitsientlari va ozod hadiga qo'yilgan ma'lum shartlarga ko'ra cheksiz ko'p xususiy yechimlarga ega bo'ladi. Shuning uchun (1) tenglamaning o'zi qaralayotgan tor tebranishini to'liq aniqlash uchun etarli emas. Masalaning fizik mohiyatidan kelib chiqqan holda qo'shimcha shartlarning bajarilishi talab qilinadi. Fizikadan ma'lumki, nuqtaning harakatini aniqlash uchun uning boshlang'ich holati va boshlang'ich tezligini bilish kifoya. Shuning uchun ham, tor harakatini aniqlash uchun $t = 0$ da uning boshlang'ich holati va boshlang'ich teziligini bilish etarli bo'ladi, ya'ni

$$u(x,t)\,|_{t=0} = \varphi(x), \quad \frac{\partial u(x,t)}{\partial t}\,|_{t=0} = \psi(x), \quad 0 \le x \le l, \tag{4}$$

Bu *boshlang'ich shartlar* yoki *Koshi shartlari* deyiladi.

Torning uchlari mahkamlangan yoki mahkamlanmagan bo'lishi mumkin. Uchlari mahkamlangan tor uchun quyidagi shartlar o'rinli bo'ladi:

$$u(x,t)\,|_{x=0} = 0, \quad u(x,t)\,|_{x=l} = 0, \quad 0 \le t \le T, \tag{5}$$

bu yerda $T > 0$, $l$ – torning uzunligi.

Bu (5) ko'rinishdagi shartlar *chegaraviy shartlar* deb yuritiladi.

Shunday qilib, uchlari mahkamlangan torning harakatini aniqlash to'g'risidagi fizikaviy masala quyidagi matematik masalaga keltirildi: *xususiy hosilali (1) differensial tenglamaning (4) boshlang'ich va (5) chegaraviy shartlarni qanoatlantiruvchi* $u(x,t)$ *yechimi topilsin.*

Bu masala tor tebranish tenglamasi uchun *birinchi aralash masala* deyiladi.

Agar torning uchlari mahkamlanmagan bo'lsa, ya'ni bu uchlar biror qoida asosida harakatlansa, u holda (5) chegaraviy shartlar quyidagi

$$u(x,t)\,|_{x=0} = \mu_1(t), \quad u(x,t)\,|_{x=l} = \mu_2(t), \quad 0 \le t \le T,$$

shartlarga almashadi.

Boshqa turdagi chegaraviy shartlarni ham olish mumkin. Chegaraviy shartlar uch xil turga bo'linadi:

1) BIRINCHI TUR CHEGARAVIY SHARTLAR:

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad 0 \le t \le T, \tag{6}$$

bu shart torning uchlari $Ox$ o'qiga vertikal holda $\mu_1(t)$ va $\mu_2(t)$ funksiyalar bilan berilgan qoida asosida harakatlanishini bildiradi.

2) IKKINCHI TURDAGI CHEGARAVIY SHARTLAR: bunday chegaraviy shartlar quyidagicha ifodalanadi:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \nu_1(t), \quad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = \nu_2(t), \quad 0 \le t \le T, \qquad (7)$$

Bu (7) shartlar torning uchlariga $\nu_1(t)$ va $\nu_2(t)$ ma'lum kuchlar qo'yilganini anglatadi.

3) UCHINCHI TURDAGI CHEGARAVIY SHARTLAR:

$$\alpha_1(t)u_x(0,t) + \beta_1(t)u(0,t) = \sigma_1(t), \quad 0 \le t \le T, \qquad (8a)$$

$$\alpha_2(t)u_x(l,t) + \beta_2(t)u(l,t) = \sigma_2(t), \quad 0 \le t \le T, \qquad (8b)$$

bu yerda $\alpha_i(t)$, $\beta_i(t)$ va $\sigma_i(t)$ $(i = 1, 2)$ – berilgan funksiyalar, ixtiyoriy $t \in [0, T]$ da yetarlicha uzluksiz va

$$\alpha_i^2(t) + \beta_i^2(t) \neq 0.$$

(8) chegaraviy shartlar torning uchlari elastik mahkamlanganligini ifodalaydi. Agar (6)–(8) chegaraviy shartlarda $\mu_i(t)$, $\nu_i(t)$ va $\sigma_i(t)$, $(i = 1, 2)$ berilgan funksiyalar nolga teng bo'lsa, u holda bunday chegaraviy shartlar *bir jinsli chegaraviy shartlar* deyiladi.

Endi ikkinchi va uchinchi tur chegaraviy shartlarni sharxlashga harakat qilaylik. Buning uchun bir uchi shiftga mahkamlangan, ikkinchi uchi erkin harakatlanuvchi sterjenning bo'ylama tebranishi haqidagi masalani qaraylik (2–shakl).

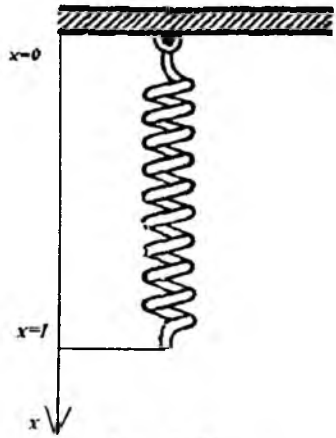
Sterjenning erkin uchining harakat qonuni berilmagan bo'lsin. Agar sterjenning mahkamlangan uchi $x = 0$ da uning og'ishi $u(0,t) = 0$, erkin uchi $x = l$ da esa uning tarangligi nolga teng, ya'ni

$$T(l,t) = k\frac{\partial u}{\partial x}(l,t) = 0.$$

Sterjenga tashqi kuchlarning ta'siri yo'qligi sababli, uning erkin harakat qiluvchi uchida chegaraviy shart quyidagi

$$u_x(x,t)\,|_{x=l} = 0$$

ko'rinishda bo'ladi.




2 — shakl.

Agar prujinaning $x = 0$ uchi ma'lum $h(t)$ qonun asosida harakatlansa, $x = l$ uchiga esa $\nu(t)$ kuch osilgan bo'lsa, u holda chegaraviy shartlar

$$u(x,t)\mid_{x=0} = h(t), \quad u_x(x,t)\mid_{x=l} = \nu(t), \quad 0 \le t \le T,$$

ko'rinishda beriladi.

Agar prujinaning $x = l$ uchi elastik mahkamlangan bo'lsa, u holda prujinaning erkin harakatlanuvchi uchida chegaraviy shart ushbu ko'rinishda beriladi:

$$ku_x(l,t) = -\alpha u(l,t) \quad \text{yoki} \quad u_x(l,t) = -hu(l,t), \quad 0 \le t \le T,$$

bu yerda $h = \alpha/k$, $k$, $\alpha > 0$. Bu holda prujinaning $x = l$ uchi ko'chishi mumkin, lekin mahkamlangan nuqtadagi elastiklik kuchi shu uchida ko'chgan uchining boshlang'ich holatga qaytarishga intiluvchi taranglik kuchini yuzaga keltiradi. Guk qonuniga ko'ra, berilgan kuch $u(l,t)$ ko'chishga proporsional, bunda proporsionallik koeffisiyenti mahkamlangan nuqtadagi *bikrlik koeffitsiyenti* deyiladi.

Agar elastik mahkamlangan $x = l$ uchi ko'chsa va uni boshlang'ich holatidan chetlanishi $\theta(t)$ funksiya bilan aniqlansa, u holda chegaraviy shart quyidagi

$$u_x(l,t) = -h[u(l,t) - \theta(t)], \quad 0 \le t \le T. \qquad (9)$$

ko'rinishda bo'ladi. Taranglik kuchi $T(0,t) = -ku_x(0,t)$ ni e'tiborga olsak, $x = 0$, ya'ni chap tomonda elastik mahkamlanganlik sharti

$$u_x(0,t) = -h[u(0,t) - \theta(t)], \quad 0 \le t \le T,$$

bo'ladi.

Ta'kidlash mumkinki, qattiq mahkamlangan holda ($\alpha$ yetarlicha katta), ya'ni uchlarning katta bo'lmagan ko'chishlarda katta taranglik vujudga keladi, u holda (9) chegaraviy shart

$$u(l,t) = \theta(t), \quad 0 \le t \le T,$$

birinchi tur chegaraviy shartga o'tadi.

Elastik mahkamlangan holda ($\alpha$ kichik), ya'ni uchlarning katta ko'chishlarda kichik taranglik vujudga keladi.

Bu holda (9) chegaraviy shart $u_x(l,t) = \theta(t)$, $0 \le t \le T$ ikkinchi tur chegaraviy shartga o'tadi (tor erkin uchining sharti).

Agar torning ikkala uchida ikkinchi yoki uchinchi chegaraviy shartlar olinsa, u holda bunday masalalar giperbolik tipdagi tenglama uchun *ikkinchi yoki uchinchi chegaraviy masala* deb yuritiladi.

Agar torning $x = 0$ va $x = l$ uchlarida turli tipdagi chegaraviy shartlar bilan birga $t = 0$ boshlang'ich shart berilsa, bunday masalalar *aralash masalalar* deyiladi.

## 3–§. Issiqliq tarqalish tenglamasini keltirib chiqarish. Asosiy boshlang'ich-chegaraviy masalalarning qo'yilishi

Qattiq jism $(x,y,z)$ nuqtasining $t$ vaqtdagi harorati $u = u(x,y,z,t)$ bo'lsin. Agar qattiq jismning turli qismlarining harorati turlicha bo'lsa, u holda qaralayotgan qattiq jismning ko'proq isigan qismidan kamroq isigan qismi tomon issiqlik harakati sodir bo'ladi. Issiqlik tarqalish tenglamasini keltirib chiqarish Fur'e qonuniga asoslanadi. Bunga ko'ra $\Delta S$ sirtdan $\Delta t$ vaqtda o'tuvchi $\Delta Q$ issiqlik miqdori quyidagi formula bilan aniqlanadi:

$$\Delta Q = -k\frac{\partial u}{\partial N}\Delta S \Delta t, \tag{1}$$

bu yerda $k$ – issiqlik o'tkazuvchanlik koeffitsiyenti, $\dfrac{\partial u}{\partial N}$ esa $\Delta S$ sirtga o'tkazilgan $N$ normal bo'yicha olingan hosila, u quyidagi formula bilan aniqlanadi:

$$\frac{\partial u}{\partial N} = \frac{\partial u}{\partial x}\cos(N, x) + \frac{\partial u}{\partial y}\cos(N, y) + \frac{\partial u}{\partial z}\cos(N, z) = (grad\, u, N),$$

ya'ni normal bo'yicha olingan hosila ikkita

$$N = i\cos\alpha + j\cos\beta + k\cos\gamma$$

$$grad\, u = \nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

vektorlarning skalyar ko'paytmasiga teng.

Bu yerda $i, j, k$ – koordinata o'qlarining yo'naltiruvchi birlik vektorlari, $\alpha$, $\beta$, $\gamma$ esa $N$ normal bilan mos ravishda $Ox$, $Oy$, $Oz$ o'qlar orasidagi burchak.

Yuqorida keltirilgan (1) formuladagi minus ishora issiqlikning jismning ko'proq isigan nuqtasidan kamroq isigan qismiga issiqlik harakatini bildiradi.

Endi faraz qilaylik, qaralayotgan jism izotrop jism bo'lsin, ya'ni jismning issiqlik o'tkazuvchanlik koeffitsienti $k$ faqat $(x, y, z)$ nuqtaga bog'liq, $u$ ga va $\dfrac{\partial u}{\partial N}$ ga bog'liq emas.

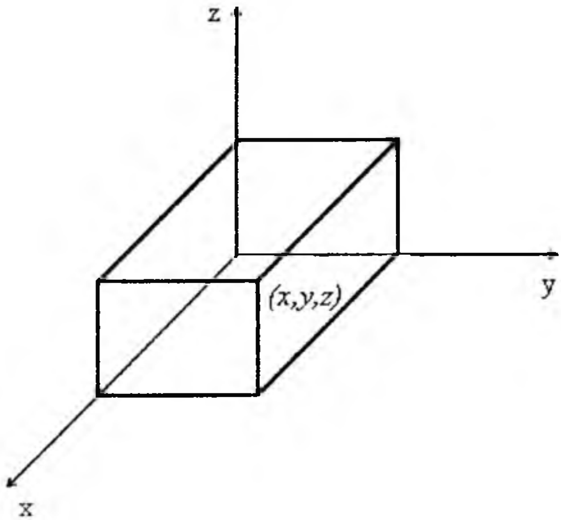
Agar qattiq jism anizotrop bo'lsa, u holda

$$k = k\left(x, y, z, N, u, \frac{\partial u}{\partial N}\right)$$

bo'ladi.

Issiqlik tarqalish tenglamasini keltirib chiqarish uchun qattiq jismdan $(z, y, z)$ nuqtani o'z ichiga olgan yetarlicha kichik ixityoriy $V$ parallelepiped ajratib olamiz, ya'ni

$$V = \{(x, y, z) : x < \xi < x + \Delta x,\ y < \eta < y + \Delta y,\ z < \zeta < z + \Delta z\}.$$




3 — shakl.

Endi $V$ parallelepiped uchun issiqlik balansni tuzaylik. Parallelepipedning $\xi = x$ yuzasi orqali $\Delta t$ vaqtda o‘tgan issiqlik miqdori (1) formulaga ko‘ra

$$Q_x = -k(x,y,z)\frac{\partial u(x,y,z,t)}{\partial x}\Delta y \Delta z \Delta t.$$

Parallelepipedning $\xi = x + \Delta x$ yuzasidan o‘tayotgan issiqlik miqdori esa

$$Q_{x+\Delta x} = -k(x+\Delta x,y,z)\frac{\partial u(x+\Delta x,y,z,t)}{\partial x}\Delta y \Delta z \Delta t$$

ga teng. U holda $V$ hajmda $Ox$ o‘qi bo‘yicha qolgan issiqlik miqdori

$$\Delta Q_x = Q_x - Q_{x+\Delta x} = \Delta y \Delta z \Delta t \times$$

$$\times\left(k(x+\Delta x,y,z)\frac{\partial u(x+\Delta x,y,z,t)}{\partial x} - k(x,y,z)\frac{\partial u(x,y,z,t)}{\partial x}\right) =$$

$$= \frac{\partial}{\partial x}\left(k(x',y,z)\frac{\partial u(x',y,z,t)}{\partial x}\right)\Delta x \Delta y \Delta z \Delta t, \quad x' \in (x, x+\Delta x).$$

bo'ladi.

Xuddi shu kabi $V$ parallelepipedning qolgan yoqlari bo'yicha issiqlik miqdori

$$\Delta Q_y = \frac{\partial}{\partial y}\left(k(x,y',z)\frac{\partial u(x,y',z,t)}{\partial y}\right)\Delta y\Delta x\Delta z\Delta t, \quad y' \in (y, y+\Delta y);$$

$$\Delta Q_z = \frac{\partial}{\partial z}\left(k(x,y,z')\frac{\partial u(x,y,z',t)}{\partial z}\right)\Delta z\Delta y\Delta x\Delta t, \quad z' \in (z, z+\Delta z).$$

ga teng bo'ladi.

U holda $V$ hajmda $\Delta t$ vaqtda oqayotgan umumiy issiqlik miqdori

$$Q_1 = \Delta Q_x + \Delta Q_y + \Delta Q_z =$$

$$= \left\{\frac{\partial}{\partial x}\left(k(x',y,z)\frac{\partial u('x,y,z,t)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y',z)\frac{\partial u(x,y',z,t)}{\partial y}\right) + \right.$$

$$\left. + \frac{\partial}{\partial z}\left(k(x,y,z')\frac{\partial u(x,y,z',t)}{\partial z}\right)\right\}\Delta x\Delta y\Delta z\Delta t, \qquad (2)$$

formula bilan aniqlanadi.

Faraz qilaylik, qaralayotgan $V$ parallelepipedning ichida issiqlik manbalari bo'lsin. Parallelepipeddagi issiqlik manbalarining zichligi $F(x,y,z,t)$ bo'lsin, ya'ni $F(x,y,z,t)$ funksiya $\Delta t$ vaqt ichida $\Delta V$ hajmdan ajralib chiqqan yoki unga singib ketgan issiqlik miqdori bo'lsin. U holda tashqi manbalar ta'sirida $V$ hajmdan $\Delta t$ vaqt oralig'ida ajralib chiqqan issiqlik miqdori

$$Q_2 = F(x,y,z,t)\Delta x\Delta y\Delta z\Delta t, \qquad (3)$$

bo'ladi.

Qaralayotgan qattiq jismning $\Delta t$ vaqtdagi haroratini o'lchash uchun $\Delta_t u$ sarflangan issiqlik miqdori

$$Q_3 = \Delta_t u\, c(x,y,z)\rho(x,y,z)\Delta x\Delta y\Delta z =$$

$$= [u(x,y,z,t+\Delta t) - u(x,y,z,t)]c(x,y,z)\rho(x,y,z)\Delta x\Delta y\Delta z.$$

Bu yerda $\rho(x,y,z,)$ qattiq jismning zichligi, $c(x,y,z)$ esa uning solishtirma issiqlik sig'imi bo'lib, ularni argumentlarining uzluksiz funksiyasi deb hisoblaymiz. Lagranj teoremasiga asosan sarf qilingan issiqlik miqdori uchun quyidagi

$$Q_3 = \frac{\partial u(x,y,z,t')}{\partial t}\Delta t\, c(x,y,z)\rho(x,y,z)\Delta x\Delta y\Delta z, \qquad (4)$$

ifodani olamiz. Bu yerda $t' \in (t, t+\Delta t)$.

Endi $V$ hajm uchun issiqlik balansi tenglamasini tuzamiz. Ma'lumki, $Q_3 = Q_1 + Q_2$, u holda (2)–(4) ifodalardan

$$\frac{\partial u(x,y,z,t')}{\partial t} c(x,y,z)\rho(x,y,z)\Delta x\Delta y\Delta z\Delta t =$$

$$= \left\{\frac{\partial}{\partial x}\left(k(x',y,z)\frac{\partial u(x',y,z,t)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y',z)\frac{\partial u(x,y',z,t)}{\partial y}\right) +\right.$$

$$\left. + \frac{\partial}{\partial z}\left(k(x,y,z')\frac{\partial u(x,y,z',t)}{\partial z}\right)\right\}\Delta x\Delta y\Delta z\Delta t +$$

$$+ F(x,y,z,t)\Delta x\Delta y\Delta z\Delta t.$$

kelib chiqadi. Oxirgi ifodani $\Delta x\Delta y\Delta z\Delta t \neq 0$ ga qisqartirib, so'ngra $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ va $\Delta t \to 0$ limitga o'tsak, ushbu

$$c(x,y,z)\rho(x,y,z)\frac{\partial u(x,y,z,t')}{\partial t} = \frac{\partial}{\partial x}\left(k(x,y,z)\frac{\partial u(x,y,z,t)}{\partial x}\right) +$$

$$+ \frac{\partial}{\partial y}\left(k(x.y,z)\frac{\partial u(x,y,z,t)}{\partial y}\right) + \frac{\partial}{\partial z}\left(k(x,y,z)\frac{\partial u(x,y,z,t)}{\partial z}\right) +$$

$$+ F(x,y,z,t) = div(k\, grad\, u) + F(x,y,z,t), \qquad (5)$$

tenglama hosil bo'ladi. Bunda vektor funksiyaning divergensiyasi quyidagicha tushuniladi:

Agar $a(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ bo'lsa, u holda

$$div\, a(x.y.z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial x}$$

bo‘ladi.

Oxirgi (5) tenglama bir jinsli bo‘lmagan izotrop qattiq jismning *issiqlik o‘tkazuvchanlik tenglamasi* deyiladi.

Agar qattiq jism bir jinsli, ya’ni

$$c(x,y,z) = const, \quad \rho(x,y,z) = const, \quad k(x,y,z) = const,$$

bo‘lsa, u holda

$$div(k\, grad\, u) = k\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right] =$$

$$= k(u_{xx} + u_{yy} + u_{zz}) = k\,\Delta u.$$

bo‘ladi. Demak, (5) tenglama quyidagi

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) + f(x,y,z,t)$$

ko‘rinishga keladi, bu yerda $a^2 = \dfrac{k}{c\rho}$, $f(x,y,z,t) = \dfrac{F(x,y,z,t)}{c\rho}$.

Agar qaralayotgan bir jinsli qattiq jismda tashqi issiqlik manbalari bo‘lmasa, ya’ni $F(x,y,z,t) \equiv 0$ bo‘lsa, u holda (5) tenglamadan ushbu

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz})$$

bir jinsli issiqlik tarqalish tenglamasini olamiz.

Agar $u$ harorat faqat $x, y, t$ koordinatalarga bog‘liq bo‘lsa, u holda bir jinsli yupqa plastinkada issiqlik tarqalish tenglamasiga ega bo‘lamiz. (5) tenglama quyidagi

$$u_t = a^2(u_{xx} + u_{yy}) + f(x,y,t). \qquad u = u(x,y,t)$$

ko‘rinishga keladi.

O‘lchamlari chiziqli bo‘lgan jismlar uchun, masalan sterjenda issiqlik tarqalish tenglamasi

$$u_t = a^2 u_{xx} + f(x,t), \qquad u = u(x,t)$$

ko‘rinishda bo‘ladi.

## Asosiy boshlang'ich-chegaraviy masalalarning qo'yilishi

Qattiq jismning ixtiyoriy vaqtdagi haroratini aniqlash uchun (5) xususiy hosilali differensial tenglamaning o'zi etarli bo'lmaydi. Buning uchun masalaning fizik xossasiga asosan jism ichida boshlang'ich vaqtdagi haroratning taqsimlanishi (boshlang'ich shart)ni va jismning sirtida issiqlik rejimi (chegaraviy shartlar)ni bilish zarur.

Chegaraviy shartlar qattiq jism sirtidagi haroratga qarab turlicha berilishi mumkin.

1) Agar qattiq jism sirtining har bir nuqtasida bir xil harorat saqlanayotgan bo'lsa, u holda chegaraviy shart

$$u(x,y,z,t)\,|_S = \mu_1(x,y,z,t), \qquad (x,y,z,t) \in S, \quad tg`eq0, \qquad (6)$$

ko'rinishda beriladi.

Bu yerda $S$ qattiq jismning sirti, $\mu_1(x,y,z,t)$ esa $S$ sirtda berilgan funksiya.

2) Qattiq jismning $S$ sirtida issiqlik oqimi berilgan bo'lsin, ya'ni $\Delta t$ vaqtda qattiq jismning $\Delta S$ sirti yuzasidan o'tuvchi issiqlik miqdori berilsa, u holda Fur'e qonuniga ko'ra (1) formulaga asosan quyidagi

$$q = \frac{Q}{\Delta S\,\Delta t} = -k\frac{\partial u}{\partial N}.$$

ifoda o'rinli bo'ladi. Bundan ushbu chegaraviy shart

$$\frac{\partial u}{\partial N} = \mu_2(x,y,z,t) = -\frac{q(x,y,z,t)}{k(x,y,z)}, \qquad (x,y,z) \in S, \quad tg`eq0, \qquad (7)$$

kelib chiqadi. $\mu_2(x,y,z,t)$ – berilgan funksiya.

3) Qattiq jism sirtida atrof muhit bilan issiqlik almashinishi sodir bo'layotgan bo'lsa, Nyuton qonuniga asosan $\Delta t$ vaqtda qattiq jismning $\Delta S$ sirtidan atrof muhitga chiqayotgan issiqlik miqdori qattiq jism sirtining haroratidan atrof muhit haroratining ayrimasiga proporsional bo'ladi, ya'ni

$$q = H(u - u_0),$$

bu yerda $H$ issiqlik almashish koeffitsiyenti bo'lib, u $u - u_0$ ayrimaga bog'liq. Energiyaning saqlanish qonuniga ko'ra, bu issiqlik miqdori Fur'e qonuni bilan aniqlangan issiqlik miqdoriga

$$q = -k\frac{\partial u}{\partial N}$$

teng bo'ladi.

U holda $S$ sirtda ushbu chegaraviy shartni

$$k\frac{\partial u}{\partial N} = H(u - u_0),$$

olamiz. yoki $h = H/k$ deb almashtirib, $S$ da quyidagi

$$\frac{\partial u}{\partial N} + hu = hu_0$$

chegaraviy shartga ega bo'lamiz.

Bundan izotrop qattiq jism uchun chegaraviy shartni quyidagi ko'rinishda

$$\left(\frac{\partial u}{\partial N} + h(x, y, z, t)u\right)\bigg|_S = \psi_3(x, y, z, t), \quad (x, y, z) \in S, \quad tg'eq0. \quad (8)$$

yozishimiz mumkin.

Shunday qilib, izotrop qattiq jismda issiqlik tarqalish tenglamasi uchun boshlang'ich–chegaraviy masalalar quyidagicha qo'yiladi:

BIRINCHI CHEGARAVIY MASALA. Issiqlik o'tkazuvchanlik tenglamasining ushbu

$$G = D \times (0, T) = \{(x, y, z, t) |\ (x, y, z) \in D \subset R^3,\ t \in (0, T)\}$$

silindrik sohada aniqlangan, uzluksiz quyidagi boshlang'ich

$$u(x, y, z, t)|_{t=0} = \varphi(x, y, z), \quad (x, y, z) \in D,$$

va

$$u(x, y, z, t)\ |_S = \mu_1(x, y, z, t), \quad (x, y, z, t) \in S, \quad tg'eq0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,y,z,t)$ yechimi topilsin.

IKKINCHI CHEGARAVIY MASALA. Issiqlik o'tkazuvchanlik tenglamasining $G = D \times (0,T)$ silindrik sohada aniqlangan, uzluksiz quyidagi boshlang'ich

$$u(x,y,z,t)|_{t=0} = \varphi(x,y,z), \qquad (x,y,z) \in D,$$

va

$$\left.\frac{\partial u}{\partial N}\right|_S = \mu_2(x,y,z,t) = -\frac{q(x,y,z,t)}{k(x,y,z)}, \qquad (x,y,z) \in S, \quad tg'eq0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,y,z,t)$ yechimi topilsin.

UCHINCHI CHEGARAVIY MASALA. Issiqlik o'tkazuvchanlik tenglamasining

$$G = D \times (0,T) = \{(x,y,z,t)|\ (x,y,z) \in D \subset R^3,\ t \in (0,T)\}$$

silindrik sohada aniqlangan, uzluksiz quyidagi boshlang'ich

$$u(x,y,z,t)|_{t=0} = \varphi(x,y,z), \qquad (x,y,z) \in D,$$

va

$$\left.\left(\frac{\partial u}{\partial N} + h(x,y,z,t)u\right)\right|_S = \psi_3(x,y,z,t), \qquad (x,y,z) \in S, \quad tg'eq0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,y,z,t)$ yechimi topilsin.

Yuqorida keltirilgan masalalar issiqlik o'tkazuvchanlik tenglamasi. ya'ni parabolik tipdagi tenglamalar uchun *asosiy boshlang'ich chegaraviy masalalar* deyiladi.

## 4–§. Puasson va Laplas tenglamalariga keladigan masalalar. Asosiy chegaraviy masalalarning qo'yilishi

Faraz qilaylik, biror $S$ sirt bilan chegaralangan $D$ jism ichida bir jinsli siqilmaydigan suyuqlik ma'lum $v(x, y, z)$ tezlik bilan statsionar harakatda bo'lsin. Agar suyuqlik bir jinsli siqilmaydigan suyuqlik, ya'ni $\rho(x, y, z) = const$ bo'lsa, u holda $\frac{\partial \rho}{\partial t} = 0$, $grad\, \rho = 0$ bo'ladi.

Agar suyuqlikning harakati uyurmali harakat bo'lmasa, u holda $v(x, y, z)$ tezlikning vektor maydoni potensial maydon bo'ladi, ya'ni biror skalyar maydonning gradienti

$$v = grad\, \varphi(x, y, z), \tag{1}$$

bu yerda $\varphi(x, y, z)$ ***tezlik potensiali*** deyiladi. Agar $D$ jism ichida suyuqlikning harakatga keltiruvchi manba bo'lmasa, u holda

$$div\, v(x, y, z) = 0, \quad \forall (x, y, z) \in D, \tag{2}$$

bo'ladi.

Endi (1) formulani (2) ifodaga qo'ysak, quyidagi

$$div(grad\varphi) \equiv \Delta\varphi = 0, \quad \forall (x, y. z) \in D$$

Laplas tenglamasiga ega bo'lamiz.

Demak, bir jinsli siqilmaydigan suyuqlikning uyurmali bo'lmagan harakatining tezlik potensiali Laplas tenglamasini qanoatlantirar ekan.

$D$ hajmli elektr o'tkazuvchan muhitda zichligi $j(x, y, z)$ bo'lgan statsionar elektr tok bo'lsin. Agar $D$ muhit ichida tok manbai bo'lmasa, u holda

$$div\, j(x, y, z) = 0, \quad \forall (x, y, z) \in D, \tag{3}$$

bo'ladi. Om qonuniga asosan elektr maydoni $E$ tok zichligi orqali

$$E = \frac{j}{\lambda}$$

formula bilan aniqlanadi. Bu erda $\lambda$ muhitning elektr o'tkazuvchanligi. Qaralayotgan $D$ muhitda tok oqimi statsionar bo'lgani uchun undagi elektr maydoni potensial (uyurmasiz) maydon bo'ladi, ya'ni $D$ jismda $\varphi(x, y, z)$ skalyar maydon mavjud va u

$$E = -grad\,\varphi(x, y, z), \tag{4}$$

formula bilan aniqlanadi. Xuddi yuqoridagi kabi (3) va (4) formulalardan

$$\Delta\,\varphi(x, y, z) = 0,$$

kelib chiqadi.

Demak, qaralayotgan muhitda elektr manbai bo'lmasa, u holda statsionar tokning elektr maydoni potensiali Laplas tenglamasini qanoatlantirar ekan.

Agar massani hisobga olmaganda tortishish maydoni potensiali ham Laplas tenglamasini qanoatlantirishini ko'rish mumkin.

Oldingi paragrafda bir jinsli izotrop qattiq jismda ushbu

$$u_t = a^2(u_{xx} + u_{yy} + u_{zz}) + f(x, y, z, t), \tag{5}$$

issiqlik tarqalish tenglamasini keltirib chiqargan edik.

Faraz qilaylik, qattiq jismning har bir nuqtasida bir xil $u(x, y, z, t)$ harorat o'rnatilgan bo'lsin va bu harorat ixtiyoriy $t$ vaqtda o'zgarmas bo'lib qolsin.

U holda $u(x, y, z, t) = u(x, y, z)$ va $\dfrac{\partial u}{\partial t} = 0$ bo'ladi va (1) tenglama quyidagi

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = -\frac{f(x, y, z)}{a^2}, \tag{6}$$

ko'rinishga keladi. Bu (6) tenglama *Puasson tenglamasi* deyiladi.

Agar qattiq jism ichida tashqi issiqlik manbalari bo'lmasa, u holda (6) tenglamada $f(x, y, z) = 0$ bo'ladi va u ushbu

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = 0, \tag{7}$$

ko'rinishga keladi. Bu tenglama *Laplas tenglamasi* deb ataladi.

Shunday qilib, bir jinsli izotrop qattiq jismda issiqlikning statsionar holati (7) *Laplas tenglamasi* orqali ifodalanar ekan. Noma'lum $u(x,y,z)$ funksiyani aniqlash uchun boshlang'ich shartni emas, balki vaqtga bog'liq bo'lmagan holda biror chegaraviy shart berish kifoya.

## Asosiy chegaraviy masalalarning qo'yilishi

1. DIRIXLE MASALASI YOKI BIRINCHI CHEGARAVIY MASALA. $R^3$ fazoda $S$ bo'lakli silliq sirt bilan chegaralangan sohani $D$ deb belgilaylik. Xususiy hosilali (7) tenglamaning $D$ sohada aniqlangan va $S$ sirtda berilgan qiymati orqali $u(x,y,z)$ yechimini topish masalasi *Dirixle masalasi* deyiladi, ya'ni (7) tenglamaning $D\cup S$ sohada uzluksiz va quyidagi shartni qanoatlantiruvchi

$$u(x,y,z)\,|_S=\varphi_1(x,y,z),\quad (x,y,z)\in S,$$

$u(x,y,z)$ yechimini toping, bu yerda $\varphi_1(x,y,z)$ – berilgan funksiya.

2. NEYMAN MASALASI YOKI IKKINCHI CHEGARAVIY MASALA. (7) tenglamaning $D$ sohada aniqlangan, $D\cup S$ da o'zining birinchi tartibli hosilalari bilan uzluksiz va ushbu chegaraviy shartni qanoatlantiruvchi

$$\left.\frac{\partial u(x,y,z)}{\partial N}\right|_S=\varphi_2(x,y,z),\quad (x,y,z)\in S,$$

$u(x,y,z)$ yechimini toping, bu yerda $\varphi_2(x,y,z)$ – berilgan funksiya, $N$ esa $S$ sirtga o'tkazilgan normal.

3. PUANKARE MASALASI YOKI UCHINCHI CHEGARAVIY MASALA. (7) tenglamaning $D$ sohada aniqlangan, $D\cup S$ da o'zining birinchi tartibli hosilalari bilan uzluksiz va ushbu chegaraviy shartni qanoatlantiruvchi

$$\left.\left(\frac{\partial u(x,y,z)}{\partial N}+\alpha(x,y,z)u(x,y,z)\right)\right|_S=\varphi_3(x,y,z),\quad (x,y,z)\in S,$$

$u(x,y,z)$ yechimini toping, bu yerda $\alpha(x,y,z)$ va $\varphi_3(x,y,z)$ – berilgan funksiyalar, $N$ esa $S$ sirtga o'tkazilgan normal.

Agar yuqorida keltirilgan masalalarning $u(x, y, z)$ yechimi $S$ sirtga nisbatan $D$ sohaning ichida (yoki tashqarisida) qidirilayotgan bo'lsa, u holda bunday masalaga mos ravishda *ichki (yoki tashqi) masala* deyiladi.

Shuni ta'kidlash muhimki, (2) va (3) tenglamalar bilan bir jinsli qattiq jismda sodir bo'ladigan issiqlik jarayonlarigina emas, balki boshqa statsionar jarayonlar ham ifodalanadi. Bunga misol sifatida siqilmaydigan suyuqliklarning potensial oqimini keltirishimiz mumkin.

## 5–§. Korrekt qo'yilgan chegaraviy masala tushunchasi. Nokorrekt chegaraviy masalalarga misollar

Biz oldingi paragraflarda xususiy hosilali tenglamalar uchun qo'yilgan chegaraviy masalalar — bu berilgan differensial tenglamaning qaralayotgan sohada ma'lum bir qo'shimcha shartlarni qanoatlantiruvchi yechimini topishdan iborat ekanligini ko'rdik.

Qo'shimcha shartlar ko'pchilik hollarda chegaraviy shartlar bo'lishi mumkin, ya'ni noma'lum funksiyaning qiymati qaralayotgan jismning sirtida yoki boshlang'ich shartlar — fizik jarayonni o'rganishda uning boshlang'ich vaqtdagi holati berilishi mumkin. Xususiy hosilali differensial tenglamalar uchun qo'yilgan chegaraviy masalalarning yechimi o'rganilayotgan fizik jarayonning taqribiy matematik ifodasini beradi. Fizikaviy jarayonlarning matematik modellarini qurishda uning ayrim parametrlari abstraktlashtiriladi. Ko'pgina ko'rsatkichlarining jarayonga ta'siri sezilarsiz deb, muhim hisoblangan parametrlar ajratib olinadi va shu parametrlar asosida fizikaviy jarayonning matematik modeli xususiy hosilali differensial tenglamalar orqali ifodalanadi. Fizikaviy jarayonlarning matematik modellashtirilishidan olingan natijalar taqribiy natijalar hisoblanadi.

Shuning qilib, xususiy hosilali differensial tenglamalar uchun qo'yilgan boshlang'ich–chegaraviy masalalarning korrektligi tushunchasini kiritamiz.

Matematik fizika masalalari real fizik jarayonlarning matematik modelini ifodalagani uchun bu masalalar quyidagi shartlarni qanoatlantirishi zarur:

A) qaralayotgan masala ma'lum bir funksiyalar ($M_1$) sinfida yechimga ega (yechimning mavjudligi);

B) qaralayotgan masalaning yechimi bir funksiyalar ($M_2$) sinfida yagona (yechimning yagonaligi);

C) yechim boshlang'ich va chegaraviy shartlarga, tenglamaning koeffitsientlariga, ozod hadiga va boshqa berilganlarga uzluksiz bog'liq (yechimining turg'unligi).

Bu shartlar bir qarashda o'rinlidek ko'rinadi, lekin ularni fizikaviy jarayonning qurilgan matematik modeli asosida isbotlash kerak.

Qo'yilgan masalaning korrektligini isbotlash — bu matematik modelning birinchi aprobatsiyasidir, ya'ni

A) qurilgan model jarayonga zid emas (masalaning yechimi mavjud);

B) model fizik jarayonni bir qiymatli ifodalaydi (masalaning yechimi yagona);

C) fizik kattaliklarning hatoliklari qurilgan modelga sezilarsiz ta'sir qiladi (yechim masalaning berilganlariga uzluksiz bog'liq. ya'ni berilganlarning ozgina o'zgarishiga yechimning ham ozgina o'zgarishi mos keladi).

Yuqoridagi A) — C) shartlarni qanoatlantiruvchi boshlang'ich–chegaraviy masala Adamar ma'nosida *korrekt qo'yilgan masala* deb ataladi.

Bo'sh bo'lmagan $M = M_1 \cap M_2$ funksiyalar sinfi boshlangich–chegaraviy masalaning *korrektlik sinfi* deyiladi.

Agar boshlang'ich–chegaraviy masala A) — S) shartlardan birortasini qanoatlantirmasa, u holda bunday masala *nokorrekt qo'yilgan yoki noto'g'ri qo'yilgan masala* deyiladi.

## Nokorrekt chegaraviy masalalarga misollar

Endi nokorrekt qo'yilgan masalalarga misollar keltiramiz:

1–MASALA (ADAMAR MISOLI). $D = \{(x, y) : x \in R,\ y > 0\}$ sohada

$$u_{xx} + u_{yy} = 0, \tag{8}$$

Laplas tenglamasining

$$u(x,0) = \tau(x), \qquad u_y(x,0) = \nu(x), \quad -\infty < x < +\infty, \tag{9}$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ yechimi topilsin.

Bu yerda $\tau(x)$, $\nu(x)$ – berilgan cheksiz differensiallanuvchi funksiyalar.

Matematik fizikada (8)–(9) masala *Laplas tenglamasi uchun Koshi masalasi* deyiladi.

YECHISH. Ushbu

$$u(x,y) = \sum_{k=0}^{\infty}(-1)^k\left[\frac{y^{2k}}{(2k)!}\Delta^k\tau(x) + \frac{y^{2k+1}}{(2k+1)!}\Delta^k\nu(x)\right] \tag{10}$$

ifoda (8) tenglamani va (9) boshlang'ich shartlarni qanoatlantirishini bevosita tekshirib, ishonch hosil qilish mumkin.

Laplas tenglamasi uchun Koshi masalasi yechimining yagonaligi (10) ifodadan kelib chiqadi, ya'ni $\tau(x) \equiv \nu(x) \equiv 0$ bo'lsa, u holda $u(x,y) \equiv 0$ bo'ladi.

Endi (8)–(9) Koshi masalasida boshlang'ich shartlardan birini ozgina o'zgartiraylik, ya'ni (8) tenglamaning

$$u(x,0) = 0, \qquad u_y(x,0) = \frac{\sin nx}{n}$$

shartlarni qanoatlantiruvchi yechimini topish kerak bo'lsin.

Bu masalaning yechimini (10) formula yordamida quyidagi

$$u(x,y) = \frac{1}{n^2}\sin nx\,\mathrm{sh}ny$$

ko'rinishda topamiz. Bu yerda $\mathrm{sh}\,ny = (e^{ny} - e^{-ny})/2$ – giperbolik sinus.

Yetarlicha katta $n$ uchun boshlang'ich funksiya

$$\frac{1}{n}\sin nx \to 0$$

bo'ladi. Lekin masalaning yechimi $n \to \infty$ da

$$u(x,y) = \frac{1}{n^2}\mathrm{sh}ny\sin nx \to \infty, \quad x \neq j\pi; \quad j = 0, \pm1, \pm2, \ldots .$$

cheksizlikka intiladi.

Shunday qilib, (8)–(9) masalaning yechimi turg'un emas, ya'ni boshlang'ich shartlarning ozgina o'zgarishi yechimning yetarlicha katta o'zgarishiga olib keldi.

Demak, Laplas tenglamasi uchun Koshi masalasi *nokorrekt qo'yilgan masala* ekan.

2–MISOL. Tomonlarining nisbati irratsional bo'lgan to'g'ri to'rtburchakli ushbu $D = \{(x,t) : 0 < x < \pi,\ 0 < t < \theta\pi\}$ sohada

$$u_{tt} - u_{xx} = 0, \tag{11}$$

tor tebranish tenglamasining

$$u(x,0) = 0, \qquad u(0,t) = 0, \qquad u(\pi,t) = 0, \tag{12}$$

$$u(x,\theta\pi) = \frac{\sin nx}{\sqrt{n}},\ qquad0 \le x \le \pi, \qquad n > 0, \tag{13}$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimi topilsin.

Noma'lum funksiyaning qiymati qaralayotgan sohaning chegarasida berilgani uchun (11)–(13) masala *tor tebranish tenglamasi uchun Dirixle masalasi* deb yuritiladi.

YECHISH. Ma'lumki [12], qaralayotgan (11)–(13) masalaning yechimi

$$u(x,t) = \frac{1}{\sqrt{n}}\frac{\sin(nt)\sin(nx)}{\sin(\theta n\pi)}, \tag{14}$$

formula bilan aniqlanadi.

(13) chegaraviy shartdan $n \to \infty$ bo'lganda $\dfrac{\sin nx}{\sqrt{n}} \to 0$ bo'lishini ko'rsatish qiyin emas.

Sonlar nazariyasidan bizga ma'lumki, ixtiyoriy berilgan $\varepsilon_n$ ratsional son uchun shunday $p_n$ va $q_n$ butun sonlar ketma-ketligi mavjud bo'lib, har qanday $\theta$ irratsional son uchun quyidagi tengsizlik

$$|\theta - r_n| = \left|\theta - \frac{p_n}{q_n}\right| < \varepsilon_n = \frac{1}{q_n^2},$$

o'rinli bo'ladi. Unga asosan

$$|\sin(\theta\pi q_n)| = |\sin(\theta\pi q_n - \pi p_n)| =$$

$$= \left|\sin\left(\pi q_n\left(\theta - \frac{p_n}{q_n}\right)\right)\right| \leq \pi q_n\left|\theta - \frac{p_n}{q_n}\right| < \pi q_n\varepsilon_n = \frac{\pi}{q_n}, \tag{15}$$

tengsizlikni hosil qilamiz. Oxirgi (15) tengsizlikka ko'ra qaralayotgan (11)–(13) masalaning $u(x,t)$ yechimi uchun quyidagi

$$|u_{q_n}(x,t)| = \frac{1}{\sqrt{q_n}}\frac{|\sin(q_n t)||\sin(q_n x)|}{|\sin(\theta\pi q_n)|} > \frac{\sqrt{q_n}}{\pi}|\sin(q_n t)||\sin(q_n x)|$$

tengsizlikka ega bo'lamiz.

Bu tengsizlikdan $q_n \to \infty$ bo'lganda $u_{q_n}(x,t) \to \infty$ bo'lishi kelib chiqadi.

Demak, tor tebranish tenglamasi uchun qo'yilgan Dirixle masalasida yechimning turg'unligi buzilar ekan.

Bundan giperbolik tipdagi tenglamalar uchun qo'yilgan Dirixle masalasining nokorrekt ekanligi kelib chiqadi.

3–MISOL. Ushbu $D = \{(x,t) : 0 < x < \pi,\ t < 0\}$ sohada

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{16}$$

issiqlik o'tkazuvchanlik tenglamasining

$$u(x,t)\Big|_{t=0} = \frac{1}{k}\sin(kx), \quad u(0,t) = u(\pi,t) = 0, \tag{17}$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

YECHISH. Bu masalaning yechimi

$$u(x,t) = \frac{1}{k}e^{-(ak)^2 t}\sin(kx), \tag{18}$$

funksiyadan iborat.

Oxirgi fo rmuladan ko'rinadiki, $k$ ning noldan farqli har qanday qiymati uchun bu funksiya o'zining boshlang'ich sharti va unga mos bo'lgan yechimi mavjud. Shu sababli (18) formulani (16)–(17) masalaning yechimlari ketma–ketligi deb qarash mumkin.

Boshlang'ich $e^{-\sqrt{k}}\sin(kx)$ funksiya $k \to \infty$ bo'lganda nolga intiladi. Lekin (18) formula bilan aniqlangan $u(x,t)$ yechim Adamar misolidagi kabi yetarlicha katta $k$ uchun $u(x,t) \to \infty$ bo'ladi.

4–MISOL. $D = \{(x,t) : -\infty < x < +\infty,\ t > 0\}$ sohada (16) tenglamaning

$$u(x,t)\Big|_{t=0} = \varphi_0(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \varphi_1(x), \quad -\infty < x < +\infty, \tag{19}$$

boshlang‘ich shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

Bu yerda $\varphi_0(x)$ va $\varphi_1(x)$ – berilgan etarlicha silliq funksiyalar.

YECHISH. Faraz qilaylik, (16) tenglamaning (19) boshlang‘ich shartlarni qanoatlantiruvchi yechimi mavjud bo‘lsin. U holda berilgan $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalar uchun quyidagi

$$\varphi_0(x) = a^2\varphi_1''(x), \quad \text{pri} \quad t = 0, \quad -\infty < x < +\infty, \tag{20}$$

shart o‘rinli. Demak, (16) tenglamaning (19) shartni qanoatlantiruvchi yechimi faqat (20) tenglik bajarilganda mavjud bo‘lishi mumkin. Berilgan $\varphi_0(x)$, $\varphi_1(x)$ funksiyalar turlicha va ular hech qanday bir–biri bilan bog‘langan emas. Shuning uchun issiqlik o‘tkazuvchanlik tenglamasining (19) shartlarni qanoatlantiruvchi yechimini topish masaslasi nokorrekt qo‘yilgan masala hisoblanadi.

Biz keyinchalik (16) tenglamaning

$$u(x,t)\Big|_{t=0} = \varphi_0(x), \qquad -\infty < x < +\infty,$$

boshlang‘ich shartni qanoatlantiruvchi yechimini topish masalasining korrekt ekanligini ko‘rsatamiz.

Nokorrekt qo‘yilgan masalalar bevosita tadqiq etish qiyin bo‘lgan ob’yektlarni o‘rganishda, masalan, Yer osti qidiruv ishlarida, georazvedka masalalarida, tomografiya va shu kabi jarayonlarni tadqiq etishda uchraydi.

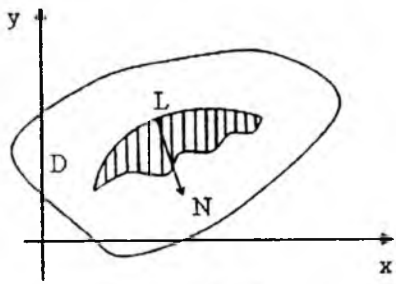
## Koshi masalasi. Koshi–Kovalevskaya teoremasi

Ikkinchi tartibli xususiy hosilalariga nisbatan chiziqli bo'lgan ushbu

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + f(x,y,u,u_x,u_y) = 0, \quad (1)$$

tenglamani biror $D$ sohada qaraylik.

Faraz qilaylik, $D$ sohada silliqlanuvchi, chekli uzunlikdagi va parametrik tenglamalari $x = x(s)$, $y = y(s)$, $0 \le s \le l$ bo'lgan $L$ egri chiziq berilgan va bu egri chiziq (1) tenglamaning xarakteristikasi bo'lmasin.




4 – shakl.

Bu yerda $s$ orqali $L$ egri chiziq yoyining, $l$ orqali esa $L$ egri chiziqning uzunligi belgilangan.

KOSHI MASALASI. Xususiy hosilali (1) differensial tenglamaning $L$ egri chiziq atrofida aniqlangan, uzluksiz va quyidagi

$$u(x,y)\Big|_L = u(x(s),\, y(s)) = \tau(s), \qquad \frac{\partial u}{\partial N}\Big|_L = \nu(s), \quad 0 \le s \le l \quad (2)$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping.

Bu yerda $\tau(s)$ va $\nu(s)$ berilgan etarlicha silliq funksiyalar, $N$ esa $L$ egri chiziqqa o'tkazilgan normal.

Xususiy hosilali differensial tenglamalar uchun qo'yilgan Koshi masalasi matematik fizikaning muhim masalalaridan biri hisoblanadi. Uni tadqiq etish ilmiy va amaliy ahamiyatga ega.

KOSHI–KOVALEVSKAYA TEOREMASI.

Agar xususiy hosilali (1) differensial tenglamaning koeffitsiyentlari va $f(\cdot)$, $x(s)$, $y(s)$, $\tau(s)$, $\nu(s)$ funksiyalar analitik bo‘lsa, u holda (1)–(2) Koshi masalasi $L$ egri chiziqning yetarlicha kichik atrofida yagona analitik yechimga ega bo‘ladi.

## Nazorat uchun savollar

1. Qanday tenglamalarga xususiy hosilali tenglamalar deyiladi?
2. Nima uchun xususiy hosilali tenglamalarni o‘rganish zarur?
3. Xususiy hosilali differensial tenglamaning tartibi nima?
4. Qanday xususiy hosilali differensial tenglamalar chiziqli, kvazichiziqli deyiladi?
5. Korrekt qo‘yilgan masala deb, qanday masalalarga aytiladi?
6. Qanday tenglamalar uchun Koshi masalasi korrekt qo‘yiladi?
7. Koshi masalasi qanday tenglamalar uchun nokorrekt qo‘yilgan bo‘ladi? Nima uchun?
8. Qanday fizikaviy masalalar giperbolik tipdagi tenglamalarga keltiriladi?
9. Parabolik tipdagi tenglamalarga qanday fizikaviy masalalar keltiriladi?
10. Qanday fizikaviy masalalar elliptik tipdagi tenglamalarga keltiriladi?

## Mustaqil yechish uchun misol va masalalar

1.1. Quyidagi tenglamalarni xususiy hosilali differensial tenglama bo‘lishi yoki bo‘lmasligini aniqlang.

1) $u_{xx}^2 + u_{xy}^2 - (u_{xx} - u_{xy})^2 = 0.$

2) $\cos(u_{xx} + u_{yy}) - \cos u_{xx} \cos u_{yy} + \sin u_{xx} \sin u_{yy} = 0.$

3) $\sin(u_x + u_{yy}) - \sin u_x \cos u_{yy} - \cos u_x \sin u_{yy} + 2u = 4.$

4) $\log u_x u_y - \log u_x - \log u_y + 5u_x - 9u = 0.$

5) $\dfrac{\partial}{\partial y}\mathrm{ctg} u_x - u_{xy}\mathrm{cosec}^2 u_x - 4u + 11 = 0.$

1.2. Tenglamalarning tartibini aniqlang.

1) $u_x u_{xy}^2 + (u_{xx}^2 - 2u_{xy}^2 + u_y^2) - 2u = 0.$

2) $\cos^2 u_{xy} - 2u_x u_{xx} + 3u_y + \sin^2 u_{xy} + u_x = 3.$

3) $\dfrac{\partial}{\partial y}(u_x - 2u)^2 + 2(u_x - 2u)u_{xy} - xy^2 = 0.$

4) $\dfrac{\partial}{\partial x}(u_{yy}^2 - u_y) - 2u_{yy}\dfrac{\partial}{\partial y}(u_{xy} - u_x) + 2u_x + 7 = 0.$

5) $2u_{xx}u_{xxy} - \dfrac{\partial}{\partial y}(u_{xx} - u_y)^2 - 2u_y u_{xxy} + u_x = 0.$

1.3. Quyidagi tenglamalardan qaysi biri chiziqli (bir jinsli yoki bir jinsli bo'lmagan), qaysi biri chiziqsiz(kvazichiziqli) ekanligini aniqlang.

1) $3u_{xx} - 6u_{xy} + 7u_y - u_x = 0.$

2) $u_x u_{xy}^2 + 2xuu_{xy} - 3xyu_y + u = 0.$

3) $u_y u_{xy} - 3x^2 u u_{xy} + 2u_x - xyu + 9x^2 y = 0.$

4) $u_{xy} + 2\dfrac{\partial}{\partial x}(u_x^2 + u) - 6x\sin y = 0.$

5) $\dfrac{\partial}{\partial y}(yu_y + u_x^2) - 2u_x u_{xy} + u_x - 6u + x^3\cos(x - y) = 0.$

1.4. Ushbu funksiya berilgan tenglamaning yechimi ekanligini isbotlang.

1) $u(x,y) = \cos y - (x - y)\sin y, \quad (x - y)u_{xy} - u_y = 0.$

2) $u(x,y) = \dfrac{x}{y}, \quad xu_{xy} = u_y.$

3) $u(x,y) = x\exp\left(\dfrac{y}{x}\right), \quad x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$

4) $u(x,y) = (x + y)^2 + \sin(3x + 2y), \quad 2u_{xx} - 5u_{xy} + 3u_{yy} = 0.$

======== ◇ ========

# II BOB
# IKKINCHI TARTIBLI XUSUSIY HOSILALI TENGLAMALARNING KLASSIFIKATSIYASI

Xususiy hosilali differensial tenglamalarni qaysi tipga tegishli bo'lishi uning yuqori tartibli hosilalari oldidagi koeffisiyentlari orqali aniqlanadi.

Ushbu bobda xususiy hosilali tenglamalarning klassifikatsiyasi hamda ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalarning kanonik ko'rinishga keltirish bayon qilingan. Kanonik tenglamani yangi noma'lum funksiya kiritish bilan yanada soddaroq ko'rinishga keltirish ko'rsatilgan.

## 6–§. Ikkinchi tartibli chiziqli xususiy hosilali differensial tenglamalarning tiplari

Quyidagi

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x)\frac{\partial u}{\partial x_i} + C(x)u = f(x) \qquad (1)$$

ikkinchi tartibli $n$ o'zgaruvchili chiziqli differensial tenglamani $R^n$ Evklid fazosidagi biror $D$ sohada qaraylik.

Bu yerda $x = (x_1, x_2, \ldots, x_n) \in D \subset R^n$, $ng'eq2$, tenglamaning koeffitsiyentlari $A_{ij}(x)$, $B_i(x)$, $C(x)$ va ozod hadi $f(x)$ yetarlicha silliq berilgan funksiyalar.

Agar $\forall x \in D$ uchun (1) tenglamada $A_{ij}(x) = 0$ bo'lsa, u holda (1) tenglama birinchi tartibli xususiy hosilali differensial tenglama bo'ladi. Shuning uchun qaralayotgan $D$ sohada tenglamaning $A_{ij}(x)$ koeffitsiyentlari bir vaqtda nolga teng bo'lmasin, deb talab qilamiz, hamda $A_{ij}(x) = A_{ji}(x)$ tenglik o'rinli bo'lsin.

Faraz qilaylik, $x_0 \in D$ ixtiyoriy nuqta bo'lsin. Chiziqli (1) tenglamaga mos ushbu xarakteristik forma

$$Q(\lambda_1, \dots, \lambda_n) = \sum_{i,j=1}^{n} A_{ij}(x_0)\lambda_i\lambda_j \tag{2}$$

*kvadratik forma* deb ataladi.

Algebra kursidan ma'lumki, $Q$ kvadratik formani $D$ sohaning har bir $x_0$ nuqtasida $\lambda_i = \lambda_i(\xi_1, \dots, \xi_n)$, $i = 1, 2, \dots, n$ xosmas almashtirishlar yordamida quyidagi

$$Q = \sum_{i=1}^{n} \alpha_i \xi_i^2 \tag{3}$$

kanonik ko'rinishga keltirish mumkin. Bu yerda $\alpha_i$ koeffitsiyentlar $-1$, 0 va 1 qiymatlarni qabul qiladi.

Qaralayotgan (1) chiziqli tenglamaning klassifikatsiyasi (3) formaning $\alpha_i$ koeffitsiyentlari qabul qiladigan qiymatlariga asoslanadi.

Agar barcha $\alpha_i = 1$ yoki $\alpha_i = -1$, $(i = \overline{1,n})$ bo'lsa, ya'ni (2) kvadratik forma musbat yoki manfiy aniqlangan bo'lsa, u holda (1) tenglama $x_0$ nuqtada *elliptik tipdagi tenglama* deyiladi.

Agar $\alpha_i$ koeffitsiyentlardan biri manfiy, qolganlari musbat (yoki aksincha) bo'lsa, u holda (1) tenglama $x_0$ nuqtada *giperbolik tipdagi tenglama* deyiladi.

Agar $\alpha_i$ koeffitsiyentlardan kamida bittasi nolga teng bo'lsa, u holda (1) tenglama $x_0$ nuqtada *parabolik tenglama* deyiladi.

Agar $\alpha_i$ koeffitsiyentlarning $l$ $(1 < l < n-1)$ tasi musbat, qolgan $n - l$ tasi manfiy bo'lsa, u holda (1) tenglama $x_0$ nuqtada *ultragiperbolik tenglama* deyiladi.

Agar $D$ sohaning har bir nuqtasida (3) kvadratik forma koeffitsiyentlarining barchasi noldan farqli va har xil ishorali, barchasi noldan farqli va bir xil ishorali hamda kamida bittasi (hammasi emas) nolga teng bo'lsa, u holda (1) tenglama $D$ sohada mos ravishda *giperbolik, elliptik* hamda *parabolik tipdagi tenglama* deyiladi.

Agar (1) tenglama qaralayotgan $D$ sohaning turli qismlarida har xil tipga tegishli bo'lsa, u holda (1) tenglama $D$ sohada *aralash tipdagi tenglama* deyiladi.

1 – MISOL. Butun $R^n$ fazoda aniqlangan $n$ o'lchovli

$$\Delta u(x) = \sum_{n=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = 0, \qquad (4)$$

Laplas tenglamasini qaraylik.

Laplas tenglamasiga mos quyidagi

$$Q(\lambda_1, \dots, \lambda_n) = \sum_{n=1}^{n} \lambda_i^2$$

kvadratik formani tuzamiz.

Agar $i = j$ bo'lsa, u holda $A_{ij}(x) = 1$ va $i \neq j$ bo'lganda $A_{ij}(x) = 0$ bo'ladi. Shuning uchun (3) kvadratik formaning koeffitsiyentlari $\alpha_i = 1, (i = \overline{1,n})$ qiymat qabul qiladi.

Demak, Laplas tenglamasi butun $R^n$ fazoda elliptik tipdagi tenglama bo'ladi.

2 – MISOL. $R^{n+1}$ fazoda aniqlangan $n$ o'lchovli

$$\Box u(x,t) \equiv u_{tt} - a^2 \sum_{n=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = u_{tt} - a^2 \Delta u = 0, \qquad (5)$$

to'lqin tarqalish tenglamasini qaraylik.

(5) tenglamaga mos kvadratik forma

$$Q(\lambda_1, \dots, \lambda_n, \lambda_{n+1}) = \sum_{n=1}^{n} (-a^2 \lambda_i^2) + \lambda_{n+1}^2 = \lambda_{n+1}^2 - \sum_{n=1}^{n} a^2 \lambda_i^2$$

bo'ladi.

Bu kvadratik forma $\xi_i = a\lambda_i$, $(i = \overline{1,n})$, $\xi_{i+1} = \lambda_{n+1}$ almashtirish yordamida quyidagi

$$Q = \xi_{n+1}^2 - \sum_{n=1}^{n} a^2 \xi_i^2$$

kanonik ko'rinishga keladi. Bunda $\alpha_i$ koeffitsiyentlardan biri musbat, qolganlari esa manfiy. Demak, (5) tenglama $R_{x,t}^{n+1}$ fazoda giperbolik tipdagi tenglama ekan.

3 – MISOL. $R^{n+1}_{x,t}$ fazoda aniqlangan $n$ o'lchovli

$$u_t - a^2 \sum_{n=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = u_t - a^2 \Delta u = 0, \tag{6}$$

issiqlik o'tkazuvchanlik tenglamasini qaraylik.

Bu tenglamaga mos kvadratik forma

$$Q(\lambda_1, \dots, \lambda_n, \lambda_{n+1}) = -a^2 \sum_{n=1}^{n} \lambda_i^2 + 0\, \lambda_{n+1}^2$$

ko'rinishda bo'ladi. Bunda $\alpha_i = -a^2 < 0$, $i = \overline{1,n}$ va $\alpha_{n+1} = 0$.

Shunday qilib, (6) issiqlik o'tkazuvchanlik tenglamasi $R^{n+1}_{x,t}$ fazoda parabolik tipdagi tenglama bo'lar ekan.

Kvadratik formaning musbat aniqlanganligi haqidagi Silvestr alomatiga asosan (2) kvadratik formani (3) kanonik ko'rinishga keltirmasdan qaralayotgan xususiy hosilali differensial tenglamaning tipini aniqlash mumkin.

Xususiy hosilali (1) differensial tenglama elliptik tipda bo'lishi uchun quyidagi

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} \tag{7}$$

simmetrik matritsaning diagonal minorlari musbat aniqlangan bo'lishi zarur va etarli.

Agar (1) tenglama ikki o'zgaruvchili $x_1 = x$, $x_2 = y$ bo'lsa, uni quyidagi

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = F(x,y,u,u_x,u_y), \tag{8}$$

ko'rinishda ifodalash mumkin.

Bu tenglamaga mos kvadratik forma

$$Q(\lambda_1, \lambda_2) = a(x,y)\lambda_1^2 + 2b(x,y)\lambda_1\lambda_2 + c(x,y)\lambda_2^2, \tag{9}$$

bo'ladi.

Agar $a(x,y) \neq 0$ bo'lsa, u holda (9) kvadratik formani ushbu

$$Q(\lambda_1, \lambda_2) = a\left(\lambda_1 + \frac{b}{a}\lambda_2\right)^2 - \frac{b^2 - ac}{a}\lambda_2^2, \qquad (10)$$

ko'rinishda ifodalash mumkin.

Agar $\delta(M_0) = b^2 - ac < 0$ bo'lsa, u holda (9) kvadratik forma $M_0 = (x_0, y_0)$ nuqtada musbat yoki manfiy aniqlangan bo'ladi. Chunki (10) ifoda quyidagi almashtirish

$$\xi_1 = \sqrt{|a|}\left(\lambda_1 + \frac{b}{a}\lambda_2\right), \qquad \xi_2 = \sqrt{\frac{ac - b^2}{|a|}}\,\lambda_2$$

yordamida ushbu kanonik ko'rinishga

$$Q = \begin{cases} \xi_1^2 + \xi_2^2, & a > 0, \\ -\xi_1^2 - \xi_2^2, & a < 0 \end{cases}$$

keladi. Bundan $\delta(M_0) = b^2 - ac < 0$ bo'lganda (8) tenglamaning $M_0 = (x_0, y_0)$ nuqtada elliptik tipda bo'lishi kelib chiqadi.

Faraz qilaylik, $M_0 = (x_0, y_0)$ nuqtada $\delta(M_0) = b^2 - ac > 0$ bo'lsin. U holda (10) kvadratik forma

$$\xi_1 = \sqrt{|a|}\left(\lambda_1 + \frac{b}{a}\lambda_2\right), \qquad \xi_2 = \sqrt{\frac{b^2 - ac}{|a|}}\,\lambda_2$$

almashtirishdan keyin

$$Q = \begin{cases} \xi_1^2 - \xi_2^2, & a > 0, \\ hfill - \xi_1^2 + \xi_2^2, & a < 0 \end{cases}$$

ko'rinishga keladi, ya'ni (3) kvadratik formaning koeffitsiyentlari $\alpha_1$ va $\alpha_2$ har xil ishorali. Bundan ko'rinadiki, qaralayotgan (8) tenglama $M_0 = (x_0, y_0)$ nuqtada giperbolik tipga tegishli ekan.

Agar $M_0 = (x_0, y_0)$ nuqtada $\delta(M_0) = b^2 - ac = 0$ bo'lsa, u holda

$$\xi_1 = \sqrt{|a|}\left(\lambda_1 + \frac{b}{a}\lambda_2\right), \qquad \xi_2 = \lambda_2$$

xosmas almashtirish (10) kvadratik formani quyidagi

$$Q = \begin{cases} \xi_1^2 + 0\xi_2^2, & a > 0, \\ -\xi_1^2 + 0\xi_2^2, & a < 0 \end{cases}$$

ko'rinishga keltiradi. Demak, (3) kvadratik formaning $\alpha_1$ va $\alpha_2$ koeffitsiyentlaridan biri nolga teng, ikkinchisi noldan farqli, ya'ni qaralayotgan (8) tenglama $M_0 = (x_0, y_0)$ nuqtada parabolik tipdagi tenglama ekan.

Yuqoridagi mulohazalardan qaralayotgan xususiy hosilali (8) differensial tenglamaning tipini aniqlash $\delta(M_0) = b^2 - ac$ diskriminantning ishorasiga bog'liq ekanligi ko'rinadi.

Agar biror $M_0 = (x_0, y_0)$ nuqtada $\delta(M_0) = b^2 - ac > 0$ bo'lsa, u holda (8) tenglama $M_0$ nuqtada *giperbolik tipdagi tenglama* deyiladi.

Agar biror $M_0 = (x_0, y_0)$ nuqtada $\delta(M_0) = b^2 - ac < 0$ bo'lsa, u holda (8) tenglama $M_0$ nuqtada *elliptik tipdagi tenglama* deyiladi.

Agar biror $M_0 = (x_0, y_0)$ nuqtada $\delta(M_0) = b^2 - ac = 0$ bo'lsa, u holda (8) tenglama shu nuqtada *parabolik tipdagi tenglama* deyiladi.

4 – MISOL. Quyidagi tenglamalarning tipini aniqlang.

$$a) \quad u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} + 3xyu = 0;$$

$$b) \quad u_{xx} + 2u_{xy} - 25u_{yz} + 2u_{yy} + 6u_{zz} + xu_z + x^2yu = 0;$$

$$c) \quad 4u_{xx} + 6u_{xy} + 2u_{yy} + 10u_{xz} + 4u_{yz} - 6u_{zz} = 0;$$

YECHISH. Berilgan tenglamalarning yuqori tartibli hosilalari oldidagi koeffitsiyentlari o'zgarmas. Shuning uchun bu tenglamalarning tipi butun fazoda aniqlanadi.

*a*) Berilgan tenglamaga mos xarakteristik forma

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 - 4\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 4\lambda_2^2 + \lambda_3^2 =$$

$$= (\lambda_1 - 2\lambda_2 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 - (\lambda_2 - \lambda_3)^2$$

ko'rinishda bo'ladi. Quyidagi

$$\lambda_1 = \xi_1 + \frac{1}{2}\xi_2 + \frac{3}{2}\xi_3; \quad \lambda_2 = \frac{1}{2}(\xi_2 + \xi_3); \quad \lambda_3 = \frac{1}{2}(\xi_2 - \xi_3)$$

xosmas almashtirish yordamida $Q(\lambda_1, \lambda_2, \lambda_3)$ forma

$$K(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 - \xi_3^2$$

kanonik ko'rinishga keladi.

$\alpha_i$ ($\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$) koeffitsiyentlar noldan farqli va har xil ishorali. Ta'rifga asosan bu tenglama giperbolik tipga tegishli.

*b*) Yuqoridagi kabi berilgan differensial tenglamaning xarakteristik formasini tuzamiz

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 + 2\lambda_2^2 + 6\lambda_3^2 \qquad (*)$$

va uni to'la kvadratlarga ajratib, kanonik ko'rinishga keltiramiz:

$$Q = \lambda_1^2 + 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2 + \lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2 + 4\lambda_3^2 =$$

$$= (\lambda_1 + \lambda_2 - \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 + 4\lambda_3^2.$$

Bundan quyidagi xosmas almashtirishlar natijasida

$$\xi_1 = \lambda_1 + \lambda_2 - \lambda_3, \qquad \xi_2 = \lambda_2 + \lambda_3, \qquad \xi_3 = 2\lambda_3.$$

$Q$ kvadratik forma ushbu

$$K(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 + \xi_3^2$$

kanonik ko'rinishga keladi.

Demak, (3) kvadratik formaning $\alpha_1$, $\alpha_2$ va $\alpha_3$ koeffitsiyentlari noldan farqli va bir xil ishorali. Shuning uchun berilgan tenglama $R^3$ fazoda elliptik tipda bo'ladi.

Endi kvadratik formaning musbat aniqlanganligi haqidagi Silvestr alomati yordamida ham berilgan tenglamaning tipini aniqlaylik.

Buning uchun berilgan tenglamaning (7) matritsaga o'xshash matritsasini tuzamiz va uning diagonal minorlarini hisoblaymiz.

$$\Delta_1 = A_{11} = 1 > 0, \qquad \Delta_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0,$$

$$\Delta_3 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{vmatrix} = 4 > 0.$$

Barcha $\Delta_i$, $i = 1, 2, 3$, determinantlar musbat bo'lgani uchun Silvestr teoremasiga asosan (*) kvadratik forma musbat aniqlangan bo'ladi. Bundan esa qaralayotgan differensial tenglamaning elliptik tipda ekanligi kelib chiqadi.

*c*) tenglamaga mos xarakteristik forma

$$Q(\lambda_1, \lambda_2, \lambda_3) = 4\lambda_1^2 + 6\lambda_1\lambda_2 + 2\lambda_2^2 + 10\lambda_1\lambda_3 + 4\lambda_2\lambda_3 - 6\lambda_3^2 =$$

$$= \frac{1}{4}(4\lambda_1 + 3\lambda_2 + 5\lambda_3)^2 - \frac{1}{4}(\lambda_2 + 7\lambda_3)^2$$

quyidagi

$$\lambda_1 = \frac{1}{2}\xi_1 - \frac{3}{2}\xi_2 + 4\xi_3; \quad \lambda_2 = 2\xi_2 - 7\xi_3; \quad \lambda_3 = \xi_3$$

xosmas almashtirish yordamida

$$K(\xi_1, \xi_2, \xi_3) = \xi_1^2 - \xi_2^2 + 0\,\xi_3^2$$

kanonik ko'rinishga keladi.

Bundan ko'rinadiki, $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 0$ ekan. Demak, berilgan tenglama *parabolik tipga tegishli* ekan.

## 7–§. Ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali tenglamalarni kanonik ko'rinishga keltirish

Biror $D \in R^2$ sohada ikki o'zgaruvchili ikkinchi tartibli ushbu

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = F(x, y, u, u_x, u_y), \qquad (1)$$

kvazichiziqli tenglamani qaraylik.

Bu yerda $a(x, y)$, $b(x, y)$, $c(x, y)$ – tenglamaning koeffitsiyentlar, $x$, $y$ o'zgaruvchilarning $D$ sohasida ikki marta uzluksiz differensiallanuvchi berilgan funksiyalari va ular $\forall(x, y) \in D$

sohada bir vaqtda nolga teng bo'lmasin, $F$ esa argumentlarining berilgan funksiyasi.

Qaralayotgan (1) tenglamani kanonik ko'rinishga keltirish uchun $x$, $y$ erkli o'zgaruvchilar o'rniga quyidagi tengliklar yordamida

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \tag{2}$$

yangi $\xi$, $\eta$ o'zgaruvchilar kiritamiz.

Bu yerda $\xi(x, y)$, $\eta(x, y)$ funksiyalar $D$ sohada ikki marta uzluksiz differensiallanuvchi va (2) almashtirishning yakobiani $D$ sohada noldan farqli, ya'ni

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0. \tag{3}$$

Agar bu tengsizlik bajarilsa, u holda (2) sistemani $\xi$, $\eta$ nuqtalarning biror sohasida $x$, $y$ o'zgaruvchilarga nisbatan bir qiymatli yechish mumkin. Topilgan $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ funksiyalar ham $\xi$, $\eta$ bo'yicha ikki marta uzluksiz differensiallanuvchi bo'ladi.

Endi $u(x, y)$ funksiyani $C^2(D)$ sinfdan deb hisoblaymiz va undan yangi $\xi$, $\eta$ o'zgaruvchilar bo'yicha xususiy hosilalarini hisoblaymiz. Murakkab funksiyalarni differensiallash haqidagi teoremaga asosan quyidagi

$$u_x = u_\xi\xi_x + u_\eta\eta_x; \quad u_y = u_\xi\xi_y + u_\eta\eta_y;$$

$$u_{xx} = (u_x)_x = (u_\xi\xi_x + u_\eta\eta_x)_x = (u_\xi\xi_x)_x + (u_\eta\eta_x)_x =$$

$$= (u_\xi)_x\xi_x + u_\xi\xi_{xx} + (u_\eta)_x\eta_x + u_\eta\eta_{xx} =$$

$$= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi\xi_{xx} + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta\eta_{xx} =$$

$$= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx};$$

$$u_{xy} = (u_x)_y = (u_\xi\xi_x + u_\eta\eta_x)_y = (u_\xi\xi_x)_y + (u_\eta\eta_x)_y =$$

$$= (u_\xi)_y\xi_x + u_\xi\xi_{xy} + (u_\eta)_y\eta_x + u_\eta\eta_{xy} =$$

$$= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy};$$

$$u_{yy} = (u_y)_y = (u_\xi \xi_y + u_\eta \eta_y)_y = (u_\xi \xi_y)_y + (u_\eta \eta_y)_y =$$
$$= (u_\xi)_y \xi_y + u_\xi \xi_{yy} + (u_\eta)_y \eta_y + u_\eta \eta_{yy} =$$
$$= (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy} + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy} =$$
$$= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.$$

tengliklarni olamiz.

Demak, $u(x, y)$ funksiyaning xususiy hosilalarini yangi $\xi$, $\eta$ o'zgaruvchilarning hosilalari bo'yicha quyidagi formulalar bilan ifodalandi:

$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x; \\ u_y = u_\xi \xi_y + u_\eta \eta_y; \\ u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}; \\ u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}; \\ u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}. \end{cases} \tag{4}$$

Bu tengliklarni (1) tenglamaga qo'yamiz va natijada (1) tenglama quyidagi ko'rinishga keladi:

$$a_1(\xi, \eta)u_{\xi\xi} + 2b_1(\xi, \eta)u_{\xi\eta} + c_1(\xi, \eta)u_{\eta\eta} = F_1(\xi, \eta, u, u_\xi, u_\eta), \tag{5}$$

bu yerda

$$\begin{cases} a_1(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\ b_1(\xi, \eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\ c_1(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2. \end{cases} \tag{6}$$

(6) tengliklarga asosan bevosita o'rniga qo'yib, ushbu tenglikning

$$b_1^2 - a_1c_1 = (\xi_x\eta_y - \xi_y\eta_x)^2(b^2 - ac)$$

bajarilishiga ishonch hosil qilishimiz mumkin.

Bundan esa $(x, y)$ o'zgaruvchilarni (2) va (3) xosmas almashtirishlar natijasida qaralayotgan (1) tenglama tipining o'zgarmasligi kelib chiqadi.

Qaralayotgan (1) tenglamada $\xi(x,y)$, $\eta(x,y)$ o'zgaruvchilarni qanday tanlaganimizda, u yanada soddaroq ko'rinishga keladi?

Bu savolga javob topish uchun $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ o'zgaruvchilarni $a_1(\xi,\eta)$, $b_1(\xi,\eta)$ va $c_1(\xi,\eta)$ koeffitsiyentlardan birini nolga aylantiradigan qilib tanlaymiz.

Buning uchun quyidagi lemmalar o'rinli.

1–LEMMA. Faraz qilaylik, $z = \varphi(x,y) \in C^1(D)$ va $D$ sohada $\varphi_y(x,y) \neq 0$ (yoki $\varphi_x(x,y) \neq 0$) bo'lsin. Agar $z = \varphi(x,y)$ funksiya ushbu

$$a(x,y)z_x^2 + 2b(x,y)z_x z_y + c(x,y)z_y^2 = 0, \tag{7}$$

birinchi tartibli xususiy hosilali chiziqsiz tenglamaning xususiy yechimi bo'lsa, u holda $\varphi(x,y) = const$ ifoda

$$a(x,y)(dy)^2 - 2b(x,y)dxdy + c(x,y)(dy)^2 = 0 \tag{8}$$

oddiy differensial tenglamaning umumiy integrali bo'ladi.

2–LEMMA. Agar $\varphi(x,y) = const$ ifoda (8) oddiy differensial tenglamaning umumiy integrali bo'lsa, u holda $z = \varphi(x,y)$ funksiya (7) chiziqsiz tenglamaning xususiy yechimi bo'ladi.

ISBOT. 1. Faraz qilaylik, $z = \varphi(x,y)$ funksiya $D$ sohada (7) tenglamaning xususiy yechimi bo'lsin. Agar $y$ o'zgaruvchi $\varphi(x,y) = const$ oshkormas funksiyadan aniqlansa, u holda $\varphi(x,y) = const$ tenglik (8) tenglamaning umumiy integrali bo'ladi.

Lemmaning shartga ko'ra $\varphi(x,y)$ funksiya $D$ sohada o'zining $\varphi_x(x,y)$ va $\varphi_y(x,y)$ xususiy hosilalari bilan birga uzluksiz va $\varphi_x(x,y) \neq 0$ (yoki $\varphi_y(x,y) \neq 0$) bo'ladi. Oshkormas funksiya haqidagi teoremadan $y$ o'zgaruvchini $\varphi(x,y) = const$ tenglikdan $y = f(x,c)$ ko'rinishda topish mumkin. Bundan

$$\frac{dy}{dx} = -\left[\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right]\Bigg|_{y=f(x,c)}$$

bo'ladi. Oxirgi tenglikni (8) tenglamaga qo'yib, ushbu

$$a(x,y)(dy)^2 - 2b(x,y)dxdy + c(x,y)(dy)^2 =$$

$$= \left[a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\left(\frac{dy}{dx}\right) + c(x,y)\right](dx)^2 =$$

$$= \left[a(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right)^2 - \right.$$

$$\left. -2b(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right) + c(x,y)\right]\Bigg|_{y=f(x,c)} (dx)^2 = 0$$

ifodaga ega bo'lamiz. Demak, barcha $(x,y) \in D$ uchun quyidagi

$$a(x,y)\varphi_x^2(x,y) + 2b(x,y)\varphi_x(x,y)\varphi_y(x,y) + c(x,y)\varphi_y^2(x,y) = 0,$$

yoki

$$a(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right)^2 - 2b(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right) + c(x,y) = 0$$

tenglik o'rinli.

**2.** Endi $\varphi(x,y) = const$ tenglik $D$ sohada (8) tenglamaning umumiy integrali bo'lsin. U holda $\forall\,(x,y) \in D$ uchun

$$a(x,y)\varphi_x^2(x,y) + 2b(x,y)\varphi_x(x,y)\varphi_y(x,y) + c(x,y)\varphi_y^2(x,y) = 0,$$

bo'lishini isbotlaymiz.

Faraz qilaylik, $(x',y')$ nuqta $D$ sohadan olingan ixtiyoriy nuqta bo'lsin. Bu nuqta orqali (8) tenglamaning biror $\varphi(x',y') = c'$ integral egri chizig'i o'tsin. U holda bu egri chiziqning tenglamasi $\varphi(x,y) = c'$ yoki $y = f(x,c')$ bo'ladi. Integral egri chiziqning barcha nuqtalari uchun $y' = f(x,c')$ bo'lganda ushbu

$$a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\left(\frac{dy}{dx}\right) + c(x,y) =$$

$$= \left[a(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right)^2 - 2b(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right) + c(x,y)\right] = 0$$

tenglik bajariladi. Bu tenglikda $x = x'$ almashtirsak,

$$a(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right)^2 - 2b(x,y)\left(-\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right) + c(x,y) = 0$$

yoki

$$a(x',y')\varphi_x^2(x',y')+2b(x',y')\varphi_x(x',y')\varphi_y(x',y')+c(x',y')\varphi_y^2(x',y')=0$$

tenglikka ega bo'lamiz.

Demak, $(x',y')$ nuqtada $z=\varphi(x,y)$ funksiya (7) tenglamani qanoatlantirishi isbotlandi. Bundan $(x',y')$ nuqtaning ixtiyoriy ekanligidan $z=\varphi(x,y)$ funksiya $D$ sohada (7) tenglamaning yechimi ekanligi kelib chiqadi.

(8) oddiy differensial tenglamaga (1) xususiy hosilali differensial tenglamaning *xarakteristik tenglamasi* deyiladi.

Xarakteristik tenglamaning umumiy yechimlari (1) xususiy hosilali differensial tenglamaning *xarakteristikalari* deb ataladi.

Ikkinchi tartibli xususiy hosilali differensial tenglamaning tipiga ko'ra uchta holni qaraymiz.

BIRINCHI HOL. Xususiy hosilali (1) differensial tenglamaning diskrminanti $D$ sohada $\delta=b^2-ac>0$ bo'lsin. U holda (1) tenglama $D$ sohada giperbolik tipdagi tenglama bo'ladi. Ixtiyoriy $(x_0,y_0)\in D$ nuqtada (1) tenglamani kanonik ko'rinishga keltiraylik. Bu nuqtada $a(x_0,y_0)\neq 0$ yoki $c(x_0,y_0)\neq 0$ bo'lsin, aks holda (1) tenglama kanonik ko'rinishdagi tenglama bo'ladi.

Faraz qilaylik, $a(x_0,y_0)\neq 0$ bo'lsin. $D$ sohada $\delta>0$ ekanligidan (8) xarakteristik tenglama quyidagi ikkita

$$\frac{dy}{dx}=\frac{b+\sqrt{b^2-ac}}{a}, \tag{9}$$

$$\frac{dy}{dx}=\frac{b-\sqrt{b^2-ac}}{a}. \tag{10}$$

birinchi tartibli oddiy differensial tenglamalarga ajraladi.

Bu tenglamalarning o'ng tomonlari ikki marta uzluksiz differensiallanuvchi va $(x_0,y_0)\in D$ nuqtaning ixtiyoriy atrofida $a(x_0,y_0)\neq 0$. Shuning uchun birinchi tartibli oddiy differensial tenglamalar uchun Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi teoremaga asosan bu tenglamalarning umumiy integrallari mavjud va ular haqiqiy va har xil

$$\varphi(x,y)=c_1=const, \qquad \psi(x,y)=c_2=const. \tag{11}$$

bo'ladi.

Demak, (11) umumiy integrallar haqiqiy va har xil bo'lgani uchun giperbolik tipdagi tenglamalar ikkita haqiqiy xarakteristikalar oilasiga ega bo'lar ekan.

Yangi o'zgaruvchilarni $\xi = \varphi(x,y)$, $\eta = \psi(x,y)$ deb olsak, yuqoridagi lemmaga ko'ra $a_1(\xi,\eta) = c_1(\xi,\eta) = 0$ va $b_1(\xi,\eta) \neq 0$ bo'ladi.

Bunda (5) tenglamani $2b_1(\xi,\eta)$ ga bo'lib, quyidagi

$$u_{\xi\eta} = Q(\xi,\eta,u,u_\xi,u_\eta), \quad Q = -\frac{F_1}{2b_1} \tag{12}$$

ko'rinishdagi tenglamani olamiz.

Bu *giperbolik tipdagi tenglamaning kanonik ko'rinishi* deyiladi.

Agar (12) tenglamada $\xi$, $\eta$ o'zgaruvchilardan yangi $\alpha$, $\beta$ o'zgaruvchilarga $\xi = \alpha + \beta$, $\eta = \alpha - \beta$ tengliklar yordamida o'tsak, u holda (12) tenglama

$$u_{\alpha\alpha} - u_{\beta\beta} = Q_1(\alpha,\beta,u,u_\alpha,u_\beta), \quad Q_1 = 4Q \tag{13}$$

ko'rinishga keladi.

Bu tenglama *giperbolik tipdagi tenglamaning ikkinchi kanonik ko'rinishi* deyiladi.

IKKINCHI HOL. Xususiy hosilali (1) differensial tenglamaning diskriminanti $D$ sohada $\delta = b^2 - ac = 0$ bo'lsin. U holda (1) tenglama $D$ sohada *parabolik tipdagi tenglama* deyiladi.

Farazimizga ko'ra, (1) tenglamaning koeffitsientlari bir vaqtda nolga aylanmaydi. U holda $\delta = b^2 - ac = 0$ shartga asosan $D$ sohaning har bir nuqtasida $a \neq 0$ va $c \neq 0$ bo'ladi. Umumiylikka ziyon qilmasdan $D$ sohaning biror $(x_0, y_0)$ nuqtasida $a \neq 0$ deb olaylik. (1) tenglamani shu $(x_0, y_0)$ nuqta atrofida kanonik ko'rinishga keltiramiz.

Bu holda (9) va (10) tenglamalar o'zaro ustma-ust tushadi va bitta $\varphi(x,y) = const$ haqiqiy umumiy integralga ega bo'ladi.

Yangi o'zgaruvchilarni $\xi = \varphi(x,y)$ bu yerda $\varphi(x,y) \in C^2(D)$ va bu funksiya lemmaga asosan (7) tenglamaning yechimi bo'ladi. Endi ikki marta uzluksiz differensiallanuvchi $\eta = \eta(x,y)$ funksiyani $D$ sohaning biror $(x_0, y_0)$ nuqtasida $J \neq 0$ bo'ladigan qilib tanlaymiz. U holda (5) tenglamada $a_1(\xi,\eta) = 0$ bo'ladi. (6) tengliklardan $b_1(\xi,\eta) = 0$ ekanligini ko'rsatish qiyin emas. Bundan $D$ sohaning biror $(x_0, y_0)$ nuqtasida $c_1(\xi,\eta) \neq 0$ ekanligi kelib chiqadi.

Agar $D$ sohaning biror $(x_0, y_0)$ nuqtasida $c_1(\xi,\eta) = 0$ bo'lsa, u holda (6) tengliklardan $b = \sqrt{ac}$ deb, $\forall a \geq 0, b \geq 0$ uchun quyidagi

$$\begin{cases} \sqrt{|a|}\xi_x + \sqrt{|c|}\xi_y = 0, \\ \sqrt{|a|}\eta_x + \sqrt{|c|}\eta_y = 0 \end{cases} \tag{14}$$

sistemani olamiz. $J \neq 0$ bo'lgani uchun (14) sistema faqat trivial $a = 0$, $c = 0$ yechimga ega. Bundan $b = 0$ ekanligi kelib chiqadi. Bu esa $|a| + |b| + |c| > 0$ shartga zid. Shuning uchun (5) tenglamani har ikki tomonini $c_1(\xi,\eta)$ ga bo'lsak, u quyidagi

$$u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta), \qquad Q = \frac{F_1}{c_1}, \tag{15}$$

kanonik ko'rinishga keladi.

Bu *parabolik tenglamaning kanonik ko'rinishi* deyiladi.

UCHINCHI HOL. Agar $D$ sohaning $(x_0, y_0)$ nuqtasida $b^2 - ac < 0$ bo'lsa, u holda (1) tenglama shu nuqtada *elliptik tipdagi tenglama* deyiladi.

Qaralayotgan (1) tenglamaning $a(x,y)$, $b(x,y)$ va $c(x,y)$ koeffisientlari $D$ sohaning biror $(x_0, y_0)$ nuqtasida analitik funksiyalar bo'lsin. U holda (9) va (10) tenglamalarning o'ng tomonlari analitik funksiyalar bo'ladi. Koshi–Kovalevskaya teoremasiga asosan (9) va (10) tenglamalar $D$ sohaning biror $(x_0, y_0)$ nuqtasi atrofida kompleks–qo'shma

$$\varphi(x,y) = \varphi_1(x,y) + i\varphi_2(x,y) = c_1,$$

$$\varphi^*(x,y) = \varphi_1(x,y) - i\varphi_2(x,y) = c_2$$

analitik yechimlarga ega. Yangi $\xi$, $\eta$ o'zgaruvchilarni ushbu

$$\xi = \frac{1}{2}\Big(\varphi(x,y) + \varphi^*(x,y)\Big) = \varphi_1(x,y),$$

$$\eta = \frac{1}{2}\Big(\varphi(x,y) - \varphi^*(x,y)\Big) = \varphi_2(x,y)$$

tengliklar yordamida kiritamiz.

Bu funksiyalar $D$ sohada

$$J = \begin{vmatrix} \varphi_{1x} & \varphi_{1y} \\ \varphi_{2x} & \varphi_{2y} \end{vmatrix} = \varphi_{1x}\varphi_{2y} - \varphi_{1y}\varphi_{2x} \neq 0$$

shartni qanoatlantiradi.

Haqiqatdan ham, $D$ sohaning biror $(x_0, y_0)$ nuqtasida atrofida

$$J = \varphi_{1x}\varphi_{2y} - \varphi_{1y}\varphi_{2x} = 0$$

bo'lsin. U holda

$$\frac{\partial\varphi_1}{\partial x} : \frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_2}{\partial x} : \frac{\partial\varphi_2}{\partial y} \quad \text{yoki} \quad \frac{\varphi_{1x}}{\varphi_{1y}} = \frac{\varphi_{2x}}{\varphi_{2y}} \tag{16}$$

tenglik o'rinli bo'ladi.

$\varphi(x, y)$ funksiya $D$ sohada analitik bo'lgani uchun bu funksiya

$$\frac{\partial\varphi_1}{\partial x} = \frac{\partial\varphi_2}{\partial y}; \qquad \frac{\partial\varphi_1}{\partial y} = -\frac{\partial\varphi_2}{\partial x}$$

Koshi–Riman shartini qanoatlantiradi.

Bundan

$$\frac{\varphi_{1x}}{\varphi_{1y}} = -\frac{\varphi_{2y}}{\varphi_{2x}}$$

kelib chiqadi va bu (16) tenglikka zid. Demak, $D$ sohaning biror $(x_0, y_0)$ nuqtasida $J \neq 0$ bo'lar ekan.

Endi $\varphi(x, y) = \xi(x, y) + i\eta(x, y)$ ifodani (7) tenglamaga qo'yib,

$$a(\xi_x + i\eta_x)^2 + 2b(\xi_x + i\eta_x)(\xi_y + i\eta_y) + c(\xi_y + i\eta_y)^2 = 0,$$

uning haqiqiy va mavhum qismlarini ajratamiz. Natijada

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_x + 2b\eta_x\eta_y + c\eta_y^2;$$

$$a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_x = 0$$

tengliklarga ega bo'lamiz.

Oxirgi tengliklardan $a_1(\xi,\eta) = c_1(\xi,\eta)$ va $b_1(\xi,\eta) = 0$ ekanligi kelib chiqadi.

(5) tenglamani har ikki tomonini $a_1(\xi,\eta)$ ga bo'lib, ushbu

$$u_{\xi\xi} + u_{\eta\eta} = Q(\xi,\eta,u,u_\xi,u_\eta), \quad Q = \frac{F_1}{a_1}, \tag{17}$$

kanonik ko'rinishdagi tenglamani olamiz. Bu tenglama *elliptik tenglamaning kanonik ko'rinishi* deyiladi.

Agar (1) tenglamada $a(x,y) = c(x,y)$ va $b(x,y) = 0$ bo'lsa, u holda berilgan tenglama (17) kanonik ko'rinishda bo'ladi.

Agar qaralayotgan sohaning barcha nuqtalarida

$$b^2 - ac > 0 \quad \text{yoki} \quad b^2 - ac = 0 \quad \text{yoki} \quad b^2 - ac < 0$$

bo'lsa, (1) tenglama shu sohada mos ravishda *giperbolik* yoki *parabolik* yoki *elliptik tipga tegishli* deyiladi.

Agar $D$ sohaning turli nuqtalarida $b^2 - ac$ ifodaning ishorasi turlicha bo'lsa, u holda (1) tenglama $D$ sohada *aralash tipdagi tenglama* deyiladi.

Endi (1) tenglamaning o'ng tomonidagi $F$ funksiya argumentlarining chiziqli funksiyasi bo'lib, uning koeffitsiyentlari o'zgarmas sonlar bo'lsin.

O'zgarmas koeffitsientli ikkinchi tartibli xususiy hosilali ushbu differensial

$$au_{xx} + 2bu_{xy} + cu_{yy} + a_1u_x + b_1u_y + c_1u = f(x,y), \tag{18}$$

tenglamani qaraylik.

Bu tenglamaning xarakteristikalari quyidagi

$$y = \frac{b + \sqrt{b^2 - ac}}{a}x + c_1, \quad y = \frac{b - \sqrt{b^2 - ac}}{a}x + c_2 \tag{19}$$

to'g'ri chiziqlardan iborat.

(19) tengliklardagi radikal ostidagi $b^2 - ac$ ifodaning ishorasiga qarab mos ravishda o'zgaruvchilarni almashtirish yordamida (18) tenglamani quyidagi kanonik ko'rinishlarga keltirish mumkin:

*a*) giperbolik tipdagi tenglama:

$$\begin{cases} u_{\xi\eta} + a_2 u_\xi + b_2 u_\eta + c_2 u = f_1(\xi,\eta), \\ u_{\xi\xi} - u_{\eta\eta} + a_2 u_\xi + b_2 u_\eta + c_2 u = f_1(\xi,\eta). \end{cases} \tag{20}$$

*b*) parabolik tipdagi tenglama:

$$\begin{cases} u_{\xi\xi} + a_2 u_\xi + b_2 u_\eta + c_2 u = f_1(\xi,\eta), \\ u_{\eta\eta} + a_2 u_\xi + b_2 u_\eta + c_2 u = f_1(\xi,\eta) \end{cases} \tag{21}$$

*c*) elliptik tipdagi tenglama

$$u_{\xi\xi} + u_{\eta\eta} + a_2 u_\xi + b_2 u_\eta + c_2 u = f_1(x,y). \tag{22}$$

Ushbu

$$u(\xi,\eta) = e^{\lambda\xi+\mu\eta} v(\xi,\eta)$$

formula bilan yangi $v(\xi,\eta)$ noma'lum funksiya kiritib, $\lambda$ va $\mu$ koeffitsientlarni tanlash hisobiga (20), (21) va (22) kanonik tenglamalarni yanada soddalashtirish mumkin.

5–MISOL. Quyidagi tenglamalarni kanonik ko'rinishga keltiring.

$$a) \quad u_{xx} - 2\cos x u_{xy} - (3 + \sin^2 x)u_{yy} - y u_y = 0;$$

$$b) \quad u_{xx} - 2u_{xy} + u_{yy} + \alpha u_x + \beta u_y + cu = 0;$$

$$c) \quad u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0.$$

YECHISH. *a*) Tenglamaning tipini aniqlaymiz:

$$\delta = b^2 - ac = \cos^2 x + 3 + \sin^2 x = 4 > 0.$$

Demak, tenglama giperbolik tipga tegishli ekan. U holda kanonik tenglamaning bosh hadi $u_{\xi\eta} = Q$ ko'rinishga ega bo'ladi.

Berilgan tenglamaning xarakteristik tenglamasini tuzamiz:

$$(dy)^2 + 2\cos x dx dy - (3 + \sin^2 x)(dx)^2 = 0$$

Bundan

$$dy = (-\cos x + 2)dx, \quad dy = (-\cos x - 2)dx$$

tenglamalarga ega bo'lamiz. Bu tenglamalarni integrallab,

$$y - \sin x - 2x = c_1, quad y + \sin x + 2x = c_2$$

umumiy yechimlarni (tenglamaning xarakteristikalarini) topamiz. Yangi

$$\xi = 2x + \sin x + y, \quad \eta = 2x - \sin x - y$$

xarakteristik o'zgaruvchilarga o'tib, berilgan tenglamada qatnashuvchi xususiy hosilalarni hisoblaymiz:

$$\begin{aligned}
&u_x = (2 + \cos x)u_\xi + (2 - \cos x)u_\eta, \quad u_y = u_\xi - u_\eta, \\
&u_{xx} = (2 + \cos x)^2 u_{\xi\xi} + 2(4 - \cos^2 x)u_{\xi\eta} + (2 - \cos x)^2 u_{\eta\eta} - \\
&\qquad - \sin x u_\xi + \sin x u_\eta, \\
&u_{xy} = (2 + \cos x)u_{\xi\xi} - 2\cos x u_{\xi\eta} - (2 - \cos x)u_{\eta\eta}, \\
&u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

Bularni tenglamaga qo'yib, soddalashtirish natijasida

$$u_{\xi\eta} + \frac{\eta - \xi}{16}(u_\xi - u_\eta) = 0$$

ko'rinishdagi kanonik tenglamaga kelamiz.

*b*) Tenglamaning tipini aniqlaymiz: $\delta = 0$. Demak, berilgan tenglama parabolik tipda ekan. Uning kanonik ko'rinishi, agar $\xi$ o'zgaruvchi ixtiyoriy tanlanganda

$$u_{\xi\xi} = Q(\xi, \eta, u, u_\xi, u_\eta)$$

yoki $\eta$ o'zgaruvchi ixtiyoriy tanlanganda

$$u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)$$

ko'rinishda bo'ladi.

Berilgan tenglamaning xarakteristik tenglamasi

$$(dy)^2 + 2dxdy + (dx)^2 = 0 \quad \text{ëki} \quad dy + dx = 0$$

bo'lib, u bitta ikki karrali $y + x = c$ yechimga ega.

Qaralayotgan tenglama bitta $y+x=c$ xarakteristikalar oilasiga ega.

Yangi o'zgaruvchilarni quyidagicha tanlaymiz:

$$\xi = x, \qquad \eta = x + y$$

bu yerda $\xi$ o'zgaruvchini ixtiyoriy tanlash mumkinligi uchun uni $x$ deb oldik, ravshanki $J \neq 0$ bo'ladi.

Hosilalarni hisoblaymiz:

$$u_x = u_\xi + u_\eta; \quad u_y = u_\eta;$$
$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_\eta;$$
$$u_{xy} = u_{\xi\eta} + u_{\eta\eta}; \quad u_{yy} = u_{\eta\eta};$$

Bu ifodalarni tenglamaga qo'yib, quyidagi

$$u_{\xi\xi} + \alpha u_\xi + (\alpha + \beta)u_\eta + cu = 0$$

kanonik tenglamani olamiz.

Agar yangi o'zgaruvchilarni $\xi = x + y$, $\eta = y$ deb tanlasak, u holda berilgan tenglamaning kanonik ko'rinishi

$$u_{\eta\eta} + (\alpha + \beta)u_\xi + \beta u_\eta + cu = 0$$

bo'ladi.

*c*) Berilgan tenglama elliptik tipga tegishli, chunki $\delta = -1$. Unga mos xarakteristik tenglama

$$(dy)^2 - 4dxdy + (dx)^2 = 0$$

ikkita kompleks-qo'shma

$$2x + ix - y = c_1, \quad 2x - ix - y = c_2$$

yechimlarga (xarakteristikalarga) ega.

Yangi xarakteristik o'zgaruvchilar sifatida

$$\xi = 2x - y, \quad \eta = x$$

funksiyalarni kiritamiz va $u(\xi, \eta)$ funksiyaning hosilalarini topamiz:

$$u_x = 2u_\xi + u_\eta, \qquad u_y = -u_\xi$$
$$u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$$
$$u_{xy} = -2u_{\xi\xi} - u_{\xi\eta}, \quad u_{yy} = u_{\xi\xi}$$

Topilgan ifodalarni tenglamaga qo'yib, ushbu

$$u_{\xi\xi} + u_{\eta\eta} + u_\eta = 0$$

kanonik tenglamaga ega bo'lamiz.

6–MISOL. Quyidagi tenglamani kanonik ko'rinishga keltiring va kanonik tenglamani soddalashtiring.

$$2u_{xx} + 2u_{xy} + u_{yy} + 4u_x + 4u_y + u = 0.$$

YECHISH. Tenglamaning tipini aniqlaymiz: $\delta = -1 < 0$ bo'lgani uchun tenglama elliptik tipga tegishli bo'ladi.

Xarakteristik tenglamasi

$$2(dy)^2 - 2dxdy + (dx)^2 = 0$$

bo'lib, u ikkita kompleks–qo'shma

$$2y - x + ix = c_1, \quad 2y - x - ix = c_2$$

yechimlarga ega.

Yangi $\xi$, $\eta$ o'zgaruvchilarni $\xi = 2y - x$, $\eta = x$ tengliklar yordamida kuritamiz. Berilgan tenglamada qatnashuvchi xususiy hosilalarni hisoblaymiz:

$$u_x = -u_\xi + u_\eta, qquadu_y = 2u_\xi,$$
$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta},$$
$$u_{xy} - 2u_{\xi\xi} + 2u_{\xi\eta}, u_{yy} = 4u_{\xi\xi}$$

Bularni tenglamaga qo'yib, kanonik tenglamani olamiz:

$$u_{\xi\xi} + u_{\eta\eta} + 2u_\xi + 2u_\eta + \frac{1}{2}u = 0.$$

Bu tenglamani soddalashtirish uchun quyidagi ko'rinishda yangi $v(\xi,\eta)$ noma'lum funksiya kiritamiz:

$$u(\xi,\eta) = e^{\lambda\xi+\mu\eta}v(\xi,\eta)$$

Xususiy hosilalarini hisoblaymiz:

$$\begin{aligned}
u_\xi &= \lambda e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\xi,\\
u_\eta &= \mu e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\eta,\\
u_{\xi\xi} &= \lambda^2 e^{\lambda\xi+\mu\eta}v + 2\lambda e^{\lambda\xi+\mu\eta}v_\xi + e^{\lambda\xi+\mu\eta}v_{\xi\xi},\\
u_{\eta\eta} &= \mu^2 e^{\lambda\xi+\mu\eta}v + 2\mu e^{\lambda\xi+\mu\eta}v_\eta + e^{\lambda\xi+\mu\eta}v_{\eta\eta}.
\end{aligned}$$

Bu ifodalarni kanonik tenglamaga qo'yib, uni soddalashtirsak, natijada ushbu

$$v_{\xi\xi} + v_{\eta\eta} + (2+2\lambda)v_\xi + (2+2\mu)v_\eta + (\lambda^2+\mu^2+2\lambda+2\mu+\frac{1}{2})v = 0$$

tenglamaga ega bo'lamiz. $\lambda$ va $\mu$ sonlarni $2+2\lambda = 0$, $2+2\mu = 0$ bo'ladigan qilib tanlaymiz. U holda $\lambda = -1$, $\mu = -1$ va

$$\lambda^2+\mu^2+2\lambda+2\mu+\frac{1}{2} = 1+1-2-2+\frac{1}{2} = -\frac{3}{2}$$

bo'lib, soddalashtirilgan tenglama

$$v_{\xi\xi} + v_{\eta\eta} - \frac{3}{2}v = 0$$

ko'rinishga ega bo'ladi.

7-MISOL. Ushbu tenglamani kanonik ko'rinishga keltiring.

$$u_{xy} + u_{xz} + u_{yz} - u_x + u_y = 0.$$

YECHISH. Berilgan tenglamaga mos xarakteristik (kvadratik) forma

$$Q(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 =$$

$$= \left(\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3\right)^2 - \left(\frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_1\right)^2 - \lambda_3^2$$

ko'rinishda bo'ladi.

$$\mu_1 = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \lambda_3, \quad \mu_2 = -\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2, \quad \mu_3 = \lambda_3$$

deb belgilaymiz va $Q$ forma

$$Q = \mu_1^2 - \mu_2^2 - \mu_3^2$$

kanonik ko'rinishga keladi. Demak, berilgan tenglama giperbolik tipga tegishli, chunki $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = -1$.

Shunday qilib, quyidagi xosmas almashtirish

$$\lambda_1 = \mu_1 - \mu_2 - \mu_3, \quad \lambda_2 = \mu_1 + \mu_2 - \mu_3, \quad \lambda_3 = \mu_3$$

$Q$ kvadratik formani kanonik ko'rinishga keltiradi.

Yuqoridagi xosmas almashtirish matritsasi

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

ga teng.

Tenglamani kanonik ko'rinishga keltiruvchi xosmas affin almashtirish matritsasi **M** matritsaga qo'shma

$$\mathbf{M}^* = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

bo'ladi. Demak, tenglamani kanonik ko'rinishga keltiruvchi xosmas affin almashtirish quyidagi

$$\xi = x + y, \quad \eta = -x + y, \quad \zeta = -x - y + z$$

ko'rinishda topiladi.

Endi $u(\xi, \eta, \zeta)$ funksiyaning xususiy hosilalarini hisoblaymiz:

$$u_x = u_\xi - u_\eta - u_\zeta, \quad u_y = u_\xi + u_\eta - u_\zeta,$$
$$u_{xy} = u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} + u_{\zeta\zeta},$$
$$u_{xz} = u_{\eta\zeta} - u_{\eta\zeta} - u_{\zeta\zeta},$$
$$u_{yz} = u_{\xi\zeta} + u_{\eta\zeta} - u_{\zeta\zeta}.$$

Topilgan ifodalarni tenglamaga qo'yib, soddalashtirsak, quyidagi

$$u_{\xi\xi} - u_{\eta\eta} - u_{\zeta\zeta} + 2u_\eta = 0$$

kanonik tenglamaga ega bo'lamiz.

8–MISOL. Quyidagi tenglamalarning

$$a)\quad u_{xx} + 3u_{xy} + 2u_{yy} + u_x + u_y = 0;$$

$$b)\quad u_{xy} - xu_x + u = 0.$$

umumiy echimlarini toping.

YECHISH. $a)$ Berilgan tenglama giperbolik tipga tegishli, chunki $\delta = b^2 - ac = 9 - 8 = 1 > 0$. Unga mos xarakteristik tenglama

$$(dy)^2 - 3dxdy + 2(dx)^2 = 0 \quad \text{yoki} \quad dy - dx = 0, \quad dy - 2dx = 0$$

bo'lib, ularni integrallasak,

$$y - x = c_1, \quad y - 2x = c_2$$

xarakteristikalar oilasiga ega bo'lamiz.

$$\xi = y - x, \quad \eta = y - 2x \tag{23}$$

tengliklar yordamida yangi o'zgaruvchilar kiritib, hosilalarni hisoblaymiz:

$$\begin{aligned}
u_{xx} &= u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta};\\
u_{xy} &= -u_{\xi\xi} - 3u_{\xi\eta} - 2u_{\eta\eta};\\
u_{yy} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta};\\
u_x &= -u_\xi - 2u_\eta; \quad u_y = u_\xi + u_\eta.
\end{aligned}$$

Bu ifodalarni berilgan tenglamaga qo'yib, ushbu

$$u_{\xi\eta} + u_\eta = 0 \tag{24}$$

kanonik tenglamaga ega bo'lamiz.

Oxirgi tenglamada

$$v(\xi,\eta) = \frac{\partial}{\partial\eta} u(\xi,\eta) \tag{25}$$

yangi noma'lum funksiya kiritib,

$$\frac{dv}{d\xi} + v = 0$$

chiziqli tenglamani olamiz.

Bu tenglamani integrallab,

$$v(\xi,\eta) = \varphi_0(\eta)e^{-\xi} \tag{26}$$

yechimni hosil qilamiz.

(26) ifodani (25) tenglikka qo'yib.

$$\frac{du}{d\eta} = \varphi_0(\eta)e^{-\xi}$$

tenglamaga ega bo'lamiz va uni integrallab, (24) tenglamaning umumiy yechimini hosil qilamiz:

$$u(\xi,\eta) = \varphi_1(\xi) + \varphi_2(\eta)e^{-\xi},$$

bu yerda $\varphi_1(\xi)$, $\varphi_2(\eta)$ – ixtiyoriy funksiyalar.

Oxirgi formulada (23) tengliklar yordamida eski $x$, $y$ o'zgaruvchilarga qaytib, berilgan tenglamaning

$$u(x,y) = \varphi_1(y-x) + e^{x-y}\varphi_2(y-2x)$$

umumiy yechimini topamiz.

Bunda $\varphi_1(y-x)$, $\varphi_2(y-2x)$ – ikki marta uzluksiz differensiallanuvchi ixtiyoriy funksiyalar.

*b*) Tenglamaning umumiy yechimini topish uchun

$$v(x,y) = u_x(x,y)$$

ko'rinishda yangi funksiya kiritamiz. U holda berilgan tenglama

$$v_y - xv + u = 0$$

ko'rinishga keladi. bu yerda $u(0, y) = -v_y(0, y)$ tenglik o'rinli.

Oxirgi tenglamani $x$ o'zgaruvchi bo'yicha differensiallaymiz va natijada $v(x, y)$ funksiyaga nisbatan quyidagi

$$v_{xy} - xv_x = 0 \qquad (27)$$

tenglamani olamiz.

Buning umumiy yechimi yuqoridagi kabi topiladi. (27) tenglamada $v_x = z(x, y)$ almashtirish bajaramiz.

U holda

$$z_y - xz = 0$$

tenglama hosil bo'ladi.

$x$ o'zgaruvchini fiksirlaymiz va hosil bo'lgan oddiy

$$\frac{dz}{z} = xdy$$

differensial tenglamani integrallaymiz. Natijada

$$z(x, y) = \varphi_0(x)e^{xy}$$

yoki $v(x, y)$ funksiyaga o'tib,

$$v_x = \varphi_0(x)e^{xy}$$

tenglamaga ega bo'lamiz.

Bundan

$$v(x, y) = \int_0^x \varphi_0(s)e^{sy}ds + \varphi_1(y)$$

hosil bo'ladi. $v = u_x$ bo'lgani uchun berilgan tenglamaning umumiy yechimi

$$u(x, y) = \int_0^x dt \int_0^t \varphi_0(s)e^{sy}ds + x\varphi_1(y) + \varphi_2(y) \qquad (28)$$

ko'rinishda topiladi.

Ikki o'lchovli integralda integrallash tartibini almashtirib, uni bir o'lchovli integralga keltiramiz:

$$u(x,y) = \int_0^x (x-s)\varphi_0(s)e^{sy}ds + x\varphi_1(y) + \varphi_2(y).$$

Bu erda $\varphi_0(x)$, $\varphi_1(y)$, $\varphi_2(y)$ — ixtiyoriy funksiyalar.

Shuni ta'kidlash muhimki, giperbolik tipdagi tenglamalarni kanonik ko'rinishga keltirish, bunday tenglamalarni integrallash usullaridan biri hisoblanadi.

## Nazorat uchun savollar

1. Ikkinchi tartibli xususiy hosilali differensial tenglamalarning tipi qanday aniqlanadi?
2. Nima uchun tenglamaning tipi nuqtada aniqlanadi?
3. Tenglamaning tipi nimalarga bog'liq bo'ladi?
4. Qanday tenglamalar giperbolik, parabolik va elliptik tipdagi tenglamalar deyiladi?
5. O'zgarmas koeffitsientli tenglamalarning tipi haqida nima deyish mumkin?
6. Qanday tenglamaga aralash tipdagi tenglama deyiladi?
7. Tenglamani kanonik ko'rinishga keltirishda yangi $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ o'zgaruvchilar qanday shartlarni qanoatlantiradi va nima uchun?
8. Xarakteristik tenglama nima va u qanday olinadi?
9. Yangi o'zgaruvchilar kiritilganda qaralayotgan giperbolik tenglamaning koeffitsiyentlari qanday ko'rinishga keladi?
10. Yangi o'zgaruvchilar kiritilganda qaralayotgan parabolik tenglamaning koeffitsiyentlari qanday ko'rinishga keladi?
11. Elliptik tenglamalarda yangi o'zgaruvchilar kiritilganda uning koeffitsiyentlari qanday ko'rinishda bo'ladi?

## Mustaqil yechish uchun misol va masalalar

2.1. Quyidagi tenglamalarning tipini aniqlang va kanonik ko'rinishga keltiring.

1) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0;$

2) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0;$

3) $u_{xx} - 2u_{xy} + u_{yy} + u_x + u = 0;$

4) $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x = 0;$

5) $u_{xx} - 6u_{xy} + 10u_{yy} + u_x - 3u_y = 0;$

6) $4u_{xx} + 4u_{xy} + u_{yy} - 2u_y = 0;$

7) $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

8) $(1 + x^2)^2 u_{xx} + u_{yy} + 2x(1 + x^2)u_x = 0.$

9) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} - 2y u_x + y e^{\frac{y}{x}} = 0.$

10) $u_{xx} - 4x u_{xy} + 8x^2 u_{yy} - \frac{1}{x} u_x + u_y = 0.$

2.2. Quyidagi tenglamalarni tipi o'zgarmaydigan sohada kanonik ko'rinishga keltiring.

1) $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} = 0.$

2) $3u_{xy} - 2u_{xz} - u_{yz} - u = 0.$

3) $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} + 3u_x = 0.$

4) $3u_{yy} - 2u_{xy} - 2u_{yz} + 4u = 0.$

5) $2u_{xx} + 5u_{yy} + 2u_{zz} - 6u_{xy} - 4u_{xz} + 6u_{yz} - 3u + y - 2z = 0.$

2.3. Berilgan tenglamalarni kanonik ko'rinishga keltiring va ularni soddalashtiring.

1) $a u_{xx} + 2a u_{xy} + a u_{yy} + b u_x + c u_y + u = 0, \quad a, b, c = const.$

2) $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0, \quad \alpha, \beta, \gamma = const.$

3) $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_x + u_y + u = 0.$

4) $u_{xy} + 2u_{yy} - u_x + 4u_y + u = 0.$

5) $u_{xy} + u_{xx} - u_y - 10u + 4x = 0.$

2.4. Quyida berilgan tenglamalarning umumiy yechimini toping.

1) $u_x - 2u_y = 0.$

2) $3u_x - 2u_y + u = 0.$

3) $2u_x - u_y + 2u = 0.$

4) $u_x + 2u_y = \sin(x + y).$

5) $3u_x - 4u_y + \sin(4x + 3y)u = 0.$

6) $u_{xy} + au_x = 0.$

7) $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

8) $3u_{xx} - 5u_{xy} - 2yu_{yy} + 3u_x + u_y = 2.$

9) $u_{xy} + au_x + bu_y + abu = 0.$

10) $\frac{\partial}{\partial x}\left(x^2 u_x\right) = x^2 \frac{\partial^2 u}{\partial y^2}.$

======== ◇ ========

# III BOB
# GIPERBOLIK TIPDAGI TENGLAMALAR

Mexanika va fizikaning tebranish jarayonlari bilan bog'liq bir qator muammolari, masalan, tor va membrananing tebranishi, gaz, elektromexanik to'lqinlarning tarqalishi kabi jarayonlar giperbolik tipdagi tenglamalar orqali ifodalanadi. Bunday tenglamalar bilan ifodalanuvchi jarayonlarinng o'ziga xos tomoni, tebranishlarning chekli tezlikda tarqalishidir.

## 8 §. Tor tebranish tenglamasi uchun Koshi masalasi. D'alamber formulasi

Mexanika va fizikaning tebranish jarayonlari bilan bog'liq bir qator masalalari giperbolik tipdagi tenglama bilan ifodalanadi. Ushbu paragrafda tor tebranish tenglamasi uchun Koshi masalasining qo'yilishi, D'alamber formulasi va uning fizikaviy talqini, Koshi masalasi yechimining turg'unligini isbotlaymiz. Shu bilan birga bir jinsli bo'lmagan tor tebranish tenglamasi uchun Koshi masalasi yechimni keltiramiz.

KOSHI MASALASINING QO'YILISHI.

Eng sodda giperbolik tipdagi tenglama ushbu

$$u_{tt} = a^2 u_{xx}, \quad a = const, \tag{1}$$

ko'rinishda bo'lib, u *tor tebranish tenglamasi* yoki *to'lqin tarqalish tenglamasi* deyiladi.

To'lqin tarqalish nazariyasida Koshi masalasi muhim o'rin egallaydi. $(x, t)$ tekislikdagi biror $D = \{(x,t) : -\infty < x < +\infty, t > 0\}$ sohada (1) tor tebranish tenglamasini qaraylik.

TA'RIF. Agar $u(x,t)$ funksiya $D$ sohada aniqlangan uzluksiz va ikki marta uzluksiz differensiallanuvchi bo'lib, shu sohada (1) tenglamani qanoatlantirsa, bu funksiya tor tebranish tenglamasining $D$ sohadagi *regulyar yechimi* deyiladi.

KOSHI MASALASI. Tor tebranish tenglamasining yopiq $\overline{D}$ sohada aniqlangan, uzluksiz va

$$u(x,0)=\varphi_0(x),\quad u_t(x,0)=\varphi_1(x),\quad -\infty<x<+\infty, \tag{2}$$

boshlang'ich shartlarni qanoatlantiruvchi regulyar yechimini toping.

Bu yerda $\varphi_0(x)$, $\varphi_1(x)$ – berilgan yetarlicha silliq funksiyalar.

Tor tenglamasi uchun Koshi masalasi cheksiz uzunlikdagi tor tebranishining matematik modeli bo'lib, uning chetki nuqtalari torning boshqa qismlarining tebranishiga ta'sir qilmaydi. Shuning uchun ham (1)–(2) Koshi masalasida chegaraviy shartlar qatnashmaydi.

TOR TEBRANISH TENGLAMASINING UMUMIY YECHIMI.

Tor tebranish tenglamasini kanonik ko'rinishga keltiramiz. Buning uchun (1) tenglamaning xarakteristik tenglamasini tuzamiz:

$$a(x,t)(dt)^2-2b(x,t)dtdx+c(x,t)(dx)^2=0,$$

bu yerda $a(x,t)=a^2$, $b(x,t)=0$, $c(x,t)=-1$ va $\delta=a^2>0$. Demak, (1) tor tebranish tenglamasi $R^2_{x,t}$ tekislikda giperbolik tipdagi tenglama ekan. U holda

$$\frac{dt}{dx}=\frac{b\pm\sqrt{\delta}}{a}=\pm\frac{1}{a}\qquad \text{yoki}\qquad dx=\pm a dt$$

bo'lib, u ikkita haqiqiy va har xil

$$x+at=c_1=const,\qquad x-at=c_2=const$$

yechimlarga ega. Bu formulalar bilan aniqlangan to'g'ri chiziqlar tor tebranish tenglamasining xarakteristikalari oilasini ifodalaydi.

Quyidagi

$$\xi=x+at,\quad \eta=x-at \tag{3}$$

tengliklarga asosan yangi $\xi$, $\eta$ o'zgaruvchilar kiritamiz va $u_{tt}$ hamda $u_{xx}$ xususiy hosilalarni

$$u_{tt}=a^2u_{\xi\xi}-2a^2u_{\xi\eta}+a^2u_{\eta\eta}$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

topamiz. Bu hosilalarni (1) tenglamaga qo'yib, soddalashtiramiz. Natijada (1) tenglama ushbu

$$u_{\xi\eta} = 0 \tag{4}$$

kanonik ko'rinishga keladi.

Oxirgi tenglamani ketma-ket integrallab, kanonik tenglamaning umumiy yechimini

$$u(\xi, \eta) = f(\xi) + g(\eta) \tag{5}$$

ko'rinishda topamiz.

Bunda $f(\xi)$, $g(\eta)$ – ixtiyoriy funksiyalar.

Agar $g(\eta) \in C^1(R)$ bo'lsa, u holda (5) funksiya (4) tenglamani qanoatlantiradi, ya'ni

$$g(\eta) \in C^1(R) \quad \text{va} \quad u_{\xi\eta} = 0$$

yoki

$$u_\eta = g'(\eta) \quad \text{va} \quad g(\eta) \in C^2(R).$$

Demak, $f(\xi)$ va $g(\eta) \in C^2(R)$ bo'lsa, u holda (5) funksiya (4) tenglamaning umumiy yechimi bo'ladi.

Endi (5) formulada $\xi$ va $\eta$ o'zgaruvchilardan eski $x$, $t$ o'zgaruvchilarga qaytib, (1) tenglamaning umumiy yechimini olamiz:

$$u(x, t) = f(x + at) + g(x - at), \tag{6}$$

bu yerda $f(x + at)$ va $g(x - at)$ – ixtiyoriy ikki marta uzluksiz hosilalarga ega funksiyalar.

Xuddi yuqoridagi kabi (6) formulada $f(x + at)$ va $g(x - at) \in C^2(R)$ bo'lsa, u holda (6) funksiya (1) tenglamaning umumiy yechimi bo'ladi.

Endi (6) umumiy yechimning fizikaviy xossasiga to'htalib o'taylik. Avvalo, $f(\xi) = 0$ bo'lsin. U holda torning siljishi

$$u_1(x, t) = g(x - at), \tag{7}$$

formula bilan aniqlanadi.

Faraz qilaylik, torning tebranishini kuzatuvchi $t = 0$ vaqtda torning $x = c$ nuqtasidan chiqib, $Ox$ o'qining musbat yo'nalishi bo'ylab $a$ tezlik bilan harakatlansin, ya'ni uning abssissasi $t$ vaqtda $x - at = c$ formula bilan aniqlanadi. Bunda kuzatuvchi uchun $u_1(x,t) = g(x - at)$ formula bilan aniqlangan torning siljishi $g(c)$ ga teng va doim o'zgarmas bo'lib qoladi. (7) funksiya bilan aniqlangan jarayon *to'g'ri to'lqinning tarqalishi* deyiladi.

Xuddi shunday $u_2(x,t) = f(x + at)$ yechim teskari to'lqinning tarqalishini ifodalaydi, ya'ni to'lqin $Ox$ o'qining manfiy yo'nalishi bo'ylab $a$ tezlik bilan tarqaladi. U holda (6) formula ikkita to'g'ri va teskari to'lqinlarning yig'indisi (superpozitsiyasi)dan iborat. Bu $t = 0$ vaqtda torning holatini grafik usulda qurish imkoniyatini beradi. Endi $t = 0$ vaqtda to'g'ri va teskari to'lqinlarni ifodalovchi $u_1(x,0) = g(x)$ va $u_2(x,0) = f(x)$ funksiyalarning grafigini yasaylik. Keyin ularning shaklini o'zgartirmasdan $a$ tezlik bilan har ikki tomonga siljitamiz, ya'ni $u_1 = g(x)$ funksiyani o'ng tomonga va $u_2 = f(x)$ ni esa chap tomonga $a$ tezlik bilan siljitamiz. Torning $t$ vaqtdagi grafigi yuqoridagi surilgan grafiklar ordinatalarining algebraik yig'indisidan iborat bo'ladi.

Koshi masalasi yechimini qurish.

Umumiy yechimdagi $f(x + at)$ va $g(x - at)$ funksiyalarni topish uchun (2) boshlang'ich shartlardan foydalanamiz

$$u(x,0) = f(x) + g(x) = \varphi_0(x),$$

$$u_t(x,0) = af'(x) - ag'(x) = \varphi_1(x)$$

va quyidagi sistemaga ega bo'lamiz:

$$\begin{cases} f(x) + g(x) = \varphi_0(x), \\ f'(x) - g'(x) = \dfrac{1}{a}\varphi_1(x). \end{cases} \tag{8}$$

Bu sistemaning ikkinchi tenglamasida $x$ ni $z$ bilan almashtiramiz va uni noldan $x$ gacha integrallaymiz. Natijada ushbu

$$\begin{cases} f(x) + g(x) = \varphi_0(x), \\ f(x) - g(x) = \dfrac{1}{a} \int\limits_0^x \varphi_1(z)dz + c. \end{cases}$$

sistemani olamiz. Bunda $c = f(0) - g(0)$ – ixtiyoriy o'zgarmas.

Oxirgi sistemadan $f(x)$ va $g(x)$ funksiyalarni topamiz:

$$\begin{cases} f(x) = \dfrac{1}{2}\varphi_0(x) + \dfrac{1}{2a} \int\limits_0^x \varphi_1(z)dz + \dfrac{c}{2} \\ g(x) = \dfrac{1}{2}\varphi_0(x) - \dfrac{1}{2a} \int\limits_0^x \varphi_1(z)dz - \dfrac{c}{2} \end{cases} \tag{9}$$

(9) formulada $f(x)$ funksiyaning $x$ argumentini $x + at$ bilan, $g(x)$ funksiyaning $x$ agumentini esa $x - at$ bilan almashtiramiz va (6) umumiy yechimga qo'yib,

$$u(x,t) = \frac{1}{2}\Big[\varphi_0(x + at) + \varphi_0(x - at)\Big] + \frac{1}{2a} \int\limits_{x-at}^{x+at} \varphi_1(z)dz, \tag{10}$$

ifodani hosil qilamiz.

Bu bir jinsli tor tebranish tenglamasi uchun Koshi masalasi yechimini ifodalovchi *D'alamber formulasi* deyiladi.

Koshi masalasining qo'yilishida $\varphi_0(x)$, $\varphi_1(x)$ funksiyalarni yetarlicha silliq bo'lsin deb talab qilgan edik. Endi bu funksiyalarning qaysi sinfga tegishli ekanligini aniqlaylik.

Agar $\varphi_0(x) \in C^2(R^1)$, $\varphi_1(x) \in C^1(R^1)$ bo'lsa, u holda (10) formula bilan aniqlangan $u(x,t)$ funksiya (1) tor tebranish tenglamasini va (2) boshlang'ich shartlarni qanoatlantirishini bevosita tekshirib ishonch hosil qilish mumkin.

Haqiqatan ham, (10) formulada $t = 0$ desak, u holda

$$u(x, 0) = \varphi_0(x), \qquad x \in R$$

bo'ladi. (10) formuladan $t$ bo'yicha hosila olamiz

$$u_t = \frac{a\varphi_0'(x + at) - a\varphi_0'(x - at)}{2} + \frac{1}{2a}\Big[a\varphi_1(x + at) + a\varphi_1(x - at)\Big],$$

keyin $t = 0$ bo'lganda

$$u_t(x,0) = \varphi_1(x), \qquad x \in R$$

ekanligini ko'ramiz. Bu tor tebranish tenglamasi uchun Koshi masalasining yechimi mavjud ekanligini ko'rsatadi. (10) formulani keltirib chiqarish tor tebranish tenglamasining (6) umumiy yechimiga asoslangan va barcha bosqichlar bir qiymatli bajarildi. Shuning uchun yechimning yagonaligi esa uning qurish usulidan ham kelib chiqadi.

KOSHI MASALASI YECHIMINING TURG'UNLIGI.

Faraz qilaylik, $u_0(x,t)$ funksiya (1) tenglamaning quyidagi

$$u(x,0) = \varphi_0^0(x), \quad u_t(x,0) = \varphi_1^0(x), \quad x \in R$$

boshlang'ich shartlarini qanoatlantiruvchi yechimi bo'lsin. Xuddi yuqoridagi kabi $u_0(x,t)$ funksiya ham (10) formula orqali quriladi.

Agar

$$|\varphi_0(x) - \varphi_0^0(x)| < \delta, \quad |\varphi_1(x) - \varphi_1^0(x)| < \delta \quad \forall x \in R$$

bo'lsa, u holda $\forall x \in R$, $t \in [0, T]$, $T$ – ixtiyoriy musbat son uchun $u(x,t)$ va $u_0(x,t)$ yechimlarning ayrimasini baholaymiz.

$$|u(x,t) - u_0(x,t)| \le \frac{1}{2}\left|\varphi_0(x+at) - \varphi_0^0(x+at)\right| +$$

$$+\frac{1}{2}\left|\varphi_0(x-at) - \varphi_0^0(x-at)\right| + \frac{1}{2a}\int\limits_{x-at}^{x+at}\left|\varphi_1(z) - \varphi_1^0(z)\right|dz <$$

$$< \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2a}\delta\int\limits_{x-at}^{x+at} dz =$$

$$= \delta + \delta\frac{1}{2a}2at = \delta(1+t) < \delta(1+T), \tag{11}$$

Faraz qilaylik, $\varepsilon$ ixtiyoriy musbat son va $\delta = \varepsilon/(1+T)$ bo'lsin. U holda (11) tengsizlikdan $\forall \varepsilon > 0$ son uchun shunday $\delta = \varepsilon/(1+T)$ son topiladiki, barcha $x \in R$ va $t \in [0, T]$ larda

$$|\varphi_0(x) - \varphi_0^0(x)| < \delta, \quad |\varphi_1(x) - \varphi_1^0(x)| < \delta$$

shartlar bajarilganda $|u(x,t) - u_0(x,t)| < \varepsilon$ tengsizlik o'rinli bo'ladi.

Bundan, tor tebranish tenglamasi uchun Koshi masalasining yechimi berilganlarga uzluksiz bog'liq ekanligi kelib chiqadi.

Shunday qilab, quyidagi teorema isbotlandi:

TEOREMA. Agar $\varphi_0(x) \in C^2(R^1)$, $\varphi_1(x) \in C^1(R^1)$ bo'lsa, u holda tor tebranish tenglama uchun Koshi masalasining yechimi mavjud, yagona va turg'un bo'ladi, ya'ni (1)–(2) masalaning $u(x,t)$ yechimi (10) formula bilan aniqlanadi.

Ma'lum bir masalalarni yechishda $\varphi_0(x)$, $\varphi_1(x)$ funksiyalar teoremaning shartlarini bajarmasligi mumkin. Bunda tor tebranish tenglamasi uchun (1)–(2) Koshi masalasining regulyar yechimi tushunchasini kiritib bo'lmaydi. Bunday hollarda umumlashgan yechim tushunchasi kiritiladi.

TA'RIF. Tor tebranish tenglamasi uchun (1)–(2) Koshi masalasining umumlashgan yechimi deb, (1) tenglamaning

$$u_n(x,0) = \varphi_{n0}(x), \quad u_{nt}(x,0) = \varphi_{n1}(x), \quad -\infty < x < +\infty,$$

boshlang'ich shartlarni qanoatlantiruvchi $u_n(x,t)$ regulyar yechimlarning tekis yakinlashuvchi ketma-ketligining limiti bo'lgan $u(x,t)$ funksiyaga aytiladi.

Bu yerda $\varphi_{n0}(x) \in C^2(R^1)$, $\varphi_{n1}(x) \in C^1(R^1)$ va bu funksiyalar sonlar o'qining ixtiyoriy $[\alpha, \beta]$ segmentida $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalarga tekis yaqinlashuvchi ketma-ketliklar, ya'ni

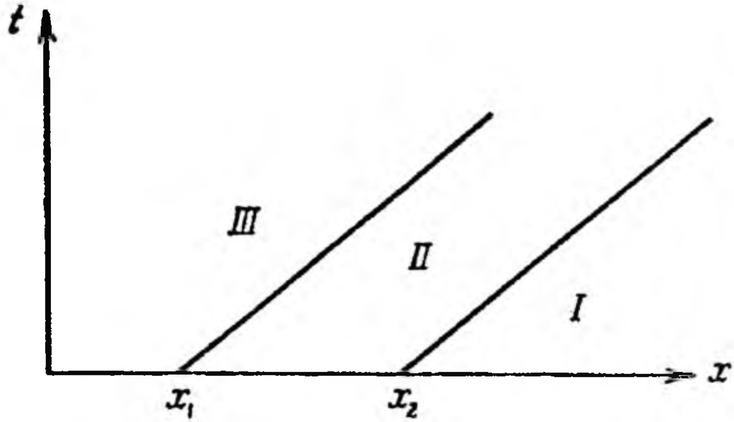
$$\lim_{n\to\infty} \varphi_{n0}(x) \rightrightarrows \varphi_0(x), \qquad \lim_{n\to\infty} \varphi_{n1}(x) \rightrightarrows \varphi_1(x).$$

Agar $\varphi_0(x)$, $\varphi_{n1}(x) \in C(R^1)$ bo'lsa, u holda (1)–(2) Koshi masalasining umumlashgan yechimi mavjud, yagona va (10) formula bilan ifodalanishini ko'rsatish qiyin emas.

KOSHI MASALASI YECHIMINING FIZIKAVIY TALQINI.

Tor tebranish tenglamasi uchun Koshi masalasining (10) formula bilan aniqlangan yechimi boshlang'ich tezlik bo'yicha torning boshlang'ich siljishini tarqalish jarayonini ifodalaydi.

Tor tebranish tenglamasining (9) umumiy yechimning fizikaviy xususiyatiga asosan (10) formula ikkita to'g'ri to'lqinning yig'indisidan iborat, ya'ni $f(x + at) + g(x - at)$, bulardan biri $a$ tezlik bilan o'ng tomonga ikkinchisi esa shu tezlik bilan chap tomonga tarqaladi.




5 — shakl.

Bu holda

$$f(x + at) = \frac{1}{2}\varphi_0(x + at) - F(x + at),$$

$$g(x - at) = \frac{1}{2}\varphi_0(x - at) = F(x - at)$$

tengliklar o'rinli bo'ladi. Bu yerda

$$F(\xi) = \frac{1}{2a}\int_0^{\xi} \varphi_1(s)ds.$$

$(x, t)$ o'zgaruvchilar tekisligida $x + at = c_1 = const$ va $x - at = c_2 = const$ to'g'ri chiziqlar (1) tenglamaning xarakteristikalari bo'lgani uchun $u(x, t) = \varphi_0(x + at)$ funksiya $x + at = c_1$ xarakteristika bo'ylab o'zgarmas va bu qiymat $\varphi_0(c_1)$ ga teng. Xuddi shunday $u(x, t) = \varphi_0(x - at)$ funksiya $x - at = c_2 = const$ xarakteristika bo'ylab o'zgarmasdir.

Faraz qilaylik, $\varphi_0(x)$ funksiya biror $(x_1, x_2)$ intervalda noldan farqli va intervaldan tashqarida nolga teng bo‘lsin. $(x_1, 0)$ va $(x_2, 0)$ nuqtalardan (1) tenglamaning $x + at = c_1$ va $x - at = c_2$ xarakteristikalarini o‘tkazamiz. Bu xarakteristikalar $t > 0$ yarim tekislikni uchta *I*, *II* va *III* bo‘lakka bo‘ladi (5–shakl).

$u(x,t) = \varphi_0(x - at)$ funksiya $II : x_1 < x - at < x_2$ sohada noldan farqli, bunda $x - at = x_1$ va $x + at = x_2$ xarakteristikalar o‘ng tomonga $a$ tezlik bilan tarqalayotgan to‘g‘ri to‘lqinning oldingi va orqa fronti deb yuritiladi.

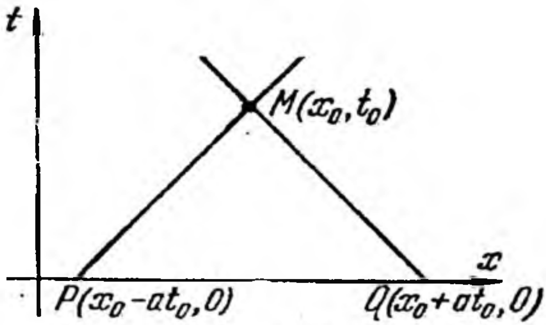
Faraz qilaylik, $M = (x_0, t_0)$ nuqta $t > 0$ yarim tekislikda fiksirlangan nuqta bo‘lsin. Bu nuqtadan (1) tenglamaning $x - at = x_0 - at_0$ va $x + at = x_0 + at_0$ xarakteristikalarini o‘tkazamiz. Bu xarakteristikalar $Ox$ o‘qi bilan $P = (x_1, 0) = (x_0 - at_0, 0)$ va $Q = (x_2, 0) = (x_0 + at_0, 0)$ nuqtalarda kesishadi. Tor tebranish tenglamasining (6) umumiy yechimining $M$ nuqtadagi qiymati $u(x_0, t_0) = g(x_1) + f(x_2)$ ga teng, ya’ni $f(x)$ va $g(x)$ funksiyalarning qiymati mos ravishda $MPQ$ uchburchak asosining $(x_1, 0)$ va $(x_2, 0)$ uchlaridagi qiymatlari orqali ifodalanadi (6–shakl).

Bu $MP$, $MQ$ xarakteristikalar va $Ox$ o‘qida $PQ$ kesmadan tashkil topgan $MPQ$ uchburchak $M$ nuktaning *xarakteristik uchburchagi* deyiladi.

Koshi masalasining yechimini ifodalovchi (10) formuladan torning $x_0$ nuqtasini $t_0$ vaqtdagi $u(x_0, t_0)$ siljishi $PQ$ kesmada berilgan boshlang‘ich holat va boshlang‘ich tezlikning $P$ va $Q$ nuqtalardagi qiymatlariga bog‘liq ekanligi ko‘rinadi. (10) formulani quyidagi

$$u(M) = \frac{\varphi_0(P) + \varphi_0(Q)}{2} + \frac{1}{2a}\int_P^Q \varphi_1(s)ds$$

shaklda yozish mumkin. $PQ$ kesmadan tashqarida berilgan boshlang‘ich shartlar $u(x,t)$ yechimning $M$ nuqtadagi qiymatiga hech qanday ta’sir ko‘rsatmaydi.




6 — shakl.

Demak, tor tebranish tenglamasi uchun boshlang'ich shartlar butun o'qda emas, balki $PQ$ kesmada berilgan bo'lsa, u holda Koshi masalasining yechimi $MPQ$ xarakteristik uchburchakning ichida aniqlanadi.

Endi Koshi masalasini bir jinsli bo'lmagan

$$u_{tt} = a^2 u_{xx} + f(x,t) \tag{12}$$

tenglama uchun qaraylik, ya'ni (12) tenglamaning

$$u|_{t=0} = \varphi_0(x), qquadu_t|_{t=0} = \varphi_1(x) \tag{2}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini topaylik.

Faraz qilaylik, $t > \tau$ da $v_f(x,t;\tau)$ quyidagi yordamchi

$$(v_f)_{tt} = a^2(v_f)_{xx}, \quad -\infty < x < \infty, \tag{13}$$

$$v_f(x,\tau;\tau) = 0, \quad \frac{\partial v_f}{\partial t}(x,\tau;\tau) = f(x,\tau), \ t = \tau, \ -\infty < x < \infty \tag{14}$$

Koshi masalasining yechimi bo'lsin. D'alamber formulasiga asosan

$$v_f(x,t;\tau) = v_f(x,t-\tau;\tau) = \frac{1}{2a}\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi. \tag{15}$$

ifodani topamiz.

Bu formulalarda $\tau$ faqat parametr, (14) boshlang‘ich shartlar $t = 0$ da emas, balki $t = \tau$ da berilgan.

U holda (6) D'alamber formulasini quyidagi

$$u(x,t) = \frac{\partial v_{\varphi_0}}{\partial t}(x,t;0) + v_{\varphi_1}(x,t;0), \tag{16}$$

ko‘rinishda yozish mumkin. Bunda

$$v_{\varphi_1}(x,t;0) = \frac{1}{2a}\int\limits_{x-at}^{x+at}\varphi_1(\xi)d\xi, \quad v_{\varphi_0}(x.t;0) = \frac{1}{2a}\int\limits_{x-at}^{x+at}\varphi_0(\xi)d\xi.$$

bo‘lib, ular (13)–(14) masalaning $\tau = 0$ bo‘lgandagi mos ravishda

$$\left.\frac{\partial v_f}{\partial t}\right|_{\tau=0} = \varphi_1(x), \qquad \left.\frac{\partial v_f}{\partial t}\right|_{\tau=0} = \varphi_0(x)$$

shartlarni qanoatlantiruvchi yechimlari. Shu bilan birga tekshirib ko‘rish osonki

$$\frac{\partial v_{\varphi_0}}{\partial t}(x,t;0) = \frac{\partial}{\partial t}\frac{1}{2a}\int\limits_{x-at}^{a+xt}\varphi_0(\xi)d\xi = \frac{\varphi_0(x+at)+\varphi_0(x-at)}{2},$$

bo‘ladi.

Bir jinsli bo‘lmagan (12) tenglamaning bir jinsli

$$u(x,0) = 0, \qquad u_t(x,0) = 0$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini (15) formuladan foydalanib, quyidagicha yozish mumkin

$$u(x,t) = a^2\int\limits_0^t v_{\varphi_0}(x,t;\tau)d\tau$$

yoki $v_{\varphi_0}$ ning qiymatini o‘rniga qo‘yib, ushbu

$$u(x,t) = \frac{a}{2}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau$$

ifodani olamiz.

Shunday qilib, (12) tenglamaning (2) boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$u(x,t) = \frac{1}{2}\left[\varphi_0(x+at) + \varphi_0(x-at)\right] +$$

$$+\frac{1}{2a}\int\limits_{x-at}^{x+at}\varphi_1(z)dz + \frac{1}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau \qquad (17)$$

formula bilan aniqlanadi. Bunda $\varphi_0''(x)$, $\varphi_1'(x)$ funksiyalar uzluksiz va $f(x,t)$ – birinchi tartibli uzluksiz hosilalarga ega bo'lgan funksiya, ya'ni $f(x,t) \in C^1(\overline{D})$.

1–MASALA. Ushbu

$$yu_{xx} - (x+y)u_{xy} + xu_{yy} = 0, \quad x > 0$$

tenglamaning

$$u(x,0) = x^2, \quad u_y(x,0) = 3x,$$

shartlarni qanoatlantiruvchi $u(x,y)$ regulyar yechimini toping.

YECHISH. Berilgan tenglamani kanonik ko'rinishga keltirib, integrallaymiz. Natijada kanonik tenglamaning umumiy yechimi hosil bo'ladi. Hosil bo'lgan yechimda $x$ va $y$ o'zgaruvchilarga qaytib, berilgan tenglamaning umumiy yechimini

$$u(x,y) = f_1(x+y) + f_2(x^2+y^2) \qquad (18)$$

ko'rinishda yozamiz.

Bu yerda $f_1(x+y)$ va $f_2(x^2+y^2)$ – ixtiyoriy ikki marta uzluksiz differensiallanuvchi funksiyalar.

(18) formuladan va boshlang'ich shartlardan foydalanib, $f_1$ va $f_2$ funksiyalarni topamiz:

$$f_1(x) = \frac{3}{2}x^2 + f_1(0),$$

$$f_2(x^2) = -\frac{1}{2}x^2 - f_1(0).$$

Topilgan funksiyalarni (18) formulaga qo'yamiz, natijada berilgan masalaning quyidagi

$$u(x,y) = \frac{3}{2}(x+y)^2 - \frac{1}{2}(x^2+y^2) = x^2 + 3xy + y^2.$$

yechimiga ega bo'lamiz:

2–MASALA. Ushbu

$$2u_{xy} - e^{-x}u_{yy} = 4x \tag{19}$$

tenglamaning

$$u(x,y)\mid_{y=x} = x^5\cos x, quad u_y(x,y)\mid_{y=x} = x^2 + 1 \tag{20}$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ regulyar yechimini toping.

YECHISH. Berilgan tenglamaning xarakteristik tenglamasi

$$-2dxdy - e^{-x}(dx)^2 = 0$$

bo'lib, u $x = c$, $2y - e^{-x} = c$ yechimlarga (xarakteristikalarga) ega. Yangi $\xi$, $\eta$ o'zgaruvchilar kiritamiz

$$\xi = x, quad \eta = 2y - e^{-x}.$$

U holda (19) tenglama

$$u_{\xi\eta} = \xi$$

kanonik ko'rinishga keladi.

Oxirgi tenglamaning umumiy yechimi

$$u(\xi,\eta) = \frac{1}{2}\xi^2\eta + f_1(\xi) + f_2(\eta)$$

bo'ladi.

Bundan eski $x, y$ o'zgaruvchilarga o'tsak, berilgan tenglamaning umumiy yechimini

$$u(x,y) = \frac{1}{2}x^2(2y - e^{-x}) + f_1(x) + f_2(2y - e^{-x}) \tag{21}$$

topamiz.

Bu yerda $f_1(x)$ va $f_2(2y-e^{-x})$ – ixtiyoriy funksiyalar. (21) ifodadagi $f_1(x)$ va $f_2(2y-e^{-x})$ funksiyalarni (20) shartlar yordamida aniqlaymiz:

$$f_1(x)+f_2(2x-e^{-x})+\frac{1}{2}x^2(2x-e^{-x})=x^5\cos x$$

$$2f_2'(2x-e^{-x})+x^2=x^2+1$$

Bundan

$$f_2(t)=\frac{1}{2}t+f_2(0)$$

$$f_1(x)=x^5\cos x-x^3+\frac{1}{2}x^2e^{-x}-\frac{1}{2}(2x-e^{-x})-f_2(0)$$

bo‘ladi.

Topilgan ifodalarni (21) formulaga qo‘yib, soddalashtirsak, (19)–(20) masalaning yechimini

$$u(x,y)=x^5\cos x+(y-x)(x^2+1)$$

ko‘rinishda topamiz.

## 9–§. Tor tebranish tenglamasi uchun aralash masala

Biz *I* bobda uchlari mahkamlangan torning tebranishi haqidagi fizik masalani ikkinchi tartibli xususiy hosilali tenglama, ya’ni giperbolik tipdagi tenglama uchun chegaraviy masalaga kelishini ko‘rdik. Ushbu paragrafda yuqorida aytilgan masalaning qo‘yilishi, yechimning yagonaligi va mavjudligini batafsil o‘rganamiz.

Masalaning qo‘yilishi. Yechimning yagonaligi.

Biror chekli $D=\{(x,t):\ 0<x<l,\ 0<t<T\}$ sohada bir jinsli torning majburiy tebranishini ifodalovchi ushbu

$$Lu\equiv\rho u_{tt}-T_0u_{xx}=f(x,t) \qquad (1)$$

bir jinsli bo‘lmagan tor tebranish tenglamasini qaraylik,
bu yerda $l$ – torning uzunligi, $T$ – musbat son, $\rho$ – torning zichligi,

$T_0$ – torning taranglik kuchi, $f(x,t)$ esa torga ta'sir qilayotgan tashqi kuchlarning yig'indisi.

ARALASH MASALA. Yopiq $D$ sohada aniqlangan va quyidagi shartlarni qanoatlantiruvchi $u(x,t)$ funksiya toping:

1) $u(x,t)$ funksiya yopiq $D$ sohada ikki marta uzluksiz differensiallanuvchi va $\forall(x,t) \in D$ da (1) tenglamani qanoatlantirsin, ya'ni

$$u(x,t) \in C^2(\overline{D}); \quad Lu(x,t) = f(x,t), \quad \forall(x,t) \in D,$$

bo'lsin;

2) $u(x,t)$ funksiya ushbu

$$u(x,t)|_{t=0} = \varphi_0(x), \quad u_t(x,t)|_{t=0} = \varphi_1(x), \quad 0 \le x \le l; \tag{2}$$

boshlang'ich shartlarni qanoatlantirsin:

3) $u(x,t)$ funksiya $D$ sohaning chegarasida quyidagi

$$u(x,t)|_{x=0} = \mu_1(t), \quad u(x,t)|_{x=l} = \mu_2(t), \quad 0 \le t \le T; \tag{3}$$

shartlarni qanoatlantirsin; bu yerda $f(x,t)$, $\varphi_0(x)$, $\varphi_1(x)$, $\mu_1(t)$ va $\mu_2(t)$ berilgan yetarlicha silliq funksiyalar.

1–TEOREMA. Agar (1)–(3) aralash masalaning yechimi mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. Faraz qilaylik, (1)–(3) masala ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimlarga ega bo'lsin. U holda bu yechimlarning ayirmasi $u(x,t) = u_1(x,t) - u_2(x,t) \in C^2(\overline{D})$ bo'lib, $v(x,t)$ funksiya bir jinsli

$$Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f(x,t) - f(x,t) = 0 \tag{4}$$

tor tebranish tenglamasini hamda bir jinsli boshlang'ich

$$u(x,t)|_{t=0} = [u_1(x,t) - u_2(x,t)]|_{t=0} = u_1(x,t)|_{t=0} - u_2(x,t)|_{t=0} =$$

$$= \varphi_0(x) - \varphi_0(x) = 0, \qquad 0 \le x \le l; \tag{5}$$

$$u_t(x,t)|_{t=0} = [u_{1t}(x,t) - u_{2t}(x,t)]|_{t=0} = u_{1t}(x,t)|_{t=0} - u_{2t}(x,t)|_{t=0} =$$

$$= \varphi_1(x) - \varphi_1(x) = 0, \qquad 0 \le x \le l; \tag{6}$$

va chegaraviy

$$u(x,t)|_{x=0} = [u_1(x,t) - u_2(x,t)]|_{x=0} = u_1(x,t)|_{x=0} - u_2(x,t)|_{x=0} =$$

$$= \mu_1(t) - \mu_1(t) = 0; \qquad 0 \le t \le T; \tag{7}$$

$$u(x,t)|_{x=l} = [u_1(x,t) - u_2(x,t)]|_{x=l} = u_1(x,t)|_{x=l} - u_2(x,t)|_{x=l} =$$

$$= \mu_2(t) - \mu_2(t) = 0; \qquad 0 \le t \le T. \tag{8}$$

shartlarni qanoatlantiradi.

Bir jinsli (4)–(8) masalaning $u(x,t)$ yechimi $\forall(x,t) \in \overline{D}$ bo'lganda aynan nolga teng ekanligini isbot qilamiz.

Buning uchun quyidagi

$$E(t) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + T_0\left(\frac{\partial u}{\partial x}\right)^2\right] dx, \tag{9}$$

integralni qaraylik.

Bu integral torning bir jinsli chegaraviy shartlar bilan erkin tebranish energiyasi saqlanish qonunining matematik ifodasi bo'lib, u torning *to'la energiyasi deyiladi.*

Chunki torning $t$ vaqtdagi $\Delta x = dx$ elementining kinetik energiyasi

$$K(t) = \frac{1}{2}\int_0^l \rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx$$

ko'rinishda bo'ladi.

Torning $t$ vaqtdagi $\Delta x = dx$ elementining potensial energiyasi taranglik kuchining bajargan ishi bo'lib, u quyidagi

$$\Pi(t) = \frac{1}{2}\int_0^l T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx$$

formula bilan aniqlanadi.

Demak, (9) formula torning ko'ndalang tebranishining to'la energiyasi bo'lib, u torning *energiyalari integrali* deyiladi.

Endi (9) integralning $t$ vaqtga bog'liq emasligini ko'raylik. Buning uchun (9) formulaning $t$ bo'yicha hosilasini hisoblaymiz:

$$\frac{dE(t)}{dt} = \int_0^l \Big(\rho u_t u_{tt} + T_0 u_x u_{xt}\Big) dx. \tag{10}$$

Bir jinsli chegaraviy shartlardan

$$u_t(0,t) = 0, \qquad u_t(l,t) = 0, \quad 0 \le t \le T,$$

ekanligi kelib chiqadi. Bu shartlarni inobatga olib (10) formulaning ikkinchi qo'shiluvchisini $x$ bo'yicha 0 dan $l$ gacha bo'laklab integrallaymiz, natijada

$$\int_0^l T_0 u_x u_{xt} dx = T_0 u_x u_t \Big|_0^l - \int_0^l T_0 u_t u_{xx} dx = -\int_0^l T_0 u_t u_{xx} dx, \tag{11}$$

ifodani olamiz. Topilgan (11) ifodani (10) formulaga qo'yib, bir jinsli (4) tor tebranish tenglamasini inobatga olsak,

$$E'(t) = \int_0^l u_t(\rho u_{tt} - T_0 u_{xx}) dx = \int_0^l u_t L u dx = 0$$

tenglikni olamiz.

Oxirgi tenglikdan $\forall t \in [0, T]$ uchun $E(t) = const$ ekanligi kelib chiqadi. Shuning uchun bir jinsli bo'lmagan boshlang'ich shartlarda

$$E(t) = E(0) = \frac{1}{2} \int_0^l \left[\rho \left(\frac{\partial u(x,0)}{\partial t}\right)^2 + T_0 \left(\frac{\partial u(x,0)}{\partial x}\right)^2\right] dx =$$

$$= \frac{1}{2} \int_0^l [\rho(\varphi_1(x))^2 + T_0(\varphi_0(x))^2] dx. \tag{12}$$

Bu formuladan ko'rinadiki, uchlari mahkamlangan torning erkin ko'ndalang tebranishining to'la energiyasi ixtiyoriy vaqtda o'zgarmas va u torning boshlang'ich energiyasiga teng bo'ladi.

Bir jinsli (5) va (6) shartlarni inobatga olib, (12) formuladan ushbu

$$E(t) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx = 0$$

tenglikni olamiz. Oxirgi tenglik faqat va faqat $\forall (x,t) \in \overline{D}$ uchun $u_t(x,t) = 0$ va $u_x(x,t) = 0$ bo'lganda o'rinli. Bundan esa $\overline{D}$ yopiq sohada $u(x,t) = const$ bo'ladi. Bir jinsli shartlarga ko'ra $\overline{D}$ sohada $u(x,t) = 0$ bo'lishi kelib chiqadi.

Demak, $u_1(x,t) \equiv u_2(x,t)$, farazimiz noto'g'ri ekan. Bu ziddiyat tor tebranish tenglamasi uchun qo'yilgan aralash masala yechimining yagona ekanligini isbotlaydi.

YECHIMNING BERILGANLARGA UZLUKSIZ BOG'LIQLIGI. Faraz qilaylik, $D$ sohada (1) tenglama bir xil chegaraviy shartlarni va

$$u_1(x,t)|_{t=0} = \varphi_0^1(x), \quad u_{1t}(x,t)|_{t=0} = \varphi_1^1(x);$$

$$u_2(x,t)|_{t=0} = \varphi_0^2(x), \quad u_{2t}(x,t)|_{t=0} = \varphi_1^2(x);$$

boshlang'ich shartlarni qanoatalantiruvchi $u_1(x,t)$ va $u_2(x,t)$ yechimlarga ega bo'lsin.

2-TEOREMA. Agar

$$\varphi_0^{(1)}(x) - \varphi_1^{(1)}(x) = \varphi_0(x), \quad \varphi_0^{(2)}(x) - \varphi_1^{(2)}(x) = \varphi_1(x)$$

funksiyalar va $\varphi_0'(x)$ funksiya $\forall x \in [0,l]$ da absolyut qiymati bo'yicha yetarlicha kichik bo'lsa, u holda $u_1(x,t) - u_2(x,t) = u(x,t)$ ayirma ham yetarlicha kichik bo'ladi.

ISBOT. Ushbu $u_1(x,t) - u_2(x,t) = u(x,t)$ ayirma $D$ sohada bir jinsli

$$Lu \equiv \rho u_{tt} - T_0 u_{xx} = 0 \tag{13}$$

tenglamani, bir jinsli chegaraviy

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = 0, \tag{14}$$

va boshlang'ich

$$u(x,t)|_{t=0} = \varphi_0(x), \quad u_t(x,t)|_{t=0} = \varphi_1(x) \tag{15}$$

shartlarni qanoatlantiradi.

Yana energiyalar integralini qaraymiz:

$$E(t) = \frac{1}{2}\int\limits_0^l \left[\rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx. \tag{16}$$

Bu integral (13) tenglamaning (14) chegaraviy shartlarni qanoatlantiruvchi ixtiyoriy yechimida o'zgarmas qiymatini saqlaydi, ya'ni $E(t) = E(0)$, $0 \le t \le T$. Bundan (15) boshlang'ich shartlarga asosan

$$\int\limits_0^l \left[\rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx =$$

$$= \int\limits_0^l [\rho(\varphi_1(x))^2 + T_0(\varphi_0(x))^2]dx$$

kelib chiqadi.

Faraz qilaylik, $M = \max\{\rho, T_0\}$ bo'lsin. U holda

$$\int\limits_0^l \left[\rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx \le$$

$$\le M\int\limits_0^l [(\varphi_1(x))^2 + (\varphi_0(x))^2]dx$$

tengsizlik o'rinli bo'ladi. Bu tengsizlikning o'ng tomonidagi funksiyalarning yetarlicha kichik ekanligidan

$$\int\limits_0^l \left[\rho\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + T_0\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx \le \delta^2$$

bo'lishi kelib chiqadi. Bundan esa

$$\int_0^l T_0 \left( \frac{\partial u(x,t)}{\partial x} \right)^2 dx \leq \frac{\delta^2}{2}$$

bo'ladi.

Quyidagi tenglikdan

$$u(x,t) - u(0,t) = \int_0^x \frac{\partial u(x,t)}{\partial x} dx$$

ushbu

$$|u(x,t)| \leq \int_0^x \left| \frac{\partial u(x,t)}{\partial x} \right| dx \leq \int_0^x \frac{1}{\sqrt{T_0}} \sqrt{T_0} \left| \frac{\partial u(x,t)}{\partial x} \right| dx \leq$$

$$\left[ \int_0^x \frac{dx}{T_0} \int_0^x T_0 \left( \frac{\partial u}{\partial x} \right)^2 dx \right]^{1/2} \leq \left[ \int_0^l \frac{dx}{T_0} \int_0^l T_0 \left( \frac{\partial u}{\partial x} \right)^2 dx \right]^{1/2} \leq$$

$$\leq \sqrt{\frac{l}{T_0}} \left[ \int_0^l T_0 \left( \frac{\partial u}{\partial x} \right)^2 dx \right]^{1/2} \leq \sqrt{\frac{l}{2T_0}} \delta = \varepsilon.$$

tengsizlik yoki $|u(x,t)| \leq \varepsilon$ ekanligi kelib chiqadi.

Demak, tor tebranish tenglamasi uchun aralash masalaning yechimi boshlang'ich funksiyalarga, ya'ni berilganlarga uzluksiz bog'liq ekan. Shunday qilib, 2–teorema isbotlandi.

## 10–§. Tor tebranish tenglamasi uchun aralash masala yechimining mavjudligi

Bu paragrafda tor tebranish tenglamasi uchun qo'yilgan aralash masala yechimining mavjudligini isbotaymiz.

Unda avval uchlari mahkamlangan bir jinsli torning erkin tebranishini, ya'ni (1)–(3) masalada $\mu_1(t) = \mu_2(t) = 0$ va $f(x,t) = 0$ bo'lgan holni, keyin uchlari mahkamlangan torning majburiy tebranishi va so'ngra uchlari qo'zg'aluvchi bo'lgan torning majburiy tebranishi, ya'ni (1)–(3) masalaning yechimini Fur'e usulida quramiz.

**Uchlari mahkamlangan torning erkin tebranishi**

Fur'e usuli, yoki o'zgaruvchilarni ajratish usuli xususiy hosilali tenglamalarni yechishda ko'p qo'llaniladigan usullardan biri hisoblanadi.

Uchlari mahkamlangan bir jinsli torning erkin tebranishi masalasi ushbu bir jinsli

$$u_{tt} = a^2 u_{xx}, \qquad a^2 = \frac{T_0}{\rho}, \tag{1}$$

tor tebranish tenglamasining quyidagi boshlang'ich

$$u(x,t)|_{t=0} = \varphi_0(x), \quad u_t(x,t)|_{t=0} = \varphi_1(x), \qquad 0 \le x \le l, \tag{2}$$

va bir jinsli chegaraviy

$$u(0,t) = 0, \qquad u(l,t) = 0, \qquad 0 \le t \le T, \tag{3}$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimini topish masalasiga keltiriladi. Bu masalaning $u(x,t)$ yechimini $C^2(\overline{D})$ funksiyalar sinfidan izlaymiz.

Buning uchun (1) tenglamaning $D$ sohada noldan farqli va bir jinsli (3) chegaraviy shartlarni qanoatlantiruvchi xususiy yechimlarini

$$u(x,t) = X(x)T(t) \neq 0, \tag{4}$$

ko'rinishda qidiramiz.

(4) ifodani (1) tenglamaga qo'yib, uning o'zgaruvchilarini ajratsak, ushbu

$$T''(t)X(x) = a^2 X''(x)T(t)$$

yoki

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{5}$$

tenglikka ega bo'lamiz. Bu tenglikning chap tomoni faqat $t$ o'zgaruvchiga, o'ng tomoni esa faqat $x$ o'zgaruvchiga bog'liq. Shuning uchun (5) tenglik, ikki tomoni ham bitta o'zgarmas $-\lambda$ songa teng bo'lgandagina o'rinli bo'ladi. U holda (5) ifodadan ikkita chiziqli ikkinchi tartibli

$$T''(t) + a^2\lambda T(t) = 0, \qquad 0 < t < T, \tag{6}$$

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < l, \tag{7}$$

oddiy differensial tenglamalarni olamiz.

Ixtiyoriy $t \in [0, T]$ da $T(t) \neq 0$ bo'lgani uchun (4) tenglikdan (3) chegaraviy shartlar asosida quyidagi

$$X(0) = 0, \qquad X(l) = 0 \tag{8}$$

shartlar kelib chiqadi.

Shunday qilib, biz $X(x)$ funksiyani topish uchun quyidagi masalaga ega bo'ldik: $\lambda$ parametrning qanday qiymatlarida (7)–(8) chegaraviy masalaning $X(x)$ yechimi noldan farqli bo'ladi. Bunday masala matematik fizikada *spektral masala yoki Shturm–Liuvill masalasi deyiladi.*

Shturm–Liuvill masalaning noldan farqli yechimini ta'minlagan $\lambda$ parametrning qiymatlari spektral masalaning *xos qiymatlari*, unga mos yechimlar esa *xos funksiyalar* deb ataladi. (7) va (8) masalaning xos qimatlari to'plami shu *masalaning spektri* deyiladi.

Endi qaralayotgan spektral masalaning xos qiymatlarini va ularga mos xos funksiyalarini topaylik.

Buning uchun uchta $\lambda < 0$, $\lambda = 0$ va $\lambda > 0$ hollarni alohida-alohida qarab chiqamiz.

1) Faraz qilaylik, $\lambda = -k^2 < 0$ bo'lsin. U holda (7) tenglamaning umumiy yechimi

$$X(x) = C_1 e^{kx} + C_2 e^{-kx},$$

ko'rinishda bo'ladi. Bu yerda $C_1$ va $C_2$ – ixtiyoriy o'zgarmas sonlar. Shu yechimni (8) bir jinsli chegaraviy shartlarga qo'ysak, $C_1$ va $C_2$ o'zgarmaslarni topish uchun quyidagi

$$\begin{cases} X(0) = C_1 + C_2 = 0, \\ X(l) = C_1 e^{kl} + C_2 e^{-kl} = 0. \end{cases}$$

tenglamalar sistemasiga ega bo'lamiz.

Bu sistemaning asosiy determinanti noldan farqli va u yagona trivial yechimga ega, ya'ni $C_1 = 0$, $C_2 = 0$.

Demak, bu holda $X(x) \equiv 0$ bo'ladi va bu yechim masala shartini qanoatlantirmaydi.

2) Endi $\lambda = 0$ bo'lsin. U holda (7) tenglamaning umumiy yechimi

$$X(x) = C_1 + C_2 x$$

va (8) shartlarga asosan $C_1 = 0$, $C_2 = 0$ bo'ladi, demak, $X(x) \equiv 0$, bu yechim ham masala shartini qanoatlantirmaydi.

3) Faraz qilaylik, $\lambda = \mu^2 > 0$, $\mu > 0$ bo'lsin. U holda (7) tenglamaning umumiy yechimi

$$X(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

ko'rinishda bo'ladi. Oxirgi ifodani (8) chegaraviy shartlarga qo'ysak,

$$\begin{cases} X(0) = C_1 + C_2\, 0 = 0, \\ X(l) = C_1 \cos \mu l + C_2 \sin \mu l = 0. \end{cases}$$

sistemaga ega bo'lamiz. Bundan $C_1 = 0$, $C_2 \sin \mu l = 0$ ekanligi kelib chiqadi.

Endi $C_2 \neq 0$ deb olamiz, aks holda yana $X(x) \equiv 0$ bo'ladi. Shuning uchun $\sin \mu l = 0$, bundan esa $\mu l = k\pi$, $k \in Z$ kelib chiqadi.

Shunday qilib, (7)–(8) masalaning noldan farqli yechimi faqatgina

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \qquad (k = 1, 2, 3, \ldots)$$

qiymatlarda mavjud va bu qiymatlar qaralayotgan masalaning *xos qiymatlari* deyiladi. Bu qiymatlarga mos *xos funksiyalar*

$$X_k(x) = \sin\frac{k\pi}{l}x$$

ko‘rinishda bo‘ladi.

Endi topilgan $\lambda = \lambda_k$ qiymatlarda (6) tenglamaning umumiy yechimi

$$T_k(t) = a_k\cos\frac{k\pi at}{l} + b_k\sin\frac{k\pi at}{l}$$

ko‘rinishda topiladi, bunda $a_k$, $b_k$ – ixtiyoriy o‘zgarmas sonlar.

Topilgan $X_k(x)$ va $T_k(t)$ funksiyalarni (4) formulaga qo‘yib, ushbu

$$u_k(x,t) = X_k(x)T_k(t)) = \left(a_k\cos\frac{k\pi at}{l} + b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l},$$

funksiyani olamiz. Bu $u_k(x,t)$, $(k = 1, 2, \ldots)$ funksiyalar $a_k$ va $b_k$ koeffitsiyentlarning ixtiyoriy qiymatlarida (1) tenglamani va (3) bir jinsli chegaraviy shartlarni qanoatlantiradi.

(1) tenglama chiziqli va bir jinsli bo‘lgani uchun yechimlarning chekli yig‘indisi ham tenglamani qanoatlantiradi va u quyidagi

$$u(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l} + b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}, \qquad (9)$$

qator uchun ham o‘rinli. Agar bu qator tekis yaqinlashuvchi bo‘lsa, u holda bu qatorni $x$ bo‘yicha va $t$ bo‘yicha ikki marta differensiallash mumkin. (9) qatorning har bir hadi (1) bir jinsli tenglamani va bir jinsli (3) chegaraviy shartlarni qanoatlantiradi. Bu shart (9) formula bilan aniqlangan qatorning yig‘indisi, ya’ni $u(x,t)$ uchun ham o‘rinli.

Endi ixtiyoriy $a_k$ va $b_k$ o‘zgarmas sonlarni topamiz. Buning uchun (9) formula bilan aniqlangan $u(x,t)$ funksiya (2) boshlang‘ich shartlarga qo‘yamiz.

(9) formulani $t$ bo‘yicha differensiallab ushbu

$$u_t = \sum_{k=1}^{\infty}\frac{k\pi a}{l}\left(-a_k\sin\frac{k\pi at}{l} + b_k\cos\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}, \qquad (10)$$

tenglikni olamiz. Endi (9) va (10) ifodalarda $t = 0$ deb, (2) boshlang'ich shartlarga asosan

$$\varphi_0(x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l}, \qquad \varphi_1(x) = \sum_{k=1}^{\infty} \frac{k\pi a}{l} b_k \sin\frac{k\pi x}{l} \tag{11}$$

ifodalarga ega bo'lamiz. Bu tengliklar $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalarning $(0, l)$ oraliqdagi sinuslar bo'yicha Fur'e qatoriga yoyilmalaridir.

U holda Fur'e qatorlari nazariyasiga asosan $a_k$ va $b_k$ koeffitsiyentlar quyidagi

$$a_k = \frac{2}{l}\int_0^l \varphi_0(x)\sin\frac{k\pi x}{l}dx, \tag{12}$$

$$b_k = \frac{2}{k\pi a}\int_0^l \varphi_0(x)\sin\frac{k\pi x}{l}dx \tag{13}$$

formulalar bilan aniqlanadi.

Shunday qilib, bir jinsli tor tebranish tenglamasi uchun aralash masalaning yechimi (9) qator ko'rinishida bo'lib, undagi $a_k$ va $b_k$ koeffitsientlar mos ravishda (12) va (13) formulalar orqali topiladi.

1–TEOREMA. Agar $\varphi_0(x)$ funksiya $[0, l]$ oraliqda uch marta uzluksiz differensiallanuvchi va quyidagi

$$\varphi_0(0) = \varphi_0(l) = 0, \qquad \varphi_0''(0) = \varphi_0''(l) = 0, \tag{14}$$

shartlarni qanoatlantirsa, $\varphi_1(x)$ funksiya esa $[0, l]$ oraliqda ikki marta uzluksiz differensiallanuvchi va

$$\varphi_1(0) = \varphi_1(l) = 0 \tag{15}$$

bo'lsa, u holda (9) formula bilan aniqlangan $u(x, t)$ funksiya yopiq $\overline{D}$ sohada ikki marta uzluksiz hosilalarga ega va $D$ sohada (1) bir jinsli tor tebranish tenglamasini hamda (2) boshlang'ich va (3) bir jinsli chegaraviy shartlarni qanoatlantiradi.

ISBOT. Teoremani isbotlash uchun avval (12) formula bilan aniqlangan integralni uch marta bo'laklab integrallaymiz. (14) tengliklarga asosan

$$a_k = -\frac{2}{l}\left(\frac{l}{\pi k}\right)^3 \int_0^l \varphi_0'''(x)\cos\frac{k\pi x}{l}\,dx = -\left(\frac{l}{\pi}\right)^3 \frac{p_k}{k^3}, \tag{16}$$

ifodani olamiz.

(13) formulaning o'ng tomonini xuddi shunday (15) tengliklarni inobatga olib, ikki matra bo'laklab integrallash natijasida

$$b_k = -\frac{2}{la}\left(\frac{l}{\pi k}\right)^3 \int_0^l \varphi_1''(x)\sin\frac{k\pi x}{l}\,dx = -\left(\frac{l}{\pi}\right)^3 \frac{q_k}{k^3}, \tag{17}$$

tenglikka ega bo'lamiz. Bu yerda

$$p_k = \frac{2}{l}\int_0^l \varphi_0'''(x)\cos\frac{k\pi x}{l}\,dx; \quad q_k = \frac{2}{la}\int_0^l \varphi_1''(x)\sin\frac{k\pi x}{l}\,dx.$$

Boshlang'ich shartlarga ko'ra $\varphi_0'''(x)$, $\varphi_1''(x)$ funksiyalar $[0, l]$ segmentda uzluksiz, u holda matematik analiz kursidan ma'lum bo'lgan Bessel tengsizligiga ko'ra ushbu qatorlar

$$\sum_{k=1}^{\infty} p_k^2 \leq \frac{2}{l}\int_0^l [\varphi_0'''(x)]^2\,dx; \quad \sum_{k=1}^{\infty} q_k \leq \frac{2}{l}\int_0^l \left[\frac{\varphi_1''(x)}{a}\right]^2 dx. \tag{18}$$

yaqinlashuvchi qatorlar bo'ladi.

Endi (16) va (17) ifodalarni (9) qatorga qo'yib, quyidagi

$$u(x,t) = -\left(\frac{l}{\pi}\right)^3 \sum_{k=1}^{\infty} \frac{1}{k^3}\left(p_k\cos\frac{k\pi at}{l} + q_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}, \tag{19}$$

qatorni olamiz.

Hosil bo'lgan (19) qatorning har bir hadi yopiq $\overline{D}$ sohaning $\forall\,(x,t)$ nuqtasida ushbu yaqinlashuvchi

$$\left(\frac{l}{\pi}\right)^3 \sum_{k=1}^{\infty} \frac{1}{k^3}\left(|p_k| + |q_k|\right),$$

sonli qatorning hadlari bilan chegaralangan. U holda Veyershtrass alomatiga ko'ra (9) qator yopiq $\overline{D}$ sohada absolyut va tekis yaqinlashadi. Demak, $u(x,t)$ funksiya yopiq $\overline{D}$ sohada yaqinlashuvchi qatorning yig'indisi sifatida uzluksiz bo'ladi.

Endi (9) qatorni $x$ va $t$ o'zgaruvchilar bo'yicha ikki marta formal ravishda hadma–had differensiallash mumkin ekanligini ko'rsatamiz. Buning uchun (9) qatorni hadma–had differensiallashdan hosil bo'lgan qatorlarning yopiq $\overline{D}$ sohada absolyut va tekis yaqinlashishini isbotlaymiz. (19) qatorni hadma–had differensiallab, ushbu

$$u_{xx}(x,t) = \frac{l}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(p_k \cos\frac{k\pi at}{l} + q_k \sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}, \qquad (20)$$

$$u_{tt}(x,t) = \frac{la^2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(p_k \cos\frac{k\pi at}{l} + q_k \sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}, \qquad (21)$$

qatorlarga ega bo'lamiz. Bu qatorlar $\forall\,(x,t) \in \overline{D}$ sohada

$$\frac{ml}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(|p_k| + |q_k|\right), \qquad m = \max\{1, a^2\}, \qquad (22)$$

qatorga majorant bo'ladi. Oxirgi (22) qatorning yaqinlashuvchi bo'lishi (18) qatorlarning yaqinlashishidan va quyidagi

$$\frac{1}{k}\,|p_k| \le \frac{1}{2}\left(\frac{1}{k^2} + p_k^2\right), \qquad \frac{1}{k}\,|q_k| \le \frac{1}{2}\left(\frac{1}{k^2} + q_k^2\right)$$

tengsizliklardan kelib chiqadi. U holda (20) va (2) qatorlar Veyershtrass alomatiga asosan yopiq $\overline{D}$ sohada absolyut va tekis yaqinlashuvi bo'ladi.

Demak, $u_{xx}(x,t)$ va $u_{tt}(x,t)$ funksiyalar yopiq $\overline{D}$ sohada uzluksiz ekan. Endi (20) va (21) qatorlarni ($1^0$) tenglamaga qo'ysak, (9) formula bilan aniqlangan $u(x,t)$ funksiya tor tebranish tenglamasini qanoatlantirishiga ishonch hosil qilish mumkin. Shunday qilib, 1–teorema isbot bo'ldi.

YECHIMNING FIZIKAVIY TALQINI. Tor tebranish tenglamasi uchun aralash masalaning (9) ko'rinishda topilgan yechimini qaraylik. Agar bu formulada

$$a_k = A_k \sin\varphi_k, \quad b_k = B_k \cos\varphi_k$$

belgilashlarni kiritsak, u holda $u_k(x,t)$ xususiy yechimlarni quyidagi

$$u_k(x,t) = A_k \sin\frac{\pi k x}{l} \sin\left(\frac{k\pi a t}{l} + \varphi_k\right), \tag{23}$$

ko'rinishda yozish mumkin. Bunda torning har bir nuqtasi bir xil $\varphi_k$ fazali, amplitudasi $A_k \sin\dfrac{k\pi x}{l}$ ga teng bo'lgan $\omega_k = \dfrac{k\pi}{l}$ chastotali garmonik tebranadi.

Tor harakatining (23) ko'rinishdagi garmonik tebranishlari *to'g'ri to'lqin* deyiladi, ya'ni (34) qatorning har bir hadi to'g'ri to'lqin deb ataladi. Bundan esa (20) ko'rinishdagi yechim cheksiz ko'p to'g'ri to'lqinlarning yig'indisini ifodalaydi.

Tor tebranish jarayonida o'zidan tovush chiqaradi. Tovushlar ikki xil – musiqiy va musiqiy bo'lmagan turlarga ajraladi. Musiqiy tovushlarni notalar, musiqiy bo'maganlarni esa *shovqinlar* deb yuritiladi. Musiqiy tovushlar ma'lum ma'noda yuqori darajada sifatli bo'lib, uni har bir kishi o'zining imkoniyati darajasida baholay oladi.

Balandligi bo'yicha kishining qulog'i ajrata olmaydigan notalar *tonlar* deyiladi. Tovushning balandligi tebranish chastotasiga bog'liq bo'ladi. Eng past tonning chastotasi *asosiy chastota* deyiladi va u

$$\omega_1 = \frac{a\pi}{l} = \frac{\pi}{l}\sqrt{\frac{T_0}{\rho}}$$

formula bilan aniqlanadi.

Asosiy chastotadan balandroq bo'lgan chastotali tonlar *obertonlar* deyiladi va obertonlar asosiy $\omega_1$ chastotaga karrali bo'lib, ular *garmonikalar* deyiladi. Birinchi garmonika deb asosiy ton hisoblanadi, ikkinchi garmonika esa chastotasi $\omega_2 = 2\omega_1$ bo'lgan tonlar va hokazo.

Demak, (9) formula bilan aniqlangan yechim alohida garmonikalarnin yig'indisi bo'lib, uning amplitudasi garmonikalarning nomeri ortib borgan sari nolga intiladi, ya'ni $k \to \infty$ koeffitsientlar $a_k \to 0$ va $b_k \to 0$ intiladi. Shu sababli garmonikaning tordan tarqalayotgan tovushga ta'siri tovush tembrini hosil qiladi.

Ushbu $x = 0, l/k, (2l)/k, \ldots, (k-1)\,l/k, l$ nuqtalarda tebranish amplitudasining $k$ – garmonikasi nolga intiladi, chunki bu nuqtalarda $\sin\dfrac{k\pi}{l}x = 0$ bo'ladi. Bu nuqtalar $k$ – garmonikaning *tugunlari* deyiladi va torning tebranish jarayonida bu tugunlar qo'zg'almaydi. $x_m = \dfrac{(2m+1)\,l}{2k}$, $(m = \overline{0, k-1})$ nuqtalarda $\sin\dfrac{k\pi x}{l} = \pm 1$ bo'lgani uchun tor maksimal $A_k$ amplitudali garmonik tebranadi va bu *to'g'ri to'lqinlar* deyiladi.

Quyidagi

$$\omega_1 = \frac{a\pi}{l} = \frac{\pi}{l}\sqrt{\frac{T_0}{\rho}} \qquad \text{yoki} \qquad T = \frac{2\pi}{\omega_1} = 2l\sqrt{\frac{\rho}{T_0}}$$

formulalar mos ravishda asosiy tonning chastotasini va davrini aniqlaydi. Bu formulalar yordamida tor tebranish qoidasini asoslashimiz mumkin:

1) Zichligi va tarangligi o'zgarmas bo'lgan torning tebranish davri uning uzunligiga to'g'ri proporsional bo'ladi.

2) Ma'lum uzunlikdagi torning tebranish davri $T_0$ taranglikning kvadrat ildiziga teskari proporsional ravishda o'zgaradi.

3) Uzunligi va tarangligi ma'lum bo'lgan torning tebranish davri tor zichligining kvadrat ildiziga to'g'ri proporsional o'zgaradi.

## Bir jinsli torning majburiy tebranishi

Uchlari mahkamlangan bir jinsli torning tashqi kuchlar ta'siridagi majburiy tebranishi quyidagi aralash masalaga ekvivalent keltiriladi

Bir jinsli bo'lmagan

$$u_{tt} - a^2 u_{xx} = g(x,t), \qquad \forall\,(x,t) \in D; \tag{24}$$

tor tebranish tenglamasining (2) boshlang'ich va bir jinsli (3) chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

Bu yerda $g(x,t) = f(x,t)/\rho \in C^2(\overline{D})$ torga ta'sir qiluvchi tashqi kuchlar yigindisi.

Bu masalaning $u(x,t)$ yechimini quyidagi

$$u(x,t) = v(x,t) + w(x,t) \tag{25}$$

ko'rinishda qidiramiz. Bu yerda $v(x,t)$ funksiya bir jinsli bo'lmagan

$$v_{tt} = a^2 v_{xx} + g(x,t) \tag{26}$$

tor tebranish tenglamaning bir jinsli boshlang'ich

$$v|_{t=0} = 0, \qquad v_t|_{t=0} = 0 \tag{27}$$

va chegaraviy

$$v|_{x=0} = 0, \qquad v|_{x=l} = 0 \tag{28}$$

shartlarni qanoatlantiruvchi yechimi;

$w(x,t)$ funksiya esa bir jinsli

$$w_{tt} = a^2 w_{xx} \tag{29}$$

tenglamaning quyidagi boshlang'ich

$$w|_{t=0} = \varphi_0(x), \qquad w_t|_{t=0} = \varphi_1(x) \tag{30}$$

va chegaraviy

$$w|_{x=0} = 0, \qquad w|_{x=l} = 0 \tag{31}$$

shartlarni qanoatlantiruvchi yechimidan iborat.

Yuqoridagi masalalardan ko'rinib turibdiki, $v(x,t)$ funksiya uchlari mahkamlangan bir jinsli torning $g(x,t)$ tashqi kuchlar ta'siridagi majburiy tebranishini, $w(x,t)$ esa shu torning erkin tebranishini ifodalaydi. $w(x,t)$ funksiyaga nisbatan (29)–(31) masala yechimining mavjudligini oldingi punktda isbot qildik.

Shuning uchun bu yerda (26)–(28) masalaning $v(x,t)$ yechimini qurish etarlidir.

Bu masalaning $v(x,t)$ yechimini quyidagi qator

$$v(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin\frac{k\pi x}{l}, \tag{32}$$

ko'rinishda izlaymiz. Bu yerda $T_k(t)$ hozircha noma'lum funksiya. (32) qatorni yopiq $\overline{D}$ sohada yaqinlashuvchi va shu sohada $x$ va $t$ o'zgaruvchilar bo'yicha hadma-had differensiallash mumkin bo'lsin. U holda (32) qator bir jinsli chegaraviy shartlarni qanoatlantiradi.

Bu qatorni (27) boshlang'ich shartlarga qo'yib, $T_k(t)$ funksiyalar uchun quyidagi

$$T_k(0) = 0, \qquad T_k^{'}(0) = 0, \quad k = 1, 2, 3, \ldots \tag{33}$$

boshlang'ich shartlarni olamiz.

Endi $g(x,t)$ funksiyani $[0,l]$ segmentda $x$ o'zgaruvchiga nisbatan sinuslar bo'yicha Fur'e qatoriga yoyilsin.

$$g(x,t) = \sum_{k=1}^{\infty} g_k(t) \sin\frac{k\pi x}{l}, \tag{34}$$

bunda $f_k$ koeffitsientlar quyidagi

$$g_k(t) = \frac{2}{l}\int_0^l g(x,t) \sin\frac{k\pi x}{l} dx \tag{35}$$

formula bilan aniqlanadi.

$T_k(t)$ funksiyalarni topish uchun (32) va (34) ifodalarni (26) tenglama qo'yib, ushbu

$$\sum_{k=1}^{\infty} \left[T_k''(t) + w_k^2 T_k(t)\right] \sin\frac{k\pi x}{l} = g(x,t), \tag{36}$$

tenglamaga ega bo'lamiz. Bu yerda $\omega_k = \frac{k\pi}{l}$.

(36) va (34) yoyilmalarni o'zaro taqqoslab, $k$ ning har bir qiymatida chiziqli o'zgarmas koeffitsiyentli

$$T_k''(t) + w_k^2 T_k(t) = g_k(t), \qquad 0 \le t \le T \tag{37}$$

oddiy differensial tenglamalarga ega bo'lamiz.

Demak, $T_k(t)$ funksiyalarni topish uchun ikkinchi tartibli (37) oddiy differensial tenglamani (44) boshlang'ich shartlarni oldik. Bu masalaning yechimini o'zgarmasni variatsiyalash usuli yordamida topamiz. Bir jinsli (37) tenglamaning umumiy yechimi

$$T_k(t) = c_1 \cos\omega_k t + c_2 \sin\omega_k t,$$

bu yerda $c_1$, $c_2$ – ixtiyoriy o'zgarmaslar.

Endi (37) tenglamaning umumiy yechimini

$$T_k(t) = c_1(t)\cos\omega t + c_2(t)\sin\omega t, \tag{38}$$

ko'rinishda izlaymiz.

Differensial tenglamalar kursidan ma'lumki, $c_1'(t)$ va $c_2'(t)$ noma'lum funksiyalarga nisbatan quyidagi

$$\begin{cases} c_1'(t)\cos\omega_k t + c_2'(t)\sin\omega_k t = 0, \\ -c_1'(t)\omega_k \sin\omega_k t + c_2'(t)\omega_k\cos\omega_k t = g_k(t), \end{cases}$$

tenglamalar sistemasiga ega bo'lamiz. Bundan $c_1'(t)$ va $c_2'(t)$ funksiyalarni

$$c_1'(t) = -\frac{g_k(t)}{\omega_k}\sin\omega_k t, \quad c_2'(t) = \frac{g_k(t)}{\omega_k}\cos\omega_k t,$$

ko'rinishda topamiz va bu tenglamalarni integrallab, $c_1(t)$ va $c_2(t)$ funksiyalarni

$$c_1(t) = -\frac{1}{\omega_k}\int_0^t g_k(\tau)\sin\omega_k\tau\, d\tau + c_1^0,$$

$$c_2(t) = \frac{1}{\omega_k}\int\limits_0^t g_k(\tau)\cos\omega_k\tau d\tau + c_2^0,$$

ko'rinishda aniqlaymiz. Bunda $c_1^0$ va $c_2^0$ – ixtiyoriy o'zgarmaslar.

Topilgan $c_1(t)$ va $c_2(t)$ funksiyalarni (38) formulaga qo'yib, (37) tenglamaning umumiy yechimini

$$T_k(t) = \frac{1}{\omega_k}\int\limits_0^t g_k(\tau)\sin[\omega_k(t-\tau)]d\tau + c_1^0\cos\omega_k t + c_2^0\sin\omega_k t, \quad (39)$$

ko'rinishda topamiz. (33) boshlang'ich shartlarni qanoatlantirib, (39) umumiy yechimdan $c_1^0 = c_2^0 = 0$ ekanligini olamiz.

Agar $g_k(t) \in C[0,T]$ bo'lsa, u holda (37) tenglamaning (33) boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$T_k(t) = \frac{1}{\omega_k}\int\limits_0^t g_k(\tau)\sin[\omega_k(t-\tau)]d\tau, \quad (40)$$

formula bilan aniqlanadi.

Endi (32) qatorning yopiq $\overline{D}$ sohada tekis yaqinlashuvchi ekanligini hamda $x$ va $t$ argumentlari bo'yicha ikki marta differensiallash mumkinligini ko'rsataylik.

Agar $g(x,t)$ funksiya yopiq $\overline{D}$ sohada uzluksiz, shu sohada $x$ o'zgaruvchi bo'yicha uzluksiz ikki marta differensiallanuvchi va $\forall t \in [0,T]$ uchun $g(0,t) = g(l,t) = 0$ bo'lsa, u holda (35) ifodani ikki marta bo'laklab integrallaymiz, natijada

$$g_k(t) = -\frac{2}{l}\left(\frac{l}{\pi k}\right)^2\int\limits_0^l g''_{xx}(x,t)\sin\frac{k\pi x}{l}dx = -\left(\frac{l}{\pi}\right)^2\frac{a_k(t)}{k^2}, \quad (41)$$

ifodani olamiz, bunda yerda

$$a_k(t) = \frac{2}{l}\int\limits_0^l g''_{xx}(x,t)\sin\frac{k\pi x}{l}dx.$$

Uzluksiz funksiyalarning kvadratidan tuzilgan $a_k(t)$ funksional qator

$$\sum_{k=1}^{\infty} a_k^2(t) < \infty, \quad \forall t \in [0,T]. \tag{42}$$

Bessel tengsizligiga asosan yaqinlashuvchi qator bo'ladi. Endi (41) ifodani (40) formulaga qo'yamiz va $T_k(t)$ ushbu

$$T_k(t) = -\left(\frac{l}{\pi}\right)^3 \frac{1}{a} \int_0^t \frac{a_k(t)}{k^3} \sin \omega_k (t-\tau)\, d\tau,$$

ko'rinishda yoziladi. Hosil bo'lgan oxirgi ifodani esa (32) formulaga qo'ysak,

$$v(x,t) = -\left(\frac{l}{\pi}\right)^3 \frac{1}{a} \sum_{k=1}^{\infty} \frac{1}{k^3} \int_0^t a_k(\tau) \sin[\omega_k(t-\tau)] d\tau \sin \frac{k\pi x}{l}, \tag{43}$$

ifodaga ega bo'lamiz. Bu qatorning har bir hadi $\forall t \in [0,T]$ bo'lganda ushbu

$$\left(\frac{l}{\pi}\right)^3 \frac{T}{a} \sum_{k=1}^{\infty} \frac{|a_k(t_0)|}{k^3}$$

sonli qatorning har bir hadi bilan chegaralangan.

Bu yerda $|a_k(t_0)| = \max\limits_{0 \le t \le T} |a_k(t)|$, $t_0$ biror fiksirlangan nuqta.

Shuning uchun (43) qator $\overline{D}$ sohada absolyut va tekis yaqinlashuvchi qator bo'ladi.

Endi (43) qatorni hadma-had ikki marta $x$ va $t$ o'zgaruvchilari bo'yicha differensiallaymiz, natijada ushbu

$$v_{xx} = \frac{l}{a\pi} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^t a_k(\tau) \sin[\omega_k(t-\tau)] d\tau \sin \frac{k\pi x}{l}, \tag{44}$$

$$v_{tt} = -\left(\frac{l}{\pi}\right)^2 \sum_{k=1}^{\infty} \frac{1}{k^2} a_k(t) \tau \sin \frac{k\pi x}{l} +$$

$$+\frac{la}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^t a_k(\tau)\sin[\omega_k(t-\tau)]d\tau\sin\frac{k\pi x}{l}, \qquad (45)$$

qatorlarni olamiz. Ma'lumki, bu qatorlar $(x,t)\in\overline{D}$ da quyidagi qatorlarga

$$\frac{l}{a\pi}T\sum_{k=1}^{\infty}\frac{|a_k(t_0)|}{\pi}, \quad \left(\frac{l}{\pi}\right)^2 T\sum_{k=1}^{\infty}\frac{|a_k(t_0)|}{k^2}+\frac{la}{\pi}T\sum_{k=1}^{\infty}\frac{|a_k(t_0)|}{k}$$

majorantlanadi. Bu qatorlarning yaqinlashishi (52) qatorning yaqinlashishidan va

$$2\frac{|a_k(t_0)|}{k}\leq\frac{1}{k^2}+a_k^2(t_0)$$

tengsizlikdan kelib chiqadi.

U holda (44) va (45) qatorlar $(x,t)\in\overline{D}$ sohada absolyut va tekis yaqinlashuvchi bo'ladi, bundan esa $\overline{D}$ da $v_{xx}(x,t)$ va $v_{tt}(x,t)$ hosilalarning uzluksiz ekanligi kelib chiqadi.

Endi (44) va (45) ifodalarni (26) tenglamaga qo'ysak, (32) formula bilan aniqlangan $v(x,t)$ funksiya bir jinsli bo'lmagan tor tebranish tenglamasini qanoatlantirishiga ishonch hosil qilish mumkin.

Shunday qilib, quyidagi teoremani isbotladik:

2–TEOREMA. Agar $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalar 1–teorema shartlarini qanoatlantirsa va $g(x,t)$ funksiya yopiq $\overline{D}$ sohada uzluksiz, shu sohada ikkinchi tartibli uzluksiz hosilalarga ega bo'lib, $g(0,t)=0$, $g(l,t)=0$ tengliklar o'rinli bo'lsa, u holda $\{(24),(2),(3)\}$ masalaning yagona yechimi mavjud va bu yechim

$$u(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin\frac{k\pi x}{l}+$$

$$+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l}+b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l};$$

formula bilan aniqlanadi. Bu yerda

$$T_k(t) = \frac{2}{k\pi}\int_0^t \sin\frac{k\pi}{l}(t-\tau)\,d\tau \int_0^l g(x,\tau)\sin\frac{k\pi x}{l}\,dx;$$

$$a_k = \frac{2}{l}\int_0^l \varphi_0(x)\sin\frac{k\pi x}{l}\,dx;$$

$$b_k = \frac{2}{k\pi a}\int_0^l \varphi_0(x)\sin\frac{k\pi x}{l}\,dx.$$

Demak, $v(x,t)$ funksiya (32) qator ko'rinishida uning koeffitsiyentlari (35), (39) formulalar orqali, $w(x,t)$ funksiya esa (9) ko'rinishda va uning koeffitsientlari (12), (13) formulalar bilan aniqlanadi.

### Uchlari qo'zg'aluvchan torning majburiy tebranishi

Torning uchlari mahkamlanmagan bo'lib, ular biror qoida asosida harakatlansin va tor biror tashqi kuch ta'sirida tebranayotgan bo'lsin. U holda bu masala bir jinsli bo'lmagan

$$u_{tt} = a^2 u_{xx} + g(x,t), \quad a^2 = T_0/\rho, \tag{46}$$

tor tebranish tenglamasining quyidagi boshlang'ich

$$u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x), \quad 0 \le x \le l, \tag{47}$$

va chegaraviy

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad 0 \le t \le T, \tag{48}$$

shartlarni qanoatlantiruvchi yechimini topish masalasiga ekvivalent bo'ladi.

Bu yerda $g(x,t) = \dfrac{f(x,t)}{\rho}$, $\mu_1(t)$, $\mu_2(t)$ $\varphi_0(x)$ va $\varphi_1(x)$ – berilgan funksiyalar.

Qaralayotgan umumiy (46)–(48) masala yechimining mavjudligini bir jinsli chegaraviy shartli masalaga keltirib isbotlash mumkin.

Buning uchun $\mu_1(t)$ va $\mu_2(t)$ funksiyalarni $C^2[0,T]$ sinfdan deb talab qilamiz. U holda (48) chegaraviy shartlarni qanoatlantiruvchi quyidagi

$$z(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)], \tag{49}$$

yordamchi funksiyani kiritamiz, ya'ni

$$z(0,t) = \mu_1(t), \qquad z(l,t) = \mu_2(t).$$

Endi (46) tenglamaning (47) boshlang'ich va (48) chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini

$$u(x,t) = v(x,t) + z(x,t), \tag{50}$$

ko'rinishda izlaymiz, bu yerda $v(x,t)$ yangi noma'lum funksiya. Boshlang'ich va chegaraviy shartlarga asosan $v(x,t)$ funksiya uchun quyidagi bir jinsli chegaraviy

$$v|_{x=0} = u(x,t)|_{x=0} - z(x,t)|_{x=0} = \mu_1(t) - \mu_1(t) = 0,$$

$$v|_{x=l} = u(x,t)|_{x=l} - z(x,t)|_{x=l} = \mu_2(t) - \mu_2(t) = 0,$$

va boshlang'ich

$$v|_{t=0} = u|_{t=0} - z|_{t=0} = \varphi_0(x) - \mu_1(0) - [\mu_2(0) - \mu_1(0)]\frac{x}{2} = \bar{\varphi}_0(x)$$

$$v_t|_{t=0} = u_t|_{t=0} - z_t|_{t=0} = \varphi_1(x) - \mu_1'(0) - [\mu_2'(0) - \mu_1'(0)]\frac{x}{2} = \bar{\varphi}_1(x)$$

shartlarga ega bo'lamiz.

(50) tenglikka ko'ra yangi noma'lum $v(x,t)$ funksiyaga nisbatan ushbu

$$v_{tt} - a^2 v_{xx} = (u - z)_{tt} - a^2(u - z)_{xx} =$$

$$= u_{tt} - a^2 u_{xx} - (z_{tt} - a^2 z_{xx}) =$$

$$= g(x,t) - \mu_1''(t) - [\mu_2''(t) - \mu_1''(t)]\frac{x}{l} = \bar{g}(x,t)$$

yoki

$$v_{tt} = a^2 v_{xx} + \bar{g}(x,t)$$

tenglamani olamiz. Bu yerda

$$\bar{g}(x,t) = g(x,t) - \mu_1''(t) - [\mu_2''(t) - \mu_1''(t)]\frac{x}{l}.$$

Shunday qilib, biz $v(x,t)$ funksiyani topish uchun quyidagi masalaga keldik: Bir jinsli bo'lmagan

$$v_{tt} = a^2 v_{xx} + \bar{g}(x,t), \tag{51}$$

tor tebranish tenglamasining quyidagi boshlang'ich

$$v(x,t)|_{t=0} = \bar{\varphi}_0(x), \qquad v_t(x,t)|_{t=0} = \bar{\varphi}_1(x), \tag{52}$$

va bir jinsli chegaraviy

$$v(x,t)|_{x=0} = 0, \qquad v(x,t)|_{x=l} = 0, \tag{53}$$

shartlarni qanoatlantiruvchi $v(x,t)$ yechimini toping.

Bu masalaning yechimini oldingi bosqichda batafsil o'rgandik. Agar $\bar{\varphi}_0(x)$, $\bar{\varphi}_1(x)$ va $\bar{g}(x,t)$ funksiyalar 2-teorema shartlarini qanoatlantirsa, u holda (51)–(53) masalaning $C^2(\overline{D})$ sinfga tegishli bo'lgan $v(x,t)$ yechimi mavjud va yagona bo'ladi.

Shunday qilib, (46)–(48 aralash masala yechimining mavjud va yagonaligi haqidagi ushbu teorema o'rinli:

3-TEOREMA. Agar berilgan funksiyalar

$$\varphi_0(x) \in C^3[0,l], \ \varphi_1(x) \in C^2[0,l]; \quad \mu_1(t), \ \mu_2(t) \in C^2[0,T];$$

$$g(x,t), \ \ g_x(x,t), \ \ g_{xx}(x,t) \in C(\overline{D})$$

bo'lib, ular uchun quyidagi tengliklar

$$\varphi_0(0) = \mu_1(0), \ \ \varphi_0(l) = \mu_2(0), \ \ \varphi_0''(0) = \varphi_0''(l) = 0,$$

$$\varphi_1(0) = \mu_1'(0), \ \ \varphi_1(l) = \mu_2'(0), \ \ g(0,t) = \mu_1''(t), \ \ g(l,t) = \mu_2''(t)$$

o'rinli bo'lsa, u holda (46)–(48) aralash masalaning yagona yechimi mavjud bo'ladi.

Endi ikkinch aralash masalani Fur'e usuli bilan yechaylik.

1-MASALA. To'g'ri to'rtburchakli $D$ sohada

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, l), \quad t > 0. \tag{54}$$

tenglamaning quyidagi

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x), \qquad 0 \le x \le l, \tag{55}$$

boshlang'ich va

$$u_x(0, t) = 0, \qquad u_x(l, t) = 0, \qquad 0 \le t \le T, \tag{56}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

YECHISH. Berilgan bir jinsli tor tebranish tenglamasining $u_x(0, t) = u_x(l, t) = 0$ chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini $u(x, t) = X(x)T(t)$ ko'rinishda izlaymiz. Bundan $X(x)$ funksiya uchun ushbu

$$X'(0) = 0, \qquad X'(l) = 0, \tag{57}$$

chegaraviy shartlarni olamiz.

Endi $u(x, t)$ ko'paytmani (54) tenglamaga qo'ysak.

$$X(x)T''(t) = a^2 X''(x)T(t)$$

tenglikka ega bo'lamiz.

Oxirgi tenglikni $a^2 X(x)T(t) \neq 0$ ifodaga bo'lib,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

ifodani olamiz. Bundan esa $X(x)$ funksiyaga nisbatan

$$X''(x) + \lambda X(x) = 0, \tag{58a}$$

$$X'(0) = 0, \; X'(l) = 0, \tag{58b}$$

Shturm–Liuvill masalaga, $T(t)$ funksiyaga nisbatan esa

$$T''(t) + a^2\lambda T(t) = 0, \quad t > 0. \tag{59}$$

tenglamaga ega bo'lamiz.

(58$a$) tenglamaning umumiy yechimi quyidagi

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}, \quad \text{agar} \quad \lambda < 0;$$
$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \quad \text{agar} \quad \lambda > 0;$$
$$X(x) = c_1 x + c_2, \quad \text{agar} \quad \lambda = 0,$$

ko'rinishda bo'ladi.

Agar $\lambda < 0$ bo'lsa, u holda $X(x) \equiv 0$ bo'lishini ko'rsatish qiyin emas.

Agar $\lambda > 0$ bo'lsa, u holda yuqoridagi umumiy yechimdan

$$X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

bo'ladi. (58$b$) chegaraviy shartlarga asosan $c_2 = 0$ yoki $X(x) = c_1\cos\sqrt{\lambda}x = 0$ kelib chiqadi. Bundan $X'(x) = -c_1\lambda\cos\sqrt{\lambda}x = 0$ bo'ladi va $X'(l) = 0$ chegaraviy shartga ko'ra $\sqrt{\lambda}l = k\pi$ yoki Shturm Liuvill masalasi cheksiz ko'p

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \quad k = 0, 1, 2, \ldots, \tag{60}$$

xos qiymatlarga ega ekanligi kelib chiqadi. Bularga mos xos funksiyalar

$$X_k(x) = \cos\frac{k\pi}{l}x, \quad k = 0, 1, 2, \ldots, \tag{61}$$

bo'ladi.

Agar $\lambda = 0$ bo'lsa, u holda (58$a$) tenglamaning umumiy yechimidan yuqoridagi kabi $c_1 = 0$ va $X(x) = c_2$ ekanligi kelib chiqadi, bundan esa $X'(l) = 0$ chegaraviy shart aynan bajariladi. Demak (58) Shturm–Liuvill masalasi uchun $\lambda = 0$ xos qiymat va unga mos xos funksiya $X_0(x) = 1$ bo'ladi, ya'ni

$$\lambda_0 = 0, \quad X_0(x) = 1.$$

(58) masalaning $\lambda_k$ xos sonlarini (60) va xos funksiyalarini esa $k = 0$ bo'lganda (61) ko'rininshda

$$\lambda_0 = \left(\frac{\pi 0}{l}\right)^2 = 0, \qquad X_0(x) = \cos\frac{\pi 0}{l}x = 1$$

yozish mumkin.

Demak, (58) Shturm–Liuvill masalasi uchun

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2, \qquad X_k(x) = \cos\frac{k\pi}{l}x, \quad k = 0, 1, 2, \ldots$$

xos qiymat va xos funksiyalarga ega bo'ldik.

Endi (59) tenglamani qaraylik. Bu tenglama $\lambda = \lambda_k$ bo'lganda ham ma'noga ega va

$$T_k''(t) + a^2\lambda_k T_k(t) = 0, \quad t > 0, \tag{62}$$

tenglamani qaraymiz. Agar $k = 0$ bo'lsa, oxirgi tenglamaning umumiy yechimi

$$T_0(t) = A_0 + B_0 t$$

bo'ladi, bu yerda $A_0$, $B_0$ – ixtiyoriy o'zgarmaslar.

Agar $k > 0$ bo'lsa, (62) tenglamaning umumiy yechimi

$$T_k(t) = A_k\cos\left(\frac{ka\pi}{l}\right)t + B_k\sin\left(\frac{ka\pi}{l}\right)t, \quad t > 0 \tag{63}$$

ko'rinishda bo'ladi, bunda $A_k$ va $B_k$ – ixtiyoriy o'zgarmaslar.

Endi qaralayotgan (54)–(56) aralash masalaning yechimini

$$u(x,t) = \sum_{k=0}^{\infty} X_k(x)T_k(t)$$

ko'rinishda izlaymiz, ya'ni

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty}\left[A_k\cos\left(\frac{ka\pi}{l}\right)t + B_k\sin\left(\frac{ka\pi}{l}\right)t\right]\cos\left(\frac{k\pi}{l}\right)x, \tag{64}$$

Qaralayotgan masalaning boshlang'ich shartlariga ko'ra

$$\varphi(x) = \sum_{k=0}^{\infty} X_k(x)T_k(0) = A_0 + \sum_{k=1}^{\infty} A_k X_k(x), \tag{65}$$

$$\psi(x) = \sum_{k=0}^{\infty} X_k(x)T_k'(0) = B_0 + \sum_{k=1}^{\infty} \frac{ka\pi}{l} B_k X_k(x), \tag{66}$$

bo'ladi.

Faraz qilaylik, $\varphi(x)$ va $\psi(x)$ funksiyalar kosinuslar bo'yicha Fur'e qatoriga yoyilsin, ya'ni

$$\varphi(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{k\pi x}{l}\right), \quad \psi(x) = \frac{\beta_0}{2} + \sum_{k=1}^{\infty} \beta_k \cos\left(\frac{k\pi x}{l}\right).$$

Bu yerda $\alpha_k$ va $\beta_k$ koeffitsiyentlar

$$\alpha_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad \beta_k = \frac{2}{k\pi a}\int_0^l \psi(x)\cos\left(\frac{k\pi x}{l}\right)dx,$$

ko'rinishda aniqlanadi.

Shunday qilib, Fur'e qatorlari uchun standart formulalardan foydalanib, (64) formuladagi $A_k$ va $B_k$ koeffitsiyentlar uchun quyidagi

$$A_k = \alpha_k = \frac{2}{l}\int_0^l \varphi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad k > 0,$$

$$B_k = \frac{l}{k\pi a}\beta_k = \frac{2}{k\pi a}\int_0^l \psi(x)\cos\left(\frac{k\pi x}{l}\right)dx, \quad k > 0,$$

$$A_0 = \frac{\alpha_0}{2} = \frac{1}{l}\int_0^l \varphi(x)dx, \quad B_0 = \frac{\beta_0}{2} = \frac{1}{k\pi a}\int_0^l \psi(x)dx,$$

formulalarni olamiz.

Endi topilgan $A_k$ va $B_k$ koeffitsiyentlarni (64) formulaga qo'yib, (54)–(56) aralash masalaning $u(x,t)$ yechimini hosil qilamiz.

## 11–§. Tor tebranish tenglamasi uchun Gursa va Darbu masalalari

Bu paragrafda tor tebranish tenglamasi uchun Gursa masalasini o'rganamiz. Bu masala fizikaviy jihatdan ham qiziqarli bo'lib, u ko'plab tadbiqiy masalalarda, ayniqsa gaz tozalash, quritish jarayonlari bilan bog'liq masalalarni o'rganishda uchraydi. Bu masalada chegaraviy shartlar tenglamaning xarakteristikalarida berilgani uchun ham *Gursa masalasi chegaraviy masala* deb yuritiladi.

Gursa masalasi chegaraviy masala bo'lib, chegaraviy shartlar tenglamaning bir nuqtadan chiquvchi xarakteristikalarida beriladi. $(x, t)$ o'zgaruvchilar tekisligida bir jinsli ushbu

$$\Box u(x,t) \equiv u_{tt} - a^2 u_{xx} = 0, \tag{1}$$

tor tebranish tenglamasini qaraylik. Umumiylikka ziyon qilmasdan (1) tenglamada $a = 1$ deb olish mumkin. Haqiqatdan ham, $(x, t)$ tekisligida quyidagicha yangi $x = x$, $y = at$ o'zgaruvchilar kiritamiz. U holda (1) tenglamada qatnashgan hosilalarni hisoblaymiz:

$$u_{tt} = u_{yy} y_t^2 + u_y y_{tt} = a^2 u_{yy} \text{ va } u_{tt} - a^2 u_{xx} = a^2 u_{yy} - a^2 u_{xx} = 0$$

Bundan esa (1) tenglamani

$$\Box u(x,y) \equiv u_{xx} - u_{yy} = 0, \tag{2}$$

ko'rinishda yozib olishimiz mumkin.

Ma'lumki, (2) tenglama ikkita haqiqiy xarakteristikalar

$$x - y = const \quad \text{va} \quad x + y = const$$

oilasiga ega. Berilgan (2) tenglamani xarakteristik to'rtburchakda qaraylik, ya'ni (2) tenglamaning $AC_1$, $C_1B$, $BC_2$ va $C_2A$ xarakteristikalari bilan chegaralangan sohani $G$ deb, belgilaylik.

Faraz qilaylik, $A = (0,0)$, $C_1 = (x_1, x_1)$, $C_2 = (x_2, -x_2)$, bo'lsin. Bu yerda $x_1 > 0$, $x_2 > 0$.

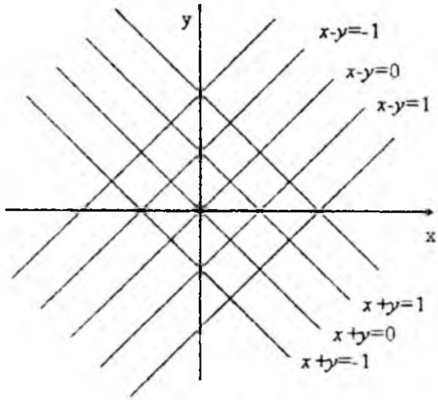
Gursa masalasi. (2) tenglamaning yopiq $G$ sohada aniqlangan, uzluksiz va quyidagi

$$u(x,y)|_{AC_1} = u(x,y)|_{y=x} = u(x,x) = \psi_1(x), \quad 0 \le x \le x_1, \tag{3}$$

$$u(x,y)|_{AC_2} = u(x,y)|_{y=-x} = u(x,-x) = \psi_2(x), \quad x \leq x \leq_2, \tag{4}$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping.

Bu yerda $\psi_1(x)$ va $\psi_2(x)$ berilgan etarlicha silliq funksiyalar bo‘lib, ular uchun $\psi_1(0) = \psi_2(0)$ tenglik o‘rinli.




7 — shakl.

Ma’lumki, (2) tenglamaning umumiy yechimi

$$u(x,y) = f(x+y) + g(x-y), \tag{5}$$

ko‘rinishda bo‘ladi. Bu yerda $f, g \in C^2(R)$. (2) tenglama uchun Gursa masalasining yechimini (5) umumiy yechim asosida quraylik. Bunda $C^2(R)$ sinfga tegishli bo‘lgan $f$, $g$ funksiyalarni topish uchun (5) formula bilan aniqlangan $u(x,y)$ funksiyani (3) va (4) shartlarga qo‘yamiz, natijada

$$u(x,y)|_{y=x} = f(2x) + g(0) = \psi_1(x),$$

$$u(x,y)|_{y=-x} = f(0) + g(2x) = \psi_2(x),$$

yoki

$$f(2x) + g(0) = \psi_1(x), \tag{6}$$

$$f(0) + g(2x) = \psi_2(x). \tag{7}$$

tengliklarni olamiz. Demak, $f$ va $g$ funksiyalarni topish uchun (6)–(7) tenglamalar sistemasiga ega bo'ldik. Bu sistemaning birinchisida $x$ ni $x/2$ ga almashtirib, $f(x)$ funksiyani

$$f(x) = \psi_1\left(\frac{x}{2}\right) - g(0), \tag{8}$$

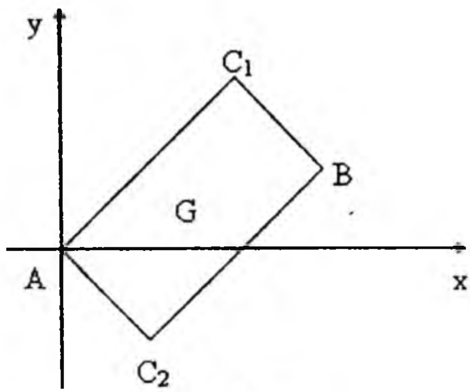
ko'rinishda topamiz. Xuddi shunday qilib, (7) tenglamadan $g(x)$ funksiyani

$$g(x) = \psi_2\left(\frac{x}{2}\right) - f(0), \tag{9}$$

topamiz. Endi (8) va (9) formulalar bilan topilgan $f(x)$ va $g(x)$ funksiyalarni (5) umumiy yechimga qo'yamiz. Natijada (2)–(4) Gursa masalasining yechimini quyidagi

$$u(x,y) = \psi_1\left(\frac{x+y}{2}\right) + \psi_2\left(\frac{x-y}{2}\right) - \psi_1(0), \tag{10}$$

ko'rinishda topamiz.




8 – shakl.

Agar $\psi_1(x)$ va $\psi_2(x)$ funksiyalar berilgan sohada ikki marta differensiallanuvchi bo'lsa, u holda (10) formula bilan aniqlangan $u(x,y)$ funksiya qaralayotgan sohada (2) tenglamani va (3), (4) chegaraviy shartlarni qanoatlantiradi.

Demak, quyidagi teorema isbotlandi.

1–TEOREMA. Agar $\psi_1(x)$ va $\psi_2(x)$ funksiyalar

$$\psi_1(x) \in C[0, x_1] \cap C^2(0, x_1), \quad \psi_2(x) \in C[0, x_2] \cap C^2(0, x_2)$$

bo‘lib, ushbu tenglikni $\psi_1(0) = \psi_2(0)$ qanoatlantirsa, u holda (2) tenglama uchun Gursa masalasining yagona yechimi mavjud bo‘ladi va bu yechim (10) formula bilan aniqlanadi.

Endi (2) tenglamaning $AC : x + y = 0$, $BC : x - y = l$ xarakteristikalari va $AB = \{(x, y) : y = 0, 0 < x < l\}$ kesma bilan chegaralangan xarakteristik uchburchakni $G$ deb belgilaylik. (2) tenglamani $G$ sohada qaraymiz.

1–DARBU MASALASI. (2) tenglamaning yopiq $\overline{G}$ sohada aniqlangan, uzluksiz va va quyidagi

$$u(x, y) \in C(\overline{G}) \cap C^2(G), \tag{11}$$

$$u(x, y)|_{y=0} = u(x, 0) = \tau(x), \quad 0 \le x \le l, \tag{12}$$

$$u(x, y)|_{y=-x} = u(x, -x) = \psi(x), \quad 0 \le x \le l/2, \tag{13}$$

shartlarni qanoatlantiruvchi $u(x, y)$ yechimini toping.

Bu yerda $\tau(x)$ va $\psi(x)$ berilgan yetarlicha silliq funksiyalar bo‘lib, ular uchun $\tau(0) = \psi(0)$ tenglik o‘rinli.

Bu masalani yechish uchun (2) tenglamaning (5) ko‘rinishdagi umumiy yechimidan foydalanamiz.

$$u(x, y)|_{y=0} = f(x) + g(x) = \tau(x),$$

$$u(x, y)|_{y=-x} = f(0) + g(2x) = \psi(x).$$

Bundan esa

$$g(x) = \psi\left(\frac{x}{2}\right) - f(0),$$

$$f(x) = \tau(x) - g(x) = \tau(x) - \psi\left(\frac{x}{2}\right) + f(0)$$

bo'ladi. Topilgan $f(x)$ va $g(x)$ funksiyalarni (5) umumiy yechimga qo'yamiz. Natijada 1–Darbu masalasining yechimini

$$u(x,y) = \tau(x+y) - \psi\left(\frac{x+y}{2}\right) + \psi\left(\frac{x-y}{2}\right), \qquad (14)$$

ko'rinishda hosil qilamiz.

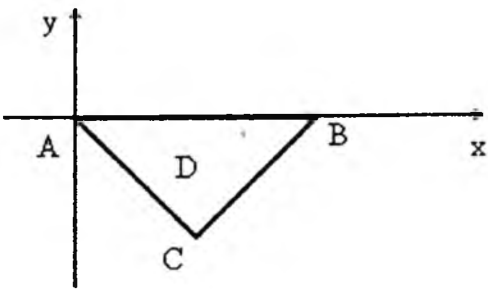
Agar $\tau(x), \psi(x) \in C^2$ bo'lsa, u holda (14) formula bilan aniqlangan $u(x,y)$ funksiya $G$ sohada (2) tenglamani va (12)–(13) shartlarni qanoatlantiradi.

Demak, quyidagi teoremaning o'rinli ekanligini isbotladik.

2–TEOREMA. Agar $\tau(x)$ va $\psi(x)$ funksiyalar

$$\tau(x) \in C[0,l] \cap C^2(0,l), \quad \psi(x) \in C[0,l/2] \cap C^2(0,l/2)$$

bo'lib, ushbu tenglikni $\tau(0) = \psi(0)$ qanoatlantirsa, u holda (2) tenglama uchun 1–Darbu masalasining yechimi $G$ sohada mavjud va yagona bo'ladi. Bu yechim (14) formula bilan aniqlanadi.




9 — shakl.

2–DARBU MASALASI. (2) tenglamaning yopiq $\overline{G}$ sohada aniqlangan uzluksiz, (11)–(13) va ushbu

$$\left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} = u_y(x,0) = \nu(x), \quad 0 \le x \le l, \qquad (15)$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping.

Bu yerda $\nu(x)$ va $\psi(x)$ berilgan etarlicha silliq funksiyalar.

2-Darbu masalasining yechimini topish uchun (2) tenglamaning umumiy yechimida qatnashgan $f$ va $g$ funksiyalarni (13) va (15) shartlardan foydalanib topamiz. Buning uchun (5) formulani ketma-ket (13) va (15) shartlarga qo'yib,

$$u(x,y)|_{y=-x} = f(0) + g(2x) = \psi(x), \tag{16}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = f'(x) - g'(x) = \nu(x), \tag{17}$$

ifodalarni olamiz. Olingan (17) ifodada $x$ ni $s$ ga almashtiramiz va hosil bo'lgan tenglikni $s$ bo'yicha noldan $x$ gacha integrallaab, ushbu

$$\int_0^x f'(s)ds - \int_0^x g'(s)ds = \int_0^x \nu(s)ds,$$

yoki

$$f(x) - g(x) = \int_0^x \nu(s)ds + f(0) - g(0) = \int_0^x \nu(s)ds + c \tag{18}$$

ifodaga ega bo'lamiz. Endi (16) formuladan ushbu $g(x)$ funksiyani

$$g(x) = \psi\left(\frac{x}{2}\right) - f(0), \tag{19}$$

olamiz. Olingan ifodani (18) ga qo'yib, $f(x)$ funksiyani

$$f(x) = \psi\left(\frac{x}{2}\right) + \int_0^x \nu(s)ds + c - f(0), \tag{20}$$

topamiz. (19) va (20) formulalar bilan topilgan $f(x)$ va $g(x)$ funksiyalarning qiymatini (5) umumiy yechimga qo'ysak, 2-Darbu masalasining yechimi

$$u(x,y) = \psi\left(\frac{x+y}{2}\right) + \int_0^{x+y} \nu(s)ds + c - f(0) + \psi\left(\frac{x-y}{2}\right) - f(0).$$

yoki

$$u(x,y) = \psi\left(\frac{x+y}{2}\right) + \psi\left(\frac{x-y}{2}\right) + \int\limits_0^{x+y} \nu(s)ds - \psi(0), \qquad (21)$$

hosil bo'ladi. Bu yerda $c - 2f(0) = \psi(0)$.

Agar $\psi(x) \in C^2(0, l/2)$ va $\nu(x)$ funksiya $[0, l]$ segmentda integrallanuvchi (ya'ni $\nu(x) \in I[0, l]$) va $\nu(x) \in C^1(0, l)$ bo'lsa, u holda (21) formula bilan aniqlangan $u(x, y)$ funksiya $G$ sohada (2) tenglamani va (13), (15) shartlarni qanoatlantiradi.

Demak, quyidagi teorema isbotlandi:

3–TEOREMA. Agar

$$\nu(x) \in I[0, l] \cap C^1(0, l), \qquad \psi(x) \in C[0, l/2] \cap C^2(0, l/2)$$

bo'lsa, u holda (2) tenglama uchun 2–Darbu masalasi $G$ sohada yagona yechimga ega bo'ladi va bu yechim (21) formula bilan aniqlanadi.

## 12–§. Chiziqli giperbolik tenglama uchun Koshi va Gursa masalalari. Ketma–ket yaqinlashish usuli

Quyidagi ikkinchi tartibli chiziqli

$$Lu \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y), \qquad (1)$$

giperbolik tipdagi tenglamani qaraylik.

Biz *II* bobda har qanday ikki o'zgaruvchili chiziqli giperbolik tipdagi tenglamani (1) kanonik ko'rinishga keltirilishi mumkin ekanligini ko'rdik. (1) tenglamaning xarakteristikalari $x = const$ va $y = const$ bo'lishini topish qiyin emas.

$R^2_{xy}$ tekislikda $\gamma$ egri chiziq shunday berilganki, bu egri chiziqni koordinat o'qlariga parallel to'g'ri chiziqlar bilan kamida bitta nuqtada kesib o'tsin.

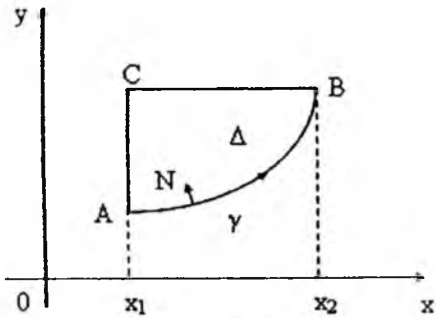
**1. Koshi masalasi.**

$\gamma$ egri chiziq atrofida (1) tenglamaning ushbu

$$u|_\gamma = \tau, \qquad \left.\frac{\partial u}{\partial n}\right|_\gamma = \nu,$$

shartlarni qanoatlantiradigan $u(x, y)$ yechimi topilsin.

Bu yerda $\tau$ va $\nu$ $\gamma$ egri chiziq ustida berilgan yetarlicha silliq funksiyalar, $n$ esa $\gamma$ egri chiziqqa o'tkazilgan normal. Bu masalada yechimning aniqlanish sohasi $\gamma$ chiziqning biror atrofidan iborat bo'ladi.




10 — shakl.

Agar $\gamma$ egri chiziq (1) tenglamaning xarakteristikalari bilan ustma-ust tushmasa, tenglamaning koeffitsiyentlari va berilgan $\tau$, $\nu$, $f(x, y)$ funksiyalar hamda $\gamma$ egri chiziq analitik funksiyalar bo'lsa, u holda Koshi–Kovalevskaya teoremasiga asosan Koshi masalaning $\gamma$ egri chiziqning yetarlicha kichik atrofida analitik yechimi mavjud va yagona bo'ladi.

Qaralayotgan (1) tenglama giperbolik tipdagi tenglama bo'lgani uchun ham Koshi masalasining yechimi mavjud bo'ladigan $\gamma$ egri chiziqning kichik atrofini aniqlash mumkin. Buning uchun $\gamma$ egri chiziqning $A$ va $B$ nuqtalaridan (1) tenglamaning $x = const$ va $y = const$ xarakteristikalarini o'tkazamiz va ular kesishgan nuqtani $C$ deb belgilaymiz. Natijada hosil bo'lgan $\Delta$ soha Koshi masalasi yechimining aniqlanish sohasi bo'ladi.

Endi (1) tenglama uchun Koshi masalasining qo'yilishini aniqlab olaylik. $\gamma$ egri chiziq $y = \gamma(x)$ tenglama bilan berilgan bo'lsin, bunda $x_A \le x \le x_B$, $\gamma(x) \in C^1[x_A, x_B]$ va $\dfrac{d\gamma(x)}{dx} > 0$.

KOSHI MASALASI. (1) tenglamaning egri chiziqli $\overline{\Delta}$ sohada aniqlangan uzluksiz va quyidagi

$$u(x,y)|_{AB} = u(x,y)|_{y=\gamma(x)} = \tau(x), \quad x_A \le x \le x_B, \tag{2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{AB} = \left.\frac{\partial u}{\partial n}\right|_{y=\gamma(x)} = \nu(x), \quad x_A \le x \le x_B, \tag{3}$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping.

Bu yerda $\tau(x)$ va $\nu(x)$ berilgan yetarlicha silliq funksiyalar.

Shuni ta'kidlash muhimki, (2) va (3) Koshi shartlari $y = \gamma(x)$ egri chiziq ustida $u_x(x,y)$ va $u_y(x,y)$ hosilalarni aniqlashga imkon beradi. Haqiqatdan ham, (2) shartni $x$ o'zgaruvchi bo'yicha differensiallab,

$$\left.\frac{\partial u}{\partial x}\right|_{y=\gamma(x)} = \left.\frac{\partial u}{\partial y}\right|_{y=\gamma(x)} \gamma'(x) - \tau'(x), \tag{4}$$

ifodani olamiz.

$u(x,y)$ funksiyadan $\gamma$ egri chiziqda normal bo'yicha olingan hosila quyidagicha

$$\left.\frac{\partial u}{\partial n}\right|_{\gamma} = \left.\frac{\partial u}{\partial x}\right|_{y=\gamma(x)} \cos(x,n) + \left.\frac{\partial u}{\partial y}\right|_{y=\gamma(x)} \cos(y,n) =$$

$$= \frac{\partial u}{\partial x}\frac{\partial y}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\gamma'(x) - \frac{\partial u}{\partial y} = \nu(x), \tag{5}$$

bo'ladi. Hosil bo'lgan (4)–(5) tenglamalar sistemasidan $u_x$ va $u_y$ hosilalarni

$$\left.\frac{\partial u}{\partial x}\right|_{y=\gamma(x)} = \frac{\tau'(x) + \nu(x)\gamma'(x)}{1 + \gamma'^2(x)} = \varphi(x), \tag{6}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=\gamma(x)} = \frac{\tau'(x)\gamma'(x) - \nu(x)}{1 + \gamma'^2(x)} = \psi(x), \tag{7}$$

bir qiymatli aniqlanlaymiz.

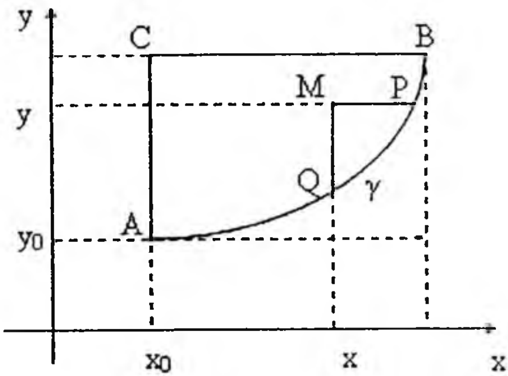
1-TEOREMA. Agar (1) tenglamaning koeffitsiyentlari va o'ng tomoni

$$a(x,y),\ b(x,y),\ c(x,y),\ f(x,y) \in C(\overline{\Delta})$$

hamda berilgan $\tau(x)$ va $\nu(x)$ funksiyalar

$$\tau(x) \in C^1[x_A, x_B], \quad \nu(x) \in C[x_A, x_B],$$

bo'lsa, u holda (1)–(3) Koshi masalasining yechimi mavjud va yagona bo'ladi




11 — shakl.

ISBOT. Agar quyidagi

$$v = \frac{\partial u}{\partial x}, \qquad w = \frac{\partial u}{\partial y}, \tag{8}$$

yordamchi funksiyalarni kiritsak, u holda (1) tenglama ushbu

$$\begin{cases} \dfrac{\partial v}{\partial y} = f(x,y) - a(x,y)v - b(x,y)w - c(x,y)u; \\ \dfrac{\partial w}{\partial x} = f(x,y) - a(x,y)v - b(x,y)w - c(x,y)u; \\ \dfrac{\partial u}{\partial y} = w(x,y). \end{cases} \tag{9}$$

uchta tenglamalar sistemasiga ekvivalent bo'ladi.

Endi $\Delta$ sohada ixtiyoriy $M(x,y)$ nuqta olamiz va shu nuqtadan chiquvchi $\gamma$ chiziq bilan $P$ va $Q$ nuqtalarda kesishuvchi (1) tenglamaning $MP$ va $MQ$ xarakteristikalarni o'tkazamiz.

(9) sistemaning birinchi va uchinchi tenglamasini $QM$ kesma $y = const$ bo'yicha, ikkinchi tenglamasini esa $PM$ kesma $x = const$ bo'yicha integrallaymiz hamda (2), (6), (7) va (8) ifodalarni hisobga olib $v(x,y)$, $u(x,y)$, $w(x,y)$ funksiyalarni

$$\begin{cases} v = \varphi(x) + \int\limits_{\gamma(x)}^{y} [f - av(x,\eta) - bw(x,\eta) - cu(x,\eta)]d\eta; \\ w = \widetilde{\psi}(y) + \int\limits_{h(y)}^{x} [f - av(\xi,y) - bw(\xi,y) - cu(\xi,y)]d\xi; \\ u = \tau(x) + \int\limits_{\gamma(x)}^{y} w(x,\eta)d\eta. \end{cases} \qquad (10)$$

ko'rinishda aniqlaymiz. Bu yerda

$$\widetilde{\psi}(y) = \left.\frac{\partial u}{\partial y}\right|_{x=h(y)},$$

$h(y)$ esa $\gamma(x)$ ga teskari bo'lgan funksiya.

Agar $u(x,y)$ funksiya (1)–(3) Koshi masalasining yechimi bo'lsa, u holda $v(x,y)$, $w(x,y)$ va $u(x,y)$ funksiyalar (10) integral tenglamalar sistemasini qanoatlantiradi va aksincha yopiq $\overline{\Delta}$ sohada (10) sistemaning ixtiyoriy $(v,w,u)$ yechimi (9) differensial tenglamalar sistemasini va $u$ funksiya esa Koshi masalasi shartlarini qanoatlantiradi.

Bundan tashqari (4), (6), (9) formulalardan va (10) sistemaning birinchi tenglamasidan

$$\frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial x}\right|_{y=\gamma(x)} + \int\limits_{\gamma(x)}^{y} \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)d\xi =$$

$$= \varphi(x) + \int\limits_{\gamma(x)}^{y} \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)d\xi = \varphi(x) + \int\limits_{\gamma(x)}^{y} \frac{\partial w}{\partial \xi}d\xi =$$

$$= \varphi(x) + \int\limits_{h(y)}^{x} [f - av(\xi, y) - bw(\xi, y) - cu(\xi, y)]d\xi = v;$$

bo‘lishi kelib chiqadi. Demak, (8) tengliklar bajariladi.

Endi (8) tengliklarni (9) sistemaning birinchi tenglamasiga qo‘yib, $u(x, y)$ funksiyaning (1) tenglamani va (2), (3) Koshi shartlarini qanoatlantirishiga ishonch hosil qilishimiz mumkin.

Shunday qilib, (1)–(3) Koshi masalasini (10) integral tenglamalar sistemasiga ekvivalent ekan. Bu integral tenglamalar sistemasini ketma–ket yaqinlashish usuli bilan echamiz.

Buning uchun nolinchi yaqinlashish sifatida

$$v_0 = \varphi(x), \qquad w_0 = \widetilde{\psi}(y), \qquad u_0 = \tau(x)$$

berilgan boshlang‘ich funksiyalarni olamiz va ketma–ketlikning keyingi hadlarini quyidagi

$$\begin{cases} \dfrac{\partial v}{\partial n} = \varphi(x) + \int\limits_{\gamma(x)}^{y} [f(x, \eta) - av_{n-1} - bw_{n-1} - cu_{n-1}]d\eta; \\ \dfrac{\partial w}{\partial n} = \widetilde{\psi}(y) + \int\limits_{h(y)}^{x} [f(\xi, y) - av_{n-1} - bw_{n-1} - cu_{n-1}]d\xi; \\ \dfrac{\partial u}{\partial n} = \tau(x) + \int\limits_{\gamma(x)}^{y} w_{n-1}(x, \eta)d\eta, \ (n = 1, 2, \ldots) \end{cases} \tag{11}$$

formulalar bo‘yicha quramiz.

Endi yopiq $\overline{\Delta}$ sohada $v_n$, $w_n$ va $u_n$ ketma-ketliklarni yaqinlashuvchi ekanligini isbotlaylik. Buning uchun quyidagi

$$v_{n+1} - v_n = -\int\limits_{\gamma(x)}^{y} [a(v_n - v_{n-1}) + b(w_n - w_{n-1}) + c(u_n - u_{n-1})]d\eta;$$

$$w_{n+1} - w_n = -\int\limits_{h(y)}^{x} [a(v_n - v_{n-1}) + b(w_n - w_{n-1}) + c(u_n - u_{n-1})]d\xi;$$

$$u_{n+1} - u_n = \int\limits_{\gamma(x)}^{y} (w_n - w_{n-1})d\eta. \qquad (12)$$

ayirmalarni tuzamiz.

Agar $K g‘eq \underset{\Delta}{\max}\{|a| + |b| + |c|\}$ va $A = const > 0$ bo‘lsa, u holda $|v_n - v_{n-1}|$, $|w_n - w_{n-1}|$ va $|u_n - u_{n-1}|$ ayrimalar quyidagi

$$\begin{cases} |v_n - v_{n-1}| \leq K^{n-1}A\dfrac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!}; \\ |w_n - w_{n-1}| \leq K^{n-1}A\dfrac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!}; \\ |u_n - u_{n-1}| \leq K^{n-1}A\dfrac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!}; \end{cases} \qquad (13)$$

tengsizliklarni qanoatlantirishini ko‘rsataylik.

Bu tengsizliklarning to‘g‘ri ekanligini matematik induksiya usuli bilan isbotlaymiz. Agar $A$ yetarlicha katta bo‘lsa, u holda $n = 1$ bo‘lganda (13) baholar to‘g‘ri bo‘ladi. Endi bu tengsizliklar $n + 1$ bo‘lganda ham o‘rinli ekanligini ko‘rsataylik. Yuqorida tuzilgan (12) ayirmalardan, masalan uning birinchisidan

$$|v_{n+1} - v_n| \leq \int\limits_{\gamma(x)}^{y} (|a| + |b| + |c|)\, K^{n-1} A\, \frac{(x + \eta - x_0 - y_0)^{n-1}}{(n-1)!}\, d\eta \leq$$

$$\leq K^n A \int\limits_{y_0}^{y} \frac{(x + \eta - x_0 - y_0)^{n-1}}{(n-1)!}\, d\eta =$$

$$= \frac{K^n}{n!} A \left[(x + y - x_0 - y_0)^n - (x - x_0)^n\right] \leq$$

$$< K^n A\, \frac{(x + y - x_0 - y_0)^n}{n!}, \qquad (x > x_0,\ y g‘eq \gamma(x) > y_0)$$

bo‘lishi kelib chiqadi.

Xuddi shu kabi $|w_{n+1} - w_n|$ va $|u_{n+1} - u_n|$ ayrimalarni baholaymiz. Natijada ushbu

$$v_0+\sum_{n=1}^{\infty}(v_n - v_{n-1}),\quad w_0+\sum_{n=1}^{\infty}(w_n - w_{n-1}),\quad u_0+\sum_{n=1}^{\infty}(u_n - u_{n-1}) \quad (14)$$

qatorlarni olamiz. Bu qatorlarning har bir hadi absolyut qiymati jihatidan yaqinlashuvchi

$$A+A\sum_{n=1}^{\infty} K^{n-1}\frac{(x+y-x_0-y_0)^{n-1}}{(n-1)!} = A\,(1+exp\{K(x+y-x_0-y_0)\})$$

qatorning hadlaridan kichik. Shuning uchun (14) qatorlar (13) tengsizliklarga asosan yopiq $\overline{\Delta}$ sohada absolyut va tekis yaqinlashuvchi bo'ladi.

Demak, $v_n$, $w_n$ va $u_n$ ketma–ket yaqinlashishlar mos ravishda yopiq $\overline{\Delta}$ sohada uzluksiz bo'lgan $v$, $w$ va $u$ limitga intiladi. (11) sistemada $n \to \infty$ limitga o'tsak, u holda bu ketma–ketliklar mos ravshida $v(x,y)$, $w(x,y)$ va $u(x,y)$ funksiyalarga intiladi. Bu limit funksiyalar (10) sistemani qanoatlantiradi, bunda esa $u_x$, $u_y$, $v_y$ va $w_x$ funksiyalarning yopiq $\overline{\Delta}$ sohada uzluksiz bo'lishi kelib chiqadi.

Endi (10) tenglamalar sistemasi yechimining yagona ekanligini isbotlaylik.

Faraz qilaylik, (10) tenglamalar sistemasi $v_1$, $w_1$, $u_1$ va $v_2$, $w_2$, $u_2$ yechimlarga ega bo'lsin. Ularning ayirmasini $V = v_1 - v_2$, $W = w_1 - w_2$ va $U = u_1 - u_2$ deb belgilaylik. U holda $V$, $W$ va $U$ funksiyalar ushbu

$$\begin{cases} V(x,y) = -\int\limits_{\gamma(x)}^{y} [aV - bW - cU]d\eta; \\ W(x,y) = -\int\limits_{h(y)}^{x} [aV - bW - cU]d\xi; \\ U(x,y) = -\int\limits_{\gamma(x)}^{y} W(x,\eta)d\eta, \end{cases} \quad (15)$$

sistemani qanoatlantiradi. Bundan $V = W = U = 0$ ekanligini isbot qilamiz. $V$, $W$ va $U$ funksiyalar egri chiziqli yopiq $\overline{\Delta}$ sohada uzluksiz va chegaralangan. Demak, shunday $B$ son mavjudki, bu son uchun

$$|V| \leq B, \qquad |W| \leq B, \qquad |U| \leq B$$

tengsizliklar o'rinli bo'ladi. U holda (15) sistemadan

$$|V(x,y)| \leq \int\limits_{\gamma(x)}^{y} (|a| + |b| + |c|)Bd\eta \leq$$

$$\leq KB(y - y_0) \leq KB\frac{(x + y - x_0 - y_0)}{1!},$$

tengsizlikni olamiz. Xuddi shunday $W$ va $U$ uchun ham

$$|W(x,y)| \leq KB\frac{(x + y - x_0 - y_0)}{1!};$$

$$|U(x,y)| \leq KB\frac{(x + y - x_0 - y_0)}{1!},$$

tengsizliklar o'rinli ekanligini ko'ramiz. Bu tengsizliklarga matematik induksiya usulini qo'llaymiz va ixtiyoriy $n$ uchun quyidagi

$$|V(x,y)| \leq K^n B\frac{(x + y - x_0 - y_0)^n}{n!};$$

$$|W(x,y)| \leq K^n B\frac{(x + y - x_0 - y_0)^n}{n!};$$

$$|U(x,y)| \leq K^n B\frac{(x + y - x_0 - y_0)^n}{n!},$$

baholar o'rinli bo'ladi.

Bundan esa $n \to \infty$ bo'lganda $V = W = U = 0$ ekanligi kelib chiqadi. Shuday qilib, 1–teorema to'liq isbot bo'ldi.

**2. Gursa masalasi.**

Agar $AB$ egri chiziq to'g'ri burchak tashkil qilsa, ya'ni $\angle ADB$ to'g'ri burchak bo'lsa, u holda $ADB$ egri chiziqda ikkita chegaraviy shart berib bo'lmaydi. Buning o'rniga quyidagi Gursa masalasini qarashimiz mumkin.

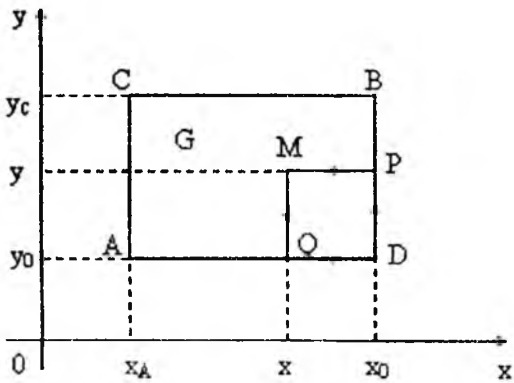
Giperbolik tipdagi (1) tenglamaning $AD$, $DB$, $BC$ va $AC$ xarakteristikalari bilan chegaralangan to'g'ri to'rtburchakli sohani $G$ deb belgilaylik. Bu yerda $A = (x_A, y_A)$, $B = (x_B, y_B)$, $C = (x_C, y_C)$, $D = (x_D, y_D)$ va $x_A = x_C$, $y_A = y_D$, $x_D = x_B$, $y_C = y_D$.

GURSA MASALASI. To'rtburchakli $G$ sohada (1) tenglamaning quyidagi

$$u(x,y)|_{AD} = u|_{y=y_D} = \varphi_1(x), \quad x_A \le x \le x_D, \tag{16}$$

$$u(x,y)|_{DB} = u|_{x=x_D} = \varphi_2(y), \quad y_D \le y \le y_C, \tag{17}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping. Bu yerda $\varphi_1(x)$ va $\varphi_2(y)$ berilgan etarlicha silliq funksiyalar va bu funksiyalar uchun $\varphi_1(x_D) = \varphi_2(y_D)$ tenglik o'rinli.




12 — shakl.

Gursa masalasi uchun ushbu teoremani isbotlaylik.

2–TEOREMA. Agar (1) tenglamaning koeffitsientlari va o'ng tomoni

$$a(x,y),\ b(x,y),\ c(x,y),\ f(x,y) \in C(\overline{G})$$

hamda berilgan funksiyalar

$$\varphi_1(x) \in C^1[x_A, x_D], \quad \varphi_2(y) \in C^1[y_D, y_B],$$

bo'lsa, u holda (1), (15)–(16) Gursa masalasining $u(x,y)$ yechimi $C^1(\overline{G})$ sinfda mavjud va yagona bo'ladi.

ISBOT. Xuddi Koshi masalasidagi kabi quyidagi

$$v = \frac{\partial u}{\partial x}, \qquad w = \frac{\partial u}{\partial y}, \tag{18}$$

yordamchi funksiyalarni kiritsak, u holda (1) tenglama ushbu

$$\begin{cases} \dfrac{\partial v}{\partial y} = f(x,y) - a(x,y)v - b(x,y)w - c(x,y)u; \\ \dfrac{\partial w}{\partial x} = f(x,y) - a(x,y)v - b(x,y)w - c(x,y)u; \\ \dfrac{\partial u}{\partial y} = w(x,y); \end{cases} \tag{19}$$

tenglamalar sistemasiga ekvivalent bo'ladi.

Endi $G$ sohada ixtiyoriy $M(x,y)$ nuqta olamiz va bu nuqta orqali (1) tenglamaning $MP$ va $MQ$ xarakteristikalarini o'tkazamiz. Bu yerda $P = (x_D, y)$, $Q = (x, y_D)$.

(19) sistemaning birinchi va uchinchi tenglamalarini $QM$ kesmada, ikkinchi tenglamasini esa $PM$ kesmada integrallab, ushbu

$$\begin{cases} v = v(x, y_D) + \int\limits_{y_D}^{y} [f - av(x,\eta) - bw(x,\eta) - cu(x,\eta)]d\eta; \\ w = w(x_D, y) + \int\limits_{h(y)}^{x} [f - av(\xi,y) - bw(\xi,y) - cu(\xi,y)]d\xi; \\ u = u(x, y_D) + \int\limits_{\gamma(x)}^{y} w(x,\eta)d\eta; \end{cases} \tag{20}$$

sistemaga ega bo'lamiz.

Bu sistemadan (16)–(18) tengliklarga ko'ra

$$v(x_0, y_D) = \left.\frac{\partial u}{\partial x}\right|_{y=y_0} = \varphi_1'(x), \quad w(x_D, y) = \left.\frac{\partial u}{\partial y}\right|_{x=x_0} = \varphi_2'(y).$$

bo'ladi. Bu tengliklarga asosan (20) sistemani quyidagi

$$\begin{cases} v(x,y) = \varphi_1'(x) + \int\limits_{y_D}^{y} [f - av(x,\eta) - bw(x,\eta) - cu(x,\eta)]d\eta; \\ w(x,y) = \varphi_2'(y) + \int\limits_{x_D}^{x} [f - av(\xi,y) - bw(\xi,y) - cu(\xi,y)]d\xi; \\ u(x,y) = \varphi_1(x) + \int\limits_{y_D}^{y} w(x,\eta)d\eta, \end{cases} \quad (21)$$

ko'rinishda yozib olish mumkin.

Aksincha, (21) sistemaning ixtiyoriy yechimi (19) sistemani qanoatlantiradi. Bundan tashqari

$$\frac{\partial u}{\partial x} = \varphi_1'(x) + \int\limits_{y_D}^{y} \frac{\partial w}{\partial x} d\eta =$$

$$= \varphi_1'(x) + \int\limits_{y_D}^{y} [f(x,\eta) - a(x,\eta)v - b(x,\eta)w - c(x,\eta)u]d\eta = v$$

bo'ladi. Demak, (18) tenglamaning birinchisi o'rinli ekan. (20) sistemadan ushbu

$$u|_{y=y_D} = \varphi_1(x),$$

$$u|_{x=x_D} = \varphi_1(x_D) + \int\limits_{y_D}^{y} w|_{x=x_D} d\eta = \varphi_1(x_D) + \int\limits_{y_D}^{y} \varphi_2'(\eta)d\eta =$$

$$= \varphi_1(x_D) + \varphi_2(y) - \varphi_2(y_D) = \varphi_2(y)$$

ifodalar kelib chiqadi.

Shunday qilib, (21) sistemaning ixtiyoriy yechimi qaralayotgan Gursa masalasining yechimi bo'lar ekan. Bundan esa (21) sistema (1) tenglama uchun qo'yilgan (16)–(17) Gursa masalasiga ekvivalent ekanligi kelib chiqadi.

Xuddi Koshi masalasidagi kabi (1), (16)–(17) Gursa masalasi (21) sistemaning uzluksiz yechimini isbotlashga keltirildi. Bu sistema yechimining mavjudligini yuqoridagi singari ketma–ket yaqinlashish usuli bilan ko‘rsatish mumkin.

## 13–§. Giperbolik tipdagi tenglamalar uchun Koshi va Gursa masalalarini Riman usuli bilan yechish

1. QO‘SHMA DIFFERENSIAL OPERATOR TUSHUNCHASI. GRIN FORMULASI. Ushbu bo‘limda chiziqli giperbolik tipdagi tenglama uchun boshlang‘ich–chegaraviy masalalar yechimlarining integral ifodasini olish uchun kerakli bo‘lgan ayrim yordamchi formulalarni keltiramiz.

Faraz qilaylik,

$$L[u] \equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u, \tag{1}$$

chiziqli giperbolik tenglamaga mos differensial operator bo‘lsin.

Bu yerda $a(x,y)$, $b(x,y)$ va $c(x,y)$ qaralayotgan operatorning koeffitsiyentlari, biror $D \subset R^2_{xy}$ sohada berilgan funksiyalar bo’lib, ular $a(x,y)$, $b(x,y) \in C^1(D)$ va $c(x,y) \in C(D)$ bo‘lsin.

$L[u]$ operatorni biror $v(x,y)$ funksiyaga ko‘paytiramiz va buning uchun quyidagi ayniyat o‘rinli:

$$vL[u] - uL^*[v] = \frac{1}{2}\left(\frac{\partial H}{\partial x} + \frac{\partial K}{\partial y}\right), \tag{2}$$

bu yerda

$$L^*[v] = v_{xy} - (av)_x - (bv)_y + cv, \tag{3}$$

va

$$H = v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} + 2auv = (uv)_y - 2u(v_y - av),$$
$$K = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} + 2buv = (uv)_x - 2u(v_x - bv).$$

(3) formula bilan aniqlangan $L^*$ operator $L$ operatorga *qo‘shma operator* deyiladi.

Agar $vL[u] - uL^*[v]$ ayirmani biror $H$ va $K$ ifodalarning mos ravishda $x$ va $y$ o'zaruvchlar bo'yicha xususiy hosilalarining yig'indisi ko'rinishida ifodalash mumkin bo'lsa, u holda ikkita $L$ va $L^*$ differensial operatorlar *o'zaro qo'shma operatorlar* deyiladi.

Agar $L[u] = L^*[u]$ bo'lsa, u holda $L[u]$ *o'z-o'ziga qo'shma operator* deyiladi.

$R^2_{xy}$ tekislikda $S$ bo'lakli silliq chiziq bilan chegaralangan soha $D$ bo'lsin. Endi (2) ayniyatni $D$ sohada integrallaymiz va unga matematik analiz kursidan ma'lum bo'lgan Grin formulasini qo'llaymiz. Natijada

$$\iint_D \Big(vL[u] - uL^*[v]\Big)dxdy = \frac{1}{2}\int_S (Hdy - Kdx), \tag{4}$$

ifodaga ega bo'lamiz. Bu formula ham ikki o'lchovli Grin formulasi deyiladi.

Riman usuli. Nemis matematigi R.Riman chiziqli giperbolik tipdagi tenglamalar uchun Koshi va Gursa masalalarining yechimini qurish usulini tavsiya qilgan.

Quyidagi Koshi masalasini qaraylik.

Koshi masalasi. Yopiq $\overline{D}$ sohada aniqlangan, uzluksiz va

$$u(x,y) \in C^1(\overline{D}), \quad u_{xy} \in C(D); \tag{5}$$

funksiyalar sinfiga tengishli

$$L[u] \equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{6}$$

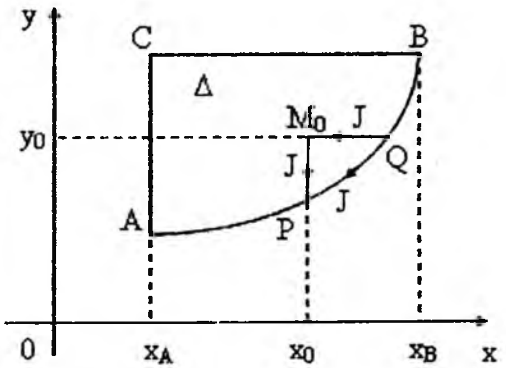
tenglamaning quyidagi

$$u|_{y=\gamma(x)} = \tau(x), \qquad \frac{\partial u}{\partial n}\bigg|_{y=\gamma(x)} = \nu(x), \tag{7}$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping. Bu yerda $a(x,y)$, $b(x,y)$ – uzluksiz va birinchi tartibli hosilalarga ega, $c(x,y)$ va $f(x,y)$ – uzluksiz funksiyalar, $\tau(x)$, $\nu(x)$ – berilgan funksiyalar, $n$ esa $\gamma(x)$ egri chiziqqa o'tkazilgan normal.

Ma'lumki, (6) tenglamaga mos xarakteristik tenglama $dxdy = 0$ bo'lib, $x = const$, $y = const$ to'g'ri chiziqlar tenglamaning xarakteristikalari bo'ladi.

Tekislikda biror $\gamma(x)$ egri chiziq berilgan bo'lib, (6) tenglamaning xarakteristikalari bu egri chiziqni bittadan ortiq nuqtalarda kesib o'tmasin.




13 — shakl.

$M(x_0, y_0)$ nuqtani belgilab, bu nuqtadan $x = x_0$, $y = y_0$ xarakteristikalarni o'tkazamiz. Bu xarakteristikalar berilgan $\gamma(x)$ chiziq bilan $P$ va $Q$ nuqtalarda kesishib, $MPQ$ egri chiziqli uchburchak hosil qiladi. $MPQ$ egri chiziqli uchburchak bilan chegaralangan soha $\Delta$ bo'lsin.

Faraz qilaylik, (5)-(7) masalaning $u(x, y)$ yechimi mavjud bo'lsin. U holda yopiq $\overline{\Delta}$ sohada aniqlangan, uzluksiz va

$$v(x, y) \in C^1(\overline{\Delta}), \quad v_{xy} \in C(\Delta)$$

shartlarni qanoatlantiruvchi ixtiyoriy $v(x, y)$ funksiya uchun (4) ayniyat o'rinli.

Noma'lum $u(x, y)$ funksiyaning $M(x_0, y_0)$ nuqtadagi qiymatini aniqlaymiz. Buning uchun (4) ifodani $\Delta$ sohada integrallab, Grin formulasini qo'llaymiz.

Natijada

$$\iint_{\Delta}(vLu - uL^*v)dxdy = \frac{1}{2}\int_{\gamma}(Hdy - Kdx)$$

hosil bo'ladi.

Bu yerda $\gamma$ kontur $PQ$ yoydan hamda $QM$ va $MP$ xarakteristikalardan iborat.

Endi (4) ifodaning o'ng tomonidagi $QM$ va $MP$ xarakteristikalar bo'yicha olingan integrallarni qaraylik.

$QM$ xarakteristikada $dy = 0$, $MP$ xarakteristikada esa $dx = 0$ bo'lgani uchun (4) tenglik quyidagi ko'rinishga keladi:

$$\iint_{\Delta}(vLu - uL^*v)dxdy = \frac{1}{2}\int_{P}^{Q}(Hdy - Kdx)-$$

$$\frac{1}{2}\int_{Q}^{M}Kdx + \frac{1}{2}\int_{M}^{P}Hdy = J_1 + J_2 + J_3, \tag{8}$$

Endi $J_i$ integrallarni alohida–alohida hisoblaymiz.

$$J_1 = \int_{PQ} u(v_x dx - v_y dy) - v(u_x dx - u_y dy) - 2uv(bdx - ady). \tag{9}$$

$$J_2 = \frac{1}{2}\int_{Q}^{M}Kdx = \frac{1}{2}\Big[(uv)_M - (uv)_Q\Big] - \int_{Q}^{M}u\left(\frac{\partial v}{\partial x} - bv\right)dx, \tag{10}$$

$$J_3 = \frac{1}{2}\int_{M}^{P}Hdy = \frac{1}{2}\Big[(uv)_P - (uv)_M\Big] - \int_{M}^{P}u\left(\frac{\partial v}{\partial y} - av\right)dy, \tag{11}$$

Topilgan $J_i$ integrallarning (9), (10) va (11) ifodalarini (8) formulaga qo'yib, mos hadlarini soddalashtirsak, quyidagi

$$u(M)v(M) = \frac{u(P)v(P) - u(Q)v(Q)}{2} -$$

$$
-\int_P^Q u(v_x dx - v_y dy) - v(u_x dx - u_y dy) - 2uv(bdx + ady) +
$$

$$
+\int_Q^M u\left(\frac{\partial v}{\partial x} - bv\right)\Bigg|_{y=y_0} dx + \int_P^M u\left(\frac{\partial v}{\partial y} - av\right)\Bigg|_{x=x_0} dy +
$$

$$
+\iint_\Delta \left[v(x,y)f(x,y) - uL^*v\right] dxdy. \tag{12}
$$

formulaga ega bo'lamiz. (12) formulaning o'ng tomonidagi ikki karrali integral va $QM$ va $MP$ xarakteristikalar bo'yicha integrallarda noma'lum $u(x,y)$ funksiya qatnashyapti. Riman usulining asosiy maqsadi, shu integrallarni nolga aylantiradigan qilib, $v$ funksiyani tanlashdan iborat.

Faraz qilaylik, ikki juft $(x,y;x_0,y_0)$ o'zgaruvchilarga bog'liq bo'lgan $v(x,y;x_0,y_0)$ funksiya quyidagi

1) $\Delta$ sohada

$$
L^*_{xy} v(x,y;x_0,y_0) = 0, \tag{13}
$$

bir jinsli tenglamani va

2) $$\left(\frac{\partial}{\partial x} v(x,y;x_0,y_0) - b(x,y)v(x,y;x_0,y_0)\right)\Bigg|_{y=y_0} = 0, \tag{14}$$

3) $$\left(\frac{\partial}{\partial y} v(x,y;x_0,y_0) - a(x,y)v(x,y;x_0,y_0)\right)\Bigg|_{x=x_0} = 0, \tag{15}$$

4) $x = x_0$ va $y = y_0$ bo'lganda

$$
v(x,y;x_0,y_0) = 1, \tag{16}
$$

shartlarni qanoatlantirsin.

Endi (13)–(16) shartlarni qanoatlantiruvchi $v(x,y;x_0,y_0)$ funksiyaning mavjudligini isbotlaymiz. Buning uchun (14) tenglikda $x$ ni $\xi$ bilan almashtirib, hosil bo'lgan ifodani $x$ dan $x_0$ gacha integrallaymiz. Natijada

$$\ln v(x,y_0;x_0,y_0)\Big|_{x_0}^{x} = \int_{x_0}^{x} b(\xi,y_0)d\xi.$$

hosil bo'ladi. Bundan (16) tenglikka asosan $QM$ xarakteristikada ushbu

$$v(x,y_0;x_0,y_0) = \exp\left\{\int_{x_0}^{x} b(\xi,y_0)d\xi\right\}, \tag{17}$$

tenglikni olamiz.

Huddi shunday qilib (15) tenglikdan $MP$ xarakteristikada esa

$$v(x_0,y;x_0,y_0) = \exp\left\{\int_{y_0}^{y} a(x_0,\eta)d\eta\right\}, \tag{18}$$

ifodani olamiz.

Yuqoridagi shartlardan ko'rinadiki, $v(x,y;x_0,y_0)$ funksiya birinchi juft argumentlari bo'yicha (13) tenglamaning (17) va (18) shartlarni qanoatlantiruvchi Gursa masalasining yechimidan iborat.

Agar (6) tenglamaning $a(x,y)$, $b(x,y)$ va $c(x,y)$ koeffitsiyentlari $C^1(\overline{\Delta})$ sinfga tegishli bo'lsa, u holda (13), (17)–(18) masalaning yagona yechimi mavjud bo'ladi. Bundan esa (13)–(16) shartlarni qanoatlantiruvchi $v(x,y;x_0,y_0)$ funksiyaning mavjud va yagonaligi kelib chiqadi.

TA'RIF. Chiziqli (6) tenglamaga qo'shma bulgan bir jinsli

$$L^*_{xy}[v] \equiv v_{xy} - (av)_x - (bv)_y + cv = 0,$$

tenglamaning

$$v(x,y_0;x_0,y_0) = \exp\left\{\int_{x_0}^{x} b(\xi,y_0)d\xi\right\};$$

$$v(x_0, y; x_0, y_0) = \exp\left\{\int_{y_0}^{y} a(x_0, \eta)d\eta\right\};$$

$$v(x_0, y_0; x_0, y_0) = 1,$$

shartlarni qanoatlantiruvchi $v(x, y; x_0, y_0)$ yechimiga *Riman funksiyasi* deb ataladi va bu funksiya $R(x, y; x_0, y_0)$ deb belgilanadi.

Endi (12) formulada $v = R(x, y; x_0, y_0)$ deb almashtirib,

$$u(M) = \frac{u(P)R(P) - u(Q)R(Q)}{2} -$$

$$-\frac{1}{2}\int_{P}^{Q} u(R_x dx - R_y dy) - R(u_x dx - u_y dy) - 2uR(bdx + ady) +$$

$$+\int_{x_0}^{\bar{y}_0}\int_{g'amma(x)}^{y_0} R(x, y; x_0, y_0)f(x, y)dxdy. \qquad (19)$$

ifodani hosil qilamiz. Bu yerda $\bar{y}_0$ qiymat $\gamma(x) = y_0$ tenglamaning yechimi, ya'ni $\bar{y}_0 = \gamma^{-1}(x)$.

Yuqorida hosil qilingan (19) formula *Riman formulasi* deyiladi. Bu formula chiziqli giperbolik tipdagi (6) tenglamaning (7) chegaraviy shartlarni qanoatlantiruvchi $u(x_0, y_0)$ yechimining Riman funksiyasi $R(x, y; x_0, y_0)$ orqali integral ifodasini beradi. Riman formulasini keltirib chiqarilishidan Koshi masalasi yechimining yagona ekanligi kelib chiqishi mumkin. Chunki $u(x, y)$ funksiyaga nisbatan uning mavjudligidan boshqa hech qanday ortiqcha shart talab qilinmadi.

Endi (5)–(7) Koshi masalasi yechimining mavjudlgini asoslaymiz. Oldingi paragrafda berilgan tenglamaning koeffitsiyentlari, berilgan $\tau$ va $\nu$ funksiyalar ma'lum shartlarni qanoatlantirganda Koshi masalasi yechimining yagonaligi va mavjudligini isbot qilingan edi. (19) formula bilan aniqlangan $u(x_0, y_0)$ funksiya (6) tenglamani va (7) shartlarni qanoatlantirishini bevosita tekshirib ishonch hosil qilish mumkin. Demak, (19) formula bilan aniqlangan $u(x_0, y_0)$ funksiya $\Delta$ sohada (6) tenglamaning yechimi ekanligidan $R(x, y; x_0, y_0)$ Riman funksiyasi ikkinchi juft $(x_0, y_0)$ argumentlariga nisbatan bir jinsli (6) tenglamani qanoatlantiradi, ya'ni

$$L\,R(x, y; x_0, y_0) = 0. \tag{20}$$

Agar $R^*(x, y; x_0, y_0)$ funksiya $L^*v = 0$ qo'shma tenglamaning Riman funksiyasi bo'lsa, u holda

$$R^*(x, y; x_0, y_0) = R(x_0, y_0; x, y)$$

tenglik o'rinli bo'ladi. Bundan esa (20) tenglikning o'rinli ekanligi kelib chiqadi.

Demak, $L$ operatorning $R(x, y; x_0, y_0)$ Riman funksiyasida $(x, y)$ argumentni $(x_0, y_0)$ ga almashtirilsa, u holda qo'shma $L^*$ operatorning Riman funksiyasi hosil bo'ladi.

Agar $L = L^*$ bo'lsa, u holda $R^*(x, y; x_0, y_0)$ Riman funksiyasi $(x, y)$ va $(x_0, y_0)$ argumentlarga nisbatan simmetrik, ya'ni

$$R(x, y; x_0, y_0) = R(x_0, y_0; x, y)$$

funksiya bo'ladi.

Shunday qilib, quyidagi teoremaning o'rinli ekanligi isbotlandi:

1–TEOREMA. Agar $\gamma(x) \in C^1[x_A, y_B]$, $\gamma'(x) > 0$, $a(x, y)$, $b(x, y)$, $c(x, y)$, $a_x(x, y)$, $b_y(x, y)$, $f(x, y) \in C(\overline{\Delta})$ va $\tau(x) \in C^1[x_A, x_B]$, $\nu(x) \in C^1[x_A, x_B]$ bo'lsa, u holda (5)–(7) Koshi masalasining yagona yechimi mavjud bo'ladi va (19) formula bilan aniqlanadi.

Bu teoremaning batafsil isboti [10] va [13] darsliklarda keltirilgan.

GURSA MASALASI. To'rtburchakli $G$ sohada

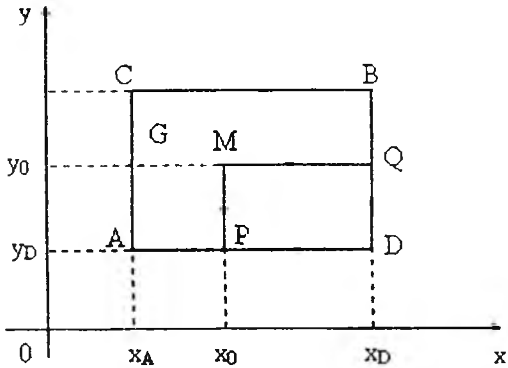
$$L[u] = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y), \tag{6}$$

tenglamaning quyidagi

$$u(x, y)|_{AD} = u|_{y=y_D} = \varphi_1(x), \quad x_A \le x \le x_D, \tag{21}$$

$$u(x, y)|_{DB} = u|_{x=x_D} = \varphi_2(y), \quad y_D \le y \le y_B, \tag{22}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, y)$ yechimini toping. Bu yerda $\varphi_1(x)$ va $\varphi_2(y)$ berilgan yetarlicha silliq funksiyalar va bu funksiyalar uchun $\varphi_1(x_D) = \varphi_2(y_D)$ tenglik o'rinli.




14 — shakl.

Shuni ta'kidlash muhimki, xuddi Koshi masalasidagi kabi chiziqli giperbolik tenglama uchun Gursa masalasi yechimining integral ifodasini Riman funksiyasi orqali topish mumkin.

Buning uchun $G$ sohada fiksirlangan $M_0 = (x_0, y_0)$ nuqtadan (6) tenglamaning $x = x_0$ va $y = y_0$ xarakteristikalarni o'tkazamiz. Ularning $AD$ va $BD$ xarakteristikalar bilan kesishgan nuqtasi $P$ va $Q$ bo'lsin.

Endi (4) formulada $u(x, y)$ funksiyani Gursa masalasining yechimi, $v(x, y)$ ni esa (6) tenglamaning $R(x, y; x_0, y_0)$ Riman funksiyasi deb, hosil bo'lgan ayniyatni $MPDQ$ to'rtburchakli sohada integrallaymiz. Natijada quyidagi

$$u(x_0, y_0) = u(P)v(P) + U(Q)v(Q) - u(D)v(D) +$$

$$+ \int_{x_0}^{x_D} \varphi_1(x)(R_x - bR)|_{y=y_0} dx - \int_{y_D}^{y_0} \varphi_2(y)(R_y - aR)|_{x=x_0} dy -$$

$$- \int_{x_D}^{x_0} \int_{y_D}^{y_0} f(x, y) R(x, y; x_0, y_0) dx dy, \tag{23}$$

Gursa masalasi yechimining integral ifodasini olamiz.

2–TEOREMA. Agar

$$a(x,y),\ b(x,y),\ c(x,y),\ a_x(x,y),\ b_y(x,y),\ f(x,y)\in C(\overline{\Delta})$$

$$\varphi_1(x)\in C^1[x_A,x_D],\ \varphi_2(y)\in C^1[y_D,y_B] \quad \text{va} \quad \varphi_1(x_D)=\varphi_2(y_D)$$

bo'lsa, u holda (6), (21), (22) Gursa masalasining yagona yechimi mavjud bo'ladi va (23) formula bilan aniqlanadi.

## 14–§. Telegraf tenglamasi uchun Koshi masalasi

O'tkazgichdan elektr toki o'tganda uning atrofida elektromagnit maydoni hosil bo'ladi. Bu maydon o'tkazgichdagi tok kuchi va kuchlanishni o'zgartiradi va bu o'zgarish o'tkazgichda tebranish jarayonini keltirib chiqaradi. Bunday tebranma jarayonlarni ifodalovchi tenglama matematik fizikada *telegraf tenglamasi* deb yuritiladi.

Biz ushbu paragrafda telegraf tenglamasini keltirib chiqarish va shu tenglama uchun Koshi masalasini o'rganamiz.

1. TELEGRAF TENGLAMASINI KELTIRIB CHIQARISH.

Uzunligi $l$ bo'lgan o'tkazgichni $Ox$ o'qi bo'ylab, koordinata boshiga o'tkazgichning bir uchini joylashtiraylik. O'tkazgichdan o'tayotgan elektr okimi $Ox$ o'kining musbat yo'nalishi bilan bir xil yo'nalgan bo'lsin. O'tkazgichning biror nuqtasidagi $i$ tok kuchi va $v$ kuchlanishi $x$ abssissa va $t$ vaqtning funksiyasi bo'ladi. Tok kuchi $i$ va uning $v$ kuchlanishi biror birinchi tartibli xususiy hosilali differensial tenglama bilan o'zaro bog'liq bo'ladi. O'tkazgichning birlik uzunligiga mos keluvchi qarshilik $R$, elektr sig'imi $C$, o'z induksiya koeffitsiyenti $L$ va tokning isrof bo'lish koeffitsiyenti $G$ o'zgarmas bo'lsin. O'tkazgichning ixtiyoriy $x=x_1$ va $x=x_2$ nuqtalar orasidagi qismini qaraymiz. Bu qismga Om qonunini qo'llaymiz. Natijada

$$v(x_1,t)-v(x_2,t)=R\int\limits_{x_1}^{x_2} i(x,t)dx+L\int\limits_{x_1}^{x_2}\frac{\partial i(x,t)}{\partial t}dx, \qquad (1)$$

ikkinchidan esa,

$$v(x_1,t) - v(x_2,t) = -\int_{x_1}^{x_2} \frac{\partial v(x,t)}{\partial x} dx$$

bo'ladi. U holda yuqoridagi tengliklardan

$$\int_{x_1}^{x_2} \left( \frac{\partial v(x,t)}{\partial x} + L\frac{\partial i(x,t)}{\partial t} + R\,i(x,t) \right) dx = 0$$

ifoda kelib chiqadi. Bundan esa $x_1$ va $x_2$ nuqtalarning ixtiyoriy ekanligidan $v(x,t)$ va $i(x,t)$ funksiyalarga nisbatan

$$\frac{\partial v(x,t)}{\partial x} + L\frac{\partial i(x,t)}{\partial t} + R\,i(x,t) = 0, \tag{2}$$

tenglamani olamiz.

Bir tomondan, birlik vaqt davomida o'tkazgichning $[x_1, x_2]$ qismidan o'tayotgan elektr miqdori

$$i(x_1,t) - i(x_2,t) = -\int_{x_1}^{x_2} \frac{\partial i(x,t)}{\partial x} dx, \tag{3}$$

ga teng bo'ladi. Ikkinchi tomondan esa, birlik vaqtda o'tkazgichning belgilangan qismidan o'tayotgan elektr miqdori o'tkazgichning shu qismni zaryadlashga ketgan va isrof bo'lgan elektr miqdorining yig'indisiga teng, ya'ni

$$i(x_1,t) - i(x_2,t) = C\int_{x_1}^{x_2} \frac{\partial v(x,t)}{\partial x} dx + G\int_{x_1}^{x_2} v(x,t)dx, \tag{4}$$

bo'ladi. U holda (3) va (4) formulalardan

$$\int_{x_1}^{x_2} \left( \frac{\partial i(x,t)}{\partial x} + C\frac{\partial v(x,t)}{\partial t} + G\,i(x,y) \right) dx = 0.$$

kelib chiqadi. Bundan esa ushbu

$$\frac{\partial i(x,t)}{\partial x} + C\,\frac{\partial v(x,t)}{\partial t} + G\,i(x,y) = 0, \qquad (5)$$

tenglamani olamiz.

Keltirib chiqarilgan (2) tenglamani $x$ bo'yicha, (5) tenglamani esa $t$ bo'yicha differensiallab, hosil bo'lgan ifodalardan $\dfrac{\partial^2 i(x,t)}{\partial x \partial t}$ hosilani ayiramiz. natijada $v(x,t)$ kuchlanishga nisbatan ikkinchi tartibli o'zgarmas koeffitsiyentli ushbu

$$\frac{\partial^2 v}{\partial x^2} = L\,C\,\frac{\partial^2 v}{\partial t^2} + (R\,C + G\,L)\frac{\partial v}{\partial t} + G\,R\,v,$$

tenglamaga ega bo'lamiz.

Xuddi shu usul bilan $i(x.t)$ tok kuchiga nisbatan quyidagi

$$\frac{\partial^2 i}{\partial x^2} = L\,C\,\frac{\partial^2 i}{\partial t^2} + (R\,C + G\,L)\frac{\partial i}{\partial t} + G\,R\,i.$$

tenglamani keltirib chiqarish mumkin.

Shunday qilib, o'tkazgichdagi $v$ kuchlanish va $i$ tok kuchi bir xil

$$\frac{\partial^2 w}{\partial x^2} = a_0\,\frac{\partial^2 w}{\partial t^2} + 2b_0\,\frac{\partial w}{\partial t} + c_0\,w, \qquad (6)$$

differensial tenglamani qanoatlantirishi kelib chiqdi.

Bu yerda $a_0 = LC$, $2b_0 = RC + GL$, $c_0 = GR$.

Agar (6) tenglamada quyidagicha yangi $u(x,t)$ noma'lum funksiya

$$t = \sqrt{a_0}\,y, \qquad w = \exp\left\{-\frac{b_0}{\sqrt{a_0}}\,y\right\}u$$

kiritsak. u holda bu tenglama soddaroq

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \lambda^2\,u = 0, \qquad \lambda^2 = \frac{b_0^2 - a_0c_0}{a_0},$$

ko'rinishga keladi.

Demak, o'tkazgichdan o'tayotgan $i$ tok kuchi va $v$ kuchlanishini qanoatlantiruvchi (6) tenglama *telegraf tenglamasi* deyiladi.

IZOH. Shuni ta'kidlash lozimki, telegraf tenglamasi uchun boshlang'ich–chegaraviy masalalar va siljishli masalalarning korrektligi [21] qo'llanmada A. Q. O'rinov tomonidan batafsil o'rganilgan. Biz bu yerda yuqoridagi qo'llanmadan foydalanib, telegraf tenglamasi uchun Koshi masalasini keltiramiz.

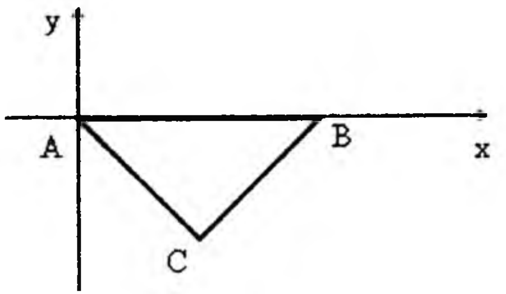
2. TELEGRAF TENGLAMASI UCHUN KOSHI MASALASI.

O'tkazgichdagi elektr tebranishlarini ifodalovchi ushbu

$$Lu \equiv u_{xx} - u_{yy} + c\,u = 0, \tag{7}$$

telegraf tenglamasini qaraylik.
Bu yerda $c = const \neq 0$, $u(x, y)$ esa tok kuchi.

(7) tenglamaning $AC : x + y = 0$, $BC : x - y = l$ xarakteristikalari va $y = 0$ o'qdagi $AB$ kesma bilan chegaralangan (xarakteristik uchburchak) soha $D$ bo'lsin. Telegraf tenglamasi uchun $D$ sohada Koshi masalasini qo'yamiz.




15 — shakl.

KOSHI MASALASI. (7) telegraf tenglamasining yopiq $\overline{D}$ sohada aniqlangan uzluksiz va quyidagi

$$u(x, y) \in C(\overline{D}) \cap C^1(D \cup AB) \cap C^2(D); \tag{8}$$

$$u(x, y)|_{y=0} = u(x, 0) = \tau(x), \quad 0 \leq x \leq l, \tag{9}$$

$$u_y(x, y)|_{y=0} = u_y(x, 0) = \nu(x), \quad 0 < x < l, \tag{10}$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping. Bu yerda $\tau(x)$ va $\nu(x)$ yetarlicha silliq berilgan funksiyalar.

Endi $xOy$ tekisligida $\xi = x+y$, $\eta = x-y$ xarakteristik o'zgaruvchilarga o'tamiz. U holda (7) tenglama

$$L_0 u \equiv u_{\xi\eta} + \frac{c}{4} u = 0, \tag{11}$$

ko'rinishga keladi. $D$ xarakteristik uchburchak esa uchlari $A_0(0,0)$, $B_0(l,l)$, $C_0(0,l)$, nuqtalarda bo'lgan $\Delta = \{(\xi,\eta) : 0 < \xi < \eta < l\}$ uchburchakli sohaga o'tadi, bu yerda $M_0 = (\xi_0, \eta_0)$, $P = (\xi_0, \xi_0)$, $Q = (\eta_0, \eta_0)$.

U holda (7)–(10) Koshi masalasi quyidagicha qo'yiladi:

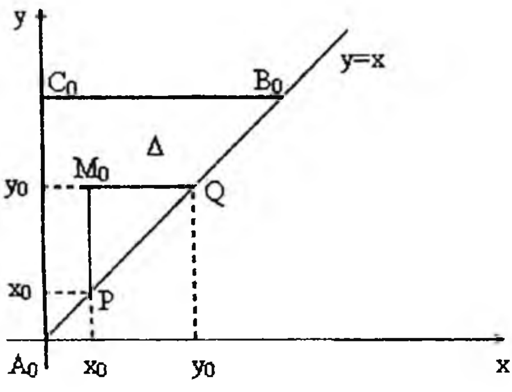
KOSHI MASALASI. (11) tenglamaning yopiq $\overline{\Delta}$ sohada aniqlangan va

$$u(\xi,\eta) \in C(\overline{\Delta}) \cap C^1(\Delta \cup A_0B_0) \cap C^2(\Delta); \tag{12}$$

$$u(x,y)|_{y=0} = u(\xi,\eta)|_{\eta=\xi} = u(\xi,\xi) = \tau(\xi), \quad 0 \le \xi \le l, \tag{13}$$

$$u_y(x,y)|_{y=0} = (u_\xi - u_\eta)|_{\eta=\xi} = \nu(\xi), \quad 0 < \xi < l, \tag{14}$$

shartlarni qanoatlantiruvchi $u(\xi,\eta)$ yechimini toping.




16 — shakl.

RIMAN FUNKSIYASINI QURISH.

Kanonik shakldagi (11) tenglamaning Riman funksiyasini

$$u = v(z) = v\left(\sqrt{c(\xi - \xi_0)(\eta_0 - \eta)}\right), \quad \xi_0 \leq \xi, \eta \leq \eta_0, \tag{15}$$

ko'rinishda izlaymiz. (15) funksiyadan kerakli tartibdagi hosilalarni hisoblaymiz:

$$u_\xi = v'(z)z_\xi, \quad u_{\xi\eta} = v''(z)z_\xi z_\eta + v'(z)z_{\xi\eta},$$

$$z_\xi = \frac{c(\eta_0 - \eta)}{2z}, \quad z_\eta = \frac{c(\xi_0 - \xi)}{2z}, \quad z_{\xi\eta} = -\frac{c}{4z}, \quad u_{\xi\eta} = -\frac{c}{4}v(z).$$

Endi (15) funksiyani va olingan hosilalarni (11) tenglamaga qo'yib, quyidagi

$$v''(z) + \frac{1}{z}v'(z) + v(z) = 0, \tag{16}$$

oddiy differensial tenglamaga ega bo'lamiz. (16) tenglamaning xususiy yechimi maxsus funksiyalar nazariyasida

$$I_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{(2 \cdot 4)^2} - \frac{z^6}{(2 \cdot 4 \cdot 6)^2} + \dots$$

nolinchi tartibli Bessel funksiyasi orqali ifodalanadi. Bundan (16) tenglamaning xususiy yechimi

$$v = I_0(z) = I_0\left(\sqrt{c(\xi - \xi_0)(\eta_0 - \eta)}\right),$$

ekanligi kelib chiqadi.

Demak, (11) tenglama uchun Riman funksiyasi

$$R(\xi, \eta; \xi_0, \eta_0) = I_0\left(\sqrt{c(\xi - \xi_0)(\eta_0 - \eta)}\right), \tag{17}$$

bo'lar ekan.

Shunday qilib, (17) formula bilan aniqlangan Riman funksiyasi quyidagi shartlarni qanoatlantiradi:

1) $L^*_{0\xi\eta}R(\xi, \eta; \xi_0, \eta_0) = 0; \qquad L^*_{0\xi_0\eta_0}R(\xi, \eta; \xi_0, \eta_0) = 0;$

$$2)\quad \left(\frac{\partial R}{\partial \xi} - bR\right)\Big|_{\eta=\eta_0} = \frac{\partial R}{\partial \xi}\Big|_{\eta=\eta_0} = I_1(z)\frac{c(\eta_0-\eta)}{2z}\Big|_{\eta=\eta_0} = 0;$$

$$3)\quad \left(\frac{\partial R}{\partial \eta} - aR\right)\Big|_{\xi=\xi_0} = \frac{\partial R}{\partial \eta}\Big|_{\xi=\xi_0} = I_1(z)\frac{c(\xi_0-\xi)}{2z}\Big|_{\xi=\xi_0} = 0;$$

hamda $\xi = \xi_0$ va $\eta = \eta_0$ bo‘lganda

$$R(\xi_0, \eta_0; \xi_0, \eta_0) = I_0(0) = 1$$

bo‘ladi.

Demak, (17) formula bilan aniqlangan $R(\xi, \eta; \xi_0, \eta_0)$ funksiya (11) tenglama uchun Riman funksiyasi bo‘ladi.

Koshi masalasi yechimini qurish. Buning uchun oldingi paragrafda topilgan (19) Riman formulasidan foydalanamiz. 12–paragrafdagi (6) tenglamada $a(x,y) = 0$, $b(x,y) = 0$, $c(x,y) = const$, $f(x,y) = 0$ bo‘lsa, telegraf tenglamasi hosil bo‘ladi. Telegraf tenglamasi uchun Koshi masalasining qo‘yilishida $AB$ egri chiziq $\xi = \eta$ tenglama bilan ifodalanadi. Xarakteristikalarning bu to‘g‘ri chiziq bilan kesishgan $P$, $Q$ nuqtalarining koordinatalari $P = (\xi_0, \xi_0)$, $Q = (\eta_0, \eta_0)$ bo‘ladi. U holda 12–§ dagi (19) Riman formulasiga asosan quyidagi

$$u(\xi_0, \eta_0) = \frac{1}{2}u(\xi_0, \xi_0)R(\xi_0, \xi_0; \xi_0, \eta_0) + \frac{1}{2}u(\eta_0, \eta_0)R(\eta_0, \eta_0; \xi_0, \eta_0))+$$

$$+\frac{1}{2}\int_{\xi_0}^{\eta_0}[u(R_\xi - R_\eta) - R(u_\xi - u_\eta)]|_{\eta=\xi}d\xi, \tag{18}$$

ifodani olamiz. Boshlang‘ich shartlarga asosan $u(\xi, \xi) = \tau(\xi)$, $(u_\xi - u_\eta)|_{\eta=\xi} = \nu(\xi)$ ekanligidan

$$R(\eta_0, \eta_0; \xi_0, \eta_0) = I_0(0) = 1, \quad R(\xi_0, \xi_0; \xi_0, \eta_0) = I_0(0) = 1;$$

$$R(\xi, \xi; \xi_0, \eta_0) = I_0(\sqrt{c(\xi - xi_0)(\eta_0 - \xi)}) = I_0(z_0);$$

$$\left(\frac{\partial R}{\partial \xi} - \frac{\partial R}{\partial \eta}\right)\Big|_{\eta=\xi} == \left[I_0'(z)\frac{c(\eta-\eta_0)}{2z} - I_0'(z)\frac{c(\xi_0-\xi)}{2z}\right]\Big|_{\eta=\xi} =$$

$$= \frac{c(\eta_0 - \xi_0)}{2}\frac{I_1(z)}{z_0} = \frac{c(\eta_0 - \xi_0)}{2}\overline{I}_1(z_0);$$

bo'ladi. Bu yerda $\overline{I}_1(z_0) = \dfrac{I_1(z)}{z_0}$, $z_0 = z|_{\eta=\xi} = \sqrt{c(\xi - \xi_0(\eta_0 - \xi))}$.

U holda (18) formula quyidagi

$$u(\xi_0, \eta_0) = \frac{\tau(\xi_0) + \tau(\eta_0)}{2} + \frac{c(\eta_0 - \xi_0)}{2}\int\limits_{\xi_0}^{\eta_0} \tau(\xi)\overline{I}_1[\sqrt{c(\xi - \xi_0)(\eta_0 - \xi)}]d\xi -$$

$$-\frac{1}{2}\int\limits_{\xi_0}^{\eta_0} \nu(\xi)\, I_0[\sqrt{c(\xi - \xi_0)(\eta_0 - \xi)}]d\xi, \tag{19}$$

ko'rinishga keladi.

Demak, telegraf tenglamasi uchun Koshi masalasining yechimi (19) formula orqali ifodalanadi.

IZOH. Riman funksiyasi $l$ egri chiziqning ko'rinishiga va boshlang'ich shartlarning berilishiga bog'liq emas.

4–MASALA. Giperbolik tipdagi

$$x^2 u_{xx} - y^2 u_{yy} = 0, \tag{20}$$

tenglamaning

$$u(x, 1) = f_1(x), \quad u_y(x, 1) = f_2(x) \tag{21}$$

shartlarni qanoatlantiruvchi regulyar yechimini Riman usuli bilan toping.

YECHISH. Berilgan tenglama

$$\xi = xy, \quad \eta = \frac{y}{x} \tag{22}$$

almashtirish yordamida

$$u_{\xi\xi} - \frac{1}{2\xi}u_\eta = 0 \tag{23}$$

kanonik ko'rinishga keladi. $y = 1$ to'g'ri chiziq tenglamasi yangi $\xi$, $\eta$ o'zgaruvchilarda $\xi\eta = 1$ ko'rinishda yoziladi. (22) tengliklarga asosan

$$x = \sqrt{\frac{\xi}{\eta}}, \quad y = \sqrt{\xi\eta}$$

ekanligidan

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi\eta=1} = \left.\left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2\xi}\frac{\partial u}{\partial y}\right)\right|_{\xi\eta=1}; \quad \left.\frac{\partial u}{\partial \eta}\right|_{\xi\eta=1} = \left.\left(-\frac{\xi^2}{2}\frac{\partial u}{\partial x} + \frac{\xi}{2}\frac{\partial u}{\partial y}\right)\right|_{\xi\eta=1};$$

tengliklarni hosil qilamiz.

Bu tengliklardan (21) boshlang'ich shartlarni e'tiborga olib,

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi\eta=1} = \frac{1}{2}f_1'(\xi) + \frac{1}{2\xi}f_2(\xi), \tag{24}$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\xi\eta=1} = -\frac{\xi^2}{2}f_1'(\xi) + \frac{\xi}{2}f_2(\xi), \tag{25}$$

$$u\,|_{\xi\eta=1} = 1, \tag{26}$$

ifodalarga ega bo'lamiz.

12–§ da keltirilgan (19) formulada integrallash tartibini o'zgartirib, $a = 0$, $2b = -\frac{1}{2\xi}$, $f = 0$ desak, masalaning yechimi quyidagi

$$u(\xi_0, \eta_0) = \frac{1}{2}\Big[(uv)_P + (uv)_Q\Big] +$$

$$+\frac{1}{2}\int_Q^P \left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi} - \frac{uv}{\xi}\right)d\xi - \left(v\frac{\partial u}{\partial \eta} - u\frac{\partial v}{\partial \eta}\right)d\eta. \tag{27}$$

ko'rinishda yoziladi.

Endi $v(\xi, \eta; \xi_0, \eta_0)$ Riman funksiyasini tuzamiz. Buning uchun

$$v_{\xi\eta} + \frac{1}{2\xi}v_\eta = 0 \tag{28}$$

tenglamaning

$$v(\xi,\eta_0;\xi_0,\eta_0)=\sqrt{\frac{\xi_0}{\xi}};\quad v(\xi_0,\eta_0;\xi_0,\eta_0)=1 \tag{29}$$

shartlarni qanoatlantiruvchi yechimini topish zarur.

Bevosita tekshirish bilan

$$v(\xi,\eta;\xi_0,\eta_0)=\sqrt{\frac{\xi_0}{\xi}}, \tag{30}$$

funksiya (28)–(29) masalaning yechimi bo'lishiga ishonch hosil qilish mumkin.

(24)–(26) tengliklardan va (30) Riman funksiyasidan foydalanib,

$$u(P)=f_1(\xi_0),\qquad u(Q)=f_1\left(\frac{1}{\eta_0}\right),$$

$$v(P)=v\left(\xi_0,\frac{1}{\xi_0};\xi_0,\eta_0\right)=1;$$

$$v(Q)=v\left(\frac{1}{\eta_0},\eta_0;\xi_0,\eta_0\right)=\sqrt{\xi_0\eta_0}$$

ekanligini inobatga olsak, masala yechimi uchun

$$u(\xi_0,\eta_0)=\frac{1}{2}f_1(\xi_0)+\frac{\sqrt{\xi_0\eta_0}}{2}f_1\left(\frac{1}{\eta_0}\right)+$$

$$+\frac{\sqrt{\xi_0}}{2}\int\limits_{\xi_0}^{1/\eta_0}\frac{f_1(\xi)}{\xi^{3/2}}d\xi-\frac{\sqrt{\xi_0}}{2}\int\limits_{\xi_0}^{1/\eta_0}\frac{f_1(\xi)}{\xi^{3/2}}d\xi, \tag{31}$$

formulaga ega bo'lamiz.

Bu yerda eski $x$, $y$ o'zgaruvchilarga qaytib, (21)–(22) masalaning yechimi

$$u(x,y)=\frac{1}{2}f_1(xy)+\frac{y}{2}f_1\left(\frac{x}{y}\right)+$$

$$+\frac{\sqrt{xy}}{4}\int\limits_{xy}^{x/y}\frac{f_1(\xi)}{\xi^{3/2}}d\xi-\frac{\sqrt{xy}}{2}\int\limits_{xy}^{x/y}\frac{f_2(\xi)}{\xi^{3/2}}d\xi \qquad (32)$$

ko'rinishda topiladi.

5–MASALA. *Tor tebranish tenglamasining*

$$u_{tt}=a^2u_{xx}, \quad a=const, \quad -\infty<x<+\infty, \quad t>0, \qquad (33)$$

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \quad -\infty<x<+\infty, \qquad (34)$$

*shartlarni qanoatlantiruvchi regulyar yechimini toping.*

YECHISH. Berilgan tenglamaning xarakteristikalari

$$x-at=c_1, \quad x+at=c_2,$$

bo'lgani uchun

$$\xi=x-at, \quad \eta=x+at,$$

almashtirish (33) tenglamani

$$u_{\xi\eta}=0, \qquad (35)$$

kanonik ko'rinishga keladi.

Berilgan masalada $l$ chiziq $t=0$, ya'ni $ox$ o'qidan iborat. $t=0$ da $\xi=x$, $\eta=\xi$ bo'lganligi uchun

$$u|_{\eta=\xi}=\varphi(\xi), \qquad \left(\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}\right)\Bigg|_{\eta=\xi}=-\frac{1}{a}\psi(\xi) \qquad (36)$$

bo'ladi.

(33) tenglama uchun Riman funksiyasi

$$v_{\xi\eta}=0$$

tenglamaning

$$v|_{\xi=\xi_0}=1, \quad v|_{\eta=\eta_0}=1$$

shartlarni qanoatlantiruvchi yechimidan iborat.

$a(x,y)=0$, $b(x,y)=0$ bo'lgani uchun Riman funksiyasi

$$v(\xi,\eta;\xi_0,\eta_0)=1$$

bo'ladi.

Yuqorida 12–§da keltirilgan (19) Riman formulasi va (36) shartlarga asosan hamda

$$H=\frac{1}{2}\frac{\partial u}{\partial\eta},\quad K=\frac{1}{2}\frac{\partial u}{\partial\xi},\quad f=0,$$

ekanligidan masalaning yechimi

$$u(\xi_0,\eta_o)=\frac{\varphi(\xi_0)+\varphi(\eta_0)}{2}+\frac{1}{2a}\int_{\xi_0}^{\eta_0}\psi(z)dz,$$

ko'rinishda hosil bo'ladi.

Bu yerda $\xi_0=x-at$, $\eta_0=x+at$ tengliklarga asosan eski $x$ va $t$ o'zgaruvchilarga qaytilsa, (33)–(34) Koshi masalasining D'alamber usuli bilan topilgan yechimi

$$u(x,t)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(z)dz$$

kelib chiqadi.

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## Nazorat uchun savollar

1. Tor tebranish tenglamasi uchun Koshi masalasi qanday qo'yiladi? Bu masala korrekt bo'ladimi?

2. Tor tebranish tenglamasi uchun Koshi masalasi qanday jarayonni ifodalaydi?

3. Yechimning berilganlarga uzluksiz bog'liqligini qanday tushunasiz va yechimning turg'unligi nima degani?

4. D'alamber formulasi tor tebranish tenglamasi uchun Koshi masalasining yechimi bo'lishi uchun $\varphi_0(x)$ va $\varphi_1(x)$ funksiyalar qanday shartlarni qanoatlantiradi?

5. Giperbolik tipdagi tenglama uchun asosiy chegaraviy masalalar qanday qo'yiladi?

6. Chegaraviy shartlar necha xil bo'ladi?

7. Tor tebranish tenglamasi uchun Koshi–Gursa masalasi qanday qo'yiladi?

8. Giperbolik tenglama uchun Darbu masalasi qanday qo'yiladi?

9. Tor tebranish tenglamasi uchun aralash masalalar qanday qo'yiladi?

10. Tor tebranish tenglamasi uchun aralash masala yechimining yagonaligi qanday isbotlanadi?

11. Aralash masala yechimining mavjudligini isbotlash sxemasini izohlab bering.

12. Agar $\varphi_0(x) = 0$ bo'lsa, u holda Fur'e $a_k$ koeffisienti haqida nima deyish mumkin?

13. Fur'e usuli qanday masalalarni yechishda qo'llaniladi?

14. Chiziqli giperbolik tipdagi tenglamalar uchun umumlashgan Koshi masalasi qanday qo'yiladi?

15. Chiziqli giperbolik tipdagi tenglamalar uchun umumlashgan Gursa masalasi qanday qo'yiladi?

16. Qo'shma operator deganda nimani tushunasiz?

17. Qanday funksiyaga Riman funksiyasi deyiladi?

18. Riman funksiyasi qanday shartlarni qanoatlantiradi?

19. Riman funksiyasining asosiy xossalarini ayting.

## Mustaqil yechish uchun misol va masalalar

3.1. Quyidagi tenglamalarning umumiy yechimini toping.

1) $4xu_{xy} - yu_{yy} + 3u_y = 0.$
2) $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0.$
3) $3xu_{xy} - yu_{yy} - 2u_y = 0.$
4) $u_{xx} - 2\cos x u_{xy} + \cos^2 x u_{yy} - 2u_x + (2\cos x + \sin x)u_y = 0.$
5) $u_{xx} - 2\frac{x}{y}u_{xy} + \frac{x^2}{y^2}u_{yy} - \frac{2}{x}u_x + \frac{y^2 - x^2}{y^3}u_y - x^3 = 0.$
6) $4u_{xx} - 4u_{xy} + u_{yy} - 2u_x + u_y = 0.$
7) $u_{xx} - 6u_{xy} + 8u_{yy} + u_x - 2u_y = 0.$
8) $\sin y u_{xy} + u_{yy} + \left(\frac{\sin x}{y} - \operatorname{ctg} y\right) u_y = 0.$

3.2. Quyidagi Koshi masalalarini yeching.

1) $u_{xy} = 3, \quad u(x,y)\,|_{y=x} = \sin x, \quad u_y(x,y)\,|_{y=x} = \cos x.$
2) $u_{xy} = 0, \quad u(x,x^2) = 0, \quad u_y(x,x^2) = \sqrt{|x|}, \quad |x| < 1.$
3) $u_{xy} + u_x = 0, \quad u(x,x) = \sin x, \quad u_x(x,x) = 1.$
4) $u_{xx} + 2u_{xy} - 3u_{yy} = 0,$
$u(x,0) = 0, \quad u_y(x,0) = x.$
5) $4xu_{xx} - yu_{yy} + 3u_y = 0,$
$u(x,1) = x^2 + 1, \quad u_y(x,1) = 4.$
6) $3xu_{xx} - yu_{xy} + 4u_x = 0,$
$u(1,y) = 1 + y^4, \quad u_x(1,y) = y^4.$
7) $3xu_{xy} - 2yu_{yy} + 2u_y = 0,$
$u(x,1) = 3x^2, \quad u_y(x,1) = 4 - x^2.$
8) $t^2u_{tt} - x^2u_{xx} = 0,$
$u(x,1) = 2\sqrt{x}, \quad u_t(x,1) = \sqrt{x}.$

9) $2xu_{xx} - yu_{xy} + 5u_x = 0,$
$u(1, y) = 0, \quad u_x(1, y) = y^5.$

10) $u_{tt} = u_{xx} + axt,$
$u(x, 0) = x, \quad u_t(x, 0) = \sin x.$

3.3. Uzunligi $l = 1$ bo'lgan uchlari mahkamlangan boshlang'ich holati $\varphi(x) = A\sin n\pi x$ va boshlang'ich tezligi $\psi(x) = 0$ bo'lgan torning tebranishi haqidagi masalani yeching.

3.4. Uzunligi $l = 1$ bo'lgan uchlari mahkamlangan boshlang'ich holati $\varphi(x) = x(1-x)$ va boshlang'ich tezligi $\psi(x) = 0$ bo'lgan torning tebranishi haqidagi masalani yeching.

3.5. Uzunligi $l = 1$ bo'lgan uchlari mahkamlangan boshlang'ich holati $\varphi(x) = 0$ va boshlang'ich tezligi $\psi(x) = \alpha_0$ bo'lgan torning tebranishi haqidagi masalani yeching.

3.6. Uzunligi $l$ bo'lgan uchlari mahkamlangan boshlang'ich holati $\varphi(x) = 0$ va boshlang'ich tezligi $\psi(x) = \sin\dfrac{2\pi x}{l}$ bo'lgan torning tebranishi haqidagi masalani yeching.

3.7. Chekli $D = \{(x, t) : 0 < x < l, \ t > 0\}$ sohada $u_{tt} = a^2 u_{xx}$ tenglamaning $u(0, t) = u_x(l, t) = 0$ chegaraviy va

$$u(x, 0) = \sin\frac{5\pi}{2l}x, \quad u_t(x, 0) = \cos\frac{\pi}{2l}x$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

3.8. Ushbu $u_{tt} = a^2 u_{xx} + f(x, t)$ tenglamaning $D = \{(x, t) : 0 < x < l, \ t > 0\}$ sohada kuyidagi

$$u(0, t) = 0, \quad u_x(l, t) + hu(l, t) = 0, \quad h > 0$$

chegaraviy va

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = 0$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

3.9. Ushbu $u_{tt} = a^2 u_{xx} + f(x, t)$ tenglamaning $D = \{(x, t) : 0 < x < l, \ t > 0\}$ sohada quyidagi

$$u(0, t) = 0, \quad u(l, t) = 0$$

chegaraviy va

$$u(x,0) = \frac{\beta - \alpha}{l}x + \alpha, \quad u_t(x,0) = 0$$

boshlang'ich shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

3.10. Ushbu $u_{xx} - u_{tt} + \lambda u = 0$ tenglamaning Riman funksiyasi

$$v(\xi,\eta;\xi_1,\eta_1) = J_o\left(\mu\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\right), \quad \mu = -\lambda^2,$$

ekanligini ko'rsating.

3.11. Quyidagi Koshi masalalarini Riman usuli bilan yeching.

1) $u_{tt} = a^2 u_{xx} + c^2 u + f(x,y), \quad -\infty < x < +\infty, quadt > 0.$
$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < +\infty.$

2) $u_{tt} = a^2 u_{xx} - c^2 u, \quad -\infty < x < +\infty, \quad t > 0,$
$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < +\infty.$

3) $u_{tt} = u_{xx} - \lambda u + 1, \quad -\infty < x < +\infty, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = 0, \quad -\infty < x < +\infty.$

4) $u_{xy} + 2u_x + u_y + 2u = 1, \quad 0 < x, y < 1;$
$u(x,y)|_{x+y=1} = x, \quad u_x(x,y)|_{x+y=1} = x.$

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# IV BOB

# PARABOLIK TIPDAGI TENGLAMALAR

Parabolik tipdagi tenglamalar issiqlik tarqalishi, diffuziya hodisalarini va boshqa ko‘plab fizikaviy jarayonlarni o‘rganishda ko‘p uchraydi.

Ushbu bobda asosan, parabolik tipdagi tenglamalarning sodda vakili bo‘lgan

$$Lu \equiv \frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t),$$

issiqlik tarqalish tenglamasi uchun boshlang‘ich–chegaraviy masalalarning qo‘yilishi va ularning yechish usullari bilan tanishamiz.

## 15–§. Parabolik tipdagi tenglamalar uchun boshlang‘ich va chegaraviy shartlar

Issiqlik tarqalish jarayoni sodir bo‘layotgan uzunligi $l$ bo‘lgan sterjenni qaraylik. Sterjenning o‘qi sifatida $Ox$ abssissa o‘qini olamiz.

Faraz qilaylik, ixtiyoriy vaqtda sterjenning barcha nuqtasida bir xil harorat saqlansin. Sterjenning ixtiyoriy $x$ nuqtasining $t$ vaqtdagi temperaturasini $u = u(x,t)$ deb belgilaylik.

Agar sterjenning issiqlik sig‘imi $c$, uning zichligi $\rho$, sterjenning issiqlik o‘tkazuvchanlik koeffitsiyenti $k$ hamda ichki issiqlik manbaining zichligi $F$ bo‘lsa, u holda $u(x,t)$ funksiya quyidagi bir o‘lchovli

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) + F, \quad 0 < x < l, \quad t > 0$$

issiqlik tarqalish tenglamasini qanoatlantirishini ko‘rsatish qiyin emas (I bob, 3–§).

Umuman olganda, $c$, $\rho$, $k$ va $F$ parametrlar $x$, $t$ va $u$ ning funksiyasi bo‘ladi. Ko‘plab tadbiqiy masalalarda bu funksiyalar sterjenda temperaturaning o'zgarishi bilan juda sekin o'zgaradi va $c$, $\rho$, $k$ funksiyalar $t$ vaqtga bog'liq bo'lmaydi. Shuning uchun ular faqat $x$ o'zgaruvchining funksiyasi, $F$ ni esa $x$ va $t$ ga bog'liq deb olish mumkin.

Agar qaralayotgan $l$ uzunlikdagi sterjen bir jinsli bo'lsa, u holda $c$, $\rho$, $k$ funksiyalar o'zgarmasga teng bo'ladi va yuqoridagi tenglamani

$$Lu \equiv \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{1}$$

ko'rinishda yozib olishimiz mumkin. Bu yerda $a^2 = \dfrac{k}{c\rho}$, $f = \dfrac{F}{c\rho}$.

Agar sterjenning harorati barcha nuqtasida bir xil bo'lmasa, u holda sterjenda issiqlik oqimi sodir bo'ladi. Bunda issiqlik oqimi sterjenning yuqori haroratli nuqtasidan past haroratli nuqtasi tomonga yo'nalgan bo'ladi.

Sterjenning $x$ ko'ndalang kesimi orqali birlik vaqtda $Ox$ o'qi bo'ylab o'tayotgan issiqlik miqdori uchun

$$q(x,t) = -k(x)\,\frac{\partial u(x,t)}{\partial x}$$

formula o'rinli bo'ladi. Bu yerda $q(x,t)$ funksiya *issiqlik oqimining zichligi* deyiladi.

Agar sterjenning $x_0$ nuqtasi orqali $(t_0, t_0+dt)$ vaqtda issiqlik $Ox$ o'qi bo'ylab tarqalayotgan bo'lsa, u holda $q(x,t)$ funksiyani $(x_0, t_0)$ nuqta atrofida musbat, aks holda manfiy deb olinadi.

Sterjenda issiqlik tarqalishini to'la aniqlash uchun (1) tenglamaning o'zi etarli bo'lmaydi. Buning uchun sterjenning boshlang'ich temperaturasini va uning uchlaridagi issiqlik rejimini bilish zarur bo'ladi.

Faraz qilaylik, $t = 0$ vaqtda sterjenning $x$ nuqtasidagi harorati $\varphi(x)$ bo'lsin. U holda

$$u(x,0) = \varphi(x), \qquad 0 \le x \le l \tag{2}$$

boshlang'ich shart beriladi.

Sterjenning $x = 0$ va $x = l$ chetlarida uning harorati yoki issiqlik oqimining zichligi ma'lum bo'lishi yoki atrof-muhit bilan issiqlik almashinish shartlarini berish mumkin.

Agar sterjenning $x = 0$ uchida $\mu_1(t)$ temperatura va $x = l$ uchida issiqlik oqimining zichligi $\nu_l(t)$ ma'lum bo'lsa, u holda

$$u(0,t) = \mu_1(t), \qquad -k(l)\,\frac{\partial u(l,t)}{\partial x} = \nu_l(t) \tag{3}$$

chegaraviy shartlar beriladi.

Agar sterjenning $x = 0$ va $x = l$ uchlarida issiqlik okimining zichligi nolga teng bo'lsa, u holda sterjenning uchlari *issiqlik o'tkazmaydigan* deyiladi.

Masalan, agar sterjenning $x = l$ uchi issiqlik o'tkazmaydigan bo'lsa, bu holda

$$\frac{\partial u(l,t)}{\partial x} = 0 \tag{4}$$

chegaraviy shart beriladi.

Agar sterjenning uchlarida atrof-muxit bilan issiqlik almashinishi sodir bo'layotgan bo'lsa, u holda birlik vaqtda sterjenning $x$ kesimidan atrof-muhitga chiqayotgan issiqlik miqdori sterjenning temperaturasidan atrof-muhit temperaturasining ayrimasiga proporsional bo'ladi, ya'ni

$$q = H(u - u_0),$$

bu yerda $H$ – issiqlik almashinish koeffitsiyenti, $u$ – sterjenning, $u_0$ esa atrof muxitning temperaturasi.

Issiqlik oqimi zichligining fizikaviy xossasiga asosan sterjenning $x = 0$ va $x = l$ uchlarida

$$k(0)\,\frac{\partial u(0,t)}{\partial x} = H_1[u(0,t) - p_1(t)], \tag{5}$$

$$k(l)\,\frac{\partial u(l,t)}{\partial x} = H_2[u(l,t) - p_2(t)], \tag{6}$$

issiqlik almashinish shartlarini olamiz.

Bu yerda $H_1$ va $H_2$ – sterjenning mos ravishda chap va o'ng uchlarining issiqlik o'tkazuvchanlik koeffitsiyenti, $p_1(t)$ va $p_2(t)$ mos ravishda sterjenning uchlari atrofining harorati.

Yuqoridagi (5)–(6) chegaraviy shartlarni quyidagi

$$\frac{\partial u(0,t)}{\partial x} - h_1 u(0,t) = \mu_1(t), \tag{7}$$

$$\frac{\partial u(l,t)}{\partial x} - h_2 u(l,t) = \mu_2(t), \tag{8}$$

ko'rinishda yozib olish mumkin. Bu yerda

$$h_1 = \frac{H_1}{k(0)} > 0, \qquad h_2 = \frac{H_2}{k(l)} > 0,$$

$$\mu_1(t) = -\frac{H_1}{k(0)} p_1(t), \qquad \mu_2(t) = \frac{H_2}{k(l)} p_2(t).$$

Issiqlik tarqalish tenglamasi uchun yuqorida keltirilgan chegaraviy shartlarni umumiy holda

$$\alpha_1 \frac{\partial u(0,t)}{\partial x} + \beta_1 u(0,t) = \mu_1(t), \quad t > 0, \tag{9}$$

$$\alpha_2 \frac{\partial u(l,t)}{\partial x} + \beta_2 u(l,t) = \mu_2(t), \quad t > 0 \tag{10}$$

yozish mumkin. Bunda $\alpha_1$, $\alpha_2$, $\beta_1$ va $\beta_2$ – berilgan o'zgarmaslar, ular uchun ushbu $\alpha_1^2 + \beta_1^2 > 0$, $\alpha_1^2 + \beta_1^2 > 0$ tengsizliklar o'rinli, $\mu_1(t)$ va $\mu_2(t)$ berilgan funksiyalar.

Agar $\alpha_i = 0$ va $\beta_i \neq 0$ bo'lsa, u holda (9), (10) shartlar

$$u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t)$$

kurinishni oladi va ular *birinchi tur chegaraviy shartlar* deyiladi.

Agar $\alpha_i \neq 0$ va $\beta_i = 0$ bo'lsa, u holda (9), (10) shartlar *ikkinchi tur chegaraviy shartlar* deyiladi va ular

$$\frac{\partial u(0,t)}{\partial x} = \nu_1(t), \qquad \frac{\partial u(0,t)}{\partial x} = \nu_2(t)$$

kurinishda ifodalanadi.

Agar $\alpha_i \neq 0$ va $\beta_i \neq 0$ bo'lsa, u holda (9), (10) shartlar *uchinchi tur chegaraviy shartlar* deyiladi.

## 16–§. Issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masala

MASALANING QO'YILISHI. YECHIMNING YAGONALIGI. Berilgan chekli $Q = \{(x,t) : 0 < x < l, 0 < t < T\}$ sohada

$$Lu \equiv \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{1}$$

tenglamaning

$$u|_{t=0} = \varphi(x), \qquad 0 \leq x \leq l; \tag{2}$$

boshlang'ich va

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \qquad 0 \leq t \leq T, \tag{3}$$

chegaraviy shartlarni qanoatlantiradigan yechimini topish masalasi *birinchi chegaraviy masala* deb yuritiladi.

Bu yerda $l$ uchi koordinat boshida bo'lgan sterjenining uzunligini, $T$ esa shu fizik jarayonni o'rganish qancha vaqt davom etishini bildiradi, $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ – berilgan funksiyalar.

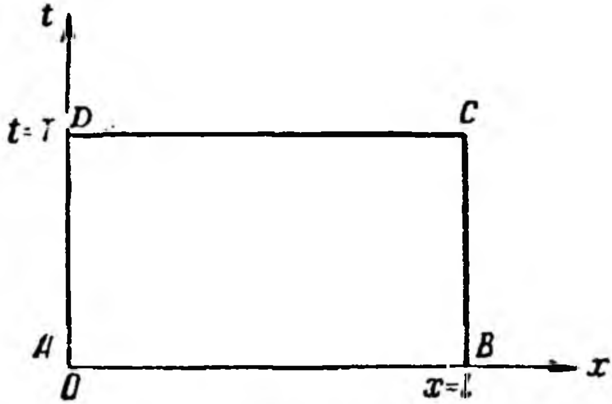
Biz izlanayotgan $u(x,t)$ yechimni $\overline{Q}$ yopiq sohada uzluksiz funksiya deb faraz qilamiz va shuning uchun berilgan $f(x,t)$, $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ funksiyalarni uzluksizligini va demak,

$$\varphi(0) = \mu_1(0), \quad \varphi(l) = \mu_2(0)$$

bo'lishini talab qilamiz.

*I* bobdan ma'lumki, (1)–(3) chegaraviy masala biror qattiq jismning boshlang'ich harorati $\varphi(x)$, uning chegarasidagi harorati ma'lum bo'lsa, bu jismning $\forall t \in [0,T]$ vaqtdagi $u(x,t)$ haroratni aniqlaydi.

Yuqorida qo'yilgan (1)–(3) masalani yechishda $t > 0$ bo'lishi juda muhim, chunki tor tebranish tenglamasidan farqli o'laroq $t$ vaqtni $-t$ ga almashtirsak, (1) tenglama tubdan o'zgarib ketadi. Shuni ta'kidlash muhimki, agar (1) tenglamada $t$ vaqt $-t$ ga almashtirilsa, qaralayotgan boshlang'ich–chegaraviy masala umuman yechimga ega bo'lmaydi. Bu fizikaviy jarayonlarga va tenglamani keltirib chiqarilishiga bevosita bog'liq.




17 — shakl.

Biz qo'yilgan (1)–(3) birinchi boshlang'ich–chegaraviy masalani to'liq o'rganish bilan cheklanamiz. Avval bu masala yechimining yagona ekanligini ekstremum prinsipi yordamida ko'rsatamiz va yechimning turg'unligini isbotlaymiz. (1)–(3) masala yechimining mavjudligi esa matematik fizikada keng ko'llaniladigan usullardan biri o'zgaruvchilarni ajratish, Fur'e usuli bilan ko'rsatamiz.

EKSTREMUM PRINSIPI.

1–TEOREMA. Yopiq $\overline{Q}$ sohada uzluksiz bo'lgan va $Q$ soha ichida bir jinsli

$$u_t = a^2 u_{xx} \tag{6}$$

issiqlik o'tkazuvchanlik tenglamasini qanoatlantiruvchi $u(x,t)$ funksiya o'zining eng katta va eng kichik qiymatlariga $\Gamma$ chiziqda erishadi.

Bu yerda $\Gamma$ qaralayotgan $Q$ to'rtburchakning $t = 0$, $x = 0$ va $x = l$ chiziqlar ustida yotgan chegaralarining yig'indisi.

ISBOT. $u(x,t)$ funksiyaning $Q$ to'rtburchakdagi eng katta qiymati $M$, ya'ni $\max\limits_{D}|u(x,t)| = M$ va chiziq ustidagi eng katta qiymati esa $m$, ya'ni $\max\limits_{\Gamma}|u(x,t)| = m$ deb belgilaymiz.

Faraz qilaylik, $Q$ to'rtburchakda shunday $(x^*, t^*)$ ichki nuqta topilsinki, bu nuqtada $M > m$ bo'lsin, bu yerda $t^* > 0$, $0 < x^* < l$.

Quyidagi yordamchi

$$v(x,t) = u(x,t) + \frac{M-m}{4l^2}(x-x^*)^2$$

funksiyani qaraylik. $Q$ to'rtburchakning asosi $t = 0$ da, yon tomonlari $x = 0$ va $x = l$ da

$$v(x,t) \le m + \frac{M-m}{4} = \frac{M}{4} + \frac{3m}{4} = \theta M < M, \quad 0 < \theta < 1$$

bo'lishini ko'rish qiyin emas.

Shu bilan birga $v(x^*, t^*) = u(x^*, t^*) = M$. Demak, yordamchi $v(x,t)$ funksiya ham $u(x,t)$ funksiya kabi o'zining eng katta qiymatiga $\Gamma$ da erishmaydi.

Shunday ekan, $v(x,t)$ funksiya o'zining eng katta qiymatiga biror $(x_1, t_1)$ $(0 < x_1 < l,\ 0 < t_1 < T)$ nuqtada erishsin.

U holda, matematik analiz kursidan ma'lumki, shu nuqtada $\dfrac{\partial^2 v}{\partial x_1^2} \le 0$ va $\dfrac{\partial v}{\partial t_1} \ge 0$ bo'ladi.

Agar $t_1 < T$ bo'lsa, u holda bu nuqtada $\dfrac{\partial v}{\partial t_1} = 0$ bo'ladi.

Agar $t_1 = T$ bo'lganda esa, $\dfrac{\partial v}{\partial t_1} \ge 0$ munosabat o'rinli bo'ladi. Demak, $(x_1, t_1)$ nuqtada

$$v_t - a^2 v_{xx} \ge 0 \tag{7}$$

tengsizlik bajariladi.

Ikkinchi tomondan esa

$$v_t - a^2 v_{xx} = u_t - a^2 u_{xx} - \frac{M-m}{2l^2} = -\frac{M-m}{2l^2} < 0$$

bo'lishi kerak. Bu esa (7) tengsizlikka zid.

Demak, $M > m$ bo'ladigan nuqta topilsin, degan farazimiz noto'g'ri ekan. Ekstremum prinsipini eng katta qiymat uchun isbotlandi. Eng kichik qiymat uchun ham ekstremum prinsipi xuddi shunday isbotlanadi.

1–XULOSA. Agar $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lib, yopiq $\overline{Q}$ sohada eng katta (eng kichik) qiymatiga ega bo'lsa, u holda bu funksiya $\overline{Q}$ sohada o'zgarmasdir.

ISBOT. Faraz qilaylik, $\forall\,(x,t) \in \overline{Q}$ da $u(x,t) \neq 0$ bo'lsin. U holda ekstremum prinsipiga asosan $u(x,t)$ funksiya $\forall\,(x,t) \in \overline{Q}$ sohada o'zining eng katta (eng kichik) qiymatiga $\Gamma$ chegarada erishadi. Bu esa shartga zid.

2–XULOSA. Agar $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lsa, u holda $\forall\,(x,t) \in \overline{Q}$ uchun quyidagi tengsizliklar o'rinli:

1) $\min\limits_{\Gamma} u(x,t) \le u(x,t) \le \max\limits_{\Gamma} u(x,t)$;

2) $|u(x,t)| \le \max\limits_{\Gamma} |u(x,t)|$.

3–XULOSA. Faraz qilaylik $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lsin. Agar $\forall\,(x,t) \in \Gamma$ uchun $u(x,t) \ge 0 (\le 0)$ bo'lsa, u holda $u(x,t) \ge 0 (\le 0)$, $\forall\,(x,t) \in \overline{Q}$ bo'ladi.

Ekstremum prinsipidan foydalanib, (1)–(3) birinchi chegaraviy masala yechimining yagonaligi va turg'unligini isbot qilaylik.

2–TEOREMA. Agar issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning yechimi mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. Haqiqatan ham, agar qaralayotgan masala bir xil boshlang'ich va chegaraviy shartlarni qanoatlantiruvchi ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimlarga ega bo'lsa, ularning ayirmasi $u(x,t) = u_1(x,t) - u_2(x,t)$ bir jinsli (6) issiqlik o'tkazuvchanlik tenglamasini hamda bir jinsli boshlang'ich

$$u(x,0) = u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0,$$

va

$$u(0,t) = u_1(0,t) - u_2(0,t) = \mu_1(t) - \mu_1(t) = 0,$$

$$u(l,t) = u_1(l,t) - u_2(l,t) = \mu_2(t) - \mu_2(t) = 0,$$

chegaraviy shartlarni qanoatlantiradi. U holda ekstremum prinsipiga ko'ra $\min\limits_{\overline{Q}} u(x,t)$ va $\max\limits_{\overline{Q}} u(x,t)$ $Q$ sohaning $\Gamma$ chegarasida erishadi.

Bir jinsli shartlarga asosan $Q$ sohaning chegarasida $u(x,t)$ funksiya nolga teng. Demak $\forall(x,t) \in \overline{Q}$ uchun $u(x,t) \equiv 0$ bo'ladi. Bundan $u_1(x,t) \equiv u_2(x,t)$ ekanligi kelib chiqadi.

3–TEOREMA. Issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning yechimi $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ funksiyalarga uzluksiz bog'liq bo'ladi.

ISBOT. Faraz qilaylik, $u_1(x,t)$ funksiya (1)–(3) birinchi chegaraviy masalaning $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ funksiyalarga bog'liq bo'lgan yechimi, $u_2(x,t)$ funksiya esa $f^*(x,t)$, $\varphi^*(x)$, $\mu_1^*(t)$ va $\mu_2^*(t)$ funksiyalarga bog'liq bo'lgan yechimi bo'lsin. Berilgan funksiyalar uchun

$$|f(x,t) - f^*(x,t)| < \varepsilon; \quad \forall(x,t) \in Q;$$

$$|\varphi(x) - \varphi^*(x)| < \varepsilon, \quad 0 \le x \le l;$$

$$|\mu_i(t) - \mu_i^*(t)| < \varepsilon, \quad i = 1,2, \quad 0 \le t \le T;$$

tengsizliklar bajarilsin. U holda $\forall(x,t) \in Q$ uchun ekstremum prinsipidan kelib chiqqan 2–xulosaga ko'ra

$$\begin{aligned}|u_1(x,t) - u_2(x,t)| &\le \max_{\Gamma} |u_1(x,t) - u_2(x,t)| = \\ &= \max\{\max_{\overline{Q}} |f(x,t) - f^*(x,t)|, \\ &\max_{x\in[0,l]} |\varphi(x) - \varphi^*(x)|, \max_{t\in[0,T]} |\mu_i(t) - \mu_i^*(t)|\}\end{aligned}$$

bo'ladi. Bundan $Q$ sohada $|u_1(x,t) - u_2(x,t)| < \varepsilon$ tengsizlikni olamiz. Bu tengsizlik (1)–(3) masala yechimining turg'un ekanligini bildiradi.

## 17–§. Birinchi chegaraviy masala yechimining mavjudligi

Cekli sterjenda issiqlik tarqalish tenglamasi uchun qo'yilgan (1)–(3) chegaraviy masala yechimining mavjudligini o'zgaruvchilarni ajratish usuli, ya'ni Fur'e usuli bilan isbotlaymiz.

CHETLARI O'ZGARMAS HARORATDA BO'LGAN STERJENDA ISSIQLIK TARQALISHI. Bu yerda biz $Q$ sohada bir jinsli

$$u_t = a^2 u_{xx} \tag{8}$$

issiqlik tarqalish tenglamasining

$$u|_{t=0} = \varphi(x), \qquad 0 \leq x \leq l, \tag{9}$$

boshlang'ich va

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \quad 0 \leq t \leq T,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

Bunda $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ – berilgan yetarlicha silliq funksiyalar.

Biz izlanayotgan $u(x,t)$ yechimni $\overline{Q}$ yopiq sohada uzluksiz funksiya, deb faraz qilamiz va shuning uchun berilgan $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ funksiyalar uzluksiz va bular uchun

$$\varphi(0) = \mu_1(0), \quad \varphi(l) = \mu_2(0)$$

tengliklar o'rinli bo'lsin.

Biz ushbu bosqichda qo'yilgan masalaning yechimini chegaraviy shartlar bir jinsli $\mu_1(t) = \mu_2(t) = 0$, ya'ni

$$u|_{x=0} = 0, \qquad u|_{x=l} = 0, \qquad 0 \leq t \leq T. \tag{10}$$

bo'lgan holda isbotlaymiz.

Qaralayotgan (8)–(10) masala bir jinsli sterjenda issiqlik tarqalish jarayonining matematik modeli bo'lib, sterjenning uchlari doim nol darajadagi haroratni saqlaydi.

Bu masalani yechish uchun Fur'e qatorlari nazariyasiga asoslangan, o'zgaruvchilarni ajratish usulini qo'llaymiz. (8) tenglamaning $Q$ sohada xususiy yechimlarini

$$u(x,t) = T(t)X(x) \neq 0, \tag{11}$$

ko'rinishda izlaymiz.

Bu ko'rinishdagi yechimni (8) tenglamaga qo'yib, ushbu tengliklarni

$$X(x)T'(t) = a^2T(t)X''(x)$$

yoki

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

hosil qilamiz. Bundan esa quyidagi

$$T'(t) + a^2\lambda T(t) = 0; \tag{12}$$

$$X''(x) + \lambda X(x) = 0 \tag{13}$$

chiziqli oddiy differensial tenglamalarga ega bo'lamiz.

Berilgan issiqlik tarqalish tenglamasining noldan farqli (11) ko'rinishdagi $u(x,t)$ yechimini topish uchun (13) oddiy differensial tenglamaning quyidagi

$$X(0) = 0, \qquad X(l) = 0, \tag{14}$$

shartlarni qanoatlantiruvchi $X(x)$ yechimini topish zarur.

Shunday qilib, noma'lum $X(x)$ funksiyani topish uchun quyidagi

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \quad X(l) = 0, \tag{15}$$

spektral masalaga ega bo'ldik. Bu masala chegaralangan bir jinsli torning ko'ndalang tebranishi haqidagi masalada o'rganilgan edik.

Ma'lumki, $\lambda$ parametrning

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad (n = 1, 2, 3, \ldots) \tag{16}$$

qiymatlarida (15) xos qiymatlar va xos funksiyalar haqidagi masalaning noldan farqli yechimi mavjud va bu yechim

$$X_n(x) = \sin\frac{n\pi}{l}x, \tag{17}$$

ko'rinishda bo'ladi.

$\lambda$ parametrning $\lambda = \lambda_n$ qiymatlariga mos (12) tenglamaning

$$T_n(t) = a_n e^{-\left(\frac{na\pi}{l}\right)^2 t} \tag{18}$$

yechimlari mos keladi, $a_n$ – ixtiyoriy o'zgarmas sonlar.

Shunday qilib, quyidagi barcha funksiyalar

$$u_n(x,t) = T_n(t)X_n(x) = a_n \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin\frac{n\pi}{l} x \tag{19}$$

qaralayotgan (8) tenglamani va (9) chegaraviy shartlarni qanoatlantiradi. Bu funksiyalardan ushbu

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin\frac{n\pi}{l} x \tag{20}$$

qatorni tuzamiz.

Endi (20) qatorni (9) boshlang'ich shartni qanoatlantirishini talab qilib,

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \tag{21}$$

ifodani olamiz. Hosil qilingan bu (21) ifoda $\varphi(x)$ funksiyaning $(0,l)$ oraliqda sinuslar bo'yicha Fur'e qatoriga yoyilmasi bo'lib, uning $a_n$ koeffitsientlari

$$a_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l} dx \tag{22}$$

formula bilan aniqlanadi.

Agar $\varphi(x)$ funksiya $(0,l)$ oraliqda uzluksiz va u yerda birinchi tartibli uzluksiz hosilalarga ega bo'lib, $\varphi(0) = \varphi(l) = 0$ bo'lsa, u holda (21) qator $(0,l)$ oraliqda $\varphi(x)$ funksiyaga absolyut va tekis yaqinlashadi.

Shu bilan birga ixtiyoriy $t \geq 0$ da

$$0 < \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \leq 1$$

bo'lgani uchun (20) qator ham tekis va absolyut yaqinlashuvchi bo'ladi. Shuning uchun (20) qator bilan aniqlangan $u(x,t)$ funksiya $\overline{Q}$ sohada uzluksiz bo'lib, boshlang'ich va chegaraviy shartlarni qanoatlantiradi.

Endi $u(x,t)$ funksiya $0 \leq x \leq l$, $t > 0$ sohada (8) issiqlik tarqalish tenglamasini qanoatlantirishini ko'rsataylik. Buning uchun (20) qatorni $t$ bo'yicha bir marta va $x$ bo'yicha ikki marta hadma–had differensiallaymiz va quyidagi

$$u_t(x,t) = -\left(\frac{a\pi}{l}\right)^2 \sum_{n=1}^{\infty} a_n n^2 \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin\frac{n\pi}{l}x \qquad (23)$$

$$u_{xx}(x,t) = -\left(\frac{\pi}{l}\right)^2 \sum_{n=1}^{\infty} a_n n^2 \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin\frac{n\pi}{l}x \qquad (24)$$

qatorlarni hosil qilamiz.

Bu qatorlarning hadlari $0 \leq x \leq l$, $tg'eqt_0 > 0$ sohada

$$\sum_{n=1}^{\infty} |a_n| n^2 \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t_0\right\} \quad \text{yoki} \quad M\sum_{n=1}^{\infty} |a_n|$$

yaqinlashuvchi sonli qatorning hadlari bilan chegaralangan va

$$0 < \left(\frac{n\pi a}{l}\right)^2 \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t_0\right\} < 1,$$

$$0 < \left(\frac{n\pi}{l}\right)^2 \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t_0\right\} < 1.$$

U holda (23) va (24) qatorlar Veyershtrass alomatiga ko'ra $t \geq t_0 > 0$ da absolyut va tekis yaqinlashuvchi bo'ladi. Bundan esa $u_t(x,t)$ va $u_{xx}(x,t)$ funksiyalarning yopiq $t \geq t_0 > 0$ sohada uzluksiz ekanligi kelib chiqadi. $t_0 > 0$ ixtiyoriy bo'lgani uchun $u_t(x,t)$ va $u_{xx}(x,t)$ funksiyalar $Q$ sohada uzluksiz bo'ladi. (23) va (24) formulalarni (8) tenglamaga qo'ysak, (20) formula bilan aniqlangan $u(x,t)$ funksiya $Q$ sohada berilgan tenglamaning yechimi bo'lishiga ishonch hosil qilamiz.

Shunday qilib, issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masala yechimining mavjudligi haqidagi ushbu teorema isbotlandi.

1–TEOREMA. Agar $\varphi(x) \in C^1[0,l]$ va $\varphi(0) = \varphi(l) = 0$ bo'lsa, u holda (8)–(10) masalaning yagona yechimi mavjud va bu yechim (20) qator bilan aniqlanadi, qatorning koeffitsientlari esa (22) formula bilan hisoblanadi.

XULOSA. Issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning $u(x,t)$ yechimi $\overline{Q}$ sohada cheksiz differensiallanuvchi, ya'ni $u(x,t) \in C^{+\infty}(\overline{Q})$ bo'ladi.

CHETKI NUQTALARI O'ZGARUVCH HARORATDA BO'LGAN STERJENDA ISSIQLIK TARQALISHI. Endi bir jinsli issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalada (3) chegaraviy shartlar bir jinsli bo'lmagan xolda yechimining mavjudligini Fur'e usulida isbotlaymiz.

Chekli $Q = \{(x,t) : 0 < x < l, 0 < t < T\}$ sohada bir jinsli

$$u_t = a^2 u_{xx} \tag{25}$$

issiqlik tarqalish tenglamasining

$$u|_{t=0} = \varphi(x), \qquad 0 \le x \le l, \tag{26}$$

boshlang'ich va

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \quad 0 \le t \le T, \tag{27}$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini topig.

Bu yerda $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ – berilgan etarlicha silliq funksiyalar.

Oxirgi masalaning yechimini

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x, \tag{28}$$

ko'rinishda izlaymiz. Bunda

$$T_n(t) = \frac{2}{l} \int_0^l u(x,t) \sin \frac{n\pi}{l} x \, dx. \tag{29}$$

Endi $T_n(t)$ noma'lum funksiyani topamiz. Buning uchun (29) ifodani ikki marta bo'laklab integrallaymiz. Natijada

$$T_n(t) = \frac{2}{n\pi}\left[u(0,t) - (-1)^n u(l,t)\right] - \frac{2l}{n^2\pi^2}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi}{l}x\,dx.$$

$u(x,t)$ funksiya (25) tenglamani va (27) chegaraviy shartlarni qanoatlantirgani uchun oxirgi formulani

$$T_n(t) = \frac{2}{n\pi}\left[\mu_1(t) - (-1)^n\mu_2(t)\right] - \frac{2l}{n^2\pi^2}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi}{l}x\,dx \tag{30}$$

ko'rinishda yozib olish mumkin.

(29) ifodani $t$ bo'yicha differensiallaymiz va

$$\frac{dT_n(t)}{dt} = \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi}{l}x\,dx. \tag{31}$$

formulani olamiz. (30) ifodadan

$$\frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi}{l}x\,dx = -\left(\frac{\pi na}{l}\right)^2 T_n(t) + \frac{2\pi na^2}{l^2}\left[\mu_1(t) - (-1)^n\mu_2(t)\right]$$

topamiz va buni (31) formulaga qo'ysak, $T_n(t)$ noma'lum funksiyaga nisbatan ushbu

$$\frac{dT_n(t)}{dt} + \left(\frac{\pi na}{l}\right)^2 T_n(t) = \frac{2\pi na^2}{l^2}\left[\mu_1(t) - (-1)^n\mu_2(t)\right]$$

tenglamaga ega bo'lamiz.

Oxirgi tenglamaning umumiy yechimi

$$T_n(t) = \exp\left\{-\left(\frac{\pi na}{l}\right)^2 t\right\}T_n(0)+$$

$$+\frac{2\pi n a^2}{l^2}\int\limits_0^t \exp\left\{-\left(\frac{\pi n a}{l}\right)^2(t-\tau)\right\}\left[\mu_1(\tau)-(-1)^n\mu_2(\tau)\right]d\tau \quad (32)$$

bo'ladi. Bu yerda

$$T_n(0)=\frac{2}{l}\int\limits_0^l \varphi(x)\sin\frac{\pi n}{l}x dx. \quad (33)$$

Shunday qilib, (25) bir jinsli issiqlik tarqalish tenglamasining (26) boshlang'ich va (27) chegaraviy shartlarni qanoatlantiruvchi yechimi (28) qator ko'rinishda aniqlanadi, bunda $T_n(t)$ koeffitsiyentlar (32) va (33) formulalar orqali topiladi.

(28) qatorning yaqinlashishi, uni $x$ o'zgaruvchi bo'yicha ikki marta va $t$ bo'yicha esa bir marta differensiallashdan hosil bo'lgan qatorlarning yaqinlashishini tekshishini o'quvchilarning o'zlariga havola qilamiz.

## 18–§. Bir jinsli bo'lmagan issiqlik tarqalish tenglamasi

Ushbu paragrafda bir jinsli bo'lmagan issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalani qaraymiz.

Chekli $Q=\{(x,t): 0<x<l,\ 0<t<T\}$ sohada

$$u_t=a^2u_{xx}+f(x,t), \quad (1)$$

issiqlik tarqalish tenglamasining

$$u|_{t=0}=\varphi(x), \qquad 0\le x\le l, \quad (2)$$

boshlang'ich va

$$u|_{x=0}=\mu_1(t), \quad u|_{x=l}=\mu_2(t), \quad 0\le t\le T, \quad (3)$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini topig.

Bu yerda $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ – berilgan uzluksiz funksiyalar.

Bu masala yechimining yagonaligi 16-§da ekstremum prinsipi yordamida isbotlangan.

Qaralayotgan (1)–(3) masala yechimining mavjudligini bir jinsli chegaraviy shartli masalaga keltirib isbotlaymiz.

Buning uchun $\mu_1(t)$, $\mu_2(t)$ funksiyalarni $C^1[0,T]$ sinfdan bo'lsin, deb talab qilamiz. U holda (3) chegaraviy shartlarni qanoatlantiruvchi

$$\omega(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)]$$

yordamchi funksiya kiritamiz, ya'ni

$$\omega(0,t) = \mu_1(t), \qquad \omega(l,t) = \mu_2(t).$$

Endi (1) tenglamaning (2) boshlang'ich va (3) chegaraviy shartlarni qanoatlantiruvchi yechimini quyidagi

$$u(x,t) = v(x,t) + \omega(x,t), \tag{4}$$

yig'indi ko'rinishida izlaymiz. Bu yerda $v(x,t)$ yangi noma'lum funksiya. (2) boshlang'ich va (3) chegaraviy shartlar asosida $v(x,t)$ funksiyaga nisbatan quyidagi

$$v(0,t) = u(0,t) - \omega(0,t) = \mu_1(t) - \mu_1(t) = 0;$$

$$v(l,t) = u(l,t) - \omega(l,t) = \mu_2(t) - \mu_2(t) = 0;$$

$$v(x,0) = u(x,0) - \omega(x,0) = \varphi(x) - \mu_1(0) + \frac{x}{l}[\mu_2(0) - \mu_1(0)] = \widetilde{\varphi}(x)$$

chegaraviy va boshlang'ich shartlarga ega bo'lamiz. Noma'lum $v(x,t)$ funksiyaga nisbatan tenglama esa

$$v_t - a^2 v_{xx} = (u-\omega)_t - a^2(u-\omega)_{xx} =$$

$$= u_t - a^2 u_{xx} - (\omega_t - a^2\omega_{xx}) =$$

$$= f(x,t) - \mu_1'(t) - \frac{x}{l}[\mu_2'(t) - \mu_1'(t)] = g(x,t),$$

yoki

$$v_t = a^2 v_{xx} + g(x,t), \tag{5}$$

bo'lib, bunda

$$g(x,t) = f(x,t) - \mu_1'(t) - \frac{x}{l}[\mu_2'(t) - \mu_1'(t)].$$

Shunday qilib. noma'lum $v(x,t)$ funksiyani topish uchun quyidagi masalaga keldik. Bir jinsli bo'lmagan (5) tenglamaning

$$v|_{t=0} = \widetilde{\varphi}(x) \tag{6}$$

boshlang'ich va bir jinsli

$$v|_{x=0} = 0, \qquad v|_{x=l} = 0 \tag{7}$$

chegaraviy shartlarni qanoatlantiruvchi $v(x,t)$ yechimi topilsin. Bu yerda $g(x,t)$ funksiya uzluksiz va $x$ bo'yicha bo'lakli uzluksiz hosilaga ega hamda barcha $t > 0$ lar uchun $g(0,t) = g(l,t) = 0$ bo'lsin, deb talab qilamiz.

Oxirgi (5)–(7) masalaning yechimini

$$v(x,t) = v_1(x,t) + v_2(x,t)$$

yig'indi ko'rinishida izlaymiz.

Bu yerda $v_1(x,t)$ funksiya bir jinsli issiqlik tarqalish tenglamasining (6)–(7) shartlarni qanoatlantiruvchi yechimi bo'lib, u avvalgi paragrafda (20) formula bilan, $a_n$ Fur'e koeffitsienti esa (22) formula bilan aniqlangan. (22) formulani inobatga olib, (20) qatorni quyidagi

$$v_1(x,t) = \frac{2}{l}\sum_{n=1}^{\infty}\int_0^l \left[e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l}\right]\varphi(\xi)d\xi,$$

ko'rinishda yozishimiz mumkin.

$v_2(x,t)$ funksiya esa (5) tenglamaning bir jinsli boshlang'ich va (7) chegaraviy shartlarni qanoatlantiruvchi echimidan iborat, ya'ni

$$\frac{\partial^2 v_2}{\partial t^2} = a^2\frac{\partial^2 v_2}{\partial x^2} + g(x,t)$$

tenglamaning bir jinsli

$$v_2(x,0)=0,\quad v_2(0,t)=0,\quad v_2(l,t)=0$$

boshlang'ich va chegaraviy shartlarni qanoatlantiruvchi yechimidan iborat.

Oxirgi masalaning echimini

$$v_2(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{n\pi x}{l} \tag{8}$$

ko'rinishida izlaymiz. $g(x,t)$ funksiyani sinuslar bo'yicha Fur'e qatoriga yoyib, ushbu

$$g(x,t)=\sum_{n=1}^{\infty}g_n(t)\sin\frac{n\pi x}{l}, \tag{9}$$

ifodani olamiz. Bu yerda

$$g_n(t)=\frac{2}{l}\int_0^l g(x,t)\sin\frac{n\pi x}{l}\,dx. \tag{10}$$

Endi (8) qatorni (5) tenglamaga qo'yamiz va (9) ifodani hisobga olib, $\forall\, x\in(0,l)$ da o'rinli bo'lgan quyidagi

$$\sum_{n=1}^{\infty}\left[T_n'(t)+\left(\frac{an\pi}{l}\right)^2T_n(t)-g_n(t)\right]\sin\frac{n\pi x}{l}=0.$$

tenglikni hosil qilamiz. Bundan esa

$$T_n'(t)+\left(\frac{an\pi}{l}\right)^2T_n(t)=g_n(t),\qquad (n=1,2,\cdots), \tag{11}$$

tenglamalarga ega bo'lamiz.

Endi (8) qatorni (6) bir jinsli boshlang'ich shartga qo'yib

$$v_2(x,0)=\sum_{n=1}^{\infty}T_n(0)\sin\frac{n\pi x}{l}=0$$

ekanligini topamiz, bundan esa $T_n(t)$ funksiyalar uchun

$$T_n(0) = 0, \qquad (n = 1, 2, \cdots) \tag{12}$$

boshlang'ich shartlarni olamiz.

Hosil bo'lgan (11)–(12) ifodalar chiziqli differensial tenglama uchun Koshi masalasi bo'lib, uning yechimini topish ortiqcha qiyinchilik tug'dirmaydi. O'zgarmasni variatsiyalash usuli yordamida bu masalaning yechimi

$$T_n(t) = \int_0^t e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} g_n(\tau) d\tau, \tag{13}$$

ko'rinishda bo'lishiga ishonch hosil qilish mumkin.

Topilgan $T_n(t)$ ifodani (8) formulaga qo'yib, (5)–(7) masalaning yechimini

$$v_2(x,t) = \sum_{n=1}^{\infty} \left[ \int_0^t e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} g_n(\tau) d\tau \right] \sin \frac{n\pi x}{l}. \tag{14}$$

korinishda hosil qilamiz.

Endi $g_n(t)$ uchun topilgan (10) ifodadan foydalanib, (14) formulani quyidagicha

$$v_2(x,t) = \int_0^t \int_0^l \left\{ \frac{2}{l} \sum_{n=1}^{\infty} \left[ e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l} \right] \right\} g(\xi, \tau) d\xi d\tau, \tag{15}$$

yozishimiz mumkin.

Yuqorida topilgan $v_1(x,t)$ va $v_2(x,t)$ funksiyalarni hisobga olib, (5)–(7) masalaning yechimini

$$v(x,t) = \frac{2}{l} \sum_{n=1}^{\infty} \int_0^l \left[ e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi \xi}{l} \sin \frac{n\pi x}{l} \right] \varphi(\xi) d\xi +$$

$$+\int_0^t\int_0^l\left\{\frac{2}{l}\sum_{n=1}^{\infty}\left[e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\right]\right\}g(\xi,\tau)d\xi d\tau;$$

yoki

$$v(x,t)=\int_0^l G(x,t;\xi,0)\varphi(\xi)d\xi+\int_0^t\int_0^l G(x,\xi;t-\tau)\,g(\xi,\tau)d\xi d\tau, \quad (16)$$

ko'rinishda topamiz. Bu yerda

$$G(x,\xi;t-\tau)=\frac{2}{l}\sum_{n=1}^{\infty}e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}.$$

(16) formuladagi $G(x,\xi;t-\tau)$ funksiya oniy issiqlik manbai yoki *Grin funksiyasi* deyiladi.

Endi mavzuga oid bir nechta masala qaraylik.

1–MASALA. Bir jinsli bolmagan ushbu

$$u_t=a^2u_{xx}+f(x,t), \quad (17)$$

issiqlik tarqalish tenglamasining $D=\{(x,t)\ :\ 0<x<l, t>0\}$ sohada aniqlangan uzluksiz va quyidagi boshlang'ich

$$u(x,0)=\varphi(x), \qquad 0\le x\le l, \quad (18)$$

va chegaraviy

$$u_x(0,t)=0, \quad u_x(l,t)+hu(l,t)=0, \quad h>0, \quad (19)$$

shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

Bu masalada qaralayotgan sterjenning $x=0$ uchida issiqlik oqimining zichligi nolga teng, ya'ni sterjenning $x=0$ uchi issiqlik o'tkazmaydi. Issiqlik oqimi zichligining xossasiga asosan sterjenning $x=l$ uchida esa atrof muhit bilan issiqlik almashish jarayoni sodir bo'ladi. Shuning uchun sterjenning $x=l$ uchida issiqlik almashish sharti berilgan.

YECHISH. Berilgan aralash masala yechimini

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

yig'indi ko'rinishida izlaymiz. Bu yerda $u_1(x,t)$ bir jinsli tenglamaning (48)–(49) shartlarni qanoatlantiruvchi yechimi, ya'ni

$$u_1(x,t) : \begin{cases} u_t = a^2 u_{xx}, \\ u(x,0) = \varphi(x), \qquad 0 \le x \le l, \\ u_x(0,t) = 0, \quad u_x(l,t) + hu(l,t) = 0, \quad h > 0; \end{cases} \tag{20}$$

$u_2(x,t)$ esa (17) tenglamaning bir jinsli boshlang'ich va chegaraviy shartlarni qanoatlantiruvchi yechimidan iborat, ya'ni

$$u_2(x,t) : \begin{cases} u_t = a^2 u_{xx} + f(x,t), \\ u(x,0) = 0, \qquad 0 \le x \le l, \\ u_x(0,t) = 0, \quad u_x(l,t) + hu(l,t) = 0, \quad h > 0; \end{cases} \tag{21}$$

$1^0$. Avval (20) masalaning yechimini topaylik.

(19) chegaraviy shartlarni qanoatlantiruvchi bir jinsli issiqlik tarqalish tenglamasining yechimini $u(x,t) = X(x)T(t)$ ko'rinishda izlaymiz. Bundan $X(x)$ funksiyaga nisbatan ushbu

$$X'(0) = 0, \qquad X'(l) + hX(l) = 0$$

chegaraviy shartlarni olamiz. $u(x,t)$ funksiyani bir jinsli tenglamaga *qo'yib, sodda almashtirishlardan so'ng $X(x)$ funksiyaga nisbatan*

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = 0, \quad X'(l) + hX(l) = 0, \quad h > 0; \end{cases} \tag{22}$$

SHturm–Liuvill masalasini, $T(t)$ funksiyaga nisbatan esa

$$T'(t) + a^2 \lambda T(t) = 0, \quad t > 0, \tag{23}$$

tenglamani olamiz.

Ma'lumki, $\lambda = 0$ va $\lambda < 0$ qiymatlarida (22) Shturm–Liuvill masalasi faqat $X(x) \equiv 0$ yechimga ega ekanligini ko'rsatish qiyin emas.

Endi $\lambda > 0$ bo'lgan holni qaraylik. (22) tenglamaning umumiy yechimi

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

ko'rinishda bo'ladi. Bundan

$$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\, x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\, x)$$

tenglikni olamiz. Bir jinsli $X'(0) = 0$ shartga ko'ra $c_2 = 0$ va $X(x) = c_1\cos(\sqrt{\lambda}\, x)$ bo'ladi. Ikkinchi $X'(l) + hX(l) = 0$ chegaraviy shartga asosan esa $-\sqrt{\lambda}\, \sin(\sqrt{\lambda}\, l) + h\cos(\sqrt{\lambda}\, l) = 0$ tenglikni olamiz. Bundan esa

$$\sqrt{\lambda}\operatorname{tg}\sqrt{\lambda}l = h \tag{24}$$

tenglamaga ega bo'lamiz. (24) tenglama cheksiz ko'p $\lambda_k$, $k = 1, 2, \ldots$ yechimlarga ega ekanligini grafik usulida ko'rsatish mumkin.

Shunday qilib, $\lambda_k > 0$ qiymatlarda $\lambda_k \operatorname{tg} \lambda_k l = h$ tenglamaning yechimlari (22) Shturm–Liuvill masalasining xos qiymatlari, ularga mos xos funksiyalar

$$X_k(x) = \cos\lambda_k\, x, \quad k = 1, 2, \ldots \tag{25}$$

boladi. (23) barcha $\lambda = \lambda_k$ qiymatlarida o'rinli bo'ladi, ya'ni

$$T_k'(t) + a^2\lambda T_k(t) = 0, \qquad t > 0.$$

Oxirgi tenglamaning umumiy yechimi

$$T_k(t) = a_k \exp\{-a^2\lambda_k\, t\}, \qquad t > 0, \tag{26}$$

bu yerda $a_k$ – ixtiyoriy o'zgarmaslar.

Endi (20) masalaning echimini $u(x,t) = \sum\limits_{k=1}^{\infty} X_k(x)T_k(t)$ ko'rinishda izlaymiz. Yuqorida topilgan (25) va (26) formulalarga asosan (20) masalaning yechimi

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x)T_k(t) = \sum_{k=1}^{\infty} a_k \exp\{-a^2\lambda_k t\}\cos\lambda_k x, \tag{27}$$

ko'rinishda aniqlanadi. Bu yerda $a_k$ koeffitsiyentlarni topish uchun (18) boshlang'ich shartdan foydalanamiz va

$$u(x,0) = \varphi(x) = \sum_{k=1}^{\infty} a_k \cos\lambda_k x, \tag{28}$$

ifodaga ega bo'lamiz. Bu boshlang'ich shartda qatnashgan $\varphi(x)$ funksiyaning kosinuslar bo'yicha Fur'e qatoriga yoyilmasini beradi.

$a_k$ koeffitsiyentlarni qanday bo'lishini aniqlash uchun (28) tenglikning har ikki tomonini $\cos\lambda_k x$ funksiyaga skalyar ko'paytiramiz va cheksiz qatorni chekli integralga almashtiramiz. Natijada

$$a_k \int_0^l \cos^2\lambda_k x dx = \int_0^l \varphi(x)\cos^2\lambda_k x dx$$

ifodaga ega bo'lamiz. Oxirgi tenglikning chap tomoni

$$a_k \int_0^l \cos^2\lambda_k x dx = \frac{a_k}{2}\int_0^l (1+\cos 2\lambda_k x)dx =$$

$$\frac{a_k}{2}\left(l + \frac{1}{2\lambda_k}\sin 2\lambda_k x\Big|_{x=0}^{x=l}\right) = \frac{a_k}{2}\left(l + \frac{1}{2\lambda_k}\sin 2\lambda_k l\right) \tag{29}$$

teng bo'ladi. (24) tenglamadan sodda almashtirishlardan keyin

$$\sin\lambda_k l = \frac{h}{\sqrt{\lambda_k^2 + h^2}}, \quad \cos\lambda_k l = \frac{\lambda_k}{\sqrt{\lambda_k^2 + h^2}}$$

kelib chiqadi. (29) ifodaning o'ng tomonini $\sin 2x = 2\sin x\cos x$ formuladan foydalanib, soddalashtiramiz

$$l + \frac{\sin(2\lambda_k l)}{2\lambda_k} = l + \frac{\sin(\lambda_k l)\cos(\lambda_k l)}{\lambda_k} =$$

$$= l + \frac{1}{\lambda_k}\cdot\frac{h}{\sqrt{\lambda_k^2 + h^2}}\cdot\frac{\lambda_k}{\sqrt{\lambda_k^2 + h^2}} = \frac{l(\lambda_k^2 + h^2) + h}{\lambda_k^2 + h^2};$$

Natijada $a_k$ koeffitsiyentlar uchun

$$a_k = \frac{2(\lambda_k^2 + h^2)}{l(\lambda_k^2 + h^2) + h} \int_0^l \varphi(x) \cos(\lambda_k x) dx$$

tengliklarni olamiz.

Topilgan $a_k$ koeffitsiyentlarni (27) formulaga qo'yib, (20) masalaning $u_1(x,t)$ yechimini

$$u_1(x,t) = \sum_{k=1}^{\infty} \left\{ \frac{2(\lambda_k^2 + h^2)}{l(\lambda_k^2 + h^2) + h} \int_0^l \varphi(x) \cos(\lambda_k x) dx \right\} \times$$
$$\times \exp\{-a^2 \lambda t\} \cos(\lambda_k x) dx \qquad (30)$$

ko'rinishda topamiz.

$2^0$. Endi (21) masalani qaraylik. Bunda $u_t = a^2 u_{xx} + f(x,t)$ tenglamaning $u_x(0,t) = u_x(l,t) + hu(l,t) = 0$ shartlarni qanoatlantiruvchi yechimini

$$u(x,t) \sum_{k=1}^{\infty} T_k(t) \cos \lambda_k x, \qquad (31)$$

ko'rinishda izlaymiz.

Faraz qilaylik, $f(x,t)$ funksiya $\forall t \in [0,T]$ da $\cos \lambda_k x$ funksiyalar bo'yicha Fur'e qatoriga yoyilsin, ya'ni

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \cos \lambda_k x, \qquad (32)$$

bu erda $f_k(t)$ Fur'e koeffitsiyentlari

$$f_k(t) = \frac{2}{l} \int_0^l f(x,t) \cos \lambda_k x dx, \qquad (33)$$

formula yordamida aniqlanadi.

Endi (31) va (32) formulalarga ko'ra (21) tenglama

$$\sum_{k=1}^{\infty}\left[T_k'(t)+a^2\lambda^2T_k(t)\right]\cos\lambda_k x=\sum_{k=1}^{\infty}f_k(t)\cos\lambda_k x$$

ko'rinishni oladi. Bundan esa

$$T_k'(t)+a^2\lambda^2T_k(t)=f_k(t),\quad k=1,2,... \tag{34}$$

chizitenglamalarga ega bo'lamiz.

(31) ifodadan $T_k(t)$ funksiyalar uchun quyidagi

$$T_k(0)=0,\qquad k=1,2,... \tag{35}$$

boshlang'ich shartlarni olamiz.

Ma'lumki, (34)–(35) Koshi masalasi $\forall f_k(t)\in C[0,T]$ da yagona yechimga ega bo'ladi. Oddiy differensial tenglamalar kursida ma'lumki, o'zgarmasni variatsiyalash usuli bilan (34)–(35) Koshi masalasining yechimini ushbu

$$T_k(t)=\int_0^t f_k(\tau)\exp\left\{-a^2\lambda^2(t-\tau)\right\}d\tau,\quad k=1,2,... \tag{36}$$

ko'rinishda topiladi. (33) formulaga asosan oxirgi echimni

$$T_k(t)=\frac{2}{l}\int_0^t\int_0^l f(\xi,\tau)\exp\left\{-a^2\lambda^2(t-\tau)\right\}\cos\lambda_k\xi d\xi d\tau,\quad k=1,2,...$$

deb yozib olish mumkin.

Endi topilgan $T_k(t)$ ifodani (31) formulaga qo'yib, (21) bir jinsli aralash masalaning $u_2(x,t)$ yechimiga

$$u_2(x,t)=\frac{2}{l}\sum_{k=1}^{\infty}\int_0^t\int_0^l f(\xi,\tau)\exp\left\{-a^2\lambda^2(t-\tau)\right\}\cos\lambda_k x\cos\lambda_k\xi d\xi d\tau,$$

ega bo'lamiz.

Yukorida topilgan $u_1(x,t)$ va $u_2(x,t)$ funksiyalarga asosan issiqlik tarqalish tenglamasi uchun (17)–(19) aralash masalaning yechimi

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \frac{2(\lambda_k^2 + h^2)}{l(\lambda_k^2 + h^2) + h} \int_0^l \varphi(x) \cos(\lambda_k x) dx \right\} \times$$
$$\times \exp\{-a^2 \lambda t\} \cos(\lambda_k x) dx +$$

$$+ \frac{2}{l} \sum_{k=1}^{\infty} \int_0^t \int_0^l f(\xi, \tau) \exp\left\{-a^2 \lambda^2 (t - \tau)\right\} \cos \lambda_k x \cos \lambda_k \xi d\xi d\tau,$$

ko'rinishda aniqlandi.

## 19–§. Chegaralanmagan sterjenda issiqlik tarqalishi (Koshi masalasi)

KOSHI MASALASI. YECHIMNING YAGONALIGI. Bu paragrafda sirti issiqlik o'tkazmaydigan (izolyasiyalangan) chegaralanmagan bir jinsli sterjenda issiqlikning tarqalishi haqidagi fizikaviy masalaning matematik qo'yilishi quyidagicha ifodalanadi.

Ushbu issiqlik tarqalish tenglamasini

$$Lu \equiv u_t - a^2 u_{xx} = 0, \tag{1}$$

$t > 0$ yarim tekislikda qaraylik.

KOSHI MASALASI. (1) tenglamaning $t > 0$ yarim tekislikda aniqlangan, uzluksiz va quyidagi boshlang'ich

$$u|_{t=0} = \varphi(x), \qquad -\infty < x < \infty, \tag{2}$$

shartni qanoatlantiruvchi $u(x,t)$ yechimi topilsin. Bu yerda $\varphi(x)$ berilgan uzluksiz va chegaralangan funksiya.

Qo'yilgan (1)–(2) Koshi masalasi yechimining yagonaligini isbotlaymiz.

1–TEOREMA. Agar (1)–(2) shartlarni qanoatlantiruvchi chegaralangan $u(x,t)$ funksiya mavjud bo'lsa, u holda bu funksiya yagona aniqlanadi.

ISBOT. Faraz qilaylik, $u(x,t)$ yechim qaralayotgan sohada chegaralangan bo'lsin, ya'ni har qanday $t \geq 0$ va $-\infty < x < \infty$ uchun shunday musbat son $M$ mavjudki, $|u(x,t) \leq M$ bo'lsin.

(1)–(2) Koshi masalaning ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimi mavjud bo'lsin. U holda $w(x,t) = u_1(x,t) - u_2(x,t)$ funksiya (1) tenglamani va bir jinsli $w(x,0) = 0$ boshlang'ich shartni qanoatlantiradi va

$$|w(x,t)| \leq |u_1(x,t)| + |u_2(x,t)| \leq 2M$$

bo'ladi.

Soha cheksiz bo'lganligi uchun $w(x,t)$ funksiyaga yuqorida isbotlangan ekstremum prinsipini bevosita qo'llab bo'lmaydi. Lekin shu prinsipdan foydalanish maqsadida quyidagi chekli

$$Q = \{(x,t) : |x| < L,\ 0 < t < T\} \tag{3}$$

sohani qaraymiz. Bu yerda $L$ va $T$ ixtiyoriy o'zgarmaslar. $Q$ sohada yordamchi

$$v(x,t) = \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

funksiya kiritamiz. Bu funksiya (1) tenglamani qanoatlantirishini ko'rsatish qiyin emas. Shu bilan birga tekshirib ko'rish osonki,

$$v(x,0) \geq w(x,0) = 0,$$

$$v(\pm L,t) \geq 2M \geq |w(\pm L,t)|$$

tengsizliklar o'rinli bo'ladi.

$Q$ sohada

$$u(x,t) = v(x,t) - |w(x,t)|$$

funksiya uchun ekstremum prinsipini qo'llaymiz.
Natijada $\forall (x,t) \in \overline{Q}$ uchun

$$v(x,t) - w(x,t) \geq 0, \qquad v(x.t) + w(x,t) \geq 0$$

tengsizliklarga ega bo'lamiz. Bundan esa yopiq $\overline{Q}$ sohada

$$-v(x,t) \le w(x,t) \le v(x,t)$$

yoki

$$|w(x,t)| \le v(x,t) = \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right)$$

ekanligi kelib chiqadi.

Oxirgi tengsizlikda $(x,t)$ nuqtani $Q$ sohada fiksirlaymiz, ya'ni $x = x_0$, $t = t_0$ va $(x_0, t_0) \in Q$ bo'lsin. $L$ ning ixtiyoriy ekanligidan shunday $\varepsilon > 0$ uchun $L_0 > 0$ mavjudki, barcha $L > L_0$ uchun $|w(x_0, t_0)| < \varepsilon$ tengsizlik o'rinli bo'ladi. Bundan $\varepsilon$ va $(x_0, t_0)$ nuqtani ixtiyoriyligini hisobga olsak, $\overline{Q}$ sohada $w(x,t) \equiv 0$ bo'ladi yoki $u_1 \equiv u_2$ kelib chiqadi.

Demak, issiqlik o'tkazuvchanlik tenglamasi uchun Koshi masalasining yechimi yagona ekan. Shu bilan 1–teorema isbotlandi.

KOSHI MASALASI YECHIMINING MAVJUDLIGI.

Bu masalani yechish uchun o'zgaruvchilarni ajratish (Fur'e) usulini qo'llaymiz, ya'ni yechimni

$$u(x,t) = T(t)X(x) \tag{4}$$

ko'rinishda izlaymiz. (4) ifodani (1) tenglamaga qo'yib, $T(t)$ va $X(x)$ funksiyalar uchun ushbu

$$T'(t) + a^2\lambda^2 T(t) = 0, \qquad X''(x) + \lambda^2 X(x) = 0$$

tenglamalarga kelamiz, $\lambda^2$ – o'zgarmas son. Bu tenglamalarni integrallab topamiz:

$$T(t) = C\,t^{-a^2\lambda^2 t}, \qquad X(x) = A\cos\lambda x + B\sin\lambda x, \tag{5}$$

bu yerda $A$, $B$ va $C$ o'zgarmas sonlar. (1)–(2) Koshi masalasida chegaraviy shartlar bo'lmaganligi uchun $\lambda$ – parametr ixtiyoriy bo'ladi. (5) yechimlarda $C = 1$, $A = A(\lambda)$, $B = B(\lambda)$ deb, uni (4) ifodaga qo'yib,

$$u_\lambda(x,t) = e^{-a^2\lambda^2 t}[A(\lambda) cos\lambda x + B(\lambda)\sin\lambda x] \tag{6}$$

formulaga ega bo'lamiz.

Bu $u_\lambda(x,t)$ funksiya ixtiyoriy $\lambda$ va $A(\lambda)$, $B(\lambda)$ koeffitsiyentlar uchun (1) tenglamaning xususiy yechimlari bo'ladi. (6) formulani $\lambda$ parametr bo'yicha $-\infty$ dan $+\infty$ gacha integrallaymiz, natijada

$$u(x,t) = \int_{-\infty}^{\infty} a^{-a^2\lambda^2 t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]d\lambda, \tag{7}$$

ya'ni (1) tenglamaning yechimi hosil bo'ladi.

Agar bu integral yaqinlashuvchi bo'lsa, u holda (7) integraldan $t$ bo'yicha bir marta, $x$ bo'yicha ikki marta hosila olish mumkin.

Endi $A(\lambda)$ va $B(\lambda)$ koeffitsientlarni shunday tanlaylikki, (7) formula (2) boshlang'ich shartni qanoatlantirsin. Buning uchun yuqoridagi (7) formulada $t = 0$ deb, ushbu

$$\varphi(x) = \int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]d\lambda \tag{8}$$

ifodani olamiz.

Ikkinchi tomondan $\varphi(x)$ funksiya uchun quyidagi Fur'e integrali

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} \varphi(\xi)\cos\lambda(x-\xi)d\xi =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\cos\lambda x\int_{-\infty}^{\infty}\varphi(\xi)\cos\lambda\xi d\xi + \sin\lambda x\int_{-\infty}^{\infty}\varphi(\xi)\sin\lambda\xi d\xi\right]d\lambda$$

ham o'rinli bo'ladi. Bu formulani (8) bilan solishtirsak,

$$A(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)\cos\lambda\xi d\xi, \qquad B(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)\sin\lambda\xi d\xi, \tag{9}$$

bo'lishi kelib chiqadi. Endi hosil bo'lgan (9) ifodalarni (7) formulaga qo'yib

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a^2\lambda^2 t}d\lambda \int_{-\infty}^{\infty} \varphi(\xi)\cos\lambda(x-\xi)d\xi,$$

yoki bu formuladagi birinchi integral ostidagi funksiya $\lambda$ ga nisbatan juft funksiya ekanligini hisobga olib,

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(\xi)d\xi \int_{0}^{\infty} e^{-a^2\lambda^2 t}\cos\lambda(x-\xi)d\lambda. \tag{10}$$

ifodaga ega bo'lamiz.

Bu yerda ichki integralni bevosita hisoblashimiz mumkin. Buning uchun

$$a\lambda\sqrt{t} = z, \qquad \lambda(x-\xi) = \mu z$$

deb belgilashlarni kiritamiz. Bulardan

$$d\lambda = \frac{dz}{a\sqrt{t}}, \qquad \mu = \frac{x-\xi}{a\sqrt{t}}$$

bo'ladi. U holda (10) formulaning o'ng tomonidagi ichki integralni

$$\int_{0}^{\infty} e^{-a^2\lambda^2 t}\cos\lambda(x-\xi)d\lambda = \frac{1}{a\sqrt{t}}\int_{0}^{\infty} e^{-z^2}\cos\mu z dz = \frac{1}{a\sqrt{t}}J(\mu), \tag{11}$$

ko'rinishda yozish mumkin.

(11) integral tekis yaqinlashuvchi bo'lgani uchun uni $\mu$ parametr bo'yicha differensiallab,

$$J'(\mu) = -\int_{0}^{\infty} e^{-z^2} z\sin\mu z dz$$

ifodaga ega bo'lamiz. Endi oxirgi ifodani bo'laklab integrallaymiz. Natijada

$$J'(\mu) = -\frac{\mu}{2}\int_{0}^{\infty} e^{-z^2}\cos\mu z dz = -\frac{\mu}{2}J(\mu)$$

tenglikni olamiz. Oxirgi tenglikdan

$$J'(\mu) = -\frac{\mu}{2} J(\mu) \qquad \text{bundan esa,} \qquad J(\mu) = c\, e^{-\frac{\mu^2}{4}}$$

kelib chiqadi.

Bu yerda $\mu = 0$ deb, $c$ o‘zgarmasni

$$c = J(0) = \int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$$

topamiz. Shuning uchun

$$J(\mu) = \frac{\sqrt{\pi}}{2} e^{-\frac{\mu^2}{4}}$$

bo‘ladi (11) formulani hisobga olib,

$$\int_0^\infty e^{-a^2\lambda^2 t} \cos\lambda(x-\xi)d\lambda = \frac{\sqrt{\pi}}{2a\sqrt{t}} \exp\left\{-\frac{(\xi - x)^2}{4a\sqrt{t}}\right\}$$

ekanligini olamiz.

Bu formulani (10) ifodaga qo‘yib, (1)–(2) Koshi masalasining $u(x,t)$ yechimi uchun quyidagi

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi. \qquad (12)$$

formulani hosil qilamiz. Bu yerdagi

$$G(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \qquad (13)$$

funksiya $(x,t)$ bo‘yicha (1) tenglamani qanoatlantiradi va shu tenglamaning *fundamental yechimi deyiladi.*

Demak, quyidagi teorema o‘rinli.

2–TEOREMA. Agar $\varphi(x)$ funksiya $(-\infty, +\infty)$ da uzluksiz va chegaralangan bo‘lsa, u holda (1)–(2) Koshi masalasining yechimi mavjud va bu yechim (12) formula bilan aniqlanadi.

ISBOT. $R = \{-\infty < x < +\infty\}$ sonlar o‘qida aniqlangan uzluksiz $\varphi(x)$ funksiya uchun (12) formula bilan aniqlangan $u(x,t)$ funksiya $Q$ sohada chegaralangan va (1)–(2) Koshi masalasining yechimi ekanligini ko‘rsatamiz.

Buning uchun (12) formulani (13) belgilash asosida quyidagicha

$$u(x,t) = \int\limits_{-\infty}^{\infty} G(x,t;\xi)\,\varphi(\xi)d\xi. \tag{14}$$

yozib olishimiz mumkin. Bu formula (1) tenglamani qanoatlantirishini bevosita hosila olib ko‘rsatish qiyin emas.

Haqiqatdan ham,

$$G_x = -\frac{1}{2\sqrt{\pi}}\frac{x-\xi}{2(a^2t)^{3/2}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\};$$

$$G_{xx} = \frac{1}{2\sqrt{\pi}}\left[-\frac{1}{2(a^2t)^{3/2}} + \frac{(x-\xi)^2}{4(a^2t)^{5/2}}\right]\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\};$$

$$G_t = \frac{1}{2\sqrt{\pi}}\left[-\frac{a^2}{2(a^2t)^{3/2}} + \frac{a^2(x-\xi)^2}{4(a^2t)^{5/2}}\right]\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\};$$

bundan

$$G_t = a^2 G_{xx}$$

ekanligi kelib chiqadi. U holda (12) integralni hamda uni $x$ va $t$ o‘zgaruvchilari bo‘yicha ixtiyoriy marta differensiallashdan hosil bo‘lgan integrallarning biror

$$Q_0 = \{(x,t) : -L < x < +L,\ 0 < t_0 \le t \le T\}$$

sohada tekis yaqinlashuvchi ekanligini ko'rsatish etarli.

(12) integral yaqinlashuvchi bo'ladi. Haqiqatdan ham,

$$M = \max_x |\varphi(x)| \quad \text{va} \quad \int_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}$$

bo'lgani uchun (12) formuladan

$$|u(x,t)| < M \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{a^2t}} \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi =$$

$$= M \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} dz = M.$$

kelib chiqadi.

Bu yerda

$$z = \frac{x-\xi}{2a\sqrt{t}}; \quad \int_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}.$$

Endi (12) integralni $x$ va $t$ o'zgaruvchilari bo'yicha bir necha marta differensiallab,

$$J(x,t) = \frac{1}{t^n} \int_{-\infty}^{+\infty} \varphi(\xi)(x-\xi)^m \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi, \tag{15}$$

kabi integrallar yig'indisini olamiz.

$$z = \frac{x-\xi}{2a\sqrt{t}}, \quad t > 0. \tag{16}$$

formula yordamida o'zgaruvchilarni almashtirish natijasida (15) integral

$$J = (2a)^{m+1} t^{\frac{m+1}{2}-n} \int_{-\infty}^{+\infty} \varphi(x - 2az\sqrt{t}) z^m e^{-z^2} dz$$

ko'rinishga keladi.

Bundan (15) integralning tekis yaqinlashuvchi bo'lishi kelib chiqadi. Chunki oxirgi ifodada integral ostidagi funksiya

$$|\varphi(x - 2az\sqrt{t})z^m e^{-z^2}| \leq M|z|^m e^{-z^2}$$

funksiyaga majorirlanadi va bu funksiya $(-\infty, +\infty)$ oraliqda integrallanuvchi bo'lganligi uchun oxirgi integral $t \geq t_0 > 0$ bo'lganda tekis yaqinlashuvchi bo'ladi, ya'ni

$$0 \leq \int_{-\infty}^{+\infty} M|z|^m e^{-z^2} dz < +\infty$$

integral $\forall m \geq 0$ yaqinlashuvchi bo'ladi. Demak, (12) formula bilan aniqlangan $u(x,t)$ funksiya $t > 0$ da uzluksiz va $x$, $t$ o'zgaruvchilar bo'yicha ixtiyoriy tartibdagi hosilalarga ega.

(12) formula bilan berilgan $u(x,t)$ yechim (2) boshlang'ich shartni ham qanoatlantirishini isbotlaymiz, ya'ni

$$\lim_{t \to 0} u(x,t) = \varphi(x), \qquad -\infty < x < +\infty$$

o'rinlidir. $\xi$ o'zgaruvchining o'rniga (16) formula asosida yangi $z$ o'zgaruvchini kiritaylik. U holda (12) integral

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2az\sqrt{t}) e^{-z^2} dz \tag{17}$$

ko'rinishiga keladi.

Bundan, $\forall x \in R$ da $|\varphi(x)| \leq M$ bo'lgani uchun $(x,t) \in Q$ sohada $|u(x,t)| \leq M$ bo'lishini ko'rsatish qiyin emas.

$$|u(x,t)| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |\varphi(x) + 2az\sqrt{t})| e^{-z^2} dz \leq$$

$$\leq \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = M,$$

chunki

$$\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} a^{-z^2}dz = 1. \tag{18}$$

Demak, $u(x,t)$ funksiyaning chegaralanganligi isbotlandi.

Endi (12) formula (2) boshlang'ich shartni qanoatlantirishini ko'rsataylik. Buning uchun (18) formulani ikki tomonini $\varphi(x)$ funksiyaga ko'paytirib, so'ngra (17) ifodadan ayirsak, natijada

$$u(x,t)-\varphi(x)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}[\varphi(x+2az\sqrt{t})-\varphi(x)]e^{-z^2}dz$$

hosil bo'ladi. Bundan $|\varphi(x+2az\sqrt{t})-\varphi(x)|\le 2M$ ekanligini hisobga olib,

$$|u(x,t)-\varphi(x)|\le\frac{2M}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}e^{-z^2}d\alpha=$$

$$=\frac{2M}{\sqrt{\pi}}\int\limits_{-\infty}^{-N}e^{-z^2}dz+\frac{2M}{\sqrt{\pi}}\int\limits_{-N}^{N}a^{-z^2}dz+\int\limits_{N}^{\infty}e^{-z^2}dz,$$

deb yozish mumkin. Bundan esa har qanday kichik $\varepsilon>0$ berilganda ham $N$ ni shunday tanlab olish mumkinki,

$$\frac{2M}{\sqrt{\pi}}\int\limits_{-\infty}^{-N}e^{-z^2}dz\le\frac{\varepsilon}{3},\quad \frac{2M}{\sqrt{\pi}}\int\limits_{N}^{\infty}e^{-z^2}dz\le\frac{\varepsilon}{3}$$

bo'ladi. Demak,

$$|u(x,t)-\varphi(x)|\le\frac{2\varepsilon}{3}+\frac{1}{\sqrt{\pi}}\int\limits_{-N}^{N}|\varphi(x+2za\sqrt{t})-\varphi(x)|e^{-z^2}dz \tag{19}$$

deb yozish mumkin. $\varphi(x)$ funksiyaning uzluksizligidan yetarli kichik $t>0$ va $|z|\le N$ uchun

$$|\varphi(x+2az\sqrt{t})-\varphi(x)|\le\frac{\varepsilon}{3}$$

bajariladi. U holda (19) tengsizlikka asosan

$$|u(x,t)-\varphi(x)| \le \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}\frac{1}{\sqrt{\pi}}\int\limits_{-N}^{N} e^{-z^2}dz \le$$

$$\le \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{-z^2}dz = \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Bundan $\varepsilon > 0$ ixtiyoriyligidan

$$\lim_{t\to 0} u(x,t) = \varphi(x)$$

ekanligi kelib chiqadi.

Xuddi shunday usul bilan Koshi masalasining yechimi turg'un ekanligini ham isbotlash mumkin. Demak, issiqlik tarqalishi tenglamasi uchun Koshi masalasi korrekt qo'yilgan masala ekan.

Agar Koshi masalasida (2) boshlang'ich shart $t = 0$ da emas, biror $t = \tau$ da berilgan bo'lsa, (12) formulada $t$ ni $t-\tau$ bilan almashtirish lozim, ya'ni

$$G(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\}. \qquad (20)$$

Bu funksiya Koshi masalasining *Grin funksiyasi* ham deb yuritiladi va chegaraviy masalalarning yechimlarini oshkor ravishda topishda keng qo'llaniladi.

## 20–§. Bir jinsli bo'lmagan issiqlik tarqalish tenglamasi uchun Koshi masalasi

Endi $t > 0$ da bir jinsli bo'lmagan issiqlik tarqalish tenglamasi

$$u_t = a^2 u_{xx} + f(x,t),$$

uchun Koshi masalasining yechimini topaylik.

(1)–(2) Koshi masalasining yechimini $u(x,t) = u_1(x,t)+u_2(x,t)$ yig'indi ko'rinishida izlaymiz, bu yerda $u_1(x,t)$ funksiya bir jinsli issiqlik tarqalish tenglamasining (2) boshlang'ich shartni qanoatlantiruvchi yechimi, $u_2(x,t)$ esa (1) tenglamaning bir jinsli boshlang'ich shartni qanoatlantiruvchi yechimi.

Yuqorida bir jinsli tenglamaning (2) boshlang'ich shartni qanoatlantiruvchi yechimini (12) formula orqali topgan edik.

Endi $u_1(x,t)$ yechimning integral ifodasini topamiz. Agar $f(x,t)$ funksiya $t > 0, -\infty < x < +\infty$ da uzluksiz va chegaralangan bo'lsa, u holda

$$v(x,t,\tau) = \frac{1}{\sqrt{2a\pi(t-\tau)}} \int\limits_{-\infty}^{+\infty} f(\xi,\tau) \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} d\xi,$$

funksiya $t > \tau$ da $v_t = a^2 v_{xx}$ tenglamani va $t = \tau$ da esa $v(x,t,t) = f(x,t)$ tenglikni qanoatlantiradi.

Ushbu

$$u(x,t) = \int\limits_0^t v(x,t,\tau)d\tau, \tag{21}$$

funksiyani qaraylik. Bu funksiya uchun quyidagi tengliklarning

$$u(x,0) = 0, \quad u_t(x,t) = v(x,t,t) + \int\limits_0^t v_t(x,t,\tau)d\tau$$

$$u_{xx} = \int\limits_0^t v_{xx}(x,t,\tau)d\tau$$

o'rinli ekanligiga ishonch hosil qilish qiyin emas. Bu tengliklardan (21) funksiya (1)–(2) masalaning yechimi ekanligi kelib chiqadi.

Demak, $v(x,t,\tau)$ funksiyaning ifodasini (21) tenglikka qo'yib, bir jinsli bo'lmagan issiqlik tarqalish tenglamasi uchun bir jinsli Koshi masalasining

$$u(x,t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t d\tau \int\limits_{-\infty}^{+\infty} \frac{1}{\sqrt{t-\tau}} \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} f(\xi,\tau)d\xi, \tag{22}$$

yechimiga ega bo'lamiz.

Endi yuqorida topilgan (12) va (22) formulalarni qo'shib, issiqlik tarqalish tenglamasi uchun (1)–(2) Koshi masalasining yechimi

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} \varphi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi +$$

$$+\int\limits_0^t \int\limits_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} f(\xi,\tau)\,d\xi\,d\tau, \qquad (23)$$

hosil qilamiz.

3–TEOREMA. Agar $\varphi(x)$ funksiya $(-\infty,+\infty)$ oraliqda, $f(x,t)$ funksiya $t>0$, $-\infty<x<+\infty$ sohada uzluksiz va chegaralangan bo'lsa, u holda (1)–(2) Koshi masalasining yechimi mavjud va yagona bo'lib, bu yechim (23) formula bilan aniqlanadi.

Bu yerda $f(x,t) \in C^2(t \geq 0)$ va uning birinchi va ikkinchi tartibli barcha hosilalari $t>0$ da chegaralangan funksiyalar.

1–NATIJA. Issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasining $u(x,t)$ yechimi $Q$ sohada cheksiz differensiallanuvchi, ya'ni $u(x,t) \in C^\infty(Q)$ bo'ladi.

2–IZOX. Yuqoridagi natijaga ko'ra qaralayotgan Koshi masalasining $u(x,t)$ yechimini $t>0$ bo'lganda $x$ va $t$ o'zgaruvchilar bo'yicha differensiallash berilgan $\varphi(x)$ funksiyaning silliqligiga bog'liq emas.

Issiqlik tarqalish tenglamasi yechimining silliqligi tor tebranish tenglamasi yechimidan tubdan farq qiladi. Qaralayotgan sohada tor tebranish tenglamasi yechimining silliqligi berilgan funksiyalarning silliqligiga bog'liq bo'ladi.

3–IZOX. (2) formuladan ko'rinadiki, issiqlik sterjen bo'ylab biror tezlik bilan emas, balki oniy tarqaladi. Haqiqatdan ham, faraz qilaylik, boshlang'ich harorat $(\alpha,\beta)$ integralda $\varphi(x)>0$ va undan tashqarida $\varphi(x)=0$ bo'lsin. U holda $u(x,t)$ temperaturaning keyingi tarqalishi uchun

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_{\alpha}^{\beta} \varphi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi$$

formulaga ega bo'lamiz.

Bu formuladan yetarlicha kichik $t > 0$ va yetarlicha katta $x$ uchun $u(x,t)$ musbat ekanligini ko'rish qiyin emas. Bundan ko'rinadiki, sterjenda issiqlik cheksiz tezlik bilan, ya'ni oniy tezlik bilan tarqaladi. Tabiiyki, bunday bo'lishi mumkin emas. Demak, (1) tenglama sterjenda issiqlik tarqalish masalasi to'liq ifodalamas ekan. Bu issiqlik tarqalish tenglamasini keltirib chiqarishdagi ayrim fizikaviy noaniqliklar bilan bog'liq.

YECHIMNING BERILGANLARGA UZLUKSIZ BOG'LIQLIGI.

Issiqlik tarqalish tenglamasi uchun Koshi masalasining yechimi berilganlarga uzluksiz bog'liq bo'ladi. Faraz qilaylik, $u(x,t)$ funksiya (1) tenglamaning (2) boshlang'ich shartni qanoatlantiruvchi yechimi, $u_\varepsilon(x,t)$ funksiya esa (1) tenglamaning

$$u_\varepsilon(x,0) = \varphi_\varepsilon(x), \tag{22}$$

shartni qanoatlantiruvchi yechimi bo'lsin.

Agar ixityoriy $x \in (-\infty, +\infty)$ bo'lganda $|\varphi(x) - \varphi_\varepsilon(x)| < \varepsilon$ bo'lsa, u holda barcha $x$ va $t > 0$ uchun $|u(x,t) - u_\varepsilon(x,t)| < \varepsilon$ bo'ladi.

Haqaqatdan ham, (1) tenglamaning (2) va (22) boshlang'ich shartlarni qanoatlantiruvchi $u(x,t)$ va $u_\varepsilon(x,t)$ yechimlari mos ravishda (12) formula bilan aniqlanadi. Ularning ayirmasi

$$u(x,t) - u_\varepsilon(x,t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} [\varphi(\xi) - \varphi_\varepsilon(\xi)] \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi$$

bo'ladi. Bu ayirmani baholaymiz, natijada

$$|u(x,t) - u_\varepsilon(x,t)| \le \frac{\varepsilon}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} d\xi$$

yoki $z = \dfrac{x-\xi}{2a\sqrt{t}}$ almashtirish yordamida

$$|u(x,t) - u_\varepsilon(x,t)| \le \varepsilon \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} e^{-z^2} dz = \varepsilon$$

kelib chiqadi.

Demak, issiqlik tarqalish tenglamasi uchun Koshi masalasining yechimi boshlang'ich funksiyaga uzluksiz bogliq ekan.

FUNDAMENTAL YECHIMNING XOSSALARI.

Mam'lumki, yuqorida keltirilgan $(x,t;\xi,\tau)$ argumentlarga bog'liq bo'lgan

$$E(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)^2}\right\}$$

funksiya issiqlik tarqalish tenglamasining *fundamental yechimi* deyiladi.

Bu funksiya quyidagi xossalarga ega:

$1^0$. $E(x,t;\xi,\tau)$ funksiya $(x,t)$ o'zgaruvchilar bo'yicha bir jinsli issiqlik tarqalish tenglamasini qanoatlantiradi.

$2^0$. $E(x,t;\xi,\tau)$ funksiya $(\xi,\eta)$ o'zgaruvchilar bo'yicha bir jinsli issiqlik tarqalish tenglamasiga qo'shma bo'lgan

$$E_t + a^2 E_{xx} = 0$$

tenglamani qanoatlantiradi.

$3^0$. Agar $x \neq \xi$ bo'lsa, u holda $\lim\limits_{\tau\to t} E(x,t;\xi,\tau) = 0$ tenglik o'rinli bo'ladi.

$4^0$. Agar $\varphi(x)$ funksiya $\forall x \in [0,l]$ bo'lganda uzluksiz, ya'ni $\varphi(x) \in C[0,l]$ bo'lsa, u holda $E(x,t;\xi,\tau)$ fundamental yechim uchun ushbu

$$\lim_{\tau\to t}\int\limits_0^l E(x,t;\xi,\tau)\varphi(\xi)d\xi = \begin{cases} \frac{1}{2}\varphi(0), & \text{agar } x = 0 \text{ b,} \\ \varphi(x), & \text{agar } x \in (0,l) \text{ b,} \\ \frac{1}{2}\varphi(l), & \text{agar } x = l \text{ b,} \end{cases} \tag{24}$$

tenglik o'rinli bo'ladi.

Issiqlik tarqalish tenglamasi uchun fundamental yechimning $1^0$ va $2^0$ xossalarining bajarilishiga bevosita tekshirib ishonch hosil qilish mumkin.

Biz bu yerda $3^0$ va $4^0$ xossalarini isbotlaymiz.

$3^0$. Ma'lumki, $E(x,t;\xi,\tau)$ funksiya $\tau \to t$ da aniqmaslikka ega. Shuning uchun bu funksiyani ushbu

$$E(x,t;\xi,\tau) = \frac{A(t)}{B(t)},$$

formal ko'rinishda ifodalab olamiz. Bu yerda

$$A(t) = \frac{1}{2a\sqrt{\pi(t-\tau)}}, \qquad B(t) = \exp\left\{\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\}.$$

Lopital qoidasiga asosan

$$\lim_{\tau\to t} E(x,t;\xi,\tau) = \lim_{\tau\to t}\frac{A'(t)}{B'(t)} =$$

$$= \frac{a}{\sqrt{\pi}}\lim_{\tau\to t}\frac{\sqrt{t-\tau}}{(x-\xi)^2}\exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} = 0$$

kelib chiqadi.

4. Bu xossani isbotlash uchun (24) formulaning chap tomonida

$$\frac{x-\xi}{2a\sqrt{t-\tau}} = z, \quad \text{ya'ni} \quad \xi = x - 2az\sqrt{t-\tau}$$

almashtirish bajaramiz. U holda

$$\xi = 0 \text{ da } z = \frac{x}{2a\sqrt{t-\tau}}; \quad \xi = l \text{ da esa } z = \frac{x-l}{2a\sqrt{t-\tau}}$$

tengliklar o'rinli bo'ladi. Bu tengliklarga asosan (24) formula

$$\int_0^l E(x,t;\xi,\tau)\varphi(\xi)d\xi = \frac{1}{\sqrt{\pi}}\int_{\frac{x-l}{2a\sqrt{t-\tau}}}^{\frac{x}{2a\sqrt{t-\tau}}} e^{-z^2}\varphi(x-2az\sqrt{t-\tau})dz, \tag{25}$$

ko'rinishga keladi. Ushbu

$$\int_{-\infty}^{+\infty} e^{-z^2}dz = \sqrt{\pi}; \quad \int_{-\infty}^{0} e^{-z^2}dz = \int_{0}^{+\infty} e^{-z^2}dz = \frac{\sqrt{\pi}}{2};$$

ayniyatlarni inobatga olib, (25) formulada $\tau \to t$ limitga o'tamiz va

$$\lim_{\tau\to t}\int_0^l E(x,t;\xi,\tau)\varphi(\xi)d\xi = \frac{1}{\sqrt{\pi}}\lim_{\tau\to t}\int_{\frac{x-l}{2a\sqrt{t-\tau}}}^{\frac{x}{2a\sqrt{t-\tau}}} e^{-z^2}\varphi(x-2az\sqrt{t-\tau})dz =$$

$$= \begin{cases} \frac{1}{2}\ \varphi(0), & \text{agar } x = 0 \text{ bo'lsa,} \\ \varphi(x), & \text{agar } x \in (0,l) \text{ bo'lsa,} \\ \frac{1}{2}\ \varphi(l), & \text{agar } x = l \text{ bo'lsa,} \end{cases}$$

tengliklarga ega bo'lamiz.

Fundamental yechimning $4^0$ xossasidan quyidagi natija kelib chiqadi:

2–NATIJA. Agar (24) formulada $\varphi(x) = 1$ bo'lsa, u holda

$$\lim_{\tau\to t}\int_0^l E(x,t;\xi,\tau)d\xi = \begin{cases} \frac{1}{2}, & \text{agar } x = 0 \text{ bo'lsa,} \\ 1, & \text{agar } x \in (0,l) \text{ bo'lsa,} \\ \frac{1}{2}, & \text{agar } x = l \text{ bo'lsa.} \end{cases}$$

tengliklar o'rinli bo'ladi.

FUNDAMENTAL YECHIMNING FIZIKAVIY XOSSASI.

Sterjenning biror $x_0$ nuqtasi atrofidan kichik segmentini ajratib olaylik, ya'ni $(x_0 - \varepsilon,\ x_0 + \varepsilon)$, bu yerda $\varepsilon > 0$. Faraz qilaylik, $\varphi(x)$ boshlang'ich harorat shu segmentda o'zgarmas $u_0 > 0$ va bu segmentdan tashqarida nolga teng bo'lsin.

Buni fizikaviy tasavvur qilish qiyin emas, boshlang'ich vaqtda sterjenning $(x_0 - \varepsilon,\ x_0 + \varepsilon)$ segmentiga $Q = 2\varepsilon c\rho u_0$ issiqlik miqdori berilgan, bunda $c$ sterjenning issiqlik sigimi, $\rho$ sterjenning zichligi, sterjenning $(x_0 - \varepsilon, x_0 + \varepsilon)$ qismida harorat ko'tarilib $u_0$ ga teng bo'ladi. Vaqtning keyingi momentlarida sterjendagi issiqlik tarqalishi (12) formula bilan aniqlanadi, biz qarayotgan holda u quyidagicha

$$u_\varepsilon(x,t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} u_0 e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi =$$

$$= \frac{Q}{2ac\rho\sqrt{\pi t}} \frac{1}{2\varepsilon} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi,$$

ifodalanadi.

Agar $\varepsilon$ ni nolga yaqin qilib kichiklashtirsak, unda sterjenning juda kichik qismiga $Q$ issiqlik miqdori taqsimlandi, deb hisoblashimiz mumkin. U holda $t = 0$ vaqtda sterjenning $x = x_0$ nuqtasida joylashgan kuchlanishi $Q$ bo'lgan oniy issiqlik manbaiga ega bo'lamiz. Bunday oniy issiqlik manbai ta'sirida sterjenda issiqlik taqsimoti quyidagi

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,t) = \frac{Q}{2ac\rho\sqrt{\pi t}} \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi, \qquad (23)$$

formula bilan aniqlanadi.

O'rta qiymat haqidagi teoremaga ko'ra

$$\frac{1}{2\varepsilon} \int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi = e^{-\frac{(x-\xi_0)^2}{4a^2 t}}$$

tenglikni olamiz. Bu yerda $x_0 - \varepsilon < \xi_0 < x_0 + \varepsilon$, agar $\varepsilon \to 0$ bo'lsa, $x_0 \to \xi_0$ bo'ladi. U holda (23) formula

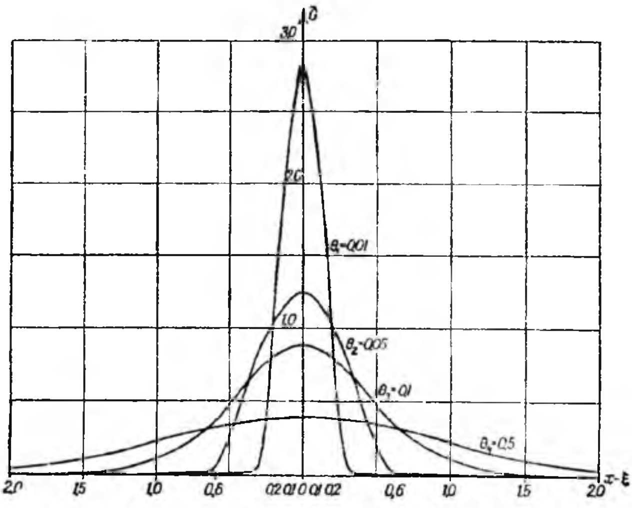
$$\frac{Q}{c\rho} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x_0-\xi_0)^2}{4a^2 t}}$$

ko'rinishga keladi.

Shunday qilib, (13) fundamental yechim sterjenning $x = \xi$ nuqtasidagi $t = 0$ boshlang'ich vaqtda joylashgan kuchlanishi $Q = c\rho$ ga teng bo'lgan oniy issiqlik manbaining taqsimotini bildiradi. Issiqlik tarqalish tenglamasining (13) formula bilan aniqlangan $G(x, t; \xi)$ fundamental yechimining

$$E(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}}$$

grafigini fiksirlangan $\xi$ uchun $x$ ning funksiyasi sifatida aloda $t$ vaqtning $0 < t_1 < t_2 < t_3 < \ldots$ momentlarida quyidagi chizmada keltirilgan.




18 — shakl.

Har bir egri chiziq ostidagi yuza birga teng, ya'ni

$$\int\limits_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} e^{-z^2} dz = 1.$$

Bu sterjendagi $Q = c\rho$ issiqlik miqdori vaqt o'tishi bilan o'zgarmas ekanligini bildiradi. 18–shakldan ko'rinadiki, agar $t > 0$ yetarlicha kichik son bo'lsa, u holda (13) egri chiziqlar bilan chegaralangan deyarli barcha yuzalar abssissalar o'qida $(x_0 - \varepsilon, x_0 + \varepsilon)$ oraliqning ustida joylashgan bo'ladi. Bu yuzaning qiymatini $c\rho$ ga ko'paytmasi boshlang'ich vaqtdagi issiqlik miqdoriga teng. Shunday qilib, $t > 0$ yetarlicha kichik bo'lganda, deyarli barcha issiqlik $x = \xi$ nuqtaning kichik atrofida uyushgan bo'lar ekan. Demak, $t = 0$ vaqtda barcha issiqlik miqdori $x = \xi$ nuqtaga joylashgan bo'ladi, ya'ni biz oniy issiqlik manbaiga ega bo'lamiz.

Endi yuqoridagi mulohazalarga asosan (12) yechimning fizik ma'nosini keltirish qiyin emas. Haqiqatdan ham, sterjenning $x = \xi$ kesimiga boshlang'ich vaqtda $\varphi(\xi)$ harorat berish uchun shu nuqtaning yetarlicha kichik $d\xi$ yuzasiga $dQ = c\rho\varphi(\xi)d\xi$ teng bo'lgan issiqlik miqdori taqsimlashimiz zarur yoki sterjenning $\xi$ nuqtasiga kuchlanishi $dQ$ bo'lgan oniy issiqlik manbaini joylashtirish zarur. Shu oniy issiqlik manbai ta'sirida issiqlikning taqsimlanishi (13) formulaga ko'ra

$$\varphi(\xi)d\xi \frac{1}{2a\sqrt{\pi t}}e^{-\dfrac{(x-\xi)^2}{4a^2t}}$$

bo'ladi. Boshlang'ich $\varphi(\xi)$ harorat ta'sirida sterjenning barcha nuqtalaridagi harorat shu yuzalarning yig'indisidan iborat bo'ladi, ya'ni (12) formula kelib chiqadi.

1–MASALA. Ushbu

$$u_t = u_{xx}, \qquad -\infty < x < +\infty, \quad t > 0,$$

tenglamaning

$$u(x, 0) = \sin x, \qquad -\infty < x < +\infty$$

boshlang'ich shartni qanoatlantiruvchi regulyar yechimini toping.

YECHISH. Masala yechimini (12) formula yordamida topamiz. Bu yerda $a = 1$, $\varphi(x) = \sin x$ bo'lgani uchun (12) formulaga ko'ra

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} \sin\xi \exp\left\{-\frac{(x-\xi)^2}{4t}\right\}d\xi \tag{24}$$

hosil bo'ladi. Ushbu

$$x - \xi = 2\sqrt{t}s, \quad d\xi = -2\sqrt{t}ds$$

almashtirish yordamida (24) integral

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-s^2} \left( \sin x \cos 2\sqrt{t}s - \cos x \sin 2\sqrt{t}s \right) ds$$

ko'rinishga keladi.

$$\int_{-\infty}^{+\infty} e^{s^2} \cos 2bs ds = \sqrt{\pi} e^{-b^2}, \quad \int_{-\infty}^{+\infty} e^{s^2} \sin 2bs ds = 0 \qquad (25)$$

tengliklarga asosan oxirgi ifodadan 1–masalaning yechimi

$$u(x,t) = e^{-t} \sin x$$

kelib chiqadi.

2–MASALA. Ushbu bir jinsli bo'lmagan

$$u_t = \frac{1}{4} u_{xx} + e^t \sin x, \quad -\infty < x < +\infty, \ t > 0;$$

tenglamaning

$$u(x,0) = e^{-x^2} \sin x, \quad -\infty < x < +\infty,$$

shartni qanoatlantiruvchi yechimi topilsin.

YECHISH. (21) formulaga

$$a = \frac{1}{2}, \quad f(x,t) = e^t \sin x, \quad \varphi(x) = e^{-x^2} \sin x$$

funksiyalarni qo'yib,

$$u(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\xi^2} \sin \xi \exp\left\{ -\frac{(x-\xi)^2}{t} \right\} d\xi +$$

$$+\int_0^t\int_{-\infty}^{+\infty}\frac{e^\tau\sin\xi}{\sqrt{\pi(t-\tau)}}\exp\left\{\frac{(x-\xi)^2}{t-\tau}\right\}d\xi d\tau=I_1+I_2; \qquad (26)$$

tenglikni olamiz.

Endi $I_1$ va $I_2$ integrallarni hisoblaymiz.

Birinchi $I_1$ integral

$$\xi=\sqrt{\frac{t}{t+1}}s+\frac{x}{t+1};\quad d\xi=\sqrt{\frac{t}{t+1}}ds$$

almashtirish natijasida

$$I_1=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-s^2}\sqrt{\frac{t}{t+1}}\exp\left\{-\frac{x^2}{t+1}\right\}\sin\left(\sqrt{\frac{t}{t+1}}s+\frac{x}{t+1}\right)ds$$

ko'rinishga keladi. Bundan

$$I_1=\frac{1}{\sqrt{\pi(t+1)}}\exp\left\{-\frac{x^2}{t+1}\right\}\left[\sin\frac{x}{t+1}\int_{-\infty}^{+\infty}e^{-s^2}\cos\sqrt{\frac{t}{t+1}}sds+\right.$$

$$\left.+\cos\frac{x}{t+1}\int_{-\infty}^{+\infty}e^{-s^2}\sin\sqrt{\frac{t}{t+1}}sds\right]$$

bo'ladi.

Bundan (25) tengliklarga ko'ra

$$I_1=\frac{1}{\sqrt{t+1}}\sin\left(\frac{x}{t+1}\right)\exp\left\{-\frac{4x^2+t}{4(t+1)}\right\}$$

ifodaga ega bo'lamiz.

Ikkinchi $I_2$ integral esa,

$$I_2=\int_0^t e^{2\tau-t}\sin x d\tau=\sin x\,\mathrm{sh}t$$

ga teng.

$I_1$ va $I_2$ integrallarning qiymatlarini (26) formulaga qo'ysak, 2–masalaning yechimi

$$u(x,t) = \sin x \operatorname{sh} t + \frac{1}{\sqrt{t+1}} \sin\frac{x}{t+1} \exp\left\{-\frac{4x^2+t}{4(t+1)}\right\}$$

hosil bo'ladi.

3–MASALA. Quyidagi

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^k \tau(x), \tag{27}$$

funksiya

$$\sum_{i=1}^{n} u_{x_i x_i} - u_t = 0, \tag{28}$$

tenglamaning

$$u(x,0) = \tau(x), \qquad -\infty < x < +\infty, \tag{29}$$

shartni qanoatlantiruvchi yechimi ekanligini isbotlang.

Bu yerda $x = (x_1, x_2, \ldots, x_n)$, $\Delta \equiv \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ – Laplas operatori, $\tau(x) = \tau(x_1, x_2, \ldots, x_n)$ cheksiz differensiallanuvchi funksiya;

Faraz qilaylik, (27) va uning hosilalaridan tuzilgan qatorlar tekis yaqinlashuvchi bo'lsin.

ISBOT. (27) qatorni $x_1, x_2, \ldots, x_n$ o'zgaruvchilari bo'yicha ikki marta, $t$ bo'yicha bir marta differensiallashdan hosil bo'lgan qatorlarning tekis yaqinlashuvchi ekanligidan $u(x,t)$ yig'indi uchun

$$\sum_{i=1}^{n} u_{x_i x_i} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^k \sum_{i=1}^{n} \frac{\partial^2 \tau(x)}{\partial x_i^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{k+1} \tau(x);$$

$$u_t = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \Delta^k \tau(x) \qquad u_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{k+1} \tau(x)$$

bo'ladi. Demak, (27) funksiya berilgan (28) tenglamani qanoatlantiradi.

(27) funksiyadan (29) shart bevosita kelib chiqadi, ya'ni

$$u(x,t) = \tau(x) + \sum_{k=1}^{\infty} \frac{t^k}{k!} \Delta^k \tau(x),$$

bundan $t \to 0$ da limitga o'tsak,

$$\lim_{t\to 0} u(x,t) = u(x,0) = \tau(x)$$

ekanligi kelib chiqadi.

Demak, (28) tenglamaning (29) boshlang'ich shartni qanoatlantiruvchi yechimi (27) formula orqali ifodalandi.

## 21–§. Birinchi chegaraviy masala yechimining integral ifodasi

Bu paragrafda issiqlik tarqalishi tenglamasi uchun birinchi chegaraviy masalani qaraymiz. $xOt$ tekisligida uchlari $A(0,0)$, $B(l,0)$, $E(l,T)$ va $F(0,T)$ nuqtalarda bo'lgan $ABEF$ to'g'ri to'rtburchakni $Q$ deb belgilaymiz.

Berilgan chekli $Q = \{(x,t) : 0 < x < l, 0 < t < T\}$ sohada

$$Lu \equiv \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{1}$$

tenglamaning

$$u|_{t=0} = \varphi(x), \qquad 0 \le x \le l, \tag{2}$$

boshlang'ich va

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \qquad 0 \le t \le T, \tag{3}$$

chegaraviy shartlarni qanoatlantiradigan $u(x,t)$ yechimini toping.

Bu yerda $f(x,t)$, $\varphi(x)$, $\mu_1(t)$ va $\mu_2(t)$ – berilgan funksiyalar bo'lib, bu funksiyalar uchun quyidagi

$$\varphi(0) = \mu_1(0), \quad \varphi(l) = \mu_2(0)$$

tengliklar o'rinli.

Bu masala yechimining yagonaligi oldingi paragrafda ekstremum prinsipi yordamida isbotlangan, echimning mavjudligi esa o'zgaruvchilarni ajratish, Fur'e usuli bilan ko'rsatilgan.

Biz bu paragrafda (1)–(3) masala yechimining mavjudligini Grin funksiyasi yordamida isbotlashga harakat qilamiz va yechimning integral ifodasini keltirib chiqaramiz.

Issiqlik tarqalish tenglamasi uchun Grin formulasi. Bu yerda issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masala yechimining integral ifodasini keltirib chiqaramiz.

Buning uchun ushbu

$$Mv \equiv a^2 v_{xx} + v_t, \tag{4}$$

operatorni qaraylik.

Faraz qilaylik, $\varphi(x,t)$, $\psi(x,t)$ – ixtiyoriy cheksiz differensiallanuvchi funksiyalar bo'lsin. $L$ va $M$ operatorlar uchun quyidagi

$$\psi L\varphi - \varphi M\psi = a^2 \frac{\partial}{\partial x}\left(\psi\,\frac{\partial \varphi}{\partial x} - \varphi\,\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial t}\left(\varphi\,\psi\right),$$

ifoda o'rinli, bu ifodani $Q_\tau$ soha bo'yicha bo'laklab integrallaymiz. Natijada Grin formulasiga asosan

$$\iint\limits_{Q_\tau}\Big[\psi L\varphi - \varphi M\psi\Big]dxdt = \int\limits_{\partial Q_\tau}\left[\varphi\,\psi dx + a^2\left(\psi\,\frac{\partial \varphi}{\partial x} - \varphi\,\frac{\partial \psi}{\partial x}\right)dt\right], \tag{5}$$

formulaga ega bo'lamiz. Bu yerda $Q_\tau = \{(x,t) : 0 < x < l, 0 < t < \tau\}$ to'g'ri to'rtburchakli soha, $\partial Q_\tau$ esa $Q_\tau$ sohaning chegarasi, ya'ni $t = 0$, $x = l$, $t = \tau$ va $x = 0$.

Agar $L\varphi = f$ va $M\psi = 0$ bo'lsa, u holda (5) formulani quyidagicha

$$\iint\limits_{Q_\tau}\psi\,L\varphi\,dxdt = \int\limits_{AB}\varphi\,\psi\Big|_{t=0}dx + \int\limits_{BE}a^2\left(\psi\,\frac{\partial \varphi}{\partial x} - \varphi\,\frac{\partial \psi}{\partial x}\right)\Big|_{x=l}dt +$$

$$+\int\limits_{EF} \varphi\psi\Big|_{t=\tau} dx + \int\limits_{FA} a^2\left(\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}\right)\Big|_{x=0} dt, \qquad (6)$$

yoki

$$\int\limits_{EF} \varphi\psi\Big|_{t=\tau} dx = \int\limits_{AB} \varphi\psi\Big|_{t=0} dx + \int\limits_{BE} a^2\left(\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}\right)\Big|_{x=l} dt-$$

$$-\int\limits_{AF} a^2\left(\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}\right)\Big|_{x=0} dt + \iint\limits_{Q_\tau} \psi\, L\varphi\, dxdt, \qquad (6)$$

yozib olish mumkin.

Faraz qilaylik, $\varphi(x,t) = u(x,t)$ funksiya (1) tenglamaning biror yechimi, ya'ni $Lu = f(x,t)$; $\psi(x,t) = E(x,t;\xi,\tau)$ esa issiqlik tarqalish tenglamasining fundamental yechimi, ya'ni

$$E(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-\dfrac{(x-\xi)^2}{4a^2(t-\tau)}} \qquad (7)$$

bo'lsin.

Ma'lumki, $E(x,t;\xi,\tau)$ funksiya $x$, $t$ o'zgaruvchilar bo'yicha $LE = 0$ tenglamani, $\xi$, $\tau$ o'zgaruvchilar bo'yicha esa $ME = 0$ tenglamani qanoatlantiradi.

Faraz qilaylik, $M(x,t)$ qaralayotgan $Q_\tau$ sohadan olingan fiksirlangan nuqta, $M_1$ esa koordinatalari $x$, $t+h$, $h > 0$ bo'lgan nuqta bo'lsin. $M$ nuqta orqali $EF$ xarakteristika o'tkazamiz, (6) formulada $x$ ni $\xi$ ga $t$ ni esa $\tau$ deb almashtiramiz. U holda (6) formulani $ABEF$ soha bo'yicha

$$\varphi(\xi,\tau) = u(\xi,\tau), \qquad \psi(\xi,\tau) = E(x,t+h;\xi,\tau),$$

funksiyalarga qo'llaymiz. Natijada

$$\int\limits_{EF} \frac{1}{2a\sqrt{\pi h}} \exp\left\{-\frac{(x-\xi)^2}{4a^2h}\right\} u(\xi,t)d\xi =$$

$$= \int\limits_{AB} u(\xi,0)E(x,t+h;\xi,0)d\xi + \int\limits_{BE} a^2\left(E\frac{\partial u}{\partial \xi} - u\frac{\partial E}{\partial \xi}\right)\Big|_{x=l} d\tau -$$

$$- \int\limits_{AF} a^2\left(E\frac{\partial u}{\partial \xi} - u\frac{\partial E}{\partial \xi}\right)\Big|_{x=0} d\tau + \iint\limits_{Q_\tau} E(x,t+h;\xi,\tau)Lu\, d\xi d\tau,$$

ifodaga ega bo'lamiz.

$FABE$ sohada $E(x,t+h;\xi,\tau)$ va $\dfrac{\partial E}{\partial \xi}$ funksiyalarning $h$ ga nisbatan uzluksiz ekanligidan hamda ushbu

$$\lim_{h\to 0} \int\limits_{EF} \frac{1}{2a\sqrt{\pi h}} \exp\left\{-\frac{(x-\xi)^2}{4a^2h}\right\} u(\xi,t)d\xi = u(x,t)$$

tenglikka asosan, oxirgi formulada $h \to 0$ bo'lganda limitga o'tamiz. Agar $(x,t)$ nuqta $EF$ kesmada yotsa, u holda quyidagi

$$u(x,t) = \int\limits_{AB} u(\xi,0)E(x,t;\xi,0)d\xi + \int\limits_{BE} a^2\left(E\frac{\partial u}{\partial \xi} - u\frac{\partial E}{\partial \xi}\right)\Big|_{x=l} d\tau -$$

$$- \int\limits_{AF} a^2\left(E\frac{\partial u}{\partial \xi} - u\frac{\partial E}{\partial \xi}\right)\Big|_{x=0} d\tau + \iint\limits_{Q_t} E(x,t;\xi,\tau)f(\xi,\tau)\, d\xi d\tau, \quad (8)$$

asosiy integral ifodaga ega bo'lamiz.

Bu formula issiqlik tarqalish tenglamasi uchun boshlang'ich-chegaraviy masala yechimini bermaydi, chunki sohaning $AF$ va $BE$ chegarasida $u$ funksiyani emas, balki $u_\xi$ ni ham bilish kerak.

Faraz qilaylik, $v$ funksiya qo'shma $Mv = 0$ tenglamaning $EF$ da nolga teng bo'lgan biror yechimi va $u$ funksiya bir jinsli issiqlik tarqalish tenglamasi $Lu = 0$ ning yechimi bo'lsin.

Yuqorida olingan (6) formulani $FABE$ sohada $u$ va $v$ funksiyalarga qo'llaymiz. U holda

$$0 = \int\limits_{AB} u(\xi,0)v(\xi,0)d\xi + \int\limits_{BE} a^2\left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi}\right)\bigg|_{x=l} d\tau -$$

$$- \int\limits_{AF} a^2\left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi}\right)\bigg|_{x=0} d\tau, \qquad (9)$$

formulani olamiz.

Endi (8) formuladan (9) formulani ayiramiz, natijada

$$u(x,t) = \int\limits_{AB} u(\xi,0)G(x,t;\xi,0)d\xi + \int\limits_{BE} a^2\left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi}\right)\bigg|_{x=l} d\tau -$$

$$- \int\limits_{AF} a^2\left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi}\right)\bigg|_{x=0} d\tau + \iint\limits_{Q_t} G(x,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau, \quad (10)$$

ifodaga ega bo'lamiz. Bu yerda

$$G(x,t;\xi,\tau) = E(x,t;\xi,\tau) - v(x,t;\xi,\tau). \qquad (11)$$

Agar qaralayotgan sohaning $FA$ va $BE$ chegarasida $v = 0$ deb tanlasak, u holda $u(x,t)$ funksiya uchun quyidagi

$$u(x,t) = \int\limits_{AB} u(\xi,0)G(x,t;\xi,0)d\xi - a^2 \int\limits_{BE} u(l,\tau)\frac{\partial G}{\partial \xi}\bigg|_{x=l} d\tau +$$

$$+a^2 \int\limits_{AF} u(0,\tau)\frac{\partial G}{\partial \xi}\bigg|_{x=0} d\tau + \iint\limits_{Q_t} G(x,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau,$$

integral ifodaga kelamiz. Oxirgi tenglikdan (2)–(3) shartlarga asosan ushbu

$$u(x,t) = \int\limits_0^l \varphi(\xi)G(x,t;\xi,0)d\xi - a^2 \int\limits_0^t \mu_1(\tau)\frac{\partial G}{\partial \xi}\bigg|_{\xi=0} d\tau +$$

$$-a^2\int_0^t \mu(\tau)\,\frac{\partial G}{\partial\xi}\bigg|_{\xi=l} d\tau+\int_0^t\int_0^l G(x,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau, \qquad (12)$$

formulaga ega bo'lamiz. Bu funksiya (1) issiqlik tarqalish tenglamasi uchun birinchi boshlang/ich–chegaraviy masalaning yechimini beradi.

Shunday qilib, issiqlik tarqalish tenglama uchun birinchi chegaraviy masala echimining integral ifodasida asosiy qiyinchilik $v(\xi,\tau)=v(x,t;\xi,\tau)$ funksiyani aniqlash hisoblanadi.

Bu funksiya quyidagi shartlar asosida aniqlanadi:

1. $v(x,t;\xi,\tau)$ funksiya $Q$ sohada aniqlangan uzluksiz hamda $v_x(x,t;\xi,\tau)$ hosila $\overline{Q}$ mavjud va uzluksiz bo'lib, $x=0$ va $x=l$ da integrallanuvchi;

2. $v(x,t;\xi,\tau)$ funksiya $\tau<t$ uchun issitarqalish tenglamasiga qo'shma bo'lgan $Mv=0$ tenglamani qanoatlantiradi;

3. $v(x,t;\xi,\tau)$ funksiya $Q$ sohaning chegarasida $x=0$ va $x=l$ da $v(x,t;\xi,\tau)$ ga teng, ya'ni

$$v|_{x=0}=E(0,t;\xi,\tau), \quad v|_{x=l}=E(l,t;\xi,\tau)$$

shartlarni qanoatlantiradi;

4. Agar $\xi\neq x\in(0,l)$ bo'lsa, u holda $\lim\limits_{\tau\to t}v(x,t;\xi,\tau)=0$ tenglik o'rinli bo'ladi.

Bu shartlarga ko'ra, $v=v(x,t;\xi,\tau)$ funksiya $x,t$ parametrlarga bog'liq bo'lib, u qo'shma $Mv=0$ tenglama uchun (1)–(3) masalaga o'xshash chegaraviy masalaning yechimi kabi aniqlanadi.

Demak, ikki juft o'zgaruvchiga bog'liq bo'lgan (11) ko'rinishdagi $G(x,t;\xi,\tau)$ funksiya issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning *Grin funksiyasi* deyiladi.

Shunday qilib, issiqlik tarqalish tenglamasining ixtiyoriy yechimi Grin funksiyasi yordamida (12) formula bilan aniqlanadi.

To'g'ri to'rtburchakda (1)–(3) masalaning Grin funksiyasi uchun quyidagi

$$G(x,t;\xi,\tau)=\sum_{n=-\infty}^{+\infty}\Big[E(x,t;\xi+2nl,\tau)-E(x,t;-\xi+2nl,\tau)\Big], \quad (13)$$

qatorni qaraymiz.

Bu qator $0 \le x \le l$, $0 < \xi < l$ va $0 < \alpha \le t - \tau < A$ sohada absolyut va tekis yaqinlashuvchi bo'ladi.

Haqiqatdan ham, $n = \pm 1, \pm 2, \pm 3, \ldots$ uchun

$$|\pm\xi - 2nl - x| \ge 2|nl| - |\pm\xi - x| > 2|nl| - 2,$$

u holda $n = 0$ dan boshqa hollarda fundamental yechim uchun

$$|E(x,t;\pm\xi + 2nl,\tau)| \le \frac{1}{2a\sqrt{\pi\alpha}}\exp\left[-\frac{(|nl|-l)^2}{4a^2A}\right]$$

tengsizlik o'rinli.

Demak, (13) qatorning har bir hadi $n = 0$ dan boshqa hollarda

$$\frac{1}{2a\sqrt{\pi\alpha}}\sum_{n=1}^{\infty}\exp\left[-\frac{l^2(n-1)^2}{4a^2A}\right]$$

yaqinlashuvchi sonli qatorning hadlari bilan chegaralangan. Bundan esa (13) qatorning qaralayotgan sohada absolyut va tekis yaqinlashuvchi ekanligi kelib chiqadi.

O'rganilayotgan masalaning Grin funksiyasi

$$G(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\times$$

$$\times\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi-2nl)^2}{4a^2(t-\tau)}\right] - \exp\left[-\frac{(x+\xi-2nl)^2}{4a^2(t-\tau)}\right]\right\}, \quad (14)$$

ko'rinishga ega bo'ladi va buni quyidagi

$$G(x,t;\xi,\tau) = E(x,t;\xi,\tau) + \sum_{n=-\infty}^{+\infty}{}' E(x,t;\xi + 2nl,\tau) -$$

$$-\sum_{n=-\infty}^{+\infty} E(x,t;-\xi + 2nl,\tau) \quad (15)$$

ko'rinishda yozib olish mumkin.

Bu yerda $\sum\limits_{n=-\infty}^{+\infty}{}'$ belgi yig'indini $n$ noldan boshqa butun qiymatlari bo'yicha olinganligini bildiradi.

Bundan (11) tenglikka asosan (1)–(3) masala uchun

$$v(x,t;\xi,\tau) = \sum_{n=-\infty}^{+\infty} E(x,t;-\xi+2nl,\tau) -$$

$$-\sum_{n=1}^{+\infty}\Big[E(x,t;\xi+2nl,\tau)+E(x,t;\xi-2nl,\tau)\Big]$$

bo'lar ekan.

Endi issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masala Grin funksiyasining asosiy xossalarini keltiramiz:

$1^0$. $G(x,t;\xi,\tau)$ funksiya $x,t$ o'zgaruvchilar bo'yicha bir jinsli

$$G_t = a^2 G_{xx}$$

issiqlik tarqalish tenglamasini qanoatlantiradi.

$2^0$. $G(x,t;\xi,\tau)$ funksiya $\xi,\tau$ o'zgaruvchilar bo'yicha qo'shma, ya'ni

$$G_\tau + a^2 G_{\xi\xi} = 0$$

tenglamani qanoatlantiradi.

Grin funksiyasining $1^0$ va $2^0$ xossalarini bevosita hisoblashlar yordamida ko'rsatish mumkin.

$3^0$. $G(x,t;\xi,\tau)$ funksiya uchun quyidagi

$$G(x,t;0,\tau)=0, \qquad G(x,t;l,\tau)=0$$

chegaraviy shartlarni o'rinli.

Tengliklarning birinchisi $G(x,t;\xi,\tau)$ Grin funksiyasining (14) ko'rinishidan bevosta kelib chiqadi, ikkinchisi esa

$$G(x,t;l,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\times$$

$$\times\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-(2n-1)l)^2}{4a^2(t-\tau)}\right]-\exp\left[-\frac{(x-(2n-1)l)^2}{4a^2(t-\tau)}\right]\right\}=0$$

tenglikka asosan o'rinli bo'ladi.

$4^0$. Agar $\xi \neq x \in (0, l)$ bo'lsa, u holda

$$\lim_{\tau \to t} G(x, t; \xi, \tau) = 0$$

tenglik o'rinli bo'ladi.

Grin funksiyasida quyidagi $t - \tau = \varepsilon$

$$\left(\frac{x - \xi - 2nl}{2a}\right)^2 = h_1 > 0, \qquad \left(\frac{x + \xi - 2nl}{2a}\right)^2 = h_2 > 0$$

belgilashlarni kiritamiz. U holda

$$\lim_{t \to \tau} G(x, t; \xi, \tau) = \lim_{\varepsilon \to 0} G(x, t; \xi, t - \varepsilon) =$$

$$= \frac{1}{2a\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left\{ \lim_{\varepsilon \to 0} \frac{1/\sqrt{\varepsilon}}{e^{h_1/\varepsilon}} - \lim_{\varepsilon \to 0} \frac{1/\sqrt{\varepsilon}}{e^{h_2/\varepsilon}} \right\}$$

bo'ladi. Bundan Loptial teoremasiga ko'ra

$$\lim_{t \to \tau} G(x, t; \xi, \tau) = \lim_{\varepsilon \to 0} G(x, t; \xi, t - \varepsilon) =$$

$$= \frac{1}{2a\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left\{ \lim_{\varepsilon \to 0} \left[ \frac{1}{2h_1} \frac{\sqrt{\varepsilon}}{e^{h_1/\varepsilon}} \right] - \lim_{\varepsilon \to 0} \left[ \frac{1}{2h_2} \frac{\sqrt{\varepsilon}}{e^{h_2/\varepsilon}} \right] \right\} = 0$$

ekanligi kelib chiqadi.

Endi issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masala yechimini beruvchi (12) formula (1) tenglamani, (2) boshlang'ich va (3) chegaraviy shartlarni qanoatlantirishini ko'rsataylik. Buning uchun quyidagi teorema o'rinli:

TEOREMA. Agar $\varphi(x) \in C[0, l]$, $\mu_1(t)$, $\mu_2(t) \in C[0, T]$ va $f(x, t) \in C(\overline{Q})$ bo'lsa, u holda (12) formula bilan aniqlangan $u(x, t)$ funksiya (1)–(3) masalaning yechimi bo'ladi.

ISBOT. (12) formulani quyidagi ko'rinishda

$$u(x, t) = \sum_{i=1}^{4} u_i(x, t),$$

yozib olamiz va har bir qo'shiluvchini alohida–alohida o'rganamiz. Bu yerda

$$u_1(x,t) = \int_0^l \varphi(\xi) G(x,t;\xi,0)\,d\xi;$$

$$u_2(x,t) = a^2 \int_0^t \mu_1(\tau) \frac{\partial G(x,t;0,\tau)}{\partial \xi}\,d\tau;$$

$$u_3(x,t) = a^2 \int_0^t \mu(\tau) \frac{\partial G(x,t;l,\tau)}{\partial \xi}\,d\tau;$$

$$u_4(x,t) = \int_0^t \int_0^l G(x,t;\xi,\tau) f(\xi,\tau)\,d\xi d\tau.$$

$1^0$. Dastlab $u_1(x,t)$ funksiyani qaraymiz. Grin funksiyasining $1^0$ xossasiga asosan $\forall (x,t) \in Q \cup EF$ uchun

$$u_{1t} - a^2 u_{1xx} = \int_0^l \varphi(\xi)[G_t - a^2 G_{xx}]\,d\xi = 0$$

bo'ladi.

Grin funksiyasining (13) ko'rinishini e'tiborga olib, $u_1(x,t)$ funksiyani ushbu

$$u_1(x,t) = \int_0^l \varphi(\xi) E(x,t;\xi,0)\,d\xi +$$

$$+ \int_0^l \varphi(\xi) \left\{ \sum_{n=-\infty}^{+\infty}{}' E(x,t;\xi+2nl,0) - \sum_{n=-\infty}^{+\infty} E(x,t;-\xi+2nl,0) \right\} d\xi$$

ko'rinishda yozib olamiz. Bundan fundamental yechimning $3^0$ va $4^0$ xossalariga asosan $\forall x \in (0,l)$ uchun

$$\lim_{t \to +0} u_1(x,t) = \varphi(x), \qquad x \in (0,l)$$

tenglik kelib chiqadi.

Agar $t > 0$ deb, $E(x,t;\xi,\tau)$ va $G(x,t;\xi,\tau)$ funksiyalarning (7) va (13) ko'rinishlarini e'tiborga olib, $u_1(x,t)$ funksiyani quyidagicha

$$u_1(x,t) = \int_0^l \varphi(\xi) \sum_{n=-\infty}^{+\infty} \Big[E(x,t;\xi+2nl,0) - E(x,t;-\xi+2nl,0)\Big] d\xi$$

yozib olamiz. Oxirgi tenglikdan

$$\lim_{x\to+0} u_1(x,t) = 0$$

ekanligi kelib chiqadi.

Xuddi shu kabi

$$\lim_{x\to l-0} u_1(x,t) = 0$$

bo'lishini ko'rsatish qiyin emas. Chunki $x = l$ da butun $n$ ning $-\infty$ dan $+\infty$ gacha o'zgarganida (13) ketma–ketlikning birinchi qo'shiluvchisi $1+2n$ va ikkinchi qo'shiluvchisi $-1+2n$ toq sonlar, ya'ni $\ldots,-3,-2,-1,+1,+2,+3,\ldots$ kabi o'zgaradi. Bundan esa yuqoridagi qatorning mos qo'shiluvchilari o'zaro qisqaradi.

$2^0$. Endi $u_2(x,t)$ funksiyani qaraylik. Agar $(x,t) \in Q \cup EF$ bo'lsa, $(x,t) \neq (0,\tau)$ bo'ladi. U hoda Grin funksiyasining $1^0$ va $4^0$ xossalariga ko'ra

$$u_{2t} - a^2 u_{2xx} = -\lim_{\tau\to t} \mu_1(\tau) G_\xi(x,t;0,\tau) +$$

$$+ \int_0^t \mu_1(\tau) \frac{\partial}{\partial\xi} \Big[G_t(x,t;\xi,0) - a^2 G_{xx}(x,t;\xi,0)\Big] d\tau = 0$$

tenglik o'rinli bo'ladi. Demak, $u_2(x,t)$ funksiya issiqlik tarqalish tenglamasini qanoatlantirar ekan.

Grin funksiyasining tuzilishiga asosan, ushbu

$$u_2(x,t) = \int_0^t \mu_1(\tau) \sum_{n=-\infty}^{+\infty} \frac{x-2nl}{t-\tau} E(x,t;2nl,\tau) d\tau =$$

$$= \frac{1}{4a\sqrt{\pi}} \int_0^t \mu_1(\tau) \frac{x}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a^2(t-\tau)}\right] d\tau +$$

$$+\frac{1}{2a\sqrt{\pi}} \int_0^t \mu_1(\tau) \sum_{n=-\infty}^{+\infty}{}' \frac{x-2nl}{(t-\tau)^{3/2}} \exp\left[-\frac{(x-2nl)^2}{4a^2(t-\tau)}\right] d\tau, \qquad (16)$$

tenglik o'rinli. U holda, agar $t > 0$ bo'lsa,

$$\lim_{x\to+0} u_2(x,t) = \lim_{x\to+0} \frac{1}{4a\sqrt{\pi}} \int_0^t \mu_1(\tau) \frac{x}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a^2(t-\tau)}\right] d\tau -$$

$$-\frac{l}{2a\sqrt{\pi}} \int_0^t \mu_1(\tau) \sum_{n=-\infty}^{+\infty}{}' \frac{n}{(t-\tau)^{3/2}} \exp\left[-\frac{n^2 l^2}{a^2(t-\tau)}\right] d\tau$$

bo'ladi. Bu yerda ikkinchi qo'shiluvchi nolga teng. Birinchi integralda

$$\alpha = \frac{x}{2a\sqrt{t-\tau}}, \qquad \tau = t - \frac{x^2}{4a^2\alpha^2}$$

almashtirish bajaramiz va

$$\lim_{x\to+0} u_2(x,t) = \lim_{x\to+0} \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{\pi}}}^{\infty} \mu_1\left(t - \frac{x^2}{4a^2\alpha^2}\right) e^{-\alpha^2} d\alpha$$

tenglikni olamiz. Oxirgi ifodada $t$ musbat bo'lgani uchun

$$\lim_{x\to+0} u_2(x,t) = \frac{2}{\sqrt{\pi}} \mu_1(t) \int_0^{\infty} e^{-\alpha^2} d\alpha = \mu_1(t)$$

bo'ladi.

(16) tenglikka asosan

$$\lim_{x\to l+0} u_2(x,t) = \int_0^t \mu_1(\tau) \sum_{n=-\infty}^{+\infty} \frac{l(1-2n)}{t-\tau} \exp\left[-\frac{(1-2n)^2 l^2}{4a^2(t-\tau)}\right] d\tau = 0.$$

kelib chiqadi.

Agar $x \in (0, l)$ bo'lsa, Loptial qoidasiga ko'ra $\forall \tau < t$ va $n = 0, \pm 1, \pm 2, \ldots$ uchun

$$\left| \frac{x - 2nl}{(t - \tau)^{3/2}} \exp\left[ -\frac{(x - 2nl)^2}{4a^2(t - \tau)} \right] \right| < +\infty$$

tenglsizlik o'rinli, $\tau \to t$ da esa bu ifoda nolga intiladi. U holda (16) formulada $t \to +0$ da limitga o'tamiz, natijada

$$\lim_{t \to +0} u_2(x, t) = 0$$

tenglikka ega bo'lamiz.

$3^0$. Xuddi yuqoridagi kabi

$$u_3(x, t) = -a^2 \int_0^t \mu_2(\tau) G_\xi(x, t; l, \tau) d\tau$$

funksiya issiqlik tarqalish tenglamasini qanoatlantirishini hamda bu funksiya uchun quyidagi

$$\lim_{t \to +0} u_3(x, t) = 0, \qquad \forall x \in (0, l);$$

$$\lim_{x \to +0} u_3(x, t) = 0, \qquad \forall t \in (0, T];$$

$$\lim_{x \to l-0} u_3(x, t) = \mu_2(t), \qquad \forall t \in (0, T]$$

tengliklarning o'rinli ekanligini ko'rsatish mumkin.

$4^0$. Endi $u_4(x, t)$ funksiyani qaraymiz: ixtiyoriy $(x, t) \in Q \cap EF$ bo'lsin. U holda

$$u_{4t} - a^2 u_{4xx} = \lim_{\tau \to t} \int_0^l f(\xi, t) G(x, t; \xi, t) d\xi +$$

$$+ \int_0^t d\tau \int_0^l f(\xi, \tau) \Big( G_t - a^2 G_{xx} \Big) d\xi$$

bo'ladi. Grin funksiyasining xossasiga asosan yuqoridagi ifodaning ikkinchi qo'shiluvchisi nolga teng va natijada

$$u_{4t} - a^2 u_{4xx} = \lim_{\tau \to t} \int_0^l f(\xi, t) G(x, t; \xi, t) d\xi$$

ega bo'lamiz. (14) formulani e'tiborga olib,

$$u_{4t} - a^2 u_{4xx} = \lim_{\tau \to t} \int_0^l f(\xi, t) E(x, t; \xi, t) d\xi +$$

$$+ \lim_{\tau \to t} \int_0^l \left[ \sum_{n=-\infty}^{+\infty} {}' E(x, t; \xi + 2nl, t) - E(x, t; -\xi + 2nl, t) \right] d\xi$$

ifodani olamiz. Bu ifodadan $E(x, t; \xi, \tau)$ funksiyaning $3^0$ va $4^0$ xossalariga ko'ra $\forall (x, t) \in Q \cap EF$ bo'lganda

$$u_{4t} - a^2 u_{4xx} = f(x, t)$$

kelib chiqadi.

Endi $t > 0$ bo'lsin. U holda (7) va (14) formulalarga asosan

$$\lim_{x \to +0} u_4(x, t) = \int_0^t d\tau \int_0^l f(\xi, \tau) \times$$

$$\times \sum_{n=-\infty}^{+\infty} \left[ E(0, t; \xi + 2nl, \tau) - E(0, t; -\xi + 2nl, \tau) \right] d\xi = 0$$

tenglik o'rinli bo'ladi. Xuddi shu kabi $t > 0$ da

$$\lim_{x \to l-0} u_4(x, t) = 0$$

ekanligini ko'rsatish mumkin.

Ma'lumki, $\forall x \in (0, l)$ va $\forall t \in (0, T)$ da

$$\int_0^l f(\xi, \tau) G(x, t; \xi, \tau) d\xi$$

funksiya uzluksiz ekanligidan

$$\lim_{t \to +0} u_4(x, t) = \lim_{t \to +0} \int_0^t d\tau \int_0^l f(\xi, \tau) G(x, t; \xi, \tau) d\xi = 0$$

bo'ladi.

Yuisbotlanganlarni va $\varphi(0) = \mu_1(0)$, $\varphi(l) = \mu_2(0)$ tengliklarni inobatga olsak, (12) formula bilan aniqlangan $u(x, t)$ funksiya (1) issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning barcha shartlarini qanoatlantirishi kelib chiqadi.

Shu bilan teorema isbotlandi.

IZOH. Issiqlik tarqalish tenglamasi uchun har xil sohalarda ikkinchi, uchinchi boshlang'ich–chegaraviy masalalar hamda turli noklassik masalalarning qo'yilishi va ularning yechish usullari [22] o'quv qo'llanmada batafsil bayon qilingan.

======== ◇ ========

## Nazorat uchun savollar

1. Qanday fizikaviy masalalar parabolik tipdagi tenglamalar orqali ifodalanadi?

2. Issiqlik tarqalish tenglamasida $u_t$ va $u_{xx}$ hosilalar qanday fizik ma'noga ega?

3. Parabolik tenglama uchun asosiy chegaraviy masalalar qanday qo'yiladi?

4. Birinchi chegaraviy masalaning korrektligi haqida nima deyish mumkin?

5. Parabolik tipdagi tenglama uchun Koshi masalasi qanday qo'yiladi?

6. Koshi masalasi yechimining mavjudligi qanday isbotlanadi va yechimning ko'rinishi qanday ifodalanadi?

7. Issiqlik tarqalish tenglamasi uchun Koshi masalasi yechimining yagonaligi qanday isbotlanadi?

8. Koshi masalasi uchun ekstremum prinsipi qanday qo'llaniladi?

9. Qanday usullarda Koshi masalasi yechimining mavjudligi isbotlanadi?

10. Issiklik tarkalish tenglamasi uchun Koshi masalasi yechimining mavjudligini isbotlashda Fur'e usuli qanday qo'llanildi?

11. Issiqlik tarqalish tenglamasi uchun maksimum prinsipi qanday ifodalanadi?

12. Ekstremum prinsipidan qanday xossalar kelib chiqadi?

13. Parabolik tipdagi tenglama echimining silliqligi haqida nima deyish mumkin?

14. Chegaraviy shartlarning qanday turlari bor?

15. Fundamental echim qanday fizikaviy xossaga ega?

## Mustaqil yechish uchun misol va masalalar

4.1. Yon sirti issiqlik o'tkazmaydigan birlik uzunlikdagi sterjen berilgan bo'lsin. Agar sterjenning uchlaridagi harorati nolga teng va uning boshlang'ich temperaturasi $u(x,0) = \varphi(x) = x(1-x)$ bo'lsa, u holda sterjenda issiqlikning $u(x,t)$ tarqalishini aniqlang.

4.2. Ushbu $u_t = a^2 u_{xx}$ tenglamaning $0 < x < l$, $t > 0$ sohada quyidagi boshlang'ich $u(x, 0) = \sin\frac{2\pi x}{l}$ va bir jinsli chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.3. Ushbu $u_t = a^2 u_{xx}$ tenglamaning $0 < x < l$, $t > 0$ sohada quyidagi

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l$$

boshlang'ich va

$$u(0, t) = u_x(l, t) = 0, \quad t > 0$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.4. Ushbu $u_t = a^2 u_{xx}$ tenglamaning $0 < x < l$, $t > 0$ sohada quyidagi

$$u(x, 0) = Ax, \quad 0 \leq x \leq l$$

boshlang'ich va

$$u(0, t) = u(l, t) = 0, \quad t > 0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.5 Ushbu $u_t = a^2 u_{xx} + f(x)$ tenglamaning $0 < x < l$, $t > 0$ sohada aniqlangan uzluksiz va quyidagi

$$u(x, 0) = A(l - x), \quad 0 \leq x \leq l$$

boshlang'ich hamda

$$u_x(0, t) = \alpha, \quad u(l, t) = \beta, \quad t > 0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.6 Ushbu $u_t = a^2 u_{xx} - \beta u$ tenglamaning $0 < x < l$, $t > 0$ sohada aniqlangan uzluksiz va quyidagi

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l$$

boshlang'ich hamda

$$u_x(0, t) = 0, \quad u_x(l, t) = 0, \quad t > 0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.7. Issiqlik tarqalish tenglamasi uchun qo‘yilgan quyidagi Koshi masalalarini yeching.

1. $u_t = u_{xx}, \quad -\infty < x < +\infty, \quad t > 0;$
$u(x,0) = \sin lx, \quad -\infty < x < +\infty.$

2. $4u_t = u_{xx}, \quad -\infty < x < +\infty, \quad t > 0;$
$u(x,0) = e^{2x-x^2}, \quad -\infty < x < +\infty.$

2. $4u_t = u_{xx}, \quad -\infty < x < +\infty, \quad t > 0;$
$u(x,0) = e^{2x-x^2}, \quad -\infty < x < +\infty.$

3. $u_t = u_{xx} + \sin t, \quad -\infty < x < +\infty, \quad t > 0;$
$u(x,0) = e^{-x^2}, \quad -\infty < x < +\infty.$

4. $u_t = u_{xx} + 3t^2, \quad -\infty < x < +\infty, \quad t > 0;$
$u(x,0) = \sin x, \quad -\infty < x < +\infty.$

5. $u_t = u_{xx} + u_{yy}, \quad -\infty < x, y < +\infty, \quad t > 0;$
$u(x,y,0) = \sin l_1 x \sin l_2 y, \quad -\infty < x, y < +\infty.$

6. $u_t = u_{xx} + u_{yy}, \quad -\infty < x, y < +\infty, \quad t > 0;$
$u(x,y,0) = \sin l_1 x + \cos l_2 y, \quad -\infty < x, y < +\infty.$

4.8. Ushbu $u_t = a^2 u_{xx}$ tenglamaning $x > 0$, $t > 0$ sohada aniqlangan uzluksiz va quyidagi

$$u(x,0) = \varphi(x), \qquad xg'eq0$$

boshlang‘ich hamda

$$u(0,t) = 0, \quad t > 0,$$

chegaraviy shartlarni qanoatlantiruvchi $u(x,t)$ yechimini toping.

4.9. Ushbu $u_t = a^2 u_{xx} - hu$ tenglamaning $x > 0$, $t > 0$ sohada aniqlangan uzluksiz va quyidagi

$$u(x,0) = \varphi(x), \qquad xg'eq0$$

boshlang‘ich hamda

$$u_x(0,t) = 0, \quad t > 0.$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ yechimini toping.

4.10. Faraz qilaylik, $f(x, t) \in C^2(tg'eq0)$ va $\forall tg'eq0$ da $x$ argument bo'yicha garmonik bo'lsin. Agar

$$u(x, t) \int_0^t f(x, \tau) d\tau$$

bo'lsa, u holda $u(x, t)$ funksiya ushbu

$$u_t = a^2 \Delta u + f(x, t); \qquad u|_{t=0} = 0$$

Koshi masalasining yechimi ekanligini isbotlang.

======== ◇ ========

# V BOB

# ELLIPTIK TIPDAGI TENGLAMALAR

Elliptik tipdagi tenglamalarga asosan statsionar holatdagi, ya'ni vaqt o'tishi bilan o'zgarmaydigan fizik jarayonlarni o'rganish masalalari keltiriladi. Eng sodda elliptik tipdagi tenglama sifatida

$$\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

Laplas tenglamasini qaraymiz. Bu tenglama yechimining asosiy xossalari erkli o'zgaruvchilar soni $n$ ga bog'liq emas.

Shuning uchun qo'llanmada $n = 2$ bo'lgan holni qaraymiz, zarur joylarda $n > 2$ bo'lganda asosiy ta'rif va tushunchalarning farqini tushuntirib o'tamiz.

## 22–§. Elliptik tenglamalar haqida umumiy ma'lumotlar

Faraz qilaylik, $R^n$ evklid fazosida biror $S$ sirt bilan chegaralangan sohani $D$ deb belgilaylik. Bu sohada quyidagi

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x), \qquad (1)$$

chiziqli xususiy hosilali differensial tenglamani qaraylik. Bu yerda $a_{ij}(x)$, $b_i(x)$, $c(x)$ tenglamaning koeffitsiyentlari, $f(x)$ esa uning ozod hadi deyiladi.

Agar (1) tenglamada $f(x) = 0$ bo'lsa, u holda berilgan tenglama bir jinsli, aks holda bir jinsli bo'lmagan tenglama deyiladi.

$D$ sohadan biror ixtiyoriy $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ nuqta olamiz va bu nuqtada (1) tenglamaga mos ushbu

$$Q(\lambda) = Q(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i \lambda_j. \qquad (2)$$

kvadratik forma tuzamiz. $\lambda_i, (i = 1, 2, \dots, n)$ haqiqiy o'zgaruvchilar.

1–TA'RIF. Agar (2) kvadratik formaning ishorasi $x_0 \in D$ nuqtada musbat yoki manfiy aniqlangan bo'lsa, u holda (1) tenglama shu *nuqtada elliptik tipdagi tenglama* deyiladi.

Agar $D$ sohaning har bir nuqtasida (1) tenglama elliptik bo'lsa, u holda bu tenglama *D sohada elliptik tenglama* deyiladi.

2–TA'RIF. Agar noldan farqli bo'lgan bir xil ishorali $k_1$ va $k_2$ haqiqiy sonlar mavjud bo'lib, barcha $x \in D$ nuqtalar uchun

$$k_0 \sum_{i=1}^{n} \lambda_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j \leq k_1 \sum_{i=1}^{n} \lambda_i^2 \qquad (3)$$

tengsizlik bajarilsa, u holda (1) tenglama *D sohada tekis elliptik tenglama* deyiladi.

Qaralayotgan tenglamaning tekis elliptikligi oddiy elliptiklik shartiga qaraganda umumiyroq, chunki tekis elliptik bo'lishidan qaralayotgan tenglamaning elliptik tenglama ekanligi kelib chiqadi, aksinchasi noto'g'ri.

1–MISOL. Ushbu

$$\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \qquad n \geq 2, \qquad (4)$$

Laplas tenglamasini qaraylik. Bunda $a_{ij} = 0$, agar $i \neq j$ bo'lsa va $a_{ij} = 1$, agar $i = j$ bo'lsa. U holda (4) tenglamaga mos kvadratik forma

$$Q(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i^2, \qquad (5)$$

bo'ladi. Demak, (4) tenglama butun $R^n$ fazoda elliptik tipga tegishli, chunki (5) kvadratik forma $\lambda \in R^n \backslash \{\lambda = 0\}$ bo'lganda musbat aniqlangan. Laplas tenglamasining $R^n$ fazoda tekis elliptik tipga tegishli bo'lishi (3) tengsizlikdan kelib chiqadi.

Bunda $k_0 = 1$, $k_1 = 1$ yoki $k_0 = \frac{1}{2}$, $k_1 = 1$ deb tanlab olish kifoya.

2–MISOL. $xOy$ tekislikdagi biror $D \subset R^2_{xy}$ sohada quyidagi ikkinchi tartibli xususiy hosilali

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u, u_x, u_y) \qquad (6)$$

differensial tenglamani qaraylik.

Bu tenglamaga mos kvadratik forma

$$Q(\lambda_1, \lambda_2) = a\lambda_1^2 + 2b\lambda_1\lambda_2 + c\lambda_2^2. \tag{7}$$

ko‘rinishda bo‘ladi.

Agar $\delta = b^2 - ac < 0$, bo‘lsa, u holda (7) kvadratik formaning ishorasi aniqlangan bo‘ladi va $D$ sohaning $\delta < 0$ bo‘lgan barcha nuqtalarida (6) tenglama *elliptik tipdagi tenglama* deyiladi.

3–MISOL. Quyidagi

$$yu_{xx} + u_{yy} = 0$$

tenglama *Trikomi tenglamasi* deyiladi. Bu tenglama $y > 0$ da elliptik tipga tegishli bo‘ladi, chunki $\delta = b^2 - ac = -y$. Lekin qaralayotgan $D$ sohada Trikomi tenglamasi tekis elliptik tenglama bo‘lmaydi.

## 23–§. Garmonik funksiyalar

$R^n$ evklid fazosida biror $S$ sirt bilan chegaralangan sohani $D$ deb belgilaylik. Bu sohada quyidagi

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0 \tag{1}$$

Laplas tenglamasini qaraylik.

Demak bundan keyin $n$ o‘lchovli fazoning $x$ nuqtasi deganda koordinatalari $x_1, x_2, \ldots, x_n$ bo‘lgan $x = (x_1, x_2, \ldots, x_n)$ nuqtani tushunamiz.

1–TA’RIF. Agar $u(x) = u(x_1, x_2, \ldots, x_n)$ chekli $D$ sohada barcha argumentlari bo‘yicha ikki marta uzluksiz hosilaga ega bo‘lgan va (1) Laplas tenglamasini qanoatlantirsa, u holda bu funksiya $D$ sohada garmonik funksiya deyiladi, ya’ni

$$u(x) \in C^2(D) \qquad \text{va} \qquad \Delta u(x) = 0, \quad \forall\, x \in D.$$

2-TA'RIF. Agar $u(x)$ funksiya cheksiz $D$ sohaning koordinat boshidan chekli masofada yotgan har bir $x$ nuqtasida garmonik bo'lib, yetarlicha katta $|x|$ nuqtalar uchun

$$|u(x)| \leq \frac{C}{|x|^{n-2}}, \tag{2}$$

tengsizlik o'rinli bo'lsa, u holda $u(x)$ funksiya cheksiz $D$ sohada garmonik deyiladi. Bu yerda $n$ fazoning o'lchovi, $C$ – biror o'zgarmas son, $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$.

Ikki o'lchovli cheksiz sohada garmonik bo'lgan funksiya (2) tengsizlikka asosan cheksizlikda chegaralangan bo'lishi kerak.

1–MISOL. Ushbu

$$u(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + b$$

chiziqli funksiya ixtiyoriy $D \subset R^n$ sohada garmonik funksiya bo'ladi, chunki

$$\frac{\partial u}{\partial x_i} = a_i \qquad \text{va} \qquad \frac{\partial^2 u}{\partial x_i^2} = 0.$$

2–MISOL. Ikki o'lchovli $(x, y)$ tekislikda ushbu

$$u(x, y) = Re\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2}, \qquad z = x + iy,$$

funksiya $(0, 0)$ nuqtani o'z ichiga olmagan ixtiyoriy sohada garmonik bo'ladi.

Haqiqatdan ham $u(x, y)$ funksiya yordamida

$$v(x, y) = Im\left(\frac{1}{z}\right) = -\frac{y}{x^2 + y^2},$$

funksiyani qurish mumkin. Koshi–Riman shartlariga ko'ra $u(x, y)$ funksiyaning garmonik ekanligini ko'rsatish qiyin emas.

3–MISOL. Ikki argumentli $u(x, y) = x^2 + y^2$ funksiya garmonik bo'lmaydi. Chunki bu funksiya Laplas tenglamasini qanoatlantirmaydi, ya'ni tekislikning ixtiyoriy nuqtasida

$$\Delta u(x, y) = \Delta(x^2 + y^2) = 4 \neq 0$$

bo'ladi.

4–MISOL. Ikki argumentli $u(x,y) = x^2 - y^2$ funksiya ixtiyoriy chekli sohada garmonik bo'ladi.

Chunki bu funksiya Laplas tenglamasini qanoatlantiradi, ya'ni tekislikning ixtiyoriy nuqtasida ushbu

$$\Delta u(x,y) = \Delta(x^2 - y^2) = 2 - 2 = 0.$$

tenglik o'rinli bo'ladi.

Agar biror $D$ sohada $u_1(x), u_2(x), \dots, u_n(x)$ funksiyalar garmonik bo'lsa, u holda Laplas tenglamasining bir jinsli va chiziqli ekanligidan bu funksiyalarning

$$\alpha_1 u_1(x) + \alpha_2 u_2(x) + \dots + \alpha_n u_n(x), \qquad \alpha_i = const$$

chiziqli kombinatsiyasi ham garmonik funksiya bo'lishi kelib chiqadi.

Agar $u(x)$ funksiya biror $D$ sohada garmonik bo'lsa, u holda

$$u(ax+b) = u(ax_1 + b_1, ax_2 + b_2, \dots, ax_n + b_n)$$

funksiya ham $D$ sohada garmonik funksiya bo'ladi.

Bunda $a$, $b_i$, $(i = \overline{1,n})$ haqiqiy o'zgarmaslar.

Haqiqatdan ham,

$$y = ax + b = (ax_1 + b_1, ax_2 + b_2, \dots, ax_n + b_n)$$

almashtirish kiritamiz va yangi o'zgaruvchilar bo'yicha $u(x)$ funksiyaning xususiy hosilalarini topib, (1) tenglamaga qo'yamiz. Natijada

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(ax+b) = \sum_{i=1}^{n} \frac{\partial^2 u(y)}{\partial y_i^2} \left(\frac{\partial y_i}{\partial x_i}\right) = a^2 \sum_{i=1}^{n} \frac{\partial^2 u(y)}{\partial y_i^2} = 0,$$

hosil bo'ladi.

## 24–§. Laplas tenglamasining fundamental yechimi

Faraz qilaylik, $(x, y)$ va $(\xi, \eta)$ ikki o'lchovli $R^2$ tekislikdan olingan fiksirlangan nuqtalar va ular orasidagi masofa $r^2 = (x-\xi)^2 + (y-\eta)^2$ bo'lsin.

Ushbu

$$q(x, y; \xi, \eta) = \ln\frac{1}{r}, \quad r^2 = (x-\xi)^2 + (y-\eta)^2,$$

funksiya $R^2$ tekislikda $(\xi, \eta)$ nuqtani o'z ichiga olmagan ixtiyoriy sohada garmonik funksiya bo'lishini ko'rsataylik.

Shuni ta'kildash muximki, $(\xi, \eta)$ nuqtani o'z ichiga olmagan ixtiyoriy sohada $q(x, y; \xi, \eta)$ funksiya va uning ixtiyoriy tartibdagi hosilasi uzluksiz bo'ladi.

Berilgan funksiyaning hosilasini hisoblash uchun uni quyidagi

$$q(x, y) = \ln\frac{1}{r} = -\frac{1}{2}\ln r^2,$$

ko'rinishda yozib olamiz.

Bu funksiyaning xususiy hosilalarini olaylik

$$q_x = -\frac{1}{2}\frac{1}{r^2}(r^2)_x = -\frac{x-\xi}{r^2};$$

$$q_{xx} = (q_x)_x = -\frac{1}{r^2} + \frac{2(x-\xi)^2}{r^4}.$$

Xuddi shu kabi berilgan funksiyaning $y$ bo'yicha ikkinchi tartibli hosilasini olamiz

$$q_{yy} = -\frac{1}{r^2} + \frac{2(y-\eta)^2}{r^4}.$$

Endi olingan $u_{xx}$ va $u_{yy}$ hosilalarni Laplas tenglamasiga qo'yamiz. Natijada barcha $(x, y) \neq (\xi, \eta)$ nuqtalarda

$$q_{xx} + q_{yy} = -\frac{2}{r^2} + \frac{2(x-\xi)^2 + 2(y-\eta)^2}{r^4} = -\frac{2}{r^2} + \frac{2}{r^2} = 0$$

ayniyatni olamiz.

Demak, $q(x,y;\xi,\eta) = \ln\frac{1}{r}$ funksiya $R^2$ tekislikning barcha $(x,y) \neq (\xi,\eta)$ nuqtalarida garmonik funksiya bo'lar ekan.

Endi $R^n$ fazoda Laplas tenglamasining fundamental yechimini keltiramiz.

Faraz qilaylik, $R^n$, $ng'eq3$ fazoda ikkita $x = (x_1, x_2, \ldots, x_n)$ va $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ nuqtalari bo'lsin. Ular orasidagi masofani

$$r = |x - \xi| = \sqrt{\sum_{k=1}^{n}(x_k - \xi_k)^2}$$

deb belgilaylik. Ushbu

$$v(x,\xi) = \frac{1}{r^{n-2}}, \qquad n \geq 3 \tag{3}$$

funksiya har qanday $x \neq \xi$ nuqtada garmonik funksiya ekanligini ko'rsatish qiyin emas.

Bu funksiya $\xi$ nuqtani o'z ichiga olmagan har qanday sohada $x$ bo'yicha Laplas tenglamasini qanoatlantiradi. Haqiqatan ham,

$$\frac{\partial v}{\partial x_k} = -\frac{n-2}{r^{n-1}}\frac{\partial r}{\partial x_k} = -\frac{(n-2)(x_k - \xi_k)}{r^n},$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{n-2}{r^n} + \frac{n(n-2)(x_k - \xi_k)^2}{r^{n+2}} =$$

$$= \frac{n-2}{r^n}\left(\frac{n(x_k - \xi_k)^2}{r^2} - 1\right).$$

Bundan

$$\Delta v = \sum_{k=1}^{n}\frac{\partial^2 v}{\partial x_k^2} = \frac{n-2}{r^n}\left[\frac{n}{r^2}\sum_{k=1}^{n}(x_k^2 - \xi_k)^2 - n\right] = 0$$

ekani kelib chiqadi.

Endi (3) funksiya (2) shartni ham qanoatlantirishini ko'rsataylik. $r = |x - \xi| g'e |x| - |\xi|$ bo'lishi ravshan. Yetarlicha katta $|x|$ lar uchun $|x| > 2|\xi|$ deb olishimiz mumkin. U holda $|\xi| < \frac{1}{2}|x|$ bo'ladi, bundan esa $r > \frac{1}{2}|x|$ kelib chiqadi. Natijada (3) formulaga asosan

$$v(x,\xi) < \frac{2^{n-2}}{|x|^{n-2}}$$

tengsizlik hosil bo'ladi. Bundan esa $v(x,\xi)$ funksiyaning cheksiz sohada ham garmonik bo'lishi kelib chiqadi.

Yuqorida ko'rilgan $q(x,y;\xi,\eta)$ va $v(x,\xi)$ funksiyalar *Laplas tenglamasining fundamental yechimi* deyiladi.

Bu funksiyalarning muhim tomoni shundaki, agar $r \to \infty$ bo'lsa, bu funksiyalar cheksizlikka intiladi.

Laplas tenglamasining fundamental yechimi $n = 2$ bo'lgan holda *logarifmik maxsuslikka ega*, $n > 2$ bo'lganda esa *darajali maxsuslikka ega* deyiladi.

Ikki o'lchovli $R^2$ tekislikda konform akslantirishlar Laplas tenglamasini o'zini o'ziga akslantiradi.

Faraz qilaylik, $z = z(\zeta) = x(\xi,\eta) + iy(\xi,\eta)$ golomorf funksiya $\zeta$ tekisligidagi $D$ sohani $z$ tekisligidagi $\Omega$ sohaga konform akslantirsin. $\Omega$ sohada $u(x,y)$ funksiya garmonik bo'lsin. U holda $\widetilde{u}(\xi,\eta) = u(x(\xi,\eta),y(\xi,\eta))$ funksiya $D$ sohada garmonik funksiya bo'ladi. Buni isbotlash uchun

$$\Delta_\zeta \widetilde{u} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$$

hosilalarni hisoblaymiz. Buning uchun

$$\widetilde{u}_\xi = u_x x_\xi + u_y y_\xi; \quad \widetilde{u}_\eta = u_x x_\eta + u_y y_\eta;$$

va

$$\widetilde{u}_{\xi\xi} = u_{xx} x_\xi^2 + 2u_{xy} x_\xi y_\xi + u_{yy} y_\xi^2 + u_x x_{\xi\xi} + u_y y_{\xi\xi};$$

$$\widetilde{u}_{\eta\eta} = u_{xx} x_\eta^2 + 2u_{xy} x_\eta y_\eta + u_{yy} y_\eta^2 + u_x x_{\eta\eta} + u_y y_\eta.$$

Oxirgi ifodalarni qo'shamiz. Bunda $x(\xi,\eta)$ va $y(\xi,\eta)$ funksiyalar $z(\zeta)$ analitik funksiyaning haqiqiy va mavhum qismlari bo'lgani uchun $x$ $y$

funksiyalar $\xi$ va $\eta$ bo'yicha garmonik va ular Koshi–Riman tenglamasi bilan bog'langan. Quyidagi munosabatlar o'rinli, ya'ni

$$\frac{\partial x}{\partial \xi}=\frac{\partial y}{\partial \eta},\quad \frac{\partial x}{\partial \eta}=-\frac{\partial y}{\partial \xi},\quad \Delta_\zeta x=\Delta_\zeta=0.$$

Bundan

$$\Delta_\zeta \tilde{u}=\left[\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2\right]\Delta_z u=|z'(\zeta)|^2\Delta_z u$$

bo'ladi. Bu erda

$$\Delta_z u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Oxirgi tenglikdan ko'rinadiki, agar $\Delta_\zeta u=0$ bo'lsa, undan $\Delta_z u=0$ bo'lishi kelib chiqar ekan. Demak, konform akslantirishlar $n=2$ da garmonik funksiyani yana garmonik funksiyaga o'tkazar ekan.

Ixtiyoriy $n>2$ bo'lganda garmonik funksiyani yana garmonik funksiyaga akslantiruvchi akslantirish mavjud, bunday almashtirish *Kelvin almashtirishi* deyiladi. Buni biz keyingi paragrafda o'rganamiz.

## 25–§. Kelvin teoremasi

Faraz qilaylik, $u(x)$ funksiya $R^n$ fazoda garmonik bo'lsin. $R^n$ fazoni biror tekislikka nisbatan simmetrik akslantirsak, u holda $u(x)$ funksiya garmoniklik xossasini saqlaydi.

Markazi $x_0=0$ nuqtada bo'lgan $R$ radiusli sfera $S_R$ bo'lsin. Agar $x$ va $\xi$ nuqtalar sfera markazidan o'tuvchi $xx_0$ nurda yotib, bu nuqtalar uchun quyidagi

$$\xi=x\,\frac{R^2}{r^2},\qquad r^2=x_1^2+x_2^2+\ldots+x_n^2=|x|^2 \tag{1}$$

yoki koordinatalari bilan

$$\xi_i=x_i\,\frac{R^2}{r^2},\qquad i=\overline{1,n}$$

munosabatlar o'rinli bo'lsa, u holda (1) tenglik $S_R$ *sferaga nisbatan inversiya* deb ataladi.

Bu holda $x$ va $\xi$ nuqtalar *garmonik qo'shma* yoki *simmetrik nuqtalar* deyiladi. (1) tenglikka asosan

$$|x| \cdot |\xi| = R^2 \tag{2}$$

tenglik o'rinli bo'ladi.

Umuman olganda $ng'eq2$ bo'lganda inversiya almashtirishi natijasida funksiyaning garmoniklik xossasi saqlanib qolmaydi. Misol tariqasida $n = 3$ bo'lgan holda ushbu

$$u(x_1.x_2,x_3) = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \tag{3}$$

garmonik funksiyani qaraylik. Bu funksiya $|x| \neq 0$ nuqtada garmonik bo'ladi. Birlik sferaga nisbatan

$$x_i = \frac{\xi}{\rho^2}, \qquad \rho = |\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2}$$

inversiya almashtirishi natijasida (3) funksiya

$$v(\xi_1,\xi_2,\xi_3) = u\left(\frac{\xi_1}{\rho^2}, \frac{\xi_2}{\rho^2}, \frac{\xi_3}{\rho^2}\right) = \rho$$

ko'rinishga keladi. Bu esa garmonik funksiya bo'lmaydi.

Inversiya almashtirishining bu kamchiligini shu almashtirishdan so'ng funksiyani $\rho^{2-n}$ ga ko'paytirish natijasida bartaraf qilish mumkin.

Buning uchun quyidagi teorema o'rinli.

KELVIN TEOREMASI. Agar $u(x)$ funksiya $D$ sohada garmonik bo'lsa, u holda

$$v(\xi) = \frac{1}{\rho^{n-2}} u\left(\frac{\xi}{\rho^2}\right) \tag{4}$$

formula bilan aniqlangan $v(\xi)$ funksiya $D'$ sohada garmonik bo'ladi.

Bu yerda $D'$ soha $D$ sohaga qo'shma bo'lib, u birlik sferaga nisbatan inversiya natijasida hosil bo'ladi.

Isbot. Faraz qilaylik, $D$ sohaning ixtiyoriy nuqtasi $x = (x_1, x_2, \ldots, x_n)$ va $D'$ sohaning ixtiyoriy nuqtasi esa $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_i = \dfrac{x_i}{r^2}$, $i = \overline{1,n}$; $x$ nuqtadan koordinata boshigacha bo'lgan masofa $r$, $\xi$ nuqtadan koordinata boshigacha bo'lgan masofa esa $\rho$ bo'lsin.

Bu nuqtalar uchun (2) formulaga asosan $r \cdot \rho = |x| \cdot |\xi| = 1$ tenglik o'rinli bo'ladi.

Quyidagi

$$\Delta v(\xi) = r^{n+2} \Delta u(x), \tag{5}$$

tenglikning to'g'ri ekanligini isbotlaymiz.

Avval ushbu hosilani hisoblaylik:

$$\frac{\partial x_k}{\partial \xi_i} = \begin{cases} \rho^{-2} - \rho^{-4} \cdot 2\xi_i^2, & k \neq i; \\ -2\rho^{-4}\,\xi_k \cdot \xi_i, & k \neq i. \end{cases} \tag{6}$$

U holda (3) formuladan

$$\frac{\partial v(\xi)}{\partial \xi_i} = (2-n)\rho^{1-n} \frac{\partial \rho}{\partial \xi_i} u(x) + \rho^{2-n} \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \frac{\partial x_k}{\partial \xi_i} =$$

$$= (2-n)\rho^{-n}\xi_i u(x) - 2\rho^{-n-2}\xi_i \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \xi_k +$$

$$+\rho^{-n} \frac{\partial u(x)}{\partial x_i} = (2-n)J_1 - 2J_2 + J_3. \tag{7}$$

Endi (6) formula asosida (7) ifodaning o'ng tomonidagi funksiyalarning $\xi_i$ bo'yicha hosilalarini topamiz:

$$\frac{\partial J_1}{\partial \xi_i} = -n\rho^{-n-2}\xi_i^2 u(x) + \rho^{-n} u(x) -$$

$$-2\rho^{-n-4}\xi_i^2 \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \xi_k + \rho^{-n-2}\xi_i^2 \frac{\partial u(x)}{\partial x_i};$$

$$\frac{\partial J_2}{\partial \xi_i} = -(n+2)\rho^{-n-4}\xi_i^2 \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \xi_k + \rho^{-n-2} \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \xi_k +$$

$$+\rho^{-n-2}\xi_i^2\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{\partial}{\partial x_m}\left(\frac{\partial u}{\partial x_k}\right)\frac{\partial x_m}{\partial \xi_i}\xi_k+\rho^{-n-2}\xi_i\frac{\partial u}{\partial x_i}=$$

$$=-(n+2)\rho^{-n-4}\xi_i^2\sum_{k=1}^{n}\frac{\partial u(x)}{\partial x_k}\xi_k+\rho^{-n-2}\sum_{k=1}^{n}\frac{\partial u(x)}{\partial x_k}\xi_k-$$

$$-2\rho^{-n-6}\xi_i^2\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{\partial}{\partial x_m}\left(\frac{\partial u}{\partial x_k}\right)\frac{\partial x_m}{\partial \xi_i}\xi_m\xi_k+$$

$$+\rho^{-n-4}\xi_i\sum_{k=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_k}\right)\xi_k+\rho^{-n-2}\xi_i\frac{\partial u}{\partial x_i};$$

$$\frac{\partial J_3}{\partial \xi_i}=-n\rho^{-n-2}\xi_i\frac{\partial u}{\partial x_i}+\rho^{-n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial x_k}{\partial \xi_i}=$$

$$=-n\rho^{-n-2}\xi_i\frac{\partial u}{\partial x_i}-2\rho^{-n-4}\xi_i\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)\xi_k+\rho^{-n-2}\frac{\partial^2 u}{\partial x_i^2}.$$

U holda topilgan hosilalarga ko'ra (5) formulaning o'rinli ekanligi kelib chiqadi, ya'ni

$$\Delta v(\xi)=(2-n)\sum_{k=1}^{n}\frac{\partial J_1}{\partial \xi_i}-2\sum_{k=1}^{n}\frac{\partial J_2}{\partial \xi_i}+\sum_{k=1}^{n}\frac{\partial J_3}{\partial \xi_i}=$$

$$=-(2-n)n\rho^{-n}u(x)+(2-n)\rho^{-n}u(x)-$$

$$-(2-n)2\rho^{-n-2}\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\xi_k+(2-n)\rho^{-n-2}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\xi_i+$$

$$+2(n+2)\rho^{-n-2}\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\xi_k-2n\rho^{-n-2}\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\xi_k+$$

$$+4\rho^{-n-4}\sum_{k,m=1}^{n}\frac{\partial^2 u}{\partial x_m\partial x_k}\xi_m\xi_k-2\rho^{-n-4}\sum_{k,i=1}^{n}\frac{\partial^2 u}{\partial x_i\partial x_k}\xi_i\xi_k-$$

$$-2\rho^{-n-2}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\xi_i-n\rho^{-n-2}\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\xi_i-$$

$$-2\rho^{-n-4}\sum_{k,i=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_k}\xi_i\xi_k+\rho^{-n-2}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=\rho^{n+2}\Delta u(x).$$

Shunday qilib, (5) tenglikning to'g'ri ekanligi isbotlandi.

Demak, (4) funksiya $\rho \neq 0$ bo'lganda $D'$ sohada garmonik funksiya bo'lar ekan.

IZOH. (4) funksiyaning ixtiyoriy $R$ radiusli $S_R$ sferaga nisbatan inversiyasi

$$v(\xi)=\left(\frac{R}{\rho}\right)^{n-2}u\left(\frac{R^2}{\rho^2}\xi\right)$$

ko'rinishda bo'ladi.

Kelvin teoremasidan foydalanib, odatda garmonik funksiyaning cheksiz uzoqlashgan nuqta atrofidagi ta'rifi beriladi.

TA'RIF. Agar

$$v(\xi)=\frac{1}{\rho^{n-2}}u\left(\frac{\xi}{\rho^2}\right)=|x|^{n-2}u(x)$$

funksiya $\rho=0$ nuqtada $\lim\limits_{\rho\to 0}v(\xi)$ sifatida qo'shimcha aniqlangan bo'lib, $\rho=0$ nuqta atrofida garmonik bo'lsa, u holda $u(x)$ funksiya cheksiz uzoqlashgan nuqta atrofida garmonik deyiladi.

Bu ta'rifga ko'ra $v(\xi)$ funksiyaning $\rho=0$ nuqta atrofida chegaralangan ekanligi kelib chiqadi.

Ushbu lemmani isbotsiz keltiramiz:

LEMMA. Agar $u(x)$ funksiya cheksiz uzoqlashgan nuqta atrofida garmonik bo'lsa, u holda ushbu

$$|u(x)|\leq\frac{A}{|x|^{n-2}},\quad \left|\frac{\partial^i u(x)}{\partial x^i}\right|\leq\frac{A}{|x|^{n-2+i}},\quad n\geq 3; \tag{8}$$

$$|u(x)\leq A,\quad \left|\frac{\partial u}{\partial x_1}\right|,\quad \left|\frac{\partial u}{\partial x_2}\right|\leq\frac{A}{|x|^2},\quad n=2, \tag{9}$$

tengsizliklar o'rinli bo'ladi. Bu yerda $A$ musbat o'zgarmas son. Yuqoridagi (8) va (9) tengsizliklar garmonik funksiyaning cheksiz uzoqlashgan nuqtada *regulyarlik sharti* ham deb yuritiladi.

## 26–§. Garmonik funksiyalar uchun ekstremum prinsipi

Bu paragrafda garmonik funksiyalarning muhim xossalaridan biri ekstremum prinsipini o'rganamiz. Bu yerda talabalarni ekstremum prinsipidan kelib chiqadigan ayrim natijalar bilan tanishtiramiz. Keyinchalik ekstremum prinsipidan foydalanib, Laplas tenglamasi uchun Dirixle masalasi yechimining yagonaligi va turg'unligini isbotlaymiz.

Faraz qilaylik, $R^n$ Evklid fazosida silliq $S$ sirt bilan chegaralangan soha $D$ bo'lsin, ya'ni $D \subset R^n$, $\overline{D} = D \cup S$.

1–TEOREMA. Agar $u(x)$ funksiya chekli $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz va o'zgarmasdan farqli bo'lsa, u holda bu funksiya $D$ sohaning ichki nuqtalarida o'zining eng katta va eng kichik qiymatiga erishmaydi, ya'ni bu funksiya o'zining eng katta va eng kichik qiymatiga $D$ sohaning $S$ chegarasida erishadi.

ISBOT. Faraz qilaylik, $\max\limits_{S} u(x) = m$, $\max\limits_{D} u(x) = M$ bo'lsin va $u(x)$ funksiya $D$ sohaning ichki nuqtalarida $m$ dan katta qiymatga erishsin. U holda $D$ sohada shunday $x_0$ nuqta topiladiki, bu nuqtada $u(x_0) = M$, $M > m$ bo'ladi.

Quyidagi ko'rinishda

$$v(x) = u(x) + \frac{M - m}{2d^2} r^2 \tag{1}$$

yordamchi funksiya kiritamiz. Bu erda $r = \rho(x, x_0)$ ikkita $x$ va $x_0$ nuqtalar orasidagi masofa,

$$r^2 = \rho^2(x, x_0) = (x_1 - x_1^0)^2 + \ldots + (x_i - x_i^0)^2 + \ldots + (x_n - x_n^0)^2;$$

$$x = (x_1, x_2, \ldots, x_i, \ldots, x_n), \qquad x_0 = (x_1^0, x_2^0, \ldots, x_i^0, \ldots, x_n^0);$$

$d$ esa $D$ sohaning diametri, ya'ni $\overline{D}$ soha nuqtalari orasidagi maksimal masofa. Agar $D$ soha chekli bo'lsa, u holda $d$ chekli musbat son bo'lib, ixtiyoriy $x \in D$ uchun $r < d$ bo'ladi.

Shuni ta'kidlashimiz mumkinki,

1) $v(x_0) = u(x_0) = M$, bunda $r = 0$, ya'ni $x = x_0$;

2) $v(x)|_S = u(x)|_S + \left.\frac{M - m}{2d^2} r^2\right|_S \le m + \frac{M - m}{2} = \frac{M + m}{2} < M$

Shunday qilib, $v(x)$ funksiya ham $D$ sohada xuddi $u(x)$ funksiya kabi xossalarga ega, ya'ni $v(x)$ funksiya $D$ sohaning ichki $x_0$ nuqtasida $M$ qiymatni qabul qiladi va sohaning $S$ chegarasida $M$ dan kichik qiymatga ega. Demak, $v(x)$ funksiya maksimum qiymatga $D$ sohaning ichki $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ nuqtalarida erishishi mumkin. U holda bu nuqtada ekstremumning zaruriy shartiga ko'ra ushbu

$$\left.\frac{\partial v}{\partial x_i}\right|_{x=\xi} = 0 \quad \text{va} \quad \left.\frac{\partial^2 v}{\partial x_i^2}\right|_{x=\xi} \leq 0, \quad i = \overline{1, n}$$

ifodalarga ega bo'lamiz. Bundan $v(x)$ funksiya uchun Laplas operatorining $x = \xi$ nuqtadagi qiymati

$$\Delta v(x) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} \leq 0, \tag{2}$$

bo'ladi.

Ikkinchi tomondan (1) tenglikka asosan

$$\frac{\partial(r^2)}{\partial x_i} = 2(x_i - x_i^0), \quad \frac{\partial^2(r^2)}{\partial x_i^2} = 2, \quad \Delta r^2 = \sum_{i=1}^{n} \frac{\partial^2(r^2)}{\partial x_i^2} = 2n$$

ekanligini inobatga olsak, $x = \xi$ nuqtada

$$\Delta v(x) = \Delta\left(u(x) + \frac{M - m}{2d^2} r^2\right) =$$

$$= \Delta u(x) + \frac{M - m}{2d^2} \Delta r^2 = n \frac{M - m}{d^2} > 0$$

tengsizlikni olamiz. Bu esa (2) tengsizlikka zid.

Bu ziddiyat garmonik funksiya o'zining maksimum qiymatiga sohaning ichki nuqtalarida erishishini isbotlaydi. Xuddi shunga o'xshash minimum bo'lgan hol isbotlanadi.

1–LEMMA. Agar $u(x)$ funksiya $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz va eng katta (eng kichik) qiymatiga $D$ sohaning ichki nuqtalarida erishsa, u holda bu funksiya o'zgarmasdir.

Isbot. Faraz qilaylik, ixtiyoriy $x \in \overline{D}$ bo'lganda $u(x) \neq const$ bo'lsin. U holda 1–teoremaga asosan $u(x)$ funksiya o'zining eng katta qiymatiga $D$ sohaning $S$ chegarasida erishadi. Bu esa $u(x) \equiv const$ shartga zid.

Yuqorida isbotlangan 1–teorema va 1–natijaga asosan quyidagi kuchsiz ekstremum prinsipi o'rinli.

2–lemma. Chekli $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz bo'lgan $u(x)$ funksiya o'zining eng katta va eng kichik qiymatiga $D$ sohaning $S$ chegarasida erishadi.

Haqiqatdan ham, agar yopiq $\overline{D}$ sohada $u(x) \neq const$ bo'lsa, u holda 1–teoremaga ko'ra bu funksiya eng katta va eng kichik qiymatiga faqat $D$ sohaning $S$ chegarasida erishadi. Agar yopiq $\overline{D}$ sohada $u(x) \equiv const$ bo'lsa, u holda $u(x)$ funksiya eng katta (eng kichik) qiymatga yopiq $\overline{D}$ sohaning ixtiyoriy nuqtasida, xususan $S$ chegarada ham erishadi.

3–lemma. Agar $u(x)$ funksiya $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz va $D$ sohaning $S$ chegarasida $u(x) \geq 0$ bo'lsa, u holda yopiq $\overline{D}$ sohada $u(x) \geq 0$ bo'ladi.

Isbot. Faraz qilaylik, $D$ sohada shunday $x' \in D$ nuqta mavjudki, bu nuqtada $u(x') < 0$ bo'lsin. U holda $u(x)$ funksiya $\overline{D}$ sohada etarlicha kichik manfiy qiymatga sohaning $S$ chegarasida erishadi. Bu esa $D$ sohaning $S$ chegarasida $u(x) \geq 0$ bo'lsin degan shartga zid.

4–lemma. (Taqqoslash xossasi) Agar $u(x)$ va $v(x)$ funksiyalar $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz va $S$ chegarada $u(x) \geq v(x)$ bo'lsa, u holda ixtiyoriy $x \in D$ uchun $u(x) \geq v(x)$ bo'ladi.

Isbot. Bu natijani isbotlash uchun quyidagi $w(x) = u(x) - v(x)$ ayirmani qaraymiz. Bu ayirma 3–natijaning shartlarini qanoatlantiradi. U holda $x \in D$ bo'lganda $w(x) \geq 0$ yoki $u(x) \geq v(x)$ ekanligi kelib chiqadi.

5–lemma. Agar $u(x)$ funksiya $D$ sohada garmonik va yopiq $\overline{D}$ sohada uzluksiz bo'lsa, u holda ixtiyoriy $x \in D$ uchun quyidagi a) $\min\limits_S u(x) \leq u(x) \leq \max\limits_S u(x)$;

b) $|u(x)| \leq \max\limits_S |u(x)|$. tengsizliklar o'rinli bo'ladi.

6–LEMMA. Agar $D$ soxada garmonik va yopiq $\overline{D}$ sohada uzluksiz funksiyalar ketma–ketligi sohaning $S$ chegarasida tekis yaqinlashuvchi bo'lsa, u holda bu funksiyalar ketma–ketligi yopiq $\overline{D}$ sohada tekis yaqinlashuvchi bo'ladi.

ISBOT. Faraz qilaylik, $u_n(x) \in C(\overline{D})$ garmonik funksiyalar ketma–ketligi bo'lib, bular uchun $u_n(x)|_S = f_n(x)$ o'rinli bo'lsin. Lemma shartiga ko'ra uzluksiz $f_n(x)$ funksiyalar ketma–ketligi $S$ sohada tekis yaqinlashuvchi, u holda Koshi alomatiga asosan ixtiyoriy $\varepsilon > 0$ bo'lganda shunday $n_0 \in N$ son mavjud bo'lsaki, barcha $x \in S$ va barcha $n, m > n_0$ lar uchun $|f_n(x) - f_m(x)| < \varepsilon$ tengsizlik bajariladi, bu yerda $n, m \in N$.

Endi garmonik funksiyalarning quyidagi $u_n(x) - u_m(x)$ ayirmasini qaraylik, bu ayirma uchun

$$[u_n(x) - u_m(x)]\Big|_S = u_n(x)\Big|_S - u_m(x)\Big|_S = f_n(x) - f_m(x)$$

tenglik o'rinli.

Yuqorida keltirilgan 5–lemmaga ko'ra ixtiyoriy $x \in \overline{D}$ va barcha $n, m > n_0$ lar uchun

$$|u_n(x) - u_m(x)| \le \max_S |u_n(x) - u_m(x)| \le \max_S |f_n(x) - f_m(x)| < \varepsilon$$

bo'ladi. Demak, $u_n(x)$ garmonik funksiyalar ketma–ketligi uchun yopiq $\overline{D}$ sohada Koshi alomati o'rinli, bundan esa $u_n(x)$ ketma–ketlikning shu sohada tekis yaqinlashuvchi ekanligi kelib chiqadi.

Puasson tenglamasi uchun qat'iy ekstremum prinsipini isbotlaymiz.

2–TEOREMA. Agar $u(x) \in C(\overline{D}) \cap C^2(D)$ va ixtiyoriy $x \in D$ uchun $\Delta u(x) = g(x)$ bo'lib, $g(x) > 0 (< 0)$ bo'lsa, u holda $u(x)$ funksiya o'zining eng katta (eng kichik) qiymatiga $D$ sohaning ichki nuqtalarida erishmaydi. Bu $u(x)$ funksiya o'zining eng katta (eng kichik) qiymatga $D$ sohaning $S$ chegarasida erishadi.

ISBOT. Faraz qilaylik, $u(x)$ funksiya $D$ sohaning ichki nuqtacida maksimumga erishsin, ya'ni $D$ sohada shunday $x_0$ nuqta mavjudki, bu nuqtada $u(x_0) = \max\limits_{\overline{D}} u(x)$ bo'lsin. U holda shu nuqtada ekstremumning zaruriy shartiga asosan

$$\Delta u(x_0) = \sum_{i=1}^{n} \frac{\partial^2 u(x_0)}{\partial x_i^2} \le 0.$$

bo'ladi. Bu esa $\Delta u(x_0) = g(x_0) > 0$ shartga zid. Bu ziddiyat esa 2–teoremani isbotlaydi.

7–LEMMA. Agar $u(x) \in C(\overline{D}) \cap C^2(D)$, ixtiyoriy $x \in D$ uchun $\Delta u(x) = g(x)$ bo'lib, $g(x) > 0\,(< 0)$ va $S$ chegarada $u(x) \le 0\,(\ge 0)$ bo'lsa, u holda $D$ sohada $u(x) \le 0\,(\ge 0)$ bo'ladi.

## 27–§. Grin formulalari

Ushbu paragrafda garmonik funksiyalar uchun Grin formulalari va undan kelib chiqadigan garmonik funksiyaning sodda xossalarini keltiramiz. Shu bilan birga $C^2$ sinfdagi va garmonik funksiyaning integral ifodasini keltirib chiqaramiz.

$xOy$ tekisligida bo'lakli silliq $S$ egri chiziq bilan chegaralangan soha $D \subset R^2$ bo'lib, unda $C^1(\overline{D}) \cap C^2(D)$ sinfga tegishli bo'lgan $u(x, y)$ va $v(x, y)$ funksiyalar berilgan bo'lsin. $D$ sohaning $S$ chegarasiga o'tkazilgan tashqi normalni $n$ deb belgilaymiz. $\overline{D} = D \cup S$.

Qaralayotgan $D$ sohada ushbu ayniyatlar o'rinli:

$$v\Delta u = (vu_x)_x + (vu_y)_y - (v_x u_x + v_y u_y);$$

$$v\Delta u = (vu_x - v_x u)_x + (vu_y - v_y u)_y + u\Delta v,$$

bu yerda $\Delta \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ – Laplas operatori. Bu ayniyatlarni $D$ sohada integrallaymiz va Gauss–Ostrogradskiy formulasini qo'llab,

$$\iint_D v\Delta u dxdy = -\iint_D (v_x u_x + v_y u_y) dxdy + \int_S v\frac{\partial u}{\partial n} ds, \tag{1}$$

$$\iint_D (v\Delta u - u\Delta v) dxdy = \int_S \left( v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds, \tag{2}$$

formulalarga ega bo'lamiz. Bu formulalar mos ravishda *birinchi va ikkinchi Grin formulalari* deyiladi.

Agar bu formulalarda $u(x,y)$ va $v(x,y)$ funksiyalar $D$ sohada garmonik bo'lsa, u holda (1) va (2) formulalar ushbu

$$\iint_D (v_x u_x + v_y u_y)dxdy = \int_S v\frac{\partial u}{\partial n}ds, \tag{3}$$

$$\int_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = 0, \tag{4}$$

ko'rinishga keladi. Oxirgi hosil bo'lgan formulalardan garmonik funksiyalarning sodda xossalari kelib chiqadi.

1) Agar $D$ sohada garmonik bo'lgan $u(x,y)$ funksiya

$$u(x,y) \in C^1(\overline{D}) \quad \text{va} \quad u(x,y)|_S = 0$$

bo'lsa, u holda barcha $(x,y) \in \overline{D}$ uchun $u(x,y) = 0$ bo'ladi.

Isbot. Haqiqatdan ham, (2) formulada $u(x,y) = v(x,y)$ bo'lsa, bundan

$$\iint_D (u_x^2 + u_y^2)dxdy = \int_S u\frac{\partial u}{\partial n}ds. \tag{5}$$

yoki

$$\iint_D (u_x^2 + u_y^2)dxdy = 0$$

tenglik kelib chiqadi. Oxirgi tenglikdan barcha $(x,y) \in \overline{D}$ uchun $u_x = 0$, $u_y = 0$ bo'ladi. Bundan esa barcha $(x,y) \in S$ da $u|_S = 0$ bo'lgani sababli va $u(x,y) \in C^1(\overline{D})$ ekanligidan $u(x,y) \equiv 0$ bo'lishi kelib chiqadi.

2) Agar $D$ sohada garmonik bo'lgan $u(x,y)$ funksiya

$$u(x,y) \in C^1(\overline{D}) \quad \text{va} \quad \left.\frac{\partial u}{\partial n}\right|_S = 0$$

bo'lsa, u holda barcha $(x,y) \in D$ uchun $u(x,y) = const$ bo'ladi.

Bu xossaning isboti barcha $(x,y) \in S$ da $u|_S = 0$ bo'lgani sababli, (5) tenglikdan kelib chiqadi.

3) Agar $u(x,y)$ funksiya $D$ sohada garmonik va $u(x,y) \in C^1(\overline{D})$ bo'lsa, u holda

$$\left.\frac{\partial u}{\partial n}\right|_S = 0$$

bo'ladi.

Haqiqatdan ham, (4) formulada $(x,y) \in \overline{D}$ uchun $v(x,y) = 1$ deb olsak,

$$\left.\frac{\partial u}{\partial n}\right|_S = 0$$

hosil bo'ladi.

Endi Grin formulalaridan foydalanib, $C^2$ sinfdagi va garmonik funksiyalarning integral ifodalarini keltirib chiqaramiz. Qaralayotgan $D$ sohaning o'zgaruvchi nuqtasini $(\xi,\eta)$ deb belgilaymiz. $u(\xi,\eta) \in C^1(\overline{D}) \cap C^2(D)$ bo'lsin. $D$ sohaning ixtiyoriy $(x,y)$ nuqtasini markaz qilib, $\varepsilon$ radiusli $\delta_\varepsilon$ doira chizamiz va uning chegarasi $S_\varepsilon$ bo'lsin. $\varepsilon$ radiusni shunday kichik qilib olamizki, $\delta_\varepsilon$ doira to'la $D$ sohada yotsin. $D_\varepsilon = D \backslash \delta_\varepsilon$ deb belgilaylik.

Ma'lumki, $D_\varepsilon$ sohada $u(x,y)$ va $v(x,y)$ funksiyalar $C^1(\overline{D_\varepsilon}) \cap C^2(D_\varepsilon)$ sinfga tegishli.

$$v(x,y;\xi,\eta) = \ln\frac{1}{r}, \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

deb, $D_\varepsilon$ sohada bu funksiyalarga ikkinchi Grin formulasini qo'llaymiz. Natijada ushbu

$$\iint\limits_{D_\varepsilon} \left(\ln\frac{1}{r}\Delta u - u\ln\frac{1}{r}\right) d\xi d\eta = \int\limits_S \left[\ln\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right)\right] ds +$$

$$+ \int\limits_{S_\varepsilon} \left[\ln\frac{1}{r}\frac{\partial u}{\partial n_\varepsilon} - u\frac{\partial}{\partial n_\varepsilon}\left(\ln\frac{1}{r}\right)\right] ds_\varepsilon, \qquad (6)$$

tenglikni olamiz.

Ma'lumki, $\ln\frac{1}{r}$ Laplas tenglamasining fundamental echimi bo'lgani uchun $\Delta\left(\ln\frac{1}{r}\right) = 0$ bo'ladi. $\delta_\varepsilon$ sohaning $S_\varepsilon$ chegarasiga o'tkazilgan tashqi $n_\varepsilon$ normal $r$ radiusga qarama-qarshi yo'nalganligi uchun

$$\frac{\partial}{\partial n_\varepsilon}\left(\ln\frac{1}{r}\right)\Bigg|_{r=\varepsilon} = -\frac{\partial}{\partial n_\varepsilon}\left(\ln\frac{1}{r}\right)\Bigg|_{r=\varepsilon} = \frac{1}{\varepsilon} \tag{7}$$

teng. $u(x,y)$ funksiyaning birinchi tartibli hosilalarining uzluksizligidan

$$\max_{S_\varepsilon}\left|\frac{\partial u}{\partial n_\varepsilon}\right| \leq C,$$

bo'ladi, bu yerda $C$ o'zgarmas $\varepsilon$ ga bog'liq emas. Shuning uchun $\varepsilon \to 0$ intilganda

$$\left|\int_{S_\varepsilon}\ln\frac{1}{r}\frac{\partial u}{\partial n_\varepsilon}ds_\varepsilon\right| = \left|\ln\frac{1}{\varepsilon}\int_{S_\varepsilon}\frac{\partial u}{\partial n_\varepsilon}ds_\varepsilon\right| \leq C\cdot 2\pi\varepsilon\cdot\ln\frac{1}{\varepsilon} \to 0$$

bo'ladi.

(7) tenglikni hisobga olib, $\varepsilon$ radiusli aylanadan birlik aylanaga o'tamiz, $x-\xi = \varepsilon\cos\varphi$, $y-\eta = \varepsilon\sin\varphi$, $ds_\varepsilon = \varepsilon\, ds_1$ natijada ushbu

$$\int_{S_\varepsilon} u\frac{\partial}{\partial n_\varepsilon}\left(\ln\frac{1}{r}\right)ds_\varepsilon = \frac{1}{\varepsilon}\int_{S_1} u(x-\varepsilon\cos\varphi, y-\varepsilon\sin\varphi)\varepsilon ds_1 = 2\pi u(x,y),$$

tenglikka ega bo'lamiz.

Endi (6) formulada $\varepsilon \to 0$ limitga o'tsak, $D_\varepsilon$ esa $D$ ga intiladi, uning o'ng tomonidagi ikkinchi integral $-2\pi u(x,y)$ ga intiladi, ya'ni

$$\lim_{\varepsilon\to 0}\int\left[\ln\frac{1}{r}\frac{\partial u}{\partial n_\varepsilon} - u\frac{\partial}{\partial n_\varepsilon}\left(\ln\frac{1}{r}\right)\right]ds_\varepsilon = -2(x,y).$$

Natijada quyidagi

$$u(x,y) = \frac{1}{2\pi}\iint_D \ln\frac{1}{r}\Delta u\, d\xi d\eta +$$

$$+\frac{1}{2\pi}\int\limits_S\left[\ln\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right)\right]ds, \tag{8}$$

$C^2$ sinfdagi funksiyalarning integral ifodasi hosil bo'ladi.

Agar $u(x,y)$ funksiya $D$ sohada garmonik bo'lsa, u holda bu funksiyaning integral ifodasi (8) formuladan

$$u(x,y)=\frac{1}{2\pi}\int\limits_S\left[\ln\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right)\right]ds, \tag{9}$$

ekanligi kelib chiqadi.

Agar $n>3$ bo'lsa, u holda $C^2$ sinfdagi funksiyalarning integral ifodasi

$$u(x)=\frac{1}{(n-2)|S_1|}\iint\limits_D\frac{1}{r^{n-2}}\,\Delta u(\xi)\,d\xi+$$

$$+\frac{1}{(n-2)|S_1|}\int\limits_S\left[\frac{1}{r^{n-2}}\frac{\partial u(\xi)}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{r^{n-2}}\right)\right]d_\xi S. \tag{10}$$

ko'rinishda bo'ladi.

Endi $n=3$ bo'lganda $|S_1|=4\pi$ ga teng bo'ladi va $C^2$ sinfdagi funksiyalarning integral ifodasi, ya'ni (10) formula

$$u(x)=\frac{1}{4\pi}\iint\limits_D\frac{1}{r}\,\Delta u(\xi)\,d\xi+$$

$$+\frac{1}{4\pi}\int\limits_S\left[\frac{1}{r}\frac{\partial u(\xi)}{\partial n}-u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]d_\xi S, \tag{11}$$

ko'rinishga keladi.

Oxirgi (10) va (11) formulalardan garmonik funksiyaning integral ifodasini qiyinchiliksiz olish mumkin.

## 28–§. Dirixle va Neyman masalalari. Yagonalik teoremalari

Bir bu paragrafda elliptik tipdagi tenglamalar uchun asosiy chegaraviy masalalar, Dirixle va Neyman masalalarni o'rganamiz. Bu masalalarni Laplas tenglamasi uchun yechimning yagonaligi va turg'unligini isbotlaymiz.

Elliptik tipdagi tenglamalar odatda ikki xil chekli va cheksiz sohalarda o'rganiladi. Har ikki holda ham qaralayotgan sohaning chegarasi chekli sondagi bo'lakli silliq chiziqlar (sfera)dan iborat, deb faraz qilinadi. Ayrim hollarda elliptik tenglamalar yarim chegaralangan sohalarda qaraladi. Bunday sohalarning chegarasi cheksiz bo'lib, unga yarim tekislik, yarim fazo misol bo'ladi.

Matematik fizika tenglamalari kursida elliptik tipdagi tenglamalar uchun ikki xil chegaraviy masalalar, ichki va tashqi masalalar o'rganiladi.

Agar noma'lum funksiyani chekli sohadan topish talab qilinsa, bunday chegaraviy masalaga *ichki masala*, agar bu funksiyani cheksiz sohadan izlansa, u holda bunday chegaraviy masala *tashqi masala* deyiladi.

Faraz qilaylik, $D \subset R^n$ bo'lakli silliq $S$ sirt bilan chegaralangan chekli soha, $f(x)$ esa $D$ sohada berilgan va uzluksiz funksiya bo'lsin.

Ushbu umumiy ko'rinishdagi

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \qquad (1)$$

elliptik tipdagi tenglamani qaraylik.

DIRIXLE MASALASI. Elliptik tipdagi (1) tenglamaning chekli $\overline{D}$ sohada aniqlangan uzluksiz va quyidagi

$$u(x) \in C(\overline{D}) \cap C^2(D);$$

$$\lim_{x \to x_0} u(x) = \varphi(x_0), \qquad x_0 \in S, \qquad (2)$$

shartlarni qanoatlantiruvchi $u(x)$ yechimini toping.

Bu yerda $\varphi(x_0)$ berilgan uzluksiz funksiya.

NEYMAN MASALASI. Elliptik tipdagi (1) tenglamaning chekli $D$ sohada aniqlangan, $D \cup S$ uzluksiz va quyidagi

$$\lim_{x \to x_0} a_{ij}(x) \frac{\partial u(x)}{\partial x_k} \cos(\nu, x_k) = \psi(x_0), \qquad x_0 \in S, \tag{3}$$

chegaraviy shartni qanoatlantiruvchi $u(x)$ yechimini toping.

Bu yerda $x_0 \in S$, $\nu$ esa $S$ sirtga o'tkazilgan normal va $\psi(x_0)$ berilgan uzluksiz funksiya.

Agar $u(x) \in C^1(D \cup S) \cap C^2(D)$ sinfga tegishli bo'lsin, deb talab qilinsa, u holda (3) chegaraviy shartni

$$a_{ij}(x) \frac{\partial u(x)}{\partial x_k} \cos(\nu, x_k) \Big|_S = \psi(x_0), \qquad x_0 \in S, \tag{3'}$$

ko'rinishda yozish mumkin.

Agar (1) tenglamada $a_{ij}(x) = \delta_{ij}$ bo'lsa, u holda tenglamaning bosh qismi Laplas operatoriga aylanadi va (1) tenglama

$$Lu \equiv -\Delta u(x) + b_i(x) u_{x_i} + c(x) u = f(x), \tag{4}$$

ko'rinishga keladi. (3') chegaraviy shart esa

$$\frac{\partial u}{\partial \nu}\Big|_S = \psi(x_0), \qquad x_0 \in S, \tag{5}$$

sodda ko'rinishni oladi.

Tashqi Dirixle va Neyman masalalarining mos ichki masalalardan farqi shundaki, noma'lum funksiyadan $|x| \to \infty$ da qo'shimcha

$$|u(x)| \leq \frac{C}{|x|^{n-2}}, \qquad C = const > 0, \tag{6}$$

shartning bajarilishi talab qilinadi.

## Yechimining yagonaligi haqidagi teoremalar

Bu yerda Laplas tenglamasi uchun Dirixle va Neyman masalalari yechimining yagonaligi haqidagi teoremalarni isbotlaymiz.

1–TEOREMA. Agar Laplas tenglamasi uchun ichki (tashqi) Dirixle masalasining yechimi $D$ sohada mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. 1) Faraz qilaylik, Laplas tenglamasi uchun ichki Dirixle masalasi $u_1(x)$ va $u_2(x)$ yechimlarga ega bo'lsin. Bu funksiyalarning $u(x) = u_1(x) - u_2(x)$ ayirmasi uchun quyidagi

1) $u(x) \in C(\overline{D}) \cap C^2(D)$;

2) $\Delta u(x) = \Delta(u_1(x) - u_2(x)) = \Delta u_1(x) - \Delta u_2(x) = 0$;

3) $u(x)\Big|_S = \Big(u_1(x) - u_2(x)\Big)\Big|_S = \varphi(x) - \varphi(x) = 0$

shartlar o'rinli.

Demak, $u(x)$ funksiya $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz va $u(x)|_S = 0$. Yuqorida isbotlangan ekstremum prinsipiga asosan bu funksiya $\min\limits_{D} u(x)$ va $\max\limits_{D} u(x)$ sohaning $S$ chegarasida erishadi. $D$ sohaning chegarasida esa $u(x) = 0$. U holda yopiq $\overline{D}$ sohada $u(x) \equiv 0$ bo'ladi. Bundan $u_1(x) \equiv u_2(x)$ ekanligi kelib chiqadi.

2) Endi 1–teoremani tashki Dirixle masalasi uchun isbot qilaylik. Buning uchun $n > 2$ bo'lgan holni qaraymiz. Tashqi Dirixle masalasining (6) shartiga asosan $u(x) = u_1(x) - u_2(x)$ funksiya cheksiz $D$ sohada garmonik bo'ladi.

$S$ sirtni markazi koordinata boshida sferasi $S_R$ bo'lgan $R$ radiusli shar bilan o'raymiz va $D_R$ halqasimon sohada funksiyani qaraylik.

Ma'lumki, $u(x)|_S = 0$, bundan tashqari koordinata boshidan etarlicha uzoqda bo'lgan nuqtalar uchun (6) shart o'rinli. Bundan shar radiusi etarlicha katta bo'lganda, ya'ni $S_R$ sferada

$$|u(x)| \leq \frac{C}{R^{n-2}}, \qquad C = const > 0,$$

tengsizlik o'rinli bo'ladi.

Ixtiyoriy $\varepsilon > 0$ uchun $R$ radiusni shunday tanlaymizki, natijada $CR^{2-n} < \varepsilon$ bo'lsin. Ekstremum prinsipiga asosan $D_R$ halqasimon sohada $u(x)$ funksiya o'zining eng katta va eng kichik qiymatlariga, yoki $S$ da, yoki $S_R$ da erishadi, lekin bu qiymatlar modul jihatdan $\varepsilon$ dan katta bo'lmaydi.

Faraz qilaylik, $x$ cheksiz $D$ sohaning ixtiyoriy nuqtasi bo'lsin. $R$ radiusni shunday tanlaymizki, natijada $x$ nuqta $D_R$ da etadi va $|u(x)| < \varepsilon$ bo'ladi. $\varepsilon$ ixtiyoriy musbat son bo'lgani uchun $u(x) = 0$ va bundan $u_1(x) \equiv u_2(x)$ ekanligi kelib chiqadi.

Endi Laplas tenglamasi uchun Dirixle masalasi yechimining turg'unligini isbotlaylik.

2–TEOREMA. Laplas tenglamasi uchun Dirixle masalasining yechimi chegaraviy funksiyaga uzluksiz bog'liq bo'ladi.

ISBOT. Faraz qilaylik, $u_1(x)$ funksiya (3)–(5) masalaning $u_1(x)|_S = \varphi_1(x)$ chegaraviy shartni qanoatlantiruvchi yechimi, $u_2(x)$ funksiya esa (3)–(5) masalaning $u_2(x)|_S = \varphi_2(x)$ shartni qanoatlantiruvchi yechimi bo'lsin. U holda $u_1(x) - u_2(x)$ ayirma qaralayotgan masalaning

$$(u_1(x) - u_2(x))|_S = \varphi_1(x) - \varphi_2(x)$$

chegaraviy shartni qanoatlantiruvchi yechimi bo'ladi.

Agar ixtiyoriy $\varepsilon > 0$ son va $x \in S$ uchun $|\varphi_1(x) - \varphi_1(x)| < \varepsilon$ bo'lsa, u holda

$$|u_1(x) - u_2(x)| \leq \max_S |u_1(x) - u_2(x)| = \max_S |\varphi_1(x) - \varphi_2(x)| < \varepsilon$$

tengsizlik o'rinli bo'ladi.

Bu Laplas tenglamasi uchun Dirixle masalasi yechimining turg'un ekanligini bildiradi.

Endi Laplas tenglamasi uchun Neyman masalasi yechimining yagonaligini isbot qilaylik.

Agar $u(x) \in C^1(\overline{D}) \cap C^2(D)$ va $S$ silliq sirt bo'lsa, u holda $u(x)$ funksiya $S$ da to'g'ri normal hosilaga ega bo'ladi.

Chekli $D$ sohada ichki Neyman masalasini qaraylik, ya'ni

$$\begin{cases} u(x) \in C^1(\overline{D}) \cap C^2(D); \\ \Delta u(x) = f(x), \qquad x \in D; \\ \left.\dfrac{\partial u(x)}{\partial \nu}\right|_S = \psi(x_0), \qquad x_0 \in S; \end{cases} \tag{7}$$

3–TEOREMA. Agar Laplas tenglamasi uchun ichki Neyman masalasi $D$ sohada ikkita yechimiga ega bo'lsa, u holda bu yechimlar bir–biridan o'zgarmasga farq qiladi.

ISBOT. Faraz qilaylik, (7) masala $D$ sohada $u_1(x)$ va $u_2(x)$ yechimlarga ega bo'lsin. Ularning ayrimasi $u(x) = u_1(x) - u_2(x)$ uchun

$$\Delta u(x) = 0, \qquad \left.\frac{\partial u(x)}{\partial \nu}\right|_S = 0, \tag{8}$$

munosabatlar o'rinli ekanligini ko'rsatish qiyin emas. $u(x)$ funksiyaga $D$ sohada Grin formulasini qo'llaymiz. Natijada

$$\iint\limits_D \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int\limits_S u \frac{\partial u}{\partial \nu} ds, \tag{9}$$

tenglikka ega bo'lamiz.

$u(x)$ funksiyaga $D$ sohada uzluksiz ekanligidan u yopiq $\overline{D}$ da chegaralangan bo'ladi va $\partial u/\partial \nu$ qiymat (8) ga ko'ra nolga tekis yaqinlashadi. (8) chegaraviy shartdan va (9) tenglikdan

$$\iint\limits_D \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = 0$$

ifodani olamiz. Oxirgi tenglikdan $\partial u/\partial x_i = 0, \quad i = 1, 2, ..., n$ va bundan esa $u(x) = const$ yoki $u_1(x) = u_2(x) + c$ kelib chiqadi.

Endi tashqi Neyman masalasini $n > 2$ bo'lgan holda qaraylik.

4–TEOREMA. Agar Laplas tenglamasi uchun tashqi Neyman masalasining yechimi $D$ sohada mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. Faraz qilaylik, cheksiz $D$ sohada Neyman masalasi $u_1(x)$ va $u_2(x)$ yechimlarga ega bo'lsin. Ularning $u(x) = u_1(x) - u_2(x)$ ayrimasi

$$\begin{cases} u(x) \in C^1(\overline{D}) \cap C^2(D); \\ \Delta u(x) = 0, \qquad x \in D; \\ \left.\dfrac{\partial u(x)}{\partial \nu}\right|_S = 0, \qquad x_0 \in S; \\ u(x) = O(|x|^{2-n}), \qquad x \to \infty, \end{cases} \tag{10}$$

munosabatlarni qanoatlantiradi.

Cheksiz $D$ sohada yotuvchi va $S$ sirtga parallel bo'lgan $S_h$ sirtni quramiz. Endi markazi koordinata boshida bo'lgan $S_h$ sirtni o'z ichiga olgan $R$ radiusli $S_R$ sfera o'tkazamiz. $D_R^{(h)}$ soha $S_R$ va $S_h$ sirtlar bilan chegaralangan va $D_R$ esa $S$ va $S_R$ sirtlar bilan chegaralangan. $D_R^{(h)}$ soha chekli va unda $u(x) \in C^2(D_R^{(h)})$ bo'lgani uchun Grin formulasini qo'llash mumkin. Unga ko'ra

$$\iint\limits_{D_R^{(h)}} u(x)\,\Delta u(x) dx = -\iint\limits_{D_R^{(h)}} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx +$$

$$+ \int\limits_{S_h} u \frac{\partial u}{\partial \nu} ds + \int\limits_{S_R} u \frac{\partial u}{\partial \nu} ds,$$

bo'ladi. Oxirgi ayniyatda $h \to 0$ limitga o'tamiz va (10) munosabatlarga asosan

$$\iint\limits_{D_R} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int\limits_{S_R} u \frac{\partial u}{\partial \nu} ds, \tag{11}$$

tenglikka ega bo'lamiz. (10) shartlarga va $u(x)$ funksiyaning $D$ sohada garmonik ekanligidan va etarlicha katta $R$ uchun quyidagi

$$\left| u(x) \right|_{|x|=R} \le \frac{C}{R^{n-2}}, \qquad C = const > 0,$$

$$\left| \frac{\partial u(x)}{\partial \nu} \right|_{|x|=R} \le \frac{C_1}{R^{n-1}}, \qquad C_1 = const > 0,$$

tengsizliklar o‘rinli bo‘ladi. Bu tengsizliklarga asosan (11) formulaning o‘ng tomoni uchun

$$\left|\int\limits_{S_R} u\,\frac{\partial u}{\partial \nu}\,ds\right| \le \frac{CC_1|S_R|}{R^{2n-3}} = \frac{CC_1|S_1|}{R^{n-2}},$$

bahoni olamiz. Bunda $n > 2$ bo‘lgani uchun $R \to \infty$ da limitga o‘tamiz. Natijada

$$\iint\limits_D \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = \lim_{R\to\infty}\int\limits_{S_R} u\,\frac{\partial u}{\partial \nu}\,ds \le \lim_{R\to\infty}\frac{CC_1|S_1|}{R^{n-2}} = 0$$

tenglikka ega bo‘lamiz.

Oxirgi tenglikdan $\dfrac{\partial u}{\partial x_i} = 0$ yoki $u(x) = const$ kelib chiqadi. Masalaning shartiga ko‘ra $u(x)$ funksiya cheksizlikda nolga intiladi. Shuning uchun $u(x) \equiv 0$ bo‘ladi. Bundan esa $u_1(x) \equiv u_2(x)$ ekanligi kelib chiqadi.

## 29–§. Doira uchun Dirixle masalasi. Puasson formulasi

Bu paragrafda yuqorida keltirilgan Dirixle masalasi yechimining mavjudligini $n = 2$ bo‘lganda, ya’ni tekislikda o‘zgaruvchilarni ajratish usuli bilan isbotlaymiz. SHu bilan birga ixtiyoriy radiusli doirada Dirixle masalasi yechimini beruvchi Puasson formulasi va uning xossalarini o‘rganamiz.

### Doirada Dirixle masalasi

$xOy$ tekislikda birlik aylana $S : x^2+y^2 = 1$ bilan chegaralangan doira $D = \{(x,y) : x^2 + y^2 < 1\}$ bo‘lsin. Birlik $D$ doirada quyidagi

$$\Delta u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (1)$$

Laplas tenglamasini qaraylik.

Dirixle masalasi. Laplas tenglamasining $\overline{D}$ yopiq sohada aniqlangan uzluksiz va quyidagi

$$u(x,y) \in C(\overline{D}) \cap C^2(D);$$

$$u(x,y)\Big|_S = f(\varphi), \quad 0 \le \varphi \le 2\pi, \tag{2}$$

shartlarni qanoatlantiruvchi $u(x,y)$ yechimini toping.

Bu yerda $f(\varphi)$ etarlicha silliq berilgan funksiya, $\varphi$ esa $Ox$ o'qi bilan $OM$ radius–vektor orasidagi burchak va $f(\varphi)$ funksiya uchun $f(0) = f(2\pi)$ tenglik o'rinli.

Doirada (1)–(2) Dirixle masalasini yechish uchun o'zgaruvchilarni ajratish usulidan foydalanamiz. Buning uchun berilgan $f(\varphi)$ funksiya $[0,2\pi]$ segmentda uzluksiz va uzluksiz hosilaga ega, ya'ni $f(\varphi) \in C^1[0,2\pi]$ bo'lsin.

$D$ sohada $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $0 < \rho < 1$, $0 \le \varphi \le 2\pi$ qutb koordinatalariga o'tamiz. U holda Laplas tenglamasi

$$\rho^2 u_{\rho\rho} + \rho u_\rho + u_{\varphi\varphi} = 0, \tag{3}$$

ko'rinishga o'tadi.

Bu tenglamaning ixtiyoriy $u(x,y) = u(\rho\cos\varphi, \rho\sin\varphi) = u(\rho,\varphi)$ yechimini

$$u(\rho,\varphi) = R(\rho)\Phi(\varphi) \neq 0, \tag{4}$$

ikki funksiyaning ko'paytmasi ko'rinishida izlaymiz.

Endi (4) formulani (3) tenglamaga qo'yib,

$$\rho^2 R''(\rho)\Phi(\varphi) + \rho R'(\rho)\Phi(\varphi) + R(\rho)\Phi(\varphi) = 0,$$

ifodani olamiz. Oxirgi ifodani $R(\rho)\Phi(\varphi)$ ko'paytmaga bo'lamiz. Natijada

$$\rho^2 \frac{R''(\rho)}{R(\rho)} + \rho\frac{R'(\rho)}{R(\rho)} = -\frac{\Phi''(\varphi)}{\Phi''(\varphi)}, \tag{5}$$

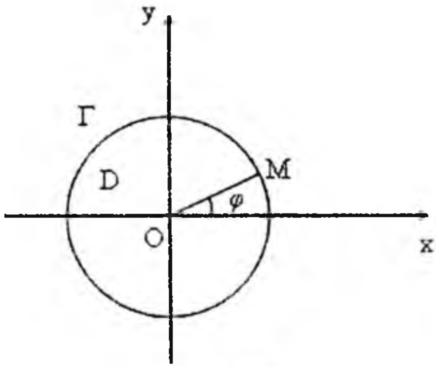
tenglikka ega bo'lamiz. Bu tenglikning chap tomoni faqat $\rho$ o'zgaruvchiga, o'ng tomoni esa $\varphi$ o'zgaruvchiga bog'liq. Bu tenglik uning o'ng va chap tomonlari faqat bitta $\lambda$ o'zgarmasga teng bo'lganda o'rinli. U holda (5) tenglama ikkita

$$\Phi''(\varphi) + \lambda\Phi(\varphi) = 0, \tag{6}$$

$$\rho^2 R''(\rho) + \rho R'(\rho) - \lambda R(\rho) = 0. \tag{7}$$

chiziqli oddiy differensial tenglamaga ekvivalent bo'ladi.

Chegaraviy funksiya $f(\varphi + 2\pi) = f(\varphi)$ ekanligidan (4) formulaga asosan $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ bo'ladi. Demak, $\Phi$ davri $T = 2\pi$ bo'lgan davriy funksiya ekan. Bu faqat $\lambda = n^2, n \in N$ bo'lganda o'rinli bo'ladi.




19–shakl.

(6) tenglamaning umumiy yechimi quyidagi

$$\Phi_n(\varphi) = a_n \cos n\varphi + b_n \sin n\varphi$$

formula bilan aniqlanadi, bunda $a_n$, $b_n$ ixtiyoriy o'zgarmaslar.

(7) tenglama esa $\lambda = n^2$ bo'lganda ikkita

$$R_1(\rho) = \rho^n, \qquad R_2(\rho) = \rho^{-n}$$

chiziqli erkli yechimlarga ega.

Dirixle masalasining yopiq $\overline{D}$ sohada uzluksiz yechimi qidirilayotgani uchun (7) tenglamaning yechimi sifatida $R(\rho) = \rho^n$ funksiyani olamiz. U holda (4) formulaga ko'ra

$$u_n(\rho, \varphi) = \rho^n(a_n \cos n\varphi + b_n \sin n\varphi)$$

formulani topamiz. Shunday qilib, (3) tenglamaning $D$ doirada garmonik bo'lgan xususiy yechimini topdik.

Agar $\lambda = 0$ bo'lsa, (3) tenglamaning yechimi $u(x, y) = const$ bo'ladi va aniqlik uchun $const = a_0/2$ deb olamiz. U holda Dirixle masalasining yechimini ushbu

$$u(\rho, \varphi) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi), \tag{8}$$

yig'indi ko'rinishda izlaymiz.

Faraz qilaylik, ushbu $a_n$, $b_n$ ketma-ketliklar chegaralangan va $M = \max\limits_n \{|a_0|/2, |a_n| + |b_n|\}$ bo'lsin. Agar $\rho < \rho_1 < 1$, ixtiyoriy $\rho_1$ fiksirlangan musbat son bo'lsa, u holda (8) qator ushbu

$$M(1 + \rho_1 + \rho_1^2 + \ldots + \rho_1^n + \ldots) = M \sum_{n=0}^{+\infty} \rho_1^n \tag{9}$$

sonli qator bilan majorantlanadi. Veyershtrass alomatiga asosan (8) qator ixtiyoriy yopiq $\rho \leq \rho_1$ doirada tekis yaqinlashuvchi va bu qatorning yig'indisi yopiq $\rho \leq \rho_1$ doirada uzluksiz bo'ladi. Budan $\rho_1 \in (0, 1)$ oraliqda ixtiyoriy ekanligidan $u(\rho, \varphi)$ funksiyaning $D$ doirada uzluksiz bo'lishi kelib chiqadi.

Endi (8) qatorni $D$ doirada uzluksiz $\rho$ va $\varphi$ o'zgaruvchilar bo'yicha $k$ marta differensiallash mumkin. Hosil bo'lgan qatorlar $\rho \leq \rho_1 < 1$ (9) kabi yaqinlashuvchi sonli qator bilan majorantlanadi.

Haqiqatdan ham, (8) qatorni $k$ marta formal differensiallash natijasida

$$\frac{\partial^k u(\rho, \varphi)}{\partial \rho^k} = \sum_{n=k}^{+\infty} n(n-1) \cdot \ldots \cdot (n-k+1) \rho^{n-k} (a_n \cos n\varphi + b_n \sin n\varphi), \tag{10}$$

$$\frac{\partial^k u(\rho, \varphi)}{\partial \varphi^k} = \sum_{n=1}^{+\infty} n^k \rho^n \left[ (a_n \cos\left(n\varphi + \frac{k\pi}{2}\right) + b_n \sin\left(n\varphi + \frac{k\pi}{2}\right) \right], \tag{11}$$

hosil bo'lgan (10) qatorning hadlari $\rho \leq \rho_1 < 1$ sohada ushbu

$$\sum_{n=k}^{+\infty} n(n-1) \cdot \ldots \cdot (n-k+1) \rho_1^{n-k} \leq M \sum_{n=k}^{+\infty} n^k \rho_1^{n-k} = M \sum_{m=0}^{+\infty} (k+m)^k \rho_1^m,$$

yaqinlashuvchi sonli qatorning hadlari bilan chegaralangan.

(11) qator esa mos ravishda

$$M\sum_{n=1}^{+\infty} n^k \rho_1^n$$

qatorga majorantlanadi.

Bundan (10)–(11) qatorlar $\rho \le \rho_1$ yopiq doirada absolyut va tekis yaqinlashuvchi, shuning uchun bu qatorlarning yig'indisi $D$ sohada uzluksiz bo'ladi.

Agar (10) qatorda $k = 1$ va $k = 2$, (11) qatorda esa $k = 2$ bo'lganda hosil bo'lgan ifodalarni (3) tenglamaga qo'ysak, (8) qator bilan aniqlangan $u(\rho, \varphi)$ funksiya $D$ doirada tenglamaning yechimi bo'lishiga ishonch hosil qilamiz.

SHunday qilib, (8) qator $D$ sohada garmonik funksiya ekan. U holda (8) qatorni (2) chegaraviy shartga qo'yamiz, natijada

$$u(\rho, \varphi)\Big|_S = f(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi), \tag{12}$$

hosil bo'ladi.

Bu $f(\varphi)$ funksiyaning $[0, 2\pi]$ segmentda sinus va kosinus bo'yicha Fur'e qatoriga yoyilmasi deyiladi.

Uning $a_n$ va $b_n$ koeffitsientlari

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)d\varphi, \tag{13}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos n\varphi d\varphi, \qquad n = 0, 1, 2, \dots, \tag{14}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin n\varphi d\varphi, \qquad n = 1, 2, 3, \dots, \tag{15}$$

formulalar yordamida aniqlanadi.

Agar $f(\varphi)$ funksiya $[0, 2\pi]$ segmentda uzluksiz va shu segmentda uzluksiz birinchi tartibli hosilalarga ega bo'lsa, u holda (12) qator $[0, 2\pi]$ segmentda $f(\varphi)$ funksiyaga tekis yaqinlashadi, bunda (12) qatorning har bir hadi ushbu

$$\frac{|a_0|}{2} + \sum_{n=1}^{\infty}(|a_n| + |b_n|), \tag{16}$$

yaqinlashuvchi sonli qatorning hadlari bilan majorantlanadi.

Demak, (8) qator ixtiyoriy $\rho \leq 1$ bo'lganda (16) qatorga majorantlansa, u holda Veyershtrass alomatiga ko'ra (8) qator yopiq $\overline{D}$ sohada absolyut va tekis yaqinlashuvchi bo'ladi.

Endi $f(\varphi)$ ixtiyoriy uzluksiz funksiya bo'lganda, koeffitsientlari (13)–(15) formulalar bilan aniqlangan (8) qator
$\rho < 1$ sohada Dirixle masalasining yechimi ekanligini ko'rsatamiz. Buning uchun $\rho = 1$ aylanada berilgan $f(\varphi)$ funksiyaga intiluvchi $f_m(\varphi)$ funksiyalar ketma-ketligini quramiz.

Faraz qilaylik, $u_m$ funksiya $u_m|_S = f_m(\varphi)$ shartni qanoatlantiruvchi Dirixle masalasining yechimi bo'lsin. 6–lemmaga asosan $u_m$ ketma-ketlik $\rho \leq 1$ doirada uzluksiz bo'lgan $u(x, y)$ funksiyaga intiladi va bu funksiya $\rho = 1$ da $f(\varphi)$ funksiyaga teng.

Faraz qilaylik, $f_m(\varphi)$ funksiyaning Fure koeffitsientlari $a_n^m$, $b_n^m$ bo'lsin. $\forall \varepsilon > 0$ olinganda barcha $n$ lar uchun etarlicha katta $m$ larda

$$|a_n - a_n^m| \leq \varepsilon, \qquad |b_n - b_n^m| \leq \varepsilon$$

bo'ladi.

Bundan

$$\left|\frac{a_0}{2} + \sum_{n=1}^{+\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi) - u_m\right| =$$

$$= \left|\frac{a_0 - a_0^m}{2} + \sum_{n=1}^{+\infty} \rho^n |(a_n - a_n^m)\cos n\varphi + (b_n - b_n^m)\sin n\varphi)|\right| \leq$$

$$\leq 2\varepsilon \sum_{n=1}^{\infty} \rho^n = 2\varepsilon \frac{1}{1-\rho}$$

kelib chiqadi.

Demak, (8) formula bilan aniqlangan $u(x,y)$ funksiya $\rho < 1$ sohada (1)–(2) Dirixle masalasining yechimi ekan.

Shunday qilib, quyidagi teoremaning o'rinli ekanligi isbotlandi.

1–TEOREMA. Agar $f(\varphi) \in C^1[0,2\pi]$ va $f(0) = f(2\pi)$ bo'lsa, u holda $D$ sohada Laplas tenglamasi uchun Dirixle masalasining yagona yechimi mavjud bo'ladi. Bu yechim (8) qator bilan, uning $a_0$, $a_n$ va $b_n$ koeffitsiyentlari mos ravishda (13), (14) va (15) formulalar bilan aniqlanadi.

### Puasson formulasi

Endi (8) qatorni (14)–(15) formulalarni inobatga olib o'zgartiramiz. Buning uchun (14) va (15) koeffitsiyentlarni (8) qatorga qo'yamiz. Natijada $\rho < 1$ sohada

$$u(\rho,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \rho^n (a_n \cos n\varphi + b_n \sin n\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt +$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} \rho^n \left\{ \int_0^{2\pi} f(t)\cos nt dt \cos n\varphi + \int_0^{2\pi} f(t)\sin nt dt \sin n\varphi \right\} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{\pi}\int_0^{2\pi} f(t)\sum_{n=1}^{\infty} \rho^n \{\cos nt \cos n\varphi + \sin nt \sin n\varphi\} dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\left[1 + 2\sum_{n=1}^{\infty} \rho^n \cos n(t-\varphi)\right] dt. \tag{17}$$

formulaga ega bo'lamiz. Oxirgi formulada $t - \varphi = w$ deb belgilaymiz va Eyler formulasidan foydalansak, qavs ichidagi ifoda quyidagi

$$1 + 2\sum_{n=1}^{\infty} \rho^n \cos n(t-\varphi) = 1 + 2\sum_{n=1}^{\infty} \rho^n \cos nw =$$

$$= -1 + 2\sum_{n=0}^{\infty} \rho^n \cos nw = -1 + 2Re\sum_{n=0}^{\infty} z^n =$$

$$= -1 + 2Re\frac{1}{1-z} = -1 + 2Re\frac{1}{1-\rho e^{iw}} = \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos w}, \quad (18)$$

ko'rinishga keladi.

U holda (18) ifodani (17) formulaga qo'yib, $\rho < 1$ bo'lganda

$$u(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{1-\rho^2}{1+\rho^2-2\rho\cos(t-w)}dt. \quad (19)$$

formulani olamiz.

Bu formula *Puasson formulasi deyiladi* va bu $\rho < 1$ sohada Laplas tenglamasi uchun Dirixle masalasi yechimini aniqlaydi.

Ixtiyoriy $R$ radiusli doira uchun Puasson formulasini topish uchun (19) formuladan $\rho$ ni $\rho/R$ ga almashtirib,

$$u(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(t)\frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(t-w)}dt,$$

$\tilde{f}(t) = f(tR) = f(s)$, $dt = ds/R$, $s = \varphi R$ ixtiyoriy $R$ radiusli aylana yoyining uzunligi. U holda

$$u(\rho,\varphi) = \frac{1}{2\pi R}\int_0^{2\pi R} f(s)\frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(s/R-w)}ds, \quad (20)$$

formulaga ega bo'lamiz.

Bu formula $\rho < R$ sohada Laplas tenglamasi uchun Dirixle masalasining

$$u(\rho,\varphi)|_{\rho=R} = u(R,\varphi) = f(R\varphi) = f(s), \quad 0 \le \varphi \le 2\pi R \quad (21)$$

shartni qanoatlantiruvchi $u(\rho,\varphi)$ yechimini beradi.

Endi (20) formulada integral ostidagi ifoda *Puasson yadrosi* deyiladi va uni

$$\Pi(\rho,\varphi,s) = \frac{R^2-\rho^2}{R^2+\rho^2-2\rho R\cos(s/R-w)}$$

orqali belgilaymiz. Puasson yadrosi quyidagi xossalarga ega:

1) $\Pi(\rho,\varphi,s) > 0, \quad \rho < R,$ bunda $R^2+\rho^2 > 2R\rho;$
2) $\Pi(\rho,\varphi,s) \to +\infty,$ agar $\rho \to R, \quad s = \varphi R;$
3) $\Pi(\rho,\varphi,s) \in C^k(\rho < R), \quad \rho < R$

doirada garmonik funksiya bo'ladi.

Shuni ta'kidlash muhimki, (20) formula $f(s) \in C^2[0,2\pi]$ shart asosida olindi. Bu formula $\rho < R$ doirada Laplas tenglamasi uchun Dirixle masalasi yechimini beradi. (20) formulani keltirib chiqarishda $f(s)$ funksiyaga qo'yilgan talabni kamaytirish mumkin, masalan $f(s) \in C[0,2\pi]$ bo'lishi ham etarli.

Haqiqatdan ham, Puasson yadrosining 3 xossasiga asosan (20) formulani $\rho < R$ doirada $\rho$ va $\varphi$ bo'yicha ixtiyoriy marta differensiallash mumkin hamda hosil bo'lgan integrallar $\rho \le R_1 < R$ yopiq doirada tekis yaqinlashuvchi bo'ladi.

Demak, (20) formula bilan aniqlangan $u(\rho,\varphi)$ funksiya $f(s) \in C[0,2\pi]$ bo'lganda $\rho < R$ doirada $\rho$ va $\varphi$ bo'yicha ixtiyoriy marta differensiallanuvchi bo'ladi va $\rho < R$ doirada Laplas tenglamasini qanoatlantiradi. Bundan esa (20) formula bilan aniqlangan $u(\rho,\varphi)$ funksiya $\rho < R$ doirada garmonik funksiya ekanligi kelib chiqadi.

Endi (20) va (21) formulalar bilan aniqlangan $u(\rho,\varphi)$ funksiya yopiq $\rho \le R$ doirada uzluksiz, ya'ni $\varphi = \varphi_0$ bo'lganda

$$\lim_{\rho\to R} u(\rho,\varphi) = f(s_0), \quad \text{bunda} \quad s_0 = R\varphi_0, \tag{22}$$

ekanligini isbotlaylik.

Buning uchun $[0,2\pi R]$ segmentda $f(s)$ funksiyaga tekis yaqinlashuvchi $C^1[0,2\pi R]$ sinfga tengishli bo'lgan $f_n(s)$ funksiyalar ketma-ketligini tuzamiz. Berilgan $f_n(s)$ funksiyalar ketma-ketligi asosida $\rho < R$ doirada garmonik bo'lgan va

$$u_n(\rho, \varphi)|_{\rho=R} = f_n(s), \quad 0 \leq \varphi \leq 2\pi R,$$

shartni qanoatlantiruvchi quyidagi

$$u_n(\rho, \varphi) = \frac{1}{2\pi R} \int\limits_0^{2\pi R} f_n(s)\,\Pi(\rho, \varphi, s)\,ds$$

Puasson formulasini ko'ramiz.

Oldingi paragrafda isbotlangan 6-lemmaga ko'ra $u_n(\rho, \varphi)$ ketma-ketlik yopiq $\rho \leq R$ doirada tekis yaqinlashuvchi bo'lib, uning limit funksiyasi

$$u(\rho, \varphi) = \frac{1}{2\pi R} \int\limits_0^{2\pi R} f(s)\,\Pi(\rho, \varphi, s)\,ds$$

$\overline{D}$ sohada uzluksiz bo'ladi. Bu yesa (22) formulaning to'g'ri ekanligini ko'rsatadi.

Shunday qilib, biz quyidagi teoremaning o'rinli ekanligini isbotladik.

2-TEOREMA. Agar $f(s) \in C[0, 2R\pi]$ va $f(0) = f(2R\pi)$ bo'lsa, u holda ixtiyoriy $\rho < R$ doirada Laplas tenglamasi uchun Dirixle masalasining yagona yechimi mavjud bo'ladi va bu yechim (20) formula bilan aniqlanadi.

NATIJA. Laplas tenglamasi uchun Dirixle masalasining $u(\rho, \varphi)$ yechimi $\rho < R$ doirada cheksiz differensiallanuvchi bo'ladi, ya'ni $u(\rho, \varphi) \in C^\infty(\rho < R)$.

## 30–§. Garmonik funksiyaning xossalari

Bu paragrafda $xOy$ tekisligida garmonik funksiyalarning ayrim xossalarini keltiramiz. Keltiriladigan xossalarni isbotlashda ixtiyoriy $R$ radiusli $D_R$ doira uchun qurilgan va yopiq doirada uzluksiz bo'lgan Puasson formulasidan foydalanamiz.

Agar $n = 1$ bo'lgan holda Laplas tenglamasi

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = 0$$

ko'rinishda bo'ladi. Bu tenglamaning umumiy yechimi

$$u(x) = ax + b, \tag{1}$$

bu yerda $a$ va $b$ ixtiyoriy haqiqiy o'zgarmas sonlar, ya'ni bu chiziqli funksiya. Ma'lumki, garmonik funksiyalarning ko'plab xossalari shu chiziqli funksiyaning xossalari kabi bo'ladi. Masalan, (1) chiziqli funksiyaning $[\alpha, \beta]$ kesmaning o'rtasidagi qiymati

$$u\left(\frac{\alpha+\beta}{2}\right) = a\frac{\alpha+\beta}{2} + b = \frac{a\alpha + b + a\beta + b}{2} = \frac{u(\alpha) + u(\beta)}{2} =$$

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta u(x)dx = \frac{1}{\beta-\alpha}\int_\alpha^\beta (ax+b)dx.$$

kesmaning uchlaridagi qiymatlarining o'rta arifmetigiga teng bo'ladi.

Garmonik funksiyalar ham xuddi shunday xossaga ega.

1–TEOREMA. Agar $u(r,\varphi)$ funksiya $D_R = \{(r,\varphi) : r < R\}$ doirada garmonik va yopiq $\overline{D}_R$ doirada uzluksiz bo'lsa, u holda bu funksiyaning $D_R$ doira markazidagi qiymati $r = R$ aylanadagi qiymatlarining o'rta arifmetigiga teng, ya'ni

$$u(0,\varphi) = \frac{1}{2\pi R}\int_0^{2\pi R} f(s)ds = \frac{1}{2\pi R}\int_0^{2\pi R} u(R, s/R)ds, \tag{2}$$

bo'ladi.

ISBOT. Teorema shartiga ko'ra, $u(r,\varphi)$ funksiya oldingi paragrafdagi 2–teorema shartini qanoatlantiradi. Shuning uchun bu funksiyani quyidagi

$$u(r,\varphi) = \frac{1}{2\pi R}\int_0^{2\pi R} f(s)\,\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(s/R - w)}\,ds, \qquad (3)$$

Puasson integrali ko'rinishida ifodalash mumkin.

Oxirgi formulada $r = 0$ deb, (2) tenglikni olamiz. Bu $u(0,\varphi)$ qiymat, garmonik funksiyaning $r = R$ aylanadagi qiymatlarining o'rta arifmetigidir.

Yuqorida keltirilgan (1) chiziqli funksiya sonlar o'qida cheksiz differensiallanuvchi va analitik funksiya bo'lgani kabi, garmonik funksiyalar ham xuddi shunday xossaga ega.

2–TEOREMA. $D$ sohadagi har qanday garmonik funksiya shu sohada cheksiz differensiallanuvchi va analitik funksiya bo'ladi.

ISBOT. Markazi $(x_0, y_0)$ nuqtada bo'lgan $R$ radiusli $D_R$ doira to'liq $D$ sohada yotsin, ya'ni $(x_0, y_0) \in D_R$ va $\overline{D}_R \subset D_R$ bo'lsin. U holda $u(x,y)$ funksiyani $D_R$ doirada (3) Puasson formulasi orqali ifodalash mumkin. 19–§ dagi 2–teoremaning natijasiga ko'ra $u(x,y)$ funksiya $D_R$ doirada, xususan $(x_0, y_0)$ nuqtada ixtiyoriy tartibdagi hosilalarga ega. $u(x,y)$ funksiyaning $D_R$ doirada analitik bo'lishi esa uni $(x_0, y_0)$ nuqtaning kichik atrofida $(x - x_0)$ va $(y - y_0)$ darajalari bo'yicha Teylor qatoriga yoyilishini anglatadi,

$$u(x,y) = \sum_{l,m=0}^{+\infty} \frac{\partial^{l+m} u(x_0, y_0)}{\partial x^l \partial y^m}\,\frac{(x - x_0)^l (y - y_0)^m}{l!\,m!}$$

Bu teoremaning to'liq isboti Puasson formulasi yordamida darsliklarda keltirilgan.

NATIJA. Garmonik funksiyaning ixtiyoriy tartibdagi hosilasi yana garmonik funksiya bo'ladi.

Buni isbotlash uchun quyidagi tenglikning

$$\Delta\left(\frac{\partial^k u}{\partial x^p\,\partial y^q}\right) = \frac{\partial^k}{\partial x^p\,\partial y^q}(\Delta u), \quad k = p + q$$

o‘rinli ekanligini ko‘rsatish etarli bo‘ladi.

3–TEOREMA. Agar $u(x, y)$ biror $D$ sohada garmonik va yopiq $\overline{D}$ sohada uzluksiz bo‘lib, $\overline{D}$ sohaning ichki nuqtalarida eng katta (eng kichik) qiymatga erishsa, u holda bu funksiya yopiq $\overline{D}$ sohada o‘zgarmas bo‘ladi.

Bu teorema [9, 10] darsliklarda batafsil isbotlangan. Shuning uchun uning isbotini bu erda keltirmadik.

Shuni ta’kidlash muhimki, sonlar o‘qida berilgan (1) chiziqli funksiya quyidan yoki yuqoridan chegaralangan bo‘lsa, u holda bu funksiya o‘zgarmas bo‘ladi.

Haqiqatdan ham, (1) chiziqli funksiyaning $R$ sonlar o‘qida quyidan yoki yuqoridan chegaralangan bo‘lishi uchun $a = 0$ bo‘lishi zarur va etarli.

Xuddi shunday xossa garmonik funksiyalar uchun ham o‘rinli.

4–TEOREMA. (Liuvill teoremasi.) Agar $u(x, y)$ funksiya butun tekislikda garmonik bo‘lib, quyidan va yuqoridan chegaralangan bo‘lsa, u holda bu funksiya o‘zgarmasdir.

ISBOT. Faraz qilaylik, $u(x, y)$ tekislikda quyidan chegaralangan bo‘lsin, ya’ni shunday $M \geq 0$ haqiqiy son mavjudki, $R^2$ tekislikdagi barcha $(x, y)$ nuqtalar uchun $u(x, y) \geq M$ bo‘lsin.

Aytaylik, $R^2$ tekislikdagi ixtiyoriy nuqta $Q = (x, y) = (\rho, \varphi)$ bo‘lsin va $u(Q) = u(0, 0)$, ya’ni $u(x, y) = const$ ekanligini ko‘rsatamiz. Markazi $(0, 0)$ nuqtada bo‘lgan $(x, y)$ nuqtani o‘z ichiga olgan $R$ radiusli doira $D_R$ bo‘lsin. U holda Puasson formulasi

$$u(r, \varphi) = \frac{1}{2\pi R} \int\limits_0^{2\pi R} f(s) \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(s/R - w)} ds, \qquad (4)$$

o‘rinli bo‘ladi.

Bu yerda $f(s) = u(R, \varphi) = u(R, s/R)$ va $-1 \leq \cos(s/R - \varphi) \leq 1$ ekanligidan barcha $(r, \varphi) \in \overline{D}_R$ nuqtalar uchun ushbu tengsizlik

$$\frac{R - r}{R + r} \leq \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(s/R - \varphi)} \leq \frac{R + r}{R - r}, \qquad (5)$$

o‘rinli. Bu tengsizlikka

$$\frac{1}{2\pi R}\int\limits_0^{2\pi R} f(s)\ldots ds$$

integral operatorni qo‘llaymiz. Yuqoridagi (2) va (4) tengliklarni inobatga olib, quyidagi

$$\frac{R-r}{R+r}u(0,0) \le u(r,\varphi) \le \frac{R+r}{R-r}u(0,0) \qquad (6)$$

tengsizlikka ega bo‘lamiz.

Bu tengsizlik *Garnak tengsizligi* deyiladi [10]. Endi (6) tengsizlikda $R \to +\infty$ da limitga o‘tsak,

$$u(0,0) \le u(r,\varphi) \le u(0,0)$$

hosil bo‘ladi. Oxirgi tengsizlikdan $u(r,\varphi) = u(0,0)$ bo‘lishi kelib chiqadi.

5–TEOREMA. (1–Garnak teoremasi.) Agar chekli $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz $\{u_n(x,y)\}$ funksiyalar ketma–ketligi sohaning $S$ chegarasida yaqinlashuvchi bo‘lsa, u holda bu ketma–ketlik yopiq $\overline{D}$ sohada tekis yaqinlashuvchi, limit funksiya $D$ sohada garmonik va $\overline{D}$ da uzluksiz bo‘ladi.

ISBOT. 18 – §da isbotlangan 6–lemmaga ko‘ra $\{u_n(x,y)\}$ funksiyalar ketma–ketligi yopiq $\overline{D}$ sohada tekis yaqinlashuvchi bo‘lgani uchun bu ketma–ketlikning limit funksiyasi yopiq $\overline{D}$ sohada uzluksiz funksiya bo‘ladi.

Faraz qilaylik, $D$ sohadagi ixtiyoriy nuqta $Q = (r,\varphi)$ bo‘lsin. U holda shu nuqtani o‘z ichiga oluvchi shunday $R$ radiusli $D_R$ doira topiladiki, bu doirada $u_n|_S = f_n(s)$ shartni qanoatlantiruvchi $u_n(x,y)$ funksiyani ushbu

$$u_n(r,\varphi) = \frac{1}{2\pi R}\int\limits_0^{2\pi R} f_n(s)\,\frac{R^2-r^2}{R^2+r^2-2rR\cos(s/R-w)}\,ds, \qquad (7)$$

Puasson integrali ko‘rinishida ifodalash mumkin.

$\{u_n(x,y)\}$ funksiyalar ketma-ketligi yopiq $\overline{D}$ sohada tekis yaqinlashuvchi ekanligidan (7) tenglikda $n \to +\infty$ limitga o'tamiz.

Agar $\lim\limits_{n\to+\infty} u_n(x,y) = u(x,y)$, $\lim\limits_{n\to+\infty} = f(s)$ bo'lsa, u holda (7) tenglikdan

$$u(r,\varphi) = \frac{1}{2\pi R}\int\limits_0^{2\pi R} f_n(s)\,\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(s/R - w)}\,ds, \qquad (8)$$

kelib chiqadi. Integral ostidagi $f(s)$ funksiya $[0, 2\pi R]$ segmentda uzluksiz bo'lgani uchun Puasson formulasining xossasiga asosan (8) integral $\overline{D}_R$ sohada garmonik funksiya bo'ladi.

$Q = (r,\varphi)$ nuqta ixtiyoriy bo'lganligidan $u(x,y)$ funksiyaning $D$ sohada garmonik funksiya ekanligi kelib chiqadi.

Xuddi yuqoridagi kabi teorema garmonik funksiyalar qatori uchun ham o'rinli ekanligini ko'rishimiz mumkin.

5*–TEOREMA. Agar chekli $D$ sohada garmonik, yopiq $\overline{D}$ sohada uzluksiz $\sum\limits_{n=1}^{+\infty} u_n(x,y)$ funksiyalar qatori sohaning $S$ chegarasida tekis yaqinlashuvchi bo'lsa, u holda bu qator yopiq $\overline{D}$ sohada tekis yaqinlashuvchi, uning yig'indisi $D$ sohada garmonik va $\overline{D}$ da uzluksiz bo'ladi.

Garnakning ikkinchi teoremasini isbotsiz keltiramiz.

6–TEOREMA. (2–Garnak teoremasi.) Agar chekli $D$ sohada garmonik funksiyalar $\{u_n(x,y)\}$ ketma-ketligi monoton o'suvchi bo'lib, sohaning kamida bitta nuqtasida yaqinlashuvchi bo'lsa, u holda bu ketma-ketlik $D$ sohada biror $u(x,y)$ funksiyaga yaqinlashuvchi bo'ladi. Shu bilan birga ixtiyoriy $\overline{D} \subset D$ sohada tekis yaqinlashuvchi bo'ladi.

Garmonik funksiyalarning chetlashtiriladigan maxsuslik to'g'risidagi, cheksizlikda regulyarligi haqidagi va boshqa xossalarini talabalar [9]–[11] kabi darsliklardan o'zlashtirib olishlari mumkin.

## 31–§. Laplas tenglamasi uchun tashqi Dirixle masalasi

Faraz qilaylik, $R^n$ evklid fazosida $\partial D = \Gamma$ sirt bilan chegaralangan soha $D$ bo'lsin. $D$ sohaning yopiq $\Gamma$ sirtga nisbatan tashqarisini $\Omega = R^n \backslash D \cup \Gamma = R^n \backslash \overline{D}$ deb belgilaylik.

Chegaralanmagan $\Omega$ sohada Laplas tenglamasi uchun Dirixle masalasini qaraymiz. Shuni ta'kidlash muhimki, chegaralanmagan sohada Dirixle masalasining korrekt bo'lishi yechimning cheksizlikdagi xususiyatiga bog'liq bo'ladi.

TASHQI DIRIXLE MASALASI. Chegaralanmagan $\Omega$ sohada quyidagi shartlarni qanoatlantiruvchi $u(x)$ garmonik funksiya toping:

$$u(x) \in C(\overline{\Omega}) \cap C^2(\Omega); \tag{1}$$

$$\Delta u(x) = 0, \quad \forall x \in \Omega; \tag{2}$$

$$u(x)|_\Gamma = f(x), \quad \forall x \in \Gamma; \tag{3}$$

$$u(x) \to 0, \qquad |x| \to \infty; \tag{4}$$

ya'ni, $n > 3$ bo'lganda $\forall x \in \Omega$ uchun $\lim\limits_{|x|\to\infty} u(x) = 0$, agar $n = 2$ bo'lsa, u holda masalaning (4) sharti

$$u(x) = O(1) \quad \text{agar} \quad |x| \to \infty, \tag{5}$$

ya'ni $u(x)$ funksiya cheksizlikda chegaralangan bo'lsin. Bu yerda $f(x)$ berilgan uzluksiz funksiya, $r^2 = |x| = x_1^2 + x_2^2 + \ldots + x_n^2$.

IZOH. Tashqi Dirixle masalasi $n \geq 3$ bo'lganda (1)–(4) shartlarni, $n = 2$ bo'lganda esa (1)–(3) va (5) shartlarni qanoatlantiradi.

Qaralayotgan tashqi Dirixle masala yechimining yagonaligi haqidagi teoremaga o'tishdan oldin chegaralanmagan sohalarda garmonik funksiyalar uchun ekstremum prinsipini isbotlaymiz.

1–TEOREMA. Agar $u(x)$ funksiya chegaralanmagan $\Omega$ sohada (1) va (2) shartlarni qanoatlantirib, $r \to \infty$ da

$$u(x) = \begin{cases} \alpha(r) \ln \dfrac{1}{r}, & \text{agar} \quad n = 2 \\ \alpha(r)\, r^{n-2}, & \text{agar} \quad n > 2 \end{cases}$$

bo'lsa, u holda barcha $x \in \Omega$ nuqtalarda $u(x)$ funksiya uchun ushbu

$$|u(x)| \leq C \cdot \max_{\Gamma} |u(x)|, \quad C = const, \tag{6}$$

tengsizlik o'rinli bo'ladi.

Bu yerda, $r \to \infty$ intilganda, $\alpha(r) \to 0$ bo'ladi.

ISBOT. Faraz qilaylik, koordinatalar boshi $\Omega$ sohada bo'lmasin. Endi markazi koordinatalar boshida bo'lgan $R$ radiusli $S_R$ sirtga nisbatan inversiya almashtirishini qaraylik. Bunda $R^n$ fazoning $\Omega$ sohada yotuvchi barcha nuqtalari koordinatalar boshini o'z ichiga olgan chekli $\Omega^*$ sohaning nuqtalariga bir qiymatli akslantiriladi, sohaning $\Gamma$ chegarasidagi nuqtalar esa $\Omega^*$ sohaning $\Gamma^*$ chegarasidagi nuqtalarga, cheksiz uzoqlikdagi $\infty$ nuqta esa koordinatalar boshiga akslanadi. U holda Kelvin teoremasiga asosan

$$v(\xi) = \left(\frac{R}{r}\right)^{n-2} u\left(\frac{R^2}{r^2}\xi\right) = \left(\frac{r}{R}\right)^{n-2} u(x), \tag{7}$$

funksiya $\Omega^* \backslash \{\xi = 0\}$ sohada garmonik bo'ladi va $r = |\xi| \to 0$ bo'lganda ushbu

$$v(\xi) = \begin{cases} \alpha^*(r) \ln \dfrac{r}{R^2}, & \text{agar} \quad n = 2 \\ \alpha^*(r) \dfrac{1}{r^{n-2}}, & \text{agar} \quad n > 2 \end{cases}$$

shartni qanoatlantiradi. Bu yerda $\alpha^*(r) = R^{n-2}\alpha\left(\dfrac{R^2}{r}\right) \to 0$. Agar $r \to 0$ intilsa, $\alpha^*(r) \to 0$ bo'ladi. U holda (7) funksiya $\xi = 0$ nuqtada garmonik bo'ladi, ya'ni $v(\xi)$ funksiya $\Omega^*$ sohaning nuqtalarida garmonik funksiya bo'lar ekan.

Shunday qilib, $v(\xi)$ funksiya chekli $\overline{\Omega^*}$ sohada uzluksiz va $\Omega^*$ sohada garmonik ekanligidan, ekstremum prinsipiga asosan bu funksiya o'zining eng katta va eng kichik qiymatiga $\Omega^*$ sohaning $G$'$amma^*$ chegarasida erishadi, ya'ni $\forall \xi \in \overline{\Omega^*}$ bo'lganda $v(\xi)$ funksiya uchun quyidagi tengsizlik

$$|v(\xi)| \leq \max_{\Gamma^*} |v(\xi)|$$

o'rinli. Bu tengsizlikdan (7) formulani inobatga olib, $\forall\, x \in \Omega$ nuqtalar uchun

$$|u(x)| = \left(\frac{R}{r}\right)^{n-2} |v(\xi)| \leq \left(\frac{R}{r}\right)^{n-2} \max_{\Gamma}\left(\frac{r}{R}\right)^{n-2} |u(x)| \leq$$

$$\leq \left(\frac{r^*}{r}\right)^{n-2} \max_{\Gamma}|u(x)| \leq C \cdot \max_{\Gamma}|u(x)|,$$

kelib chiqadi. Bu yerda $r^* = \max\limits_{\Gamma} r$, $C = \sup\limits_{\Omega}\left(\frac{r^*}{r}\right)^{n-2}$.

Shunday qilib, (6) tengsizlik isbotlandi.

2–TEOREMA. Agar tashqi Dirixle masalasining yechimi mavjud bo'lsa, u holda bu yechim yagona bo'ladi.

ISBOT. Faraz qilaylik, tashqi Dirixle masalasi $u_1(x)$ va $u_2(x)$ yechimlarga ega bo'lsin. U holda ularning ayirmasi $u(x) = u_1(x) - u_2(x)$ bir jinsli chegaraviy shartni qanoatlantiradi. Buning uchun 1–teorema shartlari bajariladi va (6) baho o'rinli. $u(x)$ funksiya qaralayotgan sohaning $\Gamma$ chegarasida nolga teng bo'lgani uchun (6) tengsizlikdan $|u(x)| \leq 0$ kelib chiqadi.

Bundan esa $\forall\, x \in \overline{\Omega}$ uchun $u(x) \equiv 0$ yoki $u_1(x) = u_2(x)$ bo'ladi. Demak, tashqi Dirixle masalasining yechimi yagona ekan.

Endi tashqi Dirixle masalasi yechimining mavjudligini $n = 2$ bo'lgan holda isbotlaymiz.

Faraz qilaylik, markazi koordinata boshida bo'lgan $R$ radiusli $\Gamma_R$ aylana bilan chegaralangan $D_R$ doiraning tashqarisi $\Omega_R = R^2 \backslash D_R$ bo'lsin. $\Omega_R$ sohada Laplas tenglamasi uchun (1)–(3) va (5) shartlarni qanoatlantiruvchi tashqi Dirixle masalasi yechimini quramiz, bunda (3) chegaraviy shart $\Gamma_R = \{(x, y) : x^2 + y^2 = R^2\}$ aylanada

$$u(x, y)|_{\Gamma_R} = u(r\cos\varphi, r\sin\varphi)|_{r=R} = f(R, \varphi), \quad 0 \leq \varphi \leq 2\pi, \qquad (3')$$

berilgan bo'ladi. Bu yerda $f$ berilgan uzluksiz funksiya va $f(0) = f(2\pi)$ tenglik o'rinli.

Xuddi avvalgi paragrafdagi kabi bu masalaning yechimini o'zgaruvchilarni ajratish ususli bilan echamiz va yechimni

$$u(r,\varphi)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left(\frac{R}{r}\right)^n(a_n\cos n\varphi+b_n\sin n\varphi), \qquad (8)$$

Fur'e qatorining yig'indisi ko'rinishida quramiz. Uning $a_n$ va $b_n$ koeffitsiyentlari

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(\varphi)\cos n\varphi d\varphi, \qquad n=0,1,2,\dots, \qquad (9)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(\varphi)\sin n\varphi d\varphi, \qquad n=1,2,3,\dots, \qquad (10)$$

formulalar yordamida aniqlanadi.

Bu (9) va (10) formulalarni (8) qatorga qo'yib, ma'lum almashtirishlarni bajargandan so'ng quyidagi

$$u(r,\varphi)=\frac{1}{2\pi R}\int_0^{2\pi R}f(s)\frac{r^2-R^2}{r^2+R^2-2rR\cos(s/R-w)}ds, \qquad (11)$$

Puasson formulasiga ega bo'lamiz. Bu yerda $r\geq R$.

Agar $f(R\varphi)$ funksiya $[0,2\pi]$ segmentda uzluksiz bo'lsa, u holda (11) qator bilan aniqlangan $u(r,\varphi)$ funksiya $\overline{D}_R$ doiradan tashqarida garmonik hamda (3') chegaraviy shartni qanoatlantirishini ko'rsatish mumkin.

## 32–§. Dirixle masalasining Grin funksiyasi

Faraz qilaylik, $R^3$ fazoda bo'lakli silliq $S$ sirt bilan chegaralangan $D\subset R^3$ soha bo'lsin. Yopiq $\overline{D}$ sohada $P(x,y,z)$, $Q(x,y,z)$ va $R(x,y,z)$ funksiyalar hamda ularning $P_x(x,y,z)$, $Q_y(x,y,z)$ va

$R_z(x, y, z)$ hosilalari uzluksiz bo'lsin. U holda bu funksiyalar uchun quyidagi

$$\iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz =$$

$$= \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS, \qquad (1)$$

Gauss–Ostrogradskiy formulasi o'rinli bo'ladi.

Bu yerda $N$ sohaning $S$ sirtiga o'tkazilgan tashqi normal, $\cos\alpha$, $\cos\beta$, va $\cos\gamma$ birlik $N$ vektorning yo'naltiruvchi kosinuslari, ya'ni $N = (\cos\alpha, \cos\beta, \cos\gamma)$, $|N| = 1$, $dS$ esa $S$ sfera yuzasining elementi.

Faraz qilaylik, $u(x, y, z)$, $v(x, y, z) \in C^2(\overline{D})$ bo'lsin. U holda (1) formulada $P = uv_x$, $Q = uv_y$ va $R = uv_z$ almashtiramiz. Natijada quyidagi

$$P_x + Q_y + R_z = (uv_x)_x + (uv_y)_y + (uv_z)_z =$$

$$= u_x v_x + u_y v_y + u_z v_z + u(v_{xx} + v_{yy} + v_{zz}) =$$

$$= u_x v_x + u_y v_y + u_z v_z + u\Delta v;$$

$$P\cos\alpha + Q\cos\beta + R\cos\gamma =$$

$$= u[v_x\cos\alpha + v_y\cos\beta + v_z\cos\gamma] = u\frac{\partial v}{\partial N};$$

tengliklarga ega bo'lamiz. Endi bu tengliklarni (1) formulaga qo'yib,

$$\iiint_D [u_x v_x + u_y v_y + u_z v_z] dxdydz =$$

$$= \iint_S u\frac{\partial v}{\partial N} dS + \iiint_D u\Delta v dxdydz, \qquad (2)$$

birinchi Grin formulasini hosil qilamiz.

Birinchi Grin formulasida $u$ va $v$ funksiyalarni o'rnini almashtirib,

$$\iiint_D [u_x v_x + u_y v_y + u_z v_z] dxdydz =$$

$$= \iint_S v\frac{\partial u}{\partial N}dS + \iiint_D v\Delta u dxdydz, \qquad (3)$$

ifodani olamiz. (2) formuladan (3) ni ayiramiz va quyidagi

$$\iiint_D (u\Delta v - v\Delta u)dxdydz = \iint_S \left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right)dS, \qquad (4)$$

Laplas operatori uchun ikkinchi Grin formulasiga ega bo'lamiz.

DIRIXLE MASALASINING GRIN FUNKSIYASI.

TA'RIF. Laplas tenglamasi uchun Dirixle masalasining Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi
$G(M, M_0)$ funksiyaga aytiladi:
bu erda $M = (x, y, z)$, $M_0 = (x_0, y_0, z_0)$

1) $\Delta_{x,y,z}G(M, M_0) = 0$, ya'ni $G(M, M_0)$ funksiya $D\backslash M_0$ sohada garmonik funksiya;

2) $G(M, M_0)|_{M\in S} = 0, quad M_0 \in D$;

3) $G(M, M_0)$ funksiya $D$ sohada quyidagi ko'rinishga ega:

$$G(M, M_0) = q(M, M_0) + g(M, M_0), \qquad (5)$$

bu yerda $q(M, M_0)$ Laplas tenglamasining fundamental yechimi va u quyidagi ko'rinishga ega:

$$q(M, M_0) = \begin{cases} \dfrac{1}{2\pi}\ln\dfrac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, & \text{agar} \quad n = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, & \text{agar} \quad n = 3; \end{cases}$$

$g(M, M_0)$ yopiq $\overline{D}$ sohada uzluksiz va $M_0 \overline{\in} D$ uchun $M$ nuqtaning koordinatalari bo'yicha $D$ sohada garmonik funksiya.

GRIN FUNKSIYASINING XOSSALARI.

$1^0$. Agar $M \to M_0$ bo'lsa, u holda $G(M, M_0) \to +\infty$ bo'ladi.

Bu xossaning to'g'ri ekanligi Grin funksiyasining (5) ko'rinishidan kelib chiqadi.

$2^0$. $G(M, M_0)$ funksiya $D$ sohada musbat, ya'ni $G(M, M_0) \geq 0$.

Haqiqatdan ham, $G(M, M_0)$ funksiyaning $M_0$ nuqtasini markaz qilib, etarlicha kichik $\delta$ radiusli $|M - M_0| \leq \delta$, $M_0 \in D$ shar chizamiz va $D$ sohaning bu shardan tashqaridagi qismini $D_\delta$ deb belgilaylik. Grin funksiyasining $1^0$ xossasiga ko'ra, etarlicha kichik $\delta$ uchun yopiq $|M - M_0| \leq \delta$ sharda $G(M, M_0) > 0$ bo'ladi. Demak, $G(M, M_0)$ funksiya $D_\delta$ sohada garmonik va yopiq $\overline{D}_\delta$ sohada uzluksiz ekanligidan $D_\delta$ sohaning chegarasida ham $G(M, M_0) \geq 0$ bo'ladi. Ekstremum prinsipiga asosan $M \in D$ nuqtalar uchun $G(M, M_0) > 0$ bo'ladi.

Bundan esa barcha $D \cup S$ sohada $G(M, M_0) \geq 0$ ekanligi kelib chiqadi.

$3^0$. $G(M, M_0)$ Grin funksiyasi uchun $M, M_0 \in D$, $M \neq M_0$ bo'lganda quyidagi

$$0 < G(M, M_0) < \frac{1}{4\pi}\frac{1}{r}, \quad \text{agar} \quad n = 3;$$

va

$$0 < G(M, M_0) < \frac{1}{2\pi}\frac{1}{r}, \quad \text{agar} \quad n = 2$$

baho o'rinli.

Haqiqatan ham, Grin funksiyasining (5) ko'rinishidan va $2^0$ xossaga asosan

$$g(M, M_0)|_{M \in S} = -\frac{1}{4\pi}\frac{1}{r} \quad \text{agar} \quad n = 3;$$

$$g(M, M_0)|_{M \in S} = -\frac{1}{2\pi}\ln\frac{1}{r} \quad \text{agar} \quad n = 2;$$

bo'ladi. Bundan esa $g(M, M_0)$ funksiya garmonik bo'lganligi uchun ekstremum prinsipiga ko'ra $D$ sohada $g(M, M_0) < 0$ bo'lishi ko'rinadi.

$4^0$. $G(M, M_0)$ Grin funksiyasi $M$ va $M_0$ nuqtalarga nisbatan simmetrik funksiya, ya'ni

$$G(M, M_0) = G(M_0, M).$$

Grin funksiyasining simmetrikligi [11] darslikda batafsil isbotlangan. Shuning uchun bu yerda uni isbotsiz keltirdik.

$5^0$. Agar Laplas tenglamasi uchun Dirixle masalasining Grin funksiyasi mavjud bo'lsa, u holda bu funksiya yagona bo'ladi.

Faraz qilaylik, ikkita $G_1(M, M_0) = q(M, M_0) + g_1(M, M_0)$ va $G_2(M, M_0) = q(M, M_0) + g_2(M, M_0)$ Grin funksiyasi mavjud bo'lsin. Ularning

$$G_1(M, M_0) - G_2(M, M_0) = g_1(M, M_0) - g_2(M, M_0)$$

ayirmasini tuzamiz. U holda

$$[G_1(M, M_0) - G_2(M, M_0)]|_{M\in S} = [g_1(M, M_0) - g_2(M, M_0)]|_{M\in S} = 0$$

yoki

$$g_1(M, M_0)|_{M\in S} = g_2(M, M_0)|_{M\in S}$$

bo'ladi.

Ekstremum prinsipiga asosan yopiq $\overline{D}$ sohada

$$g_1(M, M_0) = g_2(M, M_0),$$

bundan esa,

$$G_1(M, M_0) \equiv G_2(M, M_0)$$

bo'lishi kelib chiqadi.

Endi Laplas tenglamasi uchun $D \subset R^3$ sohada Dirixle masalasini qaraylik.

DIRIXLE MASALASI. Chekli $D$ sohada aniqlangan uzluksiz va quyidagi

$$u(x, y, z) \in C(\overline{D}) \cap C^2(D); \tag{6}$$

$$\Delta u(x, y, z) = 0, \quad (x, y, z) \in D; \tag{7}$$

$$u|_S = f(P), \quad P \in S, \tag{8}$$

shartlarni qanoatlantiruvchi $u(x, y, z)$ funksiya toping, bu yerda $S$ qaralayotgan $D$ sohaning chegarasi va $f(P)$ esa shu sirtda berilgan etarlicha silliq funksiya.

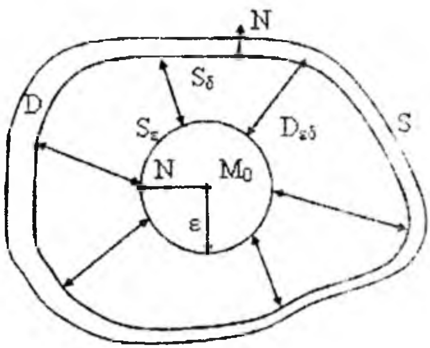
TEOREMA. Agar $D$ sohada Laplas tenglamasi uchun Dirixle masalasining $G(M, M_0)$ Grin funksiyasi mavjud, $f(P)$ funksiya $f(P) \in C(S)$ hamda qaralayotgan $D$ sohaning $S$ sirtida $u(M)$

funksiya normal bo'yicha hosilaga ega bo'lsa, u holda (6)–(8) Dirixle masalasining yechimi mavjud bo'ladi va bu echim

$$u(M_0) = -\iint_S f(M)\frac{\partial G(M, M_0)}{\partial N}\,dS, \tag{9}$$

ko'rinishda aniqlanadi. Bu yerda $G(M, M_0)$ Dirixle masalasining Grin funksiyasi.

ISBOT. Faraz qilaylik, $u(M) = u(x, y, z)$ funksiya (6)–(8) masalaning yechimi va $S$ sirtda normal bo'yicha hosilasi mavjud bo'lsin. $M_0 = (x_0, y_0, z_0) \in D$ ixtiyoriy nuqta bo'lsin. $D$ sohada $u(M) \in C^2(D)$ bo'lgani uchun bu yerda ikkinchi Grin formulasini qo'llab bo'lmaydi. Buning uchun $D$ sohaning $S$ chegarasini $\delta$ ga siljitamiz va hosil bo'lgan sohani $D_\delta$ uning chegarasini esa $S_\delta$ deb belgilaymiz, ya'ni $\overline{D}_\delta \subset D$ bo'lsin.




20 — shakl.

Grin funksiyasining ta'rifiga ko'ra, $M \to M_0$ da $G(M, M_0) \to \infty$ bo'lgani uchun $D_\delta$ sohaning $M_0$ nuqtasini etarlicha kichik $\varepsilon$ radiusli $S_\varepsilon$ sfera bilan ajratib olamiz va $D_\delta$ sohaning qolgan qismini $D_{\varepsilon\delta}$ deb belgilaymiz,
ya'ni $D_{\varepsilon\delta} = D_\delta \backslash \overline{U}(M_0, \varepsilon)$, $S_\varepsilon = \partial U(M_0, \varepsilon)$.

Endi $\overline{D}_{\varepsilon\delta}$ sohada $u(M)$ va $G(M, M_0)$ funksiyalar ikki marta uzluksiz hosilalarga ega, shuning uchun $u(M)$ va $v(M) = G(M, M_0)$

funksiyalarga ikkinchi Grin formulasini qo'llash mumkin. U holda $D_{\varepsilon\delta}$ sohada $\Delta u(M) = 0$, $\Delta_M G(M, M_0) = 0$ bo'ladi va Grin formulasidan

$$\iint\limits_{S_\delta \cup S_\varepsilon} \left( u(M) \frac{\partial G(M, M_0)}{\partial N} - G(M, M_0) \frac{\partial u(M)}{\partial N} \right) dS = 0$$

tenglikka ega bo'lamiz. Bundan esa

$$\iint\limits_{S_\delta} \left( u(M) \frac{\partial G(M, M_0)}{\partial N} - G(M, M_0) \frac{\partial u(M)}{\partial N} \right) dS +$$

$$+ \iint\limits_{S_\varepsilon} \left( u(M) \frac{\partial G(M, M_0)}{\partial N} - G(M, M_0) \frac{\partial u(M)}{\partial N} \right) dS = 0, \qquad (10)$$

bo'ladi.

Oxirgi (10) tenglikda $G(M, M_0)|_{M \in S} = 0$, $u(M)|_S = f(M)$ ekanligidan va $S$ sirtda $\dfrac{\partial u}{\partial N}$ va $\dfrac{\partial G(M, M_0)}{\partial N}$ hosilalar mavjud bo'lgani uchun $\delta \to 0$ limitga o'tamiz, natijada

$$\iint\limits_{S_\varepsilon} \left( u(M) \frac{\partial G(M, M_0)}{\partial N} - G \frac{\partial u}{\partial N} \right) dS = - \iint\limits_{S} f(M) \frac{\partial G(M, M_0)}{\partial N} dS, \qquad (11)$$

ifodani olamiz.

Endi (11) formulada $\varepsilon \to 0$ limitga o'tish uchun uning chap tomonidagi integralda ayrim almashtirishlarni bajaramiz.

$$I = \iint\limits_{S_\delta} \left( u(M) \frac{\partial G(M, M_0)}{\partial N} - G(M, M_0) \frac{\partial u(M)}{\partial N} \right) dS =$$

$$= \iint\limits_{S_\varepsilon} u(M) \frac{\partial G(M, M_0)}{\partial N} dS - \iint\limits_{S_\varepsilon} G(M, M_0) \frac{\partial u(M)}{\partial N} dS = I_1 - I_2$$

Avval birinchi integralni qaraylik.

$$I_1 = \iint\limits_{S_\delta} u(M)\left[\frac{\partial}{\partial N}\left(\frac{1}{4\pi r} + g(M, M_0)\right)\right] dS =$$

$$= \frac{1}{4\pi}\iint\limits_{S_\varepsilon} u(M)\frac{\partial}{\partial N}\left(\frac{1}{r}\right) dS + \iint\limits_{S_\varepsilon} u(M)\frac{\partial g(M, M_0)}{\partial N} dS = I_{11} + I_{12}$$

$M_0$ nuqtaning kichik $\overline{U}(M_0, \varepsilon)$ atrofida $u(M)$ va $\dfrac{\partial g(M, M_0)}{\partial N}$ funksiyalar uzluksiz bo'lgani uchun

$$\lim_{\varepsilon \to 0} I_{12} = 0. \tag{12}$$

Birinchi integral

$$I_{11} = \frac{1}{4\pi}\iint\limits_{S_\varepsilon} u(M)\frac{\partial}{\partial N}\left(\frac{1}{r}\right) dS =$$

$$= \frac{1}{4\pi\varepsilon}\iint\limits_{S_\varepsilon} u(M) dS = \frac{u(\widetilde{M})}{4\pi\varepsilon}\iint\limits_{S_\varepsilon} dS = u(\widetilde{M}),$$

bu yerda $\widetilde{M} \in S_\varepsilon$ va $u(M)$ funksiyaning $M_0$ nuqtada uzluksizligidan

$$\lim_{\varepsilon \to 0} I_{11} = \lim_{\varepsilon \to 0} u(\widetilde{M}) = u(M_0), \tag{13}$$

Endi $I_2$ integralni qaraymiz.

$$I_2 = \iint\limits_{S_\varepsilon} G(M, M_0)\frac{\partial u}{\partial N} dS = \frac{1}{4\pi}\iint\limits_{S_\varepsilon} \frac{1}{r}\frac{\partial u}{\partial N} dS +$$

$$+ \iint\limits_{S_\varepsilon} G(M, M_0)\frac{\partial u}{\partial N} dS = I_{21} + I_{22},$$

$M_0$ nuqtaning kichik $\overline{U}(M_0, \varepsilon)$ atrofida $g(M, M_0)$ va $\dfrac{\partial u}{\partial N}$ funksiyalar uzluksiz bo'lgani uchun

$$\lim_{\varepsilon \to 0} I_{12} = 0, \tag{14}$$

$\overline{U}(M_0,\varepsilon)$ sharda $\left.\dfrac{\partial u}{\partial N}\right|_{S_\varepsilon}$ funksiya chegaralangan, ya'ni $M \in \overline{U}(M_0,\varepsilon)$ uchun

$$\left|\frac{\partial u(M)}{\partial N}\right| \le K$$

tengsizlik o'rinli bo'ladi. U holda

$$|I_{21}| \le \frac{1}{4\pi}\iint\limits_{S_\varepsilon}\left|\frac{1}{r}\frac{\partial u}{\partial N}\right|dS \le \frac{1}{4\pi\varepsilon}\iint\limits_{S_\varepsilon}\left|\frac{\partial u}{\partial N}\right|dS \le$$

$$\le \frac{K}{4\pi\varepsilon}\iint\limits_{S_\varepsilon}dS = \frac{K}{4\pi\varepsilon}4\pi\varepsilon^2 = K\varepsilon.$$

bo'ladi. Buning $\varepsilon \to 0$ dagi limiti

$$\lim_{\varepsilon\to 0} I_{21} = 0, \tag{15}$$

teng.

Demak, (12) – (15) tengliklardan

$$\lim_{\varepsilon\to 0} I = \lim_{\varepsilon\to 0}(I_1 - I_2) = u(M_0), \tag{16}$$

bo'ladi.

Shunday qilib, (11) formulada (16) tenglikni inobatga olib, $\varepsilon \to 0$ da limitga o'tsak, (9) formulaning o'rinli ekanligi kelib chiqadi.

Biz yuqorida (9) formulani $u(M_0)$ funksiya (6)–(8) Dirixle masalasining yechimi bo'lsin, deb hosil qildik. Shuni aytish muhimki, (9) formula bilan aniqlangan $u(M_0)$ funksiya Laplas tenglamasini va chegaraviy shartni qanoatlantiradi.

Grin funksiyasining xossasidan $D\backslash M$ sohada

$$\Delta_{M_0}G(M,M_0) = 0,$$

bo'ladi. $D$ sohada $M_0$ nuqtaning ixtiyoriy ekanligidan va garmonik funksiyaning xossasidan

$$\Delta u(M_0) = -\iint\limits_{S} f(M)\Delta_{M_0}\left(\frac{\partial}{\partial N}G(M,M_0)\right)dS =$$

$$= -\iint_S f(M)\frac{\partial}{\partial N}\Big(\Delta_{M_0}G(M, M_0)\Big)dS = 0$$

kelib chiqadi. Demak, (9) funksiya Laplas tenglamasini qanoatlantirar ekan.

Agar $S$ sirt etarlicha silliq bo'lsa, u holda (9) funksiyadan

$$\lim_{M_0\to P_0} u(M_0) = f(P_0), \quad P_0 \in S, \tag{17}$$

bo'lishini ko'rsatish mumkin.

Agar (9) formulada $f(M) = 1$ bo'lsa, u holda Grin funksiyasining xossasi va ekstremum prinsipiga asosan $u(M_0) = 1$ bo'ladi. U holda (9) formuladan

$$\iint_S \frac{\partial G(P, M_0)}{\partial N}dS = -1. \tag{18}$$

tenglikni olamiz. (18) tenglikdan va (9) formuladan foydalanib, quyidagi

$$u(M_0) - f(P) = -\iint_S [f(M) - f(P)]\frac{\partial G}{\partial N}dS$$

ayirmani tuzamiz.

$S$ sirtdagi $P_0 \in S$ nuqtani $\delta$ radiusli shar bilan ajratib olamiz. $S$ sirtning $\delta$ radiusli shar ichidagi qismi $\sigma$ bo'lsin. $\delta$ radiusni shunday tanlaymizki, unda $M \in \sigma$ bo'lsin. $f$ funksiyaning uzluksizligidan

$$|f(M) - f(P)| \le \frac{\varepsilon}{2}, \qquad \varepsilon > 0, \tag{19}$$

tengsizlik o'rinli bo'ladi. U holda

$$u(M_0) - f(P) = -\iint_\sigma [f(M) - f(P)]\frac{\partial G}{\partial N}dS -$$

$$-\iint_{S\setminus\sigma} [f(M) - f(P)]\frac{\partial G}{\partial N}dS = J_1 + J_2$$

integrallarni baholaymiz.

Birinchi integral (18) va (19) formulalar yordamida quyidagicha

$$|J_1| \le \iint_{\sigma} |f(M) - f(P)| \left|\frac{\partial G}{\partial N}\right| dS \le$$

$$\le \frac{\varepsilon}{2} \iint_{\sigma} \left|\frac{\partial G}{\partial N}\right| dS \le \frac{\varepsilon}{2}$$

baholanadi. $f(M)$ funksiya $S$ uzluksiz bo'lganidan bu funksiya $|f(M)| \le K$ chegaralangan, u holda $J_2$ integral uchun

$$|J_2| \le 2K \iint_{S \setminus \sigma} \left|\frac{\partial G}{\partial N}\right| dS \le \frac{\varepsilon}{2}$$

baho o'rinli ekanligini ko'rishimiz mumkin. Shunday qilib,

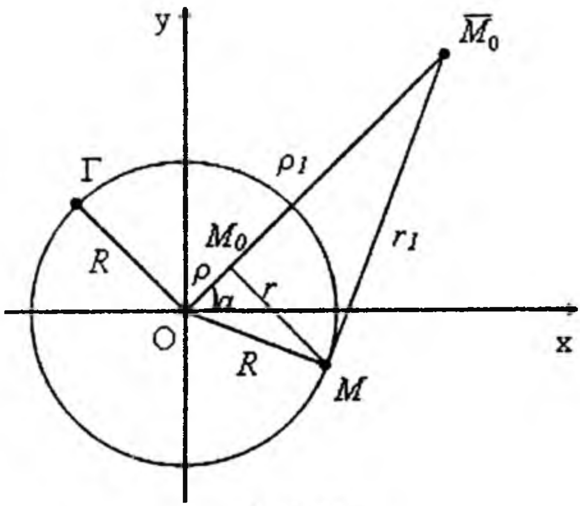
$$|u(M_0) - f(P_0)| \le \varepsilon$$

bo'lar ekan. Bundan (17) limitning o'rinli ekani kelib chiqadi.

Demak, Laplas tenglamasi uchun (6)–(8) Dirixle masalasining yechimi Grin funksiyasi yordamida (9) formula bilan ifodalanishi isbotlandi.

## Doira uchun Grin funksiyasini qurish

Faraz qilaylik, $D_R$ markazi koordinatalar boshida bo'lgan $R$ radiusli doira va uning chegarasi $S_R$ bo'lsin. Doira ichidan ixtiyoriy $M(x_0, y_0)$ nuqta olamiz va $\rho = |OM_0|$ deb belgilash kiritamiz. Endi $M_0$ nuqtaga $S_R$ aylanaga nisbatan simmetrik bo'lgan $\overline{M}_0$ nuqtani ko'ramiz. Bu nuqta $OM_0$ to'g'ri chiziqda $S_R$ aylanadan tashqarida koordinata boshidan $\rho_1$ masofada yotadi, bunda $\rho\rho_1 = R^2$.




21–shakl.

$D_R$ doirada $M_0$ nuqta bilan ustma–ust tushmaydigan ixtiyoriy $M(x, y)$ nuqta olamiz.

Faraz qilaylik, $|MM_0| = r$, $|M\overline{M}_0| = r_1$, $\rho_1 = |O\overline{M}_0|$ bo'lsin va $M$ nuqta aylanada yotgan holda $r$ va $r_1$ orasidagi munosabatni aniqlaylik. $OM_0M$ va $O\overline{M}_0M$ uchburchaklarning o'xshashligidan

$$r \cdot R = \rho \cdot r_1, \tag{20}$$

tenglik o'rinli bo'ladi. Grin funksiyasining ta'rifiga ko'ra.

$$g(M, M_0)|_{S_R} = -\frac{1}{2\pi} \ln \frac{1}{r}, \quad M \in S_R$$

shartni qanoatlantiradigan $g(M, M_0)$ regulyar qismini topishimiz zarur bo'ladi.

Yuqoridagi (20) tenglikdan $\frac{1}{r} = \frac{R}{\rho r_1}$ ekanligi kelib chiqadi. Bundan esa $g(M, M_0)$ regulyar qismini

$$g(M, M_0) = -\frac{1}{2\pi}\ln\frac{R}{\rho r_1}$$

deb olishimiz zarur bo'ladi. U holda Grin funksiyasi

$$G(M, M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r} - \ln\frac{R}{\rho r_1}\right), \tag{21}$$

ko'rinishga keladi.

Haqiqatdan ham, (21) funksiya Grin funksiyasining ta'rifida keltirilgan shartlarni qanoatlantiradi. Endi doirada Dirixle masalasining yechimini olishimiz uchun (21) Grin funksiyadan normal bo'yicha hosilani hisoblaymiz.

$$\frac{\partial G(M, M_0)}{\partial N} = \frac{\partial}{\partial N}\left(\ln\frac{1}{r}\right) - \frac{\partial}{\partial N}\left(\ln\frac{R}{\rho r_1}\right); \tag{22}$$

bu yerda

$$\frac{\partial}{\partial N}\left(\ln\frac{1}{r}\right) = -\frac{1}{r}\left[\frac{x - x_0}{r}\cos(N, x) + \frac{y - y_0}{r}\cos(N, y)\right] =$$

$$= -\frac{1}{r}[\cos(r, x)\cos(N, x) + \cos(r, y)\cos(N, y)] = -\frac{1}{r}\cos(r, N);$$

xuddi shuningdek

$$\frac{\partial}{\partial N}\left(\ln\frac{R}{\rho r_1}\right) = -\frac{1}{r_1}\cos(r_1, N)$$

bo'ladi.

Endi $OM_0M$ uchburchakdan $\rho^2 = R^2+r^2-2Rr\cos(r,N)$ tenglikni olamiz, $O\overline{M}_0M$ uchburchakda esa $\rho_1^2 = R^2+r_1^2-2Rr_1\cos(r_1,N)$ bo'ladi. Bu tengliklardan

$$\cos(r,N) = \frac{R^2+r^2-\rho^2}{2Rr}, \qquad \cos(r_1,N) = \frac{R^2+r_1^2-\rho_1^2}{2Rr_1}.$$

Topilgan ifodalarni (22) formulaga ko'yamiz va (20) tenglikni inobatga olib, uni soddalashtirsak, natijada Grin funksiyasining normal bo'yicha hosilasi

$$\frac{\partial G(M,M_0)}{\partial N} = -\frac{1}{r}\cos(r,N) + \frac{1}{r_1}\cos(r_1,N) =$$

$$= -\frac{R^2+r^2-\rho^2}{2Rr^2} + \frac{R^2+r_1^2-\rho_1^2}{2Rr_1^2} =$$

$$= \frac{2R^2r^2 - R^2r_1^2 - \rho_1^2r^2}{2Rr^2r_1^2} = -\frac{R^2-\rho^2}{Rr^2}$$

ekanligi kelib chiqadi.

Endi buni (9) formulaga qo'yib, Laplas tenglamasi uchun Dirixle masalasining yechimini quyidagi

$$u(M_0) = \frac{1}{2\pi R}\int\limits_S f(M)\,\frac{R^2-\rho^2}{r^2}\,ds \tag{23}$$

ko'rinishda hosil qilamiz. Demak, $xOy$ tekisligida Laplas tenglamasi uchun Dirixle masalasining yechimi Grin funksiyasi yordamida (23) formula bilan aniqlandi.

Xuddi $n = 3$ bo'lgandagi kabi (23) formula qaralayotgan sohada Laplas tenglamasini va chegaraviy shartni qanoatlantirishini ko'rsatish mumkin.

======== ◇ ========

## Nazorat uchun savollar

1. Qanday ko'rinishdagi tenglamalarga elliptik tipdagi tenglamalar deb ataladi?

2. Tekis elliptik tenglamalar deganda qanday tenglamalarni tushunasiz?

3. Qanday funksiyalar garmonik deyiladi?

4. Laplas tenglamasining fundamental yechimi.

5. Garmonik funksiya uchun ekstremum prinsipi.

6. Ekstremum prinsipidan kelib chiqadigan garmonik funksiyalarning qanday xossalari bor?

7. Elliptik tipdagi tenglamalar uchun asosiy chegaraviy masalalarning qo'yilishi.

8. Dirixle masalasi yechimining yagonaligi va uning turg'unligi qanday isbotlanadi?

9. Dirixle masalasi yechimini ifodalovchi Puasson formulasi qanday xossalarga ega?

10. Dirixle masalasining korrektligi qanday tushuniladi?

11. Laplas tenglamasi uchun tashqi Dirixle masalasi qanday qo'yiladi?

12. Garmonik funksiya uchun Neyman masalalari qanday qo'yiladi?

13. Laplas tenglamasi uchun Neyman masalasi qachon bir qiymatli echiladi?

14. Elliptik operatorlar uchun Grin formulalari.

15. Grin funksiyasi nima va u chegaraviy masalalarni yechishda qanday o'rin tutadi?

## Mustaqil yechish uchun misol va masalalar

5.1. Agar $u(x, y)$ funksiya biror $D$ sohada garmonik bo'lsa, u holda ushbu

$$1)\quad u_x u_y; \qquad 2)\quad x u_x - y u_y; \qquad 3)\quad u_x^2 - u_y^2;$$

funksiyalarning shu $D$ sohada garmonik bo'lishini isbotlang.

5.2. Quyidagi funksiyalar $k$ ning qanday qiymatlarida garmonik funksiya bo'ladi:

$$1)\quad \sin 3x \operatorname{ch} ky; \qquad 2)\quad \frac{1}{|x|^k}, \quad |x|^2 = \sum_{i=1}^{n} x_i^2;$$

5.3. Ushbu $u(x,y) = x^2 - y^2$ funksiyaning $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$ sohada ekstremum nuqtalarini toping.

5.4. Birlik doirada $u(1,\varphi) = f(\varphi)$ chegaraviy shartni qanoatlantiruvchi $u(r,\varphi)$ garmonik funksiyani toping:

1) $f(\varphi) = \cos^2\varphi$; 2) $f(\varphi) = \sin^3\varphi$; 3) $f(\varphi) = \sin^6\varphi + \cos^6\varphi$.

5.5. $R$ radiusli doira ichida $\left.\frac{\partial u}{\partial r}\right|_{r=R} = f(\varphi)$ shartni qanoatlantiruvchi $u(r,\varphi)$ garmonik funksiyani toping:

1) $f(\varphi) = \cos\varphi$; 2) $f(\varphi) = \cos 2\varphi$; 3) $f(\varphi) = \cos^4\varphi$.

5.6. $R$ radiusli $x^2 + y^2 = r^2 < R^2$ doirada ushbu Dirixle masalasini yeching:

$$\Delta u(x,y) = 0, \qquad 0 \leq r < R;$$

$$\left. u(x,y)\right|_{r=R} = 2(x^2 + y).$$

5.7. $R$ radiusli $x^2 + y^2 = r^2 \leq R^2$ doiradan tashqarida ushbu Dirixle masalasini yeching:

$$\Delta u(x,y) = 0, \qquad R < r < \infty;$$

$$\left. u(x,y)\right|_{r=R} = 2x^2 - x + y, \quad |u(x,y)| < \infty.$$

5.8. Chekli $D$ sohada garmonik va $C^1(D \cup S)$ bo'lgan $u(x)$ va $v(x)$ funksiyalar uchun

$$\int_S \left[ v(y)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial v(y)}{\partial \nu}\right] dS_y = 0$$

bo'lishini keltirib chiqaring.

5.9. Chekli $D$ sohada garmonik va $C^1(D \cup S)$ bo'lgan $u(x)$ va $v(x)$ funksiyalar uchun

$$\int_S \frac{\partial u(y)}{\partial \nu} dS_y = 0$$

bo'lishini keltirib chiqaring.

5.10. Laplas tenglamasi uchun Dirixle masalasining $G(x, y)$ Grin funksiyali uchun $G(x, y) = G(y, x)$ tenglikning o'rinli ekanligini isbotlang.

======== ◊ ========

## Nazorat uchun testlardan namunalar

### 1 – nazorat uchun testlar

1. Agar $1 < x < 2$, $3 < y < 4$ bo'lsa, u holda ushbu

$$u_{xx} + y^2 u_{yy} + 2yu_x - 3yu_y = 0$$

tenglamaning tipini aniqlang.
a) giperbolik, b) parabolik,
c) aralash, d) elliptik.

2. Agar $5 < x < 6$, $y = 0$ bo'lsa, u holda ushbu

$$u_{xx} + y^2 u_{yy} + 2yu_x - 3yu_y = 0$$

tenglamaning tipini aniqlang.
a) giperbolik, b) parabolik,
c) aralash, d) elliptik.

3. Agar $1 < x^2 + y^2 < 3$ bo'lsa, u holda ushbu

$$u_{xx} + 2yu_{xy} + y^2 u_{yy} - 3xu_y = 0$$

tenglamaning tipini aniqlang.
a) giperbolik, b) parabolik,
c) aralash, d) elliptik.

4. Agar $2 < x + y < 5$ bo'lsa, u holda ushbu

$$4u_{xx} - 2(x - y)u_{xy} + (1 - xy)u_{yy} - 3xu_x = 0$$

tenglamaning tipini aniqlang.
a) giperbolik, b) parabolik,

c) aralash, d) elliptik.

5. Ushbu

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} - u_x = 0$$

tenglamani qanday almashtirish kanonik ko'rinishga keltiradi?

a) $\xi = x^2 - y^2$, $\eta = x$; b) $\xi = x^2 + y^2$, $\eta = y$;
c) $\xi = xy$, $\eta = x$; d) $\xi = x/y$, $\eta = y$.

6. Ushbu

$$y^2 u_{xx} + x^6 u_{yy} - u_x + 3u_y = 0$$

tenglamani qanday almashtirish kanonik ko'rinishga keltiradi?

a) $\xi = 2y^2$, $\eta = x^2$; b) $\xi = 2y^2 + x^4$, $\eta = 2y^2 - x^4$;
c) $\xi = x^4 y^2$, $\eta = x$; d) $\xi = 2x^2$, $\eta = y^4$.

7. Ushbu

$$e^{2x} u_{xx} + 2u_{xy} + 2e^{-2x} u_{yy} + u_x = 0$$

tenglamani qanday almashtirish kanonik ko'rinishga keltiradi?

a) $\xi = 2y + e^{-2x}$, $\eta = e^{-2x}$; b) $\xi = 2y^2 + x^4$, $\eta = 2y^2 - x^4$;
c) $\xi = x^4 y^2$, $\eta = x$; d) $\xi = 2x^2$, $\eta = y^4$.

8. Ushbu

$$9y^4 u_{xx} + 6y^2 \sin x u_{xy} + \sin^2 x u_{yy} - u_y + u = 0$$

tenglamani qanday almashtirish kanonik ko'rinishga keltiradi?

a) $\xi = \cos x + y^3$, $\eta = x$; b) $\xi = 2y^2 + x^4$, $\eta = 2y^2 - x^4$;
c) $\xi = x^4 y^2$, $\eta = x$; d) $\xi = 2x^2$, $\eta = y^4$.

9. Ushbu berilgan

$$u_{xx} + 6x^3 u_{xy} + 9x^6 u_{yy} - u_x + 3u_y = 0$$

tenglamaning kanonik shaklini belgilang.

a) $\frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

b) $\frac{\partial^2 u}{\partial \xi \partial \eta} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

c) $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

d) $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0.$

10. Ushbu berilgan

$$\sin^2 x u_{xx} + \cos^4 y u_{yy} + 9x u_x - u_y - u = 0$$

tenglamaning kanonik shaklini belgilang.

a) $\frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

b) $\frac{\partial^2 u}{\partial \xi \partial \eta} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

c) $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

d) $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0.$

11. Ushbu berilgan

$$\operatorname{tg}^2 x u_{xx} - 2y \operatorname{tg} x u_{xy} + y^2 u_{yy} - u_y = 0$$

tenglamaning kanonik shaklini belgilang.

a) $\frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

b) $\frac{\partial^2 u}{\partial \xi \partial \eta} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

c) $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0;$

d) $\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0.$

12. Quyidagi tenglamalarning qaysi biri

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + f\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) = 0$$

kanonik shaklga ega bo‘ladi?

a) $x^2u_{xx} + 2xu_{xy} + u_{yy} + u_y = 0$;
b) $x^2u_{xx} + y^2u_{yy} - 2u_x = 0$;
c) $u_{xx} + \sin^2 xu_{yy} + 3u_y = 0$;
d) $u_{xx} - 4u_{xy} + 3u_{yy} - 4u_y = 0$.

13. Quyidagi tenglamalarning qaysi biri

$$\frac{\partial^2 u}{\partial\eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) = 0$$

kanonik shaklga ega bo‘ladi?

a) $x^2u_{xx} + 2xu_{xy} + u_{yy} - u_x = 0$;
b) $x^2u_{xx} + y^2u_{yy} + 3u_y = 0$;
c) $u_{xx} + \sin^2 xu_{yy} - u_x = 0$;
d) $u_{xx} - 4u_{xy} + 3u_{yy} - 4u_y = 0$.

14. Quyidagi tenglamalarning qaysi biri

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) = 0$$

kanonik shaklga ega bo‘ladi?

a) $x^2u_{xx} + y^2u_{yy} + 2xu_x + u_y = 0$;
b) $u_{xx} + 4u_{xy} - 5u_{yy} = 0$;
c) $\sin^2 xu_{xx} + 2\sin xu_{xy} + u_{yy} + \operatorname{tg} xu_x = 0$;
d) $u_{xx} - 2x^2u_{xy} + x^4u_{yy} + xu_x = 0$.

15. Quyidagi tenglamalarning qaysi biri

$$\frac{\partial^2 u}{\partial\xi\partial\eta^2} + f\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right) = 0$$

kanonik shaklga ega bo'ladi?

a) $u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y - u = 0$;
b) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$;
c) $\sin^2 x u_{xx} + 2\sin x u_{xy} + u_{yy} + \operatorname{tg} x u_x = 0$;
d) $u_{xx} - 2x^2 u_{xy} + x^4 u_{yy} + x u_x = 0$.

## 2 – nazorat uchun testlar

1. Ushbu berilgan $u_{xy} + \frac{2}{y} u_x = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = \varphi(x)e^{-2}$;
b) $u(x,y) = y^{-2}\varphi(y) + \psi(x)$;
c) $u(x,y) = y^{-2}\varphi(x) + \psi(y)$;
d) $u(x,y) = x^{-2}\varphi(y) + \psi(y)$.

2. Ushbu berilgan $u_{xy} - 3u_y = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = x^3\varphi(y) + \psi(x)$;
b) $u(x,y) = e^{3y}\varphi(x) + \psi(y)$;
c) $u(x,y) = y^{-2}\varphi(x) + \psi(y)$;
d) $u(x,y) = e^{3x}\varphi(y) + \psi(x)$.

3. Ushbu berilgan $u_{yy} = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = x\varphi(y) + \psi(x)$;
b) $u(x,y) = y\varphi(x) + \psi(y)$;
c) $u(x,y) = y\varphi(x) + \psi(x)$;
d) $u(x,y) = x\varphi(y) + \psi(y)$.

4. Ushbu berilgan $u_{yy} - \frac{2}{y} u_y = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = x^2\varphi(y) + \psi(y)$;
b) $u(x,y) = y^3\varphi(x) + \psi(x)$;

c) $u(x,y) = e^y\varphi(x) + \psi(x)$;
d) $u(x,y) = x\varphi(y) + \psi(y)$.

5. Ushbu berilgan $u_{yy} + 2u_y = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = e^{-2y}\varphi(x) + \psi(x)$;
b) $u(x,y) = y^2\varphi(x) + \psi(y)$;
c) $u(x,y) = e^{-2y}\varphi(x) + \psi(y)$;
d) $u(x,y) = y^{-2}\varphi(x) + \psi(x)$.

6. Ushbu $u(x,y) = e^{-2x}\varphi(y) + \psi(x)$ funksiya quyidagi qaysi tenglamaning umumiy yechimi bo'ladi?
a) $u_{yy} + 2u_y = 0$; b) $u_{xy} + 2u_y = 0$;
c) $u_{xy} - 2xu_y = 0$; d) $u_{xy} + 3u_y = 0$.

7. Ushbu $u(x,y) = \varphi(x) + \psi(y)$ funksiya quyidagi qaysi tenglamaning umumiy yechimi bo'ladi?
a) $u_{xy} + 2u_y = 0$; b) $u_{xy} = 0$;
c) $u_{xy} - 2xu_y = 0$; d) $u_{xy} + 3u_y = 0$.

8. Ushbu $u(x,y) = y^3\varphi(x) + \psi(y)$ funksiya quyidagi qaysi tenglamaning umumiy yechimi bo'ladi?
a) $u_{xy} + 3u_y = 0$; b) $u_{yy} - 2u_y = 0$;
c) $u_{xy} - \frac{3}{y}u_x = 0$; d) $u_{yy} - \frac{3}{y}u_y = 0$.

9. Ushbu $u(x,y) = e^{2x}\varphi(y) + \psi(x)$ funksiya quyidagi qaysi tenglamaning umumiy yechimi bo'ladi?
a) $u_{xx} - 2u_x = 0$; b) $u_{xy} - 2u_y = 0$;
c) $u_{yy} - \frac{2}{y}u_y = 0$; d) $u_{xy} + \frac{3}{x}u_y = 0$.

9. Ushbu $4xu_{xx} - yu_{xy} + 7u_x = 0$ tenglamaning umumiy yechimini toping.
a) $u(x,y) = y^{-2/3}\varphi(x^3y^2) + \psi(y)$;
b) $u(x,y) = x^{5/3}\varphi(x^2y^3) + \psi(x)$;
c) $u(x,y) = y^3\varphi(xy^4) + \psi(y)$;

d) $u(x,y) = y^{-7/3}\varphi(x^2y^3) + \psi(x)$.

10. Ushbu $u_{xy} + u_{yy} = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = \varphi(x-y) + \psi(x)$;
b) $u(x,y) = e^{-x}\varphi(x-y) + \psi(x)$;
c) $u(x,y) = x^2\varphi(x-y) + \psi(y)$;
d) $u(x,y) = e^{x}\varphi(x-y) + \psi(y)$.

11. Ushbu $u_{xy} + \frac{2}{x}u_y = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = x^2\varphi(y) + \psi(x)$;
b) $u(x,y) = x^2\varphi(y) + \psi(x)$;
c) $u(x,y) = x^{-5/6}\varphi(xy^4) + \psi(x)$;
d) $u(x,y) = y^{-6/5}\varphi(x^4y) + \psi(y)$.

12. Ushbu $u_{xx} + 2u_x = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = y^{5/2}\varphi(xy) + \psi(x)$;
b) $u(x,y) = x^{2/3}\varphi(x^2y) + \psi(y)$;
c) $u(x,y) = \varphi(y)e^{-2x} + \psi(y)$;
d) $u(x,y) = y^2\varphi(xy) + \psi(y)$.

13. Ushbu $2u_{xy} - u_{yy} - 3u_y = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = y^3\varphi(xy^2) + \psi(y)$;
b) $u(x,y) = e^{3x/2}f(x+2y) + g(x)$;
c) $u(x,y) = x^2\varphi(xy^2) + \psi(x)$;
d) $u(x,y) = y^{-2}\varphi(x^2y) + \psi(y)$.

14. Ushbu $u_{xy} - 3u_{yy} - 5u_y = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = x^2\varphi(x^3y) + \psi(x)$;
b) $u(x,y) = x\varphi(x^3y^2) + \psi(y)$;
c) $u(x,y) = e^{5x}\varphi(3x+y) + \psi(x)$;

d) $u(x,y) = y^{-2}\varphi(x^2y) + \psi(y)$.

15. Ushbu $xu_{xx} + yu_{xy} = 0$ tenglamaning umumiy yechimini toping.

a) $u(x,y) = x\varphi(xy) + \psi(x)$;
b) $u(x,y) = y\varphi(x^2y) + \psi(y)$;
c) $u(x,y) = x^2\varphi(xy^2) + \psi(x)$;
d) $u(x,y) = y\varphi\left(\frac{y}{x}\right) + \psi(y)$.

## 3 – nazorat uchun testlar

1. Ushbu

$$u(1,y) = 3y^3 + 5, \qquad u_x(1,y) = 3y + 1$$

boshlang'ich shartlarni quyidagi funksiyalardan qaysi biri qanoatlantiradi?

a) $u(x,y) = 5 - \frac{9}{2}y + \ln x + \frac{9}{2}x^{2/3}y + 3y^2$;
b) $u(x,y) = 9y + 3y^3 - 2\ln x - \frac{9}{2}x^{2/3}y + 5$;
c) $u(x,y) = 5x^2y^2 + 5 - \frac{5}{9}x^{3/2}y + 3y^2 - x^{5/9}y + 3y$;
d) $u(x,y) = 3y + 5 - x^{2/3}y + 3y^2 - 2\ln x + 5$.

2. Ushbu

$$u(x,1) = 2x^2, \qquad u_y(x,1) = 3x + 1$$

boshlang'ich shartlarni quyidagi funksiyalardan qaysi biri qanoatlantiradi?

a) $u(x,y) = 5x + \frac{3}{2}xy^2 - \frac{3}{2}xy^3 + 1 + 3x^2$;
b) $u(x,y) = \frac{3}{2}xy^2 - \frac{3}{4}y^{4/3} + 2x^2 - \frac{3}{2}x + \frac{3}{4}$;
c) $u(x,y) = 9y^2 + 5xy - 5x^3y^2 + 3xy^2 - 3x^{5/9}y + 3y$;

d) $u(x,y) = \frac{9}{5}\,y + 3y^3 - \frac{9}{5}\,x^{2/3}y + 3y^2 - 2\ln x + 5.$

3. Ushbu

$$u(x,1) = 2x^3, \qquad u_y(x,1) = 3x$$

boshlang'ich shartlarni quyidagi funksiyalardan qaysi biri qanoatlantiradi?

a) $u(x,y) = 5 - \frac{9}{2}\,y + \ln x + \frac{9}{2}\,x^{2/3}y + 3x^2;$

b) $u(x,y) = \frac{2}{9}\,x + 3x^3 - 2\ln x - \frac{9}{2}\,x^{2/3}y;$

c) $u(x,y) = 5x^2 + 12 - \frac{5}{9}\,x^{3/2}y + 3x^2 - 3x^{5/9}y + 3x;$

d) $u(x,y) = \frac{9}{4}\,xy^{4/3} + 2x^3 - \frac{9}{4}\,x.$

4. Ushbu

$$u(1,y) = 1 + 2y, \qquad u_x(1,y) = 3y^2$$

boshlang'ich shartlarni quyidagi funksiyalardan qaysi biri qanoatlantiradi?

a) $u(x,y) = 5 - \frac{9}{2}\,y + \ln x + x^{2/3}y + 3x^2;$

b) $u(x,y) = \frac{3}{2}\,x^2y^2 + 1 + 2y - \frac{3}{2}\,y^2;$

c) $u(x,y) = 5x^2 + 12 - \frac{5}{9}\,x^{3/2}y + 3x^2 - 3x^{5/9}y + 3x;$

d) $u(x,y) = \frac{9}{4}\,xy^{4/3} + 2x^3 - \frac{9}{4}\,x.$

5. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = y^{-5/3}\,f(x^3y^4) + g(y)$$

bo'lsa, u holda $u(1,y) = 3y^5$, $u_x(1,y) = 3y^4$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = 5 - \frac{9}{2}\,y + \ln x + \frac{9}{2}\,x^{2/3}y + 3y^2;$

b) $u(x,y) = 9y + 3y^3 - 2\ln x - \frac{9}{2}\,x^{2/3}y + 5;$

c) $u(x,y) = 5x^2y^2 + 5 - \frac{5}{9}\,x^{3/2}y + 3y^2 - x^{5/9}y + 3y;$

d) $u(x,y) = \frac{12}{17}\,y^4\left(x^{17/4} - 1\right) + 3y^5.$

6. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = x^{-2}\,f(x^3y^2) + g(x)$$

bo'lsa, u holda $u(x,1) = 3x^2 + 2$, $u_y(x,1) = x^4$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = \frac{1}{4}\,x^4(y^4 - 1) + 3x^2 + 2;$

b) $u(x,y) = 9y + 3y^3 - 2\ln x - \frac{9}{2}\,x^{2/3}y + 5;$

c) $u(x,y) = 5x^2y^2 + 5 - \frac{5}{9}\,x^{3/2}y + 3y^2 - x^{5/9}y + 3y;$

d) $u(x,y) = \frac{2}{7}\,y^4\left(x^{7/4} - 1\right) + 3y^5.$

7. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = \ln y\, f(xy) + g(xy)$$

bo'lsa, u holda $u(x,1) = 2x^2$, $u_y(x,1) = x^2$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x^4(y^4 - 1) + 3x^2 + 2;$

b) $u(x,y) = 9y + 3y^3 - 2\ln x - \frac{9}{2}\,x^{2/3}y + 5;$

c) $u(x,y) = 5x^2y^2 + 5 - \frac{5}{9}\,x^{3/2}y + 3y^2 - x^{5/9}y + 3y;$

d) $u(x,y) = \frac{x^2}{y} + x^2y^2.$

8. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = \frac{1}{xy}\,f(x^2y) + g(xy)$$

bo'lsa, u holda $u(x,1) = 1 + 2x^2$, $u_y(x,1) = 4x^2$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x^4(y^4 - 1) + 3x^2 + 2$;

b) $u(x,y) = 2\ln x - \dfrac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = 1 + 2x^2y^2$;

d) $u(x,y) = \dfrac{x^2}{2y} + 2x^2y^2$.

9. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = \sqrt{xy}\, f\left(\frac{x}{y}\right) + g(xy)$$

bo'lsa, u holda $u(x,1) = 2\sqrt{x}$, $u_y(x,1) = \sqrt{x}$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = 2\sqrt{xy}$;

b) $u(x,y) = 2\ln xy + 2\sqrt{xy} - 5xy$;

c) $u(x,y) = 1 + xy^2$;

d) $u(x,y) = \sqrt{xy} + 2x^2y^2$.

10. Agar biror tenglamaning umumiy yechimi

$$u(x,y) = \frac{1}{xy} f(x^2y) + g(xy)$$

bo'lsa, u holda $u(x,1) = 2x$, $u_y(x,1) = x$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x^4(y^4 - 1) + 3x^2 + 2$;

b) $u(x,y) = 2\ln x - \dfrac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = x(1 + y)$;

d) $u(x,y) = \dfrac{x^2}{2y} + 2x^2y^2$.

11. Agar $\xi = 2x + 3y$, $\eta = 5x - 4y$ bo'lganda berilgan tenglamaning umumiy yechimi

$$u(\xi,\eta) = e^{\xi\eta} f(\xi) + g(\eta)$$

bo'lsa, u holda $u(0,y) = 1$, $u_x(0,y) = 1$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x^4(y^4 - 1) + 3x^2 + 2$;

b) $u(x,y) = 2\ln x - \frac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = x(1+y)$;

d) $u(x,y) = \frac{1}{23}e^{23x} + \frac{22}{23}$.

12. Agar $\xi = 5x - 6y$, $\eta = x + 2y$ bo'lganda berilgan tenglamaning umumiy yechimi

$$u(\xi,\eta) = e^{2\xi}f(\eta) + g(\xi)$$

bo'lsa, u holda $u(0,y) = 4$, $u_x(0,y) = 1$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x(y^4 + 4) + 3x^2 + 4$;

b) $u(x,y) = 2\ln x - \frac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = x(1+y)$;

d) $u(x,y) = \frac{1}{16}e^{16x} + \frac{63}{16}$.

13. Agar $\xi = 3x - 4y$, $\eta = 5x + 6y$ bo'lganda berilgan tenglamaning umumiy yechimi

$$u(\xi,\eta) = e^{-3\eta}f(\xi) + g(\eta)$$

bo'lsa, u holda $u(0,y) = 2$, $u_x(0,y) = 3$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = \frac{40}{19} - \frac{2}{19}e^{-57x/2}$;

b) $u(x,y) = 2\ln x - \frac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = x(1+y)$;

d) $u(x,y) = x^4(y^4 - 1) + 3x^2 + 2$.

14. Agar $\xi = 2x + 3y$, $\eta = 4x - 5y$ bo'lganda berilgan tenglamaning umumiy yechimi

$$u(\xi,\eta) = e^{2\eta}f(\xi) + g(\eta)$$

bo'lsa, u holda $u(0,y) = 1$, $u_x(0,y) = 2$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x(1+y)$;

b) $u(x,y) = 2\ln x - \frac{9}{2}x^{2/3}y + 5$;

c) $u(x,y) = \frac{3}{22}e^{44x/3} + \frac{19}{22}$;

d) $u(x,y) = x^4(y^4-1) + 3x^2 + 2$.

15. Agar $\xi = x^3y^4$, $\eta = x$ bo'lganda berilgan tenglamaning umumiy yechimi

$$u(\xi,\eta) = \eta^{-4/3}f(\xi) + g(\eta)$$

bo'lsa, u holda $u(x,1) = 4x^2$, $u_y(x,1) = 6x$ boshlang'ich shartlarni qanoatlantiruvchi $u(x,y)$ funksiyani toping.

a) $u(x,y) = x(1+y)$;

b) $u(x,y) = \frac{24}{7}xy^{7/4} + 4x^2 - \frac{24}{7}x$;

c) $u(x,y) = 2\ln x - \frac{9}{2}x^{2/3}y + 5$;

d) $u(x,y) = x^4(y^4-1) + 3x^2 + 2$.

## 4 – nazorat uchun testlar

1. Laplas tenglamasining $x = r\cos\varphi$, $y = r\sin\varphi$ qutb koordinatalaridagi ifodasi qaysi javobda to'g'ri berilgan?

a) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} = 0$; b) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$;

c) $\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} = 0$; d) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$;

2. Agar $u(x) = u(x_1, x_2, \ldots, x_n)$ garmonik funksiya bo'lsa, quyidagi funksiyalar garmonik bo'ladmi?

a) $u(x+h)$, $h$ – o'zgarmas vektor;

b) $x_1u_{x_1} - x_2u_{x_2}$, $n = 2$;

c) $\dfrac{\partial u}{\partial x_1}\dfrac{\partial u}{\partial x_2}$, $n = 2$;
d) To'g'ri javob a) va v).

3. Ushbu $x^3 + axy^2$ funksiya $a$ ning qanday qiymatida garmonik bo'ladi.
a) $a = 1$; b) $a = -1$; c) $a = -3$; d) $a = -2$.

4. Ushbu $x^2 + y^2 + az^2$ funksiya $a$ ning qanday qiymatida garmonik bo'ladi.
a) $a = 1$; b) $a = -2$; c) $a = -1$; d) $a = -3$.

5. Ushbu $\sin 3x_1 \operatorname{ch} ax_2$ funksiya $a$ ning qanday qiymatida garmonik bo'ladi.
a) $a = \pm 3$; b) $a = -2$; c) $a = 1$; d) $a = \pm 2$.

6. Laplas tenglamasining fundamental yechimini aniqlang.
a) $E(x, y) = -\ln |x - y|, \quad n = 2$;
b) $E(x, y) = \dfrac{1}{2a\sqrt{\pi t}} \exp\left\{-\dfrac{(x-\xi)^2}{4a^2 t}\right\}$;
c) $E(x, y) = (n-2)^{-1} |x - y|^{2-n}, \quad n > 2$;
d) To'g'ri javob a) va v).

7. Agar $x = (x_1, x_2, x_3)$ va $y = (y_1, y_2, y_3)$ bo'lsa, u holda ushbu $E(x, y) = |x - y|^{-1}$ funksiya qaysi tenglamaning fundamental yechimi bo'ladi?

a) $\sum\limits_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2} = 0$; b) $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) = 0$;
c) $\sum\limits_{i=1}^{3} \dfrac{\partial^2 u}{\partial x_i^2} = 0$; d) $\sum\limits_{i=1}^{4} \dfrac{\partial^2 u}{\partial x_i^2} = 0$;

8. Quyidagi funksiyalardan qaysi biri garmonik bo'ladi?
a) $v(x, y) = x^2 + y^2$;
b) $v(x, y) = x^2 - y^2$;
c) $v(x, y) = 2x^2 + 3y^2 + 7$;

d) $v(x,y) = 5xy + 4y^4 - 9$.

9. Laplas tenglamasi uchun quyida keltirilgan qaysi masala nokorrekt qo'yilgan?

a) Koshi masalasi; b) Dirixle masalasi;
c) Neyman masalasi; d) Gursa masalasi.

10. Laplas tenglamasi uchun quyida keltirilgan qaysi masala korrekt qo'yilgan?

a) Koshi masalasi;
b) Dirixle masalasi;
c) Neyman masalasi;
d) To'g'ri javob b) va c).

11. Elliptik tipdagi tenglama uchun qanday masala korrekt qo'yilgan deyiladi?

a) masalaning echimi mavjud bo'lsa;
b) masalaning echimi yagona bo'lsa;
c) masalaning echimi berilganlarga uzluksiz bog'liq bo'lsa;
d) To'g'ri javob: a), b) va c).

12. Issiqlik tarqalish tenglamasining fundamental yechimini aniqlang.

a) $E(x,y) = -\ln|x-y|, \quad n=2$;
b) $E(x,y) = \dfrac{1}{2a\sqrt{\pi t}}\exp\left\{-\dfrac{(x-\xi)^2}{4a^2t}\right\}$;
c) $E(x,y) = (n-2)^{-1}\,|x-y|^{2-n}, \quad n>2$;
d) To'g'ri javob a).

13. Ushbu $u_t = u_{xx}$ tenglamaning $u(x,0) = \sin lx$ boshlang'ich shartni qanoatlantiruvchi $u(x,t)$ yechimini toping.

a) $u(x,t) = \cos lx \sin t$;
b) $u(x,t) = \sin lx$;
c) $u(x,t) = e^{-l^2t}\sin lx$;

d) To'g'ri javob a).

14. Ushbu $u_t = u_{xx} + u_{yy}$ issiqlik tarqalish tenglamasining $u(x, y, 0) = \sin kx \cos ly$ boshlang'ich shartni qanoatlantiruvchi $u(x, t)$ yechimini toping.

a) $u(x, t) = e^{-(k^2+l^2)t} \sin kx \cos ly$;
b) $u(x, t) = \sin lx$;
c) $u(x, t) = e^{-l^2 t} \sin kx \cos ly$;
d) To'g'ri javob a).

15. Ushbu $u_t = u_{xx} + u_{yy}$ issiqlik tarqalish tenglamasining $u(x, y, 0) = \sin kx + \cos ly$ boshlang'ich shartni qanoatlantiruvchi $u(x, t)$ yechimini toping.

a) $u(x, t) = \sin kx + \cos ly$;
b) $u(x, t) = e^{-k^2 t} \sin kx + e^{-l^2 t} \cos ly$;
c) $u(x, t) = e^{-l^2 t} \sin kx \cos ly$;
d) $u(x, t) = e^{-l^2 t}(\sin kx + \cos ly)$;

## Berilgan misol va masalalarning javoblari.

### I–bobda berilgan misollarning javoblari:

1.1. 1) Tenglama bo'ladi: $u_{xx} = 0, \quad u_{xy} = 0$;
2) Tenglama emas: $0 = 0$;
3) Tenglama emas: $u = 2$;
4) Tenglama bo'ladi: $5u_x - 9u = 0$;
5) Tenglama emas: $4u = -11$;

1.2. 1) Ikkinchi tartibli;
2) Ikkinchi tartibli;
3) Ikkinchi tartibli;
4) Ikkinchi tartibli;
5) Ikkinchi tartibli;

1.3. 1) Bir jinsli chiziqli tenglama;
2) Chiziqsiz tenglama;
3) Bir jinsli kvazichiziqli tenglama;
4) Bir jinsli bo'lmagan kvazichiziqli tenglama;
5) Bir jinsli bo'lmagan chiziqli tenglama;

1.5. $u(x,y) = \varphi(y); \ \ y(x) = c$; bu yerda $\varphi(y)$ — ixtiyoriy uzluksiz funksiya; $c$ — ixtiyoriy o'zgarmas.

1.6.qquad $u(x,y) = \varphi_1(y)x + \varphi_2(y); \ \ y(x) = c_1 x + c_2$; bu yerda $\varphi_1(y)$ va $\varphi_2(y)$ — ixtiyoriy uzluksiz funksiyalar, $c_1$ va $c_2$ esa ixtiyoriy o'zgarmaslar.

1.7. $u(x,y) = f(x) + g(y)$; 1.8. $u(x,y) = f(x) + g(y)e^{-5x}$;

1.9. $u(x,y) = f(x)e^{3x} + g(y)$;

1.10. 1) $u(x,y) = xy + f(x) + g(y)$;

2) $u(x,y) = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 y + f(y)x + g(y)$;

3) $u(x,y) = f(x) + xg_1(y) + g_2(y)$;

4) $u(x,y) = f(x)\exp\left\{\frac{1}{2}y^2\right\} + g(y);$

5) $u(x,y) = \frac{1}{2}xy^2 + \frac{1}{6}y^3 + yf_1(x) + f_2(x);$

6) $u(x,y) = f(x) + g(y) + \int\limits_{x_0}^{x} dt \int\limits_{y_0}^{y} F(t,\tau)d\tau.$

bu yerda va yuqoridagi 7–10 formulalarda $f$ va $g$ ixtiyoriy $C^1(R)$ sinfga tegishli funksiyalar. $(x_0, y_0)$ esa tekislikning ixtiyoriy fiksirlangan nuqtasi.

**II–bobda berilgan misollarning javoblari:**

2.1. 1) Giperbolik: $u_{\xi\eta} + \frac{1}{2}u_\xi = 0,$
$\xi = x + y, \quad \eta = 3x - y.$
2) Elliptik: $u_{\xi\xi} + u_{\eta\eta} + u_\eta = 0, \quad \xi = 2x - y, \ \eta = x.$
3) Parabolik: $u_{\eta\eta} + u_\xi + u = 0, \quad \xi = x + y, \ \eta = y.$
4) Giperbolik: $u_{\xi\eta} + u_\xi - 2u_\eta + \xi + \eta = 0,$
$\xi = 2x - y, \ \eta = x + y.$
5) Elliptik: $u_{\xi\xi} + u_{\eta\eta} + u_\xi = 0, \quad \xi = x, \ \eta = 3x + y.$
6) Parabolik: $u_{\eta\eta} + u_\xi = 0, \quad \xi = x - 2y, \ \eta = x.$
7) Giperbolik: $u_{\xi\eta} = 0, \quad \xi = x + y - \cos x,$
$\eta = -x + y - \cos x.$
8) Elliptik: $u_{\xi\xi} + u_{\eta\eta} = 0, \quad \xi = y, \ \eta = \text{arctg}x.$
9) $x \neq 0, \ y \neq 0$ da parabolik:
$$u_{\eta\eta} + 2\frac{\xi^2}{\eta^2}u_\xi + \frac{1}{\eta}e^\xi = 0, \quad \xi = \frac{y}{x}, \ \eta = y.$$
10) Elliptik: $u_{\xi\xi} + u_{\eta\eta} + \frac{1}{4\eta}u_\xi = 0,$
$\xi = y + x^2, \ \eta = x^2.$

2.2. 1) Elliptik: $u_{\xi\xi} + u_{\eta\eta} + u_{\zeta\zeta} = 0,$
$\xi = x, \ \eta = -x + y, \ \zeta = 2x - 2y + z.$
2) Giperbolik: $u_{\xi\xi} - u_{\eta\eta} + u_{\zeta\zeta} + u = 0,$
$$\xi = y + x, \ \eta = -x + y, \ \zeta = \frac{1}{\sqrt{6}}x - \frac{2}{\sqrt{6}}y + \frac{\sqrt{6}}{2}z.$$

3) Giperbolik: $u_{\xi\xi}+u_{\eta\eta}-u_{\zeta\zeta}+3u_\xi+\frac{3}{2}u_\eta-\frac{9}{2}u_\zeta=0$,

$\xi=x,\ \eta=\frac{1}{2}\left(x+y+z\right),\ \zeta=-\frac{1}{2}\left(3x+y-z\right)$.

4) Parabolik: $u_{\xi\xi}-u_{\eta\eta}+4u=0$,

$\xi=y+x,\ \eta=-y-2x,\ \zeta=x-z$.

5) Parabolik: $u_{\xi\xi}+u_{\eta\eta}-3u+\frac{1}{\sqrt{2}}(\xi+\eta)-3\zeta=0$,

$\xi=\frac{1}{\sqrt{2}}x,\ \eta=\frac{3}{\sqrt{2}}x+\sqrt{2}\,z,\ \zeta=x+z$.

2.3. 1) $v_{\eta\eta}-\frac{c-b}{a}v_\xi=0,\quad \xi=y-x,\ \eta=x;$

$u(\xi,\eta)=\exp(\alpha\xi+\beta\eta)v(\xi,\eta),$

$\alpha=\frac{b^2-4a}{4a(c-b)},\quad \beta=-\frac{b}{2a}.$

2) $v_{xx}+v_{yy}-cv=0,$

$u(x,y)=\exp\left\{-\frac{1}{2}(\alpha x+\beta y)\right\}v(x,y).$

3) $v_{\xi\xi}+v_{\eta\eta}-\frac{15}{2}v=0,\quad \xi=2x+y,\ \eta=x;$

$u(\xi,\eta)=\exp\left(\frac{5}{2}\xi+\frac{3}{2}\eta\right)v(\xi,\eta).$

4) $v_{\xi\eta}-7v=0,\quad \xi=2x-y,\ \eta=x;$

$u(\xi,\eta)=\exp(-\xi-6\eta)v(\xi,\eta).$

5) $v_{\xi\eta}+9v+4(\xi-\eta)\exp(\xi+\eta)=0,$

$\xi=y-x,\quad \eta=y;$

$u(\xi,\eta)=\exp(-\xi-\eta)v(\xi,\eta).$

2.4. 1) $u(x,y)=\varphi(2x-y)$.

2) $u(x,y)=\varphi(2x+3y)\exp\left(\dfrac{y}{2}\right)$.

3) $u(x,y)=\varphi(x+2y)e^{2y}$.

4) $u(x,y)=\varphi(2x-y)-\dfrac{1}{3}\cos(x+y)$.

5) $u(x,y)=\varphi(4x+3y)\exp\left(\dfrac{y}{4}\sin(4x+3y)\right)$.

6) $u(x,y)=f(y)+g(x)e^{-ay}$.

7) $u(x,y)=f(x+y-\cos x)+g(x-y+\cos x)$.

8) $u(x,y)=x-y+f(x-3y)+g(2x+y)\exp\left(\dfrac{3y-x}{7}\right)$.

9) $u(x,y)=[f(x)+g(y)]\exp\{-bx-ay\}$.

10) $u(x,y)=\dfrac{1}{x}\Big[f(x-y)+g(x+y)\Big]$.

**III–bobda berilgan masalalarning javoblari:**

3.1. 1) $u(x,y)=x^{-1}f(xy^4)+g(x)$;

$\xi=xy^4,\quad \eta=x;\quad u_{\xi\eta}+\eta^{-1}u_\xi=0.$

2) $u(x,y)=f(xy)\ln y+g(y)$;

$\xi=xy,\quad \eta=y;\quad u_{\eta\eta}+\eta^{-1}u_\eta=0.$

3) $u(x,y)=f(x)+x^{1/3}g(xy^3)$;

$\xi=x,\quad \eta=xy^3;\quad u_{\xi\eta}-\dfrac{1}{3\xi}u_\eta=0.$

4) $u(x,y) = f(y+\sin x) + g(y+\sin x)e^{2x};$

$$\xi = y + \sin x, \quad \eta = x; \quad u_{\eta\eta} - 2u_\eta = 0.$$

5) $u(x,y) = f(x^2+y^2) + g(x^2+y^2)x^3 + 10^{-1}x^5;$

$$\xi = x^2 + y^2, \quad \eta = x; \quad u_{\eta\eta} - 2\eta^{-1}u_\eta - \eta^3 = 0.$$

6) $u(x,y) = f(x+2y) + e^{x/2}g(x+2y);$

$$\xi = x + 2y, \quad \eta = x; \quad u_{\eta\eta} - \frac{1}{2}u_\eta = 0.$$

7) $u(x,y) = f(4x+y)\exp\left\{x + \frac{y}{2}\right\} + g(2x+y);$

$$\xi = 4x + y, \quad \eta = 2x + y; \quad u_{\xi\eta} - \frac{1}{2}u_\xi = 0.$$

8) $u(x,y) = f(x+\cos y)\frac{1}{x} + g(x);$

$$\xi = x + \cos y, \quad \eta = x; \quad u_{\xi\eta} + \frac{1}{\eta}u_\eta = 0.$$

3.2. 1) $u(x,y) = -\frac{3}{2}(x^2+y^2) + \sin y + 3xy.$

2) $u(x,y) = \frac{4}{5}(y^{3/4} - |x|^{5/2}); \quad |x| < 1, \quad 0 < y < 1.$

3) $u(x,y) = \sin y + e^{x-y} - 1.$ 4) $u(x,y) = xy + \frac{y^2}{3}.$

5) $u(x,y) = x^2 + y^4.$ 6) $u(x,y) = xy^4 + 1.$

7) $u(x,y) = 2(y^2-1) + \frac{1}{5}x^2(1-y^5) + 3x^2.$

8) $u(x,y) = 2\sqrt{xy}.$ 9) $u(x,y) = (x-1)y^5.$

10) $u(x,t) = x + \frac{axt^2}{6} + \sin x \sin t.$

3.3. $u(x,t) = A\sin\pi nx\cos\pi nax.$

3.4. $u(x,t) = \frac{8}{\pi^3}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\sin[(2k+1)\pi x]\cos[(2k+1)\pi at].$

3.5. $u(x,t) = \frac{4\alpha_0}{a\pi^2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}\sin[(2k+1)\pi x]\sin[(2k+1)\pi at].$

3.6. $u(x,t) = \frac{1}{2a\pi}\sin\frac{2a\pi}{l}t\sin\frac{2\pi}{l}x.$

3.7. $u(x,t) = \frac{2l}{a\pi}\sin\frac{a\pi}{2l}t\sin\frac{\pi}{2l}x + \cos\frac{5a\pi}{2l}t\sin\frac{5\pi}{2l}x.$

3.8. $u(x,t) = \sum_{k=1}^{\infty} a_k\cos a\lambda_k t\sin\lambda_k x +$

$$+\frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^l f_k(\tau)\sin a\lambda_k(t-\tau)\sin\lambda_k x d\tau;$$

bu yerda $a_k = \frac{2(h^2+\lambda_k^2)}{l(h^2+\lambda_k^2)+h}\int_0^l \varphi(x)\sin\lambda_k x dx;$

$$f_k(t) = \frac{2(h^2+\lambda_k^2)}{l(h^2+\lambda_k^2)+h}\int_0^l f(x,t)\sin\lambda_k x dx;$$

$\lambda_k$ esa $\lambda_k\cos\lambda_k l + h\sin\lambda_k l = 0$ tenglamaning ildizlari.

3.9. $u(x,t) = \sum_{k=1}^{\infty}\left[\frac{2}{k\pi}\left(\alpha-(-1)^k\beta\right)-\frac{l^2 f_k}{(k\pi a)^2}\right]\times$

$$\times\cos\left(\frac{k\pi a}{l}t\right)\sin\left(\frac{k\pi}{l}x\right)+\sum_{k=1}^{\infty}\frac{l^2 f_k}{(k\pi a)^2}\sin\left(\frac{k\pi}{l}x\right);$$

bu yerda

$$f_k = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi}{l}x\right)dx.$$

3.11. 1) $u(x,t) = \frac{1}{2}[\varphi(x-at)+\varphi(x+at)]-$

$$-\frac{ct}{2}\int_{x-at}^{x+at}\left(\sqrt{t^2-\frac{(x-\xi)^2}{a^2}}\right)^{-1} J_1\left(c\sqrt{t^2-\frac{(x-t)^2}{a^2}}\right)\psi(\xi)d\xi+$$

$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} J_o\left(c\sqrt{(t-\tau)^2-\frac{(x-\xi)^2}{a^2}}\right)f(\xi,\tau)d\xi d\tau.$$

2) $u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+$

$$+\frac{ct}{2}\int_{x-at}^{x+at}\left(\sqrt{t^2-\frac{(x-\xi)^2}{a^2}}\right)^{-1} I_1\left(c\sqrt{t^2-\frac{(x-\xi)^2}{a^2}}\right)\psi(\xi)d\xi.$$

3) $u(x,t)=\frac{1}{4}\int_0^{\xi}\int_0^{\eta}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\lambda}{4}\right)\frac{(\xi-\xi_1)^k(\eta-\eta_1)^k}{(k!)^2}d\eta_1 d\xi_1=$

$$=\frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\lambda}{4}\right)^k\frac{\xi^{k+1}\eta^{k+1}}{[(k+1)!]^2}=\frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\lambda}{4}\right)^k\frac{(x^2-t^2)^{k+1}}{[(k+1)!]^2}.$$

4) $u(x,y)=\frac{1}{2}+(4-3y)\exp(1-x-y)-$

$$-\left(2x+\frac{3}{2}\right)\exp[2(1-x-y)],$$

$$R(x,y;\xi,\eta)=\exp\{x-\xi+2(y-\eta)\}.$$

**IV–bobda berilgan masalalarning javoblari:**

4.1. $u(x.t)=\frac{8}{\pi}\sum_{k=0}^{+\infty}\frac{1}{2k+1}\exp\{-(2k+1)^2\pi a^2 t\}\sin(2k+1)\pi x.$

4.2. $u(x,t)=\exp\left\{-\left(\frac{a\pi}{2l}\right)^2 t\right\}\sin\frac{\pi x}{2l}.$

4.3. $u(x,t)=\sum_{k=0}^{+\infty}a_k\exp\left\{-\left[\frac{(2k+1)a\pi}{2l}\right]^2 t\right\}\sin\frac{(2k+1)\pi}{2l}x.$

Bu yerda $a_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{(2k+1)\pi}{2l}xdx.$

4.4. $$u(x,t)=\frac{2lA}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{k}\exp\left\{-\left(\frac{ak\pi}{l}\right)^2 t\right\}\sin\frac{k\pi}{l}x.$$

4.5. $u(x,t)=v(x,t)+\omega(x);$

$$v(x,t)=\sum_{k=0}^{\infty}a_k\exp\left\{-a^2\lambda_k^2 t\right\}\cos\lambda_k x;$$

$$a_k=\frac{2}{l}\int_0^l[A(l-x)-\omega(x)]\cos\lambda_k x dx;$$

$$\omega(x)=\beta-\frac{1}{a^2}\int_0^l\left(\int_0^z f(\xi)d\xi\right)dz+\frac{1}{a^2}\int_0^x\left(\int_0^z f(\xi)d\xi\right)dz$$

$$\lambda_k=\frac{\pi}{2l}+\frac{k\pi}{l},\quad k=0,1,2,\ldots$$

4.6. $$u(x,t)=\sum_{k=0}^{\infty}a_k\exp\left\{-\left[\left(\frac{ak\pi}{l}\right)^2+\beta\right]t\right\}\cos\left(\frac{k\pi}{l}\right)x;$$

$$a_0=\frac{1}{l}\int_0^l\varphi(x)dx,\quad a_k=\frac{1}{l}\int_0^l\varphi(x)\cos\left(\frac{k\pi}{l}\right)xdx,\quad k=1,2,\ldots$$

4.7. 1) $u(x,t)=e^{-l^2t}\sin lx.$

2) $$u(x,t)=\frac{1}{\sqrt{t+1}}\exp\left(-\frac{2x-x^2+t}{t+1}\right).$$

3) $$u(x,t)=1-\cos t+(1-4t)^{-1/2}\exp\left(-\frac{x^2}{1+4t}\right).$$

4) $u(x,t)=t^3+e^{-t}\sin x.$

5) $u(x,y,t)=e^{-(l_1^2+l_2^2)t}\sin l_1x\sin l_2y.$

6) $u(x,y,t)=e^{-l_1^2t}\sin l_1x+e^{-l_2^2t}\cos l_2y.$

4.8. $$u(x,t)=\int_0^{\infty}G_0(x,t;\xi)\varphi(\xi)d\xi.$$

Bu yerda

$$G_0(x,t,\xi) = \frac{1}{2a\sqrt{\pi t}}\left(\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} - \exp\left\{-\frac{(x+\xi)^2}{4a^2t}\right\}\right).$$

4.9. $u(x,t) = \int\limits_0^\infty G_1(x,t;\xi)\varphi(\xi)d\xi.$

Bu yerda

$$G_1(x,t,\xi) = \frac{e^{-ht}}{2a\sqrt{\pi t}}\left(\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\} + \exp\left\{-\frac{(x+\xi)^2}{4a^2t}\right\}\right).$$

**V–bobda berilgan masalalarning javoblari:**

5.1. 1) Garmonik; 2) Garmonik; 3) Garmonik.

5.2. 1) $k = \pm 3$; 2) $k = 0$, $k = n - 2$, $n > 0$, $|x| \neq 0$.

5.3. 1) Berilgan funksiya $(\pm 2, 0)$ nuktada maksimumga $u_{max} = 4$; va $(0, \pm 3)$ nuktada $u_{min} = -9$ minimumga erishadi.

5.4. 1) $u(r,\varphi) = \frac{1}{2}\left(1 + r^2\cos 2\varphi\right)$;

2) $u(r,\varphi) = \frac{r}{4}\left(3\sin\varphi - r^2\sin 3\varphi\right)$;

3) $u(r,\varphi) = \frac{5}{8} + \frac{5}{8}r^4\cos 4\varphi.$

5.5. 1) $u(r,\varphi) = r\cos\varphi + C$;

2) $u(r,\varphi) = \frac{r^2}{2R}\cos\varphi + C.$

5.6. $u(x,y) = x^2 - y^2 + 2y + R^2.$

5.7. $u(x,y) = R^2 + \left(\frac{R}{r}\right)^4(x^2 - y^2) - \left(\frac{R}{r}\right)^2(x - y).$

======== ◊ ========

## Nazorat testlarining to'g'ri javoblari kaliti

1–testning to'g'ri javoblar kaliti

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| g | b | v | v | v | g | a | a | a | v | a | g | a | a | v |

2–testning to'g'ri javoblar kaliti

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| v | g | v | b | a | b | b | v | v | a | a | v | b | v | g |

3–testning to'g'ri javoblar kaliti

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | g | b | g | a | g | v | a | v | g | g | a | v | b |

4–testning to'g'ri javoblar kaliti

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | g | v | b | a | g | v | b | a | g | g | b | v | a | b |

**Baholash mezoni:**
**a'lo** baho — 26 – 30 ball:
**yaxshi** baho — 21 – 25 ball;
**qoniqarli** baho — 16 – 20 ball;
Har bir to'g'ri javob — 2 balldan.

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# MUNDARIJA

ZIKIROV OBIDJAN SALIJANOVICH

# MATEMATIK FIZIKA TENGLAMALARI

*(O'quv qo'llanma)*

Toshkent – "Fan va texnologiya" – 2017


| | |
|---|---|
| Muharrir: | Sh.Kusherbayeva |
| Tex. muharrir: | F.Tishaboyeva |
| Musavvir: | D.Azizov |
| Musahhih: | N.Hasanova |
| Kompyuterda sahifalovchi: | Sh.Mirqosimova |

E–mail:tipografiyacnt@mail.ru Tel: 245–57–63, 245–61–61.
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Tiraji 350. Buyurtma №209.

"Fan va texnologiyalar Markazining bosmaxonasi"da chop etildi.
100066. Toshkent sh., Olmazor ko'chasi, 171–uy.