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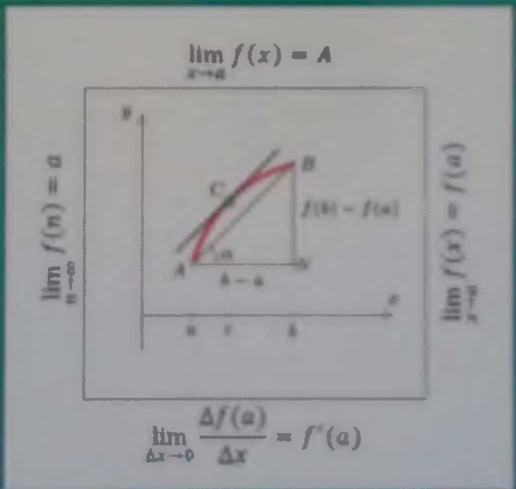
MATEMATIK ANALIZDAN MISOL VA MASALALAR



1-QISM

O'ZBEKISTON RESPUBLIKASI OLIY VA
O'RTA MAXSUS TA'LIM VAZIRLIGI

A. Gaziyev, I. Israilov, M. U. Yaxshiboyev

MATEMATIK ANALIZDAN MISOL VA MASALALAR

1-QISM

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan 5130100 — matematika, 5140300 — mexanika, 5130200 — amaliy matematika va informatika, 5140200 — fizika bakalavriat ta'lim yo'nalishlari talabalari uchun o'quv qo'llanma sifatida tavsiya etilgan

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Ushbu qo'llanma matematik analizning to'plam, to'plamlar ustida amallar, haqiqiy sonlar, yig'indi, matematik induksiya usuli, funksiya tushunchasi, funksiyaning sinflari, sonlar ketma-ketligi va uning limiti, funksiyaning limiti, funksiya grafigining asimptotalari, funksiyaning uzluksizligi va tekis uzluksizligi, funksiyaning hosilasi va differensiali, funksiyaning yuqori tartibli hosilasi va differensiallari, differensial hisobning asosiy teoremalari, Lopital qoidalari, Teylor formulasi, funksiyani hosila yordamida tekshirish mavzulari bo'yicha talabalarda misol va masalalarni mustaqil yechish ko'nikmasini hosil qilishga mo'ljallangan.

Qo'llanmaning har bir paragrafida, avvalo, mavzuning nazariy qismidan qisqacha axborot berilgan, so'ngra mavzuga mos tipik misol va masalalar batafsil yechib ko'rsatilgan hamda mustaqil ishlash uchun yetarli miqdorda misol va masalalar javoblari bilan berilgan.

O'quv qo'llanma bakalavriatning «Matematika», «Mexanika», «Amaliy matematika va informatika», «Fizika» yo'nalishlari va texnika yo'nalishlarining «Oliy matematika» chuqurlashtirilgan dastur asosida o'qitiladigan talabalari hamda o'qituvchilar uchun mo'ljallangan.

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SO'ZBOSHI

«Matematik analiz» fanining o'quv dasturiga kirgan to'plam, to'plamlar ustida amallar, haqiqiy sonlar, yig'indi, matematik induksiya usuli, funksiya tushunchasi, funksiyaning sinflari, sonlar ketma ketligi va uning limiti, funksiyaning limiti, funksiya grafigining asimptotalari, funksiyaning uzluksizligi va tekis uzluksizligi, funksiyaning hosilasi va differensiali, funksiyaning yuqori tartibli hosilasi va differensiallari, differensial hisobning asosiy teoremalari, Lopital qoidalari, Teylor formulasi, funksiyani hosila yordamida tekshirish bo'limlarini o'zlashtirishda talabalar ancha qiyinchiliklarga duch keladi, bu ularning yuqoridagi bo'limlar bo'yicha misol va masalalarni yechishida yaqqol ko'rinadi.

Mualliflar mazkur qo'llanmani yozishda o'z oldilariga talabalarda ko'rsatilgan bo'limlarga oid misol va masalalarni samarali yo'l bilan yechish ko'nikmalari hosil qilishni maqsad qilib qo'yishdi va qo'llanma talabalar uchun doimiy maslahatchi bo'lib qolishiga ishonadilar.

O'quv qo'llanma uchta bobdan iborat bo'lib, uning birinchi bobida dastlabki asosiy tushunchalar, ikkinchi bobida funksiya va uning limiti hamda funksiyaning uzluksizligi qaralgan. Qo'llanmaning uchinchi bobida hosiladan foydalanib funksiyaning grafigini chizish qaralgan. O'z navbatida har bir bob tegishli paragraflarga bo'lingan bo'lib, har bir paragraf mavzuga taalluqli asosiy ta'riflar, tasdiqlar, teoremalarni o'z ichiga oladi, shuningdek, ularning har biri an'anaviy misollarni batafsil tahlil yordamida yechish orqali namoyish qilingan. Qo'llanmada jami 304 ta misol va masalalar yechilgan, 1703 ta mustaqil yechish uchun misol va masalalar tavsiya qilingan hamda ularning javoblari berilgan. Hozirgi vaqtda amaliyotda bir necha yaxshi rivojlangan matematik dasturlar (Mathcad, Maple, Mathematica, Mathlab va h.k.) matematik masalalarni kompyuter imkoniyatlaridan foydalanib yechishda samarali natijalar bermoqda. Shu an'anadan chetda qolmaslik uchun, qo'llanmada ba'zi bo'limlar bo'yicha misol va masalalar yechishda «Maple» tizimining qo'llanilishi va uning qulayliklari namoyish etilgan.

Ushbu qo'llanmani yozishga mualliflarni undagan sabab ularning ko'p yillar mobaynida Samarqand Davlat universitetida matematik analiz kursidan olib borgan ma'ruza va amaliy mashg'ulotlarida orttirgan tajribalari natijasidir. O'ylaymizki, qo'llanma o'z o'quvchilarini topadi va boshqa mavjud o'quv adabiyotlari qatorida matematik analiz kursining aytib o'tilgan bo'limlari bo'yicha ularga bilimlarini oshirishga ko'mak beradi.

O'quv qo'llanma haqidagi fikr-mulohazalar, undagi mavjud kamchiliklar bo'yicha takliflarni mualliflar mamnuniyat bilan qabul qiladilar.

Mualliflar

I BOB. DASTLABKI ASOSIY TUSHUNCHALAR

1-§. TO‘PLAM TUSHUNCHASI. TO‘PLAMLAR USTIDA AMALLAR

1.1. To‘plam tushunchasi. To‘plam tushunchasi matematikaning ta’rifsiz qabul qilingan asosiy tushunchalaridan biri bo‘lib, ba’zi belgilariga asoslanib birgalikda qaraladigan obyektlar yoki narsalar (predmetlar) majmuasidir. To‘plamni tashkil qiluvchi har bir obyekt yoki narsa uning *elementi* deyiladi. To‘plam tushunchasi misollar yordamida tushuntiriladi. Masalan, Toshkent shahridagi oliy o‘quv yurtlarida o‘qiydigan talabalar, barcha butun sonlar, kutubxonadagi kitoblar va hokazolar to‘plamni tashkil etadi.

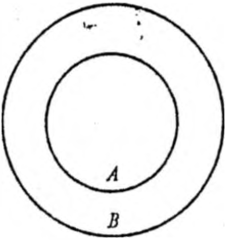
To‘plamlar lotin yoki yunon alfavitining bosh harflari bilan, uning elementlari esa kichik harflar bilan belgilanadi. Masalan, A, B, C, D, ..., X, Y, Z lar bilan to‘plamlarni, a, b, c, d, ..., x, y, z lar bilan esa to‘plamning elementlari belgilanadi.

Agar A to‘plamning elementi a bo‘lsa, bu $a \in A$ kabi yoziladi va «a element A to‘plamga tegishli» deb o‘qiladi. Aks holda, ya’ni a element A to‘plamga tegishli bo‘lmasa, unda $a \notin A$ (yoki $a \bar{\in} A$) kabi yoziladi va «a element A to‘plamga tegishli emas» deb o‘qiladi. Masalan, $A = \{2, 4, 6, 8\}$ bo‘lsa, u holda $4 \in A$, $3 \notin A$.

Chekli sondagi elementlardan tashkil topgan to‘plam *chekli*, cheksiz sondagi elementlardan tashkil topgan to‘plam esa *cheksiz* to‘plam deb ataladi. Masalan, Samarqand Davlat universiteti qoshidagi Jomiy nomli ilmiy kutubxonadagi mavjud kitoblar to‘plami *chekli* to‘plamni, natural sonlar to‘plami esa *cheksiz* to‘plamni tashkil etadi.

Bitta ham elementga ega bo‘lmagan to‘plam *bo‘sh to‘plam* deyiladi va $\varnothing$ kabi belgilanadi. Bo‘sh to‘plamlarga quyidagilar misol bo‘la oladi: a) $x^2 + 4 = 0$ tenglamaning haqiqiy ildizlari to‘plami; b) o‘zaro parallel ikkita turli to‘g‘ri chiziqning umumiy nuqtalari to‘plami; d) $|x-4| < -2$ tengsizlikning yechimlari to‘plami va h.k.

Ko‘pincha to‘plamlar, ularning elementlari chekli yoki cheksiz bo‘lishidan qat’iy nazar, simvolik ravishda doirachalar bilan tasvir-



1.1-chizma.

lanadi. Bu tasvirlash to'plamlar ustida bajariladigan amallarni tasavvur qilishda va ular orasidagi munosabatlarni o'rganishda ancha qulayliklar tug'diradi.

1.1-ta'rif. Agar A to'plamning har bir elementi B to'plamning ham elementi bo'lsa, A to'plam B to'plamning *qismi* yoki *qismiy to'plami* (to'plam osti) deb ataladi va $A \subset B$ kabi belgilanadi (1.1-chizma). Bu quyidagicha o'qiladi: «B to'plam A to'plamni o'z ichiga oladi».

1.1-eslatma. Bo'sh to'plam har qanday A to'plamning qism to'plami hisoblanadi: $\varnothing \subset A$. Har qanday A to'plam o'z-o'zining qism to'plami hisoblanadi: $A \subset A$.

1.2-eslatma. n ta elementdan iborat bo'lgan to'plamning qism to'plamlar soni 2^n ga teng.

1.3-eslatma. Agar A, B, C, ... to'plamlarning har biri J to'plamning qism to'plamlari bo'lsa, J to'plamga *universal* to'plam deyladi.

1.2-ta'rif. Agar A to'plam B to'plamning qismi, B to'plam A to'plamning qismi bo'lsa, ya'ni $A \subset B$, $B \subset A$ bo'lsa, u holda A va B to'plamlar bir-biriga *teng* deyiladi va $A = B$ kabi yoziladi.

1.1-misol. Ushbu $A = \{x : x \in N, -5 < x \leq 7\}$ to'plamning elementlarini aniqlang.

Yechilishi. Berilgan A to'plamning elementlari natural sonlardan iborat bo'lib, $-5 < x \leq 7$ tengsizlikni qanoatlantirishi kerak. Bu tengsizlikni qanoatlantiruvchi natural sonlar 1, 2, 3, 4, 5, 6 va 7 dan iborat.

Demak, $A = \{1, 2, 3, 4, 5, 6, 7\}$.

1.2-misol. Sonlar o'qida $\{M : |OM| = 3\}$ shartni qanoatlantiruvchi M nuqtalar to'plamini aniqlang va elementlarini yozing.

Yechilishi. Izlanuvchi to'plamning elementlari son o'qida joylashgan nuqtalardan iborat bo'lib, ular sanoq boshi O dan uch birlik uzoqlikda masofada joylashgan bo'ladi.

Demak, ular -3 va 3 dan iborat, ya'ni $A = \{-3, 3\}$.

1.3-misol. Ushbu $A = \{-6, -4, -2, 0, 1, 3, 5, 7\}$ va $B = \{1, 3, 5\}$ to'plamlar berilgan, $B \subset A$ ekanligini ko'rsating.

Yechilishi. B to'plamning barcha elementlari A to'plamning elementlari bo'lganligi uchun, 1.1-ta'rifga asosan $B \subset A$ bo'ladi.

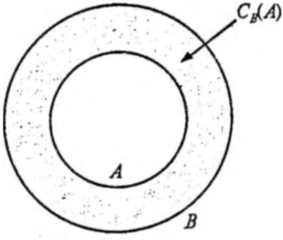
1.4-misol. a, b, c elementlardan tashkil topgan A to'plam berilgan bo'lsin. A to'plamning qism to'plamlarini aniqlang.

Yechilishi. $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, $\varnothing$ to'plamlar 1.1-ta'rifga asosan berilgan A to'plamning qism to'plamlari bo'ladi.

1.5-misol. Agar $A = \{x : x \in R, x^2 - 5x + 6 = 0\}$ va $B = \{2, 3\}$ to'plamlar berilgan bo'lsa, $A = B$ ekanligini ko'rsating.

Yechilishi. Ma'lumki, $x^2 - 5x + 6 = 0$ tenglama $x_1 = 2$, $x_2 = 3$ ildizlarga ega, shuning uchun $A = \{2, 3\}$ bo'ladi.

Demak, 1.2-ta'rifga asosan, $A = B$ bo'ladi.



1.2-chizma.

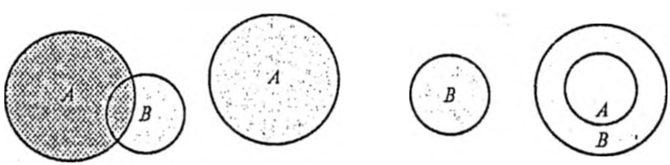
1.6-misol. Ushbu $\{a, b, c\} = \{\{a, b\}, c\}$ munosabat o'rinli bo'ladimi?

Yechilishi. Bu munosabat o'rinli bo'lmaydi, chunki tenglikning chap tomonidagi to'plam uchta a, b, c elementlarga ega, o'ng tomondagi to'plam esa ikkita: to'plam $\{a, b\}$ va c elementdan iborat. O'ng tomondagi to'plamning $\{a, b\}$ elementi chap tomondagi to'plamga tegishli emas.

1.2. To'plamlar ustida amallar

1.3-ta'rif. B ixtiyoriy to'plam bo'lib, A to'plam uning biror qismi bo'lsin. B to'plamning A ga kirmagan barcha elementlaridan tashkil topgan to'plam A ning B ga qadar *to'ldiruvchisi* deyiladi va u $C_B(A)$ kabi belgilanadi (1.2-chizma).

1.4-ta'rif. A va B ixtiyoriy to'plamlar bo'lsin. Agar C to'plam A va B to'plamlarning barcha elementlaridan iborat bo'lib, boshqa elementlari bo'lmasa, u holda C to'plam A va B to'plamlarning *yig'indisi* (birlashmasi) deyiladi va $A \cup B = C$ kabi belgilanadi (1.3-chizma).



1.3-chizma.

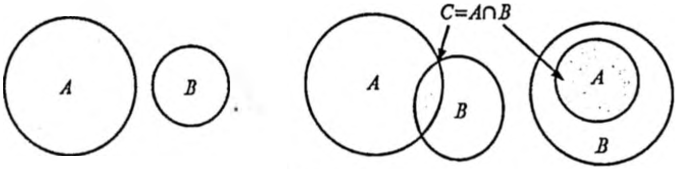
1.4-eslatma. Shuni qayd qilib o'tish kerakki, agar biror element ham A to'plamga, ham B to'plamga qarashli bo'lsa, bu element C to'plamda bir marta hisoblanadi.

Yuqoridagi 1.4-ta'rifdan to'plamlarning quyidagi xossalari kelib chiqadi:

1°. $A \cup A = A$. 2°. $A \cup B = B \cup A$. 3°. $A \cup \varnothing = A$.

4°. Agar $A \subset B$ bo'lsa, $A \cup B = B$ bo'ladi.

1.5-ta'rif. A va B to'plamlarning umumiy elementlaridan tashkil topgan C to'plam A va B to'plamlarning umumiy qismi yoki *ko'paytmasi* (kesishmasi) deyiladi va $C=A\cap B$ kabi belgilandi (1.4-chizma).



1.4-chizma.

To'plamlarning quyidagi xossalari 1.5-ta'rifdan bevosita kelib chiqadi:

5°. $A\cap A=A$. 6°. $A\cap B=B\cap A$. 7°. $A\cap\varnothing=\varnothing$.

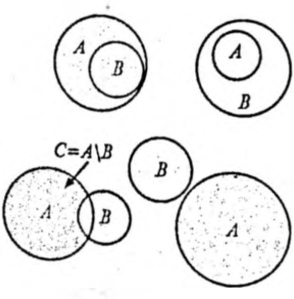
8°. Agar $A\subset B$ bo'lsa, u holda $A\cap B=A$ bo'ladi.

1.6-eslatma. Biz to'plamlarning yig'indisi hamda ko'paytmasi ta'riflarini ikkita to'plam uchun keltirdik. Agar A_1, A_2, ..., A_n to'plamlar berilgan bo'lsa, ularning yig'indisi $A_1\cup A_2\cup\ldots\cup A_n$ hamda ko'paytmasi $A_1\cap A_2\cap\ldots\cap A_n$ ham yuqoridagi 1.4- va 1.5-ta'riflarga o'xshash beriladi.

1.6-ta'rif. A to'plamning B to'plamga tegishli bo'lmagan barcha elementlaridan tuzilgan C to'plam A va B to'plamlarning *ayirmasi* deyiladi va $C=A\backslash B$ kabi belgilanadi (1.5-chizma).

To'plamlarning quyidagi xossalari 1.6-ta'rifdan bevosita kelib chiqadi:

9°. $A\backslash\varnothing=A$. 10°. $\varnothing\backslash A=\varnothing$. 11°. $A\backslash A=\varnothing$.

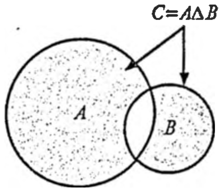


1.5-chizma.

1.7-ta'rif. A to'plamning B to'plamga tegishli bo'lmagan elementlaridan va B to'plamning A to'plamga tegishli bo'lmagan elementlaridan tuzilgan C to'plam A va B to'plamlarning *simmetrik ayirmasi* deb ataladi va $C=A\Delta B$ kabi belgilanadi, ya'ni $A\Delta B=(A\backslash B)\cup(B\backslash A)$ (1.6-chizma).

1.8-ta'rif. Birinchi element X to'plamga va ikkinchi element Y

to'plamga kirgan barcha (x, y) juftlardan iborat bo'lgan nuqtalar to'plami X va Y to'plamlarning *dekart* (to'g'ri) *ko'paytmasi* deyiladi va u $[X, Y]$ yoki $X \times Y$ kabi belgilanadi, ya'ni $C = X \times Y = \{(x, y): x \in X, y \in Y\}$.



1.6-chizma.

1.7-eslatma. A to'plamning o'z-o'ziga Dekart ko'paytmasi quyidagicha belgilanadi.

$$A \times A = A^2 = \{(x, y): x \in A, y \in A\}.$$

1.7-misol. Ushbu $A = \{x: x \in R, x^3 - 2x^2 - x + 2 = 0\}$ va $B = \{-1, 2\}$ to'plamlar berilgan bo'lsa, $C_B(A)$ ni toping.

Yechilishi. Ravshanki, $x^3 - 2x^2 - x + 2 = 0$ tenglama -1, 1 va 2 ildizlarga ega. Demak, $A = \{-1, 1, 2\}$, $B = \{-1, 2\}$. Unda 1.3-ta'rifga asosan $C_B(A) = \{1\}$.

1.8-misol. Ushbu $A = \{-4, -3, -2, -1, 0, 1, 2\}$ va $B = \{1, 2, 3, 4, 5\}$ to'plamlar berilgan bo'lsa, $C = A \cup B$ ni toping.

Yechilishi. 1.4-ta'rifga asosan $C = A \cup B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ bo'ladi.

1.9-misol. Ushbu $A = \{x: x \in R, x > 2\}$ va $B = \{x: x \in R, x < 3\}$ to'plamlar berilgan bo'lsa, $C = A \cup B$ ni toping.

Yechilishi. Ravshanki, A to'plamga $x > 2$ tengsizlikni qanoatlantiradigan, B to'plamga esa $x < 3$ tengsizlikni qanoatlantiradigan haqiqiy sonlar kiradi. Shuning uchun, 1.3-eslatmaga ko'ra $C = A \cup B = R$ bo'ladi.

1.10-misol. Ushbu $A = \{1, 2, 3\}$ va $B = \{x: x \in R, x^2 + 2 = 0\}$ to'plamlar berilgan bo'lsa, $C = A \cup B$ ni toping.

Yechilishi. Ravshanki, $x^2 + 2 = 0$ tenglama haqiqiy ildizga ega bo'lmagani uchun $B = \varnothing$ bo'ladi. To'plamlar yig'indisining 3°-xossasiga asosan $C = A \cup B = A \cup \varnothing = A = \{1, 2, 3\}$.

1.11-misol. Ushbu $A = \{1, 2, 3, 4, 5, 6, 7\}$ va $B = \{x: x \in N, -2 < x \leq 5\}$ to'plamlar berilgan bo'lsa, $C = A \cup B$ to'plamning elementlarini ko'rsating.

Yechilishi. Berilgan B to'plamning elementlari $-2 < x \leq 5$ tengsizlikni qanoatlantiruvchi 1, 2, 3, 4, 5 natural sonlardan iborat, ya'ni $B = \{1, 2, 3, 4, 5\}$. Demak, $B \subset A$ bo'lganligi uchun to'plamlar yig'indisining 4°-xossasiga asosan $A \cup B = A = \{1, 2, 3, 4, 5, 6, 7\}$.

1.12-misol. Ushbu $A = \{\pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \ldots\}$ va $B = \{\pm 5, \pm 10, \pm 15, \pm 20, \pm 25, \pm 30, \ldots\}$ to'plamlar berilgan bo'lsa, $C = A \cap B$ ni toping.

Yechilishi. A va B to'plamlarning berilishini e'tiborga olgan holda 1.5-ta'rifga ko'ra, izlanayotgan to'plam: $C = A \cap B = \{\pm 10, \pm 20, \pm 30, \ldots\}$.

1.13-misol. Ushbu $A=\{2, 4, 6, 8, ...\}$ va $B=\{1, 3, 5, 7, ...\}$ to'plamlar berilgan bo'lsa, $C=A\cap B$ ni toping.

Yechilishi. A va B to'plamlarning berilishini e'tiborga olgan holda 1.5-ta'rifga asosan, izlanayotgan to'plam: $C=A\cap B=\varnothing$.

1.14-misol. Ushbu $A=\{\pm3, \pm6, \pm9, \pm12, \pm15, \pm15, \pm18, ...\}$ va $B=\{\pm2, \pm4, \pm6, \pm8, \pm10, \pm12, ...\}$ to'plamlar berilgan bo'lsa, $C=A\backslash B$ ni toping.

Yechilishi. A va B to'plamlarning berilishini e'tiborga olgan holda 1.6-ta'rifga asosan, $C=A\backslash B=\{\pm3, \pm9, \pm15, \pm21, ...\}$ bo'ladi.

1.15-misol. Ushbu $A\backslash(B\cup C)=A\backslash B\backslash C$ ayniyatni isbotlang.

Isboti. Agar $x\in A\backslash(B\cup C)$ bo'lsa, $x\in A$ va $x\notin(B\cup C)$, $x\notin B$, $x\notin C$ ekanligi kelib chiqadi. $x\in A$ $x\notin B$ bo'lganligi uchun $x\in A\backslash B$, $x\notin C$, bundan $x\in A\backslash B\backslash C$ bo'ladi. Demak, $A\backslash(B\cup C)\subset A\backslash B\backslash C$.

Endi $x\in A\backslash B\backslash C$ bo'lsin deb faraz qilamiz, bundan $x\in A\backslash B$ va $x\notin C$. $x\in A\backslash B$ bo'lganligidan $x\in A$ $x\notin B$ ekanligi kelib chiqadi. U holda $x\notin(B\cup C)$, (chunki $x\notin B$, $x\notin C$). $x\in A$ bo'lganligi uchun $x\in A\backslash(B\cup C)$ bo'ladi. Demak, $A\backslash B\backslash C\subset A\backslash(B\cup C)$.

Shunday qilib, $A\backslash(B\cup C)=A\backslash B\backslash C$ ayniyat isbotlandi.

1.16-misol. A, B va C to'plamlar berilgan. To'plamlarning yig'indisi, ayirmasi va ko'paytmasining ta'riflaridan foydalanib quyidagi to'plamlarni yozing: elementlari: 1) uchala to'plamga ham tegishli; 2) hech bir to'plamga tegishli emas; 3) hech bo'lmaganda bitta to'plamga tegishli; 4) A to'plamga tegishli, B va C tegishli emas; 5) A va B to'plamlarga tegishli, C to'plamga tegishli emas; 6) hech bo'lmaganda berilgan to'plamlarning ikkitasiga tegishli.

Yechilishi. 1) To'plamlar ko'paytmasining ta'rifiga ko'ra, elementlari berilgan to'plamlarning uchalasiga ham tegishli. Bunda izlanayotgan to'plam, $A\cap B\cap C$ dan iborat bo'ladi; 2) 1.3-eslatmaga asosan, ya'ni universal to'plam hamda to'plamlar yig'indisining ta'rifiga asosan, elementlar berilgan to'plamlarning hech biriga tegishli emas. Bunda izlanuvchi to'plam $(J\backslash A)\cup(J\backslash B)\cup(J\backslash C)$ to'plamdan iborat; 3) element berilgan to'plamlarning hech bo'lmaganda bittasiga tegishli, to'plamlar yig'indisining ta'rifiga asosan, izlanayotgan to'plam, $A\cup B\cup C$ to'plamdan iborat; 4) to'plamlar ayirmasining ta'rifiga ko'ra, $x\in(A\backslash B\backslash C)$ bo'lsa, $x\in A$, $x\notin B$, $x\notin C$ bo'ladi. Shuning uchun izlanayotgan to'plam, $(A\backslash B\backslash C)$ to'plamdan iborat; 5) to'plamlar ko'paytmasi va ayirmasining ta'rifiga ko'ra, elementlari A va B to'plamlarga tegishli, C to'plamga tegishli emas. Bunda izlanayotgan to'plam $(A\cap B)\backslash C$ to'plamdan iborat. Haqiqatan ham, agar $x\in(A\cap B)\backslash C$ bo'lsa, $x\in A$, $x\in B$, $x\notin C$ bo'ladi; 6) elementlari hech bo'lmaganda berilgan to'plamlarning ikkitasiga tegishli, to'plamlar ko'paytmasi va yig'indisining ta'rifiga asosan, izlanayotgan to'plam $A\cap B\cup B\cap C\cup A\cap C$ to'plamdan iborat. Haqiqatan ham, agar $x\in A\cap B\cup B\cap C\cup A\cap C$ bo'lsa, $x\in A\cap B$, $x\in B\cap C$, $x\in A\cap C$ bo'ladi, bundan esa $x\in A$, $x\in B$ yoki $x\in B$, $x\in C$, yoki $x\in A$, $x\in C$ bo'ladi.

1.17-misol. Agar $A=\{x: x\in B,\ x^2-4x\leq 0\}$ $B=\{x: x\in Z,\ x^2-x-6\leq 0\}$ bo'lsa, u holda $A\Delta B$ to'plamni tuzing.

Yechilishi. Ravshanki, $x^2-4x\leq 0$ tengsizlikning natural yechimlari 1, 2, 3, 4 lardan iborat bo'lib, $A=\{1,\ 2,\ 3,\ 4\}$ to'plamni hosil qiladi.

$x^2-x-6\leq 0$ tengsizlikning butun yechimlari $-2,\ -1,\ 0,\ 1,\ 2,\ 3$ lardan iborat bo'lib, ular $B=\{-2,\ -1,\ 0,\ 1,\ 2,\ 3\}$ to'plamni tashkil etadi. 1.7-ta'rifga ko'ra, $A\Delta B=\{-2,\ -1,\ 0,\ 4\}$dan iborat bo'ladi.

1.18-misol. Ushbu $A=\{$Fakultetdagi bitta guruh talabalari$\}$, $B=\{$Fakultetdagi a'lochi talabalar$\}$ to'plamlar berilgan bo'lsa, u holda $A\Delta B$ to'plamni tuzing.

Yechilishi. Simmetrik ayirmaning ta'rifiga ko'ra, $A\Delta B=$ $=(A\backslash B)\cup(B\backslash A)=\{$berilgan guruhda a'lochi bo'lmagan talabalar va fakultetdagi A to'plamga kirmaydigan a'lochi talabalar$\}$.

1.19-misol. Ushbu $A=\{a,\ b,\ c\}$, $B=\{\alpha,\ \beta\}$ to'plamlarning dekart ko'paytmasini toping.

Yechilishi. 1.8-ta'rifga ko'ra, A va B to'plamlarning berilishini e'tiborga olgan holda, ularning dekart ko'paytmasi

$$A\times B=\{(a,\ \alpha),\ (a,\ \beta),\ (b,\ \alpha),\ (b,\ \beta),\ (c,\ \alpha),\ (c,\ \beta)\}$$

bo'ladi.

Mustaqil yechish uchun misollar va masalalar

Quyida berilgan A to'plamning elementlarini aniqlang:

1.1. $A=\{x\in Z: (x-6)(x^2-4)=0,\ x\geq 0\}$.

1.2. $A=\{x\in R: x^3-5x^2+6x=0\}$.

1.3. $A=\left\{x\in R: x+\frac{1}{x}\leq 2,\quad x>0\right\}$.

1.4. $A=\{x\in N: x^2-5x-6\leq 0\}$. **1.5.** $A=\left\{x\in Z: \frac{1}{9}\leq 3^x<10\right\}$.

1.6. $A=\{x\in R: \cos^2 2x=1,\quad 0<x\leq 2\pi\}$.

1.7. Ushbu $\{\{1;\ 2,\ \{2;\ 3\}\}=\{1;\ 2;\ 3\}$ tenglik o'rinlimi?

1.8. Ushbu $A=\{x\in R: x^3-7x^2+16x-12=0\}$ va $B=\{2;\ 3\}$ to'plamlarning tengligini ko'rsating.

1.9. $\{2; 3\} \subset \{\{1; 2; 3\}, \{1; 3\}, 1, 2\}$ munosabat o'rinlimi?

1.10. $\{2; 3\} \subseteq \{\{1; 2; 3\}, \{1; 3\}, 1, 2\}$ munosabat o'rinlimi?

1.11. Ushbu $\varnothing = \{\varnothing\}$ tenglik o'rinlimi?

1.12. 0 sondan tuzilgan to'plam bo'sh to'plam bo'ladimi?

Quyida berilgan to'plamlarning barcha qism to'plamlarini toping:

1.13. $\varnothing$. **1.14.** $\{\varnothing\}$. **1.15.** $\{1, 2\}$. **1.16.** $\{a, b, c, d\}$.

1.17. $\forall a, b, c, d$ elementlar uchun $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ bo'lsa, $a = c$ va $b = d$ ekanligini isbotlang va aksincha.

1.18. Quyidagi berilgan A va B to'plamlarga ko'ra $A \cup B$, $A \cap B$, $A \backslash B$, $B \backslash A$, $A \Delta B$ to'plamlarni toping.

1) $A = \{x \in R: x^2 + x^2 - 20 = 0\}$, $B = \{x \in R: x^2 - 7x + 12 = 0\}$;

2) $A = \{x \in R: -2 \leq x \leq 3\} = [-2; 3]$, $B = \{x \in R: 1 \leq x \leq 4\} = [1; 4]$;

3) $A = \{x \in N: x^2 - 4x \leq 0\}$, $B = \{x \in Z: x^2 - x - 6 \leq 0\}$.

1.19. Agar $A = \{x \in N: 2 < x \leq 6\}$, $B = \{x \in N: 1 < x < 4\}$, $C = \{x \in N: x^2 - 4 = 0\}$ bo'lsa, 1) $B \cup C$; 2) $A \cap B \cap C$; 3) $A \cup B \cup C$; 4) $(A \cap B) \cup (B \cap C)$ to'plamlarni toping.

Quyidagi munosabatlarning o'rinli bo'lishini ko'rsating:

1.20. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

1.21. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

1.22. $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$.

1.23. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

1.24. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

Quyida berilgan to'plamlarni koordinatalar tekisligida tasvirlang:

1.25. $\{(x, y) \in R^2: x + 3y - 6 = 0\}$. **1.26.** $\{(x, y) \in R^2: x^2 + y^2 \leq 4\}$.

1.27. $\{(x, y) \in R^2: (x^2 - 1) \cdot (y + 2) = 0\}$.

1.28. $\left\{(x, y) \in R^2:\ y > \sqrt{2x + 1},\ 2x + 1 \geq 0\right\}$.

1.29. $\left\{(x, y) \in R^2:\ \frac{1}{x} > \frac{1}{y},\ x \neq 0,\ y \neq 0\right\}$.

1.30. $\{(x, y) \in R^2: y^2 > 2x + 1\}$.

Quyida berilgan A va B to'plamlarning $A \times B$ dekart ko'paytmasini toping:

1.31. $A = \{1, 3\}$, $B = \{2, 4\}$. **1.32.** $A = R^1$, $B = R^1$.

1.33. $A = [1; 2]$, $B = [1; 2]$. **1.34.** $A = [1; 3]$, $B = [2; 4]$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

1.1. $A=\{2, 6\}$. **1.2.** $A=\{0, 2, 3\}$. **1.3.** $A=\{1\}$. **1.4.** $A=\{1, 2, 3, 4, 5, 6\}$. **1.5.** $A=\{-2, -1, 0, 1, 2\}$. **1.6.** $A=\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$. **1.7.** Yo'q. **1.9.** Yo'q. **1.10.** Ha. **1.11.** Yo'q. **1.12.** Yo'q. **1.13.** $\varnothing$. **1.14.** $\varnothing$, $\{\varnothing\}$. **1.15.** $\{1\}$, $\{2\}$, $\{1, 2\}$, $\varnothing$. **1.16.** $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{b, c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, $\varnothing$. **1.18.** 1) $A\cup B=\{-5, 3, 4\}$, $A\cap B=\{4\}$, $A\backslash B=\{-5\}$, $B\backslash A=\{3\}$, $A\Delta B=\{-5, 3\}$, 2) $A\cup B=[-2, 4]$, $A\cap B=\{1; 3\}$, $A\backslash B=[-2, 1)$, $B\backslash A=(3; 4]$, $A\Delta B=[-2; 1)\cup(3; 4]$. 3) $A\cup B=\{-2, -1, 0, 1, 2, 3, 4\}$, $A\cap B=\{1, 2, 3\}$, $A\backslash B=\{4\}$, $B\backslash A=\{-2, -1, 0\}$, $A\Delta B=\{-2; -1, 0, 4\}$. **1.19.** 1) $B\cup C=\{2, 3\}$, 2) $A\cap B\cap C=\varnothing$, 3) $A\cup B\cup C=\{2, 3, 4, 5, 6\}$, 4) $(A\cap B)\cup(B\cap C)=\{2; 3\}$. **1.31.** $A\times B=\{(1; 2), (1; 4), (3; 2), (3; 4)\}$. **1.32.** $A\times B=R^1\times R^1=R^2=\{(x, y): -\infty<x<\infty, +\infty<y<\infty\}$. **1.33.** $A\times B=\{(x, y): x\in[1; 2], y\in[1; 2]\}$. **1.34.** $A\times B=\{\{(x, y): x\in[1; 3], y\in[2; 4]\}$.

2-§. HAQIQIY SONLAR

2.1. Sonli to'plamlar. Sanoq uchun ishlatiladigan sonlar *natural sonlar* deb ataladi. Natural sonlar to'plami

$$N=\{1, 2, 3, ..., n, ...\}$$

kabi belgilanadi.

Ishorasi natural sonlarning ishorasiga qarama-qarshi bo'lgan sonlar *manfiy natural* sonlar deyiladi. Barcha manfiy natural sonlar, nol soni va barcha natural sonlardan iborat to'plam *butun sonlar to'plami* deyiladi va u odatda

$$Z=\{..., -n, ..., -3, -2, -1, 0\ 1, 2, 3, ..., n, ...\}$$

kabi belgilanadi.

Ravshanki, natural sonlar to'plami butun sonlar to'plamining qism to'plamidir: $N\subset Z$.

Qisqarmaydigan $\frac{p}{q}$, $p\in Z$, $q\in N$ kasr ko'rinishda tasvirlanadigan har bir son *rasional son* deyiladi. Barcha ratsional sonlar to'plami

$$Q=\left\{x: x=\frac{p}{q},\ p\in Z,\ q\in N\right\}$$

kabi belgilanadi.

Ravshanki, $Z\subset Q$, demak, $N\subset Q$, chunki $N\subset Z$. Jumladan, agar $q=1$ bo'lsa, $Z=Q$ bo'ladi.

2.2. Haqiqiy sonlar. Ma'lumki, haqiqiy sonlar nazariyasi asosan, ikki xil — konstruktiv va aksiomatik usullar bilan quriladi.

Konstruktiv usul. Haqiqiy sonlar nazariyasini bu usul yordamida qurishda, odatda, ratsional sonlar nazariyasi ma'lum deb faraz qilinadi va irratsional sonlarni qurish yo'llari ko'rsatiladi.

Aksiomatik usul. Haqiqiy sonlar nazariyasini bu usulda qurish jarayoni natural sonlar nazariyasini aksiomatik qurishdan boshlanadi. Bu jarayonni sxematik quyidagicha izohlash mumkin: biror bo'sh bo'lmagan N to'plam olinib, bu to'plam elementlari orasidagi munosabatlar, amallar, amallar bo'ysunadigan qonunlar, bu amallar va munosabatlarning bog'lanishi qandaydir aksiomalar sistemasi orqali beriladi va u natural sonlar to'plami, uning elementlari esa *natural sonlar* deb aytiladi.

Haqiqiy sonlar nazariyasini qurishda bir nechta konstruktiv nazariyalar mavjud bo'lib, ularga Kantor, Dedekind, Veyershtrass, Koshi tomonidan taklif qilingan nazariyalar kiradi.

Biz haqiqiy sonlar nazariyasini qurishda qisqacha Dedekind nazariyasini bayon qilish bilan chegaralanamiz. Haqiqiy sonlarni qurishda Dedekind nazariyasi ratsional sonlar to'plami Q ning tartiblanganlik xossasiga asoslangan bo'lib, u quyidagicha quriladi: Q bo'sh bo'lmagan va kesishmaydigan ikkita, quyidagi shartlarni qanoatlantiradigan, A va B sinflarga ajratiladi:

1) $A\cup B=Q$;
2) $\forall a\in A,\quad \forall b\in B \Rightarrow a<b$.

Q ni bunday tartibda sinflarga ajratishda A va B to'plamlar Q da kesim bajaradi deyiladi va u (A, B) kabi belgilanadi, bunda A — kesimning quyi sinfi, B esa uning yuqori sinfi deyiladi.

Misollar: Q — rasional sonlar to'plami berilgan bo'lib, uni quyidagicha sinflarga ajratamiz:

1) $A=\{a: a\in Q,\ a<3\},\ B=\{b: b\in Q,\ b\geq 3\}$;
2) $A=\{a: a\in Q,\ a\leq 3\},\ B=\{b: b\in Q,\ b>3\}$;
3) $A=\{a: a\in Q,\ a^2<3\},\ B=\{b: b\in Q,\ b^2>3\}$.

Osonlik bilan ko'rsatish mumkinki, Q ni yuqoridagi 1) — 3) ko'rinishlarda A va B sinflarga ajratishda A va B to'plamlar kesim bajaradi.

Haqiqatan ham, 1) da $A \neq \varnothing$, $B \neq \varnothing$, $Q = A \cup B$, $A \cap B = \varnothing$ bo'ladi. $\forall a \in A$, $\forall b \in B \Rightarrow a < b$ ekanligi kelib chiqadi.

$3 \in B$ bo'lib, u B to'plamda eng kichik son bo'ladi, lekin A da eng katta son yo'q. Teskarisini faraz qilaylik, ya'ni a son A to'plamda eng katta son bo'lsin desak, $a < 3$ bo'lgani uchun ratsional sonlarning zichlik xossasiga asosan, a va 3 sonlar orasida birorta ratsional a_1 sonni ko'rsatish mumkin, ya'ni $a < a_1 < 3$. Natijada, $a_1 \in A$ ekanligi kelib chiqadi. Demak, a soni A sinfda eng katta son bo'lsin, degan farazimiz noto'g'ri ekanligi kelib chiqadi. Shunday qilib, A da eng katta son mavjud emasligiga ishonch hosil qildik.

Xuddi shunday, Q to'plamni 2) shaklda sinflarga ajratganda ham, A va B to'plamlar Q da kesim bajarishini hamda A sinfda eng katta son mavjud va u 3 ga tengligi, B sinfda esa eng kichik son mavjud emasligi ko'rsatiladi.

3) $A = \{a : a \in Q,\ a^2 < 3\}$, $B = \{b : b \in Q,\ b^2 > 3\}$ deylik.

Q to'plamni 3) shaklda sinflarga ajratishda A va B to'plamlarning Q to'plamda kesim bajarishiga ishonch hosil qilish qiyin emas. Bu holda quyi sinf A da eng katta va yuqori sinf B da esa, eng kichik son yo'q ekanligini ko'rsatamiz. Shundan, birinchi tasdiqning to'g'riligini ko'rsatamiz.

Haqiqatan ham, $\forall a \in A$ bo'lsa, $a^2 < 3$ bo'ladi. Shunday natural n son mavjud bo'lib, $a^2 < 3$ tengsizlikni qanoatlantiruvchi a bilan birga, $a + \frac{1}{n}$ son ham A ga tegishli, ya'ni $\left(a + \frac{1}{n}\right)^2 < 3$ bo'lishligini ko'rsatamiz. Bu tengsizlikdan:

$$\frac{2a}{n} + \frac{1}{n^2} < 3 - a^2. \tag{2.1}$$

$\frac{2a}{n} + \frac{1}{n^2} < \frac{2a}{n} + \frac{1}{n}$ tengsizlik o'rinli bo'lgani uchun, agar $\frac{2a}{n} + \frac{1}{n} < 3 - a^2$ tengsizlik o'rinli bo'lsa, (2.1) tengsizlik albatta bajariladi. Keyingi tengsizlikdan $n > \frac{2a+1}{3-a^2}$ bo'ladi.

Shunday qilib, $a \in A$ son A to'plamda eng katta son bo'lsin desak, undan katta $a + \frac{1}{n}$ son topiladiki, u ham A ga tegishli bo'ladi.

Demak, A to'plamda eng katta son yo'q ekan. Xuddi yuqoridagidek, B to'plamda ham eng kichik son yo'qligi ko'rsatiladi.

Bu misollardan ko'rinadiki, 1) da A sinfda eng katta son mavjud emas, B sinfda esa, eng kichik son mavjud. 2) da A sinfda eng katta son mavjud, B sinfda esa, eng kichik son mavjud emas. 3) da esa, A sinfda eng katta son, B sinfda eng kichik son mavjud emas. Shunday qilib, Q da bajariladigan kesimlar faqat uch xil bo'lishi mumkin ekan:

1) quyi A sinfda eng katta son mavjud emas, yuqori B sinfda eng kichik son mavjud;

2) quyi A sinfda eng katta son mavjud, yuqori B sinfda eng kichik son mavjud emas;

3) quyi A sinfda eng katta son, yuqori B sinfda eng kichik son mavjud emas.

1) va 2) hollarda (A, B) kesim biror chegaraviy ratsional r sonni aniqlaydi. 3) holda A va B sinflarni chegaralovchi chegaraviy ratsional son yo'q, ya'ni kesim hech qanday ratsional sonni aniqlamaydi.

1) va 2) hollarda kesim ratsional kesim, 3) holdagi kesim esa irratsional kesim deyiladi. Irratsional kesim hosil qiluvchi sonlar to'plamini I orqali belgilaymiz.

Barcha ratsional va irratsional sonlar to'plami birgalikda *haqiqiy sonlar to'plami* deyiladi va u R bilan belgilanadi, ya'ni $R=Q\cup I$.

Osonlik bilan ko'rish mumkinki, $R\backslash Q=I$, $R\backslash I=Q$, $R\cap Q=Q$, $Q\cap I=\varnothing$, $I\subset R$, $Q\subset R$ bo'ladi.

2.1-misol. c — butun sonning kvadratiga teng bo'lmagan musbat son bo'lsin. Ushbu $A=\{a: a\in Q,\ a^2<c\}$, $B=\{a: a\in Q,\ a^2>c\}$ ko'rinishda qurilgan A va B to'plamlar Q ratsional sonlar to'plamida kesim bajaradi va bu kesim biror haqiqiy $\sqrt{c}$ sonni aniqlaydi. Quyi sinf A da eng katta son va yuqori sinf B da esa eng kichik son mavjud emasligini isbotlang.

Isboti. $a\in B$ bo'lsin, unda $a^2<c$ bo'ladi. $\left(a-\frac{1}{n}\right)^2>c$ tengsizlikni qanoatlantiradigan shunday natural n sonning mavjudligini ko'rsatamiz.

$$\left(a-\frac{1}{n}\right)^2=a^2-\frac{2a}{n}+\frac{1}{n^2}>c. \tag{2.2}$$

Agar $a^2-\frac{2a}{n}>c$ bo'lsa, u vaqtda (2.2) tengsizlik albatta bajariladi. Keyingi tengsizlikdan $n>\frac{2a}{a^2-c}$ deb olish yetarli. ($n=\left[\frac{2a}{a^2-c}\right]+1$ deb olish mumkin).

Demak, B sinfdan har qanday a son olinganda ham, B sinfda undan kichik son har doim topiladi. Shunday qilib, B sinfda eng kichik sonning mavjud emasligi isbotlandi. A sinfda eng katta son mavjud emasligi ham xuddi shunday isbotlanadi.

2.2-misol. Ushbu $\sqrt{5}+\sqrt{3}$ sonning irratsional son ekanligini isbotlang.

Isboti. Teskarisini faraz qilaylik, ya'ni $\sqrt{5}+\sqrt{3}$ son ratsional son bo'lsin. U holda

$$\sqrt{5}+\sqrt{3}=\frac{2}{\sqrt{5}-\sqrt{3}},$$

bu son ham ratsional son bo'ladi, chunki u ikkita ratsional sonning nisbatidan iborat. Ikkinchi tomondan, ravshanki,

$$\sqrt{3}=\frac{1}{2}\left[(\sqrt{5}+\sqrt{3})-(\sqrt{5}-\sqrt{3})\right]$$

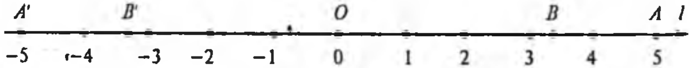
tenglik o'rinli. Bundan, $\sqrt{3}$ ikkita ratsional sonning ayirmasi sifatida ko'riladi, ya'ni ratsional son bo'ladi, lekin $\sqrt{3}$ son — irratsional sondir. Bu qarama-qarshilik farazimizning noto'g'ri ekanligini ko'rsatadi. Demak, $\sqrt{5}+\sqrt{3}$ irrasional son ekan.

2.3. Son o'qi. Biror l to'g'ri chiziqda ixtiyoriy O nuqtani belgilab (O nuqta sanoq boshi), so'ngra [0; 1] birlik kesmani tanlaymiz va yo'nalishni belgilaymiz. Bunday holda *koordinata to'g'ri chizig'i*, ya'ni *son o'qi* berilgan deyiladi (2.1-chizma). Har bir natural yoki kasr songa l to'g'ri chiziqda bitta nuqta mos keladi. Masalan, 5 soni berilgan bo'lsin. O nuqtadan (sanoq boshidan) berilgan yo'nalishda birlik kesmani 5 marta qo'yamiz. Natijada A nuqtani hosil qilamiz, bu nuqta 5 soniga mos keladi. $3\frac{1}{7}$ sonini olaylik, O nuqtadan berilgan yo'nalishda birlik kesmani 3 marta, so'ngra birlik kesmaning $\frac{1}{7}$ qismini qo'yamiz. Natijada B nuqtani hosil qilamiz. Bu nuqta $3\frac{1}{7}$ songa mos keladi.

Agar l to'g'ri chiziqning M nuqtasi biror r songa mos kelsa, bu son M nuqtaning *koordinatasi* deyiladi va u $M(r)$ kabi belgilanadi. Masalan, A va B nuqtalarning koordinatalari mos ravishda 5 va $3\frac{1}{7}$ sonlardan iborat bo'ladi, ya'ni $A(5)$, $B\left(3\frac{1}{7}\right)$. Sanoq boshi O nuqtaning koordinatasi nol sonidan iborat bo'ladi.

Endi birlik kesmani O nuqtadan boshlab berilgan yo'nalishga qarama-qarshi yo'nalishda 5 marta qo'yamiz. Sanoq boshi O ga nisbatan A nuqtaga simmetrik A' nuqtani hosil qilamiz. A' nuqtaning koordinatasi -5 son bo'ladi, ya'ni $A'(-5)$. Shunga o'xshash B nuqtaga simmetrik B' nuqtaning koordinatasi $-3\frac{1}{7}$ soni topiladi. 5 va -5, $3\frac{1}{7}$ va $-3\frac{1}{7}$ sonlarga mos ravishda *qarama-qarshi sonlar* deb ataladi. Koordinata to'g'ri chizig'ida berilgan yo'nalishda joylashgan nuqtalarga mos keluvchi sonlar *musbat sonlar* deyiladi. Koordinata to'g'ri chizig'ida berilgan yo'nalishga qarama-qarshi yo'nalishda joylashgan nuqtalarga mos keluvchi sonlar *manfiy sonlar* deyiladi.

Eslatma. Koordinata boshi O nuqtaga mos kelgan «0» (nol) soni musbat ham, manfiy ham hisoblanmaydi, u koordinata to'g'ri chizig'idagi musbat koordinatali nuqtalarni manfiy koordinatali nuqtalardan ajratib turuvchi son bo'lib hisoblanadi.



2.1-chizma.

Koordinata to'g'ri chizig'idagi berilgan yo'nalishni (odatda u o'ng tomonga yo'nalgan) *musbat*, berilgan yo'nalishga qarama-qarshi yo'nalishni esa *manfiy yo'nalish* deyiladi.

2.4. Haqiqiy sonning absolut qiymati. Haqiqiy son a ning absolut qiymati deb, agar $a \geq 0$ bo'lsa, bu sonning o'ziga, agar $a < 0$ bo'lsa, unga qarama-qarshi son $-a$ ga aytiladi. a sonning absolut qiymati $|a|$ kabi belgilanadi. Shunday qilib,

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

Geometrik nuqtayi nazardan, $|a|$ ifoda koordinata to'g'ri chizig'idagi a nuqtadan O nuqtagacha bo'lgan masofani bildiradi.

Haqiqiy sonning absolut qiymati quyidagi xossalarga ega:

1-xossa. $\forall x \in R$ uchun

$$|x| \geq 0, \ |x| = |-x|, \ x \leq |x|, \ -x \leq |x|$$

munosabatlar o'rinlidir.

2-xossa. $a > 0$, $|x| < a \Leftrightarrow -a < x < a$, $|x| \leq a \Leftrightarrow -a \leq x \leq a$ munosabatlar o'rinlidir.

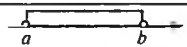
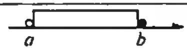
3-xossa. $\forall x,\ y \in R$

$$|x+y| \le |x|+|y|, \quad |x-y| \ge ||x|-|y||,$$
$$|x\cdot y| = |x|\cdot|y|, \quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}\ (y \ne 0)$$

munosabatlar o'rinlidir.

2.5. Sonli to'plamlarning chegaralari. $a < b$ shartni qanoatlantiradigan a va b sonlarni olamiz va ularni koordinata to'g'ri chizig'ida nuqtalar bilan belgilaymiz.

Amalda «interval», «kesma», «yarim interval», «nur» atamalaridan ko'pincha foydalanmasdan, ular bir nom bilan, «sonli oraliq» deb ishlatiladi.

Oraliqlar turi	Geometrik tasviri	Belgilanishi	Tengsizliklar yordamida yozilishi
Interval		(a, b)	$a<x<b$
Kesma		$[a, b]$	$a \le x \le b$
Yarim interval		$(a, b]$	$a < x \le b$
Yarim interval		$[a, b)$	$a \le x < b$
Nur		$[a, +\infty)$	$x \ge a$
Nur		$(-\infty, b]$	$x \le b$
Ochiq nur		$(a, +\infty)$	$x > a$
Ochiq nur		$(-\infty; b)$	$x < b$
Son o'qi		$(-\infty, \infty)$	$-\infty < x < \infty$

Biror X to'plam ($X \subset R$) berilgan bo'lsin.

2.1-ta'rif. Agar shunday M son (m son) mavjud bo'lib, $\forall x \in X$ uchun $x \le M$ ($x \ge m$) tengsizlik bajarilsa, X to'plam *yuqoridan (quyidan) chegaralangan* deyiladi. Masalan, $X = \left\{\frac{1}{2n} : n = 1, 2, 3, ...\right\}$ to'plam yuqoridan $\frac{1}{2}$ (quyidan 0) bilan chegaralangan.

2.2-ta'rif. Agar $\forall M$ son ($\forall m$ son) olinganda ham shunday $x_0 \in X$ topilsaki, $x_0 > M$ ($x_0 < m$) tengsizlik bajarilsa, X to'plam *yuqoridan (quyidan) chegaralanmagan* deyiladi.

Masalan, $Z=\{..., -n, ..., -3, -2, -1, 0, 1, 2, 3, ..., n, ...\}$ to'plam ham yuqoridan ham quyidan chegaralanmagan.

2.3-ta'rif. Agar X to'plam ham quyidan, ham yuqoridan chegaralangan bo'lsa, *X to'plam chegralangan* deyiladi.

2.1-teorema. *Har qanday yuqoridan chegaralangan to'plam uchun uni yuqoridan chegaralovchi sonlar ichida eng kichigi mavjud.*

2.4-ta'rif. Yuqoridan chegaralangan X to'plam uchun yuqoridan chegaralovchi sonlarning ichida eng kichigi X to'plamning *aniq yuqori chegarasi* deyiladi va $\sup X$ kabi belgilanadi.

2.2-teorema. *Har qanday quyidan chegaralangan to'plam uchun uni quyidan chegaralovchi sonlar ichida eng kattasi mavjud.*

2.5-ta'rif. Quyidan chegaralangan X to'plam uchun quyidan chegaralovchi sonlarning ichida eng kattasi X to'plamning *aniq quyi chegarasi* deyiladi va $\inf X$ kabi belgilanadi.

To'plamning aniq yuqori hamda aniq quyi chegaralarini mos ravishda quyidagicha ham ta'riflash mumkin, ya'ni

$$\sup X = a \Leftrightarrow \begin{cases} 1)\ \forall x \in X, & x \le a, \\ 2)\ \forall \varepsilon > 0\ \ \exists x_0 \in X, & x_0 > a - \varepsilon. \end{cases}$$

$$\inf X = b \Leftrightarrow \begin{cases} 1)\ \forall x \in X, & x \ge b, \\ 2)\ \forall \varepsilon > 0\ \ \exists x_0 \in X, & x_0 < b + \varepsilon. \end{cases}$$

To'plamning aniq quyi va aniq yuqori chegaralari quyidagi xossalarga ega:

1-xossa. Agar $X(X\subset R)$ to'plam yuqoridan chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\sup X_1 \le \sup X$ bo'ladi.

2-xossa. Agar $X(X\subset R)$ to'plam quyidan chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\inf X_1 \ge \inf X$ bo'ladi.

3-xossa. Agar $X(X\subset R)$ to'plam chegaralangan bo'lib, $X_1 \subset X$ bo'lsa, $\inf X \le \inf X_1 \le \sup X_1 \le \sup X$ bo'ladi.

4-xossa. Agar $\forall x \in X$ uchun $x \le a$ $(x \ge b)$ tengsizlik bajarilsa, $\sup X \le a$ $(\inf X \ge b)$ tengsizlik o'rinli bo'ldi.

Mustaqil yechish uchun misol va masalalar

2.1. Ushbu $\sqrt{5}+\sqrt{2}$ sonning irratsional son ekanligini isbotlang.

2.2. 1) $\sqrt{5}$; 2) $2^{\sqrt{2}}$ sonni aniqlovchi kesim tuzing.

2.3. Kesim yordamida

1) $\sqrt{2}+\sqrt{8}=\sqrt{18}$; 2) $\sqrt{3}+\sqrt{12}=\sqrt{27}$

ekanligini ko'rsating.

2.4. Kesim yordamida

1) $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$; 2) $\sqrt{3}\cdot\sqrt{5}=\sqrt{15}$

ekanligini ko'rsating.

2.5. Kesim yordamida $\sqrt{3}+(-\sqrt{3})=0$ bo'lishini ko'rsating.

2.6. Ushbu

1) $X=\left\{\frac{1}{n}: n=1, 2, 3, \ldots\right\}$; 2) $Y=\{n: n=1, 2, 3, \ldots\}$;

3) $Z=\{z: 4<z<7\}$; 4) $W=\{w: -2<w<4\}$

to'plamlarning aniq yuqori chegarasi hamda aniq quyi chegaralarini aniqlang.

2.7. Ushbu $\left\{\frac{m}{n}\right\}=\left\{\frac{m}{n}: m\in N,\ n\in N,\ m<n\right\}$ to'plamning aniq yuqori chegrasi $\sup\left\{\frac{m}{n}\right\}=1$, to'plamning aniq quyi chegarasi $\inf\left\{\frac{m}{n}\right\}=0$ bo'lishini isbotlang.

2.8. $X=\{x\}$ va $Y=\{y\}$ haqiqiy sonlar to'plamlari berilgan bo'lib, $\{x+y\}$ to'plam esa $\{x+y: x\in X,\ y\in Y\}$ yig'indilardan iborat to'plam bo'lsin. U holda

1) $\sup\{x+y\}=\sup\{x\}+\sup\{y\}$,

2) $\inf\{x+y\}=\inf\{x\}+\inf\{y\}$

bo'lishini isbotlang.

2.9. $X=\{x\}$ va $Y=\{y\}$ manfiy bo'lmagan haqiqiy sonlar to'plamlari berilgan bo'lib, $\{x\cdot y\}$ to'plam esa $\{x\cdot y: x\in X,\ y\in Y\}$ ko'paytmalardan iborat to'plam bo'lsin. U holda

1) $\sup\{x\cdot y\}=\sup\{x\}\cdot\sup\{y\}$,

2) $\inf\{x\cdot y\}=\inf\{x\}\cdot\inf\{y\}$

bo'lishini isbotlang.

2.10. $X=\{x\}$ haqiqiy sonlar to'plami berilgan bo'lib, $\{-x\}$ to'plam $-x$ sonlardan $(x\in X)$ iborat to'plam bo'lsin. U holda

1) $\sup\{-x\}=-\inf\{x\}$,

2) $\inf\{-x\}=-\sup\{x\}$

bo'lishini isbotlang.

2.11. Tengsizlikni yeching:

1) $|2x-5|\le 7$; 2) $|3x-5|>10$;

3) $\left|\frac{x-2}{x-1}\right|>2;$ 4) $|x-2|+|x+2|\le 12;$

5) $|x-1|-|x-2|+|x-3|-|x-4|+|x-5|<3.$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

2.6. 1) $\sup X=1$; $\inf X=0$; 2) $\inf Y=1$; 3) $\sup Z=7$; $\inf Z=4$; 4) $\sup W=4$; $\inf W=-2$. **2.11.** 1) $-1\le x\le 6$; 2) $x<-\frac{5}{3}$, $x>5$; 3) $(0;1)\cup(1;4)$; 4) $[-6;6]$; 5) $(0;2)\cup(2;4)\cup\cup(4;6)$.

3-§. MATEMATIK INDUKSIYA USULI

3.1. Yig'indi. $a_1, a_2, \ldots, a_n$ sonlar berilgan bo'lsin. Ularning $a_1+a_2+a_3+\ldots+a_n$ yig'indisini $\sum_{k=1}^{n} a_k$ deb belgilaymiz, ya'ni

$$\sum_{k=1}^{n} a_k = a_1+a_2+a_3+\ldots+a_n,$$

bunda k yig'indining *indeksi* deyiladi. Yig'indining indeksi qanday harf bilan belgilanishiga bog'liq emas, ya'ni

$$\sum_{k=1}^{n} a_k = \sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i .$$

Yig'indi chiziqlilik xossasiga ega, ya'ni ixtiyoriy α va β sonlar uchun

$$\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha\sum_{k=1}^{n} a_k + \beta\sum_{k=1}^{n} b_k$$

tenglik o'rinli bo'ladi.

$$S_n = \sum_{k=1}^{n} f(k)$$

yig'indini hisoblash masalasini qaraymiz, bunda $f(k)$ berilgan funksiya, odatda S_n yig'indini hisoblashda uni k ning funksiyasi deb qaraladi. Masalan, agar $f(k)=a_{k+1}-a_k$ bo'lsa (bunda $\{a_n\}$ berilgan ketma-ketlik), u holda

$$S_n = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} (a_{k+1} - a_k) = (a_2 - a_1) + (a_3 - a_2) + \dots + \qquad (3.1)$$
$$+ (a_n - a_{n-1}) + (a_{n+1} - a_n) = a_{n+1} - a_1$$

3.1-misol. Quyidagi yig'indini hisoblang:

$$1)\ \sum_{k=1}^{n} \frac{1}{k(k+2)}; \qquad 2)\ \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

Yechilishi. 1) $\frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ bo'lgani uchun (3.1) formulaga asosan

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2} - \frac{1}{2(n+1)} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right)$$

ekanligini topamiz.

$$2)\ \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right)$$

ekanligini hisobga olib, (3.1) formulaga asosan:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right).$$

3.2-misol. Ushbu yig'indini hisoblang:

$$S_n(x) = \sum_{k=1}^{n} \sin kx.$$

Yechilishi. Ushbu $S_n(x)\cdot 2\sin\frac{x}{2} = \sum_{k=1}^{n} 2\sin kx \sin\frac{x}{2}$ tenglikni qaraymiz.

$$2\sin kx \sin\frac{x}{2} = \cos\left(k - \frac{1}{2}\right)x - \cos\left(k + \frac{1}{2}\right)x$$

bo'lgani uchun (3.1) formulaga asosan:

$$S_n \cdot 2\sin\frac{x}{2} = \cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right) = 2\sin\frac{n+1}{2}x \cdot \sin\frac{n}{2}x,$$

bundan, agar $\sin\frac{x}{2}\neq 0$ bo'lsa, $S_n(x)=\dfrac{\sin\frac{n+1}{2}x\sin\frac{n}{2}x}{\sin\frac{x}{2}}$, agar $\sin\frac{x}{2}=0$ bo'lsa, unda $S_n(x)=0$. Shunday qilib,

$$S_n(x)=\begin{cases}\dfrac{\sin\frac{n+1}{2}\sin\frac{n}{2}x}{\sin\frac{x}{2}}, & \text{agar } \sin\frac{x}{2}\neq 0 \text{ bo'lsa,}\\ 0, \text{ agar } \sin\frac{x}{2}=0 \text{ bo'lsa.}\end{cases}$$

3.3-misol. $\{x_n\}$ ketma-ketlik ushbu $x_n=ax_{n-1}+b$ formula bilan berilganda: 1) x_n, 2) $S_n=\sum\limits_{k=1}^{n}x_k$ larni x_1, a, b va n orqali ifodalang.

Isboti. 1) $x_k=ax_{k-1}+b$, $x_{k-1}=ax_{k-2}+b$ bo'lgani uchun

$$x_k-x_{k-1}=a(x_{k-1}-x_{k-2})=a^2(x_{k-2}-x_{k-3})=\ldots=a^{k-2}(x_2-x_1)$$

bo'ladi, ya'ni

$$x_k-x_{k-1}=a^{k-2}(x_2-x_1).$$

Bu formulaga ketma-ket $k=2, 3, \ldots, n$ qiymatlarni qo'yib, so'ngra hosil bo'lgan tengliklarni qo'shish natijasida

$$\sum_{k=2}^{n}(x_k-x_{k-1})=(x_2-x_1)\sum_{k=2}^{n}a^{k-2} \quad \text{yoki}$$

$$x_n-x_1=(x_2-x_1)\frac{a^{n-1}-1}{a-1}=((a-1)x_1+b)\frac{a^{n-1}-1}{a-1}$$

ga ega bo'lamiz. Bundan

$$x_n=a^{n-1}x_1+b\frac{a^{n-1}-1}{a-1}, \quad a\neq 1$$

formulani hosil qilamiz. $a=1$ desak, u holda $\{x_n\}$ ketma-ketlikning ayirmasi b ga teng bo'lgan arifmetik progressiya ekanligiga ishonch hosil qilamiz, ya'ni $x_n=x_1+(n-1)b$.

2) $S_n=x_1+\sum\limits_{k=2}^{n}x_k=x_1+a\sum\limits_{k=2}^{n}x_{k-1}+(n-1)b,$

$$S_n = x_1 + a(S_n - x_n) + (n-1)b,$$

$$S_n(1-a) = x_1 - ax_n + (n-1)b = x_1 - a^n x_1 - ab\frac{a^{n-1}-1}{a-1} + (n-1)b,$$

bu yerdan

$$S_n = \frac{(n-1)b}{1-a} + \frac{ab}{(a-1)^2}(a^{n-1}-1) + \frac{a^n-1}{a-1}x_1, \quad a \neq 1$$

formulani hosil qilamiz.

3.4-misol. $\{a_n\}$ hadlari noldan farqli va ayirmasi $d \neq 0$ bo'lgan arifmetik progressiyani tashkil qilsa,

$$\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{n+1}}\right)$$

tenglikning o'rinli ekanligini isbotlang.

Isboti. $\frac{1}{a_k a_{k+1}} = \frac{1}{(a_{k+1}-a_k)}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right)$ tenglik o'rinli bo'lgani uchun hamda misolning shartini e'tiborga olgan holda

$$\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}} = \frac{1}{d}\sum_{k=1}^{n}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right)$$

ga ega bo'lamiz. Bundan (3.1) formulani e'tiborga olsak, natijada isbot qilinishi kerak bo'lgan tenglikni hosil qilamiz, ya'ni

$$\sum_{k=1}^{n}\frac{1}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{n+1}}\right).$$

3.5-misol. Ushbu yig'indilarni hisoblang:

1) $A_n(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$;

2) $B_n(x) = x + x^3 + x^5 + \ldots + x^{2n-1}$;

3) $C_n(x) = 1 + 3x^2 + 5x^4 + \ldots + (2n-1)x^{2n-2}$;

4) $D_n(x) = 1 + 4x + 9x^2 + \ldots + n^2x^{n-1}$;

5) $H_n(x) = 1 + 2^3x + 3^3x^2 + \ldots + n^3x^{n-1}$.

Yechilishi. 1) $x=1$ bo'lganda $A_n(1) = 1 + 2 + 3 + \ldots + n$.

Bundan arifmetik progressiya n ta hadining yig'indisini ifodalovchi

$$S_n = \frac{a_1 + a_n}{2}n = \frac{2a_1 + d(n-1)}{2}n \tag{3.2}$$

formulani e'tiborga olsak, $A_n(1) = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ bo'ladi. Endi $x \neq 1$ bo'lsin.

$$A_n(x) = 1 + 2x + 3x^2 + ... + nx^{n-1} \tag{3.3}$$

(3.3) tenglikning ikkala tomonini x ga ko'paytirib, hosil bo'lgan tenglikni (3.3) tenglikdan ayiramiz:

$$(1 - x)A_n = 1 + x + x^2 + ... + x^{n-1} - nx^n,$$

bundan geometrik progressiyaning dastlabki n ta hadining yig'indisi formulasini

$$S_n = b_1 \frac{1-q^n}{1-q} = \frac{b_n q - b_1}{q-1}, \quad q \neq 1 \tag{3.4}$$

e'tiborga olib,

$$(1 - x)A_n = \frac{1-x^n}{1-x} - nx^n, \quad A_n = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x} =$$

$$= \frac{1}{(1-x)^2} + \frac{nx^{n+1}-(1+n)x^n}{(1-x)^2}$$

ni topamiz. Shunday qilib, quyidagi formulaga ega bo'lamiz:

$$A_n(x) = \begin{cases} \frac{1}{(1-x)^2} + \frac{nx^{n+1}-(1+n)x^n}{(1-x)^2}, & x \neq 1, \\ \frac{n(n+1)}{2}, & x = 1. \end{cases}$$

2) $x = \pm 1$ bo'lganda $B_n(1) = 1 + 1 + ... + 1 = n,$

$$B_n(-1) = -1 - 1 - 1 - ... - 1 = -n.$$

$$B_n(x) = x + x^3 + x^5 + ... + x^{2n-1}$$

tenglikning ikkala tomonini x ga ko'paytiramiz:

$$xB_n(x) = x^2 + x^4 + x^6 + ... + x^{2n} = x^2(1 + x^2 + (x^2)^2 + ... + (x^2)^{n-1})$$

Bu tenglikning o'ng tomonidagi qavs ichidagi yig'indiga (3.4) formulani qo'llasak, natijada

$$xB_n(x) = x^2 \frac{1-(x^2)^n}{1-x^2} = x^2 \frac{1-x^{2n}}{1-x^2},$$

bundan

$$B_n(x)=\frac{x}{1-x^2}-\frac{x^{2n+1}}{1-x^2}$$

ga ega bo'lamiz. Shunday qilib, quyidagi formula o'rinli bo'ladi:

$$B_n(x)=\begin{cases}\frac{x}{1-x^2}-\frac{x^{2n+1}}{1-x^2}, & \text{agar } x\neq\pm1 \text{ bo'lsa},\\ \pm n, & \text{agar } x=\pm1 \text{ bo'lsa.}\end{cases}$$

3) $x=\pm1$ bo'lganda, $C_n(\pm1)=1+3+5+...+(2n-1)=n^2$. Endi $x\neq\pm1$ bo'lsin.

$$C_n(x)=1+3x^2+5x^4+...+(2n-1)x^{2n-2} \tag{3.5}$$

(3.5) tenglikning ikkala tomonini x^2 ko'paytirib, hosil bo'lgan tenglikni (3.5) tenglikdan ayiramiz:

$$(1-x^2)C_n(x)=1+2x^2+2x^4+2x^6+...+2x^{2n-2}-(2n-1)x^{2n}=$$
$$=1+2x^2(1+x^2+x^4+...+x^{2n-4})-(2n-1)x^{2n}.$$

(3.5) tenglikning o'ng tomonidagi qavs ichidagi ifodaga (3.4) formulani qo'llab, $(1-x^2)C_n(x)=1+2x^2\left(\frac{1-x^{2n-2}}{1-x^2}\right)-(2n-1)x^{2n}$ ga ega bo'lamiz. Bundan,

$$C_n(x)=\frac{1+x^2}{1-x^2}+\frac{(2n-1)x^{2n+2}-(2n+1)x^{2n}}{(1-x^2)^2}.$$

Shunday qilib,

$$C_n(x)=\begin{cases}\frac{1+x^2}{1-x^2}+\frac{(2n-1)x^{2n+2}-(2n+1)x^{2n}}{(1-x^2)^2}, & \text{agar } x\neq\pm1 \text{ bo'lsa},\\ \pm n, & \text{agar } x=\pm1 \text{ bo'lsa.}\end{cases}$$

4) $D_n(x)=1+4x+9x^2+...+n^2x^{n-1}$ tenglikning ikkala tomonini x ga ko'paytirib, hosil bo'lgan tenglikdan oldingi tenglikni ayiramiz:

$$D_n(x)(x-1)=\sum_{k=2}^{n+1}(k-1)^2x^{k-1}-\sum_{k=1}^{n}k^2x^{k-1}=$$
$$=\sum_{k=1}^{n}(k-1)^2x^{k-1}+n^2x^n-\sum_{k=1}^{n}k^2x^{k-1}=$$

$$= n^2x^n - \sum_{k=1}^{n}(2k-1)x^{k-1} = n^2x^n - 2\sum_{k=1}^{n}kx^{k-1} + \sum_{k=1}^{n}x^{k-1} =$$

$$= n^2x^n - 2\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} + \frac{x^n-1}{x-1}.$$

Bundan

$$D_n(x) = \frac{n^2x^n(x-1)^2-2nx^n(x-1)+(x^n-1)(x+1)}{(x-1)^3}.$$

5) $H_n(x) = 1 + 2^3x + 3^3x^2 + ... + n^3x^{n-1}$ yig'indi topish uchun 4) dagi singari ushbu

$$xH_n(x) - H_n(x) = n^3x^n - 3\sum_{k=1}^{n}k^2x^{k-1} + 3\sum_{k=1}^{n}k\,x^{k-1} - \sum_{k=1}^{n}x^{k-1}$$

ayirmani tuzamiz. Tenglikning o'ng tomonidagi yig'indilarning o'rniga 4) da olingan ifodalarni qo'ysak, natijada

$$(x-1)H_n(x) = n^3x^n - 3\frac{n^2x^n(x-1)^2-2nx^n(x-1)+(x^n-1)(x+1)}{(x-1)^3} +$$

$$+3\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} - \frac{x^n-1}{x-1}.$$

Bundan

$$H_n(x) = \frac{n^3x^n(x-1)^3-3n^2x^n(x-1)^2+3nx^n(x^2-1)-(x^n-1)(x^2+4x+1)}{(x-1)^4}.$$

Arifmetik va geometrik progressiyalar dastlabki n ta hadlari yig'indisini topish masalasi va ularga o'xshash ko'pgina masalalarni quyidagi umumiy masalaning xususiy holi deb qarash mumkin: $f(x)$ — barcha $x \geq 0$ lar to'plamida aniqlangan ixtiyoriy funksiya bo'lsa,

$$f(0) + f(1) + ... + f(n-1) \qquad (3.6)$$

yig'indini hisoblang. Oxirgi masala kabi masalalarni yechishda quyidagi teoremadan foydalanish maqsadga muvofiq bo'ladi.

3.1-teorema. *Agar berilgan $\varphi(x)$ funksiya, $x \geq 0$ lar to'plamida aniqlangan $f(x)$ funksiya uchun ushbu*

$$\varphi(x+1) - \varphi(x) = f(x) \qquad (3.7)$$

tenglamani qanoatlantirsa, u holda

$$f(0)+f(1)+...+f(n-1)=\varphi(n)-\varphi(0) \quad (3.8)$$

tenglik o'rinli.

3.1-teoremani qo'llab yechiladigan ba'zi misollarni qaraymiz.

3.6-misol. Geometrik progressiyaning dastlabki n ta hadi yig'indisini toping.

Yechilishi. $\varphi(x)=q^x$ deb olamiz. Unda

$$\varphi(x+1)-\varphi(x)=aq^{x+1}-aq^x=aq^x(q-1)$$

bo'ladi. Demak, $f(x)$ funksiya sifatida

$$f(x)=aq^x(q-1) \quad (3.9)$$

funksiyani hosil qilamiz. Endi (3.9) dan foydalanib, (3.6) yig'indini topamiz:

$$f(0)+f(1)+...+f(n-1)=a(q-1)+aq(q-1)+...+aq^{n-1}(q-1).$$

Bundan (3.8) tenglikni e'tiborga olib,

$$f(0)+f(1)+...+f(n-1)=\varphi(n)-\varphi(0)=aq^{n-1}-a$$

ni hosil qilamiz. Bundan geometrik progressiya dastlabki n ta hadi yig'indisi uchun talab qilingan

$$a+aq+...+aq^{n-1}=\frac{aq^{n-1}-a}{q-1}=\frac{a-aq^{n-1}}{1-q}$$

formulani hosil qilamiz.

3.7-misol. Arifmetik progressiyaning dastlabki n ta hadi yig'indisini toping.

Yechilishi. Bunda

$$\varphi(x)=\frac{a+[a+(x-1)d]}{2}x$$

deb olsak, (3.7) tenglama $\varphi(x+1)-\varphi(x)=a+dx=f(x)$ ko'rinishda bo'ladi. Bundan (3.6) yig'indini tuzamiz va (3.8)ni e'tiborga olib

$$f(0)+f(1)+...+f(n-1)=a+(a+d)+...+[a+(n-1)d]$$

$$a+(a+d)+...+[a+(n-1)d]=\varphi(n)-\varphi(0)=\frac{a+[a+(n-1)d]}{2}d$$

ni hosil qilamiz, bu esa talab qilingan formulani ifodalaydi.

3.8-misol. Ushbu $(n+1)^2-n^2$ ayirma 1 dan n gacha bo'lgan qanday sonlarning yig'indisini ifodalashini ko'rsating.

Yechilishi. $\varphi(x)=x^2$ deb olib, (3.7) tenglikni tuzamiz:

$$\varphi(x+1)-\varphi(x)=(x+1)^2-x^2=2x+1=f(x).$$

So'ngra, (3.6) yig'indini tuzsak, $1+3+5+\ldots+(2n-1)$ ifodani, ya'ni qaralayotgan ayirma 1 dan n gacha bo'lgan barcha toq sonlar yig'indisini ifodalashini olamiz. Qo'shimcha ravishda (3.8) formuladan foydalanib,

$$1+3+5+\ldots+(2n-1)=n^2$$

bo'lishini olamiz.

3.9-misol. Ushbu $\frac{n}{2n+1}$ ifoda qanday n ta son yig'indisini ifodalashini ko'rsating.

Yechilishi. $\varphi(x)=\frac{x}{2x+1}$ deb olib, (3.7) tenglamani tuzamiz:

$$\varphi(x+1)-\varphi(x)=\frac{x+1}{2x+3}-\frac{x}{2x+1}=\frac{1}{(2x+1)(2x+3)}=f(x).$$

Natijada izlanayotgan yig'indi uchun (3.6) ifodadan foydalanib,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}$$

ifodani hosil qilamiz. Endi (3.8) formulaga asosan,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

tenglikni olamiz.

3.10-misol. Ushbu $\frac{10^{n+1}-9n-10}{27}$ ifoda qanday n ta son yig'indisini beradi?

Yechilishi. $\varphi(x)=\frac{10^{n+1}-9n-10}{27}$ deb olib, (3.7) tenglamani tuzamiz:

$$\varphi(x+1)-\varphi(x)=f(x) \quad \text{yoki} \quad \frac{10^{x+1}-1}{3}=f(x).$$

Demak, (3.8) formulaga asosan,

$$3+33+333+...+\underbrace{333...3}_{n}=\frac{10^{n+1}-9n-10}{27}$$

tenglikni hosil qilamiz.

3.2. Matematik induksiya usuli. Har qanday matematik izlanishning asosida deduktiv va induktiv uslublar yotadi.

Umumiy xulosadan xususiy xulosalarni keltirib chiqarish usuli *deduktiv usul* deyiladi.

Xususiy tasdiqdan umumiy tasdiqni keltirib chiqarish usuli *induktiv* yoki *induksiya usuli* deyiladi.

Induksiya usuli to'la va to'la bo'lmasligi mumkin.

Agar tasdiq kuzatishga oxirigacha yetkazilmagan hollarga ham tegishli bo'lsa, bunga *to'la bo'lmagan (to'liqsiz) matematik induksiya usuli* deyiladi.

Agar mulohazalar ro'y berishi mumkin bo'lgan barcha hollarni o'z ichiga olsa va shu asosda xulosa qilinsa, bunday induksiya *to'la matematik induksiya usuli* deyiladi.

To'la matematik induksiya usuliga quyidagi tamoyil asos qilib olinadi: biror $p(n)$ tasdiq berilganda:

I) $n=1$ uchun $p(n)$ tasdiqning to'g'riligi tekshiriladi;

II) $n=k(k\in N)$ uchun $p(k)$ tasdiq to'g'ri deb faraz qilinganda, undan $n=k+1$ uchun $p(k+1)$ tasdiqning to'g'riligi kelib chiqsa, bu $p(n)$ tasdiq har qanday natural n uchun o'rinli bo'ladi, deb xulosa chiqariladi. Bu tamoyilning I) bandiga induksiya *bazisi*, II) bandiga esa *induksiya qadami* deyiladi.

Ba'zi hollarda $p(n)$ tasdiqni n ning faqat natural qiymatlari uchungina emas, balki uning Z to'plamga qarashli barcha qiymatlari uchun ham to'g'riligini isbotlash talab qilinadi. Bunday hollarda yuqoridagi to'la matematik induksiya usulidan foydalanish maqsadga muvofiq bo'ladi.

3.11-misol. Matematik induksiya usulidan foydalanib, ixtiyoriy $n(n\in N)$lar uchun ushbu

$$S_n=1+2+3+....+n=\frac{n(n+1)}{2} \qquad (3.10)$$

tenglikning to'g'riligini isbotlang.

Isboti. Bu yerda va bundan keyingi misollarda tasdiqlarni $p(n)$ orqali belgilaymiz.

1. $n=1$ bo'lganda, $S_1=1=\frac{1(1+1)}{2}=1$, demak, $p(1)$ tasdiq to'g'ri.

II. Ixtiyoriy k natural son uchun $p(k)$ tasdiq o'rinli, ya'ni

$$S_k = 1+2+3+....+k = \frac{k(k+1)}{2}$$

tenglik o'rinli bo'lsin. Bu tenglikning ikkala tomoniga $k+1$ ni qo'shib,

$$S_k + k + 1 = 1+2+3+...+k+k+1 = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

ni hosil qilamiz.

Demak,

$$S_{k+1} = 1+2+3+...+k+k+1 = \frac{(k+1)(k+2)}{2},$$

bu esa $p(k+1)$ tasdiqning o'rinli ekanligini isbotlaydi. Shunday qilib, matematik induksiya usuliga ko'ra (3.1) tenglik $\forall n(n \in N)$lar uchun to'g'ri ekan.

3.12-misol. Barcha $\forall n(n \in N)$ lar uchun ushbu

$$S_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (3.11)$$

tenglikning to'g'riligini isbotlang.

Isboti. I. $n=1$ bo'lganda $S_1 = 1 = \frac{1(1+1)(2+1)}{6} = 1$ bo'ladi. Demak, $p(1)$ tasdiq to'g'ri.

II. Endi ixtiyoriy k natural son uchun $p(k)$ tasdiq, ya'ni

$$S_k = 1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \quad (3.12)$$

tenglik o'rinli bo'lsin, deb faraz qilib, ushbu

$$S_{k+1} = 1^2 + 2^2 + 3^2 + ... + n^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \quad (3.13)$$

tenglikning to'g'riligini isbotlaymiz. (3.12) tenglikning ikkala tomoniga $(k+1)^2$ ni qo'shib,

$$S_k + (k+1)^2 = 1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad (3.14)$$

ni hosil qilamiz. Bu tenglikning o'ng tomonini quyidagicha shakl almashtiramiz:

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}. \qquad (3.15)$$

Shunday qilib, (3.14) va (3.15) dan (3.13) tenglikka ega bo'lamiz, ya'ni $p(k+1)$ tasdiqning o'rinli ekanligi isbot bo'ladi.

Demak, matematik induksiya usuliga ko'ra (3.11) tenglik $\forall\, n(n\in N)$lar uchun to'g'ri ekan.

3.13-misol. Barcha $n(n\in N)$ sonlar uchun ushbu

$$S_n=11^{n+2}+12^{2n+1}$$

sonning 133 ga karrali ekanligini isbotlang.

Isboti. I. $n=1$ bo'lganda $S_1=11^{1+2}+12^{2+1}=11^3+12^3=133\cdot 23$ bo'lib, S_1 son 133 ga karrali. Demak, $p(1)$ tasdiq to'g'ri.

II. $n=k$ natural son uchun S_k son 133 ga karrali bo'lsin, ya'ni $S_k=133\cdot m\,(m\in N)$ deb faraz qilib, $n=k+1$ bo'lganda S_{k+1} sonning 133 ga karrali ekanligini, ya'ni $S_{k+1}=133\cdot q\,(q\in N)$ bo'lishini isbotlaymiz.

Haqiqatan ham,

$$S_{k+1}=11^{k+3}+12^{2k+3}=11\cdot 11^{k+2}+12^2\cdot 12^{2k+1}=$$
$$=11\cdot 11^{k+2}+144\cdot 12^{2k+1}=11\cdot(11^{k+2}+12^{2k+1})+133\cdot 12^{2k+1}=$$
$$=11\cdot S_k+133\cdot 12^{2k+1}=11\cdot 133\cdot m+133\cdot 12^{2k+1}=$$
$$=133\cdot(11\cdot m+12^{2k+1})=133\cdot q,$$

bunda $q=11\cdot m+12^{2k+1}$.

Demak, matematik induksiya usuliga asosan, $\forall\, n(n\in N)$lar uchun S_n sonning 133 ga karrali ekanligi isbotlandi.

3.14-misol. Ushbu

$$\cos\alpha\cdot\cos 2\alpha\ldots\cos 2^n\alpha=\frac{\sin 2^{n+1}\alpha}{2^{n+1}\cdot\sin\alpha}$$

ayniyatni isbotlang.

Isboti. I. $n=0$ bo'lganda, $\cos\alpha=\frac{\sin 2\alpha}{2\cdot\sin\alpha}$ bo'ladi. Demak, $p(0)$ tasdiq to'g'ri.

II. Endi ixtiyoriy k natural son uchun $p(k)$ tasdiq, ya'ni

$$\cos\alpha\cdot\cos 2\alpha\ldots\cos 2^k\alpha=\frac{\sin 2^{k+1}\alpha}{2^{k+1}\cdot\sin\alpha}$$

tenglik o'rinli bo'lsin, deb faraz qilib, $n=k+1$ bo'lganda $p(k+1)$ tasdiqning o'rinli ekanligini isbotlaymiz. Haqiqatan ham,

$$\cos\alpha\cdot\cos 2\alpha\ldots\cos 2^k\alpha\cdot\cos 2^{k+1}\alpha=\frac{\sin 2^{k+1}\alpha}{2^{k+1}\cdot\sin\alpha}\cdot\cos 2^{k+1}\alpha=\frac{\sin 2^{k+2}\alpha}{2^{k+2}\cdot\sin\alpha}.$$

Shunday qilib, matematik induksiya usuliga asosan, $n(n\in Z_0)$lar uchun berilgan ayniyat o'rinli ekan.

3.15-misol. Barcha $n>1$ $(n\in N)$ sonlar uchun ushbu

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$$

tengsizlikning to'g'riligini isbotlang.

Isboti. Tengsizlikning chap tomonini S_n bilan belgilaymiz.

I. $n=2$ bo'lganda $\frac{7}{12}=\frac{14}{24}>\frac{13}{24}$ bo'lib, bu esa $p(2)$ tasdiqning o'rinli ekanligini isbotlaydi.

II. $n=k$ natural son uchun $S_k>\frac{13}{24}$ tengsizlik to'g'ri deb faraz qilib, $n=k+1$ bo'lganda $S_{k+1}>\frac{13}{24}$ tengsizlikning to'g'riligini isbotlaymiz. Ushbu

$$S_k=\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k} \quad \text{va}$$

$$S_{k+1}=\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

ifodalarni taqqoslaymiz. Bulardan

$$S_{k+1}-S_k=\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{1}{k+1}=\frac{1}{2(k+1)(2k+1)}.$$

$\forall k\,(k\in N)$ uchun oxirgi tenglikning o'ng tomoni musbat bo'lganligidan, $S_{k+1}>S_k$ bo'ladi. O'z navbatida, $S_k>\frac{13}{24}$ bo'lganligi uchun, $S_{k+1}>\frac{13}{24}$ tengsizlik ham o'rinli bo'ladi. Bu esa $p(k+1)$ tasdiqning o'rinli ekanligini isbotlaydi.

Shunday qilib, matematik induksiya usuliga asosan, $n>1$ $(n\in N)$lar uchun berilgan tengsizlik o'rinli ekan.

3.16-misol. $\{x_n\}$ ketma-ketlik ushbu

$$x_1=2,\ x_2=3,\ x_n=4x_{n-1}-3x_{n-2}\ (n>2)$$

shartlar bilan berilgan bo'lsa, $x_n=\frac{1}{2}(3^{n-1}+3)$ bo'lishini isbotlang.

Isboti. $x_n = 4x_{n-1} - 3x_{n-2}$ rekurrent formuladan foydalanib,

$$x_3 = 4x_2 - 3x_1 = 6,\ \ x_4 = 4x_3 - 3x_2 = 15,\ \ x_5 = 4x_4 - 3x_3 = 42$$

sonlarni topamiz. Topilgan sonlarni quyidagicha yozish mumkin:

$$x_3 = \frac{1}{2}(3^{3-1} + 3) = 6,\ \ x_4 = \frac{1}{2}(3^{4-1} + 3) = 15,\ \ x_5 = \frac{1}{2}(3^{5-1} + 3) = 42\,.$$

Ushbu qonuniyatdan $\{x_n\}$ ketma-ketlikning umumiy hadi uchun

$$x_n = \frac{1}{2}(3^{n-1} + 3) \qquad (3.16)$$

formulani yozish mumkin.

Matematik induksiya usulidan foydalanib, ixtiyoriy $n\,(n \in N)$lar uchun (3.16) formulani isbotlaymiz.

I. $n = 1$ bo'lganda $x_1 = 2 = \frac{1}{2}(3^{1-1} + 3) = 2$ bo'ladi. Demak, $p(1)$ tasdiq to'g'ri.

II. $n=k$ natural son uchun $x_k = \frac{1}{2}(3^{k-1} + 3)$ tenglik to'g'ri deb faraz qilib, $n = k + 1$ bo'lganda $x_{k+1} = \frac{1}{2}(3^k + 3)$ tenglikning to'g'riligini isbotlaymiz.

Rekurrent formula va yuqorida qilingan farazimizdan foydalansak,

$$x_{k+1} = 4x_k - 3x_{k-1} = 2(3^{k-1} + 3) - \frac{3}{2}(3^{k-2} + 3) =$$
$$= 2 \cdot 3^{k-1} - \frac{1}{2} \cdot 3^{k-1} + 6 - \frac{9}{2} = \frac{1}{2}(3^k + 3).$$

Demak, matematik induksiya usuliga asosan, $\forall\, n(n \in N)$lar uchun berilgan (3.16) formula o'rinli ekan.

3.17-misol. Har qanday n burchakli ko'pburchakning ichki burchaklari yig'indisi $(n-2)\cdot 180^\circ$ ga tengligini isbotlang.

Isboti. I. $n=3$ bo'lganda uchburchakning ichki burchaklari yig'indisi 180° ga teng. Demak, $p(3)$ tasdiq to'g'ri.

II. $p(n)$ tasdiqni — ixtiyoriy $k(k < n)$ natural son uchun $(k-2)\cdot 180^\circ$ formula to'g'ri deb faraz qilamiz.

Ma'lumki, har qanday $n(n \geq 4)$ burchakli ko'pburchakning hech bo'lmaganda bitta diagonali butunlay uning ichida yotadi. Shuning uchun, har qanday n burchakli ko'pburchakni uning ichida yo-tuvchi bitta diagonali orqali ikkita ko'pburchakka ajratish mumkin. Agar bu ko'pburchaklardan bittasining tomonlari soni $k+1$ ga teng bo'lsa, qolgan ikkinchi ko'pburchakning tomonlari soni $n-k+1$ ga teng bo'ladi, bunda ikkala ko'pburchakning tomonlar soni n dan kichik bo'ladi. Yuqoridagi mulohazalarimizga ko'ra, hosil bo'lgan ko'pburchaklarning ichki burchaklari yig'indisi, mos ravishda, $(k-1)\cdot180^\circ$ va $(n-k-1)\cdot180^\circ$ ga teng bo'ladi. n burchakli ko'pburchakning ichki burchaklari yig'indisi hosil bo'lgan ko'pburchaklar ichki burchaklari yig'indisiga teng bo'ladi, ya'ni $(k-1)\cdot180^\circ+(n-k-1)\cdot180^\circ=(n-2)\cdot180^\circ$.

Demak, har qanday n burchakli ko'pburchakning ichki burchaklari yig'indisi uchun $(n-2)\cdot180^\circ$ formula o'rinli bo'lar ekan.

3.18-misol. Har qanday a va b sonlar hamda ixtiyoriy $n(n\in N)$ lar uchun ushbu

$$(a+b)^n = a^n + C_n^1a^{n-1}b + C_n^2a^{n-2}b^2 + \ldots + \\ + C_n^m a^{n-m}b^m + \ldots + C_n^{n-1}ab^{n-1} + b^n \qquad (3.17)$$

formula o'rinli ekanligini isbotlang, bunda C_n^m — n ta turli elementdan m tadan takrorlashsiz gruppalashlar soni $C_n^m = \dfrac{n!}{m!(n-m)!}$.

Isboti. $n=1$ bo'lganda $(a+b)^1=a+b$ bo'ladi. Demak, $p(1)$ tasdiq to'g'ri.

II. $n=k$ bo'lganda $p(k)$ tasdiq o'rinli, ya'ni

$$(a+b)^k = a^k + C_k^1a^{k-1}b + C_k^2a^{k-2}b^2 + \ldots + \\ + C_k^m a^{k-m}b^m + \ldots + C_k^{k-1}ab^{k-1} + b^k \qquad (3.18)$$

formula o'rinli bo'lsin, deb faraz qilamiz. Endi $n=k+1$ bo'lganda

$$(a+b)^{k+1} = a^{k+1} + C_{k+1}^1a^kb + C_{k+1}^2a^{k-1}b^2 + \ldots + \\ + C_{k+1}^m a^{k+1-m}b^m + \ldots + C_{k+1}^k ab^k + b^{k+1} \qquad (3.19)$$

formula o'rinli ekanligini isbotlaymiz. (3.18)ning ikkala tomonini $a+b$ ga ko'paytiramiz:

$$(a+b)^{k+1} = (a+b)^k(a+b) = (a^k + C_k^1a^{k-1}b + C_k^2a^{k-2}b^2 + \\ + \ldots + C_k^m a^{k-m}b^m + \ldots + C_k^{k-1}ab^{k-1} + b^k)\cdot(a+b) = \\ = a^{k+1} + (1+C_k^1)a^kb + (C_k^1+C_k^2)a^{k-1}b^2 + \ldots + \\ + (C_k^m + C_k^{m+1})a^{k-m}b^{m+1} + \ldots + b^{k+1}.$$

Bunda $C_k^0 = 1$, $C_k^m + C_k^{m+1} = C_{k+1}^{m+1}$ ekanligini e'tiborga olsak, natijada (3.19) ga ega bo'lamiz. Demak, matematik induksiya usuliga asosan, $\forall n (n \in N)$ lar uchun berilgan (3.17) formula o'rinli ekan.

Mustaqil yechish uchun misol va masalalar

3.1. Ushbu yig'indilarni hisoblang:

1) $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)}$; 2) $\sum_{k=1}^{n} \frac{1}{(4k-3)(4k+1)}$;

3) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)(2k+3)}$; 4) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)}$.

3.2. $\{a_n\}$ — hamma hadlari va ayirmasi d noldan farqli arifmetik progressiya bo'lganda quyidagi tengliklarni isbotlang:

1) $\sum_{k=1}^{n} \frac{1}{a_k a_{k+1} a_{k+2}} = \frac{1}{2d}\left(\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}}\right)$;

2) $\sum_{k=1}^{n} \frac{1}{a_k a_{k+1} a_{k+2} a_{k+3}} = \frac{1}{3d}\left(\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1} a_{n+2} a_{n+3}}\right)$.

3.3. Ushbu tengliklarni isbotlang:

1) $\sum_{k=0}^{n} \cos(x + k\alpha) = \frac{\sin\frac{n+1}{2}\alpha \cos\left(x + \frac{n}{2}\alpha\right)}{\sin\frac{\alpha}{2}}$, $\alpha \neq 2\pi k$, $k \in Z$;

2) $\sum_{k=0}^{n} \sin(x + k\alpha) = \frac{\sin\frac{n+1}{2}\alpha \sin\left(\frac{n}{2}\alpha + x\right)}{\sin\frac{\alpha}{2}}$, $\alpha \neq 2\pi k$, $k \in Z$.

3.4. Quyidagi yig'indilarni hisoblang:

1) $\frac{1}{2} + \frac{3}{2^2} + \ldots + \frac{2n-1}{2^n}$; 2) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{(-1)^{n-1}}{2^{n-1}}$;

3) $\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}$; 4) $\sum_{k=1}^{n} \sin(2k-1)x$;

5) $\sum_{k=1}^{n} \cos^2 kx$.

3.5. Quyidagi ifodalarning qanday n ta sonlarning yig'indisini ifodalashini ko'rsating:

1) $\frac{n(n+1)(2n+1)}{6}$; 2) $\frac{n(4n^2-1)}{3}$; 3) $n^2(n+1)$;

4) $\frac{(n-1)n(n+1)}{3}$; 5) $\left(\frac{n(n+1)}{2}\right)^2$; 6) $\frac{n(n^2-1)(3n+2)}{12}$;

7) $\frac{n}{4n+1}$; 8) $\frac{n+2}{2n+2}$; 9) $\frac{10^{n+1}-9n-10}{81}$; 10) $3-\frac{2n+3}{2^n}$;

11) $\frac{1-(n+2)x^{n+1}(n+1)x^{n+2}}{(1-x)^2}$, $x \neq 1$;

12) $\frac{x^{n+2}-(n+1)x^2+nx}{(x-1)^2}$, $x \neq 1$; 13) $\frac{n(n+1)}{2(2n+1)}$;

14) $\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$; 15) $\frac{\sin 2nx}{2\sin x}$;

16) $\frac{n}{2}-\frac{\sin nx\cos(n+1)x}{2\sin x}$; 17) $\frac{\cos\frac{3(n+1)x}{2}\sin\frac{3}{2}nx}{4\sin\frac{3x}{2}}+\frac{3\cos\frac{n+1}{3}x\sin\frac{nx}{2}}{4\sin\frac{x}{2}}$.

Matematik induksiya usulidan foydalanib, ixtiyoriy $n(n\in N)$ lar uchun quyida berilgan tengliklarning to'g'riligini isbotlang:

3.6. $1+3+5+...+(2n-1)=n^2$.

3.7. $1^2+3^2+5^2+...+(2n-1)^2=\frac{n(4n^2-1)}{3}$.

3.8. $1^3+2^3+3^3...+n^3=(1+2+3+...+n)^2=\left(\frac{n(n+1)}{2}\right)^2$.

3.9. $1^2-2^2+3^2-4^2+5^2+...+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$.

3.10. $1^3+3^3+5^3+...+(2n-1)^3=n^2(2n^2-1)$.

3.11. $1\cdot2+2\cdot3+3\cdot4+...+(n-1)\cdot n=\frac{(n-1)n(n+1)}{3}$.

3.12. $1\cdot4+2\cdot7+3\cdot10+...+n\cdot(3n+1)=n\cdot(n+1)^2$.

3.13. $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+...+n\cdot(n+1)\cdot(n+2)=$
$=\frac{n(n+1)(n+2)(n+3)}{4}$.

3.14. $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$.

3.15. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+...+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}$.

3.16. $\frac{1}{1\cdot5}+\frac{1}{5\cdot9}+\frac{1}{9\cdot13}+...+\frac{1}{(4n-3)\cdot(4n+1)}=\frac{n}{4n+1}$.

3.17. $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+...+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$.

3.18. $\frac{1^2}{1\cdot3}+\frac{2^2}{3\cdot5}+\frac{3^2}{5\cdot7}+...+\frac{n^2}{(2n-1)\cdot(2n+1)}=\frac{n(n+1)}{2\cdot(2n+1)}$.

3.19. $\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)...\left(1-\frac{1}{(n-1)^2}\right)=\frac{n+2}{2n+2}$.

3.20. $x+2x^2+3x^3+...+nx^n=\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$, $x\neq1$.

3.21. $1+x+x^2+...+x^n=\frac{1-x^{n+1}}{1-x}$, $x\neq1$.

3.22. $1+2x+3x^2+...+nx^{n-1}=\frac{1+(n-1)x^n-nx^{n-1}}{(1-x)^2}$, $x\neq1$.

3.23. $x+x^3+x^5+...+x^{2n-1}=\frac{x-x^{2n+1}}{1-x^2}$, $x\neq\pm1$.

3.24. $1+3x^2+5x^4+...+(2n-1)x^{2n-2}=\frac{1+x^2+(2n-1)x^{2n+2}-(2n+1)x^{2n}}{(1-x^2)^2}$,

$x\neq\pm1$.

3.25. $\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+...+\frac{2^n}{1+x^{2^n}}=\frac{1}{x-1}+\frac{2^{n+1}}{1-x^{2^{n+1}}}$.

Matematik induksiya usulidan foydalanib, ixtiyoriy $n(n\in N)$ lar uchun quyida berilgan tengsizliklarning to'g'riligini isbotlang:

3.26. $2^n>2n+1$, $n>3$. 3.27. $2^n>n^3$, $n\geq10$.

3.28. $(1+a)^n\geq1+na$, $a>-1$.

3.29. $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}...\frac{2n-1}{2n}<\frac{1}{\sqrt{3n+1}}$, $n>1$.

3.30. $\sqrt{n}<1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}<2\sqrt{n}$, $n>1$.

3.31. $\frac{4^n}{n+1}<\frac{(2n)!}{(n!)^2}$, $n>1$. 3.32. $2\leq\left(1+\frac{1}{n}\right)^n<3$.

3.33. $n!>\left(\frac{n}{3}\right)^n$. 3.34. $n!>n^{\frac{n}{2}}$, $n>2$.

3.35. $\left(\frac{n}{e}\right)^n<n!<e\left(\frac{n}{2}\right)^2$. 3.36. $2^n\cdot n!<n^n$, $n>2$.

3.37. $\frac{x_1+x_2+...+x_n}{n}\geq\sqrt[n]{x_1, x_2, ..., x_n}$, $x_i>0$, $i=1, ..., n$.

3.38. $\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}...\frac{1}{x_n}} \le \sqrt[n]{x_1, x_2, ..., x_n} \le \frac{x_1+x_2+...+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+...+x_n^2}{n}}$,

$0 < x_1 < x_2 < ... < x_n$.

Matematik induksiya usulidan foydalanib, ixtiyoriy $n(n \in N)$ lar uchun quyidagi berilgan tengliklarning to'g'riligini isbotlang:

3.39. $\sin x + \sin 2x + ... + \sin nx = \sin\frac{(n+1)x}{2} \cdot \sin\frac{nx}{2} \cdot \left(\sin\frac{x}{2}\right)^{-1}$.

3.40. $\cos x + \cos 3x + \cos 5x + ... + \cos(2n-1)x = \frac{\sin 2(n-1)x}{2\sin x}$.

3.41. $\frac{1}{2} + \cos x + \cos 2x + ... + \cos nx = \sin\frac{2n+1}{2}x \cdot \left(2\sin\frac{x}{2}\right)^{-1}$.

3.42. $\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}...\cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\frac{x}{2^n}}$.

3.43. $\sin x + 2\sin 2x + ... + n\sin nx = \frac{(n+1)\sin nx - n\sin(n+1)x}{4\sin^2\frac{x}{2}}$.

3.44. $\cos x + 2\cos 2x + ... + n\cos nx = \frac{(n+1)\cos nx - n\cos(n+1)x - 1}{4\sin\frac{x}{2}}$.

3.45. $\frac{1}{2}\operatorname{tg}\frac{x}{2} + \frac{1}{2^2}\operatorname{tg}\frac{x}{2^2} + ... + \frac{1}{2^n}\operatorname{tg}\frac{x}{2^n} = \frac{1}{2^n}\operatorname{ctg}\frac{x}{2^n} - \operatorname{ctg} x$,

$x \ne m\pi,\ m \in Z$.

Matematik induksiya usulidan foydalanib, ixtiyoriy $n(n \in N)$ lar uchun quyidagi berilgan munosabatlarning to'g'riligini isbotlang (a sonning m ga qoldiqsiz bo'linishi $a \vdots m$ kabi belgilanadi):

3.46. $(6^{2n} - 1) \vdots 35$.

3.47. $(4^n + 15n - 1) \vdots 9$.

3.48. $(3^{2n+1} + 40n - 67) \vdots 64$.

3.49. $(n^3 + 11n) \vdots 6$.

3.50. $(7^n + 3n - 1) \vdots 9$.

3.51. $(3^{2n+3} - 3) \vdots 8$.

3.52. $(8^{n+2} + 9^{2n+1}) \vdots 73$.

3.53. Aylanaga ichki chizilgan 2^n tomonli muntazam ko'pburchakning tomoni

$$a_{2^n} = R\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}_{(n-2)\ ta\ 2}}$$

formula bilan ifodalanishini isbotlang, bunda R — aylana radiusi.

3.54. P perimetrli muntazam 2^n burchakka ichki va tashqi chizilgan aylanalarning r_n va R_n radiuslarini hisoblash qoidasini ko'rsating.

3.55. Ixtiyoriy n ta kvadrat berilgan. Ularni shunday qismlarga bo'lish mumkinligini isbotlangki, hosil bo'lgan qismlardan yangi kvadrat yasash mumkin bo'lsin.

3.56. Uchta qirrasi perpendikulyar bo'lgan parallelepipedlardan hajmi eng katta bo'lganini toping.

3.57. Bitta tekislikda yotib, bir nuqtadan o'tuvchi n ta to'g'ri chiziq bu tekislikni $2n$ qismga bo'lishini isbotlang.

3.58. Har qanday qavariq n burchak uchun $D_n = \frac{1}{2}n(n-3)$ formula o'rinli bo'lishini isbotlang, bunda D_n — ko'pburchak diagonallarining soni.

3.59. $b_1, b_2, ..., b_n, ...$ geometrik progressiyada $b_n = b_1 q^{n-1}$ bo'lishini isbotlang, bunda q — ayirma, $n \in N$.

3.60. $\{x_n\}$ ketma-ketlik ushbu

$$x_0 = 2, \ x_1 = \frac{5}{2}, \ x_n = \frac{5}{2}x_{n-1} - x_{n-2} \ (n > 1)$$

shartlar bilan berilgan bo'lsa, $x_n = 2^n + 2^{-n}$ bo'lishini isbotlang.

3.61. $\{x_n\}$ ketma-ketlik ushbu

$$x_0 = 2, \ x_1 = 3, \ x_{n+1} = 3x_n - 2x_{n-1}$$

shartlar bilan berilgan bo'lsa, $\{x_n\}$ ketma-ketlikning n ta hadi uchun formula toping.

3.62. $\{x_n\}$ ketma-ketlik ushbu

$$x_1 = 2, \ x_2 = 7, \ x_{n+1} = 3x_n + 1$$

shartlar bilan berilgan bo'lsa, $x_n = \frac{1}{2}(5 \cdot 3^{n-1} - 1)$ bo'lishini isbotlang.

3.63. $\{x_n\}$ ketma-ketlik ushbu

$$x_0 = 1, \ x_1 = 1, \ x_{n+1} = x_n + x_{n-1}$$

shartlar bilan berilgan bo'lsa, $x_{2n+1} = 1 + x_2 + x_4 + ... + x_{2n}$ tenglikni isbotlang.

3.64. $\{x_n\}$ ketma-ketlik ushbu

$$x_0 = 1,\ x_1 = 1,\ x_{n+1} = x_n + x_{n-1}$$

shartlar bilan berilgan bo'lsa, $x_n x_{n+1} = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$ tenglikni isbotlang.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

3.1. 1) $\frac{n}{3n+1}$; 2) $\frac{n}{4n+1}$; 3) $\frac{n(n+2)}{3(2n+1)(2n+3)}$; 4) $\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$.

3.4 1) $\frac{10^{n+1}-9n-10}{81}$; 2) $\frac{2}{3}+\frac{2}{3}(-1)^n$; 3) $\frac{15+(2n-1)2^{2-2n}-(2n+1)2^{4-2n}}{9}$; 4) $\frac{\sin^2 nx}{\sin x}$; 5) $\frac{n}{2}+\frac{\sin nx\cdot\cos(n+1)x}{2\sin x}$. 3.5. 1) $1^2+2^2+\ldots+n^2$; 2) $1^2+3^2+\ldots+(2n-1)^2$; 3) $1\cdot 2+2\cdot 5+\ldots+n(3n-1)$; 4) $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(n-1)n$; 5) $1^3+2^3+\ldots+n^3$; 6) $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots+(n-1)n^2$; 7) $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\ldots+\frac{1}{(2n-1)(2n+1)}$; 8) $\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot\ldots\cdot\left(1-\frac{1}{(n+1)^2}\right)$; 9) $1+11+111+\ldots+\underbrace{11\ldots1}_{n}$; 10) $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}$; 11) $1+2x+3x^2+\ldots+(n+1)x^n$; 12) $x^n+2x^{n-1}+\ldots+(n-1)x^2+nx$; 13) $\sum_{k=1}^{n}\frac{k^2}{(2k-1)(2k+1)}$; 14) $\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)(k+3)}$; 15) $\sum_{k=1}^{n}\cos(2k-1)x$; 16) $\sum_{k=1}^{n}\sin^2 kx$; 17) $\sum_{k=1}^{n}\cos^3 kx$.

II BOB. FUNKSIYA VA UNING LIMITI. FUNKSIYANING UZLUKSIZLIGI

4-§. FUNKSIYA TUSHUNCHASI

4.1. Funksiyaning ta'rifi. X va Y haqiqiy sonlar to'plamlari berilgan bo'lib, ular R ning bo'sh bo'lmagan qism to'plamlari ($X \subset R$, $Y \subset R$), x va y lar esa mos ravishda ularning elementlari ($x \in X$, $y \in Y$) bo'lsin.

4.1-ta'rif. Agar X to'plamdagi har bir x songa biror qoida yoki qonunga ko'ra Y to'plamdan bitta y son mos qo'yilsa, X to'plamda *y funksiya berilgan (aniqlangan)* deb ataladi va u simvolik ravishda $f: x \to y$ yoki $y = f(x)$ kabi belgilanadi.

Bunda x — argument yoki erkli o'zgaruvchi, y — funksiya yoki erksiz o'zgaruvchi, f — xarakteristika (qonun yoki qoida); X to'plam funksiyaning *aniqlanish sohasi*, $Y = \{y : y = f(x),\ x \in X\}$ to'plam esa, uning qiymatlari *to'plami (o'zgarish sohasi)* deyiladi. Bundan keyin biz funksiyaning *aniqlanish sohasini $D(f)$, qiymatlar to'plamini esa, $E(f)$* bilan belgilaymiz.

4.2. Funksiyaning berilish usullari. Funksiya umumiy holda *analitik, jadval, grafik* va *so'z usullari* bilan berilishi mumkin.

Analitik usul. Ko'pincha x va y o'zgaruvchilar orasidagi bog'lanish formulalar yordamida ifodalanadi. Bunda argument x ning har bir qiymatiga mos keladigan y funksiyaning qiymati, x ustida analitik amallar — qo'shish, ayirish, ko'paytirish, bo'lish, darajaga ko'tarish, ildizdan chiqarish, logarifmlash va h. k. amallarni bajarish natijasida topiladi. Odatda bunday usul funksiyaning analitik usulda berilishi deyiladi.

Funksiya analitik usulda quyidagi shakllarda berilishi mumkin:

1) $y=f(x)$ yoki $x=g(y)$ shakldagi formulalar bilan berilgan funksiyalar *oshkor shaklda* berilgan funksiyalar deyiladi. Masalan, $y = 6x - 2$, $y = x^2 + \ln x$ funksiyalar oshkor shaklda berilgan. Analitik usulda berilgan funksiya bir nechta formulalar vositasida yozilishi ham mumkin, masalan:

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0, \\ 1, & 0 < x < 1, \\ \dfrac{1}{x}, & 1 \le x \le 2. \end{cases}$$

Bu funksiyaning aniqlanish sohasi $[-\pi;\ 2]$ bo'lib, u uchta formula yordamida berilgan.

2) Agar x va y o'zgaruvchilar qandaydir $F(x, y)=0$ tenglama bilan bog'langan, ya'ni tenglama y ga nisbatan yechilmagan bo'lsa, u holda funksiya *oshkormas shaklda* berilgan deyiladi. Masalan, $x^2+y^2-R^2=0$ tenglama oshkormas shaklda berilgan funksiyani ifodalaydi, uni y ga nisbatan yechish natijasida ikkita funksiyani hosil qilamiz:

$$y=\pm\sqrt{R^2-x^2}.$$

Ba'zi bir oshkormas shakldagi funksiyalarni $y=f(x)$ (oshkor) shaklda ifodalash ham mumkin. Har qanday oshkor shakldagi $y=f(x)$ funksiyani oshkormas shaklda yozish ham mumkin: $y-f(x)=0$.

3) *parametrik shaklda*, ya'ni $\begin{cases} x=\varphi(t) \\ y=\varphi(t) \end{cases}$ $\alpha\le t\le\beta$ shaklda berilishi.

$y=f(x)$ funksiyada x ning y ga mos qo'yilishi *parametr* deb ataladigan uchinchi bir t o'zgaruvchi yordamida ifodalanishi mumkin:

$$\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} \quad \alpha\le t\le\beta,$$

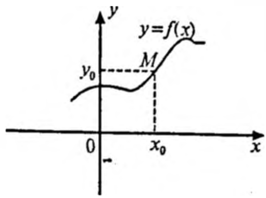
bu yerda $\varphi(t)$ va $\psi(t)$ lar ham analitik usulda berilgan funksiyalar bo'lib, $D(\varphi)\cap D(\psi)\ne\varnothing$ deb hisoblanadi. Funksiyalar berilishining eng ko'p uchraydigan usuli analitik usuldir. Bu usul matematik analizda juda ko'p ishlatiladi.

Jadval usuli. Ba'zi hollarda $x\in X$ va $y\in Y$ o'zgaruvchilar orasidagi bog'lanish formulalar yordamida berilmasdan, jadval orqali berilgan bo'lishi ham mumkin. Masalan, t — yanvar oyining birinchi dekadasi (10 kunligi) kunlari nomeri bo'lsa, T — shu nomerli kuni soat 16^{00} da Samarqand shahrida kuzatilgan havo haroratini bildirsin, natijada quyidagi jadvalga kelamiz:

t	1	2	3	4	5	6	7	8	9	10
T	-3°	-5°	$+2^\circ$	$+5^\circ$	$+1^\circ$	0°	-2°	-5°	-3°	-1°

bunda t — argument, T — funksiya bo'ladi. Bog'lanishning bunday berilishi, funksiyaning *jadval usulida* berilishi deb ataladi. Bu usuldan, ko'pincha, miqdorlar orasida tajribalar o'tkazish jarayonida foydalaniladi.

Jadval usulining qulayligi shundan iboratki, argumentning u yoki bu aniq qiymatlarida funksiyani hisoblamasdan, uning qiymatlarini aniqlash mumkin. Jadval usulining qulay bo'lmagan tomoni shundan iboratki, argumentning o'zgarishi bilan funksiyaning o'zgarish xarakterini to'liq aniqlab bo'lmaydi.



4.1-chizma.

Grafik usuli. *xOy* koordinatalar tekisligida *x* ning *X* to'plam ($X=D(f)$) dan olingan har bir qiymati uchun $M(x, y)$ nuqta yasaladi, bunda nuqtaning abssissasi *x*, ordinatasi *y* bo'lib y, *y* funksiyaning *x* ga mos kelgan qiymatiga teng. Yasalgan nuqtalarni birlashtirsak, natijada biror chiziq hosil bo'ladi, hosil bo'lgan bu chiziq berilgan funksiyaning grafigi deb qaraladi (4.1-chizma).

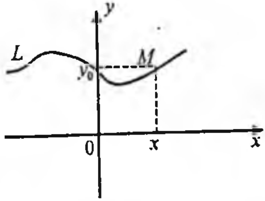
4.2-ta'rif. Tekislikning $(x, f(x))$ kabi aniqlangan nuqtalaridan iborat ushbu

$$\{(x, f(x))\} = \{(x, f(x)) : x \in X,\ \ y = f(x) \in Y\}$$

to'plam, funksiyaning grafigi deb ataladi.

xOy tekisligida shunday *L* chiziq berilgan bo'lsinki, *Ox* o'qda joylashgan nuqtalardan shu o'qqa o'tkazilgan perpendikulyar bu *L* chiziqni faqat bitta nuqtada kesib o'tsin. *Ox* o'qdagi bunday nuqtalardan iborat to'plamni *X* orqali belgilaymiz. *X* to'plamdan ixtiyoriy *x* ni olib, bu nuqtadan *Ox* o'qqa perpendikulyar o'tkazamiz. Bu perpendikulyarning *L* chiziq bilan kesishgan nuqtasining ordinatasini *y* bilan belgilaymiz. Natijada *X* to'plamdan olingan har bir *x* ga yuqorida ko'rsatilgan qoidaga ko'ra bitta *y* mos qo'yilib, funksiya hosil bo'ladi. Bunda *x* va *y* o'zgaruvchilar orasidagi bog'lanish *L* chiziq yordamida berilgan bo'ladi (4.2-chizma). Odatda funksiyaning bunday berilishi uning *grafik usulda* berilishi deb ataladi.

Funksiyaning grafik usulda berilishi ilmiy tadqiqotlarda va hozirgi zamon ishlab chiqarishi jarayonlarida keng qo'llaniladi. Masalan, tibbiyotda uchraydigan elektrokardiogramma grafigi — yurak muskullaridagi tok impulslarining vaqt bo'yicha o'zgarishini ko'rsatadi. Bu grafik analitik tarzda yo-



4.2-chizma.

4.3-chizma.

zilishi shart bo'lmagan qandaydir $y=f(x)$ funksiyaning grafigidir, bu funksiyaning formulasi tabib uchun unchalik qiziqarli emas (4.3-chizma).

Funksiyaning grafik usulda berilishining kamchiligi shundan iboratki, argumentning sonli qiymatida berilgan funksiyaning aniq shaklini har doim topib bo'lmaydi, lekin bu usulning boshqa usullardan afzalligi uning tasviri yaqqol ko'zga ko'rinib turishidadir.

So'zlar orqali ifodalanadigan usul. Bu usulda ($x \in X$, $y \in Y$) o'zgaruvchilar o'rtasidagi funksional bog'lanish faqat so'zlar orqali ifodalanadi. Masalan:

1) Har bir ratsional songa 1 ni, har bir irratsional songa 0 ni mos qo'yish natijasida ham funksiya hosil bo'ladi. Bu funksiya, odatda, Dirixle funksiyasi deyiladi va $D(x)$ kabi belgilanadi:

$$D(x)=\begin{cases}1, & \text{agar } x \text{ ratsional son bo'lsa,}\\ 0, & \text{agar } x \text{ irratsional son bo'lsa.}\end{cases}$$

2) f — har bir haqiqiy x songa uning butun qismi $[x]$ ni mos qo'yuvchi qoida bo'lsin. Demak, $f: x \to [x]$ yoki $y=[x]$ funksiyaga ega bo'lamiz;

3) f — har bir haqiqiy x songa uning kasr qismi $\{x\}$ ni mos qo'yadigan qoida bo'lsin, ya'ni $f: x \to \{x\}$. Bu holda biz $y=\{x\}$ funksiyaga ega bo'lamiz.

4.3. Funksiyaning aniqlanish sohasi

4.3-ta'rif. Argumentning funksiya ma'nosini yo'qotmaydigan (ya'ni cheksiz yoki mavhumlikka aylantirmaydigan) hamma qiymatlari to'plami shu funksiyaning *aniqlanish sohasi* deyiladi.

Agar funksiya jadval shaklida berilsa, uning aniqlanish sohasi x ning jadvalda ko'rsatilgan qiymatlaridan iborat bo'ladi.

Agar funksiya grafik shaklda berilsa, uning aniqlanish sohasi grafikdan ko'rinib turadi.

Funksiya analitik shaklda berilganda esa, x ning funksiyani aniqlaydigan formula ma'noga ega bo'ladigan qiymatlari to'plami shu funksiyaning aniqlanish sohasi bo'ladi.

Funksiyaning aniqlanish sohasini topish vaqtida funksiya ko'rinishini boshqa shaklga keltirish tavsiya etilmaydi.

4.1-misol. $f(x)=\frac{x^3-1}{x^2-6x+8}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. $\frac{x^3-1}{x^2-6x+8}$ funksiyaning maxraji nolga aylanadigan nuqtalarda funksiya ma'noga ega bo'lmaydi. Maxrajni nolga aylantiradigan nuqtalarni topish uchun $x^2-6x+8=0$ tenglamani yechamiz. Bu tenglamaning yechimi $x_1=2$, $x_2=4$ lardan iborat.

Demak, bu funksiyaning aniqlanish sohasini topishda x^2-6x+ $+8\neq 0$ yoki $x_1\neq 2$, $x_2\neq 4$ shartlarning bajarilishini talab qilish kerak. Shunday qilib, berilgan funksiyaning aniqlanish sohasi uchta oraliqlar birlashmasidan iborat, ya'ni

$$D(f)=(-\infty;2)\cup(2;4)\cup(4;+\infty).$$

Xulosa. Funksiyalar kasr shaklida berilgan bo'lsa, uning aniqlanish sohasi argumentning kasr maxrajini noldan farqli qiladigan qiymatlari to'plamidan iborat bo'ladi.

4.2-misol. Ushbu $f(x)=\sqrt{x-2}+\sqrt{7-x}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiyaning aniqlanish sohasi x ning ikkala qo'shiluvchining ham haqiqiy qiymatlarni qabul qiladigan qiymatlar to'plamidan iborat. Bu qiymatlar to'plamini topish uchun

$$\begin{cases} x-2\geq 0, \\ 7-x\geq 0 \end{cases}$$

shartlarning bajarilishini talab qilish kerak. Bu tengsizliklar sistemasini yechish natijasida $x\geq 2$; $x\leq 7$ bo'lishini topamiz.

Shunday qilib, berilgan funksiyaning aniqlanish sohasi $[2;\ 7]$ segmentdan iborat bo'lar ekan.

4.3-misol. $f(x)=\frac{\sqrt{x}}{x+1}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Funksiyaning berilishiga ko'ra, birinchidan, $x\geq 0$ bo'lishi; ikkinchidan esa, $x\neq -1$ shartning bajarilishi talab qilinadi. Bu ikki shartlardan, funksiya ma'noga ega bo'lishi uchun $x\geq 0$ shartning bajarilishi yetarli. Demak, berilgan funksiyaning aniqlanish sohasi

$$D(f)=[0;\ +\infty)$$

bo'ladi.

4.4-misol. $f(x)=\sqrt[3]{\frac{x-2}{x-3}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan ildiz ostidagi ifodaning qiymati manfiy bo'lsa ham, funksiya ma'noga ega bo'ladi, chunki ildiz ko'rsatkichi toq sondir. Bu funksiya ma'noga ega bo'lishi uchun $x \neq 3$ shart bajarilishi yetarli. Demak, berilgan funksiyaning aniqlanish sohasi

$$D(f) = (-\infty; 3) \cup (3; +\infty)$$

bo'ladi.

4.5-misol. $f(x) = \sqrt{2\sin x - 1}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya juft darajali ildiz orqali berilgani uchun $2\sin x - 1 \geq 0$ shart bajarilganda, funksiya ma'noga ega bo'ladi. Bu tengsizlikni yechamiz:

$$\sin x \geq \frac{1}{2}, \quad \frac{\pi}{6} + 2\pi n \leq x \leq \frac{5\pi}{6} + 2\pi n, \quad n \in Z.$$

Demak, berilgan funksiyaning aniqlanish sohasi:

$$D(f) = \left[\frac{\pi}{6} + 2\pi n;\ \frac{5\pi}{6} + 2\pi n\right], \quad n \in Z.$$

Xulosa. Yuqorida ko'rsatilgan misollarning yechilishidan quyidagi xulosani chiqarish mumkin: funksiya $f(x) = \sqrt[2n]{\varphi(x)}$ $(n \in N)$ shaklda berilganda, uning mavjud bo'lishi uchun $\varphi(x) \geq 0$ shart bajarilishi kerak.

4.6-misol. $f(x) = \sqrt{\lg \frac{5x - x^2}{4}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya ma'noga ega bo'lishi uchun, birinchidan, $\lg \frac{5x - x^2}{4} \geq 0$ shart bajarilishi kerak. Ikkinchidan, bu shart bajarilishi uchun $\frac{5x - x^2}{4} \geq 1$ yoki $x^2 - 5x + 4 \leq 0$ bo'lishi kerak. Oxirgi tengsizlikning yechimi $1 \leq x \leq 4$. Shunday qilib, berilgan funksiyaning aniqlanish sohasi

$$D(f) = [1;\ 4]$$

ekan.

4.7-misol. $f(x) = \frac{\sqrt{4 - x}}{\lg(x - 2)}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiyaning ko'rinishini e'tiborga olganda, uning mavjud bo'lishi uchun, birinchidan, $4 - x \geq 0$ yoki $x \leq 4$ ikkinchidan, $x - 2 > 0$ yoki $x > 2$ uchinchidan, $\lg(x-2) \neq 0$ yoki $x \neq 3$ shartlar bajarilishi kerak. Bu shartlarni e'tiborga olsak, berilgan funksiyaning aniqlanish sohasi

$$D(f)=(2;\ 3)\cup(3;\ 4]$$

bo'ladi.

4.8-misol. $f(x)=\log_{x^3}(x^2-6x+8)$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya mavjud bo'lishi uchun, birinchidan, $x^2-6x+8>0$ yoki $(x-2)(x-4)>0$ yoki $-\infty<x<2,\ 4<x<\infty$ bo'lishi; ikkinchidan, $x^3>0,\ x^3\neq 1$ yoki $x>0,\ x\neq 1$ bo'lishi kerak. x ning bu shartlarni bir vaqtda qanoatlantiradigan barcha qiymatlari to'plami

$$D(f)=(0;\ 1)\cup(1;\ 2)\cup(4;\ \infty)$$

bo'ladi.

4.9-misol. $f(x)=\log_{x^4}(6-x)$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya ma'noga ega bo'lishi uchun $6-x>0$, $x\neq 0$, $x^4\neq 1$ yoki $x<6$, $x\neq 0$ shartlar bajarilishi kerak. Bu shartlar bajarilganda, berilgan funksiyaning aniqlanish sohasi:

$$D(f)=(-\infty;\ -1)\cup(-1;\ 0)\cup(0;\ 1)\cup(1;\ 6).$$

Xulosa. Shunday qilib, yuqoridagi 4.8-, 4.9- misollarning yechilishini e'tiborga olgan holda quyidagi xulosani chiqarish mumkin: $\log_{\varphi(x)}f(x)$ funksiya ma'noga ega bo'lishi uchun $f(x)>0$, $\varphi(x)>0$ va $\varphi(x)\neq 1$ shartlarning bajarilishini talab qilish yetarli.

4.10-misol. $f(x)=\dfrac{\sqrt{x-1}}{\sqrt{x+4}-\sqrt{6x-6}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya ma'noga ega bo'lishi uchun, birinchidan, $x-1\geq 0$ yoki $x\geq 1$ bo'lishi; ikkinchidan, $x+4\geq 0$ yoki $x\geq -4$ bo'lishi; uchinchidan, $\sqrt{x+4}-\sqrt{6x-6}\neq 0$ yoki $x\neq 2$ bo'lishi kerak. Yuqoridagi shartlarni e'tiborga olgan holda, funksiyaning aniqlanish sohasi

$$D(f)=[1;\ 2)\cup(2;\ +\infty)$$

ekanligiga ishonch hosil qilamiz.

4.11-misol. $f(x)=\arccos\dfrac{3}{4+2\sin x}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Birinchidan, *arccos* funksiya ma'noga ega bo'lishi uchun uning belgisi ostidagi ifodaning absolut qiymati 1 dan katta bo'lmasligi, ya'ni $-1 \le \frac{3}{4+2\sin x} \le 1$ bo'lishi kerak. $\forall x$ uchun $4+2\sin x>0$ bo'lgani uchun, yuqoridagi shart $\frac{3}{4+2\sin x} \le 1$ shartga teng kuchli. Bundan $3 \le 4+2\sin x$ yoki $\sin x \ge -\frac{1}{2}$. Bu tengsizlikni yechish natijasida

$$-\frac{\pi}{6}+2\pi k \le x \le \frac{7\pi}{6}+2\pi k \quad (k=0, \pm 1, \pm 2, \ldots)$$

ekanligini topamiz.

4.12-misol. $f(x)=\frac{\arcsin(x-2)+\sqrt{9-x^2}}{\log_3(5-2x)}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Birinchidan, *arcsin* funksiya ma'noga ega bo'lishi uchun uning belgisi ostidagi ifodaning absolut qiymati 1 dan katta bo'lmasligi, ya'ni $|x-2| \le 1$ yoki $1 \le x \le 3$ bo'lishi; ikkinchidan, $9-x^2 \ge 0$ yoki $-3 \le x \le 3$ bo'lishi; uchinchidan esa, $5-2x>0$ yoki $\log_3(5-2x) \ne 0$ va $x<2{,}5$ yoki $x \ne 2$ bo'lishi kerak. x ning yuqoridagi uchta shartni bir vaqtda qanoatlantiradigan barcha qiymatlari to'plami

$$[1;\ 2) \cup (2;\ 2{,}5)$$

dan iborat.

Shunday qilib, berilgan funksiyaning aniqlanish sohasi

$$D(f)=[1;\ 2) \cup (2;\ 2{,}5).$$

Xulosa. $f(x)=\varphi(x) \pm \phi(x)$ funksiyaning aniqlanish sohasi $\varphi(x)$ va $\phi(x)$ funksiyalar aniqlanish sohalarining umumiy qismidan iborat.

4.13-misol. $f(x)=(4-x)^{\sqrt{x-3}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Birinchidan, $x \ge 3$, ikkinchidan, $4-x \ge 0$ yoki $x \le 4$ bo'lishi kerak. x ning bu ikki shartni qanoatlantiradigan qiymatlari to'plami $[3;\ 4]$ segmentdan iborat.

Demak, berilgan funksiyaning aniqlanish sohasi

$$D(f)=[3;\ 4]$$

bo'ladi.

Xulosa. Funksiya $U(x)^{V(x)}$ $(U(x) \ge 0)$ shaklda berilganda uning asosi va daraja ko'rsatkichi argumentning bir xil qiymatlarida bir vaqtda nolga aylanmasligi kerak (0^0 shakldagi aniqmaslik).

4.14-misol. $|f| = 9 - x^2$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiya shakliga ko'ra $|f|$ — manfiy bo'lmagan son bo'lgani uchun, $9 - x^2 \geq 0$ yoki $|x| \leq 3$ bo'lishi kerak.

Demak, berilgan funksiyaning aniqlanish sohasi:

$$D(f) = [-3;\ 3].$$

4.15-misol. $|f| = \lg(5 - x)$ funksiyaning aniqlanish sohasini toping.

Yechilishi. Berilgan funksiyaning aniqlanish sohasi

$$\begin{cases} \lg(5 - x) \geq 0, \\ 5 - x > 0 \end{cases}$$

sistemaning yechimidan iborat. Sistemaning birinchi tengsizligidan $5 - x \geq 1$ yoki $x \leq 4$ bo'lishi; ikkinchisidan esa $5 - x > 0$ yoki $x < 5$ bo'lishi kelib chiqadi.

Demak, $x \leq 4$. Berilgan funksiyaning aniqlanish sohasi:

$$D(|f|) = (-\infty;\ 4].$$

4.4. Funksiyaning o'zgarish sohasi. $y = f(x)$ funksiya $x \in X$ to'plamda berilgan bo'lsin. Funksiyaning o'zgarish sohasi diskret nuqtalardan, nuqtadan, oraliq, segment, bir necha oraliqlardan va h.k. iborat bo'lishi mumkin. Jadval yoki grafik usulda berilgan funksiyalarning o'zgarish sohalari o'z-o'zidan ma'lum. Analitik usulda, ya'ni $y = f(x)$ shaklda berilganda funksiyaning o'zgarish sohasini topish uchun y ning $f(x) = y$ tenglama haqiqiy yechimga ega bo'ladigan barcha qiymatlarini topish talab qilinadi.

Funksiyaning o'zgarish sohasini topishda quyidagi tasdiqlarni e'tiborga olish lozim:

1°. Agar berilgan funksiya (bu yerda uzluksiz funksiya nazarda tutiladi) qarayotgan sohada eng kichik va eng katta qiymatga erishsa, $f(x)$ funksiyaning o'zgarish sohasi uning eng kichik va eng katta qiymati hamda ular orasidagi barcha sonlar to'plamidan iborat bo'ladi.

4.16-misol. $[0;\ \sqrt{2}]$ kesmada $f(x) = x^4 + 9$ funksiyaning o'zgarish sohasini toping.

Yechilishi. $[0;\ \sqrt{2}]$ kesmada berilgan funksiyaning eng kichik qiymati $f(0) = 9$, eng katta qiymati $f(\sqrt{2}) = 13$ bo'lgani uchun, uning o'zgarish sohasi

$$E(f) = [9;\ 13]$$

dan iborat.

2°. Agar funksiya eng kichik (katta) qiymatga ega bo'lsa-yu, ammo eng katta (kichik) qiymatga erishmasa (ya'ni u cheksiz orta (kamaya) borsa), funksiyaning o'zgarish sohasi funksiyaning eng kichik (katta) qiymati va shu qiymatdan katta (kichik) barcha sonlar to'plamidan iborat bo'ladi.

4.17-misol. $f(x)=ax^2+bx+c$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Berilgan funksiyani

$$y=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

shaklda ifodalaymiz.

Agar $a>0$ ($a<0$) bo'lsa, berilgan funksiya $x=-\frac{b}{2a}$ nuqtada eng kichik $f_{min}=f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}$ $\left(\text{eng katta } f_{max}=f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}\right)$ qiymatiga erishadi, ammo eng katta (kichik) qiymatiga ega bo'lmaydi.

Demak, $a>0$ ($a<0$) bo'lganda berilgan funksiyaning o'zgarish sohasi

$$E(f)=\left[\frac{4ac-b^2}{4a};+\infty\right)\quad\left(E(f)=\left(-\infty;\frac{4ac-b^2}{4a}\right]\right)$$

dan iborat bo'ladi.

4.18-misol. $f(x)=\sqrt{x^2-4x+9}$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Kvadrat ildiz ostidagi x^2-4x+9 ifodada $a=1>0$ va $f(x)=\sqrt{x^2-4x+9}=\sqrt{(x-2)^2+5}$ ekanligini e'tiborga olsak, u holda (4.17-misolga qarang) funksiya $x=2$ nuqtada eng kichik $f_{min}(2)=\sqrt{5}$ qiymatiga erishadi, lekin uning eng katta qiymati yo'q.

Demak, berilgan funksiyaning o'zgarish sohasi, 2° — tasdiqqa asosan,

$$E(f)=[\sqrt{5};\infty)$$

dan iborat bo'ladi.

3°. Agar $y=f(x)$ funksiya berilgan bo'lib, uni x ga nisbatan yechish mumkin bo'lsa, ya'ni $x=\varphi(y)$ shaklda yozish mumkin bo'lsa, $y=f(x)$ funksiyaning o'zgarish sohasini topish uchun, $x=\varphi(y)$ funksiyaning aniqlanish sohasini topish yetarli. Demak, $x=\varphi(y)$

funksiyaning aniqlanish sohasi $y=f(x)$ funksiyaning o‘zgarish sohasidan iborat bo‘ladi: $D(\varphi)=E(f)$.

4.19-misol. $y=x^2-8x+7$ funksiyaning o‘zgarish sohasini toping.

Yechilishi. Ushbu $x^2-8x+7-y=0$ tenglamani x ga nisbatan yechamiz:

$$x_{1,2}=4\pm\sqrt{9+y}.$$

Berilgan funksiyaning mavjudlik sohasi $9+y\geq 0$ va $y\geq -9$ dan iborat.

Demak, 3°-tasdiqqa binoan, berilgan funksiyaning o‘zgarish sohasi $E(f)=[-9;\ +\infty)$ bo‘ladi.

4°. Umumiy holda $y=f(x)$ funksiyaning o‘zgarish sohasi $a\leq y\leq b$ $(b>0)$, $y=\varphi(x)$ funksiyaning o‘zgarish sohasi esa $c\leq y\leq d$ $(d>0)$ bo‘lganda $f(x)+\varphi(x)$ funksiyaning o‘zgarish sohasini $a+c\leq \leq y\leq b+d$ kabi aniqlash, $f(x)\cdot\varphi(x)$ funksiyaning o‘zgarish sohasini esa $ac\leq y\leq bd$ kabi aniqlash mumkin emas.

Haqiqatan ham, $\sin x$ va $\cos x$ funksiyalarning o‘zgarish sohalari $-1\leq y\leq 1$ bo‘lgani holda, $\sin x+\cos x$ funksiyaning o‘zgarish sohasini $-1-1\leq y\leq 1+1$ yoki $-2\leq y\leq 2$ kabi aniqlab bo‘lmaydi.

4.20-misol. $f(x)=a\cos x+b\sin x$ $(a^2+b^2>0)$ funksiyaning o‘zgarish sohasini toping.

Yechilishi. Berilgan funksiyani ushbu

$$f(x)=\sqrt{a^2+b^2}\cos(x-\alpha)$$

shaklda yozish mumkin, bunda

$$\cos\alpha=\frac{a}{\sqrt{a^2+b^2}},\quad \sin\alpha=\frac{b}{\sqrt{a^2+b^2}}.$$

$|\cos(x-\alpha)|\leq 1$ bo‘lgani uchun, $f(x)$ funksiyaning eng katta qiymati $f_{max}=\sqrt{a^2+b^2}$ $(\cos(x-\alpha)=1$ bo‘lganda), eng kichik qiymati $f_{min}=-\sqrt{a^2+b^2}$ $(\cos(x-\alpha)=-1$ bo‘lganda) bo‘ladi.

Demak, 1° — tasdiqqa asosan, berilgan funksiyaning o‘zgarish sohasi:

$$E(f)=[-\sqrt{a^2+b^2};\ \sqrt{a^2+b^2}].$$

4.21-misol. $f(x)=\sin x+\cos x$ funksiyaning o‘zgarish sohasini toping.

Yechilishi. Berilgan funksiyani quyidagi shaklda yozish mumkin:

$$f(x)=\sin x+\cos x=\sin x+\sin\left(\frac{\pi}{2}-x\right)=$$

$$=2\sin\frac{\pi}{4}\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right).$$

Ma'lumki, $-1\le\cos\left(x-\frac{\pi}{4}\right)\le 1$, bu yerdan,

$$-\sqrt{2}\le\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\le\sqrt{2} \text{ yoki } -\sqrt{2}\le f\le\sqrt{2}$$

bo'lgani uchun, funksiyaning o'zgarish sohasi

$$E(f)=[-\sqrt{2};\ \sqrt{2}]$$

bo'ladi.

4.22-misol. $f(x)=\frac{1}{4}\sin x\cdot\cos x$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Berilgan funksiyani quyidagi shaklda yozish mumkin:

$$f(x)=\frac{1}{4}\sin x\cdot\cos x=\frac{1}{8}\sin 2x.$$

Ma'lumki, $-1\le\sin 2x\le 1$. Bundan $-\frac{1}{8}\le\frac{1}{8}\sin 2x\le\frac{1}{8}$.

Demak, berilgan funksiyaning o'zgarish sohasi

$$E(f)=\left[-\frac{1}{8};\ \frac{1}{8}\right]$$

bo'ladi.

5°. Agar $y=f(x)$ funksiyaning o'zgarish sohasi $a\le y\le b$ bo'lsa, $g(x)=mf(x)$ funksiyaning o'zgarish sohasi $m>0$ bo'lganda $ma\le g\le mb$; $m<0$ bo'lganda esa $ma\ge g\ge mb$ bo'ladi.

Agar $y=f(x)$ funksiyaning o'zgarish sohasi $a\le y\le b$ bo'lsa, $g(x)=n+f(x)$ funksiyaning o'zgarish sohasi $n+a\le g(x)\le n+b$ dan iborat bo'ladi.

4.23-misol. $f(x)=-\frac{3}{\sqrt{5}}(\cos 2x-\sqrt{5})$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Berilgan funksiyani quyidagi shaklga keltiramiz:

$f(x)=-\frac{3}{\sqrt{5}}\cos 2x+3$, bunda $m=-\frac{3}{\sqrt{5}}<0, n=3>0$. Ma'lumki, $-1\le\cos 2x\le 1$ bo'lgani uchun $\frac{3}{\sqrt{5}}\ge-\frac{3}{\sqrt{5}}\cos 2x\ge-\frac{3}{\sqrt{5}}$ va $3+\frac{3}{\sqrt{5}}\ge$ $\ge-\frac{3}{\sqrt{5}}\cos 2x+3\ge 3-\frac{3}{\sqrt{5}}$.

Demak, berilgan funksiyaning o'zgarish sohasi

$$E(f)=\left[3-\frac{3}{\sqrt{5}};\ 3+\frac{3}{\sqrt{5}}\right].$$

4.24-misol. $f(x)=3\cos x+4\sin x-6$ funksiyaning o'zgarish sohasini toping.

Yechilishi. Berilgan funksiyani quyidagicha tasvirlaymiz (4.20-misolga qarang): $f(x)=5\cos(x-\alpha)-6$, bunda $\cos\alpha=\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$ $m=5>0$, $n=-6<0$ $-1\le\cos(x-\alpha)\le 1$ bo'lgani uchun $-5\le 5\cos(x-$ $-\alpha)\le 5$ $-6-5\le 5\cos(x-\alpha)-6\le 5-6$ yoki $-11\le 5\cos(x-\alpha)-6\le -1$.

Demak, berilgan funksiyaning o'zgarish sohasi $E(f)=[-11;\ -1]$ bo'ladi.

6°. Agar $y=f(x)$ funksiyaning o'zgarish sohasi $-a\le y\le a$ $(-\infty\le$ $\le y\le\infty)$ bo'lsa, u holda $y_1=|f(x)|$ yoki $y_2=f^2(x)$ funksiyaning o'zgarish sohasi $0\le y_1\le a$ yoki $0\le y_2\le\infty$ bo'ladi. Masalan, $y=\cos x$ ning o'zgarish sohasi $-1\le y\le 1$ bo'lgan holda, $y_1=|\cos x|$ yoki $y_2=\cos^2 x$ funksiyaning o'zgarish sohasi bir xil, ya'ni $0\le y_1\le 1$ bo'ladi. $y=\operatorname{tg}x$ ning o'zgarish sohasi $-\infty<y<\infty$ bo'lgan holda, $y_1=|\operatorname{tg}x|$ yoki $y_2(x)=\operatorname{tg}^2 x$ funksiyaning o'zgarish sohasi bir xil, ya'ni $0\le y\le\infty$ bo'ladi.

Mustaqil yechish uchun misol va masalalar

Quyidagi funksiyalarning aniqlanish sohasini toping:

4.1. $f(x)=\frac{3x}{x^2-9}$. 4.2. $f(x)=\frac{x^2-4}{x+2}$. 4.3. $f(x)=\frac{1}{x^4-x}$.

4.4. $f(x)=\frac{x}{(2^x-4)^2}$. 4.5. $f(x)=\sqrt{9x^2-1}$. 4.6. $f(x)=\sqrt[4]{\frac{3}{25-x^2}}$.

4.7. $f(x)=\sqrt[7]{\frac{x-3}{(x^2-4)(x^2+4)}}$. 4.8. $f(x)=\sqrt{x-\sqrt{x-\sqrt{x}}}$.

4.9. $f(x)=\sqrt{|x|-8}+\sqrt{x^2-16}$.

4.10. $f(x)=\dfrac{x}{\sqrt{x-4}-\sqrt{6-x}}$.

4.11. $f(x)=\sqrt{\dfrac{x^2-7x+12}{x^2-2x-3}}$.

4.12. $f(x)=\dfrac{x}{|x|}$.

4.13. $f(x)=\dfrac{x}{x^2-x-2}$.

4.14. $f(x)=\log_4(x^2-2x)$.

4.15. $f(x)=\sqrt{\log_2 x-1}+\dfrac{1}{x^2-16}$.

4.16. $f(x)=\dfrac{\sqrt{4x-x^2}}{\log_3|x-4|}$.

4.17. $f(x)=\log_3(2^{\log_{x-3}0,5}-1)+\dfrac{1}{\log_3(2x-6)}$.

4.18. $f(x)=\sqrt{\dfrac{x^2-1}{(x-3)(x+4)}}+\dfrac{1}{\log_8(x-4)}$.

4.19. $f(x)=\sqrt{\log_{0.3}\dfrac{x-1}{x+5}}$.

4.20. $f(x)=\lg(3^x-3^{-x})$.

4.21. $f(x)=\dfrac{\sqrt{9-x^2}}{\sin x-1}$.

4.22. $f(x)=\sqrt{-2\cos^2 x+3\cos x-1}$.

4.23. $f(x)=\sqrt{\sin^2 x-\sin x}$.

4.24. $f(x)=\lg\sin(x-3)-\sqrt{16-x^2}$.

4.25. $f(x)=\sqrt{\dfrac{x}{\sin x}}+\sqrt{\dfrac{10-x^2}{x^4-11x^2+18}}$.

4.26. $f(x)=\log_9\dfrac{\sin 3x}{x}-\dfrac{\sqrt{4x^2-x^4+5}}{\cos\pi x}-x$.

4.27. $f(x)=\sqrt{\dfrac{\log_{0.3}|x-2|}{|x|}}$.

4.28. $f(x)=\dfrac{x^2}{\sin x}+\dfrac{\operatorname{tg} x\cdot\sqrt{3-x^2}}{1-x^2}$.

4.29. $f(x)=\lg\left(\sqrt{8^{-2+\lg x}}-\sqrt[3]{4^{2-\lg x}}\right)$.

4.30. $f(x)=\dfrac{2}{\sin^4 x+\cos^4 x}$.

4.31. $f(x)=\dfrac{\sin x}{1-\cos x}$.

4.32. $f(x)=\sqrt{\sin(\cos x)}$.

4.33. $f(x)=\sqrt{\lg(\cos 2\pi x)}$.

4.34. $f(x)=(2x)!$.

4.35. $f(x)=\log_{\cos x}\sin x$.

4.36. $f(x)=\lg(1-\operatorname{tg} x)$.

4.37. $f(x)=\arcsin\dfrac{4}{2-x}$.

4.38. $f(x)=\arccos\dfrac{2x}{1+x^2}$.

4.39. $f(x)=\sqrt{\arcsin(\log_2 x)}$.

4.40. $f(x)=\arcsin(1+\operatorname{tg}^2\pi x)$.

4.41. $f(x)=\operatorname{ctg}\pi x+\arccos 2^x$. **4.42.** $f(x)=\arcsin\left(\lg\frac{x}{10}\right)$.

4.43. $f(x)=\arccos\sqrt{x^2-1}$. **4.44.** $f(x)=\frac{1}{x-0.5}+3^{\arcsin x}+\frac{1}{\sqrt{x-3}}$.

4.45. $f(x)=\log_{3+x}(x^2-1)$. **4.46.** $f(x)=\lg(1-\log_{1/2}(x+5))$.

4.47. $f(x)=\lg(4-x^2)\sqrt{\frac{1+\lg^2 x}{\lg x^2}-1}$.

4.48. $f(x)=\log_2\log_3\sqrt{4x-x^2-2}$.

4.49. Nechta butun son $f(x)=\sqrt{\log_{0.5}(x-2)+2}$ funksiyaning aniqlanish sohasiga tegishli?

4.50. $y=f(x)$ funksiyaning aniqlanish sohasi $[-1;\ 2]$dan iborat bo'lsa, $y=f(x+1)$ funksiyaning aniqlanish sohasini toping.

Quyidagi funksiyalarning aniqlanish sohasini toping:

4.51. $|y|=9-x^2$. **4.52.** $|y|=\lg(3-x)$. **4.53.** $y=\frac{1}{[x]}$.

4.54. $y=\frac{1}{\{x\}}$. **4.55.** $y=\sqrt{[x]-3}$. **4.56.** $y=\sqrt{\{x\}-3}$.

4.57. $y=\frac{1}{27^{x^2}-3^x}$. **4.58.** $y=(x^2-x+1)^{-\frac{3}{2}}$. **4.59.** $y=\sqrt{2^x-e^x}$.

4.60. $y=\frac{1}{\sqrt{x-|x|}}$. **4.61.** $y=\frac{1}{\sqrt{|x|-x}}$.

4.62. $y=\frac{1}{\log_2(2-x)}+\sqrt{x+3}$. **4.63.** $y=\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{1-x}{\sqrt{1+x}}}$.

4.64. $f(x)=\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x-2}}$.

Quyidagi funksiyalarning o'zgarish sohalarini toping:

4.65. $f(x)=\frac{x+4}{x-5}$. **4.66.** $f(x)=\frac{x^2-5}{2x-4}$. **4.67.** $f(x)=\frac{8}{x^2+4}$.

4.68. $f(x)=\frac{1}{\lg^2 x+1}$. **4.69.** $f(x)=x^2-4x+9$.

4.70. $f(x)=\sqrt{x-x^2}$. **4.71.** $f(x)=\lg(3x^2-4x+5)$.

4.72. $f(x)=|x-3|+|x-5|$.

4.73. $f(x)=\sqrt{x-1}+2\sqrt{3-x}$.

4.74. $f(x)=5-4\sin x-\sin^2 x$.

4.75. $f(x)=3^{2+\sin x}$.

4.76. $f(x)=3\sin x+4\cos x$.

4.77. $f(x)=\lg(1-2\cos x)$.

4.78. $f(x)=\frac{x^2+x+2}{x^2-x+2}$.

4.79. $f(x)=(-1)^x$.

4.80. $f(x)=\sqrt{-x^2+x+2}$.

4.81. $f(x)=\frac{x}{|x|}$.

4.82. $f(x)=\arcsin\sqrt{9-x^2}$.

4.83. $f(x)=\sin^4 x+\cos^4 x$.

4.84. $f(x)=4\cos x-4$.

4.85. $f(x)=(2x+1)e^{2x}$.

4.86. $f(x)=x-1+\ln(3-x)$.

4.87. $f(x)=\frac{4^x-3\cdot 2^x}{\ln 2}+x$.

4.88. $f(x)=x+\ln(x^2+1)$.

4.89. $f(x)=x+\mathrm{sign}x$.

4.90. $f(x)=ax+\frac{b}{x}$.

4.91. $f(x)=\arcsin\sqrt{\frac{1-x^2}{2}}$.

4.92. $f(x)=-x^2+5x-6$.

4.93. $f(x)=\frac{x}{1+9x^2}$.

4.94. $f(x)=2^{\arccos(1-x)}$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

4.1. $(-\infty;-3)\cup(-3;3)\cup(+3;\infty)$. 4.2. $(-\infty;-2)\cup(-2;+\infty)$. 4.3. $(-\infty;0)\cup(0;1)\cup(1;+\infty)$. 4.4. $(-\infty;2)\cup(2;+\infty)$. 4.5. $\left(-\infty;-\frac{1}{3}\right]\cup\left[\frac{1}{3};+\infty\right)$. 4.6. $(-5,5)$ 4.7. $(-\infty;-2)\cup(-2;2)\cup(2;\infty)$. 4.8. $\{0\}\cup[1;+\infty)$. 4.9. $(-\infty;-8]\cup[8;+\infty)$. 4.10. $[4;5)\cup(5;6]$. 4.11 $(-\infty;2)\cup(2;\infty)$. 4.12. $(-\infty;0)\cup(0;\infty)$. 4.13. $(-\infty;-1)\cup(-1;2)\cup\cup(2;\infty)$. 4.14. $(-\infty;0)\cup(2;\infty)$. 4.15. $[2;4)\cup(4;+\infty)$. 4.16. $[0;3)\cup(3;4)$. 4.17. $(3;3{,}5)\cup(3{,}5;4)$. 4.18. $(4;5)\cup(5;+\infty)$. 4.19. $(1;+\infty)$. 4.20. $(0;+\infty)$. 4.21. $\left[-3;\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2};3\right]$. 4.22. $\left[\frac{\pi(6n-1)}{3};\ \frac{\pi(6n+1)}{3}\right]$, $n\in Z$.

4.23. $\left\{\frac{\pi(4n+1)}{2},\ n\in Z\right\}\cup\{[\pi(2m+1);\ 2\pi(m+1)],\ m\in Z\}$. 4.24. $(3-2\pi;\ 3-\pi)\cup(3;\ 4]$. 4.25. $(-\pi;\ -3)\cup(-\sqrt{2};\ 0)\cup(0;\ \sqrt{2})\cup(3;\ \pi)$. 4.26. $0<|x|<\frac{1}{2};\ \frac{1}{2}<|x|<\frac{\pi}{3};\ \frac{2\pi}{3}<|x|\le\sqrt{5}$. 4.27. $[1;\ 2)\cup(2;\ 3]$. 4.28. $[-\sqrt{3};\ \sqrt{3}]\setminus\left\{0,\pm1,\ \pm\frac{\pi}{2}\right\}$. 4.29. $(100;\ +\infty)$. 4.30. $(\infty;\ +\infty)$. 4.31. $x\ne2\pi k,\ k\in Z$. 4.32. $-\frac{\pi}{2}+2\pi k\le x\le\frac{\pi}{2}+2\pi k,\ k\in Z$. 4.33. $x=k,\ k\in Z$. 4.34. $x=\frac{m}{2},\ m=0,1,2,\ \dots$. 4.35. $2\pi k<x<\frac{\pi}{2}(4k+1),\ k\in Z$. 4.36. $-\frac{\pi}{2}+\pi k<x<\frac{\pi}{4}+\pi k,\ k\in Z$. 4.37. $(-\infty;\ -2]\cup[6;\ +\infty)$. 4.38. $(\infty;\ +\infty)$. 4.39. $[1;\ 2]$. 4.40. $x=k,\ k\in Z$. 4.41. Butun bo'lmagan barcha manfiy sonlar. 4.42. $[1;\ 100]$. 4.43. $[-\sqrt{2};\ -1]\cup[1;\ \sqrt{2}]$. 4.44. $\varnothing$. 4.45. $(-3;\ -2)\cup(-2;\ -1)\cup(-1;\ +\infty)$. 4.46. $(4{,}5;\ +\infty)$. 4.47. $(1;\ 2)$. 4.48. $(1;\ 3)$. 4.49. 4 ta. 4.50. $[-2;\ 1]$. 4.51. $[-3;\ 3]$. 4.52. $(\infty;\ 2)$. 4.53. $(-\infty;\ 0)\cup[1;\ +\infty)$. 4.54. $\{x: x\in R\ x\notin Z\}$. 4.55. $(3;\ +\infty)$. 4.56. $\varnothing$. 4.57. $(-\infty;\ 0)\cup\left(0;\ \frac{1}{3}\right)\cup\left(\frac{1}{3};\ +\infty\right)$. 4.58. $(\infty;\ +\infty)$. 4.59. $(\infty;\ 0]$. 4.60. $\varnothing$. 4.61. $(\infty;\ 0)$. 4.62. $[-3;\ 1)\cup(1;\ 2)$. 4.63. $\varnothing$. 4.64. $[1;\ 3)\cup(3;\ +\infty)$. 4.65. $(-\infty;\ 1)\cup(1;\ +\infty)$. 4.66. $(\infty;+\infty)$. 4.67. $(0;2]$. 4.68. $(0;1]$. 4.69. $[5;\ +\infty]$. 4.70. $[0;\ 0{,}5]$. 4.71. $\left[\lg\frac{11}{3};\ +\infty\right)$. 4.72. $(2;\ +\infty)$. 4.73. $[\sqrt{2};\ \sqrt{10}]$. 4.74. $[0;\ 8]$. 4.75. $[3;\ 27]$. 4.76. $[-5;\ 5]$. 4.77. $(-\infty;\ \lg3]$. 4.78. $\left[\frac{9-4\sqrt{2}}{7};\ \frac{9+4\sqrt{2}}{7}\right]$. 4.79. $\{-1;\ 1\}$. 4.80. $\left[0;\ \frac{3}{2}\right]$. 4.81. $\{-1;\ 1\}$. 4.82. $\left[0;\ \frac{\pi}{2}\right]$. 4.83. $[0{,}5;\ 1]$. 4.84. $[-8;\ 0]$. 4.85. $[-e^{-2};\ 1]$. 4.86. $[\ln3-1;\ 1]$. 4.87. $\left[-\frac{2}{\ln2};\ 1-\frac{2}{\ln2}\right]$. 4.88. $[-3+\ln10;\ 0]$.. 4.89. $(-\infty;\ -1)\cup\{0\}\cup(1;\ +\infty)$. 4.90. $(-\infty;\ +\infty)$. 4.91. $\left[0;\ \frac{\pi}{4}\right]$. 4.92. $(-\infty;\ 0{,}25)$. 4.93. $\left[-\frac{1}{6};\ 0\right)\cup\left(0;\ \frac{1}{6}\right]$. 4.94. $[1;\ 2^{\pi}]$.

5-§. FUNKSIYALARNING SINFLARI

Odatda funksiyalar quyidagi sinflarga ajratiladi: juft va toq, davriy, bir qiymatli va ko'p qiymatli, chegaralangan va chegaralanmagan, monoton, teskari, murakkab va elementar funksiyalar.

5.1. Juft va toq funksiyalar

5.1-ta'rif. Agar istalgan $x \in X$ ($X \subset R$) uchun $-x \in X$ bo'lsa, u holda X to'plam O nuqtaga (koordinatalar boshiga) nisbatan *simmetrik to'plam* deyiladi.

Butun sonlar to'plami Z, $[-a, a]$, $(-a, a)$, $(-\infty, \infty)$ kabi to'plamlar koordinatalar boshiga nisbatan simmetrik to'plamlardir.

$y=f(x)$ funksiya O nuqtaga nisbatan simmetrik bo'lgan X to'plamda aniqlangan bo'lsin.

5.2-ta'rif. Agar istalgan $x \in X$ uchun $f(-x)=f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *juft funksiya* deyiladi.

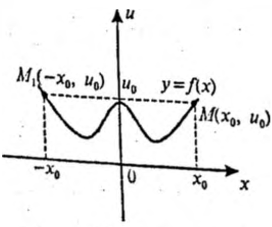
$y=x^2$, $y=\cos x$, $y=|x|$, $y=f(|x|)$ funksiyalar koordinatalar boshiga nisbatan simmetrik bo'lgan to'plamlarda qaralayotgan bo'lsa, ular juft funksiyalar bo'ladi. Ta'rifda X to'plamning koordinatalar boshiga nisbatan simmetrikligi muhimdir. Masalan, $y=x^2$, $x \in [-1,2]$ funksiya berilgan bo'lsa, u juft funksiya bo'lmaydi, chunki $[-1, 2]$ to'plam koordinatalar boshiga nisbatan simmetrik emas.

Juft funksiyalarning grafigi ordinatalar o'qiga nisbatan simmetrik bo'ladi (5.1-chizma).

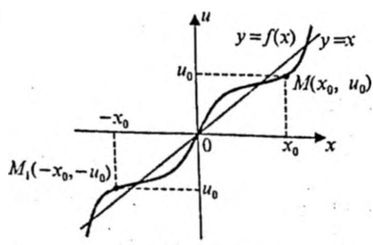
5.3-ta'rif. Agar istalgan $x \in X$ uchun $f(-x)=-f(x)$ bo'lsa, u holda $f(x)$ X to'plamda *toq funksiya* deyiladi.

$y=x^3$, $y=\text{tg}x$, $y=\dfrac{|x|}{2x}$ funksiyalar o'zlarining aniqlanish sohalarida toq funksiyalar bo'ladi.

Toq funksiyaning grafigi koordinatalar boshiga nisbatan simmetrik bo'ladi (5.2-chizma).



5.1-chizma



5.2-chizma

Agar istalgan $x\in X$, $-x\in X$ lar uchun $f(-x)\neq \pm f(x)$ shartlar o'rinli bo'lsa, u holda $y=f(x)$ funksiya X to'plamda juft ham emas, toq ham emas deyiladi.

Ushbu $f(x)=x^2-x$, $\varphi(x)=\sin x-\cos x$ funksiyalar o'zlarining aniqlanish sohasida juft ham emas, toq ham emas.

Juft funksiyaning grafigini chizishda argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni chap tomonga y o'qiga nisbatan simmetrik ravishda ko'chirish yetarli.

Toq funksiyaning grafigini chizishda esa argumentning musbat qiymatlari uchun grafikning o'ng shoxini chizib, keyin uni koordinatalar boshiga nisbatan simmetrik ko'chirish yetarli.

5.1-misol. Quyidagi berilgan funksiyalarni juft va toqlikka tekshiring:

a) $f(x)=\frac{x}{x^4+2}$, $x\in R$; b) $f(x)=x^2-3|x|+2$, $x\in R$;

d) $f(x)=\frac{x-3}{x^2-25}$, $x\in(-\infty;-5)\cup(-5;5)\cup(5;\infty)$;

e) $f(x)=|x+1|+|x-1|$, $x\in R$; f) $f(x)=\lg\frac{1-x}{1+x}$, $x\in(-1,1)$;

g) $f(x)=4-2x^4+\sin^2 x$, $x\in R$.

Yechilishi. a) $f(x)+f(-x)=0$ ekanligiga ishonch hosil qilish qiyin emas. Haqiqatan ham, $f(x)+f(-x)=\frac{x}{x^4+2}+\frac{-x}{x^4+2}=\frac{x}{x^4+2}-\frac{x}{x^4+2}=0$, demak, $f(x)+f(-x)=0$, $\forall x\in R$.

Shunday qilib, berilgan funksiya 5.3-ta'rifga asosan toq funksiya.

b) $f(x)=x^2-3|x|+2, x\in R$ bo'lganligidan, $f(-x)=(-x)^2-3|-x|+2=x^2-3|x|+2=f(x)$, ya'ni $\forall x\in R$ uchun $f(x)=f(-x)$ bo'ladi. Demak, berilgan funksiya juft funksiya ekan.

d) Berilgan funksiya $f(x)=\frac{x-3}{x^2-25}$ bo'lganligi uchun, bundan $f(-x)=\frac{-x-3}{(-x)^2-25}=\frac{-x-3}{x^2-25}=-\frac{x+3}{x^2-25}$, demak, $\forall x\in(-\infty;-5)\cup(-5;5)\cup(5;\infty)$ lar uchun $f(-x)\neq f(x)$ va $f(-x)\neq -f(x)$ bo'lgani uchun, berilgan funksiya juft ham emas, toq ham emas.

e) $f(x)=|x+1|+|x-1|$ bo'lgani uchun, bundan $f(-x)=|-x+1|+|-x-1|=|-(x-1)|+|-(x+1)|=|x-1|+|x+1|=f(x), \forall x\in R$. Demak, berilgan funksiya, 5.2-ta'rifga asosan juft funksiya bo'ladi.

f) $f(-x)=\lg\frac{1+x}{1-x}=\lg\left(\frac{1-x}{1+x}\right)^{-1}=-\lg\frac{1-x}{1+x}$. Demak, $\forall x\in(-1;\ 1)$ lar uchun $f(-x)=-f(x)$ tenglik o'rinli bo'lganligidan, 5.3-ta'rifga asosan berilgan funksiya toq funksiyadir.

g) $f(-x)=4-2(-x)^4+(\sin(-x))^2=4-2x^4+\sin^2 x$. Bundan $\forall\ x\in R$ lar uchun

$f(x)=f(-x)$ tenglikning o'rinli ekanligi kelib chiqadi. Demak, 5.2-ta'rifga ko'ra berilgan funksiya juft funksiyadir.

Juft va toq funksiyalar quyidagi xossalarga ega:

1°. Ikkita juft funksiyaning yig'indisi, ayirmasi, ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda) yana juft funksiya bo'ladi.

2°. Ikkita toq funksiyaning yig'indisi va ayirmasi yana toq funksiya bo'ladi.

3°. Ikkita toq funksiyaning ko'paytmasi va nisbati (maxraj noldan farqli bo'lganda) juft funksiya bo'ladi.

4°. Agar $y=f(x)$, $x=\varphi(t)$ toq funksiya bo'lsa, u holda $y=f(\varphi(t))$ myrakkab funksiya (8-§. 7-b.- ga q.) ham toq funksiya bo'ladi.

5°. Agar $y=f(x)$ juft funksiya, $x=\varphi(t)$ esa toq (juft) funksiya bo'lsa, u holda $y=f(\varphi(t))$ murakkab funksiya ham juft funksiya bo'ladi.

Simmetrik bo'lmagan to'plamda aniqlangan funksiyalarning juft va toqligi to'g'risida so'z yuritish ma'noga ega emas.

Aniqlanish sohasining koordinatalar boshiga nisbatan simmetrikligi funksiyaning juft va toqligi uchun zaruriy shart bo'lib, yetarli shart bo'la olmaydi. Masalan, $y=x+3$ va $y=3^x$ funksiyalar $D(f)=(-\infty;\ \infty)$ simmetrik to'plamda aniqlangan, lekin ular juft ham emas, toq ham emas.

Teorema. *Koordinatalar boshiga nisbatan simmetrik bo'lgan X to'plamda aniqlangan har qanday $f(x)$ funksiya juft va toq funksiyalar yig'indisi ko'rinishda ifodalanadi*:

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2},$$

bunda birinchi had — juft funksiya, ikkinchi had esa toq funksiyadir.

Misol.

$$e^x=\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2} \qquad (5.1)$$

funksiyani qaraymiz. Bu yerda e^x funksiya $(-\infty;\ \infty)$da aniqlangan bo'lib, u juft ham emas, toq ham emas. (5.1) tenglikning o'ng tomonidagi yig'indilarning birinchisi juft funksiya, ikkinchi esa toq funksiya.

5.2-misol. Yuqoridagi xossalar yordamida ushbu funksiyalarning juft va toqligini aniqlang:

1) $f(x)=x^3-3x$; 2) $f(x)=x^2+|x|$;

3) $f(x)=3+2x^2$; 4) $f(x)=\frac{\sin x}{x},(x\neq 0)$;

5) $f(x)=x\frac{a^x+1}{a^x-1},\ a>1$; 6) $f(x)=2\operatorname{tg}x+\sin 2x$.

Yechilishi. 1) ma'lumki, x^3 va $3x$ funksiyalar toq funksiyalar bo'lganligi uchun, berilgan $f(x)=x^3-3x$ funksiya ikkita toq funksiyaning ayirmasidan iborat. Shuning uchun, 2-xossaga binoan $f(x)$ toq funksiya bo'ladi.

2) x^2 va $|x|$ funksiyalar juft funksiyalar bo'lgani uchun, berilgan $f(x)=x^2+|x|$ funksiya ikkita juft funksiyaning yig'indisi sifatida juft funksiya bo'ladi.

3) ravshanki, 3 va $2x^2$ funksiyalar juft funksiyalar bo'lgani uchun, berilgan $f(x)=3+2x^2$ funksiya ikkita juft funksiyalar yig'indisi sifatida juft bo'ladi.

4) $\sin x$ va x funksiyalar toq funksiyalar bo'lgani uchun, berilgan $f(x)=\frac{\sin x}{x}$, $(x\neq 0)$ funksiya ikkita toq funksiyaning nisbati sifatida 1°-xossaga asosan juft bo'ladi.

5) $g(x)=\frac{a^x+1}{a^x-1}$, $g(-x)=\frac{a^{-x}+1}{a^{-x}-1}=\frac{(1+a^x)a^x}{a^x(1-a^x)}=\frac{1+a^x}{1-a^x}=-g(x)$.

Demak, $g(x)=\frac{a^x+1}{a^x-1}$ toq funksiya ekan. U holda berilgan $f(x)=$ $=x\frac{a^x+1}{a^x-1}$, $(a>1)$ funksiya 3°-xossaga asosan ikkita toq funksiyalarning ko'paytmasi sifatida juft bo'ladi.

6) Ma'lumki, $2\operatorname{tg}x$ va $\sin 2x$ funksiyalar toq bo'lgani uchun, berilgan funksiya, 2°-xossaga asosan, ikkita toq funksiyaning yig'indisi sifatida toq bo'ladi.

5.3-misol. Quyidagi funksiyalarni juft va toq funksiyalar yig'indisi shaklida tasvirlang:

1) $f(x)=(x+1)^3$; 2) $f(x)=\frac{x-3}{x^4}$; 3) $f(x)=\sin(x+1)$.

Yechilishi. 1) Berilgan $f(x)=(x+1)^3$ funksiyani ushbu ko'rinishda tasvirlaymiz: $f(x)=(x+1)^3=x^3+3x^2+3x+1=(x^3+3x)+(3x^2+1)$. Ma'-

lumki, x^3+3x, $3x^2+1$ funksiyalar, mos ravishda toq va juft funksiyalardan iborat.

Shunday qilib, berilgan funksiya juft va toq funksiyalarning yig'indisi sifatida tasvirlanadi.

2) $f(x)=\frac{x-3}{x^4}$ funksiyani ushbu ko'rinishda tasvirlaymiz:

$f(x)=\frac{x-3}{x^4}=\frac{1}{x^3}-\frac{3}{x^4}$; bunda $\frac{1}{x^3}$ — toq, $\frac{3}{x^4}$ — juft funksiyadir.

Shunday qilib, berilgan funksiya juft va toq funksiyalarning yig'indisi shaklida tasvirlanadi.

3) $f(x)=\sin(x+1)$ funksiyani quyidagicha tasvirlaymiz: $f(x)=$ $=\sin(x+1)=\sin x\cos 1+\cos x\sin 1$, bunda $\cos 1\ \sin x$ — toq funksiya, $\cos x\ \sin 1$ — juft funksiya.

5.4-misol. Quyidagi funksiyalarning juft ham, toq ham emasligini ko'rsating:

1) $f(x)=5^x$; 2) $f(x)=\frac{a^x}{x+2}$;

3) $f(x)=\sin x+\cos x$; 4) $f(x)=x^3-2x^2+9$.

Yechilishi. 1) $f(x)=5^x$, $f(-x)=5^{-x}=\frac{1}{5^x}\neq\pm f(x)$. Shunday qilib, berilgan funksiya juft ham, toq ham emas.

2) $f(x)=\frac{a^x}{x+2}$; $f(-x)=\frac{a^{-x}}{-x+2}=\frac{1}{a^x(2-x)}\neq\pm f(x)$. Shunday qilib, berilgan funksiya juft ham, toq ham emas.

3) $f(x)=\sin x+\cos x$, $f(-x)=\sin(-x)+\cos(-x)=-\sin x+\cos x\neq$ $\neq\pm f(x)$. Bu funksiya ham juft ham, toq ham emas.

4) $f(x)=x^3-2x^2+9$, $f(-x)=(-x)^3-2(-x)^2+9=-x^3-2x^2+9\neq$ $\neq\pm f(x)$. Demak, berilgan funksiya juft ham, toq ham emas.

5.2. Davriy funksiyalar. $f(x)$ funksiya $X(X\subset R)$ to'plamda aniqlangan bo'lsin.

5.4-ta'rif. Agar shunday o'zgarmas $T(T\neq 0)$ son mavjud bo'lsaki, istalgan x, $x+T\in X$ lar uchun

$$f(x+T)=f(x) \tag{5.2}$$

tenglik o'rinli bo'lsa, $f(x)$ *davriy funksiya* deyiladi, bunda T son *funksiyaning davri* deb ataladi.

(5.2) shartni qanoatlantiruvchi musbat T larning eng kichigi (agar u mavjud bo'lsa) funksiyaning *asosiy davri* deb ataladi.

Agar $y=f(x)$ funksiya T davrga ega bo'lsa, u holda $nT(n\in Z)$ ham funksiyaning davri bo'ladi.

Agar davriy funksiya T_0 – asosiy davrga ega bo'lsa, qolgan davrlarning hammasi T_0 ga karrali bo'ladi.

Funksiya eng kichik musbat davrga ega bo'lmasligi ham mumkin. Masalan, $f(x)=5$ funksiya uchun ixtiyoriy haqiqiy son davr bo'ladi, lekin u asosiy davrga ega emas. Haqiqatan ham, $f(x)=\text{const}$, $\alpha\neq 0$ ixtiyoriy haqiqiy son bo'lsin. $f(x+\alpha)=f(x)=\text{const}$. Bu yerdan kelib chiqadiki, a davr eng kichik musbat davr emas.

5.5-ta'rif. Agar

$$f(x+\omega)=-f(x),\quad (\omega\neq 0)$$

bajarilsa, u holda $f(x)$ *anti davriy funksiya* deyiladi.

Davriy funksiyalar quyidagi xossalarga ega:

1°. Ikkita T davrga ega bo'lgan funksiyaning yig'indisi, ko'paytmasi yana davriy funksiya bo'ladi va uning davri T ga teng bo'ladi.

2°. Agar $T(T\neq 0)$ $f(x)$ va $g(x)$ funksiyalarning eng kichik musbat davri bo'lsa, bu son $f(x)\pm g(x)$, $f(x)\cdot g(x)$ uchun eng kichik musbat davr bo'lmasligi ham mumkin. Masalan, 1) $f(x)=3\sin x+2$, $g(x)=2-3\sin x$ funksiyalar eng kichik musbat $T=2\pi$ davrga ega, lekin ularning yig'indisi $f(x)+g(x)=4$ esa eng kichik asosiy davrga ega emas. 2) $f(x)=\sin x+1$, $g(x)=1-\sin x$ funksiyalarning eng kichik musbat davri $T=2\pi$, lekin $f(x)\cdot g(x)=\cos^2 x=\frac{1}{2}(1+\cos 2x)$ ko'paytmaning eng kichik musbat davri $T=\pi$ bo'ladi.

3°. Agar $f(x)$ funksiya T davrga ega bo'lsa, u holda $f(ax)$, $f(ax)+b$ funksiyalar $\tau=\frac{T}{a}$ davrga ega (bunda $a\neq 0$ ixtiyoriy haqiqiy son, x, $ax\in X$) bo'ladi.

4°. Agar $f(x)$ funksiya T davrga ega bo'lsa, u holda $Af(ax+b)$ ($A=\text{const}$, $a>0$) ham davriy funksiya bo'ladi va uning davri $\tau=\frac{T}{a}$ ga teng bo'ladi.

Agar istalgan $x\in X$ va ba'zi bir T lar uchun $f(x+T)=\frac{1}{f(x)}$ ($T\neq 0$) bo'lsa, u holda $f(x)$ funksiya $2T$ davrga ega bo'ladi.

5°. $u=\varphi(x)$ davriy funksiya bo'lsin. Agar $f(x)$ funksiya qat'iy monoton bo'lsa, u holda $y=f[\varphi(x)]$ murakkab funksiya ham davriy bo'ladi va ularning davrlari bir-biriga teng bo'ladi.

6°. Agar $y=f(u)$ funksiya qat'iy monoton bo'lmasa, u holda $y=f[\varphi(x)]$ funksiyaning davri $u=\varphi(x)$ funksiyaning davridan kichik bo'lishi ham mumkin.

5.5-misol. Ushbu $f(x)=\sin 2x$ funksiyaning davriy funksiya ekanligini ko'rsating va eng kichik musbat davrini toping.

Yechilishi. Faraz qilaylik, biror $T\neq 0$ uchun $\sin 2(x+T)=\sin 2x$ tenglik o'rinli bo'lsin, bundan $\sin 2(x+T)-\sin 2x=0$ yoki $2\sin T\cos(2x+T)=0$ tenglamaga ega bo'lamiz. Bu tenglik $\sin T=0$ uchun ham o'rinli bo'ladi, bundan $T=n\mathrm{p}$, $n\in Z$. Demak, shunday $T=n\pi$, $n\in Z$ o'zgarmas son mavjud ekan. Berilgan funksiyaning eng kichik musbat davri $T_0=\pi$ ga teng bo'ladi.

5.6-misol. Quyidagi funksiyalarni davriylikka tekshiring:

1) $f(x)=x^2+x+1$; 2) $f(x)=x+\cos x$; 3) $f(x)=\sin^4 x$.

Yechilishi. 1) Faraz qilaylik, $f(x)=x^2+x+1$ davriy funksiya bo'lsin, u holda davriy funksiyaning ta'rifiga ko'ra, shunday o'zgarmas $T\neq 0$ son mavjud bo'lib, $(x+T)^2+(x+T)+1=x^2+x+1$ tenglik o'rinli bo'ladi. Oxirgi tenglikni T ga nisbatan echib, T ni topamiz:

$$x^2+2Tx+T^2+x+T+1=x^2+x+1,$$

$$T^2+(2x+1)T=0; T_1=0; T_2=-2x-1.$$

T_1 va T_2 ning shartga ko'ra topilgan qiymatlari davriy funksiyaning ta'rifini qanoatlantirmaydi (T noldan farqli o'zgarmas son bo'lishi, ya'ni x ga bog'liq bo'lmasligi kerak edi). Demak, berilgan funksiya davriy funksiya emas.

2) faraz qilaylik, berilgan funksiya T davrga ega bo'lgan davriy funksiya bo'lsin. U holda,

$x+T+\cos(x+T)=\cos x+x$ yoki $\cos(x+T)-\cos x=-T$, $-2\sin\left(x+\frac{T}{2}\right)\sin\frac{T}{2}=-T$, $\sin\left(x+\frac{T}{2}\right)=\frac{T}{2\sin\frac{T}{2}}$. Bu tenglikning o'ng tomoni o'zgarmas miqdor, chap tomoni esa x ning funksiyasidir. Bunday bo'lishi mumkin emas, ya'ni tenglikni $\forall x$ lar uchun qanoatlantiradigan $T\neq 0$ o'zgarmas son mavjud emas. Shuning uchun, berilgan funksiya davriy funksiya emas.

3) berilgan $f(x)=\sin^4 x$ funksiya davriy funksiya bo'lsin va davri T ga teng bo'lsin. U holda $\sin^4(x+T)=\sin^4 x$ yoki $\sin^4(x+T)-\sin^4 x=0$ yoki $[\sin^2(x+T)+\sin^2 x]\cdot[\sin^2(x+T)-\sin^2 x]=0$ tenglik x ning ix-

tiyoriy qiymatlarida o'rinli bo'ladi. Bunda $\sin^2(x+T)+\sin^2 x \geq 0$ bo'lib, u aynan nolga teng emas, unda $\sin^2(x+T)-\sin^2 x=0$ yoki $\frac{1-\cos(2x+2T)}{2}-\frac{1-\cos 2x}{2}=0$, $\cos 2x-\cos(2x+2T)=0$, $2\sin(2x+$ $+T)\sin T=0$ bo'ladi. Bunda $\sin(2x+T)\neq 0$ bo'lgani uchun, $\sin T=0$ bo'lishi kerak. Bu tenglikni qanoatlantiradigan eng kichik musbat son π bo'ladi. Demak, berilgan funksiya davriy funksiya bo'ladi va uning davri $T=\pi$.

5.7-misol. Davriy funksiyalarning xossalaridan foydalanib, quyidagi funksiyalarning eng kichik musbat davrini toping:

1) $f(x)=\frac{1}{3}\operatorname{tg}x+\frac{2}{5}\operatorname{ctg}x$; 2) $f(x)=\sin\frac{3}{4}x+5\cos\left(\frac{2}{3}x+5\right)$

3) $f(x)=2+\operatorname{tg}\frac{x}{3}$.

Yechilishi. 1) $\operatorname{tg}x$ va $\operatorname{ctg}x$ larning eng kichik musbat davri π bo'lganligi uchun, 1°-xossaga asosan, berilgan funksiyaning ham eng kichik musbat davri π ga teng bo'ladi.

2) $\sin x$ va $\cos x$ funksiyalarning eng kichik musbat davri 2π dan iborat bo'lgani uchun:

a) $\sin\frac{3}{4}x$ funksiyaning davri, 3°-xossaga asosan, $a=\frac{3}{4}, T_1=$ $=\frac{2\pi}{a}=\frac{2\pi}{\frac{3}{4}}=\frac{8\pi}{3}$.

b) $5\cos\left(\frac{2}{3}x+5\right)$ funksiyaning davri, 4°-xossaga asosan, $a=\frac{2}{3}$, $T_2=\frac{2\pi}{a}=\frac{2\pi}{\frac{2}{3}}=3\pi$.

Demak, berilgan funksiyaning eng kichik musbat davri $\frac{8}{3}\pi$ va 3π sonlarining eng kichik umumiy karralisidan iborat bo'ladi, ya'ni $T=24\pi$.

3) ma'lumki, $\operatorname{tg}x$ funksiyaning eng kichik musbat davri π. Demak, berilgan funksiyaning eng kichik musbat davri, 3°-xossaga asosan, $a=\frac{1}{3}$, $T=\frac{\pi}{a}=3\pi$.

5.8-misol. $f(x)$ funksiya X to'plamda aniqlangan bo'lib, shunday $T\neq 0$ soni topilsaki, $\forall x\in X$ lar uchun $x\pm T\in X$ bo'lib, $f(x+T)=$ $=\frac{f(x)+a}{bf(x)-1}$ shart bajarilsa, $f(x)$ funksiyaning davriyligini isbot qiling.

Isboti. Ixtiyoriy x lar uchun $f(x+T)=\frac{f(x)+a}{bf(x)-1}$ tenglik o'rinli bo'lgani uchun, x ni $x+T$ ga almashtirib,

$$f(x+2T)=\frac{f(x+T)+a}{bf(x+T)-1}=\frac{\frac{f(x)+a}{bf(x)-1}+a}{b\frac{f(x)+a}{bf(x)-1}-1}=\frac{f(x)(1+ab)}{1+ab}=f(x)$$

tenglikni olamiz. Demak, berilgan funksiya davriy bo'lib, uning davri $2T$ ga teng.

Eng kichik musbat davrga ega bo'lgan funksiyalar yig'indisining eng kichik musbat davri qo'shiluvchi funksiyalar davrlarining eng kichik umumiy karralisiga teng bo'ladi. Bunda, shakl almashtirishlar natijasida nolga aylanadigan qo'shiluvchi funksiyalarning davri hisobga olinmaydi.

Masalan, ushbu

$$f(x)=2\sin 4x+\text{ctg}3x+3\sin x+\sin(x-\pi)+2\sin(x+\pi)$$

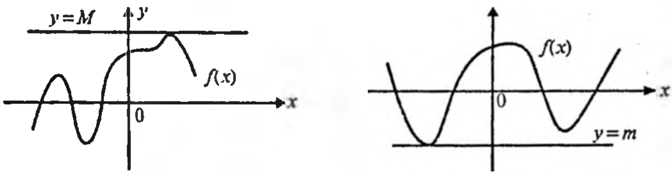
funksiyani shakl almashtirishlar natijasida quyidagi $f(x)=2\sin 4x+$ $+\text{ctg}3x$ ko'rinishda ifodalash mumkin, bunda $3\sin x+\sin(x-\pi)+$ $+2\sin(x+\pi)=0$. a) $g(x)=2\sin 4x$ funksiyaning eng kichik musbat davri, 3⁰- xossaga asosan, $T_1=\frac{2\pi}{4}=\frac{\pi}{2}$. b) $h(x)=\text{ctg}3x$ funksiyaning eng kichik musbat davri $T_2=\frac{\pi}{3}$. $\frac{\pi}{2}$ va $\frac{\pi}{3}$ sonlarning eng kichik karralisi π. Shunday qilib, yuqoridagi tasdiqqa asosan, berilgan $f(x)$ funksiyaning eng kichik musbat davri π ga teng bo'ladi.

5.3. Bir qiymatli va ko'p qiymatli funksiyalar. Agar X to'plamdagi har bir x songa biror qoida yoki qonunga ko'ra Y to'plamdan bitta y son mos qo'yilsa, u holda *u bir qiymatli funksiya* deyiladi, ya'ni $\forall\ x_1,x_2\in X, x_1\neq x_2\Rightarrow f(x_1)\neq f(x_2)$.

Agar X to'plamdagi har bir x songa biror qoida yoki qonunga ko'ra Y to'plamdan bittadan ortiq yoki cheksiz ko'p u son mos qo'yilsa, u holda funksiya *ko'p qiymatli* deyiladi. Masalan: 1) $u=\pm\sqrt{x}$ — ikki qiymatli funksiya; 2) $y=\arcsin x$ — ko'p qiymatli funksiya; 3) $y=3x+2$ — bir qiymatli funksiya.

5.4. Chegaralangan va chegaralanmagan funksiyalar. $y=f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

5.6-ta'rif. Agar shunday o'zgarmas M (o'zgarmas m) son topilib, istalgan $x \in X$ uchun $f(x) \leq M$ ($f(x) \geq m$) tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda *yuqoridan (quyidan) chegaralangan* deyiladi, aks holda funksiya *yuqoridan (quyidan) chegaralanmagan* deyiladi (5.3-chizma).

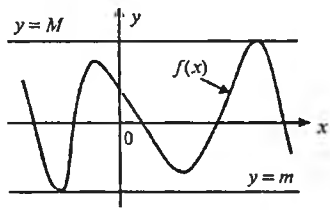


a) yuqoridan chegaralangan funksiya b) quyidan chegaralangan funksiya

5.3-chizma.

5.7-ta'rif. Agar $f(x)$ funksiya X to'plamda ham yuqoridan, ham quyidan chegaralangan bo'lsa, ya'ni shunday o'zgarmas M va m sonlar mavjud bo'lib, istalgan $x \in X$ uchun

$$m \leq f(x) \leq M \qquad (5.3)$$



5.4-chizma.

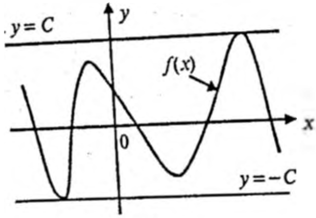
tengsizlik o'rinli bo'lsa, u holda $f(x)$ funksiya X to'plamda *chegaralangan* deyiladi (5.4-chizma).

$m = \inf\limits_{x \in X}\{f(x)\}$ son $f(x)$ funksiyaning X to'plamdagi aniq quyi chegarasi, $M = \sup\limits_{x \in X}\{f(x)\}$ son esa, $f(x)$ funksiyaning X to'plamdagi aniq yuqori chegarasi deyiladi. $M-m$ ayirma $f(x)$ funksiyaning X *to'plamdagi tebranishi* deb ataladi.

Agar $f(x)$ funksiya chegaralangan bo'lib, m va M sonlar uning aniq quyi va aniq yuqori chegaralari bo'lsa, u holda

$$|f(x)| \leq C \qquad (5.4)$$

tengsizlik o'rinli bo'ladi, bunda $C = \max\{|M|, |m|\}$. (5.3) bilan (5.4) tengsizliklar o'zaro teng kuchlidir (5.5-chizma). Demak, (5.4) tengsizlik funksiyaning chegaralanganlik shartini ifodalaydi.



5.5-chizma.

Chegaralangan funksiyalarning grafigi Ox o'qqa parallel bo'lgan $y = C$ va $y = -C$ to'g'ri chiziqlar orasida bo'ladi (5.5-chizma).

Quyidan chegaralangan ($f(x) \geq m$) funksiyaning grafigi Ox o'qqa parallel bo'lgan $y = m$ to'g'ri chiziqdan yuqorida joylashgan bo'ladi (5.3-*b* chizma).

Yuqoridan chegaralangan funksiyaning grafigi ($f(x) \leq M$) Ox o'qqa parallel bo'lgan $y = M$ to'g'ri chiziqdan pastda joylashadi (5.3-*a* chizma).

5.8-ta'rif. Agar istalgan musbat $C > 0$ son uchun shunday $x_c \in X$ topilib, $|f(x_c)| > C$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda *chegaralanmagan* deyiladi.

$f(x)$ funksiya x_0 nuqtaning biror atrofida chegaralanmagan bo'lsa, u x_0 *nuqtada chegaralanmagan* deyiladi.

$f(x)$ funksiyaning $[a; b]$ kesmada chegaralangan bo'lishi uchun, uning $[a; b]$ kesmaning har bir nuqtasida chegaralangan bo'lishi zarur va yetarlidir.

Chegaralangan funksiya quyidagi xossalarga ega:

$f(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan bo'lib, ular shu to'plamda chegaralangan bo'lsa, u holda a) $f(x) \pm g(x)$; b) $f(x) \cdot g(x)$; c) $\frac{f(x)}{g(x)}$, $(g(x) \neq 0, x \in X)$; d) $|f(x)|$, $|g(x)|$ funksiyalar ham X to'plamda chegaralangan bo'ladi.

5.9-misol. Quyidagi funksiyalarni o'z aniqlanish sohalarida chegaralanganlikka tekshiring:

1) $f(x) = -x^2 + 4x - 3$; 2) $f(x) = \frac{3x^2}{x^4 + 9}$;

3) $f(x) = 3^{\sin^2 x} + 5\cos 2x$; 4) $f(x) = \frac{1 + x^2}{1 + x^4}$; 5) $f(x) = \operatorname{tg} x$.

Yechilishi. 1) Berilgan uchhadni to'liq kvadratga keltiramiz: $f(x) = -x^2 + 4x - 3 = -(x-2)^2 + 1$. Funksiya $x = 2$ nuqtada eng katta qiymatga erishadi va u 1 ga teng. Berilgan funksiyaning qiymatlar sohasi $E(f) = (-\infty; 1]$.

Demak, funksiya yuqoridan chegaralangan, quyidan chegaralanmagan, ya'ni $-\infty < f(x) \leq 1$.

2) o'rta arifmetik va o'rta geometrik qiymatlar orasidagi munosabatlardan, ya'ni $0 \leq (x^2 - 3)^2$ yoki $6x^2 \leq x^4 + 9$ dan, ushbu $0 \leq \frac{3x^2}{x^2 + 9} \leq \frac{1}{2}$

tengsizlik kelib chiqadi. Demak, berilgan funksiya butun son o'qida chegaralangan $\left(\text{bunda } M=\frac{1}{2},\ m=0\right)$ va uning grafigi $y=M=\frac{1}{2}$ va $y=m=0$ to'g'ri chiziqlar orasida joylashadi.

3) ravshanki, berilgan funksiya butun son o'qida aniqlangan va quyidan $3^{\sin^2 x}+5\cos 2x \geq 3-5=-2$ bilan chearalangan, yuqoridan esa $|3^{\sin^2 x}+5\cos 2x| \leq |3^{\sin^2 x}|+5|\cos 2x| \leq 3+5=8$ chegaralangan. Demak, qaralayotgan funksiya chegaralangan ($C=8$, $C_1=-2$) va uning grafigi $y=C=8$ va $y=C_1=-2$ to'g'ri chiziqlar orasida joylashgan bo'ladi.

4) ravshanki, $f(x)=\frac{1+x^2}{1+x^4}>0$, bundan funksiyaning quyidan 0 bilan chegaralanganligi kelib chiqadi. $(1-x^2)^2 \geq 0$ tengsizlikdan $\frac{x^2}{1+x^4} \leq \frac{1}{2}$ tengsizlik kelib chiqadi. $f(x)=\frac{1+x^2}{1+x^4}=\frac{1}{1+x^4}+\frac{x^2}{1+x^4} \leq 1+\frac{1}{2}=\frac{3}{2}$, chunki $1+x^4 \geq 1$. Demak, son o'qidagi ixtiyoriy x larda $0<f(x) \leq \frac{3}{2}$ tengsizlik bajarilgani uchun, berilgan funksiya chegaralangan.

5) ma'lumki, $f(x)=\operatorname{tg}x$ funksiya $x=R \setminus \left\{x: x \in R;\ x=(2k+1)\frac{\pi}{2};\ k \in Z\right\}$ to'plamda aniqlangan. Ravshanki, $f(x)=\operatorname{tg}x$ funksiya davriy funksiya bo'lib, uning eng kichik musbat davri π ga teng. Berilgan funksiyani $\left(-\frac{\pi}{2};\ \frac{\pi}{2}\right)$ intervalda qaraganimizda, u $\left[-\frac{\pi}{4};\ \frac{\pi}{4}\right] \subset \left(-\frac{\pi}{2};\ \frac{\pi}{2}\right)$ segmentda chegaralangan, ya'ni $|\operatorname{tg}x| \leq 1$. Demak, $\operatorname{tg}x$ funksiya $-\frac{\pi}{2}<a<b<\frac{\pi}{2}$ shartni qanoatlantiruvchi ixtiyoriy $[a;\ b]$ segmentda chegaralangan, lekin $\left(-\frac{\pi}{2};\ 0\right]$, $\left[0;\ \frac{\pi}{2}\right)$ intervallarda esa chegaralanmagan.

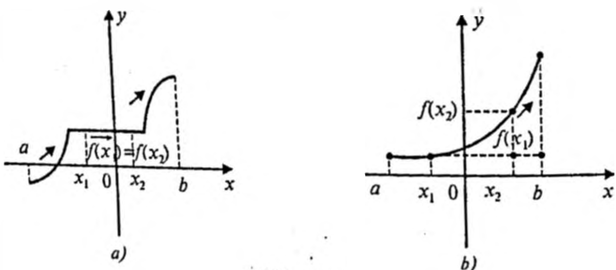
5.10-misol. Ushbu $f(x)=\frac{x}{1+x}$ funksiyaning $0 \leq x<\infty$ dagi aniq quyi, aniq yuqori chegaralarini va tebranishini toping.

Yechilishi. Pavshanki, 1) $0 \leq x<\infty$ oraliqdagi ixtiyoriy x lar uchun $\frac{x}{1+x} \geq 0$; 2) $(0;\ 1)$ intervaldan ixtiyoriy ε sonni olsak, u holda $\forall x \in \left(0;\ \frac{\varepsilon}{1-\varepsilon}\right)$ lar uchun $f(x)=\frac{x}{1+x}<\varepsilon$ tengsizlik bajariladi. Demak, $\inf\limits_{0 \leq x<+\infty}\{f(x)\}=0$. $0 \leq x<\infty$ dagi ixtiyoriy x lar uchun $f(x)=\frac{x}{1+x}<1$

tengsizlik o'rinli. U holda, yuqoridagi olingan ε lar uchun $x>\frac{1-\varepsilon}{\varepsilon}$ bo'lganda $f(x)=\frac{x}{1+x}>1-\varepsilon$ tengsizlik bajariladi. Bu tengsizlikdan $\sup\limits_{0\le x<+\infty}\{f(x)\}=1$ ekanligi kelib chiqadi. Shunday qilib, berilgan funksiyaning aniq quyi chegarasi $m=0$, aniq yuqori chegarasi $M=1$. Demak, funksiyaning $[0;+\infty)$ dagi tebranishi $w=M-m=1$ bo'ladi.

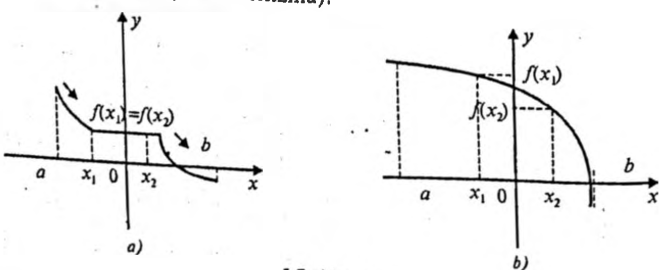
5.5. Monoton funksiyalar. $y=f(x)$ funksiya $X=[a,b]$ $(X\subset R)$ to'plamda berilgan bo'lsin.

5.9-ta'rif. Agar istalgan $x_1,x_2\in X$ lar uchun, $x_1<x_2$ bo'lganda $f(x_1)\le f(x_2)(f(x_1)<f(x_2))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda *o'suvchi* yoki *kamaymovchi (qat'iy o'suvchi)* deb ataladi (5.6-*a* chizma, 5.6-*b* chizma).



5.6-chizma.

5.10-ta'rif. Agar istalgan $x_1,x_2\in X$ lar uchun $x_1<x_2$ bo'lganda $f(x_1)\ge f(x_2)(f(x_1)>f(x_2))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya X to'plamda kamayuvchi yoki *o'smovchi (qat'iy kamayuvchi)* deb ataladi (5.7-*a* chizma, 5.7-*b* chizma).



5.7-chizma.

O'suvchi va kamayuvchi funksiyalar *monoton funksiyalar* deb ataladi.

Funksiyani monotonlikka tekshirishda quyidagi umumiy tasdiqlar muhim ahamiyatga ega:

1. Agar $f(x)$ funksiya X to'plamda o'suvchi bo'lsa, u holda $f(x)+C$ (C — ixtiyoriy o'zgarmas son) funksiya ham X to'plamda o'suvchi bo'ladi.

2. Agar $f(x)$ funksiya X to'plamda o'suvchi bo'lsa, u holda $cf(x)$ $(c>0)$ funksiya ham X to'plamda o'suvchi bo'ladi.

3. Ikkita o'suvchi (kamayuvchi) funksiyalarning yig'indisi yana o'suvchi (kamayuvchi) funksiya bo'ladi.

4. Ikkita musbat o'suvchi (kamayuvchi) funksiyalarning ko'paytmasi yana o'suvchi (kamayuvchi) bo'ladi.

5. Agar $f(x)$ funksiya X to'plamda o'suvchi, musbat va $n \in N$ bo'lsa, $f^n(x)$ funksiya ham X to'plamda o'suvchi bo'ladi.

6. Agar $f(x)$ funksiya o'suvchi bo'lsa, $-f(x)$ funksiya kamayuvchi bo'ladi va aksincha.

7. Agar $f(x)$ funksiya o'suvchi bo'lib, istalgan $x \in X$ uchun $f(x) \neq 0$ bo'lsa, $\frac{1}{f(x)}$ funksiya kamayuvchi bo'ladi.

8. Agar $f(x)$ funksiya qat'iy o'suvchi bo'lsa, $x=f^{-1}(y)$ (8-§ ning 6-b. q.) funksiya ham bir qiymatli va qat'iy o'suvchi bo'ladi.

9. Agar $x=f(t)$, $t\in[\alpha, \beta]$da o'suvchi, $y=F(x)$ funksiya esa $[f(\alpha), f(\beta)]$da o'suvchi bo'lsa, $y=F(f(x))$ funksiya ham $[\alpha, \beta]$da o'suvchi bo'ladi.

10. Agar $x=f(t)$, $t\in[\alpha, \beta]$da kamayuvchi, $y=F(x)$ funksiya esa, $[f(\alpha), f(\beta)]$da kamayuvchi bo'lsa, $y=F(f(x))$ funksiya $[\alpha, \beta]$ da o'suvchi bo'ladi.

11. Agar $x=f(t)$, $t\in[\alpha, \beta]$da o'suvchi $y=F(x)$ funksiya esa, $[f(\alpha), f(\beta)]$da kamayuvchi bo'lsa, $y=F(f(x))$ funksiya $[\alpha, \beta]$ da kamayuvchi bo'ladi.

12. Agar $\varphi(x)$, $\psi(x)$ va $f(x)$ funksiyalar o'suvchi bo'lib, $\varphi(x)\le f(x)\le \le\psi(x)$ tengsizlik o'rinli bo'lsa, $\varphi(\varphi(x))\le f(f(x))\le\psi(\psi(x))$ tengsizlik o'rinli bo'ladi.

5.11-misol. Quyidagi funksiyalarni monotonlikka tekshiring:

1) $f(x)=x^3+x$; 2) $f(x)=\sin x$, $x\in\left[\frac{-\pi}{2}; \frac{\pi}{2}\right]$;

3) $f(x)=(x^2+4x+6)\cdot\ln(x^2+4x+6)$; 4) $f(x)=\frac{4-x^2}{x}$.

Yechilishi. 1) $f(x)=x^3+x$ funksiya R da aniqlangan. R dan ixtiyoriy ikkita x_1 va x_2 nuqta olamiz. Aniqlik uchun, $x_1<x_2$ bo'lsin. $f(x_2)-f(x_1)$ ayirmani qaraymiz:

$$f(x_2)-f(x_1)=(x_2-x_1)\cdot(x_2^2+x_2x_1+x_1^2+1),$$

ikkinchi ko'paytuvchi x_1 va x_2 ning har qanday haqiqiy qiymatida musbat. Haqiqatan ham,

$$x_2^2+x_2x_1+x_1^2+1=\left(x_1+\frac{x_2}{2}\right)^2+\frac{3}{4}x_2^2+1.$$

Shartga ko'ra, $x_2-x_1>0$, u holda $f(x_2)-f(x_1)>0$, ya'ni $f(x_2)>f(x_1)$. Oxirgi tengsizlik $f(x)$ funksiyaning R da qat'iy o'suvchi ekanligini bildiradi.

2) aniqlik uchun, $x_1<x_2$ bo'lsin. $f(x_2)-f(x_1)$ ayirmani tuzamiz:

$$f(x_2)-f(x_1)=\sin x_2-\sin x_1=2\sin\frac{x_2-x_1}{2}\cdot\cos\frac{x_2+x_1}{2}.$$

$\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ kesmadan olingan $0<\frac{x_2-x_1}{2}\le\frac{\pi}{2}$ lar uchun $\sin\frac{x_2-x_1}{2}>0$ va $-\frac{\pi}{2}<\frac{x_1+x_2}{2}<\frac{\pi}{2}$ lar uchun $\cos\frac{x_1+x_2}{2}>0$. Shunday qilib, $\sin x_2>$ $>\sin x_1$. Demak, $f(x)=\sin x$ funksiya $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$da qat'iy o'suvchidir.

3) $z=x^2+4x+6=(x+2)^2+2$ bo'lsin. U holda $x<-2$ uchun z funksiya kamayuvchi, $x\ge-2$ uchun z funksiya o'suvchi bo'ladi. Bu qiymatlar uchun $z>1$ bo'ladi. Endi $y=z\ln z$ funksiyani qarasak, y $x\ge-2$ da o'suvchi, $x<-2$ da esa kamayuvchi bo'ladi, chunki, agar $x_1<x<-2$ bo'lsa, $z_1>z_2>1$ va demak, $y_1=z_1\ln z_1>z_2\ln z_2=y_2$, ya'ni berilgan $y=z\ln z$ funksiya kamayuvchidir.

4) $f(x)=\frac{4-x^2}{x}$ funksiyani $f(x)=\frac{4}{x}-x$ ko'rinishda tasvirlaymiz. Bu funksiya nolni o'z ichida saqlamaydigan har qanday intervalda $y_1=\frac{1}{x}$ va $y_2=-x$ kamayuvchi bo'lgan ikkita funksiyalar yig'indisidan iborat. 3°-xossaga ko'ra, berilgan funksiya kamayuvchidir.

5.12-misol. $f(x)=x^n$ $(n\in N)$ funksiyaning $[0;\ \infty)$ da o'suvchi ekanligini ko'rsating.

Yechilishi. Ma'lumki, $0 \le x_1 < x_2$ dan $\forall n \in N$ uchun $x_1^n < x_2^n$ tengsizligi kelib chiqadi. Demak, berilgan funksiya $[0; \infty)$ da o'suvchi.

5.13-misol. Ushbu $f(x)=\frac{1}{x^4+5x^2+7}$ funksiyaning $[0; \infty)$da kamayuvchi ekanligini ko'rsating.

Yechilishi. Ma'lumki, x^4+5x^2+7 funksiya $[0; \infty)$da manfiy emas va o'suvchi. U holda, 5) va 2) tasdiqlarga binoan, x^4 va $5x^2$ funksiyalar ham $[0; \infty)$ da o'suvchi. 1) va 3) tasdiqlarga asosan x^4+5x^2+7 funksiya ham $[0; \infty)$ da o'suvchi bo'ladi. Shunday qilib, 7) tasdiqqa asosan berilgan $f(x)=\frac{1}{x^4+5x^2+7}$ funksiya $[0; \infty)$ da kamayuvchi.

5.14-misol. Ushbu a) $f(x)=\sin x+\cos x$; b) $f(x)=\operatorname{tg}\left(x+\frac{\pi}{3}\right)$ funksiyalarning monotonlik oraliqlarini aniqlang.

Yechilishi. a) Trigonometriyadagi ma'lum formuladan foydalanib, berilgan funksiyani ushbu $f(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ ko'rinishga keltiramiz. Ma'lumki, $\cos x$ funksiya $[2n\pi; (2n+1)\pi]$ oraliqda kamayadi, $[(2n-1)\pi; 2n\pi]$, $(n=0; \pm1; \pm2, ...)$ oraliqda esa o'sadi. Shuning uchun, $f(x)=\sin x+\cos x$; funksiyaning kamayish oralig'i $\left[\frac{\pi}{4}+2n\pi; \frac{\pi}{4}+(2n+1)\pi\right]$, $(n=0; \pm1; \pm2, ...)$, o'sish oralig'i esa, $\left[\frac{\pi}{4}+(2n-1)\pi; \frac{\pi}{4}+2n\pi\right]$, $(n=0; \pm1; \pm2, ...)$ dan iborat bo'ladi.

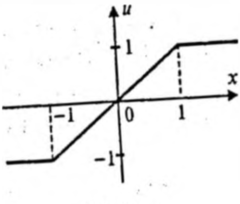
b) ma'lumki, $\operatorname{tg}x$ funksiya $\left(-\frac{\pi}{2}+k\pi; \frac{\pi}{2}+k\pi\right)$, $(n=0; \pm1; \pm2, ...)$ oraliqda o'sadi, u holda berilgan funksiya ham $\left(-\frac{5\pi}{6}+k\pi; \frac{\pi}{6}+k\pi\right)$, $(n=0; \pm1; \pm2, ...)$ oraliqda o'sadi.

5.6. Teskari funksiyalar. $f(x)$ funksiya X to'plamda aniqlangan bo'lib, funksiyaning o'zgarish (qiymatlari) sohasi Y bo'lsin.

5.11-ta'rif. $y=f(x)$ funksiyaning har bir $y \in Y$ qiymatiga biror, g munosabatga ko'ra, X dan faqat bitta x qiymat mos kelsa, Y to'plamda funksiya aniqlangan bo'ladi va u $y=f(x)$ ga nisbatan *teskari funksiya* deyiladi va $x=f^{-1}(y)=g(y)$ ko'rinishda belgilanadi.

Odatdagidek, funksiyani y bilan, argumentni esa x bilan belgilashlarga muvofiq, $x=f^{-1}(y)$ ko'rinishda yozishadi. $f^{-1}(x)=g(x)$ desak, $y=g(x)$ bo'ladi.

5.1-teorema. *$f(x)$ funksiya $D(f)$ to'plamda teskari $g(x)$ funksiyaga ega bo'lishi uchun, o'z aniqlanish sohasidagi argumentning*



5.8-chizma.

har xil qiymatiga funksiyaning ham har xil qiymati mos kelishi zarur va yetarli, ya'ni $\forall x_1, x_2 \in D(f)$ *lar uchun* $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

5.2-teorema. *Agar $y = f(x)$ funksiya X da aniqlangan qat'iy monoton o'suvchi (kamayuvchi) bo'lsa, Y da $y = f(x)$ ga teskari funksiya mavjud, bu funksiya ham qat'iy monoton o'suvchi (kamayuvchi) bo'ladi.*

5.1-eslatma. Agar funksiya monoton bo'lib, lekin qat'iy monoton bo'lmasa, bu funksiyaning teskarisi mavjud bo'lmaydi. Buni, masalan, 5.8-chizmada ko'rsatilgan

$$f(x) = \begin{cases} -1, & x < -1 \quad \text{bo'lganda}, \\ x, & -1 \le x \le 1 \quad \text{bo'lganda}, \\ 1, & x > 1 \quad \text{bo'lganda} \end{cases}$$

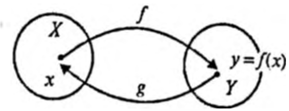
funksiya misolida ko'rish mumkin.

5.2-eslatma. Juft funksiyaning teskarisi mavjud emas. Xususiy holda, aniqlanish sohasining funksiya qat'iy monoton bo'lgan qismlarida teskari funksiya mavjud bo'ladi. Masalan, $y = x^2$ funksiya uchun $[0; +\infty)$da $y = \sqrt{x}$ teskari funksiya bo'ladi.

5.3-eslatma. Davriy funksiyaning teskarisi mavjud emas. Xususiy holda, aniqlanish sohasining funksiya qat'iy o'suvchi (kamayuvchi) bo'lgan qismlarida teskari funksiyalar mavjud bo'ladi. Masalan, $f_1(x) = \sin x \left(x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]\right)$; $f_2(x) = \cos x (x \in [0, \pi])$; $f_3(x) = \operatorname{tg} x \left(x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)\right)$; $f_4(x) = \operatorname{ctg} x (x \in (0, \pi))$ funksiyalar uchun ko'rsatilgan oraliqlarda $g_1(y) = \arcsin y \ (y \in [-1, 1])$, $g_2(y) = \arccos y \ (y \in [-1, 1])$; $g_3(y) = \operatorname{arctg} y \ (y \in (-\infty, +\infty))$; $g_4(y) = \operatorname{arcctg} y \ (y \in (-\infty, +\infty))$ teskari funksiyalar mavjud, chunki bu oraliqlarda ular qat'iy monotondir.

5.4-eslatma. $y = f(x)$ funksiya va bu funksiyaga teskari bo'lgan $x = f^{-1}(y)$ funksiyaning aniqlanish sohasi va o'zgarish sohasi o'z rollarini almashtiradi, ya'ni $y = f(x)$ funksiyaning aniqlanish sohasi teskari funksiyaning o'zgarish sohasi bo'ladi, $y = f(x)$ funksiyaning o'zgarish sohasi esa teskari funksiyaning aniqlanish sohasi bo'ladi.

$y=f(x)$ funksiya biror X to'plamda aniqlangan bo'lib, uning qiymatlari to'plami Y bo'lsin. $g(y)$ funksiya Y to'plamda aniqlangan bo'lib, X to'plam esa uning qiymatlari to'plami bo'lsin.



5.9-chizma.

5.3-teorema. $g(y)$ *funksiya* $y=f(x)$ *ga teskari funksiya bo'lishi uchun*

$$g(f(x))=x\,(x\in X)\quad (f(g(y))=y\,(y\in Y)) \qquad (5.5)$$

shartning bajarilishi zarur va yetarlidir (5.9-chizma).

Misollar. 1) $f(x)=\frac{1}{x}$, $g(x)=\frac{1}{x}$ $(x\neq 0)$ funksiyalar (5.5) shartni qanoatlantiradi. Haqiqatan ham, $f(g(x))=\frac{1}{\frac{1}{x}}=x$. Demak, ular bir-biriga teskari funksiyalar bo'ladi.

2) $f(x)=-x$, $g(x)=-x$, $x\in(-\infty,+\infty)$ funksiyalar (5.5) shartni qanoatlantiradi, ya'ni $f(g(x))=-(-x)=x$. Demak, ular bir-biriga teskari funksiyalar bo'ladi.

$y=f(x)$ to'g'ri funksiyadan $x=f^{-1}(y)$ teskari funksiyaga o'tish va uning grafigini chizish uchun, quyidagi amallarni bajarish maqsadga muvofiq:

1. $y=f(x)$ tenglama x o'zgaruvchiga nisbatan (agar tenglamani x ga nisbatan yechish mumkin bo'lsa) yechiladi:

$$x=f^{-1}(y)=g(y).$$

2. x ni y bilan, y ni x bilan almashtiriladi:

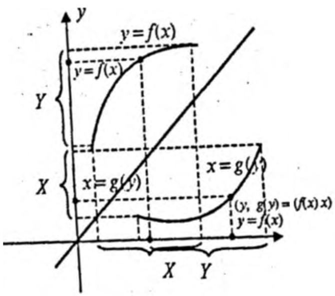
$$y=f^{-1}(x)=g(x).$$

3. $y=f(x)$ to'g'ri funksiyaning grafigi chiziladi.

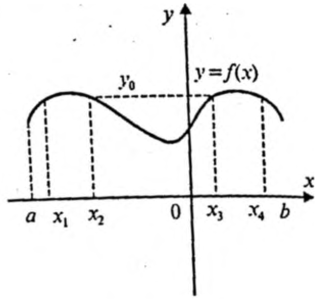
4. Hosil qilingan $y=f(x)$ funksiyaning grafigini I va III chorak koordinatalar burchaklaridan o'tuvchi bissektrisaga nisbatan simmetrik almashtirish natijasida teskari funksiya grafigi hosil qilinadi.

$f(x)$ funksiyaning grafigi $\{(x,y):x\in X, y\in Y\}$ nuqtalar to'plamidan, $g(y)$ funksiyaning grafigi esa, $\{(y,g(y))=(f(x),x)\}$ nuqtalar to'plamidan tuziladi (5.10-chizma).

5.11-chizmadagi $y=f(x)$ funksiya uchun teskari funksiya mavjud emas, chunki $x_1<x_2<x_3<x_4$ da $y_0=f(x_1)=f(x_2)=f(x_3)=f(x_4)$ bo'ladi, bu esa teskari funksiya mavjud bo'lishi shartiga ziddir.



5.10-chizma.



5.11-chizma.

5.15-misol. $y = 5x + 7$ funksiyaga teskari funksiyani toping.

Yechilishi. $y = 5x + 7$ funksiya butun son o'qida aniqlangan va qat'iy o'suvchi. Shuning uchun bu funksiyaga teskari funksiya mavjud va u ham o'suvchi. $y = 5x + 7$ tenglamani x ga nisbatan yechamiz: $x = \frac{y-7}{5}$ $(-\infty < y < \infty)$.

Endi x ni y ga va y ni x ga almashtiramiz, natijada izlanayotgan teskari funksiyaga ega bo'lamiz: $x = \frac{y-7}{5}$.

5.16-misol. Ushbu $f(x) = x^2 - x + 1, x \geq \frac{1}{2}$ va $\varphi(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$ funksiyalarning o'zaro teskari ekanligini ko'rsating.

Yechilishi. $y = f(x) = x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ funksiya $\frac{1}{2} \leq x < \infty$ da o'suvchi. x shu oraliqda o'zgarganda, $\frac{3}{4} \leq y < \infty$. Demak, berilgan funksiyaga teskari funksiya mavjud va u $x^2 - x + (1 - y) = 0$ tenglamadan topiladi:

$$x = g(y) = \frac{1}{2} + \sqrt{y - \frac{3}{4}} = \varphi(y).$$

5.17-misol. Ushbu $y = 2^x$ funksiyaga teskari funksiya mavjudmi?

Yechilishi. Berilgan funksiyaning aniqlanish sohasi — butun son o'qi; o'zgarish sohasi esa $(0; \infty)$ dan iborat bo'lib, u $(-\infty; \infty)$ to'plamda qat'iy o'suvchi bo'ladi.

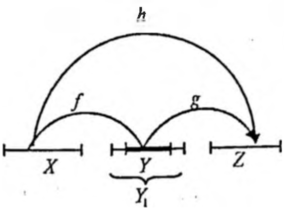
Demak, 5.2- teoremaga ko'ra, berilgan funksiyaga teskari $\log_2 x$ funksiya mavjud: $f^{-1}(f(x)) = \log_2 2^x = x \log_2 2 = x$.

5.18-misol. Ushbu $y=\cos x$ funksiyaga teskari bo'lgan funksiyani toping.

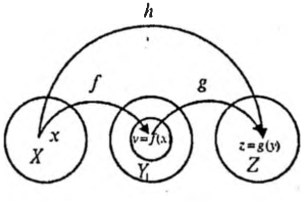
Yechilishi. Ma'lumki, $y=\cos x$ funksiyaning aniqlanish sohasi $D(y)=R=(-\infty;\infty)$, qiymatlari to'plami $E(y)=[-1;\ 1]$ dan iborat bo'lib, u teskari funksiyaning mavjudlik shartini qanoatlantirmaydi.

$R=(-\infty;\infty)$ ni ushbu $n\pi-\frac{\pi}{2}\le x\le n\pi+\frac{\pi}{2}$, $(n=0;\ \pm1;\ \pm2;\ \pm3;\ \ldots)$ oraliqlarga ajratamiz. Agar n juft bo'lsa, u holda $\cos x$ funksiya $n\pi-\frac{\pi}{2}\le x\le n\pi+\frac{\pi}{2}$, $(n=0;\ \pm1;\ \pm2;\ \pm3;\ \ldots)$ oraliqlarda o'suvchi, n toq bo'lganda esa u $n\pi-\frac{\pi}{2}\le x\le n\pi+\frac{\pi}{2}$, $(n=0;\ \pm1;\ \pm2;\ \pm3;\ \ldots)$ oraliqlarda kamayuvchi bo'lgani uchun, ko'rsatilgan oraliqlarda $[-1;\ 1]$ da aniqlangan teskari funksiya mavjud bo'ladi. Xususiy holda, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ oraliqda $\cos x$ ga teskari $x=\arccos y$ funksiya mavjud bo'ladi. $n\pi-\frac{\pi}{2}\le x\le n\pi+\frac{\pi}{2}$, $(n=0;\ \pm1;\ \pm2;\ \pm3;\ \ldots)$ oraliqda $y=\cos x$ funksiyaga teskari bo'lgan funksiyaning ko'rinishi quyidagicha bo'ladi: $x=\pm\arccos y+2\pi k,\ k\in Z$.

5.7. Murakkab funksiyalar. f va g funksiyalar, mos ravishda, X va Y_1 to'plamlarda berilgan bo'lib, f funksiyaning qiymatlari to'plami $E(f)=Y$, g funksiyaning qiymatlari to'plami $E(g)=Z$ va $Y\subseteq Y_1$ (g funksiyaning qiymatlari to'plami, g funksiyaning aniqlanish sohasida saqlansin) shart bajarilganda X to'plamda $F=g(f(x))=h(x)$ $(F=g(y),y=f(x))$ murakkab funksiya yoki g va *f funksiyalarning kompozitsiyasi aniqlangan* deyiladi va u $z=g\circ f$ kabi belgilanadi (5.12, 5.13-chizmalar). Demak, $\forall x\in X$ uchun f funksiya yordamida bitta $y\in Y$ mos qo'yiladi, so'ngra $\forall y\in Y$ uchun g funksiya yordamida bitta $z\in Z$ mos qo'yiladi. Shunday qilib, $z=g(f(x))$ funksiyaning aniqlanish sohasi $y=f(x)$ funksiya-



5.12-chizma.



5.13-chizma.

ning aniqlanish sohasiga ustma-ust tushadi yoki uning qismi bo'ladi. Bunda f funksiyaning qiymatlari sohasi g funksiyaning aniqlanish sohasida yotishi muhim, aks holda g va f funksiyalarning kompozitsiyasi aniqlanmaydi.

5.19-misol. Ushbu $z=\sin y$ va $y=x^2$ funksiyalardan murakkab funksiya tuzish mumkinmi?

Yechilishi. Berilgan $z=\sin y$ va $y=x^2$ funksiyalarning, mos ravishda, $D(z)$, $E(y)$ sohalarini topamiz. Ma'lumki, $z=\sin y$ funksiyaning aniqlanish sohasi butun son o'qidan, ya'ni $D(z)=R=(-\infty;\ \infty)$ dan iborat. $y=x^2$ funksiyaning qiymatlari to'plami: $E(y)=[0;\ \infty)$. Bundan, $E(y)\in D(z)$ ekanligi kelib chiqadi, ya'ni yuqoridagi, murakkab funksiyani tuzish sharti bajarilayapti. Shuning uchun, bu funksiyalardan murakkab funksiya tuzish mumkin: $z=\sin x^2$. Bu funksiyaning ham aniqlanish sohasi $R=(-\infty;\ \infty)$ dan iborat.

5.20-misol. $z=\sqrt{y+4}$ va $y=3^x$ funksiyalardan murakkab funksiya tuzish mumkinmi?

Yechilishi. Ma'lumki, $y=3^x$ funksiya $D(y)=(-\infty;\ \infty)$ da aniqlangan bo'lib, uning o'zgarish sohasi $E(y)=(0;\ \infty)$ dan iborat. $z=\sqrt{y+4}$ funksiyaning aniqlanish sohasi $D(z)=[-4;\ \infty)$. Ravshanki, $E(y)\in D(z)$ murakkab funksiya tuzish sharti bajarilganligi uchun, berilgan funksiyalardan $[-4;\ \infty)$da aniqlangan $z=\sqrt{3^x+4}$ murakkab funksiyani tuzish mumkin.

5.21-misol. $z=\arcsin y$ va $y=5+x^4$ funksiyalardan murakkab funksiya tuzish mumkinmi?

Yechilishi. Ma'lumki, $z=\arcsin y$ funksiya $D(z)=[-1;\ 1]$ da aniqlangan, lekin $y=5+x^4$ funksiyaning o'zgarish sohasi $E(y)$ ixtiyoriy x larda $D(z)=[-1;\ 1]$ da yotmaydi.

Demak, murakkab funksiya tuzish sharti bajarilmaydi. Shuning uchun berilgan funksiyalardan murakab funksiya tuzish mumkin emas.

5.22-misol. Ushbu $f(x)=x^2$ va $g(x)=3^x$ funksiyalar berilgan. $f(f(x))$, $f(g(x))$, $g(g(x))$ funksiyalarni toping.

Yechilishi. Funksiyalarning berilishiga ko'ra,

$$f(f(x))=f(x^2)=(x^2)^2=x^4,\quad f(g(x))=f(3^x)=(3^x)^2=3^{2x}$$

$g(g(x))=g(3^x)=3^{3^x}$ bo'ladi va ular o'z navbatida murakkab funksiyani tashkil etadi, chunki $E(f)=[0;\infty)\subset D(f)=(-\infty;\infty)$, $E(g)=(0;\infty)\subset D(f)$, $E(g)=(0;\ \infty)\subset D(g)=(-\infty;\ \infty)$.

5.23-misol. Ushbu $f(x)=3^x$ va $f^{-1}(x)=\log_3 x$ funksiyalarning $f(f^{-1}(x))$, $f^{-1}(f(x))$ kompozitsiyalarini toping.

Yechilishi. Teskari funksiyaning ta'rifiga ko'ra,

$$f(f^{-1}(x))=3^{f^{-1}(x)}=3^{\log_3 x}=x,\ f^{-1}(f(x))=\log_3 f(x)=\log_3 3^x=x.$$

5.24-misol. Ushbu $g(x)=\sqrt{3-x}$, $D(g)=(-\infty,3]$ va $f(x)=x^2-1$, $D(f)=(-\infty,\infty)$ funksiyalardan $f\circ g$, $g\circ f$ kompozitsiyalarni toping.

Yechilishi. Ta'rifga ko'ra,

$$(f\circ g)(x)=f(g(x))=(\sqrt{3-x})^2-1=3-x-1=2-x$$

va

$$D(f\circ g)=\{x\in D(g): g(x)\in D(f)\}=D(g),$$

chunki $D(f)$ barcha haqiqiy sonlardan iborat. Demak, $D(f\circ g)=(-\infty,\ 3]$ ekan.

Endi $(g\circ f)(x)$ ning ifodasini va aniqlanish sohasini topamiz:

$$(g\circ f)(x)=g(f(x))=\sqrt{3-(x^2-1)}=\sqrt{4-x^2}$$

va

$$D(g\circ f)=\{x\in D(f)\ :\ f(x)\in(-\infty,3]=\{x:\ x^2-1\le 3\}=[-2,2].$$

Demak, $D(g\circ f)=[-2,2]$.

5.25-misol. Ushbu $f(x)=\frac{1}{x}$; $g(x)=x^2+1$, $h(x)=\cos x$ funksiyalarning $(f\circ g\circ h)(x)$ kompozitsiyasini yozing.

Yechilishi. Ta'rifga ko'ra,

$$(f\circ g\circ h)(x)=f(g(h(x)))=\frac{1}{g(h(x))}=\frac{1}{[h(x)]^2+1}=\frac{1}{\cos^2 x+1}.$$

5.26-misol. $f\circ g=F$ kompozitsiyada $F(x)=(x+1)^5$ bo'lsa, f va g funksiyalarni toping.

Yechilishi. $F(x)$ funksiya x ga 1 ni qo'shib, keyin bu ifodani 5-darajaga ko'tarishdan iborat bo'lganligidan, $g(x)=x+1$ va $f(x)=x^5$ deb olish maqsadga muvofiq bo'ladi. Haqiqatan ham, yuqoridagi belgilashlarda $(f\circ g)(x)=f(g(x))=[g(x)]^5=(x+1)^5$ bo'ladi.

5.27-misol. Berilgan jadvalni to'ldiring:

№	$g(x)$	$f(x)$	$(f\circ g)(x)$
1)	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
2)	?	$1+\frac{1}{x}$	x
3)	$\frac{1}{x}$	?	x
4)	$\sqrt{x}$	?	x

Yechilishi. 1) shartga ko'ra, $f(x)=\sqrt{x-5}$, $(f\circ g)(x)=f(g(x))=$ $=\sqrt{x^2-5}$, bo'lganligi uchun $f(g)=\sqrt{g-5}$, $\sqrt{g-5}=\sqrt{x^2-5}$, bundan $g-5=x^2-5$, $g(x)=x^2$.

2) Shartga ko'ra, $f(x)=1+\frac{1}{x}$, $(f\circ g)(x)=x$ bo'lganligi uchun, $f(g)=1+\frac{1}{g}$, $1+\frac{1}{g}=x$ bo'ladi. Bundan $g(x)=\frac{1}{x-1}$;

3) shartga ko'ra, $g(x)=\frac{1}{x}$, $(f\circ g)(x)=x$ bo'lganligi uchun, $f(g)=f\left(\frac{1}{x}\right)=x$, bu yerda $\frac{1}{x}=t$ deb olsak, u holda $f(t)=\frac{1}{t}$;

4) Shartga ko'ra, $g(x)=\sqrt{x}$, $(f\circ g)(x)=x$ bo'lganligi uchun, $f(g)=f(\sqrt{x})=x$, bu yerda $\sqrt{x}=t$ deb belgilasak, u holda $x=t^2$, $f(t)=t^2$.

5.8. Elementar funksiyalar. Matematikaning ko'p masalalarida qo'llaniladigan quyidagi funksiyalarga *asosiy elementar funksiyalar* deyiladi. Ular:

1. $y=b$ — o'zgarmas funksiya ($b=\text{const}$), $b\in R$.
2. $y=x^a$ — darajali funksiya, a — haqiqiy son.
3. $y=a^x$ — ko'rsatkichli funksiya, bunda $a>0$, $a\neq 1$.
4. $y=\log_a x$ — logarifmik funksiya $a>0$, $a\neq 1$, $x>0$.
5. $y=\sin x$, $y=\cos x$, $y=\text{tg}x$, $y=\text{ctg}x$, $y=\frac{1}{\cos x}=\sec x$, $y=\frac{1}{\sin x}=$ $=\text{cosec}x$ — trigonometrik funksiyalar.
6. $y=\arcsin x$, $y=\arccos x$, $y=\text{arctg}x$, $y=\text{arcctg}x$ — teskari trigonometrik funksiyalar.

Asosiy elementar funksiyalarning xossalari umumiy o'rta ta'lim maktabi kursidan ma'lum.

Asosiy elementar funksiyalarning ustida chekli sondagi qo'shish, ayirish, ko'paytirish, bo'lish amallarini bajarish va murakkab funksiya hosil qilish natijasida yuzaga keladigan (analitik usulda berilgan) funksiyalar *elementar funksiyalar* deyiladi. Masalan, ushbu

$$y=3^{x+\mathrm{tg}\frac{1}{x}},\ y=\arcsin x^2,\ y=\log_3(x^2-4x+3)$$

funksiyalar elementar funksiyalardir.

Elementar bo'lmagan funksiyalarga misollar keltiramiz:

1. $y=\sin x+3^{4x}+\mathrm{arctg}2x^2+...$

2. $y=x+2x^2+\frac{1}{4}x^3+...$

3. Dirixle funksiyasi:

$$D(x)=\begin{cases}0, & x \text{ irratsional son bo'lganda,}\\ 1, & x \text{ ratsional son bo'lganda.}\end{cases}$$

1, 2-misollarda amallar chekli marta bajarilmagan, 3-misolda esa elementar funksiyalar qatnashgani yo'q.

Elementar funksiyalar quyidagi asosiy sinflarga bo'linadi:

1. Butun ratsional funksiya (ko'phad). Bunday funksiyaning umumiy ko'rinishi quyidagicha:

$$f(x)=a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n,$$

bunda a_0, a_1, ..., a_n – haqiqiy sonlar, $a_0\neq 0$ (ko'phadning koeffitsiyentlari), n – manfiy bo'lmagan butun son (ko'phadning darajasi). a_0, a_1, ..., a_n sonlar va x ustida chekli sondagi qo'shish va ko'paytirish amallari bajarilgan. Butun ratsional funksiyaning aniqlanish sohasi R dan iborat. Xususiy holda, $y=ax+b$ – chiziqli funksiya, va $y=ax^2+bx+c$ – kvadrat uchhad butun ratsional funksiyalardir.

2. Kasr-ratsional funksiya. Ikkita qisqarmas butun ratsional funksiyaning (ko'phadning) nisbatidan tuzilgan

$$f(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_0x^n+a_1x^{n-1}+...+a_n}{b_0x^m+b_1x^{m-1}+...+b_m}$$

funksiya *kasr-ratsional funksiya* deb ataladi. Kasr-ratsional funksiya

$$X = R \setminus \{x : x \in R,\ Q_m(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m = 0\}$$

to'plamda, ya'ni maxrajni nolga aylantiruvchi nuqtalardan farqli bo'lgan barcha haqiqiy sonlardan iborat to'plamda aniqlangan. Xususiy holda, $y = \frac{x^5+2}{x^2+3x}$ va $y = \frac{x}{x^4-1}$ — kasr-ratsional funksiyalar bo'ladi.

Butun ratsional va kasr — ratsional funksiyalar birgalikda ratsional funksiyalar sinfini tashkil qiladi.

3. Irratsional funksiya. Tarkibida irratsional ifodalar qatnashgan algebraik ifodalarning moslik qonunini ratsional ifodalar yordamida ko'rsatish mumkin bo'lmasa, bunday ifodalar *irratsional funksiyalar* deyiladi. Masalan, ushbu

$$y = \frac{3x^3 + \sqrt{x^3-4}}{\sqrt[3]{1+6x}-4},\quad y = \sqrt[3]{x+2},\quad y = \sqrt{x}$$

funksiyalar irratsional funksiyalardir.

4. Algebraik funksiya. Ratsional va irratsional funksiyalar sinflari birgalikda algebraik funksiyalar sinfini tashkil qiladi. Ushbu

$$A_0(x)y^n + A_1(x)y^{n-1} + \ldots + A_{n-1}(x)y + A_n(x) = 0$$

tenglamani qanoatlantiradigan har qanday $y = f(x)$ funksiya *algebraik funksiya* deyiladi. Bunda $A_0(x), A_1(x), \ldots, A_n(x)$ — berilgan butun ratsional funksiyalar, $A_0(x) \neq 0$ va n — butun musbat son.

Misollar. 1. $y = \sqrt[3]{x^2+1}$ — algebraik funksiyadir, chunki u $y^3 - x^2 - 1 = 0$ tenglamani qanoatlantiradi.

2. Ushbu $xy^2 - 2(x^2-1)y - 4x = 0$ tenglama bilan aniqlangan funksiya ikkita algebraik funksiyadan iborat bo'ladi, ya'ni

$$y = \frac{x^2 - 1 + \sqrt{(x^2-1)^2 + 4x^2}}{x} = 2x,$$

$$y = \frac{x^2 - 1 - \sqrt{(x^2-1)^2 + 4x^2}}{x} = -\frac{2}{x}$$

3. Ushbu $x^4y^5 + xy - x^3 + 4 = 0$ tenglama bilan aniqlangan funksiya ham algebraik funksiya bo'ladi, lekin bu funksiyani oshkor

ko'rinishda yozish mumkin emas, chunki uni radikalga nisbatan echib bo'lmaydi. Bu holda biz oshkormas algebraik funksiyaga ega bo'lamiz.

5. Transsendent funksiyalar. Barcha algebraik bo'lmagan funksiyalar *transsendent funksiyalar* deb ataladi (ko'rsatkichli, logarifmik, trigonometrik funksiyalar va teskari trigonometrik funksiyalar —transsendent funksiyalardir). Masalan: $y=\sin 3x$, $y=2^{x+3}$, $y=\mathrm{arctg}5x$ funksiyalar transsendent funksiyalardir.

Shunday qilib, barcha elementar funksiyalar algebraik va *transsendent funksiyalar sinflariga bo'linadi.*

5.9. Funksiya grafigi ustida elementar shakl almashtirishlar

Biz quyida elementar funksiyalarning ba'zi sinflarini alohida qaraymiz va ularning asosiy elementar funksiyalarning ma'lum xossalariga tayangan holda kelib chiqadigan asosiy (bosh) xossalarini hamda funksiya ustida ma'lum bir amallar bajarish natijasida uning grafigi shaklini almashtirish qoidalarini keltiramiz.

Faraz qilaylik, X to'plamdan aniqlangan $y=f(x)$ funksiyaning grafigi chizilgan bo'lsin. Quyidagi jadvalda $f(x)$ funksiya yoki uning argumentini ma'lum bir shakl almashtirish natijasida bu grafik qanday o'zgarishi bayoni berilgan.

Funksiya	$y=f(x)$ funksiya grafigi ustida xOy tekislikda amalga oshirishi lozim bo'lgan (shakl) almashtirishlar
$f(x)+A,\ A\neq 0$	$y=f(x)$ funksiya grafigini Oy o'q bo'ylab: $A>0$ bo'lgan A birlik yuqoriga; $A<0$ bo'lganda $\lvert A\rvert$ birlik quyiga (pastga) siljitish (ko'chirish)
$f(x-a),\ a\neq 0$	$y=f(x)$ funksiya grafigini Ox o'q bo'ylab: $a>0$ bo'lganda a birlik o'ngga; $a<0$ bo'lganda $\lvert a\rvert$ birlik chapga siljitish (ko'chirish)
$kf(x),\ k>0,\ k\neq 1$	$y=f(x)$ funksiya grafigini Oy o'q bo'ylab Ox o'qqa nisbatan: $k>1$ bo'lganda k marta cho'zish; $0<k<1$ bo'lganda $\frac{1}{k}$ marta qisish.
$f(kx),\ k>0,\ k\neq 1$	$y=f(x)$ funksiya grafigini Ox o'q bo'ylab Oy o'qqa nisbatan: $k>1$ bo'lganda k marta qisish; $0<k<1$ bo'lganda $\frac{1}{k}$ marta cho'zish.

$-f(x)$	$y=f(x)$ funksiya grafigining Ox o'qqa nisbatan simmetrik akslantirish.
$\|f(x)\|$	$y=f(x)$ funksiya grafigining Ox o'qdan pastda joylashgan qismi shu o'qqa nisbatan simmetrik akslantirilib, grafikning qolgan qismi o'zgarmasdan qoldiriladi.
$f(-x)$	$y=f(x)$ funksiya grafigini Oy o'qqa nisbatan simmetrik akslantirish.
$f(\|x\|)$	$y=f(x)$ funksiya grafigining Oy o'qdan chapda joylashgan qismini o'chirib tashlash; grafikning Oy o'qdan o'ngda va o'qning ustida joylashgan qismini o'z holida qoldirish; funksiya grafigining $x\geq 0$ sohada yotgan qismini $x<0$ sohaga Oy o'qqa nisbatan simmetrik akslantirish.

Umumiy holda, $y=Cf(ax+b)+D$ funksiya grafigini chizish $y=f(x)$ funksiya grafigi ustida bir necha shakl almashtirishlar (siljitish, qisish, akslantirish va hokazo) bajarishga keltiriladi.

Avvalo, uni quyidagi $y=Cf\left[a\left(x+\frac{b}{a}\right)\right]+D$ ko'rinishda tasvirlaymiz. Bundan, bu funksiya grafigini (yasash) chizish uchun $y_1=Cf\left[a\left(x+\frac{b}{a}\right)\right]$ funksiya grafigini chizish yetarli ekanligi ko'rinadi. y_1 funksiya grafigini chizish uchun esa, $y_2=f\left[a\left(x+\frac{b}{a}\right)\right]$ funksiya grafigini chizish yetarli. O'z navbatda, y_2 funksiyaning grafigini chizish uchun $y_3=f(ax)$ funksiya grafigini chizish yetarli. Demak, berilgan $y=Cf\left[a\left(x+\frac{b}{a}\right)\right]+D$ funksiya grafigini chizish uchun $f(x)$ funksiya grafigi ustida quyidagi (shakl) almashtirishlarni bajarish zarur bo'ladi:

1. $f(x)$ funksiya grafigini $a>0$ bo'lganda Ox o'q bo'ylab Oy o'qqa nisbatan qisish yoki cho'zish; $a<0$ bo'lganda Oy o'qqa nisbatan simmetrik akslashtirish va Ox o'q bo'ylab Oy o'qqa nisbatan qisish yoki cho'zish.

2. $f(ax)$ funksiyaning hosil bo'lgan grafigini Ox o'q bo'ylab, $\frac{b}{a}>0$ bo'lganda $\frac{b}{a}$ birlikka chapga siljitish; $\frac{b}{a}<0$ bo'lganda $\left|\frac{b}{a}\right|$ birlikka o'ngga siljitish.

3. $f\left[a\left(x+\frac{b}{a}\right)\right]$ funksiyaning hosil bo'lgan grafigini, $C>0$ bo'lganda Oy o'q bo'ylab Ox o'qqa nisbatan qisish va cho'zish; $C<0$ bo'lganda Ox o'qqa nisbatan simmetrik akslantirish va Oy o'q bo'ylab Ox o'qqa nisbatan qisish yoki cho'zish.

4. $Cf\left[a\left(x+\frac{b}{a}\right)\right]$ funksiyaning hosil bo'lgan grafigini, $D>0$ bo'lganda D birlik yuqoriga, $D<0$ bo'lganda esa, $|D|$ birlik pastga siljitish.

$y=Cf(ax+b)+D$ funksiya grafigini chizishda amalga oshirilgan shakl almashtirishlar ketma-ketligini quyidagi ko'rinishda ifodalash mumkin bo'ladi:

$$f(x)\to f(ax)\to f\left[a\left(x+\frac{b}{a}\right)\right]=f(ax+b)\to Cf(ax+b)\to Cf(ax+b)+D.$$

Unda berilgan funksiya grafigini yasashda qanday funksiya grafigi asosiy rol o'ynashi ko'rinib turibdi.

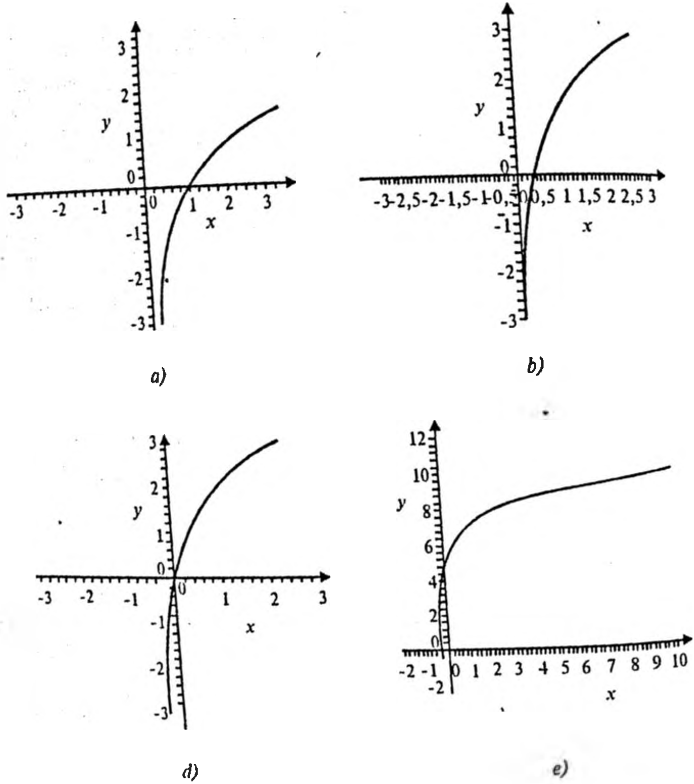
5.22-misol. Ushbu $y=\log_2(3x+1)+5$ funksiya grafigi eskizini chizing.

Yechilishi. $y=\log_2(1+3x)+5$ funksiya grafigi eskizini chizishda amalga oshiriladigan shakl almashtirishlar ketma-ketligini quyidagi ko'rinishda ifodalaymiz:

$$\log_2 x\to\log_2 3x\to\log_2\left[3\left(x+\frac{1}{3}\right)\right]\equiv\log_2(3x+1)+5.$$

Demak, berilgan funksiya grafigi eskizini chizish uchun avvalo: 1) $y_1=\log_2 x$ funksiya grafigi (5.14-*a* chizma) chiziladi; 2) so'ngra chizilgan grafik Ox o'q bo'ylab, Oy o'qqa nisbatan 3 birlikka qisiladi (5.14-*b* chizma); 3) $\log_2 3x$ funksiyaning hosil bo'lgan grafigi chapga $\left|-\frac{1}{3}\right|$ birlikka siljitiladi (5.14-*d* chizma); 4) $\log_2\left[3\left(x+\frac{1}{3}\right)\right]$ funksiyaning hosil bo'lgan grafigi Oy o'q bo'ylab yuqoriga 5 birlikka siljitiladi (5.14-*e* chizma). Natijada berilgan $y=\log_2(1+3x)+5$ funksiya grafigi eskizi hosil bo'ladi.

5.23-misol. Ushbu $y=-2\cos\left(2x+\frac{\pi}{3}\right)$ funksiya grafigi eskizini chizing.

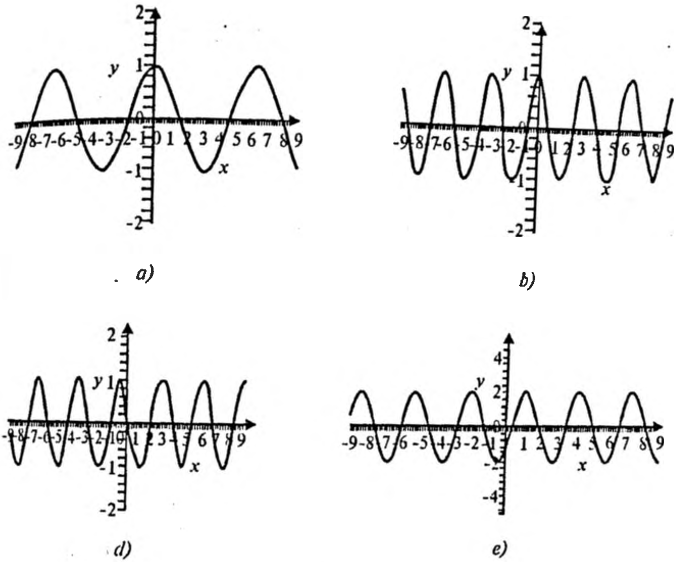


5.14-chizma.

Yechilishi. Avvalo, $y=-2\cos\left(2x+\frac{\pi}{3}\right)$ funksiya grafigining eskizini chizishda amalga oshiriladigan shakl almashtirishlar ketma-ketligini tuzamiz:

$$\cos x \to \cos 2x \to \cos\left[2\left(x+\frac{\pi}{6}\right)\right]=\cos\left(2x+\frac{\pi}{3}\right)\to -2\cos\left(2x+\frac{\pi}{3}\right).$$

Shunday qilib, $y=-2\cos\left(2x+\frac{\pi}{3}\right)$ funksiya grafigi eskizini chizish uchun avvalo: 1) $y_1=\cos x$ funksiya grafigi (5.15-*a* chizma) chizi-



5.15-chizma.

ladi; 2) $y_1 = \cos x$ funksiya grafigi Ox o‘q bo‘ylab, Oy o‘qqa nisbatan 2 birlikka qisiladi (5.15-*b* chizma); 3) $y_2 = \cos 2x$ funksiyaning hosil bo‘lgan grafigi chapga $\frac{\pi}{6}$ birlik siljitiladi (5.15-*d* chizma); 4) $y_3 = \cos\left[2\left(x+\frac{\pi}{6}\right)\right]$ funksiyaning hosil bo‘lgan grafigi Ox o‘qqa nisbatan simmetrik akslantirilib, grafik Oy o‘q bo‘ylab Ox o‘qqa nisbatan $|-2|$ birlikka cho‘ziladi (5.15-*e* chizma). Natijada berilgan funksiya grafigining eskizi hosil bo‘ladi.

5.24-misol. Ushbu $y = \frac{1}{2}x^2 + 3x + \frac{7}{2}$ funksiya grafigining eskizini chizing.

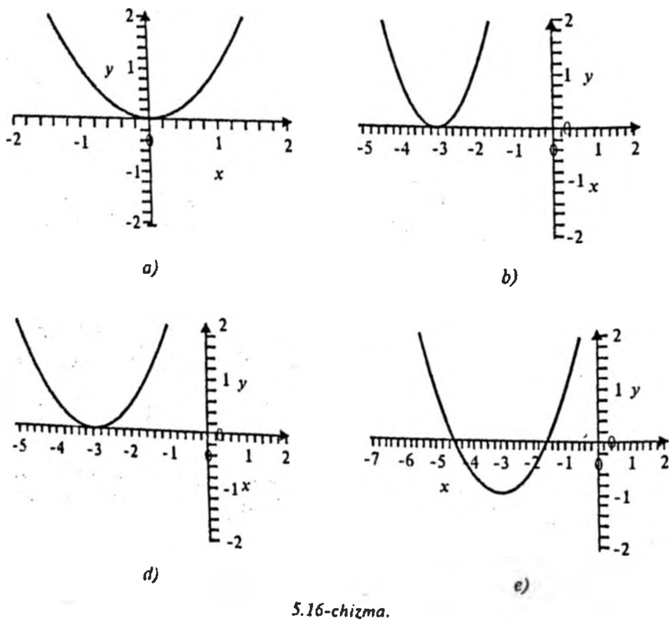
Yechilishi. Berilgan funksiya grafigining eskizini chizish uchun avvalo, grafikni chizishda amalga oshiriladigan shakl almashtirishlar ketma-ketligini tuzib olamiz:

$$x^2 \to (x+3)^2 \to \frac{1}{2}(x+3)^2 \to \frac{1}{2}(x+3)^2 - 1 = \frac{1}{2}x^2 + 3x + \frac{7}{2}.$$

Bundan ko'rinadiki, $y=\frac{1}{2}x^2+3x+\frac{7}{2}$ funksiya grafigining eskizini chizish uchun avvalo: 1) $y_1=x^2$ funksiya grafigi chiziladi (5.16-*a* chizma); 2) $y_1=x^2$ funksiyaning chizilgan grafigi Ox o'q bo'ylab, chapga $|-3|$ birlik siljitiladi (5.15-*b* chizma); 3) $y_2=(x+3)^2$ funksiyaning hosil bo'lgan grafigi Oy o'q bo'ylab 2 birlikka qisiladi (5.15-*d* chizma); 4) $y_3=\frac{1}{2}(x+3)^2$ funksiyaning hosil bo'lgan grafigi Oy o'q bo'ylab pastga $|-1|$ birlikka siljitiladi (5.15-*d* chizma). Natijada berilgan $y=\frac{1}{2}x^2+3x+\frac{7}{2}$ funksiya grafigining eskizini hosil qilamiz (5.15-*e* chizma).

5.25-misol. Ushbu $y=2^{\frac{1}{3}x-2}$ funksiya grafigini eskizini chizing.

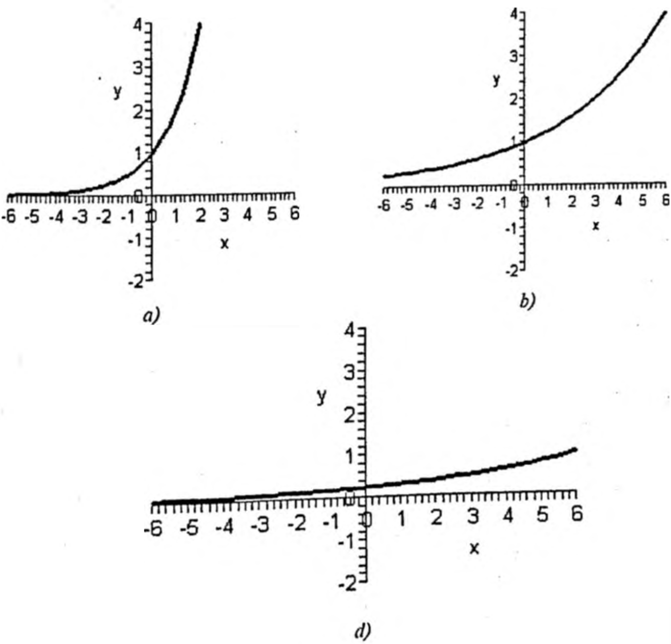
Yechilishi. Berilgan funksiya grafigini chizishda amalga oshiriladigan shakl almashtirishlar ketma-ketligini tuzamiz:



5.16-chizma.

$$2^x \to 2^{\frac{1}{3}x} \to \frac{1}{4}2^{\frac{1}{3}x} = 2^{\frac{1}{3}x-2}.$$

$y = 2^{\frac{1}{3}x-2}$ funksiya grafigining eskizini chizish uchun avvalo: 1) $y_1 = 2^x$ funksiya grafigi (5.17-*a* chizma) chiziladi; 2) so'ngra chizilgan $y_1 = 2^x$ funksiya grafigi *Ox* o'q bo'ylab, *Oy* o'qqa nisbatan 3 birlikga cho'ziladi (5.17-*b* chizma); 3) $y = 2^{\frac{1}{3}x}$ funksiyaning hosil bo'lgan grafigi *Oy* o'q bo'ylab, *Ox* o'qqa nisbatan 4 birlikka qisiladi (5.17-*d* chizma). Natijada berilgan $y = 2^{\frac{1}{3}x-2}$ funksiya grafigining eskizini (5.17-*d* chizma) hosil qilamiz.



5.17-chizma.

Mustaqil yechish uchun misol va masalalar

Quyidagi funksiyalarning qaysi biri juft, qaysi biri toq va qaysilari juft ham, yoki toq ham emasligini aniqlang:

5.1. $y=2x^2-3x$.   5.2. $y=3x^2+4$.   5.3. $y=\sqrt[3]{x}$.

5.4. $y=x^2-x^4$.   5.5. $y=x|x|$.   5.6. $y=x^2+\mathrm{ctg}\,x$.

5.7. $y=x^2|x|+3$.   5.8. $y=\sqrt[4]{2+x+x^2}-\sqrt[4]{2-x+x^2}$.

5.9. $y=\dfrac{e^x+e^{-x}}{2}$.   5.10. $y=x^4-\cos x$.

5.11. $y=2\sin 4x+3\cos 4x$.   5.12. $y=\ln\dfrac{2+x}{2-x}$.

5.13. $y=\sqrt[5]{(x+1)^2}+\sqrt[5]{(x-1)^2}$.   5.14. $y=\dfrac{x^3-x}{x^2+1}$.

5.15. $y=\dfrac{x^4+x^2+1}{x^2+1}$.   5.16. $y=\dfrac{x+4}{x-4}+\dfrac{x-4}{x+4}$.

5.17. $y=\dfrac{x^3+1}{x^2+1}$.   5.18. $y=\dfrac{x^2+4}{3x^6+x^4+7}$.

5.19. $y=ax+b,\ a,\ b\in R$.   5.20. $y=\log_2(2^x+2^{-x})$.

5.21. $y=\begin{cases} x, x\geq 0 \\ -x, x<0 \end{cases}$.   5.22. $y=x^2\cdot\dfrac{a^x-1}{a^x+1}$.

5.23. $y=\dfrac{\mathrm{tg}\,x}{\cos x}-x^3$.   5.24. $y=\sin x+\dfrac{x^3+1}{x^3-1}$.

5.25. $y=x^3+\cos x$.   5.26. $y=\sin x\cdot\mathrm{tg}\,x$.

5.27. $y=(1-x^4)\cdot\cos x$.   5.28. $y=\sin 7x+\cos 5x$.

5.29. $y=\arccos|x|$.   5.30. $y=\arcsin+\arccos x$.   5.31. $y=\dfrac{1+\sin x}{\sin x}$.

5.32. Quyidagi funksiyalarni juft va toq funksiyalar yig'indisi ko'rinishida tasvirlang:

1. $f(x)=5x^2-x+8$. 2. $f(x)=\dfrac{x^2+1}{x^2+x+1}$. 3. $f(x)=\dfrac{x^2+6}{x^3+5}$.

4. $f(x)=\dfrac{x^3+5}{x^2+6}$. 5. $f(x)=\dfrac{a^x-a^{-x}}{2}$.

Quyidagi funksiyalarni davriylikka tekshiring. Agar davriy bo'lsa, u holda uning eng kichik musbat davrini toping:

5.33. $f(x)=x^2-3x+4$.   5.34. $f(x)=\sin 2x-2\,\mathrm{tg}\,\dfrac{x}{2}$.

5.35. $f(x)=\sin|x|$.   5.36. $f(x)=\cos 2x\cdot\cos 6x$.

5.37. $f(x)=\begin{cases}1, & x \text{ ratsional son bo‘lganda,}\\ 0, & x \text{ irratsional son bo‘lganda.}\end{cases}$

5.38. $f(x) = 6\sin(0{,}25\pi x)$.

5.39. $f(x)=\cos\left(\frac{3}{2}x-18^\circ\right)$.

5.40. $f(x) = \sin\left(\frac{E}{2}+\frac{\pi}{2}\right)+5\,\mathrm{tg}\left(3x-\frac{\pi}{6}\right)$.

5.41. $f(x) = \mathrm{tg}\,\frac{E}{3}+\sin 2\pi x$.

5.42. $f(x) = 13\sin^2 3x$.

5.43. $f(x)=\cos x\cdot\cos\sqrt{3}x$.

5.44. $y = \sin\frac{x}{3}+\mathrm{tg}\frac{x}{7}$.

5.45. $y = \mathrm{tg}\frac{x}{3}+\sin 2\pi x$.

5.46. $y = \sin\frac{\pi x}{5}+\mathrm{tg}\frac{\pi x}{20}$.

5.47. $y=\sin\pi x+\cos 2x$.

5.48. $y=\mathrm{tg}2x+\mathrm{ctg}3x+\cos 5x$.

5.49. $y=\sin\left(\frac{4}{5}x-17^\circ\right)$.

5.50. $y=6\cos\frac{x}{2}-4\sin\left(\frac{2}{3}x+\frac{\pi}{2}\right)$.

5.51. $y= 15\sin^2 12x+ 12\sin^2 15x$.

5.52. $y=\sin x+\cos\frac{x}{3}+\sin\frac{x}{5}$.

5.53. $y=\sqrt{1+\cos 4x}$.

5.54. $y=\frac{2\sin 6x-\cos 4x}{3\sin 6x+\cos 4x}$.

5.55. $y=\sqrt{\cos 5{,}3x-\cos 11x+4}$.

5.56. $y=\cos\sqrt{|x|}$.

5.57. $y=\cos x+\sin(x\sqrt{2})$.

5.58. $y=\cos x^3$.

5.59. $y=\left\{\frac{x}{3}\right\}+3\left\{\frac{x}{5}\right\}$.

5.60. $y=\{3x\}+8\{5x\}$.

5.61. $y=\frac{2\{6x\}-\{4x\}}{3\{6x\}+\{4x\}}$.

5.62. $y=\left\{2x+\frac{1}{2}\right\}+\left\{5x+\frac{1}{4}\right\}$

5.63. $y=\sqrt{\{5{,}3x\}-\left\{11x+\frac{4}{5}\right\}+4}$.

5.64. $y=\left\{\sqrt{|x|}\right\}$.

5.65. $y=\{x\}+\{x\sqrt{2}\}$.

5.66. $y=\{x^2\}$

5.67. $y=x^2-5x+6$

5.68. $f(x)=[2x+5]-2x$, bu yerda $[a]$ — qaralayotgan sonning butun qismini bildiradi.

5.69. $f(x)$ funksiya X to‘plamda aniqlangan bo‘lib, shunday $T\neq 0$ son topilsaki, har qanday x lar uchun $x+T\in X$, $x-T\in X$ bo‘lib, quyidagi shartlardan bittasi bajarilsa, uning davriyligini isbotlang:

1) $f(x+T)=-f(x)$; 2) $f(x+T)=1/f(x)$; 3) $f(x+T)=\frac{f(x)+a}{bf(x)-1}$;

4) $f(x+T)=\frac{1}{1-f(x)}$; 5) $f(x+T)=f(x)$.

5.70. Butun sonlar o'qidan bitta nuqta chiqarib tashlangan to'plamda aniqlangan funksiyaning davriy emasligini isbotlang.

Quyidagi funksiyalarni o'z aniqlanish sohalarida chegaralanganlikka tekshiring:

5.71. $y=\frac{1}{x+14}$, $x\in[0;5]$. 5.72. $y=x^2-6x+9$.

5.73. $y=\frac{2}{\sin 3x}$. 5.74. $y=\lg(x^2-6x+8)$.

5.75. $f(x)=\frac{x^3}{x^6+1}$. 5.76. $f(x)=\frac{1}{x^2+6}$.

5.77. $y=\frac{\sin x}{x^4+4}$. 5.78. $y=\frac{1}{\sqrt{4-x^2}}$.

5.79. $y=x\cdot\cos^2 x$. 5.80. $y=\frac{1+x}{1+x^2}$.

5.81. $y=\frac{1}{x-x^2}$. 5.82. $y=\frac{e^x}{e^x-1}$.

5.83. $y=\arcsin\frac{4x}{4+x^2}$. 5.84. $y=\sqrt[3]{2-x-x^2}$.

5.85. $y=\sqrt{x^2-4x+5}$. 5.86. $y=2^{x^2-2x+2}$.

5.87. $y=\frac{\sin x}{x}$. 5.88. $f(x)=\operatorname{arctg}\frac{x^2}{x^4+1}$, $x\in R$.

5.89. $f(x)=\frac{x^2-1}{|x^2-1|}$, $x\in R$, $x\neq 1$ 5.90. $f(x)=\frac{\sqrt[3]{x^4+10}}{x^2+1}$.

5.91. $f(x)=-0{,}5\sin x\cos x$. 5.92. Ushbu $f(x)=\frac{1}{x}\cos\frac{1}{x}$ funksiya $x=0$ nuqtaning atrofida chegaralanmagan bo'lishi, x nolga intilganda esa cheksiz katta bo'lmasligini isbotlang.

5.93. $(0;\varepsilon)(\varepsilon>0)$da $f(x)=\ln x\sin^2\frac{\pi}{x}$ funksiyani chegaralanganlikka tekshiring.

5.94. $f(x)=\frac{x^2}{1+x^2}$ funksiyaning $(-\infty;\ +\infty)$da chegaralanganligini ko'rsating.

5.95. $f(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan bo'lib, chegaralangan bo'lsa, u holda
a) $f(x)+g(x)$; b) $f(x)-g(x)$; c) $f(x)\ g(x)$; d) $|f(x)|$ funksiyalarning ham X to'plamda chegaralanganligini ko'rsating.
e) qanday shart bajarilganda $\frac{f(x)}{g(x)}$ funksiya ham X to'plamda chegaralangan bo'ladi?

5.96. $f(x)$ funksiya X to'plamda aniqlangan va chegaralangan bo'lsa, ushbu funksiyalarning X da chegaralanganligini ko'rsating:
1) $\sqrt[n]{f(x)}$; 2) $2^{f(x)}$; 3) $\cos^{f(x)}$; 4) $\sin f(x)$; 5) $\arcsin f(x)$; 6) $\arccos f(x)$; 7) $\mathrm{arctg} f(x)$; 8) $\mathrm{arcctg} f(x)$.

5.97. Quyidagi funksiyalarning o'z aniqlanish sohalarida chegaralanmaganligini ko'rsating:

3) $f(x)=\sqrt[5]{x^6}$; 4) $f(x)=4^{\sqrt[6]{x}}+3$; 5) $f(x)=x+\frac{1}{x}$;
6) $f(x)=\frac{1}{\sin x}$; 7) $f(x)=|x-3|+|4x-1|$;
8) $f(x)=x\cos x$; 9) $f(x)=\log(x^2+4)$; 10) $f(x)=\left(\frac{1}{4}\right)^x x$;
11) $f(x)=\frac{1}{\mathrm{arcctg}\, x}$; 12) $f(x)-|x-3|+2$.

5.98. $f(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan va chegaralangan bo'lsin. $f(x)$ va $g(x)$ funksiyalarning ayirmasi X to'plamda chegaralangan bo'lishi mumkinmi? Misollar keltiring.

5.99. $f(x)$ va $g(x)$ funksiyalar X to'plamda aniqlangan bo'lib, $f(x)$ funksiya X to'plamda chegaralangan, $g(x)$ chegaralanmagan bo'lsin. $f(x)$ va $g(x)$ funksiyalar orasida arifmetik amallar bajarilishi natijasida hosil bo'lgan funksiyalarning chegaralanganligi haqida nima deyish mumkin? Misollar keltiring.

5.100. Ixtiyoriy funksiyaning kvadrati quyidan chegaralanganligini isbotlang.

Quyidagi funksiyalarni o'z aniqlanish sohalarida monotonlikka tekshiring:

5.101. $y=x+\frac{1}{x}$. **5.102.** $y=x^2-6x+10$. **5.103.** $y=\frac{x^2}{1+x^2}$.

5.104. $y=x^3+x$. 5.105. $y=x^4-2x^2-3$. 5.106. $y=4^{\frac{1}{x}}$.

5.107. $y=2\cdot 3^{1-x}-9^{-x}$. 5.108. $y=x-\sqrt{x^2-1}$.

5.109. $y=5^{|x-2|+|x+1|}$. 5.110. $y=\operatorname{arctg}x-x$.

5.111. $y=\log_{\frac{1}{2}}\frac{x}{x+1}$. 5.112. $y=\frac{x-2}{|x|+2}$.

5.113. $y=\frac{1-|x-1|}{1+|x|}$. 5.114. $y=\frac{|x-2|-|x+2|}{|x-2|+|x+2|}$.

5.115. $y=\sqrt[6]{x^2-4}$. 5.116. $y=\sqrt[4]{x-3}+\sqrt[4]{x+3}$.

5.117. $y=\sqrt[4]{x-5}-\sqrt[4]{x+5}$. 5.118. $y=\frac{1}{\sqrt[3]{x^2-27}}$.

5.119. $y=\frac{1}{x^3-8}$, $x\neq 2$. 5.120. $y=\lg(x^2-6x+10)$.

5.121. $y=\frac{2x-5}{x-1}$. 5.122. $y=5^{x+1}$

5.123. $y=\log_{0,5}\frac{x}{x+1}$. 5.124. $y=|x|+x$.

5.125. $y=\frac{1}{e^x-1}$. 5.126. $y=e^{\frac{1}{x-2}}$. 5.127. $y=\sqrt[5]{(1-x)^2}$.

5.128. $y=\begin{cases} 2x, x\leq -2 \text{ bo‘lganda.} \\ 4, -2<x<2 \text{ bo‘lganda.} \\ 2x,\ x\geq 2 \text{ bo‘lganda.} \end{cases}$

5.129. $y=\sin 4x+\cos 4x$, $x\in[0;\ \pi]$. 5.130. $y=\sqrt{x-2}+\sqrt{x+2}$.

5.131. $y=\frac{1}{\sqrt{x^2-3}-1}$. 5.132. $y=2\cdot 3^{1-x}-9^{-x}$. 5.133. $y=7^{-|x|}$.

5.134. Ushbu $f(x)=x-\sin x$ funksiyani $\left(0;\ \frac{\pi}{2}\right)$da o‘sish va kamayishga tekshiring.

5.135. Ushbu $f(x)=\frac{x}{1+x^2}$ funksiyani o‘sish va kamayishga tekshiring.

5.136. Ushbu $f(x)=\operatorname{ctg}x+\operatorname{tg}x$ funksiyani $\left(0;\ \frac{\pi}{2}\right)$da o‘sish va kamayishga tekshiring.

5.137. Ushbu $f(x)=(x^2-1)^2$ funksiya $(-1, 0)$da o'suvchi, $(0, 1)$da kamayuvchi bo'lishini isbotlang.

5.138. Ushbu $f(x)=5^{(x^2-1)^3+1}$ funksiyani o'sish va kamayishga tekshiring.

5.139. Ushbu $f(x)=\sin^4 x+\cos^4 x$ funksiyani $\left(0; \frac{\pi}{2}\right)$da o'sish va kamayishga tekshiring.

5.140. Ushbu $f(x)=\frac{2-\sin x}{2+\sin x}$ funksiyani $[0; 2\pi]$da o'sish va kamayishga tekshiring.

5.141. Ushbu $f(x)=x-\varepsilon\cdot\sin x$, $(0<\varepsilon\le 1)$ funksiyani qat'iy o'suvchiligini isbotlang.

5.142. Ushbu $f(x)=x^3+x^2$ funksiya: a) $(0; +\infty)$da o'suvchi, b) $[-1; 0]$da esa, monoton emasligini isbotlang.

Quyidagi funksiyalar uchun o'z aniqlanish sohasida (agar teskari funksiya mavjud bo'lsa) teskari funksiyalarni toping:

5.142. $y=\frac{6-x}{6+x}$.

5.143. $y=\frac{32+x^5}{32-x^5}$.

5.144. $y=2x-x^2$, $x\ge 1$.

5.145. $y=2x-x^2$, $x\le 1$.

5.146. $y=\sqrt{x-6}$.

5.147. $y=\sqrt[3]{x^3-125}$.

5.148. $y=\frac{2x}{1-x^2}$, $x\le -1$.

5.149. $y=\frac{2x}{1+x^2}$, $-1\le x\le 1$.

5.150. $y=\sin^5 x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

5.151. $y=\sin^5 x$, $\frac{\pi}{2}\le x\le\frac{3\pi}{2}$.

5.152. $y=\cos^3 x$, $0\le x\le\pi$.

5.153. $y=\arccos x^2$, $0\le x\le 1$.

5.154. $f(x)=\begin{cases} x^2-4x+6, & x\le 2, \\ -x+4, & x>2. \end{cases}$

5.155. $f(x)=\begin{cases} x, & x\in(-\infty; 0]; \\ 2x, & x\in(0; \infty) \end{cases}$

5.156. $y=\cos 2x$; a) $x\in\left[0; \frac{\pi}{2}\right]$; b) $x\in\left[\frac{\pi}{2}; \pi\right]$; c) $x\in\left[\pi; \frac{3\pi}{2}\right]$.

5.157. $y=(x+1)^2$, $x\in[-1; +\infty)$.

5.158. $y=\ln\frac{e^x+1}{e^x-1}$, $x\in(0; +\infty)$.

5.159. $y=5x-2$.

5.160. $y=x^3+5$, $x\in R$.

5.161. $y=\sqrt{1-x^3}$, $x\in[-1; 0]$.

5.162. $y=\sqrt{1-x^2}$, $x\in[0;1]$.

5.163. $y=|x|-x$, $x\in(-\infty;+\infty)$.

5.164. $y=\operatorname{arctg}3x$.

5.165. $x\le 0$ bo'lganda,
$x>0$ bo'lganda.

5.166. [illegible]
$-x$, $\le x\le 1$ bo'lganda,
1, $x>1$ bo'lganda.

5.167. [illegible]
x^2, $1\le x\le 4$ bo'lganda,
2^x, $x>4$ bo'lganda.

Tuzatishlar:
98-betidagi misollar
quyidagicha o'qilsin

5.165. $y=\begin{cases} x, & x\le 0 \text{ bo'lganda,} \\ x^2, & x>0 \text{ bo'lganda.} \end{cases}$

5.166. $y=\begin{cases} 1, & x\le -1 \text{ bo'lganda,} \\ -x, & \le x\le 1 \text{ bo'lganda,} \\ -1, & x>1 \text{ bo'lganda.} \end{cases}$

5.167. $y=\begin{cases} x, & x<1 \text{ bo'lganda,} \\ x^2, & 1\le x\le 4 \text{ bo'lganda,} \\ 2^x, & x>4 \text{ bo'lganda.} \end{cases}$

5.168. $y=2^x-1$.

5.169. $y=\arccos x^3$.

5.170. $y=\frac{1-x}{1+x}$, $x\in(-\infty;-1)\cup(-1;+\infty)$.

5.171. $y=\lg\frac{x}{2}$.

5.172. $f(x)=x^2$ va $g(x)=2^x$ funksiyalar berilgan bo'lsa, u holda $f(f(x))$, $f(g(x))$, $g(f(x))$, $g(g(x))$ murakkab funksiyalarni toping.

5.173. Agar $f(x)=\frac{x}{\sqrt{1+x^2}}$ bo'lsa, $f_3(x)=f(f(f(x)))$ funksiyani toping.

5.174. $f(x)=\frac{5x^2+1}{2-x}$ funksiya uchun $f(3x)$, $f(x^3)$, $(f(x))^2$ funksiyalarni toping.

5.175. Agar $f(x+1)=x^3-3x+2$ bo'lsa, $f(x)$ funksiyani toping.

Quyidagi murakkab funksiyalarning kompozitsiyalarini yozing:

5.176. $f(x)=x^2$, $g(x)=\sqrt{x}$.

5.177. $f(x)=x^4$, $g(x)=\sqrt[4]{x}$.

5.178. $f(x)=1-x^3$, $g(x)=x^4$.

5.179. $f(f(f(x)))=?$ $f(x)=\frac{1}{1-x^2}$.

5.180. $g(y)=y^2$ va $y=f(x)=\lg x$.

5.181. $g(y)=y^2$ va $y=f(x)=1-g^2$.

5.182. $g(y)=y^2$ va $y=\sin x$.

5.183. $g(y)=\begin{cases} 2y, & y\le 0 \text{ bo'lganda,} \\ 0, & y>0 \text{ bo'lganda.} \end{cases}$ va $y=x^2=1$.

5.184. $g(y)=9^y$ va $y=x$ 5.185. $g(y)=\sqrt{y}$ $(y\ge 0)$, $y=1+x^2$.

5.186. $f(x)=5^x$ va $f^{-1}(x)=\log_5 x$ funksiyalar berilganda $f(f^{-1}(x))$, $f^{-1}(f(x))$ larni toping.

5.187. $z=\arccos y$ va $y=3+x^4$ funksiyalar uchun murakkab funksiya mavjudmi?

5.188. $z=\sqrt{y}$ va $y=9-x^2$ funksiyalar uchun murakkab funksiya mavjudmi?

5.189. Agar $u(x)=4x-5$, $v(x)=x^2$ va $f(x)=\frac{1}{x}$ bo'lsa, quyidagilar uchun ifodalarni yozing.
a) $u(v(f(x)))$; b) $u(f(v(x)))$;
d) $v(u(f(x)))$; e) $v(f(u(x)))$;
f) $f(u(v(x)))$; g) $f(v(u(x)))$.

5.190. Ushbu $f\circ g=F(x)$ kompozitsiyada
$g(x)=\frac{1+x^2}{1+x^4}$, $F(x)=\frac{1+x^4}{1+x^2}$ berilganda $f(x)$ funksiyani toping.

5.191. Ushbu $f\circ g=F(x)$ kompozitsiyada $g(x)=3x$, $F(x)=2\sin 3x$ berilganda $f(x)$ funksiyani toping.

5.192. Ushbu $f\circ g=f(x)$ kompozitsiyada $f(x)=x^3$, $F(x)=\left(1-\frac{1}{x^4}\right)^2$ berilganda $g(x)$ funksiyani toping.

5.193. Quyidagi f, g funksiyalar berilganda, mos ravishda, $f\circ g$, $g\circ f$ kompozitsiyalarni toping.
a) $f(x)=1-x^2$; $g(x)=\sin x$. b) $f(x)=x^3+1$; $g(x)=\sqrt[3]{x-1}$.

5.194. Quyidagi f, g funksiyalarning $f\circ g$ kompozitsiyasini tuzing va uning aniqlanish sohasini toping:
a) $f(x)=\sqrt{x}$; $g(x)=x^2+5$;

b) $f(x)=\frac{1}{x}$; $g(x)=\frac{x-2}{x}$;

d) $f(x)=\sqrt{1-x^2}$; $g(x)=\cos 2x$;

5.195. Quyidagi funksiyalar grafigining eskizini chizing:

1) $y=1-2x^2$. 2) $y=\operatorname{tg}\frac{3\pi x-4\pi}{6}$. 3) $y=\frac{1}{3}\sin\left(2x+\frac{\pi}{4}\right)+1$.

4) $y=\frac{1}{4}x^2\frac{1}{2x-1}$. 5) $y=3^{\frac{1}{2}|x|-1}$. 6) $y=\log_{\frac{1}{2}}|2x-1|$.

7) $y=|\log_4 x|$. 8) $y=|\log_3|x||$.

9) $y=\left(\frac{1}{2}\right)^{|x|}$. 10) $y=\frac{1}{3}\cos\left(3x+\frac{\pi}{4}\right)-1$.

11) $y=|\arcsin 2x|$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

5.1. Juft ham, toq ham emas. 5.2. Juft. 5.3. Toq. 5.4. Juft. 5.5. Toq. 5.6. Toq. 5.7. Juft. 5.8. Toq. 5.9. Juft. 5.10. Juft. 5.11. Juft ham, toq ham emas. 5.12. Toq. 5.13. Juft. 5.14. Toq. 5.15. Juft. 5.16. Juft. 5.17. Juft ham, toq ham emas. 5.18. Juft. 5.19. $a\neq 0$, $b\neq 0$, juft ham, toq ham emas, $a\neq 0$, $b=0$ toq, $a=0$, $b\neq 0$ juft. 5.20. Juft. 5.21. Juft. 5.22. Toq. 5.23. Toq. 5.24. Juft ham, toq ham emas. 5.25. Juft ham, toq ham emas. 5.26. Juft. 5.27. Juft. 5.28. Juft ham, toq ham emas. 5.29. Juft. 5.30. Juft ham, toq ham emas. 5.31. Juft ham, toq ham emas. 5.33. Davriy emas. 5.34. Davriy, $T_0=2\pi$. 5.35. Davriy emas. 5.36. Davriy, $T_0=\frac{\pi}{2}$. 5.37. Davriy, eng kichik musbat davri yo'q. 5.38. Davriy, $T_0=8$. 5.39. Davriy, $T_0=\frac{4}{3}\pi$. 5.40. Davriy, $T_0=4\pi$. 5.41. Davriy emas. 5.42. Davriy, $T_0=\frac{\pi}{3}$. 5.43. Davriy emas. 5.44. Davriy, $T_0=42\pi$. 5.45. Davri mavjud emas. 5.46. Davriy, $T_0=20$. 5.47. Davri mavjud emas. 5.48. Davriy, $T_0=2\pi$. 5.49. Davriy, $T_0=\frac{5\pi}{2}$. 5.50. Davriy, $T_0=12\pi$. 5.51. Davriy, $T_0=\frac{\pi}{3}$. 5.52. Davriy, $T_0=30\pi$. 5.53. Davriy, $T_0=\frac{\pi}{2}$. 5.54. Davriy, $T_0=\pi$. 5.55. Davriy, $T_0=20\pi$. **5.56.** Davriy emas. **5.57.** Davriy emas. **5.58.** Davriy emas. **5.59.** Davriy, $T_0=15$. **5.60.** Davriy, $T_0=1$. **5.61.** Davriy, $T_0=0{,}5$. **5.62.** Davriy, $T_0=1$. **5.63.** Davriy, $T_0=10$. **5.64.** Davriy emas. **5.65.** Davriy, emas. **5.66.** Davriy emas. **5.67.** Davriy emas. **5.68.** Davriy, $T_0=1$. **5.71.** Chegaralangan. **5.72.** Quyidan chegaralangan, yuqoridan chegaralanmagan. **5.73.** Chegaralanmagan. **5.74.** Chegaralanmagan. **5.75.** Chegaralangan. **5.76.** Chegaralangan. **5.77.** Chegaralangan. **5.78.** Quyidan chegaralangan, yuqoridan chegaralanmagan. **5.79.** Chegaralanmagan. **5.80.** Chegaralangan. **5.81.** Chegaralanmagan. **5.82.** Chegaralanmagan. **5.83.** Chegaralangan. **5.84.** Yuqoridan chegaralangan, quyidan chegaralanmagan. **5.85.** Quyidan chegaralangan, yuqoridan chegaralanmagan. **5.86.** Quyidan chegaralangan, yuqoridan chegaralanmagan. **5.87.** Chegaralangan. **5.88.** Chegaralangan. **5.89.** Chegaralangan. **5.90.** Chegaralangan. **5.91.** Chegaralangan. **5.93.** Quyidan chegaralanmagan, yuqoridan chegaralangan. **5.98.** $f(x)$ va $g(x)$ funksiyalar X to'plamda chegaralangan va chegaralanmagan bo'lishi ham, bo'lmasligi ham mumkin. Masalan: $X=(0;\ 1)$ to'plamda 1) $f(x)=x+\frac{1}{x}$; $g(x)=\frac{1}{x}$ funksiyalarning ayirmasi chegaralangan; 2) $f(x)=x^2-\frac{1}{x}$; $g(x)=\frac{2}{x}$ funksiyalarning ayirmasi esa chegaralanmagan. **5.99.** Masalan: $X=(0;\ 1)$ to'plamda: 1) $f(x)=x$; $g(x)=\frac{1}{x}$ funksiyalarning yig'indisi va ayirmasi chegaralanmagan; 2) $f(x)=x$; $g(x)=\frac{1}{x}$ funksiyalarning ko'paytmasi va nisbati chegaralangan; 3) $f(x)=x^2$; $g(x)=\frac{1}{x^4}$ funksiyalarning ko'paytmasi chegaralanmagan. **5.101.** $(-\infty;-1]\cup[1;\infty)$ da qat'iy o'suvchi, $[-1;0)\cup(0;\infty)$ da kamayuvchi. **5.102.** $(\infty;\ 3]$ da qat'iy kamayuvchi, $[3;\ \infty)$da qat'iy o'suvchi. **5.103.** $(\infty;\ 0]$da qat'iy kamayuvchi, $[0;\ \infty)$ da qat'iy o'suvchi. **5.104.** $(-\infty;\ \infty)$da o'suvchi. **5.105.** $(-\infty;-1]\cup(0;1)$ da kamayuvchi, $(-1;0)\cup[1;\infty)$ da o'suvchi. **5.106.** $(-\infty;0)\cup(0;\infty)$ qat'iy kamayuvchi. **5.107.** $(-\infty;\ -1]$ da o'suvchi, $[-1;\ \infty)$da kamayuvchi. **5.108.** $(\infty;\ -1]$da o'suvchi, $[1;\ \infty)$da kamayuvchi. **5.109.** $(\infty;\ -1]$da qat'iy kamayuvchi, $[-1;\ 2]$ da o'zgarmas, $[2;\ \infty)$da qat'iy o'suvchi. **5.110.** $(-\infty;\ \infty)$ da kamayuvchi. **5.111.** $(-\infty;-1)\cup(0;+\infty)$ da kamayuvchi. **5.112.** $(-\infty;\ 0]$da o'zgarmas, $[0;\ \infty)$da o'suvchi. **5.113.** $(-\infty;\ 1]$da qat'iy o'suvchi, $[1;$

$+\infty)$ da kamayuvchi. **5.114.** $(-\infty;-1]\cup[1;\infty)$ da qat'iy o'suvchi, $[-1;1]$da qat'iy kamayuvchi. **5.115.** $(-\infty;-2]$da kamayuvchi, $[2;+\infty)$ da o'suvchi. **5.116.** $[3;\infty)$da o'suvchi. **5.117.** $[5;\infty)$da o'suvchi. **5.118.** $(-\infty;-3\sqrt{3})\cup(-3\sqrt{3};0]$ da o'suvchi, $[0;3\sqrt{3})\cup(3\sqrt{3};\infty)$ da kamayuvchi. **5.119.** $(-\infty;2)\cup(2;+\infty)$ da qat'iy kamayuvchi. **5.120.** $(-\infty;3]$da qat'iy kamayuvchi, $[2;+\infty)$ qat'iy o'suvchi. **5.121.** $(-\infty;1)\cup(1;+\infty)$ qat'iy o'suvchi. **5.122.** $(-\infty;+\infty)$ da qat'iy o'suvchi. **5.123.** $(-\infty;-1)\cup(0;+\infty)$ qat'iy kamayuvchi. **5.124.** $(-\infty;+\infty)$da o'suvchi. **5.125.** $(-\infty;0)\cup(0;+\infty)$ qat'iy kamayuvchi. **5.126.** $(-\infty;2)\cup(2;+\infty)$ qat'iy kamayuvchi. **5.127.** $(-\infty;2]$ da kamayuvchi, $[2;+\infty)$ da o'suvchi. **5.128.** $(-\infty;+\infty)$ da o'suvchi. **5.129.** $\left[0;\frac{\pi}{4}\right]$ va $\left[\frac{\pi}{2};\frac{3\pi}{4}\right]$da kamayuvchi, $\left[\frac{\pi}{4};\frac{\pi}{2}\right]$ va $\left[\frac{3\pi}{4};\pi\right]$da o'suvchi. **5.130.** $[5;+\infty)$ da o'suvchi. **5.131.** $(-\infty;-2)$ va $\left(-2;-\sqrt{3}\right]$ da o'suvchi, $\left[\sqrt{3};2\right)$ va $(2;+\infty)$ da kamayuvchi. **5.132.** $(-\infty;-1]$da o'suvchi, $[1;+\infty)$ da kamayuvchi. **5.133.** $(-\infty;0]$ qat'iy o'suvchi, $[0;+\infty)$ qat'iy kamayuvchi. **5.134.** $\left(0;\frac{\pi}{2}\right)$da o'suvchi. **5.135.** $(-\infty;-1]\cup[1;\infty)$ da kamayuvchi, $[-1;1]$da qat'iy o'suvchi. **5.136.** $\left(0;\frac{\pi}{4}\right]$da kamayuvchi, $\left[\frac{\pi}{4};\frac{\pi}{2}\right)$ da o'suvchi. **5.138.** $(-\infty;0]$ qat'iy kamayuvchi, $[0;+\infty)$ da qat'iy o'suvchi. **5.139.** $\left[0;\frac{\pi}{4}\right]$ da o'suvchi, $\left[\frac{\pi}{4};\frac{\pi}{2}\right]$da kamayuvchi. **5.140.** $\left[0;\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2};2\pi\right]$ da kamayuvchi, $\left[\frac{\pi}{2};\frac{3\pi}{2}\right]$ da o'suvchi. **5.142.** $f^{-1}(x)=2\sqrt[5]{\frac{x-1}{x+1}},\ x\neq-1$. **5.143.** $f^{-1}(x)=2\sqrt[5]{\frac{x-1}{x+1}},\ x\neq-1$. **5.144.** $f^{-1}(x)=1+\sqrt{1-x},\ x\geq1$. **5.145.** $f^{-1}(x)=1-\sqrt{1-x}, x\leq1$. **5.146.** $f^{-1}(x)=x^2+6, x\geq6$. **5.147.** $f^{-1}(x)=\sqrt[3]{x^3+125}$. **5.148.** $f^{-1}(x)=\frac{1+\sqrt{1-x^2}}{x},\ -1\leq x<0$.

5.149. $f^{-1}(x)=\begin{cases}\frac{1-\sqrt{1-x^2}}{x}, 0<|x|\leq1\\ 0, x=0\end{cases}$. **5.150.** $f^{-1}(x)=$

$=\arcsin\sqrt[5]{x}, -1\le x\le 1$. 5.151. $f^{-1}(x)=\pi-\arcsin\sqrt[3]{x}, -1\le x\le 1$.
5.152. $f^{-1}(x)=\arccos\sqrt[3]{x}, -1\le x\le 1$. 5.153. $f^{-1}(x)=\sqrt{\cos x},\ 0\le x\le\frac{\pi}{2}$.

5.154. $f(x)=\begin{cases}x^2-4x+6, x\le 2\\ -x+4, x>2\end{cases}$. 5.155. $f(x)=\begin{cases}x, x\in(-\infty;0]\\ \frac{x}{2}, x\in(0;\infty)\end{cases}$.

5.156. a) $y=\frac{1}{2}\arccos(2x-1),\ 0\le x\le 1$; b) $y=\pi-\frac{1}{2}\arccos(2x-1)$, $0\le x\le 1$; d) $y=\pi+\frac{1}{2}\arccos(2x-1),\ 0\le x\le 1$. 5.157. $g(y)=\sqrt{y}+1$.

5.158. $g(y)=\ln\frac{e^y+1}{e^y-1}$. 5.159. $g(y)=\frac{y+2}{5}$. 7.160. $g(y)=\sqrt[3]{y-5}$.

5.161. $g(y)=\sqrt[3]{1-y^2},\ y\in[0;1]$. 5.162. $g(y)=\sqrt{1-y^2},\ y\in[0;1]$.
5.163. Teskari funksiya mavjud emas. 5.164. $g(y)=\frac{1}{3}\operatorname{tg} y$, $y\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$. 5.165. $g(y)=\begin{cases}y,\ y\le 0 \text{ bo‘lganda},\\ \sqrt{y},\ y>0 \text{ bo‘lganda}.\end{cases}$ 5.166. Teskari funksiya mavjud emas. 5.167. $g(y)=\begin{cases}y,\ y<1 \text{ bo‘lganda},\\ \sqrt{y},\ 1\le y\le 4 \text{ bo‘lganda},\\ \log_2 y,\ y>4 \text{ bo‘lganda}.\end{cases}$

5.168. $g(y)=\log_2(y+1),\ y\in(-1;+\infty)$. 5.169. $g(y)=\sqrt[3]{\cos y}$.

5.170. $g(y)=\frac{1-y}{1+y},\ y\ne -1$. 5.171. $g(y)=2\cdot 10^y,\ y\in(-\infty;+\infty)$.
5.172. $f(f(x))=x^4,\ f(g(x))=4^x,\ g(f(x))=2^{x^2}$; $g(g(x))=2^{2^x}$. 5.173. $f_3(x)=\frac{x}{\sqrt{1+3x^2}}$. 5.174. $f(3x)=\frac{45x^2+1}{2-3x}$, $f(x^3)=\frac{5x^6+1}{2-x^3}$, $(f(x))^2=\frac{25x^4+10x^2+1}{4-4x+x^2}$. 5.175. x^2-5x+6. 5.176. $(f\cdot g)=x$; $(g\cdot f)=|x|$. 5.177. $(f\cdot g)(x)=x$; $(g\cdot f)(x)=|x|$. 5.178. $(f\cdot g)(x)=1-x^{12}$; $(g\cdot f)(x)=(1-x^3)^4$. 5.179. $f_3(x)=\frac{x}{1-3x^2}$. 5.180. $\lg^2 x$. 5.181. $\sin(1-x^2)$.

5.182. $\sin^2 x$. 5.183. $g(f)=\begin{cases}2\cdot(x^2-1),\ x\in[-1;1] \text{ bo‘lganda},\\ 0,\ x\in(-\infty;-1)\cup[1;+\infty) \text{ bo‘lganda}.\end{cases}$

5.184. 9^x. 5.185. $\sqrt{1+x^2}$. 5.186. $f(f^{-1}(x))=x, \quad f^{-1}(f(x))=x$.
5.187. Murakkab funksiya mavjud emas. 5.188. Murakkab funksiya mavjud emas. 5.189. a) $\frac{4}{x^2}-5$; b) $\frac{4}{x^2}-5$; d) $\left(\frac{4}{x}-5\right)^2$; e) $\frac{1}{(4x-5)^2}$; f) $\frac{1}{4x^2-5}$;
g) $\frac{1}{(4x-5)^2}$. 5.190. $f(x)=\frac{1}{x}$. 5.191. $f(x)=2\sin x$. 5.192. $g(x)=\left(1-\frac{1}{x^4}\right)^{2/3}$.
5.193. a) $\cos^2 x$; $\sin(1-x^2)$; b) x; x. 5.194. a) $\sqrt{x^2+5}$; $(-\infty;\infty)$.
b) $\frac{x}{x-2}$; $(-\infty;\, 0)\cup(0;\, 2)\cup(2;\, +\infty)$. d) $|\sin 2x|$, $(-\infty;\infty)$.

6-§. SONLAR KETMA-KETLIGI VA UNING LIMITI

6.1. Nuqtaning atrofi. Bizga $a\in R$ hamda ixtiyoriy musbat $\varepsilon>0$ va c sonlari berilgan bo'lsin.

6.1-ta'rif. Quyidagi

$$U_\varepsilon(a)=\{x:\ x\in R,\ a\text{-}\varepsilon<x<a+\varepsilon\}$$

to'plam a nuqtaning *ε atrofi* deyiladi, ε son esa *atrofning radiusi* deyiladi (6.1-chizma).

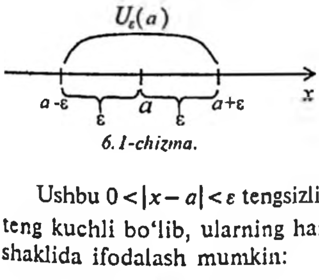
Ushbu

$$U_\varepsilon^+(a)=\{x:\ x\in R,\ a<x<a+\varepsilon\}$$

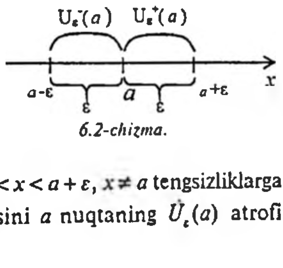
to'plam a nuqtaning *o'ng atrofi*,

$$U_\varepsilon^-(a)=\{x:\ x\in R,\ a\text{-}\varepsilon<x<a\},$$

to'plam esa, a nuqtaning *shap atrofi* deyiladi (6.2-chizma).



6.1-chizma.



6.2-chizma.

Ushbu $0<|x-a|<\varepsilon$ tengsizlik, $a-\varepsilon<x<a+\varepsilon,\ x\neq a$ tengsizliklarga teng kuchli bo'lib, ularning har ikkalasini a nuqtaning $\dot{U}_\varepsilon(a)$ atrofi shaklida ifodalash mumkin:

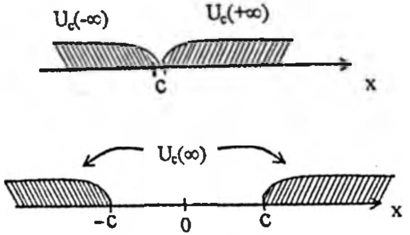
$$\dot{U}_\varepsilon(a)=\{x: x\in R,\ a-\varepsilon<x<a+\varepsilon, x\neq a\}.$$

Ba'zi hollarda $\dot{U}_\varepsilon(a)$ atrof a nuqtaning *teshik atrofi* deb ham yuritiladi.

Haqiqiy sonlar to'plami R tarkibiga $-\infty$ va $+\infty$ belgilarni $\forall x\in R$ uchun $-\infty<x$ va $x<+\infty$ xususiyatlar bilan qo'shib, $\overline{R}$ to'plamni hosil qilamiz:

$$\overline{R}=R\cup\{-\infty\}\cup\{+\infty\}.$$

$\overline{R}$ da $-\infty$ va $+\infty$ «nuqta»larning atrofi tushunchalari quyidagicha kiritiladi (6.3-chizma):



6.3-chizma.

$U_c(+\infty)=\{x:x\in R,\ c\in R,\ c<x<+\infty\}$, $U_c(-\infty)=\{x:x\in R,\ c\in R,\ -\infty<x<c\}$, $U_c(\infty)=\{x:x\in R, c\in R,\ |x|>c\}$.

6.2. Natural argumentli funksiya va uning limiti. N va R to'plamlar berilgan bo'lib, f — har bir natural $n(n\in N)$ songa biror haqiqiy x_n $(x_n\in R)$ sonni mos qo'yuvshi qoida yoki usul bo'lsin: $f:n\to x_n$. Bu holda N to'plamda *natural argumentli funksiya* aniqlangan deyiladi va u $x_n=f(n)$ kabi belgilanadi.

Agar $x_n=f(n)$ funksiya berilgan bo'lsa, u holda uning argumenti yoki n indeksini x_n o'zgaruvshi mos qiymatining nomeri deb qarash mumkin. Shunday qilib, $x_1=f(1)$ — funksiyaning birinchi qiymati, $x_2=f(2)$ — ikkinshi qiymati, $x_3=f(3)$ — ushinshi qiymati va h.k. Biz har doim qiymatlar to'plami $E(x_n)=\{x_n\}$ ni natural 1, 2, 3, ..., n, ... ketma-ketlikka o'xshash nomerlarning ortishi bo'yisha tartiblangan, ya'ni

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots \qquad (6.1)$$

sonlar ketma-ketligi shaklida tasavvur qilamiz. (6.1) ketma-ketlik qisqasha $\{x_n\}$ kabi belgilanadi. Ketma-ketlikning qiymatlar to'plami chekli yoki cheksiz bo'lishi mumkin, lekin shunday bo'lsada, ketma-ketlikning elementlari har doim cheksiz ko'p bo'ladi.

Biz bundan keyin, qulaylik uchun, natural argumentli $x_n=f(n)$ funksiyani sonli ketma-ketlik deb qaraymiz. Masalan, agar $x_n=f(n)$ funksiya

1) $x_n=1$; 2) $x_n=\frac{1}{n}$; 3) $x_n=(-1)^{n+1}$; 4) $x_n=\frac{1+(-1)^n}{2}$;

5) $x_n = \sin n\pi$; 6) $x_n = x_{n-1} + x_{n-1}$, $n \geq 3$, $x_1 = 1$, $x_2 = 1$;
7) x_n — (n — nomerli tub sonlar to'plami) formulalardan birortasi bilan berilgan bo'lsa, ularga mos ketma-ketliklar quyidagi shaklda bo'ladi:

1) $1, 1, 1, ..., 1, ...;$ 2) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...;$

3) $1, -1, 1, -1, ..., (-1)^n, ...$ 4) $0, 1, 0, 1, ..., \frac{1+(-1)^n}{2}, ...;$

5) $\sin\pi, \sin 2\pi, \sin 3\pi, ..., \sin n\pi, ...;$

6) $1, 1, 2, 3, 5, 8, 13, 21, ...;$ 7) $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 7$ va hokazo.

Yuqorida keltirilgan misollardan ko'rinadiki, ba'zi ketma-ketliklarning umumiy hadlari aniq formulalar orqali ifodalanib, ularning hamma hadlari shu formulalar orqali topilsa, ba'zi ketma-ketlikning hadlarini ma'lum bir qoidalar yordamida topish mumkin, Ba'zi hollarda ketma-ketliklarning berilishi, uning hadlarining nomini aytish bilan amalga oshiriladi.

Masalan, 1) – 5) misollarda ketma-ketlikning umumiy hadi aniq formula orqali ifodalangan, 6) misolda ketma-ketlikning dastlabki x_1 va x_2 hadlari berilgan holda, qolgan hadlarini $x_n = x_{n-2} + x_{n-1}$, $n \geq 3$ rekurrent formula orqali topiladi. 6) misolda ketma-ketlik rekurrent formula orqali topilgan 1, 1, 2, 3, 5, 8, 13, 21, ... sonlar Fibonachchi sonlari deyiladi. 7) misolda ketma-ketlikning umumiy hadi so'zlar orqali ifoda qilingan.

Ushbu

$$\{x_n\}: x_1, x_2, x_3, ..., x_n, ...; \tag{6.2}$$

$$\{y_n\}: y_1, y_2, y_3, ..., y_n, ... \tag{6.3}$$

ketma-ketliklar berilgan bo'lsin. (6.2) va (6.3) ketma-ketliklarning yig'indisi (ayirmasi), ko'paytmasi, bo'linmasi (nisbati) deb, mos ravishda

$$x_1 + y_1, x_2 + y_2, x_3 + y_3, ..., x_n + y_n, ... \ (x_1 - y_1, x_2 - y_2, x_3 - y_3, ..., x_n - y_n, ...);$$

$$x_1 \cdot y_1, x_2 \cdot y_2, ..., x_n \cdot y_n, ...; \ \frac{x_1}{y_1}, \frac{x_2}{y_2}, ..., \frac{x_n}{y_n}, ...$$

ketma-ketliklarga aytiladi va ular, mos ravishda,

$$\{x_n+y_n\}\ (\{x_n-y_n\}),\quad \{x_n\cdot y_n\},\ \left\{\frac{x_n}{y_n}\right\}$$

kabi belgilanadi. (6.2) va (6.3) ketma-ketliklar nisbatining ta'rifida $\forall n\in N$ uchun $y_n\neq 0$ deb faraz qilinadi. Agar $\{y_n\}$ ketma-ketlikning chekli sondagi elementlari nolga teng bo'lsa, u holda $\left\{\frac{x_n}{y_n}\right\}$ ketma-ketlikni, y_n ning nolga teng bo'lmagan hadlaridan boshlab aniqlash kerak bo'ladi.

Endi $\{x_n\}$, $n = 1, 2, \ldots$ ketma-ketlikning chegaralanganligi tushunchalari bilan tanishamiz.

6.2-ta'rif. Agar shunday o'zgarmas M son mavjud bo'lib, $\{x_n\}$ ketma-ketlikning har bir hadi shu M sondan katta bo'lmasa, ya'ni $\forall n\in N$ uchun $x_n\le M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ *ketma-ketlik yuqoridan chegaralangan* deb ataladi. Masalan, $\{x_n\}=\left\{\frac{n-1}{n}\right\}$, $\{x_n\}=\{-(2n-1)\}$ ketma-ketliklar yuqoridan chegaralangan, chunki, birinchi ketma-ketlikning har bir hadi 1 dan, ikkinshi ketma-ketlikning har bir hadi esa 0 dan katta emas, ya'ni $\forall n\in N$ uchun $x_n=\frac{n-1}{n}\le 1$, $\forall n\in N$ uchun $x_n=-(2n-1)<0$.

6.3-ta'rif. Agar shunday o'zgarmas m son mavjud bo'lib, $\{x_n\}$ ketma-ketlikning har bir hadi shu m sondan kichik bo'lmasa, ya'ni $\forall n\in N$ uchun $m\le x_n$ tengsizlik bajarilsa, $\{x_n\}$ *ketma-ketlik quyidan chegaralangan* deyiladi.

Masalan, $\{x_n\}=\{n^2+1\}$, $\{x_n\}=\left\{\frac{1}{2^{n-1}}\right\}$ ketma-ketliklar quyidan chegaralangan, chunki birinchi ketma-ketlikning har bir hadi 2 dan, ikkinshi ketma-ketlikning har bir hadi esa 0 dan kichik emas.

6.4-ta'rif. Agar $\{x_n\}$ ketma-ketlik ham quyidan, ham yuqoridan chegaralangan bo'lsa, ya'ni shunday M va m o'zgarmas sonlar mavjud bo'lib, $\forall n\in N$ uchun $m\le x_n\le M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ *ketma-ketlik chegaralangan* deyiladi. Masalan, $\{x_n\}=\left\{\frac{1}{n}\right\}$; $\{x_n\}=\{2^{(-1)^n}\}$ ketma-ketliklar chegaralangan ketma-ketliklardir.

6.1-teorema. *$\{x_n\}$ ketma-ketlik chegaralangan bo'lishi uchun, shunday $A>0$ son mavjud bo'lib, $\forall n\in N$ uchun*

$$|x_n|\le A \tag{6.4}$$

tengsizlikning bajarilishi zarur va yetarlidir.

Odatda (6.4) shart ketma-ketlikning chegaralanganlik sharti deb ham yuritiladi.

6.5-ta'rif. Agar ixtiyoriy $A>0$ (istalgancha katta) son olinganda ham, $\{x_n\}$ ketma-ketlikning hesh bo'lmaganda bitta x_{n_0} elementi topilib,

$$|x_{n_0}| \geq A \tag{6.5}$$

tengsizlik bajarilsa, $\{x_n\}$ *ketma-ketlik chegaralanmagan ketma-ketlik* deyiladi. Masalan, 1, 3, 1, 6, ..., 1, $3n$, ... ketma-ketlik chegaralanmagan bo'ladi, chunki ixtiyoriy musbat haqiqiy A sonni qanday qilib olmaylik, ketma-ketlikning juft nomerdagi elementlari ichidan A dan katta bo'lgani, ya'ni (6.5) tengsizlikni qanoatlantiruvchisi topiladi.

6.1-misol. Ushbu

$$x_1 = a,\ x_2 = b,\ x_{n+2} = x_{n+1} + 2x_n,\quad n \in N$$

shartlarni qanoatlantiruvchi ketma-ketlikning umumiy hadini toping.

Yeshilishi. Berilgan shartlarni qanoatlantiruvshi ketma-ketlikni $\{x_n\} = \{\lambda^n\}$ shaklda izlaymiz, $\lambda \neq 0$ $x_n = \lambda^n$ ni $x_{n+2} = x_{n+1} + 2x_n$ shartga qo'yish natijasida $\lambda^2 - \lambda - 2 = 0$ xarakteristik tenglamani hosil qilamiz. Xarakteristik tenglama $\lambda_1 = 2$, $\lambda_2 = -1$ ildizlarga ega. $\{2^n\}$, $\{(-1)^n\}$ ketma-ketliklarning har biri $x_{n+2} = x_{n+1} + 2x_n$ shartni qanoatlantiradi. Umumiy hadi $x_n = \alpha 2^n + \beta(-1)^n$ bo'lgan (α va β — ixtiyoriy o'zgarmas sonlar) ketma-ketlik ham berilgan shartlarni qanoatlantiradi:

$$x_1 = 2\alpha - \beta = a,$$
$$x_2 = 4\alpha - \beta = b.$$

Bu sistemadan $\alpha = \frac{a+b}{6}$, $\beta = \frac{b-2a}{3}$. λ_1, λ_2, α va β larning qiymatlarini $x_n = \alpha\lambda_1^n + \beta\lambda_2^n$ formulaga qo'yish natijasida, izlanayotgan ketma-ketlikning umumiy hadi

$$x_n = \frac{((a+b)2^{n-1} + (b-2a)(-1)^n)}{3}$$

ekanligini topamiz.

6.2-misol. Ketma-ketlikning dastlabki bir neshta hadlarini bilgan holda uning umumiy hadining ko'rinishini yozing:

a) $\frac{2}{3}, \frac{5}{8}, \frac{10}{13}, \frac{17}{18}, \frac{26}{23}$; b) $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}$.

Yechilishi. a) ketma-ketlikning berilgan hadlarining surati ketma-ketlik hadi nomerining kvadratiga birni qo'shishdan hosil bo'ladi, ya'ni n^2+1, maxraji esa, birinchi hadi $a_1 = 3$, ayirmasi $d = 5$ bo'lgan 3, 8, 13, 18, ... arifmetik progressiyadan iborat. Arifmetik progressiya umumiy hadini topish formulasiga asosan

$$x_n = a_1 + d(n-1) = 3 + 5(n-1) = 5n-2$$

ekanligini topamiz.

Demak, izlanayotgan ketma-ketlik umumiy hadining ko'rinishi

$$x_n = \frac{n^2+1}{5n-2}$$

shaklida bo'ladi.

b) izlanayotgan ketma-ketlikning umumiy hadini quyidagi ko'rinishda yozish mumkin:

$$x_n = \begin{cases} k, & n = 2k-1, \ k \in N, \\ \frac{1}{k+1}, & n = 2k, \ k \in N. \end{cases}$$

Bu formulani bitta umumiy formula orqali ham yozish mumkin:

$$x_n = \frac{n+1}{4}[1-(-1)^n] + \frac{1}{n+2}[1+(-1)^n]$$

6.3-misol. Quyidagi ketma-ketliklarning chegaralanganligini isbotlang:

1) $x_n = \frac{20+(-1)^n \cdot n}{\sqrt{n^2+13}}$, $n \in N$; 2) $x_n = \frac{n}{3^n}$, $n \in N$.

Yechilishi. 1) $|20+(-1)^n \cdot n| \leq 20 + |(-1)^n \cdot n| = 20+n$, $\sqrt{n^2+13} > n$ ekanligini e'tiborga olgan holda

$$|x_n|=\frac{|20+(-1)^n\cdot n|}{\sqrt{n^2+13}}\le\frac{20+n}{n}=1+\frac{20}{n}\le 21.$$

Demak, ketma-ketlik chegaralangan, chunki (6.4) shart bajariladi ($A=21$).

2) ravshanki, $\forall n\in N$ uchun $\frac{n}{3^n}>0$. Bernulli tengsizligiga asosan $3^n=(1+2)^n\ge 2n$. Bundan $\frac{n}{3^n}\le\frac{1}{2}$. Demak, $\forall n\in N$ uchun $0<\frac{n}{3^n}\le\frac{1}{2}$. Demak, berilgan ketma-ketlik chegaralangan.

6.4-misol. Ushbu

a) $x_n=\frac{8n^3-5}{3n^3+11}$; b) $x_n=(-1)^n\frac{6n^3-1}{2n^3+3}\cos n$;

d) $x_n=\frac{2n^2}{n-1}\cos\pi n,\ n\ge 2$

ketma-ketliklarning qaysi biri chegaralangan, qaysi biri chegaralanmagan?

Yechilishi. a) ravshanki, $0<\frac{8n^3-5}{3n^3+11}<\frac{8n^3}{3n^3}<\frac{8}{3}=2\frac{2}{3}$, $\forall n\in N$.

Demak, $x_n=\frac{8n^3-5}{3n^3+11}$ ketma-ketlik chegaralangan.

b) $x_n=(-1)^n\frac{6n^3-1}{2n^3+3}\cos n$ ketma-ketlik ham chegaralangan, chunki $|x_n|=\left|(-1)^n\frac{6n^3-1}{2n^3+3}\cos n\right|=|(-1)^n|\cdot\frac{6n^3-1}{2n^3+3}\cdot|\cos n|<\frac{6n^3}{2n^3+3}<3$.

Demak, berilgan ketma-ketlik chegaralangan.

d) $|x_n|=\left|\frac{2n^2}{n-1}\cdot\cos\pi n\right|=\frac{2n^2}{n-1}\cdot|\cos\pi n|=\frac{2n^2}{n-1}>2n$.

Ixtiyoriy $A>0$ son uchun $n>\frac{A}{2}$ deb olsak, $\left(n=\left[\frac{A}{2}\right]+1\right)$ $|x_n|>2n>A$ tengsizlik bajariladi. Demak, berilgan ketma-ketlik chegaralanmagan ekan.

6.6-ta'rif. Agar $\forall A>0$ (A — istalgancha katta son bo'lganda ham) son uchun $\exists n_0(A)$ nomer mavjud bo'lib, $n\ge n_0(A)$ dan boshlab $\{x_n\}$ ketma-ketlikning barsha elementlari

$$|x_n|>A \qquad (6.6)$$

tengsizlikni qanoatlantirsa, $\{x_n\}$ ketma-ketlik *cheksiz katta ketma-ketlik* deyiladi.

Ma'lumki, $\{x_n\}$ ketma-ketlik cheksiz katta bo'lsa, u chegaralanmagan ketma-ketlik ham bo'ladi, lekin buning aksi har doim o'rinli bo'lmaydi, ya'ni chegaralanmagan ketma-ketlik cheksiz katta ketma-ketlik bo'lmasligi ham mumkin. Masalan, 1, 3, 1, 6, ..., 1, $3n$, ... ketma-ketlik chegaralanmagan ketma-ketlik bo'lsada, u cheksiz katta ketma-ketlik bo'la olmaydi, chunki $A>1$ uchun (6.6) tengsizlik x_n ning toq nomerdagi elementlari uchun bajarilmaydi.

6.7-ta'rif. Agar $\forall\varepsilon>0$ (ε — istalgancha kichik son olinganda ham) uchun $\exists n_0(\varepsilon)$ nomer mavjud bo'lib, $n\geq n_0(\varepsilon)$ dan boshlab

$$|x_n|<\varepsilon \tag{6.7}$$

tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik *cheksiz kichik ketma-ketlik* deyiladi.

6.6-misol. q, q^2, ..., q^n, ... ketma-ketlikning $|q|>1$ bo'lganda cheksiz katta, $|q|<1$ bo'lganda esa cheksiz kichik ketma-ketlik ekanligini isbot qiling.

Isboti. 1) $|q|>1$ bo'lsin. U holda, $|q|=1+\delta$ $(\delta>0)$ deb tasvirlash mumkin. Bundan $|q|^n=(1+\delta)^n$ ni hosil qilamiz. Bernulli tengsizligidan foydalanib,

$$|q|^n=(1+\delta)^n>1+n\delta>n\delta$$

tengsizlikni hosil qilamiz. $\forall A>0$ (istalgancha katta) son olinganda ham

$$\delta n>A \tag{6.8}$$

tengsizlik $n_0=\left[\frac{A}{\delta}\right]+1=\left[\frac{A}{|q|-1}\right]+1$ nomerdan boshlab, ya'ni $\forall n\geq n_0(A)$ uchun bajariladi.

Shunday qilib, $\forall A>0$ uchun shunday $n_0=\left[\frac{A}{|q|-1}\right]+1$ nomer topiladiki, $\forall n\geq n_0$ uchun

$$|q|^n>A$$

tengsizlik bajariladi. Bundan esa, $|q|>1$ bo'lganda, 6.6-ta'rifga ko'ra berilgan ketma-ketlikning cheksiz katta ketma-ketlik ekanligi kelib shiqadi.

2) $|q|<1$ $(q\neq 0)$ bo'lsin, bundan $\frac{1}{|q|}>1$, buni $\frac{1}{|q|}=1+\delta$ $(\delta>0)$, $\frac{1}{|q|^n}=(1+\delta)^n$, $n\in N$ deb qarash mumkin. Bernulli tengsizligidan foydalanib, $|q|^n\leq\frac{1}{1+n\delta}$ tengsizlikni hosil qilamiz. $\forall\varepsilon>0$ sonni olib, unga bog'liq $n_0(\varepsilon)$ nomerni $n_0(\varepsilon)=\left[\frac{\frac{1}{\varepsilon}-1}{\delta}\right]+1$ deb olsak, u holda, $\forall n\geq n_0(\varepsilon)$ uchun

$$|q|^n\leq\frac{1}{1+n\delta}<\frac{1}{1+n_0\delta}=\frac{1}{1+\left(\left[\frac{\frac{1}{\varepsilon}-1}{\delta}\right]+1\right)\delta}<\frac{1}{1+\frac{\frac{1}{\varepsilon}-1}{\delta}\cdot\delta}=\varepsilon$$

tengsizlik, ya'ni $|q|^n<\varepsilon$ bajariladi. Bundan esa, $|q|<1$ bo'lganda, berilgan ketma-ketlikning cheksiz kichik ketma-ketlik ekanligi kelib shiqadi, $\{x_n\}=\{q^n\}$ ketma-ketlik uchun $q=0$ bo'lsa, $\forall n\in N$ uchun $x_n=0$ bo'ladi. Bu esa, $\{x_n\}=\{0\}$ ketma-ketlikning cheksiz kichik ketma-ketlik ekanligini anglatadi.

6.7-misol. Ushbu 1) $x_n=\frac{(-1)^n 4}{5\sqrt{n}+3}$; 2) $x_n=\frac{1}{n}\sin\left[(2n-1)\frac{\pi}{2}\right]$; 3) $x_n=2^{\sqrt{n}}$ ketma-ketliklarning qaysi biri $n\to\infty$ da cheksiz kichik, qaysi biri cheksiz katta ketma-ketlik bo'ladi?

Yechilishi. 1) $\forall\varepsilon>0$ sonni olamiz.

$$|x_n|=\left|\frac{(-1)^n\cdot 4}{5\sqrt{n}+3}\right|<\frac{4}{5\sqrt{n}+3}<\frac{4}{5\sqrt{n}}<\frac{4}{4\sqrt{n}}=\frac{1}{\sqrt{n}},$$

$n>\frac{1}{\varepsilon^2}$ bo'lganda, $|x_n|<\varepsilon$ tengsizlik bajariladi $\left(n_0(\varepsilon)=\left[\frac{1}{\varepsilon^2}\right]+1\right)$. $\varepsilon=\frac{1}{10}$ deb olinganda, $|x_n|<\frac{1}{10}$ bo'lishi uchun $\frac{1}{\sqrt{n}}<\frac{1}{10}$ yoki $n>100$ bo'lishi kerak. Shuning uchun $n_0(\varepsilon)=100$ deb olish mumkin. Shunday qilib, $x_n=\frac{(-1)^n\cdot 4}{5\sqrt{n}+3}$ ketma-ketlik cheksiz kichik ketma-ketlik ekan.

2) $\forall\varepsilon>0$ olamiz. $|x_n|=\left|\frac{1}{n}\sin\left[(2n-1)\frac{\pi}{2}\right]\right|<\frac{1}{n}$. $n>\frac{1}{\varepsilon}$ bo'lganda $|x_n|<\varepsilon$ tengsizlik bajariladi. $n_0(\varepsilon)=\left[\frac{1}{\varepsilon}\right]+1$ deb olish mumkin.

Demak, $n \geq n_0(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$ dan boshlab $|x_n| < \varepsilon$ bajariladi. Shunday qilib, 6.7-ta'rifga ko'ra, berilgan ketma-ketlik cheksiz kichik ketma-ketlik ekan.

3) $\forall A > 0$ (istalgancha katta) son olamiz va $2^{\sqrt{n}} > M$ tengsizlikni yechamiz: $\sqrt{n} > \log_2 M$, $n > (\log_2 M)^2$.

Agar $n_0(A) = [(\log_2 M)^2]$ deb olinsa, $n > n_0(A)$ dan boshlab $|x_n| > M$ bo'ladi. Demak, berilgan ketma-ketlik cheksiz katta ketma-ketlik ekan.

6.3. Cheksiz kichik ketma-ketliklarning asosiy xossalari

6.1-xossa. Ikkita $\{x_n\}$ va $\{y_n\}$ cheksiz kichik ketma-ketliklarning yig'indisi va ayirmasi $\{x_n \pm y_n\}$, yana cheksiz kichik ketma-ketlik bo'ladi. Bu xossadan ushbu natija kelib shiqadi:

6.1-natija. Chekli sondagi cheksiz kichik ketma-ketliklarning algebraik yig'indisi yana cheksiz kichik ketma-ketlik bo'ladi.

6.2-xossa. Cheksiz kichik ketma-ketlik bilan chegaralangan ketma-ketliklarning ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

6.3-xossa. Har qanday cheksiz kichik ketma-ketlik chegaralangan ketma-ketlik bo'ladi.

6.2 va 6.3-xossalardan quyidagi natija kelib shiqadi.

6.2-natija. Chekli sondagi cheksiz kichik ketma-ketliklarning ko'paytmasi cheksiz kichik ketma-ketlik bo'ladi.

6.4-xossa. Agar $\{x_n\}$ cheksiz katta ketma-ketlik bo'lsa, u holda biror n_0 nomerdan boshlab $\left\{\frac{1}{x_n}\right\}$ ketma-ketlik aniqlangan bo'ladi va u cheksiz kichik ketma-ketlik bo'ladi. Agar $\{y_n\}$ cheksiz kichik ketma-ketlikning hamma hadlari noldan farqli bo'lsa, u holda $\left\{\frac{1}{y_n}\right\}$ ketma-ketlik cheksiz katta ketma-ketlik bo'ladi.

6.4. Yaqinlashuvchi ketma-ketliklar va ularning xossalari. $\{x_n\}$ ketma-ketlik berilgan bo'lsin.

6.8-ta'rif. Agar shunday haqiqiy a son mavjud bo'lib, $\{x_n - a\}$ ketma-ketlik cheksiz kichik ketma-ketlik bo'lsa, $\{x_n\}$ ketma-ketlik *yaqinlashuvchi ketma-ketlik* deyiladi. Bunda, a soniga $\{x_n\}$ ketma-ketlikning limiti deyiladi va u $\lim\limits_{n\to\infty} x_n = a$ kabi yoziladi.

Cheksiz kichik ketma-ketlikning ta'rifidan foydalanib, yaqinlashuvchi ketma-ketlikning 6.8-ta'rifiga ekvivalent bo'lgan quyidagi ta'rifni berish mumkin:

6.9-ta'rif. Agar shunday haqiqiy a son mavjud bo'lib, $\forall\varepsilon>0$ uchun shunday $n_0(\varepsilon)$ nomer topilib, $\forall n\geq n_0(\varepsilon)$ lar uchun

$$|x_n-a|<\varepsilon \tag{6.9}$$

tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik *yaqinlashuvchi ketma-ketlik* deyiladi. Bunda a soniga $\{x_n\}$ ketma-ketlikning *limiti* deyiladi. Bu ta'rifni qisqasha quyidagisha ifodalash mumkin:

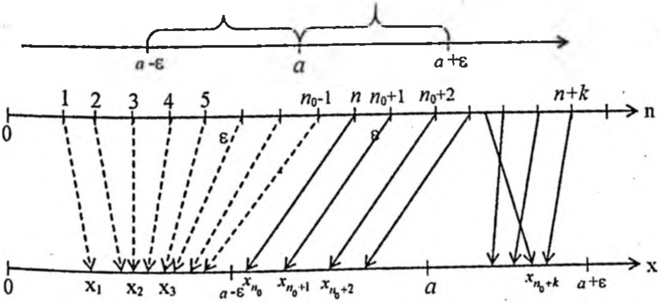
$$\forall\varepsilon>0\ \exists n_0=n_0(\varepsilon)\in N:\forall n>n_0\Rightarrow|x_n-a|<\varepsilon.$$

(6.9) tengsizlik ushbu $-\varepsilon<x_n-a<\varepsilon$ yoki $a-\varepsilon<x_n<a+\varepsilon$ tengsizlikka ekvivalent bo'ladi.

Agar $(a-\varepsilon,\ a+\varepsilon)$ oraliq (6.4-chizma) a nuqtaning ε-atrofi, ya'ni $U_\varepsilon(a)$ to'plamdan iborat ekanligini e'tiborga olsak, u holda $\{x_n\}$ ketma-ketlik limitining yuqoridagi 6.8 va 6.9-ta'riflarga ekvivalent bo'lgan quyidagi geometrik ta'rifini berish mumkin.

6.10-ta'rif. Agar a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olinganda ham, $\{x_n\}$ ketma-ketlikning biror hadidan keyingi barsha hadlari shu atrofga joylashsa, *a son $\{x_n\}$ ketma-ketlikning limiti* deyiladi.

Demak, agar a nuqta $\{x_n\}$ ketma-ketlikning limiti bo'lsa, a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofining tashqarisida ketma-ketlikning birorta ham hadi bo'lmasligi, agar bo'lsa, chekli sondagi hadlari bo'lishi mumkin.



6.4-chizma.

Bu ta'rifni qisqasha

$$(\lim_{n\to\infty} x_n = a) \Leftrightarrow (\forall\varepsilon>0,\quad \exists n_0 = n_0(\varepsilon)\in N : \forall n \geq n_0 \to |x_n - a| < \varepsilon)$$

kabi ifodalash ham mumkin.

6.11-ta'rif. Agar ixtiyoriy a son va ixtiyoriy n_0 son olinganda ham shunday ε_0 son va shunday $n > n_0$ natural son topilib,

$$|x_n - a| \geq \varepsilon_0$$

bo'lsa, $\{x_n\}$ ketma-ketlik *limitga ega emas* deyiladi.

Bu ta'rifni qisqacha quyidagisha ifodalash mumkin:

$$\forall n_0 \in N,\ \exists\varepsilon_0,\ \exists n\in N : n > n_0 \Rightarrow |x_n - a| \geq \varepsilon_0 .$$

6.12-ta'rif. Agar $\{x_n\}$ ketma-ketlik limitga ega bo'lmasa, u *uzoqlashuvchi* ketma-ketlik deyiladi.

Ketma-ketlik limitining ta'rifidan foydalanib, cheksiz kichik va cheksiz katta ketma-ketliklarning ta'riflarini quyidagisha ifodalash mumkin.

6.13-ta'rif. Limiti nolga teng $(\lim_{n\to\infty} x_n = 0)$ bo'lgan $\{x_n\}$ ketma-ketlik, *cheksiz kichik ketma-ketlik* deyiladi.

6.14-ta'rif. Limiti cheksiz $(\lim_{n\to\infty} x_n = \infty)$ bo'lgan $\{x_n\}$ ketma-ketlik, *cheksiz katta ketma-ketlik* deb ataladi.

Cheksiz kichik ketma-ketlik o'z navbatida yaqinlashuvchi, cheksiz katta ketma-ketlik esa uzoqlashuvshi ketma-ketlik bo'ladi.

6.2-teorema. *$\{x_n\}$ ketma-ketlik a limitga ega bo'lishi uchun, uni $x_n = a + \alpha_n$ shaklda tasvirlanishi zarur va yetarli.*

Bunda $\{\alpha_n\}$ — cheksiz kichik ketma-ketlik.

6.8-misol. Ushbu $x_n = \frac{2n-1}{3n+1}$ ketma-ketlikning limiti $\frac{2}{3}$ ga teng ekanligini ta'rif bo'yisha isbot qiling va quyidagi jadvalni to'ldiring:

ε	0,1	0,01	0,001	0,0001	....
$n_0(\varepsilon)$					......

Yechilishi. Ixtiyoriy musbat ε sonni olamiz. Bu songa ko'ra shunday $n_0(\varepsilon)$ nomerning mavjudligini ko'rsatish kerakki, ular uchun (6.9) tengsizlik o'rinli bo'lsin. Buning uchun

$$\left|x_n - \frac{2}{3}\right| = \left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| = \left|-\frac{5}{3(3n+1)}\right| = \frac{5}{3(3n+1)} < \varepsilon$$

tengsizlikni n ga nisbatan yechish kerak:

$$\frac{5}{3\varepsilon} < 3n+1, \quad \frac{5}{3\varepsilon} - 1 < 3n, \quad \frac{5-3\varepsilon}{9\varepsilon} < n, \quad .$$

$n_0(\varepsilon)$ natural son (izlanayotgan nomer) sifatida $\left[\frac{5-3\varepsilon}{9\varepsilon}\right] = n_0(\varepsilon)$ son olinsa, u holda $\forall n > n_0$ uchun $\left|\frac{2n-1}{3n+1} - \frac{2}{3}\right| < \varepsilon$ tengsizlik bajariladi. Endi berilgan ε ga ko'ra, $n_0(\varepsilon)$ ni topib, jadvalni to'ldiramiz:

ε	0,1	0,01	0,001	0,0001
n_0	5	55	555	5555

6.9-misol. Ushbu

$$x_n = \frac{1}{n!}$$

ketma-ketlikning limiti 0 ga teng ekanligini ta'rif bo'yicha isbot qiling va quyidagi jadvalni to'ldiring:

ε	0,1	0,01	0,001	0,0001	...
n_0					

Yechilishi. Ixtiyoriy musbat ε sonni olamiz va bu songa ko'ra shunday $n_0(\varepsilon) \in N$ nomerning topilishini ko'rsatish kerakki, $\forall n > n_0(\varepsilon)$ uchun

$$|x_n| = \left|\frac{1}{n!}\right| < \varepsilon$$

tengsizlik bajarilsin. Buning uchun

$$\left|\frac{1}{n!}\right|=\frac{1}{n!}=\frac{1}{1\cdot2\cdot3\cdots n}<\varepsilon,$$

tengsizlikni n ga nisbatan yechish kerak:

$$\frac{1}{1\cdot2\cdot3\cdots n}<\frac{1}{1\cdot2\cdot2\cdots2}=\frac{1}{2^{n-1}}<\varepsilon$$

$$\frac{1}{\varepsilon}<2^{n-1},\ \lg\frac{1}{\varepsilon}<(n-1)\lg2,\ n>1+\lg\frac{1}{\varepsilon}\cdot\lg^{-1}2.$$

Izlanayotgan nomer $n_0(\varepsilon)$ natural son sifatida $1+\left[\lg\frac{1}{\varepsilon}\cdot\lg^{-1}2\right]=n_0(\varepsilon)$ son olinsa, $\forall n>1+\left[\lg\frac{1}{\varepsilon}\cdot\lg^{-1}2\right]$ uchun $\left|\frac{1}{n!}\right|<\varepsilon$ tengsizlik bajariladi. Endi berilgan ε ga ko‘ra $n_0(\varepsilon)$ ni topib, jadvalni to‘ldiramiz:

ε	0,1	0,01	0,001	0,0001
n_0	4	7	11	14

6.10-misol. Ushbu

$$x_n=3^{\sqrt[3]{n}}$$

ketma-ketlikning limiti $n\to\infty$ da ∞ ga teng ekanligini (ya’ni cheksiz katta ekanligini) ta’rif bo‘yicha isbot qiling va quyidagi jadvalni to‘ldiring:

A	10	100	1000	10000
n_0				

Yechilishi. Ixtiyoriy $A>0$ sonni olaylik va bu songa ko‘ra shunday $n_0(A)\in N$ nomer topilishini ko‘rsatish kerakki, $\forall n>n_0$ nomerdan boshlab berilgan ketma-ketlikning hamma hadlari $|x_n|=3^{\sqrt[3]{n}}>A$ tengsizlikni qanoatlantirsin. Buning uchun $3^{\sqrt[3]{n}}>A$ tengsizlikni n ga nisbatan yechish kerak: izlanayotgan $n_0(A)$ nomer sifatida $\left[\frac{(\lg A)^3}{(\lg 3)^3}\right]=n_0(A)$ son olinsa, $\forall n>(\lg A)^3$ dan boshlab $3^{\sqrt[3]{n}}>A$

tengsizlik bajariladi. Berilgan $A>0$ songa ko'ra n_0 ni topib, jadvalni to'ldiramiz:

A	10	100	1000	10000
n_0	9	73	244	572

Yaqinlashuvchi ketma-ketliklar quyidagi xossalarga ega:

6.5-xossa. Yaqinlashuvchi ketma-ketlik yagona limitga ega bo'ladi.

6.6-xossa. Har qanday yaqinlashuvchi ketma-ketlik chegaralangan ketma-ketlik bo'ladi, aks holda ketma-ketlik chegaralanmagan bo'ladi.

6.7-xossa. $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, ular, mos ravishda, a va b limitlarga ega bo'lsa, u holda $\{x_n \pm y_n\}$, $\{x_n \cdot y_n\}$, $\left\{\frac{x_n}{y_n}\right\}$ ketma-ketliklar ham yaqinlashuvchi bo'ladi va ushbu

$$\lim_{n\to\infty}(x_n \pm y_n) = a \pm b; \quad \lim_{n\to\infty}(x_n \cdot y_n) = a \cdot b; \quad \lim_{n\to\infty}\frac{x_n}{y_n} = \frac{a}{b}\,(b \neq 0)$$

munosabatlar o'rinli bo'ladi.

6.8-xossa. $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$ bo'lsin. Agar $\forall n\,(n\in N)$ uchun $x_n \le y_n$ $(x_n \ge y_n)$ bo'lsa, u holda $a \le b$ $(a \ge b)$ bo'ladi.

6.9-xossa. $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi bo'lib, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$ bo'lsin. Agar $\forall n(n\in N)$ uchun $x_n \le z_n \le y_n$ tengsizlik o'rinli bo'lsa, u holda $\{z_n\}$ ketma-ketlik ham yaqinlashuvchi va $\lim\limits_{n\to\infty} z_n = a$ bo'ladi.

6.1-eslatma. Agar yaqinlashuvchi ketma-ketlikning hamma elementlari $x_n > b$ qat'iy tengsizlikni qanoatlantirsa, u holda, bu ketma-ketlikning limiti x uchun har doim ham $x > b$ bo'lmaydi.

Masalan, $x_n = \frac{1}{n}$ bo'lsin. Bundan $\forall n\,(n\in N)$ uchun $x_n > 0$ bo'ladi, lekin $\lim\limits_{n\to\infty}\frac{1}{n} = 0$ bo'lib, u $x > 0$ tengsizlikni qanoatlantirmaydi.

6.10-xossa. Agar yaqinlashuvchi $\{x_n\}$ ketma-ketlikning hamma hadlari $[a;\, b]$ segmentning ishida joylashsa, u holda uning x limiti ham $[a;\, b]$ segmentning ishida joylashadi.

6.2-eslatma: Ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketliklarning yig'indisi, ayirmasi, ko'paytmasi va nisbatidan iborat bo'lgan ketma-ketlikning yaqinlashuvchi bo'lishidan bu $\{x_n\}$ va $\{y_n\}$ ketma-ketliklarning har birining yaqinlashuvchi bo'lishi har doim kelib shiqavermaydi. Masalan:
1) $\{\sqrt{2n+1}-\sqrt{n-1}\}$ ketma-ketlik yaqinlashuvchi. Haqiqatdan ham, $\lim\limits_{n\to\infty}[\sqrt{2n+1}-\sqrt{n-1}]=\lim\limits_{n\to\infty}\frac{2}{\sqrt{3n+1}+\sqrt{n-1}}=0$, lekin $\{\sqrt{2n+1}\}$, $\{\sqrt{n-1}\}$ ketma-ketliklar uzoqlashuvchi.

6.11-misol. Ushbu

$$x_n=\sqrt[n]{n}$$

ketma-ketlikning limiti 1 ga teng ekanligini isbot qiling.

Yechilishi. Berilgan ketma-ketlikning $\sqrt[n]{n}=1+\alpha_n$ ko'rinishda tasvirlanishini ko'rsatamiz. Bunda α_n — cheksiz kichik ketma-ketlik, ya'ni $\lim\limits_{n\to\infty}\alpha_n=0$. $\sqrt[n]{n}-1=\alpha_n$ deb belgilaymiz, $\alpha_n\geq 0$.

$$\sqrt[n]{n}=1+\alpha_n,\quad n=(1+\alpha_n)^n=1+n\cdot\alpha_n+\frac{n(n-1)}{2!}\cdot\alpha_n^2+\ldots+\alpha_n^n.$$

Bu tenglikdan $\forall n>1$ uchun

$$n>1+\frac{n(n-1)}{2!}\alpha_n^2,\quad n-1>\frac{n(n-1)}{2!}\alpha_n^2,\quad 1>\frac{n}{2}\alpha_n^2,$$
$$n\geq 2 \text{ uchun } n-1\geq\frac{n}{2},\quad n\geq\frac{n^2}{4}\alpha_n^2,\quad 0\leq\alpha_n\leq\frac{2}{\sqrt{n}}$$

tengsizlik o'rinli bo'ladi. Bundan $\lim\limits_{n\to\infty}\sqrt{\frac{2}{n}}=0$ bo'lgani uchun, oxirgi tengsizlikdan 6.7-xossasiga ko'ra $\lim\limits_{n\to\infty}\alpha_n=0$ ekanligi, ya'ni α_n ning cheksiz kichik ketma-ketlik ekanligi kelib shiqadi.

Demak, $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ ekan.

6.12-misol. Ushbu $x_n=\sqrt[n]{a}$ ketma-ketlikning limiti 1 ga teng ekanligini limit ta'rifidan foydalanib ko'rsating.

Yechilishi. $a=1$ bo'lganda berilgan ketma-ketlik limiti 1 ga teng ekanligi ravshan. Faraz qilaylik, $a>1$ bo'lsin. U holda $\sqrt[n]{a}>1$, $\sqrt[n]{a}-1>0$. Bernulli tengsizligini e'tiborga olgan holda $a=[1+(\sqrt[n]{a}-1)]^n>n\cdot(\sqrt[n]{a}-1)$ tengsizlikka ega bo'lamiz. Bu tengsizlikdan $0<\sqrt[n]{a}-1<\frac{a}{n}$, $n>\left[\frac{a}{\varepsilon}\right]+1$ $(\varepsilon>0)$ uchun $0<\sqrt[n]{a}-1<\frac{a}{n}<\varepsilon$ tengsizlik bajariladi. Bundan $a>1$ bo'lganda $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$ ekanligi kelib shiqadi. Endi $0<a<1$ bo'lsin, u holda $\frac{1}{a}>1$ bo'ladi. Yuqorida isbot qilinganga asosan: $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{a}}=1$.

Demak, yaqinlashuvchi ketma-ketliklarning 6.7-xossasiga binoan

$$\lim_{n\to\infty}\sqrt[n]{a}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{\frac{1}{a}}}=1.$$

6.13-misol. $a>1$ bo'lganda $\lim\limits_{n\to\infty}\frac{n}{a^n}=0$ ekanligini isbotlang.

Yechilishi. Shartga ko'ra $a>1$ bo'lgani uchun, $a-1>0$ bo'ladi.

$$0\le\frac{n}{a^n}=\frac{n}{(1+a-1)^n}=\frac{n}{1+\frac{n(a-1)}{1!}+\frac{n(n-1)}{2!}(a-1)^2+...+(a-1)^n}<\frac{n}{\frac{n(n-1)}{2}(a-1)^2}\le$$

$\le\frac{4}{n(a-1)^2}$, chunki, $\forall n\ge2$ uchun $\frac{n(n-1)}{2}(a-1)^2\ge\frac{n^2}{4}(a-1)^2$ tengsizlik o'rinli. Bundan

$$0\le\frac{n}{a^n}\le\frac{4}{n(a-1)^2},\quad \lim_{n\to\infty}\frac{4}{n(a-1)^2}=0.$$

Demak, yaqinlashuvchi ketma-ketliklarning 6.9-xossasiga ko'ra, berilgan ketma-ketlik limiti 0 bo'ladi.

6.14-misol. $a>1$ bo'lganda $\lim\limits_{n\to\infty}\frac{\log_a n}{n}=$ ekanligini isbotlang.

Yechilishi. $\forall\varepsilon>0$ olaylik, $b>1$ bo'lganda $\lim\limits_{n\to\infty}\frac{n}{b^n}=0$ bo'ladi (6.13-misolga qarang). U holda istalgancha katta $n(n\in N)$ lar uchun $\frac{1}{b^n}<\frac{n}{b^n}<1$ tengsizlik o'rinli. $b=a^\varepsilon(a>1)$ deb olsak, u holda $\frac{1}{a^{\varepsilon n}}<\frac{n}{a^{\varepsilon n}}<1$ yoki $1<n<a^{\varepsilon n}$. Bu tengsizlikni logarifmlash natijasida $0<\log_a n<\varepsilon n$ tengsizlikni hosil qilamiz. Bundan esa, $0<\frac{\log_a n}{n}<\varepsilon$ bo'ladi.

ε ning istalgancha kichikligini e'tiborga olsak, yaqinlashuvchi ketma-ketliklarning 6.9°-xossasiga asosan, $\lim\limits_{n\to\infty}\frac{\log_a n}{n}=0$ bo'ladi.

Yuqoridagi 6.13 va 6.14-misollarni solishtirganda, ushta $\{a^n\}$, $\{n\}$, $\{\log_a^n\}$, $a>1$ ketma-ketliklarning birinchisi $n\to\infty$ da qolganlariga nisbatan tezroq o'sishini, ushinshisi esa qolganlariga nisbatan sekinroq o'sishini ko'rish qiyin emas.

6.15-misol. Ushbu

$$x_n=\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1}$$

ketma-ketlikning limiti $\frac{2}{3}$ ga tengligini ko'rsating.

Yechilishi. Ravshanki, $\prod_{k=2}^{n}\frac{k^3-1}{k^3+1}=\prod_{k=2}^{n}\frac{k-1}{k+1}\cdot\prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1}$,

$$\prod_{k=2}^{n}\frac{k-1}{k+1}=\frac{1\cdot2\cdot3\ldots(n-1)}{3\cdot4\cdot5\ldots(n+1)}=\frac{2}{n(n+1)},$$

$$\prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1}=\frac{7\cdot13\cdot21\ldots(n^2+n+1)}{3\cdot7\cdot13\ldots(n^2-n+1)}=\frac{n^2+n+1}{3}.$$

Bu munosabatlarni e'tiborga olsak,

$$x_n=\frac{2}{3}\cdot\frac{n^2+n+1}{n^2+n}$$

ekanligi kelib shiqadi. Bundan yaqinlashuvchi ketma-ketliklarning 6.7-xossasiga binoan, $\lim\limits_{n\to\infty}x_n=\frac{2}{3}\lim\limits_{n\to\infty}\frac{n^2+n+1}{n^2+n}=\frac{2}{3}$ ekanligini topamiz.

6.16-misol. Quyidagi $\{x_n\}$ ketma-ketliklarning limitlarini hisoblang:

1) $x_n=\frac{2n^3}{2n^2+3}+\frac{1-5n^2}{5n+1}$; 2) $x_n=\sqrt[n]{10n}$;

3) $x_n=\sqrt{3n+5}-\sqrt{n-1}$; 4) $x_n=n^2\left(n-\sqrt{n^2+1}\right)$;

5) $x_n=\sqrt[3]{n^2-n^3}+n$.

Yechilishi. 1) kasrlarni qo'shib,

$$x_n=\frac{2n^3-13n^2+3}{10n^3+2n^2+15n+3}$$

ni hosil qilamiz. Bundan, yaqinlashuvchi ketma-ketliklarning xossalarini e'tiborga olib,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^3\left(2-\frac{13}{n}+\frac{3}{n^3}\right)}{n^3\left(10+\frac{2}{n}+\frac{5}{n^2}+\frac{3}{n^3}\right)} = \frac{2}{10} = \frac{1}{5}$$

ekanligini topamiz. Agar $y_n = \frac{2n^3}{2n^2+3}$, $z_n = \frac{1-5n^2}{5n+1}$ deb belgilasak, u holda bu ketma-ketliklar yig'indisining limiti, ya'ni $\lim_{n\to\infty}(y_n+z_n)=\frac{1}{5}$, lekin qo'shiluvshilarning har biri uzoqlashuvchi (cheksiz katta) ketma-ketliklar (6.2-eslatmaga qarang).

Misolni MAPLE tizimidan foydalanib yechish:

```
> Limit((2*n^3-
13*n^2 + 3)/(10*n^3 + 2*n^2 + 15*n + 3), n = infinity)
= limit((2*n^3-
13*n^2 + 3)/(10*n^3 + 2*n^2 + 15*n + 3), n = infinity);
```

$$\lim_{n\to\infty}\left(\frac{2n^3-13n^2+3}{10n^3+2n^2+15n+3}\right)=\frac{1}{5}.$$

2) yuqoridagi 6.10 va 6.11-misollarni e'tiborga olgan holda

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sqrt[n]{10\,n} = \lim_{n\to\infty}\sqrt[n]{10}\cdot\lim_{n\to\infty}\sqrt[n]{n} = 1\cdot 1 = 1$$

ekanligini topamiz.

3) $x_n = \sqrt{3n+5}-\sqrt{n-1} = \sqrt{n}\left(\sqrt{3+\frac{5}{n}}-\sqrt{1-\frac{1}{n}}\right)$.

Bunda $\lim_{n\to\infty}\left(\sqrt{3+\frac{5}{n}}-\sqrt{1-\frac{1}{n}}\right)=\lim_{n\to\infty}\sqrt{3+\frac{5}{n}}-\lim_{n\to\infty}\sqrt{1-\frac{1}{n}}=\sqrt{3}-1$ ekanligini e'tiborga olsak, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{3+\frac{5}{n}}-\sqrt{1-\frac{1}{n}}\right)=+\infty$ ekanligini topamiz.

Misolni MAPLE tizimidan foydalanib yechish:

```
> Limit(sqrt(3*n + 5)-sqrt(n-1), n = infinity) =
limit(sqrt(3*n + 5)-sqrt(n-1), n = infinity);
```

$$\lim_{n\to\infty}\left(\sqrt{3n+5}-\sqrt{n-1}\right)=\infty.$$

4) $x_n=n^2\left(n-\sqrt{n^2+1}\right)$, bundan ikkinshi ko'paytuvchini $n+\sqrt{n^2+1}$ ga ko'paytirib, bo'lsak, $x_n=-\frac{n^2}{n+\sqrt{n^2+1}}=-\frac{n}{1+\sqrt{1+\frac{1}{n^2}}}=-n\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}}$,

$\lim\limits_{n\to\infty}x_n=-\lim\limits_{n\to\infty}n\cdot\frac{1}{1+\sqrt{1+\frac{1}{n^2}}}=-\infty$ ekanligini topamiz.

Misolni MAPLE tizimidan foydalanib yechish:

> Limit(n^2*(n-sqrt(n^2 + 1)), n = infinity) = limit(n^2*(n-sqrt(n^2 + 1)), n = infinity);

$\lim\limits_{n\to\infty}\left(n^2\left(n-\sqrt{n^2+1}\right)\right)=-\infty.$

5) $x_n=\sqrt[3]{n^2-n^3}+n=\frac{n^2}{(n^2-n^3)^{2/3}-n\sqrt[3]{n^2-n^3}+n^2}$ ekanligini ko'rish qiyin emas, bundan

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}-\left(\frac{1}{n}-1\right)^{1/3}+1}=\frac{1}{3}.$$

Misolni MAPLE tizimidan foydalanib yechish:

> Limit(sqrt(n^2−n^3,3) + n, n = infinity) = limit(sqrt(n^2−n^3,3) + n, n = infinity);

$\lim\limits_{n\to\infty}\left(\sqrt[3]{n^2-n^3}+n\right)=\frac{1}{3}.$

6.17-misol. Ushbu

$$\{y_n\}=\left\{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right\}$$

ketma-ketlikning limitini toping.

Yechilishi. Ravshanki, barcha $n\in N$ lar uchun

$$x_n=\frac{n}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}<\frac{n}{\sqrt{n^2+1}}=z_n$$

tengsizlik bajariladi va $\{x_n\}$, $\{z_n\}$ ketma-ketliklar

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1,\quad \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$$

bir xil limitga ega. U holda 6.9-xossaga asosan $\{y_n\}$ ketma-ketlik ham limitga ega bo'lib,

$$\lim_{n\to\infty} y_n = 1$$

bo'ladi.

6.18-misol. $\{x_n\}=\{(-1)^n\}$ ketma-ketlikni yaqinlashishga tekshiring.

Yechilishi. Faraz qilaylik, a son $\{x_n\}=\{(-1)^n\}$ ketma-ketlikning limiti bo'lsin, ya'ni $\forall\varepsilon>0$ soni olinganda ham $\exists n_0(\varepsilon)$ $(n_0\in N)$, $\forall n\geq n_0(\varepsilon)$ lar uchun

$$|x_n-a|<\varepsilon \tag{1}$$

bajarilsin. Agar $\varepsilon=\frac{1}{2}$ deb olsak, (1) tengsizlik n ning juft qiymatlarida

$$|1-a|<\frac{1}{2} \tag{2}$$

ko'rinishga, n ning toq qiymatlarida esa

$$|-1-a|<\frac{1}{2} \text{ yoki } |1+a|<\frac{1}{2} \tag{3}$$

ko'rinishga ega bo'ladi. (2) va (3) tengsizliklardan $2=|(1-a)+(1+a)|\leq$ $\leq|1-a|+|1+a|<1$, ya'ni $2<1$ bo'lishi mumkin emas. Bu qarama-qarshilik, farazimizning noto'g'ri ekanligini isbotlaydi. Demak, $\{(-1)^n\}$ ketma-ketlik uzoqlashuvchi.

6.5. Monoton ketma-ketlikning ta'riflari. $\{x_n\}$ ketma-ketlik berilgan bo'lsin.

6.15-ta'rif. Agar $\{x_n\}$ ketma-ketlikning elementlari $\forall n\in N$ uchun $x_n\leq x_{n+1}$ $(x_n<x_{n+1})$ tengsizlikni qanoatlantirsa, $\{x_n\}$ *o'suvchi (qat'iy o'suvchi) ketma-ketlik* deyiladi.

6.16-ta'rif. Agar $\{x_n\}$ ketma-ketlikning elementlari $\forall n\in N$ uchun $x_n\geq x_{n+1}$ $(x_n>x_{n+1})$ tengsizlikni qanoatlantirsa, $\{x_n\}$ *kamayuvchi (qat'iy kamayuvchi) ketma-ketlik* deyiladi.

6.17-ta'rif. O'suvchi va kamayuvchi ketma-ketliklar umumiy nom bilan *monoton ketma-ketliklar* deb ataladi.

Monoton ketma-ketlik bir tomonlama (yuqoridan yoki quyidan) chegaralangan bo'ladi. Masalan, o'suvchi ketma-ketlik quyidan chegaralangan (quyi chegarasi sifatida uning birinchi hadini olish mumkin) bo'ladi, kamayuvchi ketma-ketlik esa yuqoridan chegaralangan (yuqori chegarasi sifatida ham uning birinchi hadini olish mumkin) bo'ladi. Bu mulohazalardan ko'rinadiki, yuqoridan chegaralangan o'suvchi ketma-ketlik chegaralangan bo'ladi, quyidan chegaralangan kamayuvchi ketma-ketlik ham chegaralangan bo'ladi.

Masalan: 1) $1, 1, 2, 2, \ldots, n, n, \ldots$ ketma-ketlik o'suvchi ketma-ketlik bo'lib, u quyidan 1 bilan chegaralangan;

2) $3, 3^2, 3^3, \ldots, 3^n, \ldots$ ketma-ketlik qat'iy o'suvchi va u quyidan 3 bilan chegaralangan;

3) $1, 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n}, \ldots$ ketma-ketlik kamayuvchi ketma-ketlik bo'lib, u yuqoridan 1 bilan, quyidan nol bilan chegaralangan, ya'ni u chegaralangan ketma-ketlik bo'ladi;

4) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ ketma-ketlik qat'iy kamayuvchi ketma-ketlik bo'lib, u chegaralangan bo'ladi.

6.6. Monoton ketma-ketlikning yaqinlashishi haqidagi teoremalar.

6.2-teorema. *Agar $\{x_n\}$ ketma-ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, u yaqinlashuvchi (chekli limitga ega) bo'ladi; agar $\{x_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo'lsa, u holda u uzoqlashuvchi (limiti $-\infty$) bo'ladi.*

6.3-teorema. *Agar $\{x_n\}$ ketma-ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, u yaqinlashuvchi (chekli limitga ega) bo'ladi; agar $\{x_n\}$ ketma-ketlik quyidan chegaralanmagan bo'lsa, u holda u uzoqlashuvchi (limiti $-\infty$) bo'ladi.*

Bu teoremalardan quyidagi natijalar kelib shiqadi.

6.3-natija. O'suvchi ketma-ketlik yaqinlashuvchi bo'lishi uchun uning yuqoridan chegaralangan bo'lishi zarur va yetarli.

6.4-natija. Kamayuvchi ketma-ketlik yaqinlashuvchi bo'lishi uchun uning quyidan chegaralangan bo'lishi zarur va yetarli.

Yuqoridagi teoremalarni birlashtirib, uni quyidagisha ham ifoda qilish mumkin.

6.4-teorema. *Monoton $\{x_n\}$ ketma-ketlik yaqinlashuvchi (chekli limitga ega) bo'lishi uchun uning chegaralangan bo'lishi zarur va yetarli.*

6.3-eslatma. Har qanday yaqinlashuvchi ketma-ketlik monoton ketma-ketlik bo'lavermaydi. Masalan, $\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots, \frac{1}{n}, -\frac{1}{n}, \ldots$ ketma-ketlik monoton emas.

6.4-eslatma. Yuqoridagi teoremalardan quyidagi xulosani shiqarish mumkin. Yuqoridan chegaralangan o'suvchi $\{x_n\}$ ketma-ketlikning hamma hadlari uning limiti $\overline{x}$ dan katta ($x_n \le \overline{x}$) bo'la olmaydi. Xuddi shunday, quyidan chegaralangan kamayuvchi $\{x_n\}$ ketma-ketlikning hamma hadlari uning limiti $\underline{x}$ dan kichik ($\underline{x} \le x_n$) bo'la olmaydi.

6.5-teorema. *Ikkita $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar berilgan bo'lsin: Agar: 1) $\{x_n\}$ o'suvchi, $\{y_n\}$ kamayuvchi ketma-ketlik; 2) $\forall n \in N$ uchun $x_n < y_n$; 3) $\lim\limits_{n\to\infty}(y_n - x_n) = 0$ bo'lsa, $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n$ tenglik o'rinli bo'ladi.*

Bu teoremadan natija sifatida, quyidagi muhim, ichma-ich joylashgan segmentlar haqidagi teorema kelib shiqadi.

6.6-teorema. *Agar $[a_1; b_1] \supset [a_2; b_2] \supset [a_3; b_3] \supset \ldots \supset [a_n; b_n] \supset \ldots$ munosabatda bo'lgan $[a_1; b_1]$, $[a_2; b_2]$, $[a_3; b_3]$, ..., $[a_n; b_n]$ ... segmentlar ketma-ketligi uchun $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ shart o'rinli bo'lsa, u holda $\{a_n\}$ va $\{b_n\}$ ketma-ketliklar bitta limitga ega bo'ladi hamda bu limit barcha segmentlarga tegishli bo'lgan yagona nuqta bo'ladi.*

Ko'p hollarda, $\left\{\frac{x_n}{y_n}\right\}$ ko'rinishdagi ketma-ketliklarni yaqinlashishga tekshirishda quyidagi teorema muhim rol o'ynaydi.

6.7-teorema (Shtols). *Agar $\{y_n\}$ — o'suvchi cheksiz katta ketma-ketlik bo'lib,*

$$\left\{\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right\}$$

ketma-ketlik yaqinlashuvchi va uning limiti a bo'lsa, u holda

$$\left\{\frac{x_n}{y_n}\right\}$$

ketma-ketlik ham yaqinlashuvchi bo'ladi va u ham a limitga ega bo'ladi, ya'ni

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a.$$

6.19-misol. Ushbu

$$\{x_n\}=\left\{\frac{1^k+2^k+...+n^k}{n^{k+1}}\right\},\ k \text{ — butun, musbat son,}$$

ketma-ketlikning limiti $\frac{1}{k+1}$ ekanligini isbotlang.

Yechilishi. $y_n=1^k+2^k+...+n^k$, $z_n=n^{k+1}$ deb belgilab, $\left\{\frac{y_n-y_{n-1}}{z_n-z_{n-1}}\right\}$ ketma-ketlikning yaqinlashuvchi ekanligini ko'rsatamiz:

$$\frac{y_n-y_{n-1}}{z_n-z_{n-1}}=\frac{n^k}{n^{k+1}-(n-1)^{k+1}}=\frac{n^k}{(k+1)n^k-\frac{(k+1)}{2!}n^{k-1}+...+(-1)^{k+1}}=$$

$$=\frac{1}{(k+1)+\frac{1}{n}\cdot\left[-\frac{(k+1)k}{2!}+\frac{(k+1)k(k-1)}{3!}\cdot\frac{1}{n}-...+\frac{(-1)^{k+1}}{n^{k-1}}\right]}$$

Ravshanki, $\lim\limits_{n\to\infty}\left[-\frac{(k+1)k}{2!}+\frac{(k+1)k(k-1)}{3!}\cdot\frac{1}{n}-...+\frac{(-1)^{k+1}}{n^{k-1}}\right]=-\frac{(k+1)k}{2!}$ — chekli.

Shunday qilib, yaqinlashuvchi ketma-ketlikning xossalarini e'tiborga olgan holda

$$\lim_{n\to\infty}\frac{y_n-y_{n-1}}{z_n-z_{n-1}}=\frac{1}{k+1}$$

ekanligini topamiz.

Demak, Shtols teoremasiga asosan:

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{1^k+2^k+...+n^k}{n^{k+1}}=\frac{1}{k+1}.$$

6.20-misol. Ushbu

$$\lim_{n\to\infty}\frac{a^n}{n!}=0\quad (a \text{ — ixtiyoriy haqiqiy son})\qquad (*)$$

tenglikni isbotlang.

Yechilishi. Ravshanki, $|a|\le 1$ bo'lganda $\lim\limits_{n\to\infty}\frac{a^n}{n!}=0$ tenglik o'rinli. Faraz qilaylik, $a>1$ bo'lsin. $x_n=\frac{a^n}{n!}$ deb belgilaylik, u holda $\frac{x_{n+1}}{x_n}=\frac{a}{n+1}\xrightarrow[n\to\infty]{}0$. Bundan $\forall n>n_0$ (n_0 — istalgancha katta) uchun $\frac{x_{n+1}}{x_n}<1$ yoki $x_{n+1}<x_n$. Shunday qilib, $\{x_n\}$ ketma-ketlik $n>n_0$ uchun kamayuvchi va $0<x_n$ ekan. Demak, 6.3-teoremaga ko'ra, $\{x_n\}$ ketma-ketlik chekli limitga ega, ya'ni $\lim\limits_{n\to\infty}x_n=A\ge 0$. Ikkinshi tomondan $A=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}x_n\frac{a}{n+1}=A\cdot 0=0$.

Shunday qilib, (*) tenglik $a\ge 0$ uchun isbotlandi. $a<0$ bo'lganda ham (*) tenglik o'rinli bo'ladi, chunki $\left|\frac{a^n}{n!}\right|=\frac{|a|^n}{n!}\xrightarrow[n\to\infty]{}0$.

6.21-misol. Ikkita $x_0, x_1, x_2, ..., x_n, ...; y_0, y_1, y_2, ..., y_n, ...$ ($x_0>y_0>0$) ketma-ketliklar berilgan bo'lib, ularning umumiy hadlari ushbu $x_n=\frac{x_{n-1}+y_{n-1}}{2}$, $y_n=\sqrt{x_{n-1}\cdot y_{n-1}}$ formulalar orqali topiladi. Bu ketma-ketliklarning limiti mavjudligini va bu limitlarning bir-biriga tengligini isbotlang.

Yechilishi. Avvalo, $x_n>y_n$ ekanligini ko'rsatamiz. Haqiqatan ham,

$$x_n-y_n=\frac{x_{n-1}+y_{n-1}}{2}-\sqrt{x_{n-1}\cdot y_{n-1}}=\left(\sqrt{\frac{x_{n-1}}{2}}-\sqrt{\frac{y_{n-1}}{2}}\right)^2>0,$$

$$x_n-x_{n-1}=\frac{x_{n-1}+y_{n-1}}{2}-x_{n-1}=\frac{y_{n+1}-x_{n-1}}{2}<0,$$

bundan $x_n<x_{n-1}$, demak, $\{x_n\}$ ketma-ketlik kamayuvchi ekan.

$$y_n-y_{n-1}=\sqrt{x_{n-1}\cdot y_{n-1}}-y_{n-1}=\sqrt{y_{n-1}}\left(\sqrt{x_{n-1}}-\sqrt{y_{n-1}}\right)>0,$$

bundan esa $y_n>y_{n-1}$ ekanligi, ya'ni $\{y_n\}$ — o'suvchi ketma-ketlik ekanligi kelib chiqadi. Monoton ketma-ketliklarning limiti haqidagi 6.2-, 6.3-teoremalarga asosan $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar chekli limitga ega, ya'ni $\lim\limits_{n\to\infty}x_n=x$, $\lim\limits_{n\to\infty}y_n=y$. $\{x_n\}$ ketma-ketlikning aniqlanishiga ko'ra $x_n=\frac{x_{n-1}+y_{n-1}}{2}$. Bundan $x=\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\frac{x_{n-1}+y_{n-1}}{2}=\frac{x+y}{2}$, $x=y$ ekanligi kelib chiqadi.

6.23-misol. Ushbu $x_n = \frac{n!}{n^n}$ ketma-ketlikning yaqinlashuvchi ekanligini isbotlang va uning limitini toping.

Yechilishi.

$$x_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n} = \frac{n!}{n^n} \cdot \frac{n^n}{(n+1)^n} = \frac{n^n}{(n+1)^n} x_n < x_n,$$

chunki $\frac{n^n}{(n+1)^n} < 1$. Demak, $\{x_n\}$ ketma-ketlik monoton kamayuvchi, ikkinshi tomondan, ravshanki, $x_n > 0$. Bundan $\{x_n\}$ ketma-ketlikning quyidan chegaralanganligi kelib shiqadi.

Shunday qilib, monoton ketma-ketliklarning limiti haqidagi 6.3-teoremaga asosan berilgan ketma-ketlik yaqinlashuvchi, ya'ni chekli limitga ega. Uning limitini x bilan belgilaymiz, ya'ni $0 \le \lim\limits_{n\to\infty} x_n = x$. Endi x ni topamiz. Ravshanki, $\frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \ge 1 + n\frac{1}{n} = 2$, bundan $\frac{n^n}{(1+n)^n} < \frac{1}{2}$, $x_{n+1} < \frac{1}{2} x_n$ ekanligi kelib chiqadi. Keyingi tengsizliklarda limitga o'tsak, $x \le \frac{1}{2} x$. Bu yerdan $x \ge 0$ ekanligini e'tiborga olsak, $x = 0$ degan xulosaga kelamiz.

6.24-misol. Ushbu $x_n = \left(1+\frac{1}{n}\right)^n$, $y_n = \left(1+\frac{1}{n}\right)^{n+1}$ $(n = 1, 2, ...)$ ketma-ketliklarning mos ravishda, monoton o'suvchi va yuqoridan chegaralangan, monoton kamayuvchi va quyidan chegaralanganligini hamda $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e$ ekanligini isbotlang.

Yechilishi. $\frac{x_{n+1}}{x_n}$ nisbatni qaraymiz:

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}\left(1+\frac{1}{n}\right) = \left(\frac{n(n+2)}{(n+1)(n+1)}\right)^{n+1}\frac{(n+1)}{n} =$$

$$= \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^{n+1}\frac{(n+1)}{n} = \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{(n+1)}{n}.$$

Bundan Bernulli tengsizligiga asosan,

$$\frac{x_{n+1}}{x_n} > \left[1-\frac{1}{(n+1)^2}(n+1)\right]\frac{(n+1)}{n}=1,\quad \frac{x_{n+1}}{x_n}>1$$

tengsizlikni hosil qilamiz, ya'ni $\{x_n\}$ ketma-ketlik monoton o'suvchi ekanligi kelib chiqadi. Xuddi shunday:

$$\frac{y_n}{y_{n-1}}=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^n}=\frac{1}{\left(1+\frac{1}{n^2-1}\right)^n}\frac{n+1}{n}<\frac{1}{1+\frac{n}{n^2-1}}\frac{n+1}{n}=$$

$$=\frac{n^3+n^2-n-1}{n^3+n^2-n}<1,\quad y_n<y_{n-1}.$$

Demak, $\{y_n\}$ ketma-ketlik monoton kamayuvchi ekan. Ravshanki, $\{x_n\}$ ketma-ketlikning berilishidan: $x_n \geq 2$. Endi uning yuqoridan chegaralanganligini ko'rsatamiz:

$$\left(1+\frac{1}{n}\right)^n=1+\frac{n}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+...+\frac{n(n-1)...(n-n+1)}{n!}\frac{1}{n^n}=$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+...+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right)<$$

$$<1+1+\frac{1}{2!}+...+\frac{1}{n!}<1+1+\frac{1}{2}+...+\frac{1}{2^{n-1}}+...=1+\frac{1}{1-\frac{1}{2}}=3,$$

$2 \leq x_n < 3$. Ravshanki, $0<x_n<y_n$.

Demak, monoton ketma-ketliklarning limiti haqidagi 6.2- va 6.3-teoremalarga asosan, $\{x_n\}$ va $\{y_n\}$ ketma-ketliklar limitga ega: $\lim\limits_{n\to\infty} x_n=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$ deb belgilaymiz, bunda $e=2{,}718281828459045...$

$$0<y_n-x_n=\left(1+\frac{1}{n}\right)^n\frac{1}{n}<\frac{e}{n}.$$

$\lim\limits_{n\to\infty}\frac{e}{n}=0$ ekanligini e'tiborga olsak, u holda $\lim\limits_{n\to\infty}(y_n-x_n)=0$ yoki $\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty} x_n=e$ ekanligi kelib chiqadi.

6.25-misol. Ushbu $x_1=0,\ x_{n+1}=\sqrt{6+x_n},\ n\in N$ ko'rinishda berilgan $\{x_n\}$ ketma-ketlikning yaqinlashuvchi ekanligini isbotlang va uning limitini toping.

Yechilishi. $x_1=0$ bo'lganda $x_2=\sqrt{6+x_1}=\sqrt{6}$. Faraz qilaylik, $\forall n\in N$ uchun $x_{n+1}>x_n$ o'rinli bo'lsin. $\forall n\in N$ uchun $x_{n+2}^2=6+x_{n+1},\ x_{n+1}^2=6+x_n$, bu tengliklarni ayirish natijasida

$$x_{n+2}^2-x_{n+1}^2=x_{n+1}-x_n \qquad (*)$$

ni hosil qilamiz. Bundan, $x_{n+2}^2>x_{n+1}^2$, $x_{n+2}>0$ va $x_{n+1}>0$ bo'lgani uchun, $x_{n+2}>x_{n+1}$ tengsizlik o'rinli bo'ladi.

Demak, matematik induksiya usuliga asosan $\{x_n\}$ ketma-ketlik qat'iy o'suvchi. Endi $\{x_n\}$ ketma-ketlikning o'suvchi, yuqoridan chegaralangan ekanligini ko'rsatamiz. Ravshanki, $x_n\geq 0,\ x_n^2<x_{n+1}^2=6+x_n$, bundan $x_n^2-x_n-6<0$, $x_n<3$ ekanligi kelib chiqadi.

Shunday qilib, $\{x_n\}$ ketma-ketlik o'suvchi va yuqoridan chegaralangan ekan.

Demak, monoton ketma-ketliklarning limiti haqidagi 6.2-teoremaga asosan, $\{x_n\}$ ketma-ketlik limitga ega, ya'ni $\exists \lim\limits_{n\to\infty}x_n=x$. Ravshanki, $x>0$.

$$x_{n+1}^2=6+x_n$$

tenglikda limitga o'tsak, $x^2=6+x$ ni hosil qilamiz. Bundan $x=3$ ekanligini topamiz.

Demak, $\{x_n\}$ ketma-ketlik yaqinlashuvchi, ya'ni $\lim\limits_{n\to\infty}x_n=3$.

6.26-misol. Ushbu

$$x_n=\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{1}{2^n}\right)$$

ketma-ketlikning yaqinlashuvchi ekanligini isbotlang.

Yechilishi. $\frac{x_{n+1}}{x_n}=1+\frac{1}{2^{n+1}}>1$. Demak, $\{x_n\}$ ketma-ketlik o'suvchi. Endi berilgan ketma-ketlikning chegaralanganligini ko'rsatamiz:

$$\ln x_n = \ln\left(1+\frac{1}{2}\right)+\ln\left(1+\frac{1}{4}\right)+...+\ln\left(1+\frac{1}{2^n}\right)<\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}<$$

$$<\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2}+...=\frac{1}{2}\frac{1}{1-\frac{1}{2}}=1.$$

Bu tengsizlikdan $x_n > e$.

Demak, monoton ketma-ketliklarning limiti haqidagi 6.2-teoremaga asosan u yaqinlashuvchi bo'ladi.

6.7. Ixtiyoriy ketma-ketliklarning quyi va yuqori limitlari. Ixtiyoriy $\{x_n\}: x_1, x_2, ..., x_n, ...$ ketma-ketlik bilan birga ixtiyoriy o'suvchi $k_1, k_2, ..., k_n, ...$ butun musbat sonlar ketma-ketligi berilgan bo'lsin. $\{x_n\}$ ketma-ketlikning elementlari ichidan $k_1, k_2, ..., k_n, ...$ nomerdagilarini ajratib olib, ularni ko'rsatilgan nomerlarning o'sish tartibida joylashtirib chiqamiz. Natijada, $x_{k_1}, x_{k_2}, ..., x_{k_n}, ...$ ketma-ketlik hosil bo'ladi. Hosil bo'lgan bu ketma-ketlik berilgan $\{x_n\}$ ketma-ketlikning *qismiy ketma-ketligi* deb ataladi va $\{x_{k_n}\}$ kabi belgilanadi.

Xususiy holda, $\{x_n\}$ ketma-ketlikning o'zini ham $k_n = n$ nomerli qismiy ketma-ketlik deb qarash mumkin. $\{x_n\}$ ketma-ketlikdan umumiy holda cheksiz ko'p usul bilan qismiy ketma-ketliklar ajratish mumkin.

Masalan: 1) ushbu

$$2, 4, 6, 8, ..., 2n, ...;$$
$$1, 3, 5, 7, ..., 2n-1, ...;$$
$$1, 4, 9, 16, ..., n^2, ...;$$
$$1, 8, 27, 64, ..., n^3, ...$$

ketma-ketliklar $\{n\}: 1, 2, 3, ..., n, ...$ ketma-ketlikning qismiy ketma-ketliklar bo'ladi.

2) ushbu

$$1, \frac{1}{4}, \frac{1}{9}, ..., \frac{1}{n^2}, ...;$$
$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ..., \frac{1}{2^n}, ...;$$
$$\frac{1}{5}, \frac{1}{5^3}, \frac{1}{5^5}, ..., \frac{1}{5^{2n-1}}, ...$$

ketma-ketliklar $\left\{\frac{1}{n}\right\}: 1, \frac{1}{2}, ..., \frac{1}{n}, ...$ ketma-ketlikning qismiy ketma-ketliklardir.

3) $\{(-1)^n\}: -1, 1, -1, ..., (-1)^n, ...$ ketma-ketlikdan

$$1, 1, ..., 1, ...;$$
$$-1, -1, ..., -1, ...$$

ketma-ketliklarni ajratish mumkin.

Ketma-ketlik limiti bilan uning qismiy ketma-ketliklari limiti orasidagi munosabat ushbu teorema orqali ifodalanadi:

6.8-teorema. *Agar $\{x_n\}$ ketma-ketlik a limitga ega bo'lsa,u holda uning har qanday qismiy ketma-ketligi ham shu a limitga ega bo'ladi.*

6.5-eslatma. $\{x_n\}$ ketma-ketlik qismiy ketma-ketliklarining limitga ega bo'lishidan, berilgan $\{x_n\}$ ketma-ketlikning limitga ega bo'lishi har doim ham kelib chiqavermaydi. Masalan. $\{(-1)^n\}$ ketma-ketlikning ushbu

$$1, 1, ..., 1, ...;$$
$$-1, -1, ..., -1, ...$$

qismiy ketma-ketliklari, mos ravishda, 1 va -1 limitlarga ega bo'lsada, berilgan $\{(-1)^n\}$ ketma-ketlik limitga ega emas.

Shunday qilib, ketma-ketlik limitga ega bo'lmasa ham, uning qismiy ketma-ketliklari limitga ega bo'lishi mumkin ekan.

6.18-ta'rif. $\{x_n\}$ ketma-ketlik qismiy ketma-ketligining limiti $\{x_n\}$ ketma-ketlikning *qismiy limiti* deyiladi.

6.9-teorema. Agar $\{x_n\}$ ketma-ketlikning hamma qismiy ketma-ketliklari yaqinlashuvchi bo'lib, ularning har biri bitta *a* limitga ega bo'lsa, u holda $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, u ham shu *a* limitga ega bo'ladi.

6.19-ta'rif. Agar x $(x \in (-\infty; +\infty))$ nuqtaning ixtiyoriy ε atrofida $\{x_n\}$ ketma-ketlikning cheksiz ko'p elementlari joylashsa, x nuqta $\{x_n\}$ ketma-ketlikning *limit nuqtasi* deyiladi.

6.10-teorema. *Har qanday haqiqiy sonlar (chegaralangan va chegaralanmagan) ketma-ketligidan chekli songa, $+\infty$ yoki $-\infty$ ga intiluvchi qismiy ketma-ketlik ajratish mumkin.*

6.11-teorema (Bolsano-Veyershtrass). *Har qanday chegaralangan $\{x_n\}$ ketma-ketlikdan yaqinlashuvchi $\{x_{n_k}\}$ qismiy ketma-ketlik ajratish mumkin.*

6.20-ta'rif. Agar $\{x_n\}$ ketma-ketlikdan x $(x \in (-\infty; +\infty))$ nuqtaga yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, x nuqta $\{x_n\}$ ketma-ketlikning *limit nuqtasi* deyiladi.

6.19-ta'rif bilan 6.20-ta'rif o'zaro ekvivalent bo'lishiga ishonsh hosil qilish qiyin emas.

6.21-ta'rif. $\{x_n\}$ ketma-ketlik limit nuqtalarining eng kattasi bu ketma-ketlikning *yuqori limiti* deyiladi va $\overline{x} = \overline{\lim\limits_{n\to\infty}} x_n = \lim\limits_{n\to\infty} \sup\limits_{k>n} x_k$ kabi belgilanadi.

6.22-ta'rif. $\{x_n\}$ ketma-ketlik limit nuqtalarining eng kichigi bu ketma-ketlikning *quyi limiti* deyiladi va $\underline{x} = \underline{\lim\limits_{n\to\infty}} x_n = \lim\limits_{n\to\infty} \inf\limits_{k>n} x_k$ kabi belgilanadi.

Agar $\{x_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo'lsa, $\overline{\lim\limits_{n\to\infty}} x_n = +\infty$ bo'ladi.

Agar $\{x_n\}$ ketma-ketlik quyidan chegaralanmagan bo'lsa, $\underline{\lim\limits_{n\to\infty}} x_n = -\infty$ bo'ladi.

Masalan: 1) $\{x_n\} = \{(-1)^n\}$ ketma-ketlikning yuqori limiti, ya'ni $\overline{\lim\limits_{n\to\infty}} x_n = 1$, quyi limiti $\underline{\lim\limits_{n\to\infty}} x_n = -1$ bo'ladi. 2) $\{x_n\} = \{n^{(-1)^n}\} = \left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots\right\}$ ketma-ketlik yuqoridan chegaralanmagan, shuning uchun $\overline{\lim\limits_{n\to\infty}} x_n = +\infty$.

Ravshanki, berilgan $\{x_n\}$ ketma-ketlikning quyi va yuqori limitlari $\underline{\lim\limits_{n\to\infty}} x_n \le \overline{\lim\limits_{n\to\infty}} x_n$ munosabatni qanoatlantiradi.

6.12-teorema. *Har qanday $\{x_n\}$ ketma-ketlik yuqori (quyi) (chekli, $+\infty$ yoki $-\infty$) limitga ega.*

Natija. Agar $\{x_n\}$ ketma-ketlik chegaralangan bo'lsa, uning quyi va yuqori limitlari chekli bo'ladi.

Ketma-ketlikning quyi va yuqori limitlari quyidagi xossalarga ega:

Ixtiyoriy $\{x_n\}$ ketma-ketlik uchun $\overline{\lim\limits_{n\to\infty}} x_n = \overline{x}$ bo'lsin. U holda $\forall \varepsilon > 0$ son olinganda ham:

1°. Shunday $n_0 \in N$ son topiladiki, $\forall n > n_0$ uchun $x_n < \overline{x} + \varepsilon$ bo'ladi.

2°. $\forall n_1 \in N$ son uchun ε va n_1 larga bog'liq shunday natural son $n' > n_1$ topiladiki, $x_{n'} > \overline{x} - \varepsilon$ bo'ladi.

Agar $\overline{x}$ son 1° va 2° shartlarni qanoatlantirsa, u holda $\overline{x} = \lim\limits_{n\to\infty} \sup\limits_{n>k}\{x_k\} = \overline{\lim\limits_{n\to\infty}} x_n$ bo'ladi.

Endi $\{x_n\}$ ketma-ketlik uchun $\underline{x} = \underline{\lim\limits_{n\to\infty}} x_n$ bo'lsin. U holda $\forall \varepsilon > 0$ son olinganda ham:

1°. $\exists n_0 \in N$ son topiladiki, $\forall n > n_0$ uchun $x_n > \underline{x} - \varepsilon$ bo'ladi.

2°. $\forall n_1 \in N$ son uchun ε va n_1 larga bog'liq natural son $n' > n_1$ topiladiki, $x_{n'} < \underline{x} + \varepsilon$ bo'ladi.

Agar $\underline{x}$ son 1° va 2° shartlarni qanoatlantirsa, u holda $\underline{x} = \lim\limits_{k\to\infty} \inf\limits_{k>n}\{x_k\} = \underline{\lim\limits_{n\to\infty}} x_n$ bo'ladi.

6.13-teorema. *$\forall\{x_n\}$ ketma-ketlik a limitga ega bo'lishi uchun, $\underline{\lim\limits_{n\to\infty}} x_n = \overline{\lim\limits_{n\to\infty}} x_n = a$ tenglikning o'rinli bo'lishi zarur va yetarli.*

6.14-teorema (Koshi kriteriysi). *$\forall\{x_n\}$ ketma-ketlik yaqinlashuvchi (chekli limitga ega) bo'lishi uchun, $\forall \varepsilon > 0$ son olinganda ham, shunday $n_0 \in N$ son mavjud bo'lib, $\forall n > n_0$ va $\forall m > n_0$ lar uchun*

$$|x_m - x_n| < \varepsilon \qquad (*)$$

tengsizlikning bajarilishi zarur va yetarli.

Agar (*) shart bajarilmasa, ya'ni

$$\exists \varepsilon_0 > 0: \ \forall k \in N, \ \exists n \geq k, \ \exists m \geq k: \ |x_m - x_n| \geq \varepsilon_0$$

bo'lsa, $\{x_n\}$ ketma-ketlik uzoqlashuvshi bo'ladi.

6.27-misol. Ushbu $x_n = 1 + \frac{n}{1+n}\cos\frac{\pi n}{2}$ ketma-ketlikning $\inf\{x_n\}$, $\sup\{x_n\}$, $\underline{\lim\limits_{n\to\infty}} x_n$, $\overline{\lim\limits_{n\to\infty}} x_n$ toping.

Yechilishi. Ravshanki, $x_{4n-2} < x_{2n-1} < x_{4n}$ tengsizlik o'rinli. Bunda osonlik bilan ko'rsatish mumkinki, $\{x_{4n-2}\}$ ketma-ketlik kamayuvchi, $\{x_{4n}\}$ ketma-ketlik o'suvchi. Shuning uchun

$$\inf\{x_n\} = \underline{\lim\limits_{n\to\infty}} x_n = \lim\limits_{n\to\infty}\left(1 - \frac{4n-2}{4n-1}\right) = 0.$$

$$\sup\{x_n\} = \overline{\lim\limits_{n\to\infty}}\left(1 + \frac{4n}{4n+1}\right) = 2.$$

6.28-misol. $\{x_n\}=\left\{1-\frac{1}{n}\right\}$ ketma-ketlikning $\inf\{x_n\}$ $\sup\{x_n\}$ $\underline{\lim}_{n\to\infty} x_n$, $\overline{\lim}_{n\to\infty} x_n$ larini toping.

Yechilishi. Ravshanki, $\{x_n\}=\left\{1-\frac{1}{n}\right\}$ ketma-ketlik monoton o'suvchi va u yuqoridan 1 bilan chegaralangan. Shuning uchun u yaqinlashuvchi. Berilgan $\{x_n\}$ ketma-ketlikning eng kichik elementi $x_1=0=\inf\{x_n\}$. Bu ketma-ketlikning yuqori va quyi limitlari bir-biriga teng va u $\sup\{x_n\}$ ga teng.

Shuning uchun $\underline{\lim}_{n\to\infty} x_n=\overline{\lim}_{n\to\infty} x_n=\sup\{x_n\}=1$.

6.29-misol. $x_k=k^{(-1)^k}$, $k\in N$ ketma-ketlikning quyi va yuqori limitlarini toping.

Yechilishi.

$$\underline{\lim}_{k\to\infty} k^{(-1)^k}=\lim_{n\to\infty}\inf_{k\geq n} k^{(-1)^k}=\lim_{n\to\infty} 0=0,\quad k=2m-1,\ m\in N,$$

$$\overline{\lim}_{k\to\infty} k^{(-1)^k}=\lim_{n\to\infty}\sup_{k\geq n} k^{(-1)^k}=\lim_{n\to\infty}(+\infty)=+\infty,\quad k=2m,\ m\in N.$$

6.30-misol. Ushbu $x_n=1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$ ketma-ketlikning yaqinlashuvchiligini Koshi kriteriysi orqali ko'rsating.

Yechilishi. $\forall\varepsilon>0$ sonni olaylik. U holda

$$|x_{n+p}-x_n|=\left|\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+p)^2}\right|<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\ldots+$$

$$+\frac{1}{(n+p-1)(n+p)}=\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\ldots+\left(\frac{1}{n+p-1}-\frac{1}{n+p}\right)=$$

$$=\frac{1}{n}-\frac{1}{n+p}=\frac{p}{n(n+p)}=\frac{\frac{p}{n+p}}{n}<\frac{1}{n}.$$

$n_0(\varepsilon)=\left[\frac{1}{\varepsilon}\right]+1$ deb olinsa, $|x_{n+p}-x_n|<\varepsilon$ tengsizlik $\forall p\in N$ uchun bajariladi.

Shunday qilib, berilgan ketma-ketlik Koshi kriteriysiga ko'ra yaqinlashuvchi bo'ladi.

6.31-misol. Ushbu

$$x_n=\frac{1}{\ln 2}+\frac{1}{\ln 3}+\ldots+\frac{1}{\ln n}$$

ketma-ketlikning uzoqlashuvchi ekanligini ko'rsating.

Yechilishi. $\forall n\in N$ berilgan bo'lsin. $\varepsilon_0=\frac{1}{4}$ deb olsak, u holda $\forall n\in N$ uchun

$$|x_m-x_n|=\frac{1}{\ln(n+1)}+\frac{1}{\ln(n+2)}+\ldots+\frac{1}{\ln 2n}>\frac{n}{\ln 2n}>\frac{n}{2n}=\frac{1}{2}>\varepsilon_0$$

tengsizlik bajariladi. Demak, berilgan ketma-ketlik Koshi kriteriysini qanoatlantirmaydi. Shuning uchun ketma-ketlik uzoqlashuvchi bo'ladi.

Mustaqil yechish uchun misol va masalalar

Quyidagi ketma-ketliklarning dastlabki beshta hadini yozing:

6.1. $x_n=2+(-1)^n\frac{2}{n+1}$. 6.2. $x_n=n(3-3(-1)^n)$.

6.3. $x_n=\frac{4n+3}{3n-2}$. 6.4. $x_n=(-1)^n\arcsin\frac{\sqrt{3}}{2}+\pi n$.

6.5. $x_n=\cos\frac{n\pi}{2}$. 6.6. $x_n=2^{(-1)^n}$.

6.7. $x_n=\operatorname{sign}\left(\sin\frac{n\pi}{3}\right)$. 6.8. $x_n=\left[\sqrt{2n}\right]$, $[a]$ — a sonining butun qismi.

6.9. $x_n=\frac{2^n}{n!}$. 6.10. $x_n=\frac{\sin\frac{n\pi}{2}}{n}$.

Quyidagi ketma-ketliklarning umumiy hadini yozing:

6.11. $-\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \frac{1}{5},\ \ldots$. 6.12. $0,\ 4,\ 0,\ 4,\ \ldots$.

6.13. $1,\ 0,\ -3,\ 0,\ 5,\ 0,\ -7,\ 0,\ \ldots$.

6.14. $2,\ \frac{4}{3},\ \frac{5}{6},\ \frac{8}{7},\ \ldots$. 6.15. $-3,\ \frac{5}{3},\ -\frac{7}{5},\ \frac{9}{7},\ -\frac{11}{9},\ \ldots$.

6.16. $0,\ \frac{\sqrt{2}}{2},\ 1,\ \frac{\sqrt{2}}{2},\ 0,\ -\frac{\sqrt{2}}{2},\ -1,\ -\frac{\sqrt{2}}{2},\ 0,\ \ldots$.

6.17. $\frac{2}{3};\ \frac{5}{8};\ \frac{10}{13};\ \frac{17}{18};\ \frac{26}{23};\ \ldots$. 6.18. $1;\ \frac{1}{2};\ 2;\ \frac{1}{3};\ 3;\ \frac{1}{4};\ 2;\ \frac{1}{5}\ \ldots$.

Quyida berilgan $\{x_n\}$ ketma-ketlikning x_4, x_5, x_{n+1} hadlarini yozing.

6.19. $x_n=\frac{(-2)^n}{2n}$. 6.20. $x_n=\frac{(-1)^n}{2n+1}$. 6.21. $x_n=\frac{(-1)^n}{3n-1}$.

6.22. $x_n=\frac{2n-1}{2n}$. 6.23. $x_n=(-10)^n\sin\frac{n\pi}{2}$. 6.24. $x_n=n^{(-1)^n}$.

6.26. $x_n = \frac{n}{n+1}\sin^2\frac{n\pi}{4}$. 6.27. $x_n = \cos^n n\pi$.

Quyida $\{x_n\}$ ketma-ketlik rekurrent formula bilan berilgan. Bu ketma-ketlikning dastlabki to'rtta hadini yozing:

6.28. $x_1 = -2,\ x_{n+1} = x_n + 3$. 6.29. $x_1 = -3,\ x_{n+1} = x_n + 5$.

6.30. $x_1 = -3,\ x_{n+1} = x_n - 2$. 6.31. $x_1 = -1,\ x_{n+1} = x_n + 4$.

Quyidagi $\{x_n\}$ ketma-ketlikning umumiy hadini yozing.

6.32. $x_1 = \frac{1}{2},\ x_{n+1} = \frac{1}{2 - x_n},\ n \in N$.

6.33. $x_1 = \frac{1}{2},\ x_{n+1} = \frac{2}{3 - x_n},\ n \in N$.

6.34. $x_1 = 0,\ x_2 = 1,\ x_{n+2} = (3x_{n+1} - x_n)/2\ \ n \in N$; .

6.35. $x_1 = a,\ x_2 = b,\ x_{n+2} = x_{n+1} + 2x_n,\ n \in N$.

6.36. $x_1 = x_2 = 1,\ x_{n+2} = 0{,}5\ (x_{n+1} + x_n) + 1,\ n \in N$

6.37. $x_1 = x_2 = 1,\ x_{n+2} = x_{n+1} + 2x_n + 2,\ n \in N$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarni chegaralanganlikka tekshiring:

6.38. $\left\{\frac{3n+40}{n}\right\}$. 6.39. $\left\{\frac{5n}{6n-7}\right\}$. 6.40. $\left\{\frac{4n^2-3}{4n^2}\right\}$. 6.41. $\left\{\frac{n^2-5}{2+3n^2}\right\}$.

6.42. $\{(-1)^n(n^2+1)\}$. 6.43. $\{(-1)^{n+1}[1+(-1)^n]\cdot n\}$. 6.44. $\{5n^2\}$.

6.45. $\{-(2n-1)^3\}$. 6.46. $\left\{\frac{1^2+2^2+3^2+...+n^2}{n^3}\right\}$. 6.47. $\{\mathrm{tg}\,n\}$.

6.48. $\{n\log_{1/3}n\}$. 6.49. $\{(-1)^n n\}$. $\{(-1)^n n\}$. 6.50. $\{n^{(-1)^n}\}$.

6.51. $\{\sin an + \cos an\},\ a \in R$. 6.52. $\{\sqrt[3]{n-1} - \sqrt[3]{n}\}$. 6.53. $\{2^n\}$.

Quyidagi ketma-ketliklarning chegaralanganligini isbotlang:

6.54. $\left\{\frac{4n^2-3}{3+n^2}\right\}$. 6.55. $\left\{\frac{2n+(-1)^n}{3n+2}\right\}$. 6.56. $\left\{\frac{n^2+5n-6}{(n+2)^2}\right\}$.

6.57. $\left\{\frac{4n^5+3}{(n^3+1)(n+1)^2}\right\}$. 6.58. $\left\{\frac{2-n}{\sqrt{n^2+3}}\right\}$. 6.59. $\left\{\frac{\sin^2 n}{n^2+3}\right\}$.

6.60. $\left\{\frac{n^2}{e^n+3}\right\}$. 6.61. $\left\{\sum_{k=1}^{n}\frac{1}{n+k}\right\}$. 6.62. $\left\{x_n = 1 + \sum_{k=1}^{n}\frac{1}{k!}\right\}$.

6.63. Agar $a_1 = 1,\ a_{n+1} = (n+1)(a_n + 1),\ n \in N$, bo'lsa,

$x_n=\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)...\left(1+\frac{1}{a_n}\right)$ ketma-ketlikning chegaralanmaganligini isbotlang.

6.64. $\left\{\sqrt{n^4+2}-n^2\right\}$. 6.65. $\left\{\sqrt{n^2-1}-\sqrt{n^2+1}\right\}$.

6.66. $\left\{\sqrt[3]{n^3+1}-\sqrt{n^2-1}\right\}$. 6.67. $\left\{\sqrt[3]{9n-n^3}+\sqrt[3]{9n+n^3}\right\}$.

6.68. $\left\{\sqrt{\frac{n^4+n^3}{n^2+1}}-\sqrt{n^2-1}\right\}$. 6.69. $\left\{\sqrt[n]{n}\right\}$. 6.70. $\left\{\frac{3^n+2}{4^n-3}\right\}$.

6.71. $\left\{\frac{7^{2n+1}+3^n}{2-49^n}\right\}$. 6.72. $\left\{\frac{n+\ln n}{n+2}\right\}$. 6.73. $\left\{\lg\frac{3n+5}{n+2}\right\}$.

6.74. $\left\{n\ln\frac{n+1}{n}\right\}$. 6.75. $\left\{\ln\frac{n+1}{n}\ln(n^2+n)\right\}$. 6.76. $\left\{\frac{n}{5^n}\right\}$.

6.77. $\left\{\frac{n^2}{2^n}\right\}$. 6.78. $\left\{nq^n\right\}$, $|q|<1$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarning chegaralanmaganligini isbotlang:

6.79. $\{n+\cos n\pi\}$. 6.80. $\left\{n^{(-1)^n}\right\}$. 6.81. $\{n^3-n^2\}$.

6.82. $\left\{\frac{n^4}{n^3+3}\right\}$. 6.83. $\left\{(n+1)^{\cos\frac{n\pi}{2}}\right\}$. 6.84. $\left\{\frac{n-n^4}{(n+2)^3}\right\}$.

6.85. $\left\{\sqrt{n^4+n^2+1}-\sqrt{n^4-n^2+1}\right\}$. 6.86. $\{6^n-5^n\}$.

6.87. $\left\{\sqrt[n]{(2n)!}\right\}$. 6.88. $\left\{\frac{4^n}{n^2}\right\}$. 6.89. $\left\{\frac{n+1}{\lg(n+1)}\right\}$.

6.90. $\left\{\frac{7^n-5^n}{3^n+1}\right\}$. 6.91. $x_1=x_2=1$, $x_{n+2}=x_{n+1}+6x_n$.

6.92. $x_1=-4$, $x_2=3$, $x_{n+2}=x_{n+1}+\frac{3}{4}x_n$.

6.93. $x_1=3$, $x_{n+1}=\frac{1}{2}x_n^2-1$.

Quyida berilgan $\{x_n\}$ ketma-ketlik yuqoridan (quyidan) chegaralangan bo'lsa, uning eng katta (eng kichik) elementini toping:

6.94. $x_n=6n-n^2-3$. 6.95. $x_n=e^{4n-n^2-3}$. 6.96. $x_n=\frac{\sqrt{n}}{9+n}$.

6.97. $x_n=n^2-10n+5$. 6.98. $x_n=-\frac{n^2}{2^n}$. 6.99. $x_n=\frac{2}{n^2-6n+1}$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarning cheksiz kichik ketma-ketlik ekanligini ta'rif bo'yisha ko'rsating:

6.100. $x_n=\frac{3}{n}$. 6.101. $x_n=\frac{(-1)^{n+1}}{n}$. 6.102. $x_n=\frac{1+(-1)^n}{3^n}$.

6.103. $x_n=\frac{1}{n}\cos\frac{n\pi}{2}$. 6.104. $x_n=\frac{1}{\sqrt{n}}$.

6.105. $x_n=\frac{2}{\sqrt{2n-1}}$. 6.106. $x_n=\frac{1}{\sqrt[5]{8n-11}}$. 6.107. $x_n=(-0,6)^n$.

6.108. $x_n=(0,99)^n$. 6.109. $x_n=\frac{\operatorname{sign}(\operatorname{tg}n)}{n}$. 6.110. $x_n=\frac{3^n}{n!}$.

Quyida berilgan $\{x_n\}$ ketma-ketliklarning cheksiz katta ketma-ketlik ekanligini ta'rif bo'yisha ko'rsating:

6.111. $x_n=n$. 6.112. $x_n=4-3\cdot n$. 6.113. $x_n=2^n$.
6.114. $x_n=\ln n$. 6.115. $x_n=n^{1/p}(p\in N)$.
6.116. $x_n=\log_a n(0<a<1)$. 6.117. $x_n=(-n)^n$.

6.118. $x_n=5\sqrt[3]{n}-n$. 6.119. $x_n=\frac{n^2}{n+3}$.

6.120. $x_n=\frac{2}{1-\sqrt[n]{n}}$. 6.121. $x_n=\frac{3^n}{n^2}$. 6.122. $x_n=\frac{n\sqrt{n}}{n+1}$.

Ketma-ketlik limiti ta'rifidan foydalanib, quyidagi tengliklarni isbotlang:

6.123. $\lim\limits_{n\to\infty}\frac{4n-3}{4n+5}=1$. 6.124. $\lim\limits_{n\to\infty}\frac{5n-2}{3n+4}=\frac{5}{3}$.

6.125. $\lim\limits_{n\to\infty}\frac{n^2+3}{n^2+2n+1}=1$. 6.126. $\lim\limits_{n\to\infty}\frac{5n^2+4}{8n^2+7}=\frac{5}{8}$.

6.127. $\lim\limits_{n\to\infty}\frac{\sqrt{n^2+n}}{n}=1$. 6.128. $\lim\limits_{n\to\infty}\frac{5n+1}{7-9n}=\frac{-5}{9}$.

$\lim\limits_{n\to\infty}x_n=a$ ni toping va qaysi $n_0(\varepsilon)$ nomerdan boshlab $\forall n>n_0(\varepsilon)$ uchun $|x_n-a|<\varepsilon$ tengsizlik o'rinli bo'lishini aniqlang.

6.129. $x_n=0,\underbrace{33...3}_{n\ ta}$, $\varepsilon=0,001$. 6.130. $x_n=\frac{2n-5}{n}$, $\varepsilon=0,005$.

6.131. $x_n=\frac{1}{n}\sin\frac{n\pi}{2}$, $\varepsilon=0,001$. 6.132. $x_n=\frac{\sqrt[3]{n^2+1}}{n}$, $\varepsilon=0,005$.

6.133. $x_n=\frac{1}{n^2}$, $y_n=\frac{2}{n^3}$, $z_n=\frac{n-1}{n^3}$ cheksiz kichik ketma-ketliklar yig'indisining yana cheksiz kichik ketma-ketlik ekanligini ko'rsating.

6.134. $x_n=\frac{1}{n^4}$, $y_n=\frac{-9}{n^4}$, $z_n=\frac{(2n-1)^2}{n^4}$ cheksiz kichik ketma-ketliklar yig'indisining yana cheksiz kichik ketma-ketlik ekanligini ko'rsating.

6.135. $x_n=\frac{1}{n^3}$, $y_n=\frac{4}{n^3}$, $z_n=\frac{(n-1)^2}{n^3}$ cheksiz kichik ketma-ketliklar ko'paytmasining yana cheksiz kichik ketma-ketlik ekanligini ko'rsating.

6.136. $x_n=\frac{1}{3^n}$, $y_n=\frac{-2}{3^n}$, $z_n=\frac{2^{n-1}}{3^n}$ cheksiz kichik ketma-ketliklar ko'paytmasining yana cheksiz kichik ketma-ketlik ekanligini ko'rsating.

6.137. $x_n=\frac{2n}{n^2-1}$, $y_n=\frac{2}{n+1}$ cheksiz kichik ketma-ketliklar ayirmasining yana cheksiz kichik ketma-ketlik ekanligini ko'rsating.

6.138. $\{x_n\}$ cheksiz kichik ketma-ketlik bilan $\{y_n\}$ chegaralangan ketma-ketlik ko'paytmasi cheksiz kichik ketma-ketlik ekanligini isbotlang:

1) $x_n=\frac{1}{n}$, $y_n=\sin n$; 2) $x_n=\frac{1}{n^2}$, $y_n(-1)^n+1$;

3) $x_n=\frac{1}{n^3}$, $y_n=\frac{2n^2}{n^4+4}$; 4) $x_n=\frac{1}{n!}$, $y_n=\cos\frac{n\pi}{2}$;

5) $x_n=\frac{1}{n}$, $y_n=\text{sign}(\text{tg}\, n)$; 6) $x_n=\frac{1}{n^2}$, $y_n=\frac{1}{2+(-1)^n}$.

6.139. $\{x_n\}$ cheksiz kichik ketma-ketlik uchun $\left\{\frac{1}{x_n}\right\}$ cheksiz katta ketma-ketlik ekanligini isbotlang:

1) $x_n=0,\underbrace{99\cdots9}_{n\ ta}$; 2) $x_n=\frac{\sin^2 n+1}{n^2}$;

3) $x_n=\frac{2^n+(-3)^n}{5^n}$; 4) $x_n=\frac{\sqrt{n^2+4}}{n^3}$.

6.140. $\{y_n\}$ cheksiz katta ketma-ketlik bo'lsa, $\left\{\frac{1}{y_n}\right\}$ ketma-ketlik cheksiz kichik ketma-ketlik ekanligini isbotlang:

1) $y_n=\frac{3^n+2}{2^n}$; 2) $y_n=n\left(\sin^2\frac{n\pi}{2}+1\right)$; 3) $y_n=\frac{\sqrt{n^4+a^2}}{n}$;

4) $y_n=2^{\sqrt{n}}$; 5) $y_n=(-1)^n n$; 6) $y_n=\ln(\ln n)$ $n\geq 2$.

6.141. Quyida berilgan $\{x_n\}$ ketma-ketliklarning yaqinlashuvchi ekanligini isbotlang:

1) $x_n=\frac{3+(-1)^n}{n}$; 2) $x_n=\frac{n+1}{n}$;

3) $x_n=\sqrt[n]{2}\quad (a>0)$; 4) $x_n=\frac{1}{\sqrt[n]{n!}}$;

5) $x_n=\frac{2n^2+5n+1}{8n^2+3n+2}$; 6) $x_n=\sqrt{\frac{8n^2+1}{2n^2+2}}$;

7) $x_n=\frac{(n+1)^{10}+(n+2)^9}{(n-1)^{12}+1}$; 8) $x_n=\frac{a^n+3b^n}{5a^n+7b^n}\quad (a>0,\ b>0)$;

9) $x_n=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{(n+1)(n+2)}$;

10) $x_n=\frac{1\cdot 2+2\cdot 3+...+n(n+1)}{n^3}$.

6.142. Quyida berilgan $\{x_n\}$ ketma-ketliklarning uzoqlashuvchi ekanligini isbotlang:

1) $x_n=(-1)^n\cdot 2^n$; 2) $x_n=n^{(-1)^n}$; 3) $x_n=\frac{n^2-10}{n}$.;

4) $x_n=(-1)^n+\frac{1}{n}$; 5) $x_n=\sin n^0$; 6) $x_n=\frac{n}{n+1}\cos\frac{2\pi n}{3}$;

7) $x_n=\left(\frac{2}{3}\right)^{((-1)^n-1)\cdot n}$; 8) $x_n=n^2\sin\frac{\pi n}{2}$.

6.143. a soni $\{x_n\}$ ketma-ketliklarning limiti emasligini ta'rif yordamida ko'rsating:

1) $x_n=(-1)^n\cdot 2+2,\quad a=0$; 2) $x_n=\frac{n^2-1}{n^2}\quad a=1$;

3) $x_n=\frac{(-1)^n}{n},\quad a=-1$; 4) $x_n=\cos\frac{\pi n}{3},\quad a=\frac{1}{2}$;

5) $x_n=2^{(-1)^n\cdot n},\quad a=0$; 6) $x_n=\sin\frac{2\pi n}{3},\quad a=0$;

7) $x_n=\sqrt{n^2+1}-n,\quad a=1$; 8) $x_n=\frac{n^2-2n}{n=1},\quad a=1$;

9) $x_n=n^{(-1)^n},\quad a=0$; 10) $x_n=n\cdot[1+(-1)^n],\quad a=0$.

Limitlarni toping:

6.144. $\lim\limits_{n\to\infty}\left(\frac{1}{6}+\frac{1}{12}+...+\frac{1}{n^2+3n+2}\right)$.

6.145. $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+...+\frac{1}{(3n-2)(3n+1)}\right)$.

6.146. $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+...+\frac{1}{(4n-3)(4n+1)}\right)$.

6.147. $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+...+\frac{1}{(2n-1)(2n+1)(2n+3)}\right)$.

6.148. $\lim_{n\to\infty}\left(\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\ldots+\frac{1}{n(n+1)(n+2)(n+3)}\right)$.

6.149. $\lim_{n\to\infty}\left(\frac{1}{1\cdot3}+\frac{4}{3\cdot5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}\right)$.

6.150. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\ldots+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right)\right)$.

$\{a_n\}$ ketma-ketlik arifmetik progressiya bo'lib, uning ayirmasi $d\neq0$ va barsha xadlari $a_n\neq0$ $(n\in N)$ bo'lsin. Limitlarni toping:

6.151. $\lim_{n\to\infty}\left(\frac{1}{a_1\cdot a_2}+\frac{1}{a_2\cdot a_3}+\ldots+\frac{1}{a_n\cdot a_{n+1}}\right)$.

6.152. $\lim_{n\to\infty}\left(\frac{1}{a_1a_2a_3}+\frac{1}{a_2a_3a_4}+\ldots+\frac{1}{a_na_{n+1}a_{n+2}}\right)$.

6.153. $\lim_{n\to\infty}\left(\frac{1}{a_1a_2a_3a_4}+\frac{1}{a_2a_3a_4a_5}+\ldots+\frac{1}{a_na_{n+1}a_{n+2}a_{n+3}}\right)$.

6.154. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n+1}}}\right)$, $(a_n>0,\ n\in N)$.

6.155. $\lim_{n\to\infty}\left(\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{n(n+1)(n+2)}\right)$.

6.156. $\lim_{n\to\infty}\frac{(n+1)^2-(n-1)^2}{2n+1}$.

6.157. $\lim_{n\to\infty}\frac{(n^2+1)^2-(n^2-1)^2}{(n+1)^4+(n-1)^4}$.

6.158. $\lim_{n\to\infty}\left(\frac{n^4}{n^3+1}-\frac{n^5}{n^4+1}\right)$.

6.159. $\lim_{n\to\infty}\left(\frac{n^5}{n^4+n^2+1}-\frac{n^3}{n^2+n+1}\right)$.

6.160. $\lim_{n\to\infty}\frac{(n^2+3n+4)^3-(n^2+3n-4)^3}{(n^2+5n+6)^3-(n^2+5n-6)^3}$.

6.161. $\lim_{n\to\infty}\left(n-\frac{3}{\frac{3}{n}-\frac{3}{n^2}+\frac{1}{n^3}}\right)$.

6.162. $\lim_{n\to\infty}\left(\frac{(2+n)^{100}-n^{100}-200n^{99}}{n^{99}-10n^2+1}\right)$.

6.163. $\lim_{n\to\infty}\frac{\ln(n^2-n+1)}{\ln(n^4+n^2+1)}$.

6.164. $\lim_{n\to\infty}\frac{(1+\ln n)^2}{\ln^2 n}$.

6.165. $\lim_{n\to\infty}\frac{5^n+7^{-n}}{5^{-n}-7^n}$.

6.166. $\lim_{n\to\infty}\frac{a^n-b^n}{a^n+b^n}$ $(a>1,\ b>1)$.

6.167. $\lim_{n\to\infty}\frac{a^n}{1+a^n}$ $(a\neq-1)$.

6.168. $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

6.169. $\lim_{n\to\infty}\left(\sqrt{4n^2+3n}-2n\right)$.

6.170. $\lim_{n\to\infty}\left(\sqrt[3]{n^2+2}-\sqrt[3]{n^2-2}\right)$.

6.171. $\lim_{n\to\infty}\left(\sqrt{n^2+3n+2}-\sqrt{n^2-n}\right)$.

6.172. $\lim_{n\to\infty}\left(\sqrt[3]{n^3+n^2+1984}-n\right)$. 6.173. $\lim_{n\to\infty}\frac{1}{4}\left(\sqrt[5]{n^{25}+4n^{20}}-n^5\right)$.

6.174. $\lim_{n\to\infty} n^{3/2}\left(\sqrt{n+1}+\sqrt{n-1}-2\sqrt{n}\right)$.

6.175. $\lim_{n\to\infty}\left(\sqrt{(n+a_1)(n+a_2)}-n\right)$.

6.176. $\lim_{n\to\infty}\left(\sqrt[5]{(n+a_1)(n+a_2)(n+a_3)(n+a_4)(n+a_5)}-n\right)$.

6.177. $\lim_{n\to\infty}\left(\sqrt[p]{(n+a_1)(n+a_2)....(n+a_p)}-n\right)$, $p\in N$.

6.178. a ning qanday qiymatlarida $\left\{\sqrt{an^2+bn+2}-n\right\}$ ketma-ketlik limitga ega? Bu limit nimaga teng?

6.179. $x_n=(n+1)^a-n^a$, $n\in N$ ketma-ketlik berilgan bo'lsin. Ushbu

1) $0<a<1$ uchun $\lim_{n\to\infty} x_n=0$; 2) $a>1$ uchun $\lim_{n\to\infty} x_n=\infty$

tengliklarni isbotlang:

6.180. $\lim_{n\to\infty}\frac{\sqrt{4n^2+2}-2n}{\sqrt{n+2}-\sqrt{n}}$. 6.181. $\lim_{n\to\infty}\frac{\sqrt{n^2+7}-\sqrt{n^2-7}}{\sqrt{n^2+1}-n}$.

6.182. $\lim_{n\to\infty}\frac{\sqrt[3]{n}-\sqrt[3]{n+1}}{\sqrt[4]{n+1}-\sqrt[4]{n}}$. 6.183. $\lim_{n\to\infty}\frac{\sqrt[4]{n^3+n}-\sqrt{n}}{n+2+\sqrt{n+1}}$.

6.184. $\lim_{n\to\infty}\frac{\sqrt{2n}}{\sqrt{3n+\sqrt{3n+\sqrt{3n}}}}$. 6.185. $\lim_{n\to\infty}\frac{n^2-\sqrt{n}}{\left(\sqrt{n^3}+n+\sqrt{n}\right)(n-1)}$.

6.186. $\lim_{n\to\infty}\frac{\sqrt{n-2\sqrt{n-1}}+1}{\sqrt{n-1}-1}$, $(n>2)$. 6.187. $\lim_{n\to\infty}\frac{1-n}{2\sqrt{n^2-1}-n+1}$.

6.188. $\lim_{n\to\infty}\frac{\left(\sqrt{n+1}-\sqrt{n-1}\right)(n+2)}{\sqrt{(n+1)^3}-\sqrt{(n-1)^3}}$. 6.189. $\lim_{n\to\infty}\frac{\sqrt{n-4\sqrt{n-4}}+2\sqrt{n}}{\sqrt{n+4\sqrt{n-4}}-2\sqrt{n}}$.

6.190. $\lim_{n\to\infty}\frac{1+2+3+...+n}{n^2+n+2}$.

6.191. $\lim_{n\to\infty}\frac{\left[\left(2+\frac{1}{2}\right)^2+\left(4+\frac{1}{4}\right)^2+...+\left(2^n+\frac{1}{2^n}\right)^2-2n\right]}{4^n}$.

6.192. $\lim_{n\to\infty}\frac{1\cdot3+3\cdot9+5\cdot27+...+(2n-1)\cdot3^n}{3^n(n+2)}$.

6.193. $\lim_{n\to\infty}\frac{1^2+2^2+...+n^2}{n^3}$.

6.194. $\lim_{n\to\infty}\frac{1\cdot2+2\cdot3+...+n(n+1)}{n^3}$.

6.195. $\lim_{n\to\infty}\frac{1-2+3-4+...+(2n-1)-2n}{\sqrt{n^2+1}}$.

6.196. $\lim_{n\to\infty}\frac{1^3+2^3+...+n^3}{n^4+n^3+n+1}$.

6.197. $\lim_{n\to\infty}\left(\frac{1+2^2+...+n^2}{n^2}-\frac{n}{3}\right)$.

6.198. $\lim_{n\to\infty}\left(1-\frac{2}{2\cdot3}\right)\left(1-\frac{2}{3\cdot4}\right)...\left(1-\frac{2}{(n+1)(n+2)}\right)$.

6.199. $\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}$.

6.200. Agar $\{x_n\}$ va $\{y_n\}$ uzoqlashuvchi ketma-ketliklar bo'lsa, quyidagi:

1) $\{x_n+y_n\}$; 2) $\{x_n\cdot y_n\}$; 3) $\left\{\frac{x_n}{y_n}\right\}$ ketma-ketliklar yaqinlashuvchi bo'lishiga misollar tuzing.

6.201. $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lib, $\{y_n\}$ ketma-ketlik uzoqlashuvchi bo'lsin. $b\neq0$ uchun $\{ax_n+by_n\}$ ketma-ketlik uzoqlashuvchi ekanligini isbotlang.

6.202. $\{y_n\}$ ketma-ketlik uzoqlashuvchi bo'lib, $\{x_n\}$ esa yaqinlashuvchi ketma-ketlik va $\lim_{n\to\infty}x_n=x$, $x\neq0$ bo'lsa, $\{x_n\ y_n\}$ ketma-ketlik yaqinlashuvchi emasligini isbotlang.

6.203. $\{x_n\}$ yaqinlashuvchi, $\{y_n\}$ esa uzoqlashuvchi ketma-ketlik bo'lsin. $\{x_n\ y_n\}$ ketma-ketlik yaqinlashuvchi va uzoqlashuvchi bo'lishiga misol tuzing.

6.204. $\forall n\in N$ uchun $x_n\neq0$ va $\lim_{n\to\infty}x_n=0$ bo'lsa: 1) $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$ limit mavjudmi?

2) agar bu limit mavjud va q ga teng bo'lsa, u holda $|q|\leq1$ bo'lishini isbotlang.

3) $\left\{\frac{x_{n+1}}{x_n}\right\}$ ketma-ketlik chegaralanmagan bo'lishi mumkinmi?

Quyidagi limitlarni toping:

6.205. $\lim_{n\to\infty}\left(1+\frac{p}{m}\right)^{n\,q}$, $p,q\in N$. 6.206. $\lim_{n\to\infty}\left(\frac{3^n+1}{3^n}\right)^{3^n}$

6.207. $\lim\limits_{n\to\infty}\left(\frac{n+2}{n+5}\right)^n$. **6.208.** $\lim\limits_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)^{n^2}$.

6.209. $\lim\limits_{n\to\infty}\left(1+\frac{1}{k+n}\right)^n$, $k\in N$. **6.210.** $\lim\limits_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)$.

6.211. Quyida berilgan $\{x_n\}$ ketma-ketliklarni Koshi kriteriysidan foydalanib, yaqinlashuvchi ekanligini isbotlang:

1) $x_n=\frac{\cos a}{3}+\frac{\cos 2a}{3^2}+\frac{\cos 3a}{3^3}+...+\frac{\cos na}{2^n}$, $a\in R$.

2) $x_n=1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$.

3) $x_n=a_1q+a_2q^2+...+a_nq^n$, bunda $|q|<1$, $\forall n\in N$ uchun $|a_n|\le C$, $C=\text{const}$.

4) $x_n=\frac{\operatorname{arctg} a}{2}+\frac{\operatorname{arctg} 2a}{2^2}+...+\frac{\operatorname{arctg} na}{2^n}$, $a\in R$.

5) $x_n=\frac{\ln(1+1)}{0!}+\frac{\ln\left(1+\frac{1}{2}\right)}{1!}+\frac{\ln\left(1+\frac{1}{3}\right)}{2!}+...+\frac{\ln\left(1+\frac{1}{n}\right)}{(n-1)!}$.

6) $x_n=\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+...+\frac{(-1)^{n-1}}{n(n+1)}$.

6.212. Quyida berilgan ketma-ketliklarni o'suvchi ekanligini isbotlang:

1) $x_n=6n-11$. 2) $x_n=n^2+2n-3$. 3) $x_n=5\cdot 2^n-3$.

4) $x_n=5^n-4^n$. 5) $x_n=\frac{3n+4}{n+2}$. 6) $x_n=\frac{2n-4}{n}$.

7) $x_n=\frac{2n-1}{n}$. 8) $x_n=\frac{a^n-1}{n}$ $(a\ne 1, a>0)$.

9) $x_n=n(1-a^{1/n})$ $(a\ne 1,\ a>0)$. 10) $x_n=\left(1+\frac{1}{2n}\right)^n$.

6.213. Quyida berilgan ketma-ketliklarning kamayuvchi ekanligini isbotlang:

1) $x_n=810-30n$. 2) $x_n=-9n^2+10n+25$. 3) $x_n=\frac{4^n+1}{2^{2n}}$.

4) $x_n=\frac{2n+9}{n+3}$. 5) $x_n=\frac{n}{n^2+2}$. 6) $x_n=\frac{n+1}{n^2+2n+5}$.

7) $x_n=\frac{n+2}{n}$. 8) $x_n=\left(1+\frac{1}{n}\right)^{n+1}$. 9) $x_n=\lg\left(\frac{3}{4}\right)^n$.

10) $x_n=\frac{n^2+2n-3}{n^2+2n+2}$.

6.214. Quyida berilgan $\{x_n\}$ ketma-ketliklar qandaydir nomerdan boshlab monoton ekanligini isbotlang:

1) $x_n=3n^2-n.$ 2) $x_n=n^2-3n.$ 3) $x_n=n^3-6n^2.$

4) $x_n=\frac{n+1}{2n-1}.$ 5) $x_n=\frac{n^3}{n^2-3}.$ 6) $x_n=\frac{n^2}{n^3+32}.$

7) $x_n=\frac{n^2+26}{n+2}.$ 8) $x_n=\sqrt{n+3}-\sqrt{n+2}.$

9) $x_n=\sqrt[3]{n^3-1}-n.$ 10) $x_n=4^n-10n.$ 11) $x_n=5^n-3^n.$

12) $x_n=\frac{7^{n+1}-4^{n+1}}{6^n+4^n}.$ 13) $x_n=\frac{10^n}{n!}.$

14) $x_n=\ln(n+1)-\ln n.$ 15) $x_n=\ln n-n.$

16) $x_n=\frac{5^n}{n}.$ 17) $x_n=\ln\frac{n+9}{n}.$ 18) $x_n=\frac{n}{6^n}.$

19) $x_n=\sqrt[n]{n}.$ 20) $x_n=\frac{(3n+1)^2}{3^n}.$

6.215. Quyida berilgan $\{x_n\}$ ketma-ketliklarni o‘suvchi ham kamayuvchi ham emasligini isbotlang:

1) $x_n=\frac{(-1)^n}{n+2}.$ 2) $x_n=3n^2-17n+1.$ 3) $x_n=\frac{2n-7}{2n-9}.$

4) $x_n=3+12n-\frac{1}{9}n^3.$ 5) $x_n=(-1)^n\cdot 7-5.$

6) $x_n=\frac{1}{n}\sin\frac{n\pi}{2}.$ 7) $x_n=\sin n.$ 8) $x_n=\frac{1}{n}\cos n\pi.$

9) $x_n=7n-n^2.$ 10) $x_n=\mathrm{sign}(\mathrm{tg}n).$

6.216. Quyida berilgan $\{x_n\}$ ketma-ketliklarni Koshi kriteriysining inkoridan foydalanib, uzoqlashuvchiligini isbotlang:

1) $x_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n};$ 2) $x_n=\frac{1}{2^2}+\frac{2}{3^2}+...+\frac{n}{(n+1)^2};$

3) $x_n=(-1)^n\left(\frac{4^n+1}{4^n}\right)^n;$ 4) $x_n=\left(1+\frac{(-1)^n}{n}\right)^n;$

5) $x_n=\left(\frac{2}{3}\right)^{(-1)^n\cdot n};$ 6) $x_n=\frac{n\sin\frac{(2n-1)\pi}{2}-1}{3n};$

7) $x_n=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}k.$

6.217. Quyida berilgan $\{x_n\}$ ketma-ketliklarning yaqinlashuvchiligini isbotlang va ularning limitini toping:

1) $x_1=4,\quad x_{n+1}=\sqrt{6+x_n};$ 2) $x_n=3,\quad x_{n+1}=\sqrt{12+x_n};$

3) $x_1=2,\quad x_{n+1}=2-\frac{1}{x_n};$ 4) $x_1=2,\quad x_{n+1}=\frac{6}{x_n-1};$

5) $x_1=1,\quad x_2=1,\quad x_{n+2}=\frac{x_{n+1}+x_n}{2}$;

6) $x_1=\sqrt{2},\quad x_{n+1}=\sqrt{2+x_n}$.

6.218. Quyida berilgan $\{x_n\}$ ketma-ketliklar uchun $\inf x_n$, $\sup x_n$, $\varliminf_{n\to\infty} x_n$ va $\varlimsup_{n\to\infty} x_n$ larni toping:

1) $x_n=2-\frac{3}{n}$. 2) $x_n=\frac{3}{n-4,5}$. 3) $x_n=(-1)^{n-1}\left(4+\frac{7}{n}\right)$.

4) $x_n=(-1)^n n$. 5) $x_n=\frac{(-1)^n}{n}+\frac{1+(-1)^n}{2}$.

6) $x_n=-n[4+(-1)^n]$. 7) $x_n=3^{(-1)^n}\cdot n$. 8) $x_n=5^{(-1)^n\cdot n}$.

9) $x_n=2^{(-1)^n-n}\cdot n$. 10) $x_n=\sqrt[n]{25^{(-1)^n}+5}$.

11) $x_n=(-n)^{\sin\frac{n\pi}{2}}$. 12) $x_n=\cos n\pi$.

6.219. Quyida berilgan $\{x_n\}$ ketma-ketliklarning quyi va yuqori limitlarini toping.

1) $x_n=1+\cos\frac{n\pi}{2}$. 2) $x_n=3\left(1-\frac{1}{n}\right)+2\cdot(-1)^n$.

3) $x_n=\left(\sin\frac{n\pi}{2}\right)^n$. 4) $x_n=\frac{n^4}{1+n^3}\cos\frac{2n\pi}{3}$.

5) $x_n=\frac{n}{n+1}\sin^2\frac{n\pi}{2}$. 6) $x_n=\frac{\left(1-(-1)^n\right)\cdot 4^n+1}{4^n+5}$.

7) $x_n=\left(1+\frac{1}{n}\right)^n(-1)^n+\sin\frac{n\pi}{4}$; 8) $x_n=\cos^n\frac{2n\pi}{3}$.

6.220. Agar $\{x_n\}$ ketma-ketlik $x_1=0,\quad x_{2k}=\frac{x_{2k-1}}{2},\quad x_{2k+1}=1+x_{2k}$, $k\in N$ rekurrent formula bilan berilgan bo'lsa, $\varliminf_{n\to\infty} x_n$ va $\varlimsup_{n\to\infty} x_n$ larni toping.

6.221. Quyidagi tengsizliklarni isbotlang:

a) $\varliminf_{n\to\infty} x_n+\varliminf_{n\to\infty} y_n\le\varliminf_{n\to\infty}(x_n+y_n)\le\varliminf_{n\to\infty} x_n+\varlimsup_{n\to\infty} y_n$;

b) $\varliminf_{n\to\infty} x_n+\varlimsup_{n\to\infty} y_n\le\varlimsup_{n\to\infty}(x_n+y_n)\le\varlimsup_{n\to\infty} x_n+\varlimsup_{n\to\infty} y_n$;

c) $\varliminf_{n\to\infty} x_n\cdot\varliminf_{n\to\infty} y_n\le\varliminf_{n\to\infty}(x_n\cdot y_n)\le\varliminf_{n\to\infty} x_n\cdot\varlimsup_{n\to\infty} y_n$ $(x_n\ge 0$ va $y_n\ge 0;\ n=1,2,3,\ldots)$;

d) $\varliminf_{n\to\infty} x_n\cdot\varlimsup_{n\to\infty} y_n\le\varlimsup_{n\to\infty}(x_n\cdot y_n)\le\varlimsup_{n\to\infty} x_n\cdot\varlimsup_{n\to\infty} y_n$.

f) agar x_n va y_n ketma-ketliklar: 1) yuqoridan chegaralangan bo'lsa, u holda,

$\overline{\lim\limits_{n\to\infty}}(x_n+y_n)\le\overline{\lim\limits_{n\to\infty}}x_n+\overline{\lim\limits_{n\to\infty}}y_n$;

2) quyidan chegaralangan bo'lsa, u holda

$\underline{\lim\limits_{n\to\infty}}(x_n+y_n)\ge\underline{\lim\limits_{n\to\infty}}x_n+\underline{\lim\limits_{n\to\infty}}y_n$.

g) agar $x_n>0,\ n\in N$ bo'lsa, u holda

1) $\underline{\lim\limits_{n\to\infty}}\frac{x_{n+1}}{x_n}\le\underline{\lim\limits_{n\to\infty}}\sqrt[n]{x_n}$;   2) $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{x_n}\le\overline{\lim\limits_{n\to\infty}}\frac{x_{n+1}}{x_n}$.

6.222. Quyida berilgan $\{x_n\}$ ketma-ketliklarning qismiy limitlar to'plamini toping:

1) $\left\{\frac{1}{2},\frac{1}{2},\frac{2}{2},\frac{3}{2},\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4},\frac{5}{4},\ldots,\frac{1}{2^n},\ldots,\frac{2^n+1}{2^n},\ldots\right\}$;

2) $\left\{1,\frac{1}{2},1+\frac{1}{2},\frac{1}{3},1+\frac{1}{3},\frac{1}{2}+\frac{1}{3},\frac{1}{4},1+\frac{1}{4},\frac{1}{2}+\frac{1}{4},\frac{1}{3}+\frac{1}{4},\frac{1}{5},\ldots,\right.$

$\left.\ldots,\frac{1}{n},1+\frac{1}{n},\frac{1}{2}+\frac{1}{n},\ldots,\frac{1}{n-1}+\frac{1}{n},\frac{1}{n+1},\ldots\right\}$;

3) $\left\{1,\frac{1}{2},\frac{2}{2},\frac{3}{2},\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4},\ldots,\frac{9}{4},\ldots,\frac{1}{2^n},\frac{2}{2^n},\ldots,\frac{3^n}{2^n},\ldots\right\}$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

6.1. $1,\ \frac{8}{3},\ \frac{3}{2},\ \frac{12}{5},\ \frac{5}{3},\ldots$ **6.2.** $6,\ 0,\ 18,\ 0,\ 30,\ \ldots$. **6.3.** $7,\ \frac{11}{4},\ \frac{15}{7},\ \frac{19}{10},\ldots$ **6.4.** $\frac{2\pi}{3},\ \frac{7\pi}{3},\ \frac{8\pi}{3},\ \frac{13\pi}{3},\ \frac{14\pi}{3},\ldots$ **6.5.** $0;\ -1;\ 0;\ 1;\ 0;\ \ldots$. **6.6.** $\frac{1}{2};\ 2,\ \frac{1}{2},\ 2,\ \frac{1}{2},\ldots$ **6.7.** $1,1,0,-1,-1,\ldots$ **6.8.** $1,\ 2,\ 2,\ 2,\ 3,\ \ldots$. **6.9.** $2;\ 2;\ \frac{4}{3};\ \frac{4}{9},\ldots$ **6.10.** $1;\ 0;\ -\frac{1}{3};\ 0;\ \frac{1}{5};\ldots$ **6.11.** $x_n=\frac{(-1)^n}{n+1}$. **6.12.** $x_n=2(1+(-1)^n)$. **6.13.** $x_n=n\cos\frac{\pi(n-1)}{2}$. **6.14.** $x_n=\frac{2n}{2n-1}$. **6.15.** $x_n=(-1)^n\frac{2n+1}{2n-1}$. **6.16.** $x_n=\sin\frac{(n-1)\pi}{4}$. **6.17.** $x_n=\frac{n^2+1}{5n-2}$. **6.18.** $x_n=\frac{n+1}{4}[1-(-1)^n]+\frac{1}{n+2}[1+(-1)^n]$. **6.19.** $2;\ -\frac{16}{5};\ \frac{(-2)^{n+1}}{2(n+1)}$. **6.20.** $\frac{1}{9};\ -\frac{1}{11};\ \frac{(-1)^{n+1}}{2n+3}$. **6.21.** $\frac{1}{11};\ \frac{-1}{14};\ \frac{(-1)^{n+1}}{3n+2}$.

6.22. $\frac{7}{8}$; $\frac{9}{10}$; $\frac{2n+1}{2(n+1)}$. 6.23. 0; -10^5; $(-10)^{n+1}\cos\frac{\pi n}{2}$. 6.24. 4; $\frac{1}{5}$; $(n+1)^{(-1)^{n+1}}$. 6.26. 0; $\frac{5}{12}$; $\frac{n+1}{n+2}\sin^2\frac{(n+1)\pi}{4}$. 6.27. 1; −1; $(-1)^{(n+1)^2}$. 6.28. −2; 1; 4; 7; 9. 6.29. −3; 2; 7; 12; 17. 6.30. −3; −5; −7; −9; −11. 6.31. −1; 3; 7; 11; 15. 6.32. $x_n=\frac{n}{n+1}$. 6.33. $x_n=\frac{3\cdot 2^{n-1}-2}{3\cdot 2^{n-1}-1}$. 6.34. $x_n=2-2^{2-n}$. 6.35. $x_n=\frac{(a+b)2^{n-1}+(b-2a)(-1)^n}{3}$. 6.36. $x_n=\frac{6n-1-8(-0,5)^n}{9}$. 6.37. $x_n=\frac{2^{n+1}-2\cdot(-1)^n}{3}-1$. 6.38. Chegaralangan. 6.39. Chegaralangan. 6.40. Chegaralangan. 6.41. Chegaralangan. 6.42. Chegaralanmagan. 6.43. Yuqoridan chegaralangan, quyidan chegaralanmagan. 6.44. Quyidan chegaralangan, yuqoridan chegaralanmagan. 6.45. Yuqoridan chegaralangan quyidan chegaralanmagan. 6.46. Chegaralangan. 6.47. Chegaralanmagan. 6.48. Chegaralanmagan. 6.49. Chegaralanmagan. 6.50. Chegaralanmagan. 6.51. Chegaralangan. 6.52. Chegaralangan. 6.53. Quyidan chegaralangan, yuqoridan chegaralanmagan. 6.94. Eng kattasi $x_3=6$. 6.95. Eng kattasi $x_2=e$. 6.96. Eng kattasi $x_9=\frac{1}{6}$. 6.97. Eng kichigi $x_5=-20$. 6.98. Eng kichigi $x_3=-\frac{9}{8}$. 6.99. Eng kattasi $x_1=1$. 6.129. $a=\frac{1}{3}$, $n_0(\varepsilon)=3$. 6.130. $a=2$, $n_0(\varepsilon)=1000$ 6.131. $a=0$, $n_0(\varepsilon)=999$. 6.132. $a=1$, $n_0(\varepsilon)=10$. 6.144. $\frac{1}{2}$. 6.145. $\frac{1}{3}$. 6.146. $\frac{1}{4}$. 6.147. $\frac{1}{12}$. 6.148. $\frac{1}{18}$. 6.149. ∞. 6.150. $\frac{1}{\sqrt{2}}$. 6.151. $\frac{1}{d\cdot a_1}$. 6.152. $\frac{1}{2d\cdot a_1\cdot a_2}$. 6.153. $\frac{1}{3d\cdot a_1\cdot a_2\cdot a_3}$. 6.154. $\frac{1}{\sqrt{d}}$. 6.155. $\frac{1}{4}$. 6.156. 2. 6.157. 0. 6.158. 0. 6.159. 1. 6.160. $\frac{2}{3}$. 6.161. −1. 6.162. 19800. 6.163. $\frac{1}{2}$. 6.164. 1. 6.165. 0. 6.166. 1, $a>b$ bo'lganda, −1, $a\le b$ bo'lganda. 6.167. 1, $|a|>1$ bo'lganda, $\frac{1}{2}$, $a=1$ bo'lganda, 0, $|a|>1$ bo'lganda. 6.168. 0. 6.169. $\frac{3}{4}$. 6.170. 0. 6.171. 2. 6.172. $\frac{1}{3}$. 6.173. $\frac{1}{5}$. 6.174. $-\frac{1}{4}$. 6.175. $\frac{a_1+a_2}{2}$. 6.176. $\frac{a_1+a_2+a_3+a_4+a_5}{5}$. 6.177. $\frac{a_1+a_2+\ldots+a_p}{p}$. 6.178. $a=1$, $\lim\limits_{n\to\infty}x_n=\frac{b}{2}$. 6.180. 0. 6.181. 14. 6.182. −∞. 6.183. 0. 6.184. $\sqrt{\frac{2}{3}}$. 6.185. 0. 6.186. 1. 6.187. −1. 6.188. $\frac{1}{3}$. 6.189. −3. 6.190. $\frac{1}{2}$. 6.191. $\frac{4}{3}$. 6.192. 3. 6.193. $\frac{1}{3}$.

6.194. $\frac{1}{3}$. 6.195. -1. 6.196. $\frac{1}{4}$. 6.197. $\frac{1}{2}$. 6.198. $\frac{1}{3}$. 6.199. 0. 6.200. Masalan: 1) $x_n = n+\frac{1}{n}$, $y_n = -n+\frac{2n}{n+3}$; 2) $x_n=(-1)^n+2$, $y_n=(-1)^n-2$; 3) $x_n=n^2+1$, $y_n=n^2+3$. 6.203. Masalan: 1) $\{x_n\}=\left\{\frac{(-1)^n}{n}\right\}$, $\{y_n\}=\{(-1)^n\}$ bo'lsa, $\{x_n y_n\}$ ketma-ketlik yaqinlashuvchi; 2) $\{x_n\}=\left\{\frac{(-1)^n}{n^2}\right\}$, $\{y_n\}=\{n^2\}$ bo'lsa, $\{x_n y_n\}$ ketma-ketlik uzoqlashuvchi.

6.204. 1) Shart emas; 3) bo'lishi mumkin. 6.205. e^{pq}. 6.206. e. 6.207. e^{-3}. 6.208. e^{-1}. 6.209. e. 6.210. 1. 6.217. 1) 3; 2) 4; 3) 1; 4) -2; 5) 1; 6) 2. 6.218. 1) $\inf x_n=-1$, $\sup x_n=2$, $\underline{\lim}_{n\to\infty} x_n=2$, $\overline{\lim}_{n\to\infty} x_n=2$; 2) $\inf x_n=-6$, $\sup x_n=6$, $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=0$; 3) $\inf x_n=-7{,}5$, $\sup x_n=11$, $\underline{\lim}_{n\to\infty} x_n=-4$, $\overline{\lim}_{n\to\infty} x_n=1$; 4) $\inf x_n=-\infty$, $\sup x_n=+\infty$, $\underline{\lim}_{n\to\infty} x_n=-\infty$, $\overline{\lim}_{n\to\infty} x_n=+\infty$; 5) $\inf x_n=-1$, $\sup x_n=1{,}5$, $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=1$; 6) $\inf x_n=-\infty$, $\sup x_n=-3$, $\underline{\lim}_{n\to\infty} x_n=-\infty$, $\overline{\lim}_{n\to\infty} x_n=-\infty$; 7) $\inf x_n=\frac{1}{3}$, $\sup x_n=\infty$, $\underline{\lim}_{n\to\infty} x_n=\infty$, $\overline{\lim}_{n\to\infty} x_n=\infty$; 8) $\inf x_n=0$, $\sup x_n=+\infty$, $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=\infty$; 9) $\inf x_n=0$, $\sup x_n=+\infty$, $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=+\infty$; 10) $\inf x_n=1$, $\sup x_n=\sqrt{30}$, $\underline{\lim}_{n\to\infty} x_n=1$, $\overline{\lim}_{n\to\infty} x_n=1$; 11) $\inf x_n=-\infty$, $\sup x_n=1$, $\underline{\lim}_{n\to\infty} x_n=-\infty$, $\overline{\lim}_{n\to\infty} x_n=1$; 12) $\inf x_n=-1$, $\sup x_n=1$, $\underline{\lim}_{n\to\infty} x_n=-1$, $\overline{\lim}_{n\to\infty} x_n=1$.

6.219. 1) $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=2$; 2) $\underline{\lim}_{n\to\infty} x_n=1$, $\overline{\lim}_{n\to\infty} x_n=5$; 3) $\underline{\lim}_{n\to\infty} x_n=-1$, $\overline{\lim}_{n\to\infty} x_n=1$; 4) $\underline{\lim}_{n\to\infty} x_n=-\frac{1}{2}$, $\overline{\lim}_{n\to\infty} x_n=1$; 5) $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=1$; 6) $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=2$; 7) $\underline{\lim}_{n\to\infty} x_n=-e-\frac{\sqrt{2}}{2}$, $\overline{\lim}_{n\to\infty} x_n=e+1$; 8) $\underline{\lim}_{n\to\infty} x_n=0$, $\overline{\lim}_{n\to\infty} x_n=1$. 6.220. $\underline{\lim}_{n\to\infty} x_n=1$, $\overline{\lim}_{n\to\infty} x_n=2$.

6.222. 1) $[0;\ 1]$; 2) $1;\ \frac{1}{2};\ \frac{1}{3};\ \ldots;\ 0$; 3) $[0;+\infty)$ va $+\infty$.

7-§. FUNKSIYANING LIMITI

7.1. Ixtiyoriy argumentli funksiyaning limiti. $f(x)$ funksiya $X(X\subset R)$ — biror haqiqiy sonlar to'plamida aniqlangan bo'lsin. a nuqta biror son yoki $-\infty$, $+\infty$, ∞ simvollarning biri bo'lsin.

7.1-ta'rif. Agar $U_\varepsilon(a)(\varepsilon>0)$ atrofda X to'plamning a dan farqli hech bo'lmaganda bitta nuqtasi joylashsa, a nuqta X to'plamning *limit nuqtasi (quyuqlanish nuqtasi)* deb ataladi.

Misollar: 1) ushbu $X=[0;\ 3]=\{x: x\in R,\ 0\le x\le 3\}$ to'plamning har bir nuqtasi uning limit nuqtasi bo'ladi;

2) $X=N=\{1, 2, 3, ..., n, ...\}$ to'plam limit nuqtaga ega emas;

3) $X=(0,2)=\{x: x\in R,\ 0<x<2\}$ to'plamning har bir nuqtasi shu to'plamning limit nuqtasi bo'ladi va yana 0 va 2 nuqtalar ham (0; 2) to'plamning limit nuqtalari bo'ladi.

Demak, yuqoridagi misollardan ko'rinadiki, to'plamning limit nuqtasi to'plamga qarashli bo'lishi ham, qarashli bo'lmasligi ham mumkin ekan.

Limit nuqta quyidagi xossalarga ega:

1°. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, a nuqtaning ixtiyoriy atrofida X to'plamning cheksiz ko'p nuqtalari (elementlari) joylashgan bo'ladi.

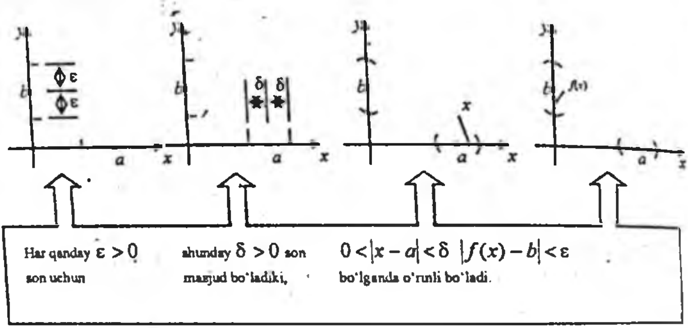
2°. Agar a nuqta X to'plamning limit nuqtasi bo'lsa, X to'plam nuqtalaridan (elementlaridan) har doim a ga intiluvchi $\{x_n\}$ $(x_n\in X,\ x_n\ne a;\ n=1,2,...)$ ketma-ketlik tuzish mumkin.

7.2-ta'rif (Geyne ta'rifi). Agar X to'plamning elementlaridan tuzilgan va a ga intiluvchi har qanday $\{x_n\}$ $(x_n\in X,\ x_n\ne a;\ n=1,2,...)$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b (chekli yoki cheksiz) limitga intilsa, shu b songa $f(x)$ funksiyaning *a nuqtadagi (yoki $x\to a$ dagi) limiti* deb ataladi va u $\lim\limits_{x\to a} f(x)=b$ yoki $x\to a$ da $f(x)\to b$ kabi belgilanadi.

7.3-ta'rif (Koshi ta'rifi). Agar istalgan $\forall\varepsilon>0$ son uchun shunday $\delta(\varepsilon)>0$ son topilsaki, argument x ning $0<|x-a|<\delta$, $x\in X$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning *a nuqtadagi ($x\to a$ dagi) limiti* deb ataladi.

Ma'lumki, 7.2- va 7.3-ta'riflar o'zaro ekvivalent ta'riflardir.

$f(x)$ funksiyaning $x=a$ nuqtadagi limiti b bo'lganda, uning Koshi ta'rifini quyidagi chizmadagi kabi yozish mumkin:



7.1-chizma.

$(+\infty)$ limit nuqta bo'lgan holda bu limit nuqtaning atrofi $x > C$ $(C > 0)$ nurdan, $(-\infty)$ limit nuqta bo'lgan holda esa, bu limit nuqtaning atrofi $x < -C$ nurdan iborat bo'ladi. (∞) limit nuqta bo'lganda, uning atrofi: $\{x < -C\} \cup \{x > C\}$ nurlar yig'indisidan iborat bo'ladi.

Limit nuqta a xosmas, ya'ni $-\infty$, $+\infty$, ∞ bo'lgan hollarda funksiya limitining Koshi ta'riflari, mos ravishda, qisqacha quyidagicha ifodalanadi:

Agar

$$\forall \varepsilon > 0 \quad \exists C = C(\varepsilon) > 0: \ \forall x, \ x > C, \ x \in X, \ |f(x) - b| < \varepsilon;$$

$$\forall x, \ x < -C, \ x \in X, \ |f(x) - b| < \varepsilon;$$

$$\forall x, \ |x| > C, \ x \in X, \ |f(x) - b| < \varepsilon$$

va mos ravishda $\lim\limits_{x \to +\infty} f(x) = b$; $\lim\limits_{x \to -\infty} f(x) = b$; $\lim\limits_{x \to \infty} f(x) = b$ kabi yoziladi.

7.4-ta'rif ($x \to \infty$ **da funksiya limitining Geyne ta'rifi**). Agar X to'plamning nuqtalaridan (elementlaridan) tuzilgan har qanday cheksiz katta $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b songa intilsa, shu b songa $f(x)$ funksiyaning $x \to \infty$ *dagi limiti* deyiladi va u $\lim\limits_{x \to \infty} f(x) = b$ kabi belgilanadi.

7.5-ta'rif $(x \to +\infty)(x \to -\infty)$ **da funksiya limitining Geyne ta'rifi).** Agar X to'plamning musbat (manfiy) elementlaridan tuzilgan har qanday cheksiz katta $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b songa intilsa, shu b ga $f(x)$ funksiyaning $x \to +\infty$ *($x \to -\infty$) dagi limiti* deyiladi va u $\lim_{x\to+\infty} f(x) = b$ $(\lim_{x\to-\infty} f(x) = b)$ kabi belgilanadi.

7.6-ta'rif (funksiyaning chekli nuqtadagi cheksiz limitlari). Agar istalgan $\forall E > 0$ son uchun shunday $\exists\delta = \delta(E) > 0$ mavjud bo'lib, $\forall x \in \dot{U}_\delta(a) \cap X$ uchun $|f(x)| > E$ tengsizlik o'rinli, ya'ni $f(x) \in U_E(\infty) = \{x : x \in R,\ |x| > E\}$ bo'lsa, $f(x)$ funksiya $x \to a$ $(x = a)$*da* ∞ *limitga ega* deyiladi va $\lim_{x\to a} f(x) = \infty$ kabi yoziladi.

7.7-ta'rif. Agar $\forall E > 0$ son uchun shunday $\exists\delta = \delta(E) > 0$: $\forall x \in \dot{U}_\delta(a) \cap X$ uchun $f(x) > E(f(x) < -E)$ tengsizlik o'rinli, ya'ni $f(x) \in U_E(+\infty) = (E;\ +\infty)$ $f(x) \in U_E(-\infty) = (-\infty; -E)$ bo'lsa, $f(x)$ funksiya $x \to a$ $(x = a)$ da $+\infty(-\infty)$ *limitga ega* deyiladi va $\lim_{x\to a} f(x) = +\infty$ $(\lim_{x\to a} f(x) = -\infty)$ kabi yoziladi.

$X(X \subset R)$ to'plam berilgan bo'lib, a nuqta uning o'ng (chap) limit nuqtasi bo'lsin. Shu to'plamda $f(x)$ funksiya aniqlangan.

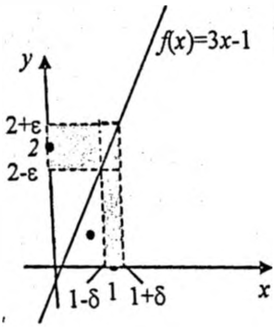
7.8-ta'rif (Geyne ta'rifi). Agar X to'plamning nuqtalaridan (elementlaridan) tuzilgan va har bir hadi a dan katta (kichik) bo'lib, a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt yagona b songa intilsa, shu b songa $f(x)$ funksiyaning *a nuqtadagi o'ng (chap)* limiti deb ataladi.

7.9-ta'rif (Koshi ta'rifi). Agar istalgan $\forall\varepsilon > 0$ son uchun shunday $\delta = \delta(\varepsilon) > 0$ son topilsaki, argument x ning $U_\delta^+(a)$ $(U_\delta^-(a))$ atrofdagi barcha qiymatlarida $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning a nuqtadagi *o'ng (chap) limiti* deb ataladi va u, mos ravishda, quyidagicha yoziladi:

$$\lim_{x\to a+0} f(x) = b \quad \text{yoki} \quad f(a+0) = b \quad (\lim_{x\to a-0} f(x) = b \quad \text{yoki} \quad f(a-0) = b).$$

Ma'lumki, 7.8- va 7.9-ta'riflar o'zaro ekvivalent ta'riflardir.

7.1-eslatma. Funksiyaning biror nuqtada bir tomonli limitlari mavjud bo'lishidan uning shu nuqtada limitga ega bo'lishi har doim ham kelib chiqavermaydi.



7.2-chizma.

7.1-teorema. *$f(x)$ funksiyaning a nuqtada b limitga ega bo'lishi uchun uning shu nuqtada o'ng va chap limitlari mavjud bo'lib,*

$$f(a+0)=f(a-0)=b$$

tengliklarning o'rinli bo'lishi zarur va yetarli.

7.1-misol. $\lim\limits_{x\to 1}(3x-1)=2$ bo'lishini Koshi ta'rifi bo'yicha ko'rsating (7.2-chizma) va ε=0,03 uchun δ ni toping.

Yechilishi. $\forall\varepsilon>0$ berilgan bo'lsin. Biz avvalo a) berilgan ε ga ko'ra δ ni topish bilan shug'ullanamiz. Biz shunday δ ni izlashimiz kerakki, x ning $0<|x-1|<\delta$ tengsizlikni qanoatlantiradigan barcha qiymatlarida

$$|(3x-1)-2|<\varepsilon$$

tengsizlik o'rinli bo'lsin. Buning uchun avvalo $|(3x-1)-2|$ va $|x-1|$ ifodalar orasidagi bog'lanishni o'rnatish zarur, bu bog'lanishni topish uchun esa, bu ifodalardan birinchisining shaklini o'zgartiramiz:

$$|(3x-1)-2|=|3x-3|=3|x-1|. \qquad (7.1)$$

Endi $|(3x-1)-2|$ ni berilgan ε dan kichik qilish uchun, biz $3|x-1|<\varepsilon$ tengsizlikka ega bo'lishimiz lozim. Bu tengsizlikni $|x-1|<\frac{\varepsilon}{3}$ ko'rinishda yozamiz. Bundan, $\delta=\frac{\varepsilon}{3}$ deb olishimiz zarurligi kelib chiqadi.

b) topilgan δ ning «*ishlash*»*ini* ko'rsatish. Agar $0<|x-1|<\frac{1}{3}\varepsilon$ bo'lsa, u holda $3|x-1|<\varepsilon$ munosabatga ega bo'lamiz va (7.1) dan

$$|(3x-1)-2|<\varepsilon$$

tengsizlik o'rinli bo'lishiga ishonch hosil qilamiz. Endi $\varepsilon = 0{,}03$ desak, $\delta = \frac{0{,}03}{3} = 0{,}01$. $0 < |x-1| < 0{,}01$ bo'lganda, $|(3x-1)-2| < 0{,}03$ tengsizlik bajariladi.

7.2-misol. $\lim\limits_{x\to-2} x^2 = 4$ bo'lishini Koshi ta'rifi bo'yicha ko'rsating.

Yechilishi. a) δ ni topish. Faraz qilaylik, ixtiyoriy $\varepsilon>0$ berilgan bo'lsin. Biz shunday $\delta = \delta(\varepsilon) > 0$ sonni izlaymizki, x ning $0 < |x-(-2)| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|x^2-4| < \varepsilon$ tengsizlik o'rinli bo'lsin. Buning uchun avvalo $|x^2-4|$ va $|x-(-2)|$ ifodalar orasidagi bog'lanishni topish zarur. Bu bog'lanishni topish uchun esa, ularning ikkalasini ham soddalashtiramiz:

$$|x^2-4| = |x-2||x+2| \text{ va } |x-(-2)| = |x+2|.$$

$|x-2|$ ko'paytuvchi sonlar o'qida chegaralanmagan. Shuning uchun ko'paytuvchini sodda holda baholash uchun -2 nuqtani o'z ichida saqlaydigan biror oraliqni ajratamiz. Masalan, $a=-2$ nuqtaning $\delta=1$ atrofi $(-3;\ -1)$ni qaraylik. $\forall x \in (-3;-1)$ uchun $|x-2|<5$ tengsizlik o'rinli bo'ladi.

Shunday qilib,

$$|x^2-4| < 5|x+2| \qquad (7.2)$$

tengsizlik o'rinli bo'ladi. $a=-2$ nuqtaning δ atrofi bo'lgan $(-2-\delta;\ -2+\delta)$ oraliq $(-3;\ -1)$ atrofdan chiqib ketmasligi kerak, buning uchun $\delta = \min\left(1; \frac{\varepsilon}{5}\right)$ deb olish yetarli.

b) topilgan *δning «ishlash»ini* ko'rsatamiz. Agar $0 < |x-(-2) < \frac{\varepsilon}{5}$ bo'lsa, bundan $5|x-(-2)| < \varepsilon$ bo'lishi va (7.2) ga muvofiq, $|x^2-4| < \varepsilon$ tengsizlik kelib chiqadi.

Shunday qilib,

$$\lim_{x\to-2} x^2 = 4.$$

7.3-misol. $\lim\limits_{x \to 9} \sqrt{x} = 3$ bo'lishini Koshi ta'rifi bo'yicha ko'rsating.

Yechilishi. a) δ ni topish. $\varepsilon > 0$ son berilgan bo'lsin. Biz shunday $\delta = \delta(\varepsilon) > 0$ sonni izlaymizki, x ning $0 < |x - 9| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|\sqrt{x} - 3| < \varepsilon$ tengsizlik bajarilsin.

Dastlab, biz $\sqrt{x}$ ifoda ma'noga ega bo'lishi uchun $x \geq 0$ munosabatning bajarilishini talab qilamiz. Buni ta'minlash uchun biz $\delta \leq 9$ deb olishga majburmiz, aks holda $\begin{cases} x \geq 0 \\ -\delta + 9 < x \end{cases}$ tengsizlik yechimga ega bo'lmasdan qoladi.

$|\sqrt{x} - 3|$, $|x - 9|$ ifodalar orasidagi bog'lanishni topish uchun ularning ikkinchisining shaklini o'zgartiramiz:

$$x - 9 = (\sqrt{x} + 3)\ (\sqrt{x} - 3),$$
$$|x - 9| = |\sqrt{x} + 3|\ |\sqrt{x} - 3|. \tag{7.3}$$

Ravshanki, $|\sqrt{x} + 3| > 1$. U holda (7.3) dan

$$|\sqrt{x} - 3| < |x - 9| < \delta \tag{7.4}$$

tengsizlik kelib chiqadi. Oxirgi (7.4) tengsizlikda $\delta = \varepsilon$ deb olish yetarli, lekin, biz yuqoridagi $\delta \leq 9$ shartni e'tiborga olsak, u holda $\delta = \min\{9 : \varepsilon\}$ deb olishimiz yetarli.

b) *δ ning «ishlash»ini* ko'rsatish. Agar $0 < |x - 9| < \varepsilon$ bo'lsa, (7.4) tengsizlikka ko'ra, $|\sqrt{x} - 9| < \varepsilon$ tengsizlikning bajarilishi kelib chiqadi. Demak, funksiyaning Koshi ta'rifiga ko'ra limiti $\lim\limits_{x \to 9} \sqrt{x} = 3$ bo'ladi.

7.4-misol. $\lim\limits_{x \to \infty} \dfrac{x^2 \cos x}{x^3 - 300x^2 + 500} = 0$ ekanligini ko'rsating.

Yechilishi. a) C ni topish. $\forall \varepsilon > 0$ son berilgan bo'lsin. C ni shunday izlaymizki, x ning $x > C$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$\left|\frac{x^2\cos x}{x^3-300x^2+500}\right|<\varepsilon$$

tengsizlik bajarilishi kerak. Buning uchun, avvalo,

$$\left|\frac{x^2\cos x}{x^3-300x^2+500}\right|,\quad x>C$$

ifodalar o'rtasidagi bog'lanishni topishimiz lozim. Birinchi ifodaning shaklini quyidagicha o'zgartiramiz: $+\infty$ nuqtaning biror atrofini ($x>C$ nurni) ajratamiz, masalan, $x>600$ tengsizlikni qanoatlantiruvchi barcha x lar uchun

$$x^3-300x^2+500>x^3-300x^2=x^2(x-300)>\frac{x^3}{2}$$

tengsizlik o'rinli.

Demak, $\left|\frac{x^2\cos x}{x^3-300x^2+500}\right|<\frac{x^2}{\frac{x^3}{2}}=\frac{2}{x}$.

Shunday qilib, $C=\max\left\{600;\ \frac{2}{\varepsilon}\right\}$ deb olinsa, $x>C$ uchun $\left|\frac{x^2\cos x}{x^3-300x^2+500}\right|<\varepsilon$ tengsizlik bajariladi.

b) *C ning «ishlash»ini* ko'rsatish. 1) $600<\frac{\varepsilon}{2}$, 2) $600>\frac{\varepsilon}{2}$ hollarni qaraymiz. 2) $600>\frac{\varepsilon}{2}$ bo'lsin. U holda, $C=600$ deb olsak, $x>600$ tengsizlikni qanoatlantiruvchi x larni qaraymiz. Bu tengsizlikdan $x>600>\frac{2}{\varepsilon}$, $\varepsilon>\frac{2}{x}$ tengsizlikni hosil qilamiz. $\cos x\le 1$ ekanligini hamda $x\ne 0$ ligini e'tiborga olgan holda oxirgi tengsizlikdan

$$\varepsilon>\frac{2}{x}>\frac{2\cos x\cdot x^2}{x\cdot x^2}=\frac{x^2\cos x}{\frac{x^3}{2}}$$

ga ega bo‘lamiz. $x>600$ ni qanoatlantiruvchi barcha x lar uchun $\frac{x^3}{2}<x^3-300x^2+500$ tengsizlik o‘rinli ekanligini hisobga olgan holda, oxirgi tengsizlikdan

$$\left|\frac{x^2\cos x}{x^3-300x^2+500}\right|<\varepsilon$$

tengsizlikni hosil qilamiz.

Demak, $\lim\limits_{x\to+\infty}\frac{x^2\cos x}{x^3-300x^2+500}=0$ bo‘lar ekan. 1) hol ham xuddi shunday ko‘rsatiladi.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(x^2*cos(x)/(x^3-300*x^2+500), x=infinity);
```

7.5-misol. Ushbu

$$f(x)=\sin\frac{1}{x}\quad(x\neq 0)$$

funksiyaning $x\to 0$ da limitga ega emasligini ko‘rsating.

Yechilishi. Nol nuqtaning atrofidan nolga intiluvchi va noldan farqli ikkita har xil

$$\{x_n'\}=\left\{\frac{1}{n\pi}\right\},\ \{x_n''\}=\left\{\frac{2}{(4n+1)\pi}\right\}$$

ketma-ketliklarni olaylik. U holda, ularga mos ketma-ketliklar:

$$f(x_n')=\sin\frac{1}{x_n'}=\sin n\pi=0,$$

$$f(x_n'')=\sin\frac{1}{x_n''}=\sin\frac{(4n+1)\pi}{2}=\sin\left(2n\pi+\frac{\pi}{2}\right)=1.$$

bo‘lib, $\lim\limits_{x\to\infty}f(x_n')=0$, $\lim\limits_{x\to\infty}f(x_n'')=1$ bo‘ladi. Bu esa $f(x)=\sin\frac{1}{x}$ funksiyaning $x=0$ nuqtada limiti mavjud emasligini isbotlaydi.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(sin(((1)/x)), x=0);
-1..1.
```

7.6-misol. $\lim\limits_{x\to+\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)=0$ ekanligini isbotlang.

Yechilishi. a) $C>0$ ni topish. $\forall\varepsilon>0$ son berilgan bo'lsin. Shunday $C=C(\varepsilon)$ sonni izlaymizki, $x>C$ tengsizlikni qanoatlantiruvchi x ning barcha qiymatlarida

$$\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|<\varepsilon \tag{7.5}$$

tengsizlik o'rinli bo'lsin. Buning uchun dastlab

$$\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|,\quad x>C$$

ifodalar orasidagi bog'lanishni topamiz. Yuqoridagi ifodalarning birinchisining shaklini almashtiramiz:

$$\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|=\left|2\cdot\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cdot\cos\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|\le$$

$$\le 2\cdot\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|=2\left|\sin\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}\right|<\frac{1}{\sqrt{x+1}+\sqrt{x}}<\frac{1}{2\sqrt{x}}.$$

Bunda, $C(\varepsilon)=\frac{1}{4\varepsilon^2}$ deb olinsa, (7.5) tengsizlik $\forall x>C$ uchun bajariladi. $C(\varepsilon)=\frac{1}{4\varepsilon^2}$ ni tanlaymiz.

b) *C ning «ishlash»ini* ko'rsatamiz. $x>\frac{1}{4\varepsilon^2}$ bo'lsin. Bundan $\varepsilon>\frac{1}{2\sqrt{x}}$. Ko'rsatilgan x ning barcha qiymatlarida

$$2\sqrt{x}<\sqrt{x+1}+\sqrt{x},\quad \left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|<\frac{\sqrt{x+1}-\sqrt{x}}{2},\quad \left|\cos\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|<1$$

tengsizliklarni e'tiborga olgan holda, keyingi tengsizlikdan

$$\varepsilon>\frac{1}{2\sqrt{x}}>\frac{1}{\sqrt{x+1}+\sqrt{x}}=2\,\frac{\sqrt{x+1}-\sqrt{x}}{2}>2\cdot\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cos\frac{\sqrt{x+1}-\sqrt{x}}{2}=$$

$$=\sin\sqrt{x+1}-\sin\sqrt{x}$$

ni topamiz, ya'ni $\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right|<\varepsilon$.

Shunday qilib, talab qilingan limitning 0 ga tengligi isbot bo'ldi.

7.7-misol. $\lim\limits_{x\to a}\frac{1}{(a-x)^2}=+\infty$ ekanligini Koshi ta'rifi bo'yicha ko'rsating va ushbu

E	10	100	1000	10000	....
δ					.....

jadvalni to'ldiring.

Yechilishi. a) δ ni topish. $\forall E>0$ son berligan bo'lsin. Biz shunday $\delta=\delta(E)>0$ sonni izlashimiz kerakki, x ning $0<|a-x|<\delta$ tengsizlikni qanoatlantiruvchi qiymatlarida $\frac{1}{(a-x)^2}>E$ tengsizlik bajarilsin. Avvalo, $\frac{1}{(a-x)^2}$, $|a-x|$ ifodalar orasidagi bog'lanishni topish kerak. Buning uchun $\frac{1}{(a-x)^2}>E$ ning shaklini o'zgartiramiz: $\frac{1}{E}>(a-x)^2$, bundan $|a-x|<\frac{1}{\sqrt{E}}$, bunda $\delta=\frac{1}{\sqrt{E}}$ deb olish yetarli.

b) δ ning «ishlash»ini ko'rsatish. $\delta=\frac{1}{\sqrt{E}}$ bo'lsin. Agar $0<|a-x|<\frac{1}{\sqrt{E}}$ bo'lsa, bundan $\sqrt{E}<\frac{1}{|a-x|}$ yoki $E<\frac{1}{(a-x)^2}$ tengsizilk kelib chiqadi. 7.7-ta'rifga ko'ra, bu tengsizlikdan $\lim\limits_{x\to a}\frac{1}{(a-x)^2}=+\infty$ ekanligi kelib chiqadi. Endi topilgan δ ga ko'ra jadvalni to'ldiramiz:

E	10	100	1000	10000	...
δ	$1/\sqrt{10}$	$1/10$	$1/10\sqrt{10}$	$1/100$	...

7.8-misol. Ushbu

$$f(x)=\begin{cases}x+1, & x\le 0\\ x^3, & x>0\end{cases}$$

funksiyaning $x\to 0$ da limitga ega emasligini ko'rsating.

Yechilishi. Berilgan funksiyaning o'ng va chap limitlarini topamiz:

1. *a) O'ng limit. δ ni topish.* $\forall\varepsilon>0$ son berilgan bo'lsin. δ ni shunday izlaymizki, $\forall x\in U_\delta^+(0)=(0,\delta)$ uchun $x^3<\varepsilon$ tengsizlik bajarilsin. Bundan $x<\sqrt[3]{\varepsilon}$, bunda $\delta=\sqrt[3]{\varepsilon}$ deb olish yetarli.

b) δ ning «ishlash»ini ko'rsatish. Agar $x\in U_{\sqrt[3]{\varepsilon}}^+(0)=(0,\sqrt[3]{\varepsilon})$ bo'lsa, $x\in U_{\sqrt[3]{\varepsilon}}^+(0)$ uchun $x<\sqrt[3]{\varepsilon}$ tengsizlik o'rinli, bundan esa $x^3<\varepsilon$ bo'lishi kelib chiqadi. Bu tengsizlikdan 7.9-ta'rifga ko'ra $\lim\limits_{x\to0+0}x^3=0$ ekanligi kelib chiqadi.

2. *Chap limit. a) δ ni topish.* $\forall\varepsilon>0$ son berilgan bo'lsin. Bu holda δ ni shunday izlaymizki, $\forall x\in U_\delta^-(0)=(-\delta,0)$ uchun $|f(x)-f(1-0)|=|(x+1)-1|<\varepsilon$ tengsizlik o'rinli bo'lsin. Bunda $\delta=\varepsilon$ deb olish yetarli.

b) *δ ning «ishlash»ini* ko'rsatish. Agar $\forall x\in U_\delta^-(0)$ uchun $|x|<\delta=\varepsilon$ bo'lsa, bundan $|(x+1)-1|<\varepsilon$ tengsizlik kelib chiqadi. 7.9-ta'rifga ko'ra

$$\lim_{x\to0-0}(x+1)=1.$$

Demak, berilgan funksiyaning $x=0$ dagi o'ng va chap limitlari bir-biriga teng bo'lmaganligi uchun funksiya $x\to0$ da limitga ega emas.

7.9-misol. Ushbu

$$f(x)=\begin{cases}1+x^3, & x<1\\ 3, & x=1\\ 4-2x, & x>1\end{cases}$$

funksiyaning $\lim\limits_{x\to1\pm0}f(x)=2$ ekanligini ko'rsating.

Yechilishi. Berilgan funksiyaning o'ng va chap limitlarini topamiz:

1. *O'ng limit. a) δ ni topish.* $\forall\varepsilon>0$ berilgan bo'lsin. δ ni shunday izlaymizki, $\forall x\in U_\delta^+(1)=(1;\,1+\delta)$ uchun $|(4-2x)-2|<\varepsilon$

tengsizlik bajarilsin. Buning uchun $|(4-2x)-2|$, $|x-1|$ ifodalar orasidagi bog'lanishni topamiz. Bu ifodalardan birinchisining shaklini o'zgartiramiz:

$$|(4-2x)-2|=|2-2x|=2|x-1|,$$

bundan $\delta=\frac{\varepsilon}{2}$ deb olish yetarli.

b) *δ ning «ishlash»ini* ko'rsatish. $\delta=\frac{\varepsilon}{2}$ bo'lsin. Agar $|x-1|<\frac{\varepsilon}{2}$ bo'lsa, bundan $|2x-2|=|(4-2x)-2|<\varepsilon$ tengsizlik kelib chiqadi. Bu tengsizlikdan

$$\lim_{x\to 1+0}(4-2x)=2$$

2. *Chap limit. a) δ ni topish.* $\forall\varepsilon>0$ son berilgan bo'lsin. δ ni shunday izlaymizki, $\forall x\in U_{\delta}^{-}(1)=(1-\delta;\,1)$ uchun $|(1+x^3)-2|<\varepsilon$ bo'lishi lozim.

Dastlab, $|1-x|$ va $|(1+x^3)-2|$ orasidagi munosabat (bog'liqlik)ni topishimiz zarur. Barcha munosabatlarda $x<1$ shartning bajarilishini talab qilamiz. Yuqoridagi munosabatlarning ikkinchisida shakl almashtiramiz:

$$|(1+x^3)-2|=|x^3-1|=|1-x|\cdot|x^2+x+1| \qquad (*)$$

$|x^2+x+1|$ ifoda sonlar o'qida chegaralanmaganligi uchun, keyingi tenglikning o'ng tomonini qulay holda baholash uchun $U_{\delta}^{-}(1)$ atrofni o'z ichida saqlaydigan biror atrofni olamiz, masalan, $U_{1}^{-}(1)$ atrofni qaraymiz. $U_{\delta}^{-}(1)\subset U_{1}^{-}(1)$ $\forall x\in U_{1}^{-}(1)$ uchun $|x^2+x+1|<3$ tengsizlik o'rinli bo'ladi.

Shunday qilib, (*) dan $|(1+x^3)-2|<3\cdot|1-x|$ tengsizlikka ega bo'lamiz. Bundan $\delta=\frac{\varepsilon}{3}$ deb olish yetarli. 1 nuqtaning $(1-\delta;\,1)$ atrofi $U_{1}^{-}(1)=(0;1)$ atrofdan chiqib ketmasligi uchun esa, $\delta=\min\{1;\varepsilon/3\}$ deb olish zarur.

b) *δ ning «ishlash»ini* ko'rsatish $\delta=\min\{1;\varepsilon/3\}$ ni tanlaymiz va $|x-1|<\delta\le\frac{\varepsilon}{3}$ bo'lsin, deb faraz qilamiz. $\forall x\in U_{1}^{-}(1)=(0;\,1)$ uchun $x^2+x+1<3$ bo'lganligi uchun:

$$\frac{\varepsilon}{3} > \frac{|1-x|(x^2+x+1)}{(x^2+x+1)} > \frac{|1-x|(x^2+x+1)}{3} = \frac{|x^3-1|}{3} = \frac{|(1+x^3-2)|}{3}.$$

Bundan, $|(1+x^3-2)| < \varepsilon$ bajariladi. Chap limitning ta'rifiga asosan, $\lim\limits_{x\to 1-0}(1+x^3)=2$ bo'lar ekan.

Shunday qilib, $\lim\limits_{x\to 1+0} f(x)=2$, $\lim\limits_{x\to 1-0} f(x)=2$ bo'lgani uchun $\lim\limits_{x\to 1} f(x)=2$, lekin $f(1)\neq 2$.

7.2. Funksiya limitga ega bo'lishining zaruriy va yetarli sharti (Koshi kriteriysi). $f(x)$ funksiya $X(X\subset R)$ to'plamda aniqlangan, a (chekli yoki cheksiz) nuqta X to'plamning limit nuqtasi bo'lsin.

7.10-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilib, argument x ning

$$0<|x'-a|<\delta, \quad 0<|x''-a|<\delta \qquad (7.6)$$

tengsizliklarni qanoatlantiruvchi ixtiyoriy x' va x'' $(x'\in X;\ x''\in X)$ qiymatlarida

$$|f(x')-f(x'')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya uchun a nuqtada *Koshi sharti bajariladi* deyiladi.

$f(x)$ funksiya uchun a nuqtada Koshi shartining bajarilmasligi quyidagicha ta'riflanadi: $\forall\delta>0$ son olganimizda ham, shunday $\varepsilon>0$ va $0<|x'-a|<\delta$, $0<|x''-a|<\delta$ tengsizliklarni qanoatlantiruvchi $\exists x'$, x'' $(x'\in X,\ x''\in X)$ qiymatlar topiladiki,

$$|f(x')-f(x'')|\geq\varepsilon$$

tengsizlik o'rinli bo'ladi.

7.2-teorema (Koshi). *$f(x)$ funksiyaning a nuqtada chekli limitga ega bo'lishi uchun, uning uchun a nuqtada Koshi shartining bajarilishi zarur va yetarli.*

7.3-teorema (Monoton funksiyaning limiti). *Agar $f(x)$ funksiya X to'plamda o'suvchi (kamayuvchi) bo'lib, yuqoridan (quyidan) chegaralangan bo'lsa, $f(x)$ funksiya a nuqtada chekli limitga ega* *bo'ladi va agar $f(x)$ funksiya yuqoridan (quyidan) chegaralanmagan bo'lsa, uning limiti $\infty(-\infty)$ bo'ladi.*

a nuqtada o'ng (chap) limitlar uchun hamda $x\to\infty$ dagi $x\to+\infty$ $(x\to-\infty)$ dagi limitlar uchun Koshi shartining ta'rifi yuqoridagi 7.10-ta'rif singari ifodalanadi, faqat (7.6) shart, mos ravishda, ushbu

$$a<x'<a+\delta,\ a<x''<a+\delta\ \ (a-\delta<x''<a,\ \ a-\delta<x''<a);$$
$$|x'|>\delta,\ |x''|>\delta,\ \ x'>\delta,\ x''>\delta\ \ (x'<-\delta,\ x''<-\delta)$$

shartlarga almashtiriladi.

7.10-misol. Ushbu

$$f(x)=x^3\sin\frac{1}{x^2}$$

funksiyaning $a=0$ nuqtada Koshi shartini qanoatlantirishini ko'rsating.

Yechilishi. a) δ ni topish. $\forall\varepsilon>0$ son berilgan bo'lsin. Biz shunday δ ni izlaymizki, x ning $0<|x'|<\delta,\ \ 0<|x''|<\delta$ tengsizlikni qanoatlantiruvchi qiymatlarida

$$\left|x''^3\sin\frac{1}{x''^2}-x'^3\sin\frac{1}{x'^2}\right|<\varepsilon \tag{7.7}$$

tengsizlik bajarilishi kerak. Buning uchun biz dastlab $\left|x''^3\sin\frac{1}{x''^2}-\right.$ $\left.-x'^3\sin\frac{1}{x'^2}\right|$ va $0<|x'|<\delta,\ \ 0<|x''|<\delta$ ifodalar orasidagi bog'lanishni topishimiz kerak. Bu bog'lanishni topish maqsadida, yuqoridagi birinchi ifodaning shaklini almashtiramiz:

$$\left|x''^3\sin\frac{1}{x''^2}-x'^3\sin\frac{1}{x'^2}\right|\le\left|x''^3\sin\frac{1}{x''^2}\right|+\left|x'^3\sin\frac{1}{x'^2}\right|\le|x''^3|+|x'^3|.$$

Bunda (7.7) tengsizlik bajarilishi uchun $\delta=\sqrt[3]{\frac{\varepsilon}{2}}$ deb olish yetarli.

b) *δ ning «ishlash»ini* ko'rsatish. $0<|x'|<\sqrt[3]{\frac{\varepsilon}{2}}$, $0<|x''|<\sqrt[3]{\frac{\varepsilon}{2}}$ bo'lsin. Unda bu tengsizliklardan: $1\geq\left|\sin\frac{1}{x'^2}\right|$, $1\geq\left|\sin\frac{1}{x''^2}\right|$larni e'tiborga olgan holda

$$\frac{\varepsilon}{2}>|x'|^3\cdot\left|\sin\frac{1}{x'^2}\right|,\quad \frac{\varepsilon}{2}>|x''|^3\cdot\left|\sin\frac{1}{x''^2}\right|$$

tengsizliklarga ega bo'lamiz. Bu tengsizliklarni hadma-had qo'shish natijasida

$$\varepsilon>|x'|^3\cdot\left|\sin\frac{1}{x'^2}\right|+|x''|^3\cdot\left|\sin\frac{1}{x''^2}\right|>\left|x'^3\sin\frac{1}{x'^2}-x''^3\sin\frac{1}{x''^2}\right|$$

tengsizlikni hosil qilamiz.

Shunday qilib, berilgan funksiya $a=0$ nuqtada Koshi shartini qanoatlantirar ekan.

7.11-misol. $\lim\limits_{x\to 0-0} a^{\frac{1}{x}}=0$, $a>1$ ekanligini isbotlang.

Yechilishi. a) δ ni topish. $\forall\varepsilon>0$ berilgan bo'lsin. Berilgan ε songa ko'ra δ ni shunday izlaymizki, x ning $x<0$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$a^{\frac{1}{x}}<\varepsilon \qquad (7.8)$$

tengsizlik bajarilsin.

Agar $\varepsilon\geq 1$ bo'lsa, (7.8) tengsizlik barcha $x<0$ uchun bajariladi. Shuning uchun har bir $\varepsilon\geq 1$ uchun δ sifatida ixtiyoriy musbat sonni olish mumkin, masalan, $\delta=1$.

Agar $\varepsilon<1$ bo'lsa, (7.8) tengsizlikning ikkala tomonini logarifmlash natijasida $\frac{1}{x}\ln a<\ln\varepsilon$ tengsizlikka ega bo'lamiz. Bundan, $x>\frac{\ln a}{\ln\varepsilon}$. Bu holda $-\delta<x<0$ tengsizlikni qanoatlantiruvchi ixtiyoriy x larda (7.8) tengsizlikning bajarilishi uchun $\delta=-\frac{\ln a}{\ln\varepsilon}$ deb olish yetarli.

b) δ *ning «ishlash»ini* ko'rsatish. $\frac{\ln a}{\ln \varepsilon} < x < 0$ bo'lsin. Bundan, $\frac{\ln a}{x} < \ln \varepsilon$, $\ln a^{\frac{1}{x}} < \ln \varepsilon$, $a^{\frac{1}{x}} < \varepsilon$ kelib chiqadi. Demak, $\lim\limits_{x \to -0} a^{\frac{1}{x}} = 0$ bo'ladi.

7.12-misol. $\lim\limits_{x \to 0+0} \operatorname{arctg} \frac{1}{x} = \frac{\pi}{2}$ ekanligini isbotlang.

Yechilishi. a) δ ni topish. $\forall \varepsilon > 0$ berilgan bo'lsin. δ ni shunday izlaymizki, barcha $x > 0$ lar uchun

$$\left|\operatorname{arctg} \frac{1}{x} - \frac{\pi}{2}\right| < \varepsilon \tag{7.9}$$

tengsizlik bajarilsin. Buning uchun $\left|\operatorname{arctg} \frac{1}{x} - \frac{\pi}{2}\right|$, $x > 0$ ifodalar o'rtasidagi bog'lanishni topamiz.

Agar $\varepsilon \geq \frac{\pi}{2}$ bo'lsa, (7.9) tengsizlik barcha $x > 0$ uchun bajariladi. Shuning uchun δ sifatida ixtiyoriy musbat sonni olish mumkin, masalan $\delta = 1$.

Agar $\varepsilon < \frac{\pi}{2}$ bo'lsa, u holda (7.9) dan:

$$-\varepsilon < \operatorname{arctg} \frac{1}{x} - \frac{\pi}{2} < \varepsilon, \quad \frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} < \varepsilon, \quad \operatorname{arctg} \frac{1}{x} > \frac{\pi}{2} - \varepsilon,$$
$$\frac{1}{x} > \operatorname{tg}\left(\frac{\pi}{2} - \varepsilon\right), \quad x < \operatorname{tg} \varepsilon.$$

Shunday qilib, $\varepsilon < \frac{\pi}{2}$ bo'lganda $x > 0$ uchun (7.9) tengsizlik bajarilishi uchun $\delta = \operatorname{tg} \varepsilon$ deb olish yetarli. Demak, $\forall \varepsilon > 0$ uchun $0 < x < \delta$ ni qanoatlantiruvchi $\exists \delta(\varepsilon)$, $\left|\operatorname{arctg} \frac{1}{x} - \frac{\pi}{2}\right| < \varepsilon$ bajariladi, bu esa, $\lim\limits_{x \to 0+0} \operatorname{arctg} \frac{1}{x} = \frac{\pi}{2}$ ekanligini anglatadi.

b) δ *ning «ishlash»ini* ko'rsatish. $0 < x < \operatorname{tg} \varepsilon$ bo'lsin. Bundan $\frac{1}{x} > \frac{1}{\operatorname{tg} \varepsilon}$ yoki $\frac{1}{x} > \operatorname{tg}\left(\frac{\pi}{2} - \varepsilon\right)$, $\varepsilon < \frac{\pi}{2}$ bo'lsa, $\operatorname{arctg} \frac{1}{x} > \operatorname{arctg}\left(\operatorname{tg}\left(\frac{\pi}{2} - \varepsilon\right)\right)$ tengsizlik o'rinli. Bundan $\operatorname{arctg} \frac{1}{x} > \frac{\pi}{2} - \varepsilon$ yoki $\frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} < \varepsilon$. Bu tengsizlikdan

$$-\varepsilon < \operatorname{arctg} \frac{1}{x} - \frac{\pi}{2} < \varepsilon \quad \text{yoki} \quad \left|\frac{\pi}{2} - \operatorname{arctg} \frac{1}{x}\right| < \varepsilon$$

kelib chiqadi.

7.13-misol. $f(x)=\frac{x^3}{x^2+1}-x$ funksiyaning $x\to\infty$ da chekli limitga ega ekanligini isbotlang.

Yechilishi. Berilgan funksiya uchun $x\to\infty$ da Koshi shartining bajarilishini ko'rsatamiz:

a) C ni topish. $\forall\varepsilon>0$ son berilgan bo'lsin. Bu berilgan songa ko'ra $C=C(\varepsilon)$ni shunday izlaymizki, argument x ning $|x'|>C$, $|x''|>C$ shartlarni qanoatlantiruvchi $\forall x', x''$ $(x'\in R, x''\in R)$ qiymatlarida

$$\left|\frac{x'^3}{x'^2+1}-x'-\left(\frac{x''^3}{x''^2+1}-x''\right)\right|<\varepsilon \tag{7.10}$$

tengsizlik bajarilsin. Buning uchun avvalo

$$\left|\frac{x'^3}{x'^2+1}-x'-\left(\frac{x''^3}{x''^2+1}-x''\right)\right|,\quad |x'|>C,\ |x''|>C$$

munosabatlar orasidagi bog'lanishni topamiz. Buning uchun esa yuqoridagi munosabatlardan birinchisining shaklini almashtiramiz:

$$\left|\frac{x'^3}{x'^2+1}-x'-\left(\frac{x''^3}{x''^2+1}-x''\right)\right|=\left|\frac{x'}{x'^2+1}+\frac{x''}{x''^2+1}\right|\le\left|\frac{x'}{x'^2+1}\right|+\left|\frac{x''}{x''^2+1}\right|\le$$

$$\le\left|\frac{x'}{x'^2}\right|+\left|\frac{x''}{x''^2}\right|=\frac{1}{|x'|}+\frac{1}{|x''|}.$$

Bundan, (7.10) tengsizlik bajarilishi uchun $C=\frac{2}{\varepsilon}$ deb olish yetarli bo'ladi.

b) C ning «ishlash»ini ko'rsatamiz. $|x'|>\frac{2}{\varepsilon}$, $|x''|>\frac{2}{\varepsilon}$ bo'lsin. Bundan,

$$\frac{\varepsilon}{2}>\frac{1}{|x'|},\quad \frac{\varepsilon}{2}>\frac{1}{|x''|}.$$

Bu tengsizliklarni hadma-had qo'shamiz:

$$\varepsilon > \frac{1}{|x'|}+\frac{1}{|x''|}=\frac{|x'|}{x'^2}+\frac{|x''|}{x''^2} > \frac{|x'|}{x'^2+1}+\frac{|x''|}{x''^2+1} > \left|-\frac{x'}{x'^2+1}+\frac{x''}{x''^2+1}\right| =$$

$$=\left|\frac{x'^3-x'-x'^3}{x'^2+1}+\frac{x''-x''^3+x''^3}{x''^2+1}\right|=\left|\frac{x'^3}{x'^2+1}-x'-\left(\frac{x''^3}{x''^2+1}-x''\right)\right|.$$

Demak, $|x'|>\frac{2}{\varepsilon}$, $|x''|>\frac{2}{\varepsilon}$ tengsizliklarni qanoatlantiruvchi $\forall x', x''$ $(x'\in R,\ x''\in R)$ lar uchun

$$\left|\frac{x'^3}{x'^2+1}-x'-\left(\frac{x''^3}{x''^2+1}-x''\right)\right|<\varepsilon$$

tengsizlik, ya'ni berilgan funksiya uchun $x\to\infty$ da Koshi sharti bajarilar ekan. Bundan esa, berilgan funksiyaning $x\to\infty$ da chekli limitga ega ekanligi kelib chiqadi.

7.14-misol. Ushbu $f(x)=\sqrt{x^2+1}-x$ funksiya uchun $x\to\pm\infty$ da Koshi shartining bajarilishini isbotlang.

Yechilishi. $x\to+\infty$ da berilgan funksiya uchun Koshi shartining bajarilishini ko'rsatamiz. Xuddi shunday $x\to-\infty$ da ham Koshi shartining bajarilishi ko'rsatiladi.

a) C ni topish. $\forall\varepsilon>0$ son berilgan bo'lsin. Berilgan ε ga ko'ra C sonni shunday izlaymizki, $x'>C$, $x''>C$ tengsizliklarni qanoatlantiruvchi $\forall x', x''$ $(x'\in R,\ x''\in R)$ lar uchun

$$\left|\sqrt{x'^2+1}-x'-\left(\sqrt{x''^2+1}-x''\right)\right|<\varepsilon \qquad (7.11)$$

tengsizlik o'rinli bo'lishi lozim. Buning uchun biz avvalo

$$\left|\sqrt{x'^2+1}-x'-(\sqrt{x''^2+1}-x'')\right|,\quad x'>C,\quad x''>C$$

munosabatlar orasidagi bog'lanishni topamiz. Qaralayotgan munosabatning birinchisining shaklini o'zgartirib, ushbu

$$\left|\frac{1}{\sqrt{x'^2+1}+x'}-\frac{1}{\sqrt{x''^2+1}+x''}\right|\le\left|\frac{1}{\sqrt{x'^2+1}+x'}\right|+\left|\frac{1}{\sqrt{x''^2+1}+x''}\right|=$$
$$=\frac{1}{\sqrt{x'^2+1}+x'}+\frac{1}{\sqrt{x''^2+1}+x''}<\frac{1}{2x'}+\frac{1}{2x''}<\frac{1}{x'}+\frac{1}{x''}$$

tengsizlikka ega bo'lamiz. $x'>C$, $x''>C$ tengsizlikni qanoatlantiruvchi $\forall x', x''$ $(x'\in R,\ x''\in R)$ larda (7.11) tengsizlikning bajarilishi uchun $C=\frac{2}{\varepsilon}$ deb olish lozim bo'ladi.

b) C ning «ishlash»ini ko'rsatamiz. $x'>\frac{2}{\varepsilon}$, $x''>\frac{2}{\varepsilon}$ tengsizliklar o'rinli bo'lsin. Bundan, $\frac{\varepsilon}{2}>\frac{1}{x'}$, $\frac{\varepsilon}{2}>\frac{1}{x''}$ tengsizliklarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shish natijasida

$$\varepsilon>\frac{1}{x'}+\frac{1}{x''}>\frac{1}{2x'}+\frac{1}{2x''} \qquad (7.12)$$

bo'lishini topamiz. Ravshanki, $\sqrt{x'^2+1}+x'>2x'$, $\sqrt{x''^2+1}+x''>2x''$ tengsizliklar o'rinli. Keyingi tengsizliklarni e'tiborga olgan holda (7.12)dan

$$\varepsilon>\frac{1}{\sqrt{x'^2+1}+x'}+\frac{1}{\sqrt{x''^2+1}+x''}>\left|\frac{1}{\sqrt{x'^2+1}+x'}-\frac{1}{\sqrt{x''^2+1}+x''}\right|=$$
$$=\left|\sqrt{x'^2+1}-x'-\left(\sqrt{x''^2+1}-x''\right)\right|$$

ekanligini olamiz.

Shunday qilib, $x'>\frac{2}{\varepsilon}$, $x''>\frac{2}{\varepsilon}$ tengsizliklarni qanoatlantiruvchi $\forall x', x''$ $(x'\in R,\ x''\in R)$ uchun $\left|\sqrt{x'^2+1}-x'-(\sqrt{x''^2+1}-x'')\right|<\varepsilon$ tengsizlik bajarilar ekan. Bu tengsizlik berilgan funksiyaning $x\to+\infty$ da Koshi shartini qanoatlantirishini ifodalaydi.

7.15-misol. Ushbu

$$f(x)=\sin\frac{1}{x} \quad (x\ne 0)$$

funksiya uchun $x=0$ nuqtada Koshi shartining bajarilmasligini ko'rsating.

Isboti. $\forall\delta > 0$ son berilgan bo'lsin. Bu son bo'yicha $\exists\varepsilon_0 > 0$ va $|x'|<\delta$, $|x''|<\delta$ tengsizliklarni qanoatlantiruvchi

$$\exists x', x'' \ (x' \in R\setminus\{0\}, \ \ x'' \in R\setminus\{0\})$$

nuqtalarni topish kerakki,

$$\left|\sin\frac{1}{x'} - \sin\frac{1}{x''}\right| \geq \varepsilon_0$$

tengsizlik bajarilsin. $\varepsilon_0 = \frac{1}{2}$ va $x' = \frac{1}{2k\pi}$, $x'' = \frac{2}{(4k+1)\pi}$ $(k \neq 0)$ nuqtalar uchun $k > \left[\frac{1}{2\delta\pi}\right]$ deb olsak, u holda $|x'|<\delta$, $|x''|<\delta$ tengsizliklar bajariladi.

$$\left|\sin\frac{1}{x'} - \sin\frac{1}{x''}\right| = \left|\sin 2k\pi - \sin(4k+1)\frac{\pi}{2}\right| = 1 > \varepsilon_0 = \frac{1}{2}.$$

Demak, berilgan funksiya uchun $x = 0$ nuqtada Koshi sharti bajarilmas ekan.

7.3. Chekli limitga ega bo'lgan funksiyalar ustida arifmetik amallar. $X(X \subset R)$ to'plam berilgan bo'lib, a uning limit nuqtasi, $f(x)$, $h(x)$ va $g(x)$ lar X to'plamda aniqlangan funksiyalar bo'lsin.

7.4-teorema. *Agar $f(x)$ va $g(x)$ funksiyalar a nuqtada limitga ega va ularning limitlari mos ravishda b va c bo'lsa, u holda $f(x) \pm g(x)$, $f(x)\cdot g(x)$, $kf(x)$, $\frac{f(x)}{g(x)}$ $(c \neq 0)$ funksiyalar ham shu a nuqtada chekli limitga ega bo'ladi va ushbu*

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = b \pm c, \qquad \text{(I)}$$

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = b\cdot c, \qquad \text{(II)}$$

$$\lim_{x\to a}(kf(x)) = k\lim_{x\to a} f(x) = kb, \quad (k \text{ — o'zgarmas}) \qquad \text{(III)}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{b}{c}. \qquad \text{(IV)}$$

munosabatlar o'rinli bo'ladi.

7.2-eslatma. Yuqoridagi (I) va (II) munosabatlar qo'shiluvchilar va ko'paytuvchilar soni ixtiyoriy chekli bo'lganda ham o'rinli.

7.3-eslatma. (I), (II) va (IV) larda $g(x)$ va $g(x)$ funksiyalarning yig'indisi, ko'paytmasi va nisbatidan iborat bo'lgan funksiyalarning limitga ega bo'lishidan, bu funksiyalar har birining limitga ega bo'lishi har doim ham kelib chiqavermaydi. Masalan, $f(x)=x$, $g(x)=\sin\frac{1}{x}$ funksiyalar ko'paytmasi $f(x)\cdot g(x)=x\sin\frac{1}{x}$ bo'lib, $x\to 0$ da $f(x)\cdot g(x)\xrightarrow[x\to 0]{}0$. Ammo $x\to 0$ da $g(x)=\sin\frac{1}{x}$ funksiya limitga ega emas.

7.5-teorema. *Agar a nuqtaning biror $U_\delta(a)$ atrofidan olingan x ning barcha qiymatlarida $g(x)\le f(x)\le h(x)$ tengsizlik o'rinli bo'lib, $x\to a$ da $f(x)$ va $g(x)$ funksiyalar limitga ega bo'lib, $\lim\limits_{x\to a} g(x)=\lim\limits_{x\to a} h(x)=b$ bo'lsa, u holda $f(x)$ funksiya ham a nuqtada limitga ega va $\lim\limits_{x\to a} f(x)=b$ bo'ladi.*

7.4. Aniqmas ifodalar. Yuqoridagi 7.4-teoremada $f(x)$ va $g(x)$ funksiyalardan quyidagi ikki shartni talab qilgan edik: 1) $f(x)$ va $g(x)$ funksiyalar a nuqtada chekli limitga ega; 2) $\frac{f(x)}{g(x)}$ ning limitiga doir mulohazalarda esa, $\lim\limits_{x\to a} g(x)=c\ne 0$ bo'lsin deb faraz qilingan edi. Agar $x\to a$ da bu shartlarning birortasi bajarilmasa, ya'ni $x\to a$ da $f(x)$ va $g(x)$ funksiyalarning har birining limiti cheksiz yoki $\frac{f(x)}{g(x)}$ ning limiti qaralganda, $\lim\limits_{x\to a} g(x)=0$ bo'lib qolsa. u holda 6-§ da batafsil o'rganilgan aniqmasliklar kabi turli xil aniqmas ifodalarga kelamiz. Jumladan:

1) $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=0$ bo'lsa, ularning $\frac{f(x)}{g(x)}$ nisbati $\frac{0}{0}$ ko'rinishdagi aniqmaslikni ifodalaydi; 2) $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} g(x)=\infty$ bo'lsa, ularning $\frac{f(x)}{g(x)}$ nisbati $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslikni ifodalaydi; 3) $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=\infty$ bo'lsa, ularning $f(x)\cdot g(x)$ ko'paytmasi $0\cdot\infty$ ko'rinishdagi aniqmaslikni ifodalaydi. 4) $\lim\limits_{x\to a} f(x)=+\infty\ (-\infty)$, $\lim\limits_{x\to a} g(x)=-\infty\ (+\infty)$ bo'lsa, u holda $f(x)+g(x)$ ifoda $\infty-\infty$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu hollarda $x\to a$ da $f(x)$ va $g(x)$ funksiyalarning o'z limitlariga qanday intilishi xususiyatlariga qarab $\frac{f(x)}{g(x)}$, $f(x)\cdot g(x)$, $f(x)+g(x)$ ifodalarning xarakterini aniqlash *aniqmaslikni ochish* deb yuritiladi.

7.16-misol. Ushbu

a) $\lim_{x\to 1}\left(\frac{x^2+x+5}{x^2-3}+3\sin^2\frac{\pi x}{2}+2\cos^3\frac{\pi x}{2}\right)$; b) $\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}$;

d) $\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}$; e) $\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}$

limitlarni hisoblang.

Yechilishi. a) $\lim_{x\to 1}\left(\frac{x^2+x+5}{x^2-3}+3\sin^2\frac{\pi x}{2}+2\cos^3\frac{\pi x}{2}\right)$. Qo'shiluvchilarning har birining $x\to 1$ dagi limitini hisoblaymiz. Birinchi qo'shiluvchi kasr funksiya bo'lib, uning surat va maxraji ham $x\to 1$ da chekli limitga ega bo'lgani uchun, uning limitini I—IV qoidalar bo'yicha hisoblaymiz, ya'ni

$$\lim_{x\to 1}\frac{x^2+x+5}{x^2-3}=\frac{\lim\limits_{x\to 1}(x^2+x+5)}{\lim\limits_{x\to 1}(x^2-3)}=\frac{\lim\limits_{x\to 1}x\cdot\lim\limits_{x\to 1}x+\lim\limits_{x\to 1}x+\lim\limits_{x\to 1}5}{\lim\limits_{x\to 1}x\lim\limits_{x\to 1}x-\lim\limits_{x\to 1}3}=-\frac{7}{2}.$$

Ikkinchi va uchinchi qo'shiluvchilarning $x\to 1$ dagi limitlari mavjud bo'lgani uchun, ularning limitlarini II va III qoidalar bo'yicha hisoblaymiz:

$$\lim_{x\to 1}\left(3\sin^2\frac{\pi}{2}x\right)=3\lim_{x\to 1}\sin\frac{\pi}{2}x\cdot\lim_{x\to 1}\sin\frac{\pi}{2}x=3;$$

$$\lim_{x\to 1}\left(2\cos^3\frac{\pi}{2}x\right)=2\lim_{x\to 1}\cos\frac{\pi}{2}x\cdot\lim_{x\to 1}\cos\frac{\pi}{2}x\cdot\lim_{x\to 1}\cos\frac{\pi}{2}x=0.$$

Shunday qilib, 7.4- teorema va 7.2-eslatmaga ko'ra:

$$\lim_{x\to 1}\left(\frac{x^2+x+5}{x^2-3}+3\sin^2\frac{\pi}{2}x+2\cos^3\frac{\pi}{2}x\right)=\lim_{x\to 1}\frac{x^2+x+5}{x^2-3}+$$

$$+\lim_{x\to 1}3\sin^2\frac{\pi}{2}x+\lim_{x\to 1}2\cos^3\frac{\pi}{2}x=-\frac{7}{2}+3+0=-\frac{1}{2}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2 + x + 5)/(x^2-
3)) + 3*(sin(x*Pi/2))^2 + 2*(cos(x*Pi/2))^3, x = 1);
```

$$\frac{-1}{2}.$$

b) $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}$. Bunda $\lim\limits_{x\to 0}(x^2-1)=-1$, $\lim\limits_{x\to 0}(2x^2-x-1)=-1\neq 0$ bo'lgani uchun, IV qoidani qo'llash mumkin:

$$\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=\frac{\lim\limits_{x\to 0}(x^2-1)}{\lim\limits_{x\to 0}(2x^2-x-1)}=\frac{-1}{-1}=1.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2-1)/(2*x^2-x-1)), x=0);
1
```

d) $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1}$. Bu limitni hisoblashda 7.4-teoremani qo'llash mumkin emas, chunki $\lim\limits_{x\to 1}(x^2-1)=0$, $\lim\limits_{x\to 1}(2x^2-x-1)=0$, ya'ni 7.4-teoremadagi $\lim\limits_{x\to a}g(x)=c\neq 0$ shart bajarilmayapti. Berilgan ifodaning $x\to 0$ dagi limiti $\frac{0}{0}$ aniqmaslikni ifodalaydi. Bu aniqmaslikni ochish uchun $\frac{x^2-1}{2x^2-x-1}$ ifodaning shaklini o'zgartiramiz: $\frac{(x-1)(x+1)}{2(x-1)\left(x+\frac{1}{2}\right)}$. U holda

$$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{1}{2}\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)\left(x+\frac{1}{2}\right)}=\frac{1}{2}\lim_{x\to 1}\frac{x+1}{x+\frac{1}{2}}=\frac{2}{2\cdot\frac{3}{2}}=\frac{2}{3}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2-1)/(2*x^2-x-1)), x=1);
2
-
3
```

e) $\lim\limits_{x\to\infty}\frac{x^2-1}{2x^2-x-1}$. Bu limitni hisoblashda ham 7.4-teoremani qo'llab bo'lmaydi, chunki berilgan kasr ifodaning surati va maxraji chekli limitga ega emas, ya'ni $\lim\limits_{x\to\infty}(x^2-1)=\infty$, $\lim\limits_{x\to\infty}(2x^2-x-1)=\infty$. Shunday qilib, $\frac{x^2-1}{2x^2-x-1}$ ifoda $x\to\infty$ da $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu limitni hisoblash, ya'ni aniqmaslikni ochish uchun kasr ifodaning surat va maxrajini x^2 ga bo'lamiz:

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}}.$$

Endi shakl o'zgartirish natijasida xosil bo'lgan kasr ifodaning limitini hisoblashda 7.4-teoremani qo'llash mumkin. Shunday qilib,

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}}=\frac{1-0}{2-0-0}=\frac{1}{2}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2-1)/(2*x^2-x-1)), x=infinity);
```

$\frac{1}{2}$.

7.17-misol. Ushbu

$$a)\ f(x)=\sqrt{1+\frac{1}{x^2}};\qquad b)\ f(x)=x^2\sin\frac{1}{x}$$

funksiyalarning, mos ravishda, $x\to\infty$ da va $x\to 0$ dagi limitlarini hisoblang.

Yechilishi. a) Barcha $x\neq 0$ lar uchun $0<\sqrt{1+\frac{1}{x^2}}<1+\frac{1}{x^2}$ tengsizlik bajariladi. $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)=1$, chunki, $\lim_{x\to\infty}\frac{1}{x^2}=0$.

Demak, 7.5-teoremaga ko'ra $\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}=1$ bo'ladi.

b) barcha $x\neq 0$ bo'lgan x lar uchun $-x^2<x^2\sin\frac{1}{x}\leq x^2$ tengsizlik o'rinli va $\lim_{x\to 0}(-x^2)=0$, $\lim_{x\to 0}x^2=0$ bo'lganligi uchun, 7.5-teoremaga asosan $\lim_{x\to 0}x^2\sin\frac{1}{x}=0$ bo'ladi.

7.18-misol. Ushbu

A) $\lim_{x\to a}\sin x=\sin a$; B) $\lim_{x\to a}\cos x=\cos a$; D) $\lim_{x\to a}\operatorname{tg}x=\operatorname{tg}a$ $\left(a\neq\frac{2n-1}{2}\pi;\ n=0,\pm1,\pm2,...\right)$; E) $\lim_{x\to x_0}a^x=a^{x_0}$ $(a>0)$;

F) $\lim_{x\to x_0}\ln x=\ln x_0$

tasdiqlarni isbotlang.

Yechilishi. A) a) *δ ni topish.* $\forall\varepsilon>0$ son berilgan bo'lsin. $\delta=\delta(\varepsilon)$ ni shunday izlashimiz kerakki, $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha x lar uchun

$$|\sin x-\sin a|<\varepsilon \tag{7.13}$$

tengsizlik bajarilishi lozim. Avvalo,

$$|\sin x-\sin a|,\quad |x-a|$$

munosabatlar orasidagi bog'lanishni topamiz. Buning uchun yuqoridagi munosabatlarning birinchisida shakl almashtiramiz:

$$0\le|\sin x-\sin a|=\left|2\sin\frac{x-a}{2}\cos\frac{x+a}{2}\right|\le 2\left|\sin\frac{x-a}{2}\right|\le 2\frac{|x-a|}{2}=|x-a|.$$

Bunda $\delta=\varepsilon$ deb olinsa, $0<|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha x lar uchun (7.13) tengsizlik bajariladi, ya'ni $|\sin x-\sin a|<\varepsilon$ bo'ladi.

b) *δ ning «ishlash»ini* ko'rsatish. $|x-a|<\varepsilon$ tengsizlikni qanoatlantiruvchi x larni qaraymiz. Keyingi tengsizlikdan $1\ge\left|\cos\frac{x+a}{2}\right|$, $\frac{|x-a|}{2}\ge\left|\sin\frac{x-a}{2}\right|$ tengsizliklarni e'tiborga olsak,

$$\varepsilon>|x-a|=2\frac{|x-a|}{2}\ge 2\left|\sin\frac{x-a}{2}\right|\ge 2\left|\sin\frac{x-a}{2}\right|\left|\cos\frac{x+a}{2}\right|=$$

$$=2\left|\sin\frac{x-a}{2}\cos\frac{x+a}{2}\right|=|\sin x-\sin a|$$

tengsizlik o'rinli ekanligini ko'ramiz.

Demak, limitning Koshi ta'rifiga ko'ra $\lim\limits_{x\to a}\sin x=\sin a$.

Xuddi shunday B) $\lim\limits_{x\to a}\cos x=\cos a$ ekanligi ko'rsatiladi.

d) (IV) formulani e'tiborga olsak,

$$\lim_{x\to a}\mathrm{tg}x=\lim_{x\to a}\frac{\sin x}{\cos x}=\frac{\lim\limits_{x\to a}\sin x}{\lim\limits_{x\to a}\cos x}=\frac{\sin a}{\cos a}=\mathrm{tg}a,$$

$\left(\cos a \neq 0, \text{ ya'ni } a \neq \frac{2n-1}{n}\pi, \; n = 0; \pm 1, \pm 2, \ldots\right)$.

E) $a > 1$ bo'lgan holni qarash yetarli. $\forall \varepsilon > 0$ son berilgan bo'lsin. $|a^x - a^{x_0}|$, $|x - x_0|$ ifodalar orasidagi bog'lanishni topish uchun $|a^x - a^{x_0}|$, ifodaning shaklini almashtiramiz:

$$|a^x - a^{x_0}| = a^{x_0} |a^{x-x_0} - 1|.$$

Ma'lumki, $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = \lim\limits_{n\to\infty} a^{-\frac{1}{n}} = 1$. Bundan berilgan $\varepsilon > 0$ bo'yicha shunday n_0 topiladiki, $1 - \frac{\varepsilon}{a^{x_0}} < a^{-\frac{1}{n_0}} < a^{\frac{1}{n_0}} < 1 + \frac{\varepsilon}{a^{x_0}}$ tengsizlik o'rinli bo'ladi. $|x - x_0| < \frac{1}{n_0}$ deb olinsa, tengsizlikni qanoatlantiruvchi barcha x lar uchun $1 - \frac{\varepsilon}{a^{x_0}} < a^{-\frac{1}{n_0}} < a^{x-x_0} < a^{\frac{1}{n_0}} < 1 + \frac{\varepsilon}{a^{x_0}}$ yoki $|a^x - a^{x_0}| = a^{x_0} |a^{x-x_0} - 1| < \varepsilon$ tengsizlik bajariladi. Bundan esa, $\lim\limits_{x\to x_0} a^x = a^{x_0}$ bo'ladi.

F) Ravshanki, $n > 1$ bo'lganda

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}, \quad -\frac{1}{n-1} < \ln\left(1 - \frac{1}{n}\right) < \frac{1}{n}.$$

$\forall \varepsilon > 0$ berilgan bo'lib, $\varepsilon \leq \frac{1}{2}$ bo'lsin. U holda shunday n_0 mavjud bo'lib,

$$-\varepsilon < \ln\left(1 - \frac{1}{n_0}\right) < \ln\left(1 + \frac{1}{n_0}\right) < \varepsilon$$

tengsizlik bajariladi.

Agar $-\frac{1}{n_0} < \frac{x - x_0}{x_0} < \frac{1}{n_0}$ deb olinsa, u holda $\ln x - \ln x_0 = \ln\left(1 + \frac{x - x_0}{x_0}\right)$ ayirma uchun quyidagi baho o'rinli bo'ladi:

$$-\varepsilon < \ln\left(1 - \frac{1}{n_0}\right) < \ln\left(1 + \frac{x - x_0}{x_0}\right) < \ln\left(1 + \frac{1}{n_0}\right) < \varepsilon$$

yoki

$$|\ln x-\ln x_0|<\ln\left(1+\frac{x-x_0}{x_0}\right)<\ln\left(1+\frac{1}{n_0}\right)<\varepsilon$$

tengsizlik bajariladi. Bundan esa $\lim\limits_{x\to x_0}|\ln x-\ln x_0|=0$, $\lim\limits_{x\to x_0}\ln x=\ln x_0$ ekanligi kelib chiqadi.

7.5. Murakkab funksiyaning limiti. Limitlarni hisoblashda ko'pincha quyidagi murakkab funksiyaning limiti haqidagi teorema qo'llaniladi.

$t=\varphi(x)$ funksiya X to'plamda aniqlangan bo'lib, bu funksiyaning qiymatlaridan tuzilgan T to'plamda esa $y=f(t)$ funksiya aniqlangan, ular yordamida murakkab $y=f(\varphi(x))$ funksiya hosil qilingan bo'lsin. Bu murakkab funksiya X to'plamda aniqlangan, a nuqta X to'plamning limit nuqtasi bo'lsin.

7.6-teorema. *Agar: 1)* $\lim\limits_{x\to a}\varphi(x)=c$ *o'rinli bo'lib, a nuqtaning shunday* $\dot{U}_\delta(a)$ *atrofi mavjud bo'lsinki, barcha* $x\in\dot{U}_\delta(a)$ *lar uchun* $\varphi(x)\neq c$ *bo'lsa; 2) c nuqta T to'plamning limit nuqtasi bo'lib,* $\lim\limits_{t\to c}f(t)=b$ *limit mavjud bo'lsa, u holda* $x\to a$ *da* $y=f(\varphi(x))$ *murakkab funksiya ham limitga ega va*

$$\lim_{x\to a}f(\varphi(x))=\lim_{t\to c}f(t) \tag{7.14}$$

bo'ladi.

f(t) funksiya c nuqtada uzluksiz bo'lgan holda (7.14) tenglikni

$$\lim_{x\to a}f(\varphi(x))=f(\lim_{x\to a}\varphi(x)) \tag{7.15}$$

tenglik ko'rinishida yozish mumkin.

7.4-eslatma. Teoremadagi a nuqtaning $\dot{U}_\delta(a)$ atrofida $\varphi(x)\neq c$, bo'lsin, degan shartni $f(t)$ funksiya $t=c$ nuqtada aniqlangan va $\lim\limits_{t\to c}f(t)=f(c)=b$ tengliklar o'rinli bo'lsin, degan shart bilan almashtirish mumkin.

7.5-eslatma. Yuqoridagi a, c va b larning biri chekli, ikkinchisi ∞ yoki barchasi cheksiz bo'lganda ham 7.6-teorema o'rinli bo'ladi.

7.19-misol. Ushbu

$$\lim_{x\to 2\pi}\ln\left(3+\sqrt{1+\mathrm{tg}^2\frac{x}{2}}\right)$$

limitni hisoblang.

Yechilishi. Quyidagi ketma-ket almashtirishlar olamiz:

$$y_1=\frac{x}{2},\quad y_2=\operatorname{tg} y_1,\quad y_3=y_2^2,\quad y_4=1+y_3,\quad y_5=\sqrt{y_4},$$

$$y_6=3+y_5,\quad y_7=\ln y_6$$

Yuqoridagi ifodalarga ketma-ket 7.6-teoremani va 7.4-eslatmani qo'llash natijasida

$$\lim_{x\to 2\pi} y_1(x)=\lim_{x\to 2\pi}\frac{x}{2}=\frac{1}{2}\cdot 2\pi=\pi;\qquad \lim_{y_1\to\pi} y_2(y_1)=\lim_{y_1\to\pi}\operatorname{tg} y_1=0,$$

$$\lim_{y_2\to 0} y_3(y_2)=\lim_{y_2\to 0} y_2^2=\lim_{y_2\to 0} y_2\cdot\lim_{y_2\to 0} y_4=0;\qquad \lim_{y_3\to 0} y_4(y_3)=\lim_{y_3\to 0}(1+y_3)=1,$$

$$\lim_{y_4\to 1} y_5(y_4)=\lim_{y_4\to 1}\sqrt{y_4}=1;\qquad \lim_{y_5\to 1} y_6(y_5)=\lim_{y_5\to 1}(3+y_5)=4,\ .$$

$$\lim_{x\to 2\pi}\ln\left(3+\sqrt{1+\operatorname{tg}^2\frac{x}{2}}\right)=\lim_{y_6\to 4} y_7(y_6)=\lim_{y_6\to 4}\ln y_6=2\ln 2$$

ekanligini olamiz.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(ln(3+sqrt(1+(tan(x/2)^2))), x=2*Pi);
2ln(2).
```

7.20-misol. Ushbu

$$\lim_{x\to 2}\left(\sqrt[3]{\frac{2x^2-8}{x-2}}+3\right)$$

limitni hisoblang.

Yechilishi. Quyidagi belgilashlarni kiritamiz: $t=\varphi(x)=\frac{2x^2-8}{x-2}$, $y=f(t)=\sqrt[3]{t}+3$. $y=f(\varphi(x))=\sqrt[3]{\frac{2x^2-8}{x-2}}+3$ funksiya $(-\infty;\,2)\cup(2;\,+\infty)$ to'plamda aniqlangan. 7.6-teorema shartlarini tekshiramiz:

$$x\in \dot{U}_\delta(2)\Rightarrow \varphi(x)\neq 8,\quad \lim_{x\to 2}\varphi(x)=\lim_{x\to 2}\frac{2x^2-8}{x-2}=8=c,$$

$$\lim_{t\to c} f(t)=\lim_{t\to 8}\left(\sqrt[3]{t}+3\right)=5$$

Demak, 7.6-teoremaga asosan

$$\lim_{x\to 2}\left(\sqrt[3]{\frac{2x^2-8}{x-2}}+3\right)=5$$

ekanligi kelib chiqadi.

Misolni MAPLE tizimidan foydalanib yechish:

```
>limit(surd((2*x^2-8)/(x-2), 3)+3, x=2);
```

$8^{(1/3)}+3.$

Eslatma. Murakkab funksiyaning limiti mavjudligi haqidagi 7.6-teorema yetarli shart bo'lib, zaruriy shart emas. $y=f(\varphi(x))$ murakkab funksiyaning limiti mavjud bo'lib, lekin $f=\varphi(x)$, $y=f(t)$ funksiyalardan har birining limiti mavjud bo'lmasligi ham mumkin. Masalan,

$$D(x)=\begin{cases}1, x\in Q,\\ 0, x\in I\end{cases}$$

bo'lsa, $D(D(x))=1$ bo'ladi. $\lim_{x\to 0} D(x)$ mavjud emas, lekin $\lim_{x\to 0} D(D(x))=1$ mavjud.

7.6. Ajoyib limitlar. Funksiyaning limitini hisoblashda quyidagi ajoyib limitlar muhim rol o'ynaydi:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{\sin x}=1; \qquad \text{(V)}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{y\to 0}(1+y)^{1/y}=e \quad (e\approx 2,71828...) \qquad \text{(VI)}$$

7.6-eslatma. Agar biror $\dot{U}_\delta(x_0)$ atrofda $\alpha(x)\neq 0$ va $\lim_{x\to x_0}\alpha(x)=0$ bo'lsa, u holda

$$\lim_{x\to x_0}(1+\alpha(x))^{1/\alpha(x)}=e \qquad \text{(V')}$$

bo'ladi.

7.7-eslatma. Agar biror $\dot{U}_\delta(x_0)$ atrofda $\alpha(x)\neq 0$, $\beta(x)\neq 0$, $\lim_{x\to x_0}\alpha(x)=\lim_{x\to x_0}\beta(x)=0$ va $\exists\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)}=\lambda$ bo'lsa, u holda

$$\lim_{x\to x_0}(1+\alpha(x))^{1/\beta(x)}=e^{\lambda} \qquad (V'')$$

bo‘ladi.

Xususiy holda,

$$\lim_{x\to x_0}(1+\mu\alpha(x))^{1/\alpha(x)}=e^{\mu}, \qquad \mu=\text{const}. \qquad (V''')$$

(VI) formuladan natija sifatida kelib chiqadigan ushbu

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e, \qquad (VII)$$

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\frac{1}{\ln a}, \quad a>0,\ a\neq 1, \qquad (VIII)$$

$$\lim_{x\to 0}\frac{(1+x)^{\mu}-1}{x}=\mu, \qquad (IX)$$

$$\lim_{x\to 0}\frac{a^{x}-1}{x}=\ln a, \quad a>0, \qquad (X)$$

(xususiy holda $a=e$ bo‘lganda)

$$\lim_{x\to a}\frac{\ln(1+x)}{x}=1, \qquad (XI)$$

$$\lim_{x\to 0}\frac{e^{x}-1}{x}=1 \qquad (XII)$$

formulalar funksiya limitini hisoblashda ko‘p qo‘llaniladi.

7.21-misol. Ushbu

$$1)\ \lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}, \quad (\beta\neq 0), \qquad 2)\ \lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{\sin^{3}x},$$

$$3)\ \lim_{x\to\infty}x\sin\frac{\pi}{x}, \qquad 4)\ \lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{\cos x}-2x\operatorname{tg}x\right)$$

limitlarni hisoblang.

Yechilishi. 1) berilgan $\frac{\sin\alpha x}{\sin\beta x}$ kasr funksiyaning shaklini almashtiramiz:

$$\alpha\frac{\sin\alpha x}{\alpha x}\frac{1}{\beta\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}.$$

U holda (III), (IV) va (V) formulalarga asosan,

$$\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\frac{\alpha}{\beta}\lim_{x\to 0}\frac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\lim\limits_{x\to 0}\frac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(sin(alpha*x)/sin(beta*x), x=0);
```

$\frac{a}{b}$.

2) $$\frac{\operatorname{tg}x-\sin x}{\sin^3 x}=\frac{\sin x(1-\cos x)}{\cos x\sin^3 x}=\frac{2\sin^2\frac{x}{2}}{\cos x\sin^2 x}=\frac{1}{2}\frac{\frac{\sin\frac{x}{2}}{x/2}\frac{\sin\frac{x}{2}}{x/2}}{\cos x\frac{\sin x}{x}\frac{\sin x}{x}}$$

$\frac{x}{2}=y$ deb olib, (II)—(V) formulalarni hamda 7.18-misolning B) bandini e'tiborga olsak, u holda

$$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{\sin^3 x}=\frac{1}{2}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit((tan(x)-sin(x))/(sin(x))^3, x=0);
```

$\frac{1}{2}$

3) bunda $\frac{1}{x}=y$ almashtirishni bajaramiz. $x\to\infty$ da $y\to 0$. $x\sin\frac{\pi}{x}=\pi\frac{\sin\pi y}{\pi y}$ va (II), (V) formulalarni e'tiborga olsak,

$$\lim_{x\to\infty}x\sin\frac{\pi}{x}=\lim_{y\to 0}\pi\frac{\sin\pi y}{\pi y}=\pi$$

ekanligini topamiz.

Misolni MAPLE tizimidan foydalanib yechish:

> limit(x*sin(Pi/x), x=infinity);

π.

4) $\frac{\pi}{2}-x=y$ almashtirishni bajaramiz:

$$\frac{\pi}{\cos x}-2x\operatorname{tg}x=\frac{\pi-2x\sin x}{\cos x}=\frac{\pi-2\left(\frac{\pi}{2}-y\right)\cos y}{\cos\left(\frac{\pi}{2}-y\right)}=\frac{2y\cos y}{\sin y}=\frac{2\cos y}{\frac{\sin y}{y}}.$$

B) tasdiqni hamda (IV), (V) formulani e'tiborga olgan holda:

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{\cos x}-2x\operatorname{tg}x\right)=\lim_{y\to0}\frac{2\cos y}{\frac{\sin y}{y}}=2.$$

Misolni MAPLE tizimidan foydalanib yechish:

> limit(Pi/cos(x)-2*x*tan(x), x=Pi/2);

2.

7.22-misol. Ushbu

$$1)\ \lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{x^2},\qquad 2)\ \lim_{x\to0}(\cos x)^{-\frac{1}{x^2}}$$

limitlarni hisoblang.

Yechilishi. 1) $\left(\frac{x^2+4}{x^2-4}\right)^{x^2}=\frac{\left(1+\frac{4}{x^2}\right)^{x^2}}{\left(1-\frac{4}{x^2}\right)^{x^2}}$. Bu kasrning surat va maxrajiga (VI''') formulani qo'llasak, natijada, (IV) formulaga asosan

$$\lim_{x\to\infty}\left(\frac{x^2+4}{x^2-4}\right)^{x^2}=\lim_{x\to\infty}\frac{\left(1+\frac{4}{x^2}\right)^{x^2}}{\left(1-\frac{4}{x^2}\right)^{x^2}}=\frac{\lim\limits_{x\to\infty}\left(1+\frac{4}{x^2}\right)^{x^2}}{\lim\limits_{x\to\infty}\left(1-\frac{4}{x^2}\right)^{x^2}}=\frac{e^4}{e^{-4}}=e^8$$

ekanligini topamiz.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2+4)/(x^2-4))^(x^2), x=infinity);
```

e^8.

2) $\cos x = 1 - 2\sin^2\frac{x}{2}$ ekanligini e'tiborga olib, (VI'') formulaga asosan:

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}} = \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{-\frac{1}{x^2}} = e^{\lambda},$$

bunda

$$\lambda = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{-x^2} = \frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}.$$

Shunday qilib, izlanayotgan limit $e^{\frac{1}{2}}$ ga teng bo'lar ekan, ya'ni

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}} = e^{\frac{1}{2}} = \sqrt{e}.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit((cos(x))^(-1/(x^2)), x=0);
```

$\sqrt{e}$.

7.23-misol. Ushbu

1) $\lim\limits_{x\to a}\frac{\ln x - \ln a}{x-a}$ $(a>0)$; 2) $\lim\limits_{x\to\infty}[\sin\ln(x+1) - \sin\ln x]$;

3) $\lim\limits_{x\to 0}\frac{\operatorname{sh} x}{x}$; 4) $\lim\limits_{x\to 0}\frac{\operatorname{ch} x - 1}{x^2}$;

5) $\lim\limits_{h\to 0}\frac{a^{x+h}+a^{x-h}-2a^x}{h^2}$, $(a>0)$; 6) $\lim\limits_{x\to 0}\frac{(1-\cos^{\mu} x)}{x^2}$ (μ — haqiqiy son) limitlarni hisoblang.

Yechilishi. 1) berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$\frac{\ln x - \ln a}{x-a} = \frac{\ln\frac{x}{a}}{x-a} = \frac{\ln\left[1+\frac{(x-a)}{a}\right]}{x-a} = \ln\left[1+\left(\frac{x-a}{a}\right)\right]^{\frac{1}{x-a}}.$$

E) tasdiqni e'tiborga olgan holda, (VII) formulaga asosan

$$\lim_{x\to a}\frac{\ln x-\ln a}{x-a}=\lim_{x\to a}\ln\left[1+\frac{(x-a)}{a}\right]^{\frac{1}{x-a}}=\ln e^{\frac{1}{a}}=\frac{1}{a}$$

bo'lishi kelib chiqadi.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit((ln(x)-ln(a))/(x-a), x=a);
```

$\frac{1}{a}$.

2) $\sin\ln(1+x)-\sin\ln x=2\sin\frac{\ln(1+x)-\ln x}{2}\cos\frac{\ln(1+x)+\ln x}{2}=$

$$=\frac{2}{x}\sin\frac{\ln\left(1+\frac{1}{x}\right)^x}{2}\cos\frac{\ln x(x+1)}{2}\le\frac{2}{x}\sin\frac{\ln\left(1+\frac{1}{x}\right)^x}{2}$$

tengsizlik o'rinli, chunki $\cos\frac{\ln x(x+1)}{2}\le 1$.

A) tasdiqni, (VI) formulani e'tiborga olsak,

$$\lim_{x\to\infty}\sin\frac{\ln\left(1+\frac{1}{x}\right)^x}{2}=\sin\frac{1}{2},\quad \lim_{x\to\infty}\frac{2}{x}=0$$

bo'lgani uchun izlanayotgan limit 0 ga teng bo'ladi, ya'ni

$$\lim_{x\to\infty}[\sin\ln(1+x)-\sin\ln x]=0.$$

3) $\operatorname{sh}x=\frac{e^x-e^{-x}}{2}$ formulani e'tiborga olgan holda, berilgan funksiyaning shaklini o'zgartiramiz:

$$\lim_{x\to 0}\frac{\operatorname{sh}x}{x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^{2x}-1}{2xe^x}=\lim_{x\to 0}\left(\frac{1}{e^x}\frac{e^{2x}-1}{2x}\right).$$

Ravshanki, $\lim_{x\to 0}\frac{1}{e^x}=1$. (XII) formulaga asosan:

$$\lim_{x\to 0}\frac{e^{2x}-1}{2x}=1.$$

Demak, (II) formulaga asosan

$$\lim_{x\to 0}\frac{\operatorname{sh} x}{x}=1$$

ekanligiga ishonch hosil qilamiz.

4), 3) ga asosan:

$$\lim_{x\to 0}\frac{\operatorname{ch} x-1}{x^2}=\lim_{x\to 0}\frac{2\operatorname{sh}^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\operatorname{sh}\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

5) berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$\frac{a^{x+h}+a^{x-h}-2a^x}{h^2}=\frac{a^x\left(a^h-1+\frac{1}{a^h}-1\right)}{h^2}=\frac{a^x}{a^h}\frac{(a^h-1)}{h}\frac{(a^h-1)}{h}.$$

Bunda, $\lim\limits_{h\to 0}\frac{a^x}{a^h}=a^x$ ekanligini hamda (*X*) formulani e'tiborga olgan holda, $\lim\limits_{h\to 0}\frac{a^{x+h}+a^{x-h}-2a^x}{h^2}=a^x\ln^2 a$ ekanligini topamiz.

6) limiti izlanayotgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$\frac{1-\cos^{\mu} x}{x^2}=-\frac{(-1+\cos^{\mu} x)}{x^2}=-\frac{[1+(\cos x-1)]^{\mu}-1}{x^2}=$$

$$=\frac{[1+(\cos x-1)]^{\mu}-1}{\cos x-1}\cdot\frac{1-\cos x}{x^2}=\frac{[1+(\cos x-1)]^{\mu}-1}{\cos x-1}\cdot\frac{2\sin^2\frac{x}{2}}{x^2}.$$

Bunda $\lim\limits_{x\to 0}(\cos x-1)=0$ ekanligini hamda (V) va (IX) formulalarni hisobga olgan holda, (II) formulaga binoan

$$\lim_{x\to 0}\frac{(1-\cos^{\mu} x)}{x^2}=\frac{\mu}{2}$$

ekanligini topamiz.

Ikkita $f(x)$ va $g(x)$ funksiyalar $X\,(X\subset R)$ to‘plamda berilgan bo‘lib, $f(x)>0,\ x\in X$ hamda a nuqta X to‘plamning limit nuqtasi bo‘lsin. Bu holda daraja – ko‘rsatkichli funksiyaning limiti *D)* va *E)* tasdiqlarni hisobga olgan holda

$$\lim_{x\to a}[f(x)]^{g(x)}=\lim_{x\to a}e^{g(x)\ln f(x)}=e^{\lim\limits_{x\to a}[g(x)\ln f(x)]} \qquad \text{(XIII)}$$

formula orqali topiladi, ya’ni daraja-ko‘rsatkichli funksiyaning limitini topish masalasi $\lim\limits_{x\to a}[g(x)\ln f(x)]$ limitni topishga olib kelinar ekan. Bu limitni hisoblashda quyidagi hollar bo‘lishi mumkin:

I. Agar $\lim\limits_{x\to a}g(x)=A,\quad \lim\limits_{x\to a}\ln f(x)=B$ bo‘lsa, u holda $\lim\limits_{x\to a}f(x)=e^B$ va $e^{\lim\limits_{x\to a}[g(x)\ln f(x)]}=e^{AB}=(e^B)^A=[\lim\limits_{x\to a}f(x)]^{\lim\limits_{x\to a}g(x)}$ formula o‘rinli bo‘ladi.

II. Agar $\lim\limits_{x\to a}[g(x)\ln f(x)]=+\infty$ bo‘lsa, u holda $e^{\lim\limits_{x\to a}[g(x)\ln f(x)]}=+\infty$, $\lim\limits_{x\to a}[g(x)\ln f(x)]=-\infty$ bo‘lganda esa, $e^{\lim\limits_{x\to a}[g(x)\ln f(x)]}=0$ bo‘ladi.

Agar $\lim\limits_{x\to a}[g(x)\ln f(x)]=\infty$ va biror $\dot{U}_\delta(a)$ atrofda $g(x)\ln f(x)$ ko‘paytma funksiyaning ishorasi saqlanmasa, u holda $[f(x)]^{g(x)}=$ $=e^{g(x)\ln f(x)}$ funksiya $x\to a$ da limitga ega bo‘lmaydi.

III. $g(x)\ln f(x)$ ko‘paytmaning $x\to a$ da birining limiti nol, ikkinchisining limiti esa cheksiz bo‘lsa, bu holda quyidagi uch hol bo‘lishi mumkin:

a) $\lim\limits_{x\to a}g(x)=0,\quad \lim\limits_{x\to a}f(x)=+\infty\quad (\infty^0)$,

b) $\lim\limits_{x\to a}g(x)=0,\quad \lim\limits_{x\to a}f(x)=0\quad (0^0)$,

d) $\lim\limits_{x\to a}g(x)=\infty,\quad \lim\limits_{x\to a}f(x)=1\quad (1^\infty)$.

Bu hollarda daraja-ko‘rsatkichli funksiyaning limitini topishda birdaniga 7.4-teoremani qo‘llab bo‘lmaydi.

Bu hollarda $\infty^0,\ 0^0,\ 1^\infty$ ko‘rinishdagi aniqmasliklar paydo bo‘ladi.

7.24-misol. Ushbu

$$\lim_{x\to 1}\left(\frac{x^2}{x^2+1}\right)^{\operatorname{ctg}\pi x}$$

limitni toping.

Yechilishi. $\lim\limits_{x\to 1}\ln\frac{x^2}{x^2+1}=\ln\frac{1}{2}$, $\lim\limits_{x\to 1}\operatorname{ctg}\pi x=\infty$ bo'lgani uchun

$$\lim_{x\to 1}\left[\ln\frac{x^2}{x^2+1}\operatorname{ctg}\pi x\right]=\infty.$$

$0<x<1$ bo'lganda $\ln\frac{x^2}{x^2+1}\operatorname{ctg}\pi x>0$, $1<x<2$ bo'lganda esa,

$$\ln\frac{x^2}{x^2+1}\operatorname{ctg}\pi x<0.$$

Demak, $\ln\frac{x^2}{x^2+1}\operatorname{ctg}\pi x$ funksiya $\dot{U}_\delta(1)$ atrofda har xil ishorali qiymatlarni qabul qiladi.

Shuning uchun $\left(\frac{x^2}{x^2+1}\right)^{\operatorname{ctg}\pi x}=e^{\operatorname{ctg}\pi x\ln\frac{x^2}{x^2+1}}$ funksiya $x\to 1$ da limitga ega emas, lekin $\lim\limits_{x\to 1+0}\left(\frac{x^2}{x^2+1}\right)^{\operatorname{ctg}\pi x}=+\infty$, $\lim\limits_{x\to 1-0}\left(\frac{x^2}{x^2+1}\right)^{\operatorname{ctg}\pi x}=0$.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x^2)/(x^2 + 1))^(1/(tan(Pi*x))), x = 1, right);
0
> limit(((x^2)/(x^2 + 1))^(1/(tan(Pi*x))), x = 1, left);
∞
```

7.25-misol. Ushbu

$$\lim_{x\to\infty}\left(\frac{x+2}{2x+1}\right)^{x^2}$$

limitni toping.

Yechilishi. D), E) tasdiqlar hamda (I) formulaga asosan

$$\lim_{x\to\infty}\left(\frac{x+2}{2x+1}\right)^{x^2}=\lim_{x\to\infty}e^{x^2\ln\frac{x+2}{2x+1}},$$

$$\lim_{x\to\infty}\ln\frac{x+2}{2x+1}=\ln\frac{1}{2}=-\ln 2,\quad \lim_{x\to\infty}x^2=+\infty,\quad \lim_{x\to\infty}\left[x^2\ln\frac{x+2}{2x+1}\right]=-\infty$$

bo'lgani uchun, II holni e'tiborga olgan holda

$$\lim_{x\to\infty}\left(\frac{x+2}{2x+1}\right)^{x^2}=0$$

bo'ladi.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((x+2)/(2*x+1))^(x^2), x=infinity);
0
```

7.8-eslatma. Agar $\lim_{x\to a}f(x)=1$, $\lim_{x\to a}g(x)=\infty$ bo'lsa, u holda 1^∞ ko'rinishdagi aniqmaslik (I) formulani e'tiborga olgan holda

$$\lim_{x\to a}[f(x)]^{g(x)}=\lim_{x\to a}\left\{[1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right\}^{(f(x)-1)g(x)}=e^{\lim\limits_{x\to a}(f(x)-1)g(x)}\qquad\text{(XIV)}$$

formula yordamida ochiladi.

7.26-misol. Ushbu

$$\lim_{x\to\infty}\left(\frac{x^4+4}{x^4-5}\right)^{x^4}$$

limitni toping.

Yechilishi. Bu holda:

$$f(x)=\frac{x^4+4}{x^4-5},\quad g(x)=x^4,\quad [f(x)-1]g(x)=\left[\frac{x^4+4}{x^4-5}-1\right]x^4=\frac{9x^4}{x^4-5}.$$

Shunday qilib, (XIV) formulaga binoan

$$\lim_{x\to\infty}\left(\frac{x^4+4}{x^4-5}\right)^{x^4}=e^{\lim\limits_{x\to\infty}\frac{9x^4}{x^4-5}}=e^9.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit((((x^4)+4)/(x^4-5))^(x^4), x=infinity);
0.
```

7.27-misol. Ushbu

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}}$$

limitni toping.

Yechilishi. Xuddi 7.26-misoldagidek,

$$f(x)=\cos x, \quad g(x)=-\frac{1}{x^2},$$

$$-[f(x)-1]\frac{1}{x^2}=\frac{(1-\cos x)}{x^2}=\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2.$$

(XIV) formulaga asosan:

$$\lim_{x\to 0}(\cos x)^{-\frac{1}{x^2}}=e^{\frac{1}{2}\lim\limits_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x/2}\right)^2}=e^{\frac{1}{2}}=\sqrt{e}.$$

7.28-misol. Ushbu

$$\lim_{x\to 0}\frac{\operatorname{arctg}2x}{x}$$

limitni hisoblang.

Yechilishi. Bunda $y=\operatorname{arctg}2x$ almashtirishni bajaramiz, bundan $2x=\operatorname{tg}y$, $x=\frac{\operatorname{tg}y}{2}$, demak,

$$\lim_{x\to 0}\frac{\operatorname{arctg}2x}{x}=\lim_{y\to 0}\frac{2y}{\operatorname{tg}y}=\lim_{y\to 0}\frac{2\cos y}{\frac{\sin y}{y}}$$

ega bo'lamiz. 7.6-teorema bilan 7.4-eslatmani hamda $\lim\limits_{y\to 0}\frac{\sin y}{y}=1$ ekanligini e'tiborga olsak, 7.4-teoremaga asosan,

$$\lim_{x\to 0}\frac{\operatorname{arctg}2x}{x}=2\frac{\lim\limits_{y\to 0}\cos y}{\lim\limits_{y\to 0}\frac{\sin y}{y}}=2$$

ekanligi kelib chiqadi.

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(arctan(2*x)/x, x=0);
2
```

7.7. Cheksiz katta va cheksiz kichik funksiyalar. X to'plam berilgan bo'lib, a uning limit nuqtasi bo'lsin. X to'plamda $\alpha(x)$ va $\beta(x)$ funksiyalar berilgan bo'lsin.

7.11-ta'rif. Agar $\lim\limits_{x\to a}\alpha(x)=0$ bo'lsa, $\alpha(x)$ funksiya $x\to a$ da *cheksiz kichik funksiya* deyiladi.

Masalan, $\alpha(x)=(a-x)^m$ (m — ixtiyoriy butun musbat son) funksiya $x\to a$ da cheksiz kichik bo'ladi, chunki $\lim\limits_{x\to a}(a-x)^m=0$.

Agar X to'plamda aniqlangan $f(x)$ funksiya $x\to a$ da b limitga ega bo'lsa, u holda $\alpha(x)=f(x)-b$ funksiya $x\to a$ da cheksiz kichik funksiya bo'ladi, chunki $\lim\limits_{x\to a}\alpha(x)=\lim\limits_{x\to a}(f(x)-b)=\lim\limits_{x\to a}f(x)-b=0$.

Demak, b limitga ega bo'lgan har qanday $f(x)$ funksiyani

$$f(x)=b+\alpha(x) \tag{1}$$

ko'rinishda tasvirlash mumkin, bu yerda $\alpha(x)$ cheksiz kichik funksiya.

7.12-ta'rif. Agar $\lim\limits_{x\to a}\beta(x)=\infty$ bo'lsa, $\beta(x)$ funksiya $x\to a$ da *cheksiz katta funksiya* deyiladi.

Masalan, $\beta(x)=\frac{1}{(1-x)^4}$ funksiya $x\to 1$ da cheksiz katta funksiya bo'ladi.

Cheksiz kichik va cheksiz katta funksiyalar quyidagi xossalarga ega:

1°. Cheksiz kichik funksiyalarning yig'indisi (ayirmasi) cheksiz kichik funksiya bo'ladi.

2°. Cheksiz kichik funksiya bilan chegaralangan funksiyalarning ko'paytmasi cheksiz kichik bo'ladi.

3°. Agar $\alpha(x)$ $(\alpha(x)\neq 0)$ cheksiz kichik funksiya bo'lsa, $\frac{1}{\alpha(x)}$ cheksiz katta funksiya bo'ladi.

4°. Agar $\beta(x)$ cheksiz katta funksiya bo'lsa, $\frac{1}{\beta(x)}$ cheksiz kichik funksiya bo'ladi.

5°. Agar $x \to a$ da $f(x)$ cheksiz katta funksiya bo'lsa, $g(x)$ esa biror $U_\delta(a)$ da $|g(x)| > c$ ($x \neq a$, c — biror musbat son) bo'lsa, u holda $f(x)g(x)$ cheksiz katta funksiya bo'ladi.

Eslatma. Cheksiz katta funksiyalarning yig'indisi (ayirmasi) va nisbati cheksiz katta funksiya bo'lmasligi ham mumkin. Masalan, $x \to 0$ da $f(x) = \frac{1}{x}$ va $g(x) = -\frac{1}{x}$ funksiyalarning har biri cheksiz katta funksiyalar bo'lsa ham, ularning yig'indisi $x \to 0$ da cheksiz katta funksiya bo'lmaydi, chunki $\lim\limits_{x\to 0}(f(x)+g(x)) = \lim\limits_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) = 0$. $x \to 0$ da $f(x)$va $g(x)$ funksiyalar cheksiz katta funksiyalar bo'lsa, $\lim\limits_{x\to 0}(f(x) \pm g(x))$ ni hisoblashda I qoidani qo'llab bo'lmaydi. Bu holda, bu limit $\infty - \infty$ ko'rinishdagi aniqmaslikni ifodalaydi. Xuddi shunday, $x \to a$ da $\frac{f(x)}{g(x)}$ ifodaning limiti ham $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslikni ifodalaydi.

Agar $x \to a$ da $f(x)$ va $g(x)$ funksiyalar cheksiz kichik funksiyalar bo'lsa, ularning nisbati $\frac{0}{0}$ ko'rinishdagi aniqmaslikni ifodalaydi.

7.29-misol. Quyidagi funksiyalarning qaysi biri cheksiz kichik funksiya bo'ladi:

1) $f(x) = \frac{x^2-2x+1}{x^3-x}$, $x \to 1$; 2) $f(x) = \sqrt{x^2+1} - x$, $x \to +\infty$;

3) $f(x) = \frac{\ln(x^2-x+1)}{\ln(x^4+x^2+1)}$, $x \to \infty$;

4) $f(x) = \frac{1}{1+2^x}$; *a*) $x \to +\infty$, *b*) $x \to -\infty$.

Yechilishi. 1) $f(x) = \frac{x^2-2x+1}{x^3-x} = \frac{(x-1)^2}{x(x^2-1)} = \frac{(x-1)^2}{x(x-1)(x+1)} = \frac{(x-1)}{x}\cdot\frac{1}{x+1}$.

Demak, berilgan $f(x)$ funksiya ikkita $\frac{x-1}{x}$ va $\frac{1}{x+1}$ funksiyalarning ko'paytmasi shaklida tasvirlandi. Bulardan biri $\frac{1}{x+1}$ funksiya $U_\delta(1)$ atrofda chegaralangan. Masalan, $U_{\frac{1}{2}}(1)$ atrofda $\frac{1}{x+1} < 1$. Ikkinchisi $\frac{x-1}{x}$, $x \to 1$ da cheksiz kichik funksiya bo'ladi.

Shunday qilib, berilgan funksiya 2°-xossaga asosan, $x \to 1$ da cheksiz kichik funksiya bo'lar ekan.

2) berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$\sqrt{x^2+1}-x=\frac{1}{\sqrt{x^2+1}+x}=\frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)}.$$

Bundan, $\lim\limits_{x\to\infty}\frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)}=0.$

Demak, 7.11-ta'rifga ko'ra berilgan funksiya $x\to\infty$ da cheksiz kichik funksiya bo'ladi.

3) ushbu $f(x)=\frac{\ln(x^2-x+1)}{\ln(x^4+x^2+1)}$ berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$f(x)=\frac{\ln(x^2-x+1)}{\ln(x^4+x^2+1)}=\frac{\ln(x^2-x+1)}{\ln(x^2-x+1)+\ln(x^2+x+1)}=\frac{1}{1+\frac{\ln(x^2+x+1)}{\ln(x^2-x+1)}}.$$

D) va E) tasdiqlarni inobatga olgan holda quyidagiga ega o'lamiz:

$$\lim_{x\to\infty}\frac{\ln(x^2+x+1)}{\ln(x^4-x+1)}=\lim_{x\to\infty}\frac{\ln(x^2+x+1)^{\frac{1}{x^2}}}{\ln(x^2-x+1)^{\frac{1}{x^2}}}=1.$$

Shunday qilib, $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\frac{1}{1+\frac{\ln(x^2+x+1)}{\ln(x^2-x+1)}}=\frac{1}{2}.$

Demak, berilgan funksiya $x\to\infty$ da cheksiz kichik funksiya bo'lmas ekan.

4) a) $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to\infty}\frac{1}{1+2^x}=0,$ chunki $\lim\limits_{x\to\infty}2^x=+\infty$.

Demak, bu holda berilgan funksiya $x\to+\infty$ da cheksiz kichik funksiya bo'ladi.

b) $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\frac{1}{1+2^x}=1,$ chunki $\lim\limits_{x\to\infty}2^x=0.$

Bu holda berilgan funksiya $x \to -\infty$ da cheksiz kichik funksiya emas.

7.30-misol. Ushbu

$$f(x) = \frac{x^3-8}{x^2-5x+6}$$

funksiyaning $x \to 3$ da cheksiz katta funksiya ekanligini ko'rsating.

Yechilishi. Berilgan funksiyani $f(x) = \frac{1}{x-3}$ va $g(x) = \frac{x^3-8}{x-2}$ ko'rinishda tasvirlaymiz. Bunda $f(x) = \frac{1}{x-3}$ funksiya $x \to 3$ da cheksiz katta funksiya, $g(x) = \frac{x^3-8}{x-2}$ funksiya esa $x = 2$ nuqtaning biror atrofida, masalan, $\left(\frac{5}{2}, \frac{7}{2}\right)$ atrofda

$$|g(x)| = \left|\frac{x^3-8}{x-2}\right| = |x^2+2x+4| > \frac{61}{4}$$

shartni qanoatlantiradi.

Shunday qilib, 5°-xossaga ko'ra berilgan funksiya $x \to 3$ da cheksiz katta funksiya bo'lar ekan.

7.31-misol. Ushbu funksiyalarning qaysi birlari cheksiz katta funksiya bo'ladi:

1) $f(x) = \frac{1}{x^2-5x+6} - \frac{1}{x^2+x-12}, \quad x \to 3;$

2) $f(x) = \frac{\cos x}{\sqrt[3]{(1-\sin x)^2}}, \quad x \to \frac{\pi}{2};$

3) $f(x) = (1-x)^{1/x^2}$, a) $x \to +0$, b) $x \to -0$.

Yechilishi. 1) berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$x^2-5x+6 = (x-3)(x-2) \quad \text{va} \quad x^2+x-12 = (x-3)(x+4)$$

ekanligini e'tiborga olgan holda berilgan funksiya uchun

$$f(x) = \frac{1}{(x-3)}\left[\frac{1}{x-2} - \frac{1}{x+4}\right] = \frac{1}{(x-3)} \frac{6}{(x-2)(x+4)}$$

ifodaga ega bo'lamiz. Bundan

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{1}{(x-3)}\cdot\lim_{x\to 3}\frac{6}{(x-2)(x+4)}=\infty.$$

Demak, berilgan funksiya $x\to 3$ da cheksiz katta funksiya bo'ladi.

2) berilgan funksiyani ushbu

$$f(x)=\frac{\cos x}{\sqrt[3]{(1-\sin x)^2}}=\frac{\sqrt{1-\sin x}\sqrt{1+\sin x}}{\sqrt[3]{(1-\sin x)^2}}=\frac{\sqrt{1+\sin x}}{(1-\sin x)^{1/6}}$$

ko'rinishda tasvirlaymiz va uning $x\to\frac{\pi}{2}$ dagi limitini hisoblaymiz:

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{1+\sin x}}{(1-\sin x)^{1/6}}=\infty.$$

Demak, berilgan funksiya $x\to\frac{\pi}{2}$ da cheksiz katta funksiya bo'ladi.

3) (XIV) formulaga asosan, a) va b) hollarda berilgan funksiyaning limitini topamiz:

a) $\lim\limits_{x\to 0+0}(1-x)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0+0}\left(-x\frac{1}{x^2}\right)}=0,$

b) $\lim\limits_{x\to 0-0}(1-x)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to 0-0}\left(-x\frac{1}{x^2}\right)}=e^{\infty}=\infty.$

Demak, a) holda berilgan funksiya cheksiz katta emas, b) holda esa cheksiz katta funksiya bo'ladi.

Misolni MAPLE tizimidan foydalanib yechish:

a)

```
> Limit((1-x)^(1/x^2),x=0,right)=limit((1-
x)^(1/x^2),x=0,right);
```

$$\lim_{x\to +0}(1-x)^{\frac{1}{x^2}}=0.$$

b)

```
> Limit((1-x)^(1/x^2),x=0,left)=limit((1-
```

x)^(1/x^2),x = 0,left);

$$\lim_{x\to -0}(1-x)^{\frac{1}{x^2}} = \infty.$$

7.8. Funksiyalarni solishtirish. $O(f)$ **va** $o(f)$ **belgilar.** X to'plamda $f(x)$ va $g(x)$ funksiyalar aniqlangan bo'lsin. Biror a nuqtaning $U_\delta(a)$ atrofida $f(x)$ va $g(x)$ funksiyalarni solishtiramiz.

7.13-ta'rif. Agar shunday $\delta > 0$ va $C > 0$ o'zgarmas sonlar mavjud bo'lib, $\forall x \in U_\delta(a)$ uchun

$$|f(x)| \le C|g(x)| \qquad (1)$$

tengsizlik o'rinli bo'lsa, u holda $x \to a$ da $f(x)$ funksiya $g(x)$ *funksiyaga nisbatan chegaralangan* deyiladi va $f(x) = O(g(x))$ kabi yoziladi.

Xuddi shunday $x \to a+0$, $x \to a-0$, $x \to \infty$, $x \to -\infty$ da ham $f(x) = O(g(x))$ kabi yozuv saqlanadi.

Xususiy holda, $f(x)$ funksiya $U_\delta(a)$ atrofda chegaralangan bo'lsa, u $x \to a$ da $f(x) = O(1)$ kabi yoziladi.

7.14-ta'rif. Agar $x \to a$ da $f(x)$ va $g(x)$ funksiyalar uchun $f(x) = O(g(x))$ va $g(x) = O(f(x))$ munosabatlar o'rinli bo'lsa, u holda $x \to a$ da $f(x)$ va $g(x)$ funksiyalar *bir xil tartibli funksiyalar* deb ataladi va $f(x) \succ\!\!\prec g(x)$, $x \to a$ kabi belgilanadi.

7.7-teorema. *Agar* $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = k$ *mavjud bo'lib,* $k \ne 0$ *bo'lsa, u holda* $f(x)$ *va* $g(x)$ *funksiyalar* $x \to a$ *da bir xil tartibli funksiyalar bo'ladi.*

7.32-misol. Quyidagi funksiyalar juftining qaysi birlari $x \to 0$ da bir xil tartibli funksiyalar bo'ladi:

1) $f(x) = 2\cos x$, $g(x) = x^2 + 4$;

2) $f(x) = \frac{2}{x}$, $g(x) = \frac{1}{2^x - 1}$;

3) $f(x) = x^2\left(3 + \cos\frac{1}{x}\right)$, $g(x) = x^2$;

4) $f(x) = 3^{\sin x} - 1$, $g(x) = \sin x$.

Yechilishi. 1) $\lim\limits_{x\to 0}\frac{2\cos x}{x^2+4}=\frac{2}{4}=\frac{1}{2}\neq 0$ bo'lgani uchun, 7.4-teoremaga asosan $x\to 0$ da $2\cos x \succ\!\prec x^2+4$.

2) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{\frac{2}{x}}{\frac{1}{2^x-1}}=2\lim\limits_{x\to 0}\frac{2^x-1}{x}=2\ln 2\neq 0$ bo'lgani uchun, 7.4-teoremaga asosan $x\to 0$ da $\frac{2}{x}\succ\!\prec\frac{1}{2^x-1}$.

3) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{x^2\left(3+\cos\frac{1}{x}\right)}{x^2}=\lim\limits_{x\to 0}\left(3+\cos\frac{1}{x}\right)$. Bu holda limit mavjud emas, lekin berilgan funksiyalar bir xil tartibli bo'ladi.

Haqiqatdan ham,

$$\left|\frac{f(x)}{g(x)}\right|=3+\cos\frac{1}{x}\leq 4,\quad x\neq 0.$$

Bundan, $|f(x)|\leq 4|g(x)|$.

Demak, (1) shart $C=4$ bo'lganda bajariladi. Shunday qilib, $f(x)=x^2\left(3+\cos\frac{1}{x}\right)$ funksiya $g(x)=x^2$ funksiyaga nisbatan $x\to 0$ da chegaralangan bo'ladi.

Ikkinchi tomondan, $\left|\frac{g(x)}{f(x)}\right|=\frac{1}{3+\cos\frac{1}{x}}\leq 1,\quad x\neq 0.$

Bundan, $|g(x)|\leq|f(x)|$. Bu holda (1) shart $C=1$ bo'lganda bajariladi.

Demak, 7.14-ta'rifga ko'ra $x\to 0$ da $x^2\left(3+\cos\frac{1}{x}\right)\succ\!\prec x^2$.

Bu misoldan ko'rinadiki, 7.7-teorema berilgan funksiyalarning $x\to a$ da bir xil tartibli bo'lishi uchun yetarli shart bo'lib, zaruriy shart bo'la olmas ekan.

4) (X) formulaga asosan: $\lim\limits_{x\to 0}\frac{3^{\sin x}-1}{\sin x}=\ln 3\neq 0$.

Demak, 7.7-teoremaga binoan, $x\to 0$ da $3^{\sin x}-1\succ\!\prec\sin x$ bo'ladi.

7.15-ta'rif. Agar biror $\dot{U}_\delta(a)$ atrofda aniqlangan $f(x)$ va $g(x)$ cheksiz kichik funksiyalar uchun

$$f(x)=\varphi(x)g(x)$$

tenglik o'rinli bo'lib, bunda $\lim\limits_{x\to a}\varphi(x)=0$ bo'lsa, u holda $x\to a$ da $f(x)$ funksiya $g(x)$ funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya deyiladi va u

$$f(x)=o(g(x))$$

kabi belgilanadi.

Xususiy holda, $g(x)=1$ bo'lsa, $x\to a$ da $f(x)=o(1)$ ifoda $f(x)$ funksiyaning cheksiz kichik funksiya ekanligini anglatadi ($x\to a$ da $f(x)\to 0$).

Xuddi yuqoridagidek, $f(x)=o(g(x))$ simvolik ifodaning mazmuni $x\to a-0$, $x\to a+0$, $x\to\infty$, $x\to+\infty$, $x\to-\infty$ da ham saqlanadi.

7.8-teorema. *$x\to a$ da $f(x)$ funksiya $g(x)$ cheksiz kichik funksiyaga nisbatan yuqori tartibli cheksiz kichik funksiya bo'lishi uchun, $\forall x\in \dot{U}_\delta(a)$ uchun $g(x)\neq 0$ bo'lib,*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0$$

tenglik o'rinli bo'lishi zarur va yetarli.

7.33-misol. Quyidagi tasdiqlarning qaysi birlari to'g'ri, qaysi birlari noto'g'ri ekanligini isbotlang:

1) $1-\cos x=o(x)$, $x\to 0$; 2) $1+x^2=o\left(\frac{1}{x^2}\right)$, $x\to 0$;

3) $x=o(\sin^2 x)$, $x\to 0$; 4) $x^2D(x)=o(xD(x))$, $x\to 0$. ($D(x)$ – Dirixle funksiyasi)

Yechilishi. 1) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\lim\limits_{x\to 0}\frac{x\sin^2\frac{x}{2}}{\frac{x}{2}\cdot\frac{x}{2}}\cdot\frac{x}{2}=\lim\limits_{x\to 0}\frac{x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=0$

bo'lgani uchun, 7.8-teoremaga asosan $1-\cos x=o(x)$ tasdiq o'rinli bo'ladi.

2) $\lim\limits_{x\to 0}\frac{(1+x^2)}{\frac{1}{x^2}}=\lim\limits_{x\to 0}x^2(1+x^2)=0$. Demak, tasdiq to'g'ri.

3) $\lim\limits_{x\to 0}\frac{x}{\sin^2 x}\neq 0$. Bu holda tasdiq noto'g'ri.

4) bu holda $g(x)=xD(x)$ funksiya $x\to 0$ da nolga teng. Shunday $\varphi(x)=x$ funksiya mavjudki, $f(x)=\varphi(x)g(x)$ tenglik o'rinli bo'lib, $\lim\limits_{x\to 0}\varphi(x)=0$ bo'ladi.

Demak, 7.15-ta'rifga asosan 4) tasdiq to'g'ri.

7.16-ta'rif. Agar biror $U_\delta(a)$ atrofda aniqlangan $f(x)$ va $g(x)$ funksiyalar uchun

$$f(x)=\varphi(x)g(x)$$

tenglik o'rinli bo'lib, $\lim_{x\to a}\varphi(x)=1$ bo'lsa, $x\to a$ da $f(x)$ va $g(x)$ funksiyalar *o'zaro ekvivalent funksiyalar* deb ataladi va $x\to a$ da $f(x)\sim g(x)$ deb belgilanadi.

7.6-teorema. *$x\to a$ da $f(x)$ va $g(x)$ funksiyalar $U_\delta(a)$ da ($g(x)\neq 0$, $f(x)\neq 0$) o'zaro ekvivalent bo'lishi uchun*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{g(x)}{f(x)}=1$$

bo'lishi zarur va yetarli.

Odatda, ekvivalentlik tushunchasi $f(x)$ va $g(x)$ funksiyalar cheksiz kichik va cheksiz katta bo'lgan hollarda ishlatiladi.

Funksiyalarning ekvivalentligi tushunchasi quyidagi sodda xossalarga ega:

1°. $x\to a$ da $f(x)\sim f(x)$.

2°. Agar $f(x)\sim g(x)$ bo'lsa, $g(x)\sim f(x)$ ham bo'ladi.

3°. $x\to a$ da $f(x)\sim g(x)$ va $g(x)\sim h(x)$ bo'lsa, u holda $f(x)\sim h(x)$ bo'ladi.

4°. Agar $f(x)\sim g(x)$ bo'lsa, u holda $f(x)=O(g(x))$ bo'ladi.

5°. Agar $f(x)\sim g(x)$ va $h(x)\sim S(x)$ bo'lsa, $f(x)h(x)\sim g(x)S(x)$ bo'ladi.

6°. Agar $\lim f(x)=k\neq 0$ bo'lsa, $f(x)\sim k$ bo'ladi.

Bu xossalardan quyidagi munosabat kelib chiqadi: agar $f(x)\sim g(x)$ bo'lsa, u holda

$$f(x)-g(x)=o(g(x)) \quad \text{yoki} \quad f(x)=g(x)+o(g(x)) \qquad (*)$$

o'rinli bo'ladi. Agar $f(x)$ funksiya (*) ko'rinishda tasvirlangan bo'lsa u holda $g(x)$ funksiya $x\to a$ da $f(x)$ ning bosh qismi deb ataladi

Funksiyalarning limitini hisoblashda quyidagi $x\to 0$ da ekvivalen funksiyalar ko'proq ishlatiladi.

$$x \sim \sin x \sim \operatorname{tg} x \sim \arcsin x \sim \operatorname{arctg} x \sim \ln(1+x) \sim e^x - 1. \qquad \text{(XVII)}$$

7.34-misol. Quyidagi $f(x)$ va $g(x)$ funksiyalar juftining qaysi birlari ekvivalent:

1) $f(x)=\sqrt{2x+\sqrt{x+\sqrt{x}}}$; *a)* $g(x)=x^{\frac{1}{8}}$, $x\to+0$,

b) $g(x)=\sqrt{2x}$, $x\to+\infty$;

2) $f(x)=1-\cos\left(1-\cos\frac{1}{x}\right)$, $g(x)=\frac{1}{8}x^{-4}$, $x\to\infty$;

3) $f(x)=x^3-x^2-x+1$, $g(x)=x^3-x$, a) $x\to1$, b) $x\to+\infty$;

4) $f(x)=\frac{2x+1}{x^2+2}$, $g(x)=\frac{3}{x}$; a) $x\to1$, b) $x\to\infty$.

Yechilishi. 1) a) $\lim\limits_{x\to0+0}\frac{\sqrt{2x+\sqrt{x+\sqrt{x}}}}{x^{1/8}}=\lim\limits_{x\to0+0}\frac{x^{1/8}\sqrt{2x^{\frac{3}{4}}+\sqrt{\sqrt{x}+1}}}{x^{1/8}}=1.$

Demak, $x\to0+0$ da $\sqrt{2x+\sqrt{x+\sqrt{x}}}\sim x^{1/8}$.

b) $\lim\limits_{x\to+\infty}\frac{\sqrt{2x+\sqrt{x+\sqrt{x}}}}{\sqrt{2}x^{\frac{1}{2}}}=\lim\limits_{x\to+\infty}\frac{\sqrt{2+\sqrt{\frac{1}{x}+\frac{1}{x^3}}}}{\sqrt{2}}=1.$

7.9-teoremaga asosan, $x\to+\infty$ da $\sqrt{2x+\sqrt{x+\sqrt{x}}}\sim\sqrt{2}\sqrt{x}$.

2) A) tasdiqni hamda

$$1-\cos\frac{1}{x}=2\sin^2\frac{1}{2x},\quad 1-\cos\left(2\sin^2\frac{1}{2x}\right)=2\sin^2\left(\sin^2\frac{1}{2x}\right)$$

formulalarni e'tiborga olganda,

$$\lim_{x\to\infty}\frac{1-\cos\left(1-\cos\frac{1}{x}\right)}{\frac{1}{8}x^{-4}}=\lim_{x\to\infty}\left(\frac{\sin\left(\sin^2\frac{1}{2x}\right)}{\sin^2\frac{1}{2x}}\right)^2\left(\frac{\sin\frac{1}{2x}}{\frac{1}{2x}}\right)^4=1$$

ega bo'lamiz.

Demak, berilgan funksiyalar $x\to+\infty$ da ekvivalent bo'lar ekan.

3) a) $\lim\limits_{x\to1}\frac{x^3-x^2-x+1}{x^3-x}=\lim\limits_{x\to1}\frac{x^2(x-1)-(x-1)}{x(x^2-1)}=\lim\limits_{x\to1}\frac{(x-1)(x^2-1)}{x(x^2-1)}=0.$

Demak, $\lim\limits_{x\to 1}\frac{x^3-x^2-x+1}{x^3-x}=0\neq 1$ bo'lgani uchun, berilgan funksiyalar o'zaro ekvivalent emas, ya'ni ekvivalentlik shartini qanoatlantirmaydi.

b) $\lim\limits_{x\to\infty}\frac{x^3-x^2-x+1}{x^3-x}=\lim\limits_{x\to\infty}\frac{x-1}{x}=1.$

Bu holda berilgan funksiyalar $x\to\infty$ da ekvivalent bo'lar ekan.

4) a) $\lim\limits_{x\to 1}\frac{(2x+1)x}{3(x^2+2)}=\frac{1}{3}\neq 1.$

Shunday qilib, $x\to 1$ da $\frac{2x+1}{x^2+2}$ va $\frac{3}{x}$ funksiyalar ekvivalent emas. $x\to 1$ da bir xil tartibli funksiyalar bo'ladi.

b) $\lim\limits_{x\to\infty}\frac{(2x+1)x}{3(x^2+2)}=\lim\limits_{x\to\infty}\frac{2+\frac{1}{x}}{3+\frac{6}{x^2}}=\frac{2}{3}\neq 1.$

Bu holda ham berilgan funksiyalar ekvivalent emas. 7.7-teoremaga ko'ra ular $x\to\infty$ da bir xil tartibli funksiyalar bo'ladi.

Ikkita funksiya nisbatining limitini topishda quyidagi teorema muhim rol o'ynaydi.

7.10-teorema. *Agar $x\to a$ da $f(x)\sim f_1(x)$ va $g(x)\sim g_1(x)$ bo'lib, quyidagi*

$$\lim_{x\to a}\frac{f_1(x)}{g_1(x)}$$

limit mavjud bo'lsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ limit ham mavjud bo'ladi va

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f_1(x)}{g_1(x)}$$

tenglik o'rinli bo'ladi.

7.34-misol. Ushbu limitni hisoblang:

$$\lim_{x\to 0}\frac{\sin^2 2x+\arcsin^2 x+2\,\mathrm{arctg}\,x^2}{7x^2}.$$

Yechilishi. $x \to 0$ da $\sin 2x \sim 2x$, $\arcsin x \sim x$, $\operatorname{arctg} x^2 \sim x^2$ bo'lgani uchun (*) formulani e'tiborga olsak, u holda,

$$\sin 2x = 2x + o(2x), \quad \arcsin x = x + o(x), \quad \operatorname{arctg} x^2 = x^2 + o(x^2).$$

Shunday qilib,

$$\lim_{x \to 0} \frac{\sin^2 2x + \arcsin^2 x + 2\operatorname{arctg} x^2}{7x^2} =$$

$$= \lim_{x \to 0} \frac{4x^2 + o(4x^2) + x^2 + o(x^2) + 2x^2 + o(x^2)}{7x^2} =$$

$$= \lim_{x \to 0} \frac{7 + \frac{o(4x^2)}{x^2} + \frac{o(x^2)}{x^2} + \frac{o(x^2)}{x^2}}{7} = 1.$$

Misolni MAPLE tizimidan foydalanib yechish:

```
> limit(((sin(2*x))^2 +
(arcsin(x))^2 + 2*arctan(x^2))/(7*x^2), x = 0);
1
```

7.9. Funksiyaning yuqori va quyi limiti. $f(x)$ funksiya $X(X \subset R)$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin.

7.17-ta'rif. Agar X to'plamning elementlaridan tuzilgan va $x_n \to a$ da shunday $\{x_n\}$ $(x_n \neq a,\ n = 1,2,...)$ ketma-ketlik mavjud bo'lib, $\lim_{n \to \infty} f(x_n) = b$ bo'lsa, b son $f(x)$ funksiyaning $x \to a$ dagi *qismiy limiti* deb ataladi.

Cheksiz va bir tomonli qismiy limitlar ham xuddi shunday ta'riflanadi.

Funksiyaning qismiy limitlari ichida har doim eng kattasi va eng kichigi topiladi. Ular mos ravishda funksiyaning yuqori va quyi limiti deb ataladi va $\overline{\lim_{x \to a}} f(x)$, $\underline{\lim_{x \to a}} f(x)$ kabi belgilanadi.

$\overline{\lim_{x \to a}} f(x) = \underline{\lim_{x \to a}} f(x)$ tenglikning bajarilishi funksiyaning limitga ega bo'lishi uchun zaruriy va yetarli shart bo'lib hisoblanadi.

7.35-misol. Ushbu

$$f(x) = \sin^2 \frac{1}{x} + \frac{2}{\pi} \operatorname{arctg} \frac{1}{x}$$

funksiyaning $x \to 0$ da yuqori va quyi limitini toping.

Yechilishi. $x = x_n = -\frac{1}{n\pi}$ $(n = 1, 2, \ldots)$ qiymatda

$$\inf\left\{\sin^2\frac{1}{x}\right\} = 0, \quad \lim_{n\to\infty}\frac{2}{\pi}\operatorname{arctg}\frac{1}{x_n} = \inf\left\{\frac{2}{\pi}\operatorname{arctg}\frac{1}{x}\right\} = -1$$

bo'lgani uchun

$$\underline{\lim}_{x\to 0} f(x) = \underline{\lim}_{x\to 0}\left[\sin^2\frac{1}{x} + \frac{2}{\pi}\operatorname{arctg}\frac{1}{x}\right] =$$
$$= \lim_{n\to\infty}\left[\sin^2 n\pi + \frac{2}{\pi}\operatorname{arctg}(-n\pi)\right] = -1.$$

Xuddi shunday, $x = x_n = -\frac{2}{\pi(1+2n)}$ $(n = 1, 2, \ldots)$ bo'lganda

$$\sup\left\{\sin^2\frac{1}{x}\right\} = 1, \quad \lim_{x\to 0}\frac{2}{\pi}\operatorname{arctg}\frac{1}{x_n} = \sup\left\{\frac{2}{\pi}\operatorname{arctg}\frac{1}{x}\right\} = 1$$

bo'lgani uchun:

$$\overline{\lim_{x\to a}} f(x) = \overline{\lim_{x\to a}}\left[\sin^2\frac{1}{x} + \frac{2}{\pi}\operatorname{arctg}\frac{1}{x}\right] =$$
$$= \lim_{n\to\infty}\left[\sin^2\frac{\pi(1+2n)}{2} + \frac{2}{\pi}\operatorname{arctg}\frac{\pi(1+2n)}{n}\right] = 1 + 1 = 2.$$

Mustaqil yechish uchun misol va masalalar

δ ning qanday qiymatlarida $0 < |x - x_0| < \delta$ ekanligidan $|f(x) - b| < \varepsilon$ tengsizlikning o'rinliligi kelib chiqadi?

7.1. $f(x) = 3x - 2;$ $x_0 = 1;$ $b = 1;$ $\varepsilon = 0{,}001.$

7.2. $f(x) = x^2;$ $x_0 = 2;$ $b = 4;$ $\varepsilon = 0{,}001.$

7.3. $f(x) = \sqrt{x};$ $x_0 = a;$ $b = \sqrt{a};$ $\varepsilon = 0{,}01.$

7.4. $f(x) = 3x^2 - 2;$ $x_0 = 2;$ $b = 10;$ $\varepsilon = 0{,}01.$

7.5. $f(x) = \frac{x^2 - 4x + 3}{x^2 - 2x - 3};$ $x_0 = 3;$ $b = \frac{1}{2};$ $\varepsilon = 0{,}01.$

7.6. $f(x) = \operatorname{sign} x;$ $x_0 = 0;$ $b = 1;$ $\varepsilon = 1{,}5.$

7.7. $f(x) = \sin x;$ $x_0 = \frac{\pi}{2};$ $b = 1;$ $\varepsilon = 0{,}01.$

7.8. $f(x)=\frac{x^2-1}{x^2+1}$; $x_0=2$; $b=\frac{3}{5}$; $\varepsilon=0,1$.

7.9. $f(x)=\frac{x-1}{2(x+1)}$; $x_0=3$; $b=\frac{1}{4}$; $\varepsilon=0,01$.

7.10. $f(x)=|1-3x|$; $x_0=2$; $b=5$; $\varepsilon=0,01$.

$x\to x_0$ da $f(x)$ funksiyaning cheksiz katta ekanligi ma'lum. $|f(x)|>E$ tengsizlik o'rinli bo'lishi uchun x qanday bo'lishi lozim:

7.11. $f(x)=\frac{2+3x}{x}$; $x_0=0$; $E=10^3$.

7.12. $f(x)=\frac{x+2}{x-4}$; $x_0=4$; $E=1000$.

7.13. $f(x)=\frac{1}{e^x-1}$; $x_0=0$; $E=1000$.

7.14. $f(x)=\frac{1}{x-2}$; $x_0=2$; $E=100$.

7.15. $f(x)=\lg x$; $x_0=+\infty$; $E=100$.

Funksiya limitining Geyne ta'rifidan foydalanib, quyidagi limitlarni toping:

7.16. $\lim\limits_{x\to 1}\frac{2x+1}{5x+3}$.

7.17. $\lim\limits_{x\to 0} x\cos\frac{1}{x}$.

7.18. $\lim\limits_{x\to 0} x^2\sin\frac{1}{x}$.

7.19. $\lim\limits_{x\to 0} x\,\mathrm{arcctg}\frac{1}{x}$.

Funksiya limitining Geyne ta'rifidan foydalanib, quyidagi limitlarning mavjud emasligini isbotlang:

7.20. $\lim\limits_{x\to 2}\sin\frac{1}{x-2}$.

7.21. $\lim\limits_{x\to\infty}\cos x$.

7.22. $\lim\limits_{x\to 0}\cos\frac{1}{x}$.

7.23. $\lim\limits_{x\to\infty}\sin x$.

7.24. $\lim\limits_{x\to 0}\mathrm{arcctg}\frac{1}{x}$.

7.25. $\lim\limits_{x\to 0}\mathrm{sign}\left(\sin\frac{1}{x}\right)$.

Quyidagi munosabatlarni ta'rif yordamida yozing va tegishli misollar keltiring.

7.26. $\lim\limits_{x\to a} f(x)=b$.

7.27. $\lim\limits_{x\to a-0} f(x)=b$.

7.28. $\lim\limits_{x\to a+0} f(x)=b$.

7.29. $\lim\limits_{x\to\infty} f(x)=b$.

7.30. $\lim\limits_{x\to+\infty} f(x)=b$.

7.31. $\lim\limits_{x\to-\infty} f(x)=b$.

7.32. $\lim\limits_{x\to a} f(x)=\infty$.

7.33. $\lim\limits_{x\to a} f(x)=-\infty$.

7.34. $\lim\limits_{x\to a} f(x)=+\infty$.

7.35. $\lim\limits_{x\to a-0} f(x)=-\infty$.

7.36. $\lim\limits_{x\to a-0} f(x)=\infty$.

7.37. $\lim\limits_{x\to a-0} f(x)=+\infty$.

7.38. $\lim_{x\to a+0} f(x)=\infty$.

7.39. $\lim_{x\to a+0} f(x)=-\infty$.

7.40. $\lim_{x\to a+0} f(x)=+\infty$.

7.41. $\lim_{x\to\infty} f(x)=\infty$.

7.42. $\lim_{x\to\infty} f(x)=-\infty$.

7.43. $\lim_{x\to\infty} f(x)=+\infty$.

7.44. $\lim_{x\to-\infty} f(x)=\infty$.

7.45. $\lim_{x\to-\infty} f(x)=-\infty$.

7.46. $\lim_{x\to-\infty} f(x)=+\infty$.

7.47. $\lim_{x\to+\infty} f(x)=\infty$.

7.48. $\lim_{x\to+\infty} f(x)=-\infty$.

7.49. $\lim_{x\to+\infty} f(x)=+\infty$.

7.50. Ushbu $\lim_{x\to c} f(x)=3$, $\lim_{x\to c} g(x)=-2$, $\lim_{x\to c} h(x)=0$ limitlar berilganda quyidagi mavjud limitlarni hisoblang; agar limit mavjud bo'lmasa, nima uchun mavjud emasligini izohlang:

1) $\lim_{x\to c}[f(x)-g(x)]$; 2) $\lim_{x\to c}[f(x)]^2$; 3) $\lim_{x\to c}\frac{f(x)}{g(x)}$;

4) $\lim_{x\to c}\frac{h(x)}{g(x)}$; 5) $\lim_{x\to c}\frac{f(x)}{h(x)}$; 6) $\lim_{x\to c}[f(x)\cdot h(x)]$;

7) $\lim_{x\to c}\frac{g(x)}{f(x)-h(x)}$; 8) $\lim_{x\to c}\frac{1}{f(x)-g(x)}$.

7.51. Ushbu $\lim_{x\to c} f(x)=5$, $\lim_{x\to c} g(x)=0$, $\lim_{x\to c} h(x)=-5$ limitlar berilganda quyidagi limitlarni hisoblang; agar limit mavjud bo'lmasa, nima uchun mavjud emasligini izohlang:

1) $\lim_{x\to c}[2f(x)-3h(x)]$; 2) $\lim_{x\to c}\frac{f(x)}{x-c}$; 3) $\lim_{x\to c}[h(x)]^2$;

4) $\lim_{x\to c}\frac{3}{f(x)+h(x)}$; 5) $\lim_{x\to c}[3+g(x)]^3$.

Quyidagi limitlarni hisoblang:

7.52. $\lim_{x\to c} 2$.

7.53. $\lim_{x\to-2}(x^2+2x-5)$.

7.54. $\lim_{x\to-3} 5|x-2|$.

7.55. $\lim_{x\to 0}\left(2x-\frac{5}{x}\right)$.

7.56. $\lim_{x\to 2}\frac{3-x^3}{5x}$.

7.57. $\lim_{x\to 0}\frac{x^3}{x^2+5}$.

7.58. $\lim_{t\to 0} t\left(2-\frac{3}{t}\right)$.

7.59. $\lim_{x\to 3}\frac{x^3-27}{x-3}$.

7.60. $\lim_{x\to-2}\frac{(x^2-x-6)^2}{x+2}$.

7.61. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$.

7.62. $\lim_{t\to 0}\frac{1-1/t^2}{1-\frac{1}{t}}$.

7.63. $\lim_{t\to 0}\frac{1-\frac{1}{t}}{1+\frac{1}{t}}$.

7.64. $\lim_{t\to 0}\frac{t+\frac{a}{t}}{t+\frac{b}{t}}$.

7.65. $\lim_{x\to 1}\frac{x^7-1}{x^5-1}$.

7.66. $\lim_{h\to 0} h^2\left(1+\frac{1}{h^3}\right)$.

7.67. $\lim_{x\to-3}\left(\frac{3x}{x+3}+\frac{9}{x+3}\right)$.

7.68. $\lim\limits_{x\to 4}\left[\left(\frac{1}{x}-\frac{1}{4}\right)\frac{1}{x-4}\right]$. 7.69. $\lim\limits_{x\to 2}\left[\left(\frac{1}{x}-\frac{1}{2}\right)\left(\frac{1}{x-2}\right)^2\right]$.

7.70. $f(x)=x^2-3x$ funksiya berilganda quyidagi limitlarni hisoblang:

1) $\lim\limits_{x\to 3}\frac{f(x)-f(3)}{x-3}$. 2) $\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}$.

7.71. $\lim\limits_{x\to 0} f(x)$; $f(x)=\begin{cases} x^2, & x<0, \\ 1+x, & x>0. \end{cases}$

7.72. $\lim\limits_{x\to 0} f(x)$; $f(x)=\begin{cases} 3, & x-\text{ratsional bo‘lganda}, \\ -3, & x-\text{irratsional bo‘lganda}. \end{cases}$

7.73. $\lim\limits_{x\to 2} f(x)$; $f(x)=\begin{cases} 3x, & x<1, \\ x+2, & x\geq 1. \end{cases}$ 7.74. $\lim\limits_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{x-1}$.

Misollarda $\frac{f(x)-f(c)}{x-c}$ nisbatni tuzing va $\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$ limit mavjud yoki mavjud emasligini aniqlab, agar mavjud bo'lsa, $(c, f(s))$ nuqtada $f(x)$ funksiya grafigiga o‘tkazilgan urinma tenglamasini tuzing.

7.75. $f(x)=x^2$, $c=2$.

7.76. $f(x)=1-2x+x^2$, $c=-1$.

7.77. $f(x)=\sqrt{x}$, $c=1$.

Quyida berilgan funksiyalarning bir tomonli limitlarini toping:

7.78. $f(x)=2^{\frac{1}{x-1}}$, $x\to 1\pm 0$. 7.79. $f(x)=\frac{4}{(x-2)^3}$, $x\to 2\pm 0$.

7.80. $f(x)=\frac{x^2-1}{|x-1|}$, $x\to 1\pm 0$.

7.81. $f(x)=\frac{\sqrt{1-\cos 2x}}{x}$, $x\to 0\pm 0$.

7.82. $f(x)=\frac{\sqrt{1-\cos 2x}}{x}$, $x\to 0\pm 0$.

7.83. $f(x)=\cos\frac{\pi}{x}$, $x\to 0\pm 0$.

7.84. $f(x)=\frac{1}{1+e^{\frac{1}{x}}}$, $x\to 0\pm 0$. 7.85. $f(x)=\frac{\ln(1+e^x)}{x}$, $x\to\pm\infty$.

7.86. $f(x)=\begin{cases} 3x+1, & x\leq 1, \\ -2x+3, & x>1; \end{cases}$ $x\to 1\pm 0$.

7.87. $f(x)=\dfrac{9x+2}{4x^2-3}, \quad x\to 0\pm 0$

7.88. $f(x)=\begin{cases} x+1, & 0\le x<1, \\ 3x+2, & 1<x<3; \end{cases} \quad x\to 1\pm 0.$

7.89. $f(x)=\dfrac{|\sin x|}{x}, \quad x\to 0\pm 0.$

7.90. $f(x)=\operatorname{sign}(\cos x), \quad x\to \dfrac{\pi}{2}\pm 0.$

7.91. $f(x)=\dfrac{1}{x-[x]}$, a) $x\to -1+0$, b) $x\to -1-0$.

7.92. $f(x)=\dfrac{x-|x|}{2x}$, a) $x\to 0+0$, b) $x\to 0-0$.

7.93. $f(x)=x+[x^2]$, a) $x\to 10+0$, b) $x\to 10-0$.

7.94. $f(x)=\lim\limits_{n\to\infty}\dfrac{x^n}{1+x^n}$, a) $x\to 1+0$, b) $x\to 1-0$.

7.95. $f(x)=\lim\limits_{n\to\infty}\dfrac{2x^n-3}{x^n-1}$, a) $x\to 1+0$, b) $x\to 1-0$.

7.96. $f(x)=3x-\sqrt{9x^2-6x+8}$, a) $x\to +\infty$, b) $x\to -\infty$.

7.97. $f(x)=\sqrt{x^2+6x+3}-\sqrt{x^2+3x+3}$ a) $x\to +\infty$, b) $x\to -\infty$.

7.98. $f(x)=x-\sqrt{\dfrac{x^5+2x^4}{x^3+1}}, \quad x\to -\infty.$

7.99. $f(x)=\operatorname{ctg} x$, a) $x\to 0+0$, b) $x\to 0-0$.

7.100. $f(x)=\dfrac{x}{x+4^{\frac{1}{2-x}}}$, a) $x\to 2+0$, b) $x\to 2-0$.

Quyidagi limitlarni hisoblang:

7.101. $\lim\limits_{x\to 4}\dfrac{x^2-16}{x^2+x-20}$.

7.102. $\lim\limits_{x\to -1}\dfrac{7x^2+4x-3}{2x^2+3x+1}$.

7.103. $\lim\limits_{x\to -5}\dfrac{x^2-2x-35}{2x^2+11x+5}$.

7.104. $\lim\limits_{x\to 0}\dfrac{3x^2+x}{4x^2-5x+1}$.

7.105. $\lim\limits_{x\to -8}\dfrac{2x^2+15x-8}{3x^2+25x+8}$.

7.106. $\lim\limits_{x\to 1}\dfrac{2x^2-3x+1}{4x-3x^2-1}$.

7.107. $\lim\limits_{x\to 7}\dfrac{3x^2-17x-28}{x^2-9x+14}$.

7.108. $\lim\limits_{x\to 3}\dfrac{2x^2-3x-9}{3x^2-5x-10}$.

7.109. $\lim\limits_{x\to 5}\dfrac{2x^2-17x+35}{x^2-x-20}$.

7.110. $\lim\limits_{x\to -1}\dfrac{3x^2+x-2}{3x^2+4x+1}$.

7.111. $\lim\limits_{x\to\infty}\dfrac{5x^3-4x^2+3}{4x^3+2x^2+x}$.

7.112. $\lim\limits_{x\to\infty}\dfrac{7x^5-2x^2+4x}{2x^5+9}$.

7.113. $\lim\limits_{x\to\infty}\dfrac{-x^2+3x+1}{3x^2+x+5}$.

7.114. $\lim\limits_{x\to\infty}\dfrac{5-4x^2-3x^4}{x^4+6x^2+3}$.

7.115. $\lim\limits_{x\to\infty}\dfrac{3x^4-2x^2-7}{3x^4+3x+5}$.

7.116. $\lim\limits_{x\to\infty}\dfrac{18x^2+7x}{14-3x-9x^2}$.

7.117. $\lim\limits_{x\to\infty}\dfrac{3x+14x^2}{1+2x+7x^2}$.

7.118. $\lim\limits_{x\to\infty}\dfrac{7+5x^6+x^8}{6x+4x^4-2x^8}$.

7.119. $\lim\limits_{x\to\infty}\dfrac{5x^4+3x^2+3}{15+2x-2x^4}$.

7.120. $\lim\limits_{x\to\infty}\dfrac{14x^2+7}{3-7x^2}$.

7.121. $\lim\limits_{x\to-\infty}\dfrac{7x^2+5x+9}{12+4x-x^3}$.

7.122. $\lim\limits_{x\to-\infty}\dfrac{3x^4+2x+4}{8x^3-5x+3}$.

7.123. $\lim\limits_{x\to-\infty}\dfrac{4x^2+7x-10}{2x^4-3x+7}$.

7.124. $\lim\limits_{x\to-\infty}\dfrac{5x^4-2x^3+3}{2x^2+5x-7}$.

7.125. $\lim\limits_{x\to-\infty}\dfrac{2x^3+3x^2+5}{3x^2-4x+1}$.

7.126. $\lim\limits_{x\to0}\dfrac{\sqrt{9+x}-3}{x^2+x}$.

7.127. $\lim\limits_{x\to0}\dfrac{3-\sqrt{x^2+9}}{5x^2}$.

7.128. $\lim\limits_{x\to9}\dfrac{\sqrt{2x+7}-5}{3-\sqrt{x}}$.

7.129. $\lim\limits_{x\to-1}\dfrac{\sqrt{5+x}-2}{\sqrt{8-x}-3}$.

7.130. $\lim\limits_{x\to10}\dfrac{\sqrt{x-1}-3}{\sqrt{x+6}-4}$.

7.131. $\lim\limits_{x\to-4}\dfrac{\sqrt{x+12}-\sqrt{4-x}}{x^2+2x-8}$.

7.132. $\lim\limits_{x\to5}\dfrac{x^2-3x-10}{\sqrt{x-1}-\sqrt{9-x}}$.

7.133. $\lim\limits_{x\to0}\dfrac{\sqrt{x^2+3}-\sqrt{3}}{\sqrt{x^2+1}-1}$.

7.134. $\lim\limits_{x\to2}\dfrac{\sqrt{4x+1}-3}{x^3+8}$.

7.135. $\lim\limits_{x\to0}\dfrac{3x}{\sqrt{1+x}-\sqrt{1-x}}$.

7.136. $\lim\limits_{x\to1}\dfrac{2x^2-2}{\sqrt{8+x}-3}$.

7.137. $\lim\limits_{x\to3}\dfrac{\sqrt{x-1}-\sqrt{2}}{\sqrt{2x+3}-3}$.

7.138. $\lim\limits_{x\to0}\dfrac{\sqrt{7-x}-\sqrt{7+x}}{3x}$.

7.139. $\lim\limits_{x\to4}\dfrac{x-4}{\sqrt{5x+5}-5}$.

7.140. $\lim\limits_{x\to2}\dfrac{x^3-8}{\sqrt{4x+1}-3}$.

7.141. $\lim\limits_{x\to0}\dfrac{1-\cos5x}{4x^2}$.

7.142. $\lim\limits_{x\to0}\dfrac{\cos x-\cos^3x}{3x^2}$.

7.143. $\lim\limits_{x\to0}\dfrac{\sin7x+\sin3x}{x\cos2x}$.

7.144. $\lim\limits_{x\to0}\dfrac{\sin4x-\sin2x}{3x}$.

7.145. $\lim\limits_{x\to0}\dfrac{\cos^2x-\cos^22x}{x^2}$.

7.146. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$.

7.147. $\lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2}-x}{\operatorname{tg} x}$.

7.148. $\lim_{x\to 0} \frac{1-\cos 8x}{1-\cos 2x}$.

7.149. $\lim_{x\to 0} \sin 3x \operatorname{ctg} 2x$.

7.150. $\lim_{x\to 0} \frac{\arcsin^2 3x}{2x \sin 5x}$.

7.151. $\lim_{x\to 0} \frac{\arcsin 8x}{\sin 5x}$.

7.152. $\lim_{x\to 0} \frac{\operatorname{tg} 2x - \sin 2x}{x^2}$.

7.153. $\lim_{x\to 0} \frac{x \sin 3x}{\cos x - \cos^3 x}$.

7.154. $\lim_{x\to 0} \frac{\sin^2 3x}{\operatorname{arctg}^2 5x}$.

7.155. $\lim_{x\to\infty} \left(\frac{x+5}{x+10}\right)^{-4x}$

7.156. $\lim_{x\to\infty} \left(\frac{2x}{2x-3}\right)^{5x}$

7.157. $\lim_{x\to\infty} \left(\frac{4x}{3+4x}\right)^{-3x}$.

7.158. $\lim_{x\to\infty} \left(\frac{3x-4}{3x+2}\right)^{5x}$.

7.159. $\lim_{x\to\infty} \left(\frac{1-x}{2-x}\right)^{3x}$

7.160. $\lim_{x\to\infty} \left(\frac{2x-1}{2x+8}\right)^{-2x}$.

7.161. $\lim_{x\to-\infty} \left(\frac{2x-1}{4x+1}\right)^{3x-1}$.

7.162. $\lim_{x\to-\infty} \left(\frac{x+3}{4x-5}\right)^{2x}$.

7.163. $\lim_{x\to-\infty} \left(\frac{3x-1}{2x+5}\right)^{3x}$.

7.164. $\lim_{x\to-\infty} \left(\frac{3x+7}{x+4}\right)^{4x}$.

7.165. $\lim_{x\to\infty} \left(\frac{1-x}{2-10x}\right)^{5x}$.

7.166. $\lim_{x\to-\infty} \left(\frac{5x-7}{x+6}\right)^{2x}$.

7.167. $\lim_{x\to-\infty} \left(\frac{6x+1}{12x+9}\right)^{3x}$.

7.168. $\lim_{x\to-\infty} \left(\frac{x-2}{3x+1}\right)^{5x}$.

7.169. $\lim_{x\to+\infty} \left(\frac{3x+4}{3x+5}\right)^{x+1}$.

Birinchi ajoyib limitga doir misollarni yeching:

7.170. $\lim_{x\to 0} \frac{\sin 20x}{x}$.

7.171. $\lim_{x\to 0} \frac{\sin 9x}{\sin 6x}$.

7.172. $\lim_{x\to 0} \frac{1-\cos 4x}{2x^2}$.

7.173. $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$.

7.174. $\lim_{x\to 0} \frac{\operatorname{tg} 8x}{3x}$.

7.175. $\lim_{x\to a} \frac{\sin x - \sin a}{x-a}$.

7.176. $\lim_{x\to 0} \frac{\sin^2 x - \sin^2 a}{x-a}$.

7.177. $\lim_{x\to 0} \frac{1-\cos^3 x}{x \cdot \sin 4x}$.

7.178. $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2}$.

7.179. $\lim_{x\to 0} \frac{\sqrt{4+\sin x} - \sqrt{4-\sin x}}{3x}$.

7.180. $\lim_{x\to 0} \frac{\sqrt[3]{1+\sin x} - \sqrt[3]{1-\sin x}}{x}$.

7.181. $\lim_{x\to 0} \frac{\sqrt{4+\cos x} - \sqrt{5}}{2x^2}$.

7.182. $\lim_{x\to 0}\frac{5x}{\sin x+\sin 7x}$. 7.183. $\lim_{x\to 0}\frac{\frac{\pi}{2}-x}{\operatorname{tg} x}$. 7.184. $\lim_{x\to 0}\frac{8^x-1}{5^x-1}$.

7.185. $\lim_{x\to\infty} x\left(5^{1/x}-1\right)$. 7.186. $\lim_{x\to 0}\frac{e^{\sin 5x}-e^{\sin x}}{\ln(1-2x)}$.

7.187. $\lim_{x\to 0}\frac{5^{6x}-5^{3x}}{\sin x}$. 7.188. $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{\sin^2 x}$.

7.189. $\lim_{x\to\infty} x^2\left(4^{\frac{1}{x}}-4^{\frac{1}{x+1}}\right)$. 7.190. $\lim_{x\to 0}(1+5x^4)^{\frac{1}{\sin^2 x}}$.

7.191. $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\operatorname{tg}^2 x}$. 7.192. $\lim_{x\to 0}(\ln(e+x))^{\operatorname{ctg} x}$.

7.193. $\lim_{x\to 0}\left(\frac{xe^x+1}{x\pi^x+1}\right)^{\frac{1}{x^2}}$. 7.194. $\lim_{x\to 0}\frac{\operatorname{ch}2x-1}{\cos x-1}$.

7.195. $\lim_{x\to 0}\frac{4^{\operatorname{sh}3x}-4^{\operatorname{sh}x}}{\operatorname{th}x}$. 7.196. $\lim_{x\to 0}(x^2-\ln\operatorname{ch} x^2)$.

7.197. $\lim_{x\to a}\frac{a^{a^x}-a^{x^a}}{a^x-x^a}$ $(a>0)$. 7.198. $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x^2}}$ $(a>0,\ b>0)$.

7.199. $\lim_{x\to\infty}\left(1-\frac{2}{x^4}\right)^{x^2}$. 7.200. $\lim_{x\to 0}(1-\sin x)^{\frac{1}{x^3}}$.

7.201. $\lim_{x\to\infty}\left(\frac{x^2+4x+9}{x^2+3x+5}\right)^{3x}$. 7.202. $\lim_{x\to+0} x^{\frac{1}{1+\ln x}}$.

7.203. $\lim_{x\to\infty}\left(x+\sqrt{x^2+1}\right)^{\frac{3}{\ln x}}$. 7.204. $\lim_{x\to+0} x^x$.

7.205. $\lim_{x\to\frac{\pi}{4}}(\operatorname{tg} x)^{\operatorname{tg}2x}$. 7.206. $\lim_{x\to\frac{\pi}{2}-0}(\operatorname{tg} x)^{\cos x}$.

7.207. $\lim_{x\to\frac{\pi}{2}-0}(\cos x)^{\frac{1}{\ln\operatorname{tg}x}}$. 7.208. $\lim_{x\to+0} x^{\frac{1}{3\ln x+1}}$.

7.209. $\lim_{x\to 0}\sqrt[x]{1-2x}$. 7.210. $\lim_{n\to\infty}\left(\cos\frac{a}{n}\right)^{n^2}$.

7.211. $\lim_{x\to 0}\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)^{\frac{1}{\sin x}}$. 7.212. $\lim_{x\to 0}\left[\operatorname{tg}\left(\frac{\pi}{4}-x\right)\right]^{\operatorname{ctg}x}$.

7.213. $\lim_{x\to 0}\frac{\ln(x^2-x+1)}{\ln(x^{10}+x+1)}$. 7.214. $\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}$.

7.215. $\lim\limits_{x\to 0}(x+e^x)^{\frac{1}{x}}$.   7.216. $\lim\limits_{x\to a}\frac{x^x-a^a}{x-a}$.

7.217. $\lim\limits_{h\to 0}\frac{a^{x+h}+a^{x-h}-2a^x}{h^2}$ $(a>0)$.

7.218. $\lim\limits_{h\to a}n^2\left(\sqrt[n]{x}-\sqrt[n+1]{x}\right)(x>0)$.

7.219. $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ $(a>0,\ b>0,\ c>0)$.

7.220. $\lim\limits_{x\to 0}\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{\frac{1}{x}}$, $(a>0,\ b>0)$.

7.221. a) $\lim\limits_{x\to -\infty}\frac{\ln(1+3^x)}{\ln(1+2^x)}$. b) $\lim\limits_{x\to +\infty}\frac{\ln(1+3^x)}{\ln(1+2^x)}$.

7.222. $\lim\limits_{x\to 1}(1-x)\log_x 2$.   7.223. $\lim\limits_{x\to 0}\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$.

7.224. $\lim\limits_{x\to +\infty}\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$.

7.225. Isbotlang:

a) $\lim\limits_{x\to +\infty}\frac{x^n}{a^x}=0$ $(a>1,\ n>0)$. b) $\lim\limits_{x\to +\infty}\frac{\log_a x}{x^\varepsilon}=0$ $(a>1,\ \varepsilon>0)$.

7.226. $\lim\limits_{x\to 1}\frac{x^4-2x+1}{x^8-2x+1}$.   7.227. $\lim\limits_{x\to 1}\frac{x^{101}-101x+100}{x^2-2x+1}$.

7.228. $\lim\limits_{x\to 1}\left(\frac{3}{1-x^3}+\frac{1}{x-1}\right)$.

7.229. $\lim\limits_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)$ $(m,n\in N)$.

7.230. $\lim\limits_{x\to 1}\left(\frac{x^2-4x+6}{x^2-5x+4}+\frac{x-4}{3x^2-9x+6}\right)$.

7.231. $\lim\limits_{x\to 1}\frac{x^n-1}{x^m-1}$ $(n,m\in N)$.

Quyidagi funksiyalarning qaysi biri cheksiz kichik bo‘ladi:

7.232. $f(x)=\frac{x^2-6x+9}{x^3-27}$, $x\to 3$.

7.233. $f(x)=\frac{1}{1+4^x}$; a) $x\to+\infty$; b) $x\to-\infty$.

7.234. $f(x)=\sqrt{x^4+4}-x^2$; a) $x\to+\infty$, b) $x\to-\infty$.

7.235. $f(x)=\sin\ln(x^2+1)-\sin\ln(x^2-1)$, $x\to\infty$.

7.236. $f(x)=\frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}}$, $x\to+0$.

7.237. $f(x)=\frac{x}{\sqrt[3]{1+x}-1}$, $x\to 0$.

Quyidagi funksiyalardan qaysi biri cheksiz katta bo'ladi?

7.238. $f(x)=\frac{4x^3-8x+4}{x^3-3x^2+3x-1}$, $x\to 1$.

7.239. $f(x)=\frac{1}{x^3-6x^2+9x}-\frac{1}{x^2-2x-3}$, $x\to 3$.

7.240. $f(x)=x\left(\sqrt{x^2+1}-x\right)$, a) $x\to+\infty$, b) $x\to-\infty$.

7.241. $f(x)=\left(\frac{2x+1}{3x-1}\right)^x$, a) $x\to+\infty$, b) $x\to-\infty$.

7.242. $f(x)=(1-x)^{1/x^2}$, a) $x\to+0$, b) $x\to-0$.

Quyidagi tasdiqlardan qaysilari $x\to+\infty$ da to'g'ri ekanligini ko'rsating:

7.243. $20x^2+x\sin x=O(x^2)$.

7.244. $x^2=O(20x^2+x\sin x)$.

7.245. $e^x+x^2=O(e^x)$.

7.246. $e^x=O(e^x+x^2)$.

7.247. $\sqrt{x^2+3x+1}-|x|=O\left(\frac{1}{x}\right)$.

7.248. $\frac{1}{x}=O(\sqrt{x^2+3x+1}-|x|)$.

7.249. $\frac{\operatorname{arctg} x}{1+x^2}=O\left(\frac{1}{x^2}\right)$.

7.250. $\frac{x+1}{x^2+1}=O\left(\frac{1}{x}\right)$.

7.251. $x^p\cdot e^{-x}=o\left(\frac{1}{x^2}\right)$.

7.252. $\ln x=o(x^\varepsilon)$, $\varepsilon>0$.

7.253. $(x+1)^2=o(x^2)$, $x\to+\infty$.

7.254. $\frac{1}{x^2}=o\left(\frac{1}{x}\right)$, $x\to+\infty$.

7.255. $\operatorname{sh}x=O(e^x)$, $x\to+\infty$.

7.256. $x^2=O(e^x)$, $x\to+\infty$.

7.257. $\left(x^2-\frac{1}{2}\right)^{1/2}=O(x)$, $x\to+\infty$.

7.258. $x+o(x)=O(x)$ ekanligini isbotlang.

7.259. $\{O(x)\}^2=O(x^2)=o(x^3)$ ekanligini isbotlang.

7.260. $O(\varphi)O(\psi)=O(\varphi\psi)$

7.261. $O(\varphi)o(\psi)=o(\varphi\psi)$

7.262. $O(\varphi+\psi)=O(|\varphi|+|\psi|)$.

7.263. $\cos\{O(x^{-1})\}=O(1)$.

7.264. $O(o(f))=o(f)$.

7.265. $O(O(f))=O(f)$.

7.266. $o(O(f))=o(f)$.

$x\to 0+0$ da quyidagi tengliklarni isbotlang:

7.267. $4x^2-x^3=O(x^2)$.

7.268. $x\sin\sqrt[5]{x}=O(x^{\frac{6}{5}})$.

7.269. $x\sin\frac{1}{x}=O(|x|)$.

7.270. $\ln x=o\left(\frac{1}{x^{\varepsilon}}\right)$, $\varepsilon>0$.

7.271. $\cos x\,\mathrm{sh}^2 x=o(x)$.

7.272. $\mathrm{tg}^3 x\sin\frac{1}{x}=o(x)$.

7.273. $\dfrac{1}{\sqrt{x}\sqrt[3]{|x-1|(x-2)^2}}=O\left(\dfrac{1}{\sqrt{x}}\right)$.

$l=\underline{\lim}_{x\to 0} f(x)$ va $L=\overline{\lim}_{x\to 0} f(x)$ ni toping.

7.274. $f(x)=e^{\cos\frac{1}{x^2}}$.

7.275. $f(x)=\sin^2\frac{1}{x}+\frac{2}{\pi}\,\mathrm{arctg}\,\frac{1}{x}$.

7.276. $f(x)=(3-x^4)\cos\frac{1}{x}$.

7.277. $f(x)=\frac{1}{2}\,\mathrm{arcctg}\,\frac{1}{x}$.

7.278. $f(x)=\sqrt{\frac{1}{x^2}+\frac{1}{x}}-\frac{1}{x}$.

$l=\underline{\lim}_{x\to\infty} f(x)$ va $L=\overline{\lim}_{x\to\infty} f(x)$ ni toping.

7.279. $f(x)=\cos x$.

7.280. $f(x)=(x^2+2)\cos^4 x$.

7.281. $f(x)=\frac{\pi}{2}\cos^2 x+\mathrm{arctg}\,x$;

7.282. $f(x)=(\sqrt{4x^2+x+1}-\sqrt{4x^2-x+1})(1+\cos 2x)$;

7.283. $f(x)=(1+\sin^2 x)^{\frac{1}{\sin^2 x}}$.

Quyidagi funksiyalarning qaysi biri cheksiz kichik?

7.284. $f(x)=\dfrac{x^3-3x^2+3x-1}{x^3-2x^2+x}$, $x\to 1$.

7.285. $f(x)=\sqrt{4x^2+7}-2x$, a) $x\to+\infty$, b) $x\to-\infty$.

7.286. $f(x)=\dfrac{2^x}{2^x+3^x}$ a) $x\to+\infty$ b) $x\to-\infty$.

7.287. $f(x)=\sin\ln(x^2+1)-\sin\ln(x^2-1)$ $x\to\infty$.

7.288. $f(x)=\frac{\ln(x^2-x+1)}{\ln(x^4+x+1)}$ $x\to+\infty$.

Quyidagi funksiyalarning qaysi birlari cheksiz katta?

7.289. $f(x)=\frac{1}{x^3-6x+9x}-\frac{1}{x^2-4x+3}$, $x\to3$.

7.290. $f(x)=x\left(\sqrt{4x^2+7}-2x\right)$; a) $x\to+\infty$, b) $x\to-\infty$.

7.291. $f(x)=\frac{\sin x}{\sqrt[3]{1-\cos2x}}$, $x\to0$.

7.292. $f(x)=\mathrm{sh}2x-\mathrm{ch}2x$; a) $x\to+\infty$, b) $x\to-\infty$.

7.293. $f(x)=\left(\frac{x+2}{3x+5}\right)^x$; a) $x\to+\infty$, b) $x\to-\infty$.

7.294. $f(x)=(1-x)^{1/x^2}$; a) $x\to+0$, b) $x\to-0$.

7.295. $f(x)=4^{\frac{x+3}{x-2}}$; a) $x\to2+0$, b) $x\to2-0$.

7.296. $f(x)=e^{-\frac{1}{x}}$; a) $x\to+0$, b) $x\to-0$.

α va β larning qanday qiymatlarida $f(x)$ funksiya cheksiz kichik bo'ladi?

7.297. $f(x)=\frac{x^2(x-1)}{(x^2+1)^2}-\alpha x-\beta$, $x\to+\infty$.

7.298. $f(x)=\sqrt{4x^2+x+1}-\alpha x-\beta$; a) $x\to+\infty$, b) $x\to-\infty$.

7.299. $f(x)=x^2\,\mathrm{arctg}\,x^2-\alpha x^2-\beta$; a) $x\to+\infty$, b) $x\to-\infty$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

7.1. $\delta\le\frac{0,001}{3}=\frac{1}{3000}$. 7.2. $\delta\le\sqrt{4,001}-2\approx0,00025$. 7.3. $\delta=\min(a;0,01\sqrt{a})$. 7.4. $\delta=\frac{1}{1500}$. 7.5. $\delta\le\frac{4}{51}$. 7.6. Mavjud emas. 7.7. $\delta\le\frac{\pi}{2}-\arcsin0,92\approx0,14$. 7.8. $\delta<2-\sqrt{3}$. 7.9. $\delta\le\frac{2}{13}$. 7.10. $\delta\le\frac{1}{300}$. 7.11. $\left(-\frac{2}{1003};0\right)\cup\left(0;\ \frac{2}{9997}\right)$. 7.12. $\left(\frac{3996}{1001};\ 4\right)\cup\left(4;\ \frac{4003}{999}\right)$. 7.13. $\left(\ln\frac{999}{1000};\ 0\right)\cup\left(0;\ \ln\frac{1001}{1000}\right)$. 7.14. $\left(\frac{199}{100};\ 2\right)\cup\left(2;\ \frac{201}{100}\right)$. 7.15. (0;

$10^{-100})\cup(10^{100};+\infty)$. 7.16. $\frac{3}{8}$. 7.17. 0. 7.18. 0. 7.19. 0. 7.50. 1) 5. 2) 9. 3) $-\frac{3}{2}$. 4) 0. 5) mavjud emas, chunki maxrajning limiti nolga teng. 6) 0. 7) $-\frac{2}{3}$. 8) $\frac{1}{5}$. 7.51. 1) 25. 2) mavjud emas, chunki maxraj nolga aylanadi. 3) 25. 4) mavjud emas, maxrajning limiti nolga aylanadi. 5) 27. 7.52. 2. 7.53. −5. 7.54. 25. 7.55. Mav-jud emas. 7.56. $-\frac{1}{2}$. 7.57. 0. 7.58. −3. 7.59. 27. 7.60. 0. 7.61. $\frac{1}{4}$. 7.62. Mavjud emas. 7.63. −1. 7.64. $\frac{a}{b}$. 7.65. $\frac{7}{5}$. 7.66. Mavjud emas. 7.67. 3. 7.68. $-\frac{1}{16}$. 7.69. Mavjud emas. 7.70. 1) 3, 2) Mavjud emas. 7.71. Mavjud emas. 7.72. Mavjud emas. 7.73. 4. 7.74. $\frac{1}{\sqrt{2}}$. 7.75. 4; $y=4x-4$. 7.76. −4; $y=-4x$. 7.77. $\frac{1}{2}$; $y=\frac{1}{2}x+\frac{1}{2}$. 7.78. $f(1-0)=0$, $f(1+0)=+\infty$. 7.79. $f(0-0)=-\infty$, $f(0+0)=+\infty$. 7.80. $f(1-0)=-2$, $f(1+0)=2$. 7.81. $f(0-0)=-\sqrt{2}$, $f(0+0)=\sqrt{2}$. 7.82. $f(1-0)=\frac{\pi}{2}$, $f(1+0)=-\frac{\pi}{2}$. 7.83. Mavjud emas. 7.84. $f(0-0)=1$, $f(0+0)=0$. 7.85. $f(-\infty)=0$, $f(+\infty)=1$. 7.86. $f(1-0)=4$, $f(1+0)=1$. 7.87. $f(0-0)=2$, $f(0+0)=-3$. 7.88. $f(1-0)=2$, $f(1+0)=5$. 7.89. $f(0-0)=-1$ $f(0+0)=1$. 7.90. $f\left(\frac{\pi}{2}-0\right)=1$, $f\left(\frac{\pi}{2}+0\right)=-1$. 7.91. a) ∞, b) 1. 7.92. a) 0, b) 1. 7.93. a) 110, b) 109. 7.94. a) 1, b) 0. 7.95. a) 2, b) 3. 7.96. a) 1, b) ∞. 7.97. a) $\frac{3}{2}$, b) $-\frac{3}{2}$. 7.98. −1. 7.99. a) $+\infty$, b) $-\infty$. 7.100. a) 1, b) 0. 7.101. $\frac{8}{9}$. 7.102. 10. 7.103. $\frac{4}{3}$. 7.104. 0. 7.105. $\frac{17}{23}$. 7.106. −0,5. 7.107. 5. 7.108. $\frac{8}{13}$. 7.109. $\frac{1}{3}$. 7.110. 2,5. 7.111. 1,25. 7.112. 3,5 7.113. $-\frac{1}{3}$. 7.114. −3. 7.115. 1. 7.116. −2. 7.117. 2. 7.118. −0,5 7.119. −2,5. 7.120. −2. 7.121. 0. 7.122. $-\infty$. 7.123. 0. 7.124. ∞ 7.125. $-\infty$. 7.126. $\frac{1}{6}$. 7.127. $\frac{1}{30}$. 7.128. $-\frac{6}{5}$. 7.129. $-\frac{3}{2}$. 7.130. $\frac{4}{3}$ 7.131. $-\frac{1}{12\sqrt{2}}$. 7.132. 14. 7.133. $\frac{1}{\sqrt{3}}$. 7.134. $\frac{1}{18}$. 7.135. $\frac{3\sqrt{2}}{2}$ 7.136. 24. 7.137. $\frac{2}{2\sqrt{2}}$. 7.138. $\frac{\sqrt{7}}{21}$. 7.139. 2. 7.140. 18. 7.141. 3,125 7.142. $\frac{1}{3}$. 7.143. 10. 7.144. $-\frac{2}{3}$. 7.145. 3. 7.146. $\frac{1}{2}$. 7.147. 0

7.148. 16. 7.149. $\frac{3}{2}$. 7.150. 0,9. 7.151. $\frac{8}{3}$. 7.152. 0. 7.153. 3. 7.154. $\frac{9}{25}$. 7.155. e^{20}. 7.156. $e^{7,5}$. 7.157. $e^{\frac{9}{4}}$. 7.158. e^{-10}. 7.159. e^{3}. 7.160. $e^{1,5}$. 7.161. $+\infty$. 7.162. $+\infty$. 7.163. 0. 7.164. 0. 7.165. 0. 7.166. 0. 7.167. $+\infty$. 7.168. $+\infty$. 7.169. $e^{-\frac{1}{3}}$. 7.170. 20. 7.171. $\frac{9}{8}$. 7.172. 4. 7.173. $\frac{a}{b}$. 7.174. $\frac{8}{3}$. 7.175. $\cos\alpha$. 7.176. $\cos a \sin a = \sin 2a$. 7.177. $\frac{3}{8}$. 7.178. -6. 7.179. $\frac{1}{6}$. 7.180. $\frac{2}{3}$. 7.181. $-\frac{\sqrt{5}}{40}$. 7.182. $\frac{5}{8}$. 7.183. 0. 7.184. $\frac{\ln 8}{\ln 5}$. 7.185. ln5. 7.186. 2. 7.187. ln125. 7.188. $\frac{3}{2}$. 7.189. ln4. 7.190. e^{5}. 7.191. $\frac{1}{\sqrt{e}}$. 7.192. $e^{\frac{1}{e}}$. 7.193. $\frac{e}{\pi}$. 7.194. -4. 7.195. ln16. 7.196. ln2. 7.197. $a^{a^a}\ln a$. 7.198. $\sqrt{ab}$. 7.199. 1. 7.200. 0. 7.201. e^{3}. 7.202. e. 7.203. e^{3}. 7.204. 1. 7.205. e^{-1}. 7.206. 1. 7.207. e^{-1}. 7.208. $e^{\frac{1}{3}}$. 7.209. e^{-2}. 7.210. $e^{-\frac{a^2}{2}}$. 7.211. 1. 7.212. e^{-2}. 7.213. $\left(\frac{a}{b}\right)^2$. 7.214. $\frac{3}{2}$. 7.215. e^{2}. 7.216. $a^{a}\ln(e\cdot a)$. 7.217. $a^{x}\ln^2 a$. 7.218. $\ln x$. 7.219. $\sqrt[3]{ab\cdot c}$. 7.220. $\frac{1}{\sqrt{ab}}$. 7.221. a) 0. b) $\frac{\ln 3}{\ln 2}$.

7.222. $-\ln 2$. 7.223. $\frac{1}{2}$. 7.224. $\frac{1}{2}$. 7.226. $\frac{1}{3}$. 7.227. 5050. 7.228. 1. 7.229. $\frac{m-n}{2}$. 7.230. 1. 7.231. $\frac{m}{n}$. 7.232. Ha. 7.233. a) ha; b) yo'q. 7.234. a) ha; b) ha. 7.235. Ha. 7.236. Ha. 7.237. Yo'q. 7.238. Ha. 7.239. Ha. 7.240. b) cheksiz katta. 7.241. b) cheksiz katta. 7.242. b) cheksiz katta. 7.274. $l=\frac{1}{e}$; $L=e$. 7.275. $l=-1$; $L=2$. 7.276. $l=-3$; $L=3$. 7.277. $l=0$; $L=+\infty$. 7.278. $l=\frac{1}{2}$; $L=+\infty$. 7.279. $l=-1$; $L=1$ 7.280. $l=0$; $L=+\infty$. 7.281. $l=-\frac{\pi}{2}$; $L=\pi$. 7.282. $l=-1$; $L=1$. 7.283. $l=2$; $L=e$. 7.284. Cheksiz kichik. 7.285. a) cheksiz kichik; b) cheksiz kichik emas. 7.286. a) cheksiz kichik; b) cheksiz kichik emas. 7.287. Cheksiz kichik emas. 7.288. cheksiz kichik emas. 7.289. Cheksiz katta. 7.290. a) cheksiz katta emas; b) cheksiz katta. 7.291. Cheksiz katta emas. 7.292. a) cheksiz katta emas; b) cheksiz katta. 7.293.

a) cheksiz katta emas; b) cheksiz katta. **7.294.** a) cheksiz katta emas. b) cheksiz katta. **7.295.** a) cheksiz katta; b) cheksiz katta emas. **7.296.** a) cheksiz katta; b) cheksiz katta emas. **7.297.** $\alpha=1$, $\beta=-3$. **7.298.** a) $\alpha=2$, $\beta=\frac{1}{4}$; b) $\alpha=-2$, $\beta=-\frac{1}{4}$. **7.299.** a) $\alpha=\frac{\pi}{2}$, $\beta=-1$ b) $\alpha=\frac{\pi}{2}$, $\beta=-1$.

8-§. FUNKSIYANING UZLUKSIZLIGI

8.1. Uzluksiz funksiyaning ta'riflari. $f(x)$ funksiya $X(X\subset R)$ to'plamda aniqlangan bo'lib, a X to'plamning limit nuqtasi va $a\in X$ bo'lsin

8.1-ta'rif. Agar $x\to a$ da $f(x)$ funksiyaning limiti mavjud va u $f(a)$ ga teng, ya'ni

$$\lim_{x\to a} f(x) = f(a) \tag{8.1}$$

bo'lsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

$a=\lim\limits_{x\to a} x$ ekanligini e'tiborga olgan holda (8.1) tenglikni quyidagicha ham yozish mumkin: $\lim\limits_{x\to a} f(x) = f(\lim\limits_{x\to a} x)$.

Demak, funksiya a nuqtada uzluksiz bo'lsa, «lim» belgisi bilan funksiyaning xarakteristikasi «f»ning o'rnini almashtirish mumkin.

8.2-ta'rif (Geyne). Agar X to'plamning elementlaridan tuzilgan va a ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganda ham funksiyaning unga mos qiymatlaridan tuzilgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt $f(a)$ ga intilsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

8.3-ta'rif (Koshi). Agar istalgan $\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon, a)>0$ son topilsaki, funksiya argumenti x ning $|x-a|<\delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$|f(x)-f(a)|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi.

8.4-ta'rif. Agar istalgan $\varepsilon>0$ son uchun shunday $\delta>0$ son topilsaki, funksiya argumenti x ning barcha $x\in U_\delta(a)$ qiymatlarida $f(x)$ funksiyaning mos qiymatlari $f(x)\in U_\varepsilon(f(a))$ bo'lsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi (8.1-chizma).

Matematik belgilardan foydalanib, 8.2-, 8.3-, 8.4-ta'riflarni, mos ravishda, quyidagi ko'rinishda yozish mumkin:

$$\forall\{x_n\}:\lim_{n\to\infty} x_n = a \to \lim_{n\to\infty} f(x_n) = f(a),$$

$$\forall\varepsilon>0 \quad \exists\delta>0: \quad \forall x: |x-a|<\delta \to |f(x)-f(a)|<\varepsilon,$$

$$\forall\varepsilon>0 \quad \exists\delta>0: \quad \forall x\in U_\delta(a) \to f(x)\in U_\varepsilon(f(a)).$$

Bunda, $x-a$ ayirmaga *argument orttirmasi*, $f(x)-f(a)$ ayirmaga esa, *funksiyaning a nuqtadagi orttirmasi* deyiladi. Ular mos ravishda Δx va Δy yoki $\Delta f(a)$ kabi belgilanadi:

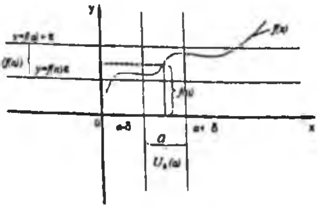
$$\Delta x = x-a, \quad \Delta y = \Delta f(a) = f(x) - f(a).$$

Argument va funksiya orttirmasini quyidagi ko'rinishda ham yozish mumkin:

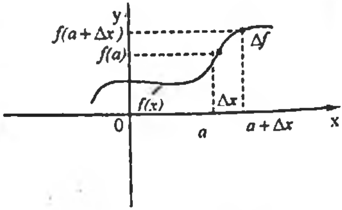
$$x = a+\Delta x, \quad \Delta f(a) = f(a+\Delta x) - f(a). \qquad (8.2)$$

Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, (8.1) va (8.2) munosabatlardan $\lim\limits_{\Delta x\to 0} \Delta f = 0$ kelib chiqadi. Bu esa funksiya uzluksizligini quyidagicha ta'riflash ham mumkinligini ko'rsatadi.

8.5-ta'rif. Agar argumentning a nuqtadagi orttirmasi Δx nolga intilganda $f(x)$ funksiyaning unga mos orttirmasi Δf ham nolga intilsa, ya'ni $\lim\limits_{\Delta x\to 0} \Delta f = 0$ bo'lsa, $f(x)$ funksiya *a nuqtada uzluksiz* deyiladi (8.2-chizma).



8.1-chizma.



8.2-chizma.

8.1-eslatma. X to'plamda $f(x)$ funksiya aniqlangan bo'lib, $a\in X$ to'plamning limit nuqtasi bo'lmasa, ya'ni $\lim\limits_{x\to a} f(x)$ ma'noga ega bo'lmasa, $f(x)$ funksiyaning a nuqtada uzluksizligi haqida gapirishning ma'nosi yo'q.

$X\subset R$ to'plamda $f(x)$ funksiya aniqlangan bo'lib, $a\in X$ esa X to'plamning o'ng (chap) limit nuqtasi bo'lsin.

8.6-ta'rif. Agar $x\to a+0$ $(x\to a-0)$ da $f(x)$ funksiyaning o'ng (chap) limiti mavjud va u $f(a)$ ga teng, ya'ni

$$\lim_{x\to a+0} f(x) = f(a+0), \quad f(a+0) = f(a)$$
$$(\lim_{x\to a-0} f(x) = f(a-0), \quad f(a-0) = f(a))$$

bo'lsa, $f(x)$ funksiya *a nuqtada o'ngdan (chapdan) uzluksiz* deyiladi.

8.7-ta'rif (Geyne). Agar X to'plamning elementlari (nuqtalari) dan tuzilgan va har bir hadi $x_n > a$ $(x_n < a)$ $(n = 1,2,...)$ bo'lib, a ga intiluvchi har qanday $\{x_n\}$ olinganda ham, unga mos kelgan $\{f(x_n)\}$ ketma-ketlik hamma vaqt $f(a)$ qiymatga intilsa, $f(x)$ funksiya *a nuqtada o'ng (chap)dan uzluksiz* deyiladi.

8.8-ta'rif (Koshi). Agar $\forall\varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topilib, argument x ning $a < x < a+\delta$ $(a-\delta < x < a)$ tengsizliklarni qanoatlantiruvchi qiymatlarida $|f(x) - f(a)| < \varepsilon$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya *a nuqtada o'ng (chap) dan uzluksiz* deyiladi.

Ma'lumki, funksiya limitining Geyne va Koshi ta'riflari o'zaro ekvivalent bo'lgani singari, funksiyaning nuqtadagi uzluksizligining Geyne va Koshi ta'riflari ham o'zaro ekvivalent bo'ladi.

8.9-ta'rif. Agar $f(x)$ funksiya $X(X \subset R)$ to'plamning har bir nuqtasida uzluksiz bo'lsa, $f(x)$ funksiya *X to'plamda uzluksiz* deyiladi.

8.10-ta'rif. Agar $f(x)$ funksiya $(a;\ b)$ oraliqda uzluksiz bo'lib, $x = a$ nuqtada o'ngdan, $x = b$ nuqtada esa chapdan uzluksiz bo'lsa, $f(x)$ funksiya $[a;\ b]$ *kesmada uzluksiz* deyiladi.

8.1-teorema. $f(x)$ funksiyaning a nuqtada uzluksiz bo'lishi uchun $f(a+0) = f(a-0)$ tenglikning bajarilishi zarur va yetarlidir.

8.1-misol. Ushbu funksiyaning berilgan a nuqtada uzluksiz ekanligini isbotlang:

1) $f(x) = (x^2 - 5)^3, \quad a = 2;$ 2) $f(x) = \frac{1}{x^3}, \quad x = a, \quad a \neq 0;$

3) $f(x) = \sqrt{x}, \quad x = a, \quad a > 0;$ 4) $f(x) = \begin{cases} x\cos\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad a = 0$

5) $f(x) = x^4, \quad x = a.$

Yechilishi. 1) birinchidan, $x \to 2$ da $f(x) = (x^2 - 5)^3$ funksiyaning limiti, chekli limitga ega bo'lgan funksiyalar ustidagi arifmetik amallarga asosan (I—II formulalarga q.), mavjud, ya'ni

$$\lim_{x\to 2}(x^2-5)^3=-1,$$

ikkinchidan, bu funksiyaning $x=2$ nuqtadagi qiymati ham, -1 ga teng.

Demak, berilgan $f(x)=(x^2-5)^3$ funksiya, 8.1-ta'rifga asosan $a=2$ nuqtada uzluksizdir:

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}(x^2-5)^5=f(2)=-1.$$

2) $x\to a$ $(a\neq 0)$ da (III) formuladan foydalanib,

$$\lim_{x\to a}\frac{1}{x}=\frac{1}{a}, \qquad \lim_{x\to a}\frac{1}{x^3}=\frac{1}{a^3}$$

ekanligini topamiz, ya'ni $\exists \lim\limits_{x\to a}\frac{1}{x^3}=\frac{1}{a^3}$. $f(x)=\frac{1}{x^3}$ funksiyaning $x=a$ dagi qiymati ham, $f(a)=\frac{1}{a^3}$ bo'lgani uchun, 8.1-ta'rifga asosan $f(x)=\frac{1}{x^3}$ funksiya $x=a$ da uzluksiz.

3) $f(x)=\sqrt{x}$ funksiyaning $x=a>0$ nuqtada uzluksizligini Koshi ta'rifi, ya'ni 8.3-ta'rif bo'yicha ko'rsatamiz: $\forall\varepsilon>0$ son berilgan bo'lsin.

a) berilgan $\varepsilon>0$ son bo'yicha $\delta=\delta(\varepsilon)>0$ *ni topamiz.* Buning uchun avvalo $|\sqrt{x}-\sqrt{a}|$, $|x-a|$ ifodalar orasidagi bog'lanishni topish kerak. Birinchi ifoda shaklini quyidagicha o'zgartiramiz:

$$|\sqrt{x}-\sqrt{a}|=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}.$$

$x>0$ lar uchun $|\sqrt{x}-\sqrt{a}|<\frac{1}{\sqrt{a}}\cdot|x-a|$ tengsizliklar o'rinli bo'ladi.

Demak, $|\sqrt{x}-\sqrt{a}|$ ifoda berilgan ε dan kichik bo'lishi uchun $\delta=\varepsilon\sqrt{a}$ deb olish yetarli.

b) Endi topilgan δ *ning «ishlash»ini* ko'rsatamiz. $|x-a|<\varepsilon\sqrt{a}$ bo'lsin. Bundan $\varepsilon>\frac{|x-a|}{\sqrt{a}}$, $x>0$ bo'lganda $\frac{1}{\sqrt{a}}>\frac{1}{\sqrt{x}+\sqrt{a}}$ ekanligini e'tiborga olsak,

$$\varepsilon > \frac{|x-a|}{\sqrt{a}} > \frac{|x-a|}{\sqrt{x}+\sqrt{a}} = \frac{|x-a||\sqrt{x}-\sqrt{a}|}{|x-a|} = |\sqrt{x}-\sqrt{a}|$$

ga ega bo'lamiz.

Demak, $|x-a|<\delta = \sqrt{a}\cdot\varepsilon$ tengsizlikni qanoatlantiruvchi barcha x lar uchun $|\sqrt{x}-\sqrt{a}|<\varepsilon$ tengsizlik o'rinli bo'lar ekan.

Shunday qilib, 8.3-ta'rifga asosan, $f(x)=\sqrt{x}$ funsiya $x=a>0$ nuqtada uzluksiz ekan.

Endi, $\sqrt{x}$ funksiyaning $a=0$ nuqtada o'ngdan uzluksizligini ko'rsatamiz.

$|\sqrt{x}-\sqrt{0}|<\varepsilon$ tengsizlik $0\le x<\varepsilon^2$ tengsizlikka teng kuchli. Bunda $\delta=\varepsilon^2$ deb olsak, $0\le x<\delta$ ni qanoatlantiruvchi barcha x ning qiymatlarida $\sqrt{x}<\varepsilon$ tengsizlik bajariladi. Shunday qilib, $\lim\limits_{x\to 0+0}\sqrt{x}=0$ bo'lgani uchun $\sqrt{x}$ funksiya $a=0$ nuqtada o'ngdan uzluksiz ekan.

4) $f(x)$ funksiya $\forall x\in R$ uchun aniqlangan. $\forall\varepsilon>0$ son berilgan bo'lsin.

a) berilgan $\varepsilon>0$ son bo'yicha $\delta=\delta(\varepsilon)>0$ sonni shunday izlashimiz kerakki, natijada $|x|<\delta$ tengsizlikni qanoatlantiruvchi barcha x lar uchun

$$|f(x)-f(0)| = \left|x\cos\frac{1}{x}\right| < \varepsilon \qquad (8.3)$$

tengsizlik bajarilishi kerak. Buning uchun biz, avvalo

$$\left|x\cos\frac{1}{x}\right|, \quad |x|$$

ifodalar orasidagi bog'lanishni topamiz: $\left|x\cos\frac{1}{x}\right|\le|x|$, chunki $x\ne 0$, $\forall x\in R$ uchun $\left|\cos\frac{1}{x}\right|\le 1$.

Demak, $|x|<\delta$ ni qanoatlantiruvchi barcha x lar uchun (8.3) tengsizlikning bajarilishi uchun $\delta=\varepsilon$ deb olish yetarli bo'ladi.

b) endi tanlangan δ ning *«ishlash»ini* ko'rsatamiz. $|x|<\varepsilon$ bo'lsin. Bundan $\forall x\in R(x\ne 0)$ uchun $1\ge\left|\cos\frac{1}{x}\right|$, $f|0|=0$ ekanligini e'tiborga olgan holda, $\varepsilon>|x|\cdot 1\ge|x|\cdot\left|\cos\frac{1}{x}\right|$, $\left|x\cos\frac{1}{x}-0\right|<\varepsilon$

tengsizlik o'rinli ekanligiga ishonch hosil qilamiz. Shunday qilib, $f(x)=x\cos\frac{1}{x}$ funksiya $x=0$ da uzluksiz.

5) $f(x)=x^4$ funksiyaning $\forall a\in R$ da uzluksizligini 8.5-ta'rifdan foydalanib isbotlaymiz. $\forall a\in R$ va $\forall\Delta x$ lar uchun:

$$\Delta f=(a+\Delta x)^4-a^4=4a^3\Delta x+6a^2(\Delta x)^2+4a(\Delta x)^3+(\Delta x)^4.$$

(I—III) formulalarni e'tiborga olgan holda,

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}(4a^3\Delta x+6a^2(\Delta x)^2+4a(\Delta x)^3+(\Delta x)^4)=0.$$

ekanligini topamiz. Demak, 8.5-ta'rifga asosan, $f(x)=x^4$ funksiya har bir $a\in R$ nuqtada uzluksiz.

8.2-misol. Ushbu $f(x)=x^2\cdot D(x)$ funksiyaning $x=0$ nuqtada uzluksiz ekanligini ko'rsating. Bu yerda, $D(x)$ – Dirixle funksiyasi.

Yechilishi. Ixtiyoriy nolga intiluvchi $\{x_n\}$ ketma-ketlik olamiz. U holda unga mos $\{f(x_n)\}$ ketma-ketlik $f(x_n)=x_n^2D(x_n)$ ko'rinishga ega bo'ladi.

Ma'lumki, Dirixle funksiyasi chegaralangan. $\{x_n\}$ ketma-ketlik esa cheksiz kichik bo'lgani uchun $\{f(x_n)\}$ ketma-ketlik ham cheksiz kichik bo'ladi. Demak, $n\to\infty$ da $f(x_n)=x_n^2D(x_n)\to 0$ bo'ladi. Bu esa qaralayotgan funksiyaning, 8.2-ta'rifga asosan $a=0$ nuqtada uzluksiz ekanligini bildiradi.

8.3-misol. Ushbu $f(x)=\sqrt{x+6}$ funksiyaning $a=3$ nuqtada uzluksiz ekanligini ko'rsating.

Yechilishi. Berilgan funksiya $a=3$ nuqtada aniqlangan. a) $\forall\varepsilon>0$ son olib, bu ε songa ko'ra, $\delta>0$ sonni $\delta=3\varepsilon$ bo'lsin deb qaralsa, u holda, $|x-3|<\delta$ bo'lganda

$$|f(x)-f(3)|=\left|\sqrt{x+6}-3\right|=\frac{|x-3|}{\sqrt{x+6}+3}<\frac{|x-3|}{3}<\frac{\delta}{3}=\varepsilon$$

bo'ladi.

b) endi topilgan δ ning *«ishlash»ini* ko'rsatamiz. $|x-3|<\delta=3\varepsilon$ bo'lsin. Bundan $\varepsilon>\frac{|x-3|}{3}=\frac{\left|\sqrt{x+6}-3\right|\left|\sqrt{x+6}+3\right|}{3}>\left|\sqrt{x+6}-3\right|$, demak, $|x-3|<\delta=3\varepsilon$ tengsizlikni qanoatlantiruvchi x ning qiymatlarida $|\sqrt{x+6}-3|<3$ tengsizlik bajariladi.

Bu esa 8.3-ta'rifga asosan, qaralayotgan funksiyaning $a=3$ nuqtada uzluksiz ekanligini bildiradi.

8.4-misol. 1) $f(x)=\sin x$; 2) $f(x)=a^x$ $(a>1)$ funksiyalarning $\forall a\in R$ nuqtada uzluksiz bo'lishini ko'rsating.

Yechilishi. 1) $\forall a\in R$ nuqtani olib, unga $\Delta x=x-a$ orttirma beramiz. Natijada $f(x)=\sin x$ funksiya ham ushbu $\Delta u=\sin(a+\Delta x)-\sin a$ orttirmaga ega bo'ladi. $|\cos t|\le 1$, $|\sin t|\le|t|$ tengsizliklarni e'tiborga olgan xolda

$$|\Delta y|=|f(a+\Delta x)-f(a)|=\left|2\sin\frac{\Delta x}{2}\cos\frac{a+\Delta x+a}{2}\right|\le 2\frac{|\Delta x|}{2}=|\Delta x|$$

munosabatga ega bo'lamiz. Bundan esa $\Delta x\to 0$ da $\Delta y\to 0$ bo'lishi kelib chiqadi. Demak, $f(x)=\sin x$ funksiya, 8.5-ta'rifga asosan, $\forall a\in R$ nuqtada uzluksizdir.

2) ma'lumki, $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$, umumiy holda $\lim\limits_{x\to 0}a^x=1$. Demak, $f(x)=a^x$ funksiya $x=0$ da uzluksiz, chunki $f(0)=a^0=1$. Endi $f(x)=a^x$ funksiyaning $\forall x_0\in R$ nuqtada uzluksiz ekanligini ko'rsatamiz. x_0 nuqtaga ixtiyoriy $\Delta x\ne 0$ orttirma beramiz, $x_0+\Delta x\in R$.

$$\Delta f=a^{x_0+\Delta x}-a^{x_0}=a^{x_0}(a^{\Delta x}-1)=a^{x_0}\left(\frac{a^{\Delta x}-1}{\Delta x}\right)\cdot\Delta x.$$

$\lim\limits_{\Delta x\to 0}\dfrac{a^{\Delta x}-1}{\Delta x}=\ln a$ ekanligini e'tiborga olgan holda, keyingi tenglikdan $\lim\limits_{\Delta x\to 0}\Delta f=0$ bo'lishi kelib chiqadi. Demak, $f(x)=a^x$ funksiya R da uzluksiz.

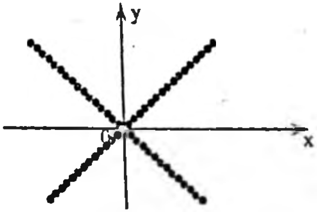
8.5-misol. Ushbu

$$f(x)=\begin{cases}x, & x - \text{ratsional son bo'lganda},\\ -x, & x - \text{irratsional son bo'lganda},\end{cases}$$

funksiyani ixtiyoriy $x=a$ nuqtada uzluksizlikka tekshiring.

Yechilishi. Ma'lumki, $f(x)$ funksiya a nuqtaning atrofida ham musbat, ham manfiy qiymatlar qabul qiladi. Agar $x\in U_\delta(0)$ bo'lsa, $f(x)\in U_\varepsilon(0)$ bo'ladi, ya'ni $\lim\limits_{x\to 0}f(x)=f(0)=0$. Demak, 8.4-ta'rifga asosan, $f(x)$ funksiya $x=0$ nuqtada uzluksiz bo'ladi.

Agar $a \neq 0$ bo'lsa, $f(a) \neq 0$. x — ratsional son bo'lganda $\lim_{x \to a} f(x) = a$, x — irratsional son bo'lganda esa, $\lim_{x \to a} f(x) = -a$ bo'ladi. Bu yerdan $f(x)$ funksiyaning $x = a$ nuqtada uzluksiz bo'lmasligi kelib chiqadi (8.3-chizma).



8.3-chizma.

8.6-misol. Ushbu

$$f(x) = \begin{cases} \dfrac{1}{1+8^{\frac{1}{x}}}, & x \neq 0 \text{ bo'lganda}, \\ 0, & x = 0 \text{ bo'lganda} \end{cases}$$

funksiyani $x = 0$ nuqtada o'ngdan va chapdan uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiyaning $x \to 0$ da o'ng va chap limitlarini topamiz:

$$\lim_{x \to 0+0} \cdot f(x) = \lim_{x \to 0+0} \frac{1}{1+8^{\frac{1}{x}}} = 0 = f(0+0) = f(0),$$

$$\lim_{x \to 0-0} \cdot f(x) = \lim_{x \to 0-0} \frac{1}{1+8^{\frac{1}{x}}} = 1 = f(0-0) \neq f(0).$$

Demak, 8.6-ta'rifga ko'ra, berilgan funksiya $x = 0$ nuqtada o'ngdan uzluksiz bo'lib, chapdan uzluksiz emas.

8.7-misol. Ushbu

1) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0,\\ 1, & x=0;\end{cases}$ 2) $f(x)=\cos\frac{\pi}{x}$;

3) $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0;\end{cases}$ 4) $f(x)=\begin{cases}3\cdot 2^x, & x<0,\\ 2a+x, & x\geq 0;\end{cases}$

5) $f(x)=(x^3+1)/(x+1)$

funksiyalarni uzluksizlikka tekshiring.

Yechilishi. 1) $\forall a\in R$ $(a\neq 0)$ olaylik. $f(x)=\frac{\sin x}{x}$ funksiyaning a nuqtada uzluksizligini 8.1-ta'rifga ko'ra ko'rsatamiz. 8.4-misolni hisobga olgan holda, (IV) formulaga asosan,

$$\lim_{x\to a}\frac{\sin x}{x}=\frac{\lim\limits_{x\to a}\sin x}{\lim\limits_{x\to a}x}=\frac{\sin a}{a}.$$

Demak, berilgan funksiya $\forall a\in R$ $(a\neq 0)$ da uzluksiz ekan. Endi $a=0$ bo'lsin. Funksiyaning berilishiga ko'ra, $f(0)=1$; $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. Shunday qilib, $f(x)=\frac{\sin x}{x}$ funksiya $a=0$ nuqtada ham uzluksiz bo'lar ekan.

2) $f(x)=\cos\frac{\pi}{x}$ funksiya R ning $x=0$ dan tashqari barcha nuqtalarida aniqlangan va uzluksiz. $x=0$ nuqtada uzluksizlikka tekshirish uchun

$$x_n=\frac{1}{2n},\quad x_n'=\frac{1}{2n+1}\quad (n=1,2,\ldots)$$

ketma-ketliklarni olamiz. Ravshanki,

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_n'=0,\quad \lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}\cos 2\pi n=1;$$

$$\lim_{n\to\infty}f(x_n')=\lim_{n\to\infty}\cos(2n+1)\frac{\pi}{2}=0.$$

Demak, $f(x)=\cos\frac{\pi}{x}$ funksiya $x=0$ nuqtada o'ng limitga ega emas, $f(x)$ funksiyaning juftligini e'tiborga olganda, u $x=0$ nuqtada chap limitga ham ega bo'lmaydi. Shuning uchun, berilgan funksiya $x=0$ nuqtada uzluksiz bo'lmaydi.

3) $\lim\limits_{x\to0} x\sin\frac{1}{x}=0$, chunki $\left|x\sin\frac{1}{x}\right|\le|x|$ Shartga ko'ra, $x=0$ da $f(0)=0$ bo'lgani uchun 8.1-ta'rifga asosan, $f(x)=x\sin\left(\frac{1}{x}\right)$ funksiya R da uzluksiz.

4) $\lim\limits_{x\to0-0} 3\cdot2^x=3$, $f(0-0)=3$; $\lim\limits_{x\to0+0}(2a+x)=2a$, $f(0+0)=2a$.

Agar $a=\frac{3}{2}$ bo'lsa, berilgan funksiya $x=0$ nuqtada uzluksiz bo'ladi.

5) $f(x)=\frac{x^3+1}{x+1}$ funksiyaning $x=-1$ nuqtadagi chap va o'ng limitlarini topamiz:

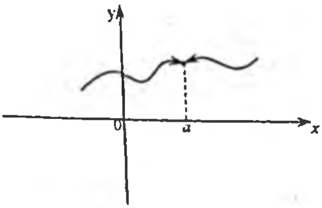
$$\lim_{x\to-1-0} f(x)=\lim_{x\to-1-0}\frac{x^3+1}{x+1}=\lim_{x\to-1-0}(x^2-x+1)=3,$$

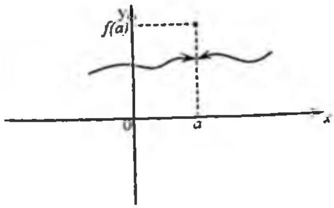
$$\lim_{x\to-1+0} f(x)=\lim_{x\to-1+0}(x^2-x+1)=3.$$

Demak, berilgan funksiyaning $x=-1$ nuqtadagi chap va o'ng limitlari mavjud va ular bir-biriga teng, lekin funksiya $x=-1$ nuqtada aniqlanmagan, shuning uchun u bu nuqtada uzluksiz bo'lmaydi.

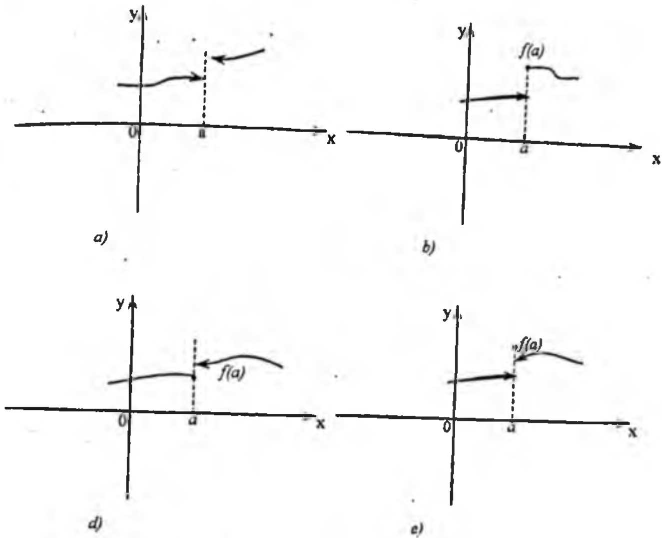
8.2. Funksiya uzilish nuqtalarining turlari. $f(x)$ funksiya $X(X\subset R)$ to'plamda aniqlangan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin, $a\in X$.

8.9-ta'rif. Agar $x\to a$ da $f(x)$ funksiyaning:





8.4-chizma.



8.5-chizma.

1) limiti mavjud va chekli bo'lib, $\lim\limits_{x\to a} f(x) = b \neq f(a)$;

2) $\lim\limits_{x\to a} f(x) = \infty$ ($\lim\limits_{x\to a} f(x) = \pm\infty$);

3) limiti mavjud bo'lmasa, $f(x)$ funksiya *a nuqtada uzilishga* ega deyiladi.

Funksiyaning berilgan nuqtada uzilishga ega bo'lish hollarini alohida qarab o'tamiz:

1-hol. Agar $f(x)$ funksiyaning a nuqtadagi o'ng $f(a+0) = \lim\limits_{x\to a+0} f(x)$ va chap $f(a-0) = \lim\limits_{x\to a-0} f(x)$ limitlari mavjud bo'lib, $f(a+0) = f(a-0) \neq f(a)$ munosabat o'rinli bo'lsa, $f(x)$ funksiya *a nuqtada yo'qotilishi mumkin bo'lgan uzilishga* ega deyiladi (8.4-*a*, *b* chizmalar).

2-hol. Agar $x \to a$ da $f(x)$ funksiyaning o'ng va chap limitlari mavjud va chekli bo'lib, ular bir-biriga teng bo'lmasa ($f(a-0) \neq f(a+0)$), $f(x)$ funksiya *a nuqtada birinchi tur uzilishga* ega deyiladi (8.5-a, b, d, e chizmalar).

Ushbu $|f(a+0) - f(a-0)|$ ayirmaga $f(x)$ funksiyaning a nuqtadagi *sakrashi* deyiladi va u $\omega = |\Delta f(a)|$ kabi belgilanadi.

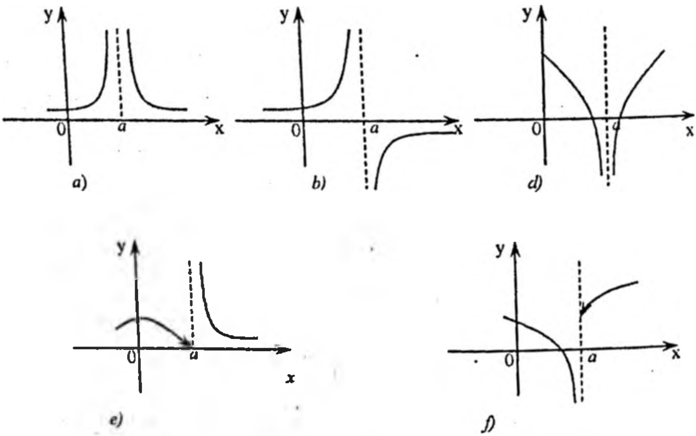
3-hol. Agar $x \to a$ da $f(x)$ funksiyaning:

1) limiti cheksiz (funksiyaning o'ng va chap limitlari cheksiz) bo'lsa;

2) o'ng va chap limitlaridan hech bo'lmaganda bittasi mavjud bo'lmasa;

3) o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlari turli ishorali cheksizdan iborat bo'lsa, $f(x)$ funksiya *a nuqtada ikkinchi tur uzilishga* ega deyiladi (8.6-*a*, *b*, *d*, *e*, *f* chizmalar).

Agar chap $f(a-0)$ yoki o'ng $f(a+0)$ limitlardan hech bo'lmaganda biri ∞ ga teng bo'lsa, $x=a$ nuqta *cheksiz uzilish nuqtasi* deyiladi (8.6-chizma).



8.6-chizma.

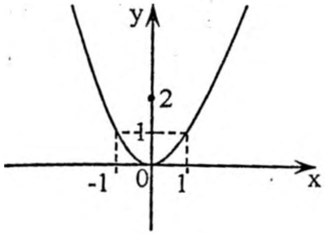
8.8-misol. Ushbu

$$f(x)=\begin{cases} x^4, & x \neq 0 \text{ bo'lganda}, \\ 2, & x=0 \text{ bo'lganda} \end{cases}$$

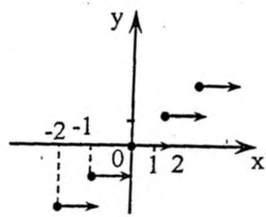
funksiyani uzluksizlikka tekshiring.

Yechilishi. Ravshanki, berilgan funksiyaning $x=0$ nuqtadagi chap limiti $\lim\limits_{x\to 0-0} x^4 = 0 = f(0-0)$ va o'ng limiti $\lim\limits_{x\to 0+0} f(x) = \lim\limits_{x\to 0+0} x^4 = 0 = f(0+0)$ mavjud bo'lib, lekin $f(0-0)=f(0+0) \neq 2 = f(0)$

teng emas. Demak, bu funksiya $x=0$ nuqtada yo'qotilishi mumkin bo'lgan uzilishga ega (8.7-chizma).



8.7-chizma.



8.8-chizma.

8.9-misol. $f(x)=[x]$ funksiyani uzluksizlikka tekshiring.

Yechilishi. $x=n(n\in Z)$ nuqtada berilgan funksiyaning xos qiymatini va o'ng hamda chap limitlarini topamiz:

$$f(n)=[n]=n,\quad \lim_{x\to n+0} f(x)=n=f(n+0),\quad \lim_{x\to n-0} f(x)=n-1=f(n-0).$$

Demak, berilgan funksiya $x=n$ nuqtada birinchi tur uzilishga ega bo'ladi (8.8-chizma).

8.10-misol. Ushbu

$$f(x)=\begin{cases} \frac{1}{7}(2x^2+5), & -\infty<x\le 1 \text{ bo'lganda,}\\ 5-4x, & 1<x<3 \text{ bo'lganda,}\\ x-5, & 3\le x<\infty \text{ bo'lganda,}\end{cases}$$

funksiyani uzluksizlikka tekshiring.

Yechilishi. Berilgan funksiya $(-\infty;\ \infty)$ da aniqlangan va $(-\infty;\ 1)$, $(1;\ 3)$, $(3;\ \infty)$ larda uzluksiz bo'lib, u faqat $x=1$ va $x=3$ nuqtalarda uzilishga ega bo'lishi mumkin. Berilgan funksiyaning $x=1$ nuqtadagi bir tomonli limitlarini hisoblaymiz:

$$f(1-0)=\lim_{x\to 1-0}\frac{1}{7}(2x^2+5)=1,\quad f(1+0)=\lim_{x\to 1+0}(5-4x)=1.$$

Ma'lumki, $x=1$ nuqtada berilgan funksiyaning qiymati birinchi analitik ifoda bilan aniqlanadi:

$$f(1)=\frac{1}{7}(2\cdot 1^2+5)=1.$$

Demak, $f(1-0)=f(1+0)=f(1)=1$ bo'lgani uchun, 8.1-teoremaga asosan, berilgan funksiya $x=1$ nuqtada uzluksiz bo'ladi.

Endi $f(x)$ funksiyaning $x=3$ nuqtada chap va o'ng limitlarini hisoblaymiz:

$$f(3-0)=\lim_{x\to 3-0}(5-4x)=-7,\ f(3+0)=\lim_{x\to 3+0}(x-5)=-2.$$

Bu yerdan $f(3-0)\neq f(3+0)=f(3)$.

Demak, $f(x)$ funksiya $x=3$ nuqtada 1-tur uzilishga ega bo'lib, uning $x=3$ nuqtadagi sakrash kattaligi

$$\omega=|f(3+0)-f(3-0)|=|-2+7|=5$$

bo'ladi.

8.11-misol. Ushbu

$$f(x)=\operatorname{arctg}(\operatorname{tg}x)\left(x\in\left(-\frac{\pi}{2}+k\pi;\frac{\pi}{2}+\pi k\right),k\in Z\right)$$

funksiyani $x=\frac{\pi}{2}$ nuqtada uzluksizlikka tekshiring.

Yechilishi. $f(x)$ funksiyaning $x=\frac{\pi}{2}$ nuqtada chap va o'ng limitlarini hisoblaymiz:

$$\lim_{x\to\frac{\pi}{2}-0}f(x)=\lim_{x\to\frac{\pi}{2}-0}\operatorname{arctg}(\operatorname{tg}x)=\operatorname{arctg}(+\infty)=\frac{\pi}{2},$$

$$\lim_{x\to\frac{\pi}{2}+0}f(x)=\lim_{x\to\frac{\pi}{2}+0}\operatorname{arctg}(\operatorname{tg}x)=\operatorname{arctg}(-\infty)=-\frac{\pi}{2},$$

bu yerdan $\lim\limits_{x\to\frac{\pi}{2}-0}f(x)\neq\lim\limits_{x\to\frac{\pi}{2}+0}f(x)$, demak $\lim\limits_{x\to\frac{\pi}{2}}f(x)$ mavjud emas.

Shunday qilib, $x=\frac{\pi}{2}$ nuqta berilgan funksiya uchun birinchi tur uzilish nuqtasi bo'lar ekan.

8.12-misol. Ushbu $f(x)=4^{\frac{1}{x-3}}+2$ funksiyani $x_1=3$ va $x_2=4$ nuqtalarda uzluksizlikka tekshiring.

Yechilishi. $f(x)$ funksiyaning $x_1=3$ nuqtada chap va o'ng limitlarini hisoblaymiz:

$$\lim_{x\to 3-0} f(x)=\lim_{x\to 3-0}(4^{\frac{1}{x-3}}+2)=2,$$

$$\lim_{x\to 3+0} f(x)=\lim_{x\to 3+0}(4^{\frac{1}{x-3}}+2)=\infty.$$

Demak, funksiya nuqtada uzilishga ega bo'lishining 3-holiga asosan berilgan funksiya $x_1=3$ nuqtada 2-tur uzilishga ega bo'ladi. Endi $f(x)$ funksiyaning $x_2=4$ nuqtada chap va o'ng limitlarini hisoblaymiz:

$$\lim_{x\to 4-0} f(x)=\lim_{x\to 4-0}(4^{\frac{1}{x-3}}+2)=6,$$

$$\lim_{x\to 4+0} f(x)=\lim_{x\to 4+0}(4^{\frac{1}{x-3}}+2)=6, \quad f(4)=4^{\frac{1}{x-3}}+2=4+2=6.$$

Demak, 8.1-teoremaga asosan: $f(4-0)=f(4+0)=f(4)$, ya'ni berilgan $f(x)$ funksiya $x_2=4$ nuqtada uzluksiz bo'ladi.

8.13-misol. $f(x)=\cos^2\frac{1}{x}$ funksiyani uzluksizlikka tekshiring.

Yechilishi. Berilgan $f(x)$ funksiya $x=0$ nuqtada aniqlanmagan, shuning uchun $x=0$ nuqtaning atrofidan nolga intiluvchi ixtiyoriy ikkita ketma-ketlik olamiz:

$$x'_n=\frac{1}{n\pi}\quad (n\in N),\qquad x''_n=\frac{2}{\pi(1+2n)}\quad (n\in N).$$

Bu ketma-ketliklarning har biri $n\to\infty$ da $x'_n\to 0$, $x''_n\to 0$. Funksiyaning bu ketma-ketliklarga mos kelgan qiymatlari ketma-ketliklari $n\to\infty$ da har xil limitlarga intiladi, ya'ni

$$\lim_{n\to\infty}\cos^2\frac{1}{x'_n}=1,\quad \lim_{n\to\infty}\cos^2\frac{1}{x''_n}=0$$

Demak, 8.2-ta'rifga asosan, $f(x)=\cos^2\frac{1}{x}$ funksiyaning $x\to 0$ da limiti mavjud emas. Shuning uchun, berilgan funksiya $x=0$ nuqtada 2-tur uzilishga ega bo'ladi.

8.14-misol. a ning qanday qiymatlarida

$$f(x)=\begin{cases}\cos x, \ x\le 0 \ \text{bo'lganda}, \\ a(x-1)^2, \ x>0 \ \text{bo'lganda},\end{cases}$$

funksiya $x_0=0$ nuqtada uzluksiz bo'ladi?

Yechilishi. Berilgan funksiya $x_0=0$ nuqtada uzluksiz bo'lishi uchun, quyidagi $\lim\limits_{x\to 0-0} f(x)=\lim\limits_{x\to 0+0} f(x)=f(0)$ tenglik bajarilishi kerak. Buni tekshirish uchun funksiyaning chap va o'ng limitlarni hisoblaymiz:

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\cos x=1,$$

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0} a(x-1)^2=a, \ f(0)=1.$$

Demak, $a=1$ bo'lganda berilgan funksiya $x_0=0$ nuqtada uzluksiz bo'lar ekan, $a\ne 1$ qiymatlarda funksiya uzilishga ega.

8.15-misol. a va b ning qanday qiymatlarida berilgan funksiya uzluksiz bo'lishini *MAPLE tizimidan foydalanib* ko'rsating:

$$f(x)=\begin{cases}(x-1)^2, \ x\le 0, \\ ax+b, \ -x<0 \ \text{va} \ x<1, \\ \sqrt{x}, \ 1\le x.\end{cases}$$

Yechilishi.

```
>f:=x->piecewise(x<=0, (x-1)^2, x>0 and x<1, a*x+b,
x>=1, sqrt(x));
```

$$f:=x\to piecewise(x\le 0,(x-1)^2, 0<x \ and \ x<1, \ ax+b, \ 1\le x, \sqrt{x})$$

```
>s1:=limit(f(x), x=0, left)=limit(f(x), x=0, right);
```

$$s1:=1=b$$

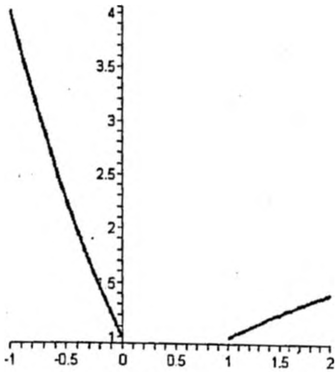
```
>s2:=limit(f(x), x=1, left)=limit(f(x), x=1, right);
```

$$s2:=a+b=1$$

```
>solve({s1, s2}, {a, b});
```

$$\{b=1, \ a=0\}$$

```
>plot(f(x), x=-1...2, thickness=2, color=black);
```



8.9-chizma.

8.16-misol. $f(x)=\frac{x^2-1}{x-1}$ funksiyaning $x_0=1$ nuqtadagi qiymatini shunday tanlash kerakki, natijada berilgan funksiya uzluksiz bo'lsin.

Yechilishi. Berilgan funksiyaning $x\to 1\pm 0$ da limitlarini hisoblaymiz:

$$\lim_{x\to 1\pm 0} f(x)=\lim_{x\to 1\pm 0}\cdot\frac{x^2-1}{x-1}=\lim_{x\to 1\pm 0}(x+1)=2.$$

Agar $f(1)=2$ deb olsak, $x=1$ da funksiya uzluksiz bo'ladi, ya'ni

$$f(x)=\begin{cases}\frac{x^2-1}{x-1}, & x\neq 1 \text{ bo'lganda,}\\ 2, & x=1 \text{ bo'lganda.}\end{cases}$$

8.17-misol. Ushbu

$$1)\ f(x)=\begin{cases}\frac{1}{x-1}, & x<0,\\ (x+1)^2, & 0\le x\le 2,\\ 1-x, & 2<x;\end{cases} \qquad 2)\ f(x)=\frac{|x+3|}{x+3};$$

$$3)\ f(x)=\frac{1}{x^2-4}; \qquad 4)\ f(x)=\frac{|x-1|}{x^2-x^3};$$

$$5)\ f(x)=\frac{(x+1)^2-(x-1)^2}{x^2-x}.$$

funksiyalarning uzilish nuqtalari va ularning turlarini aniqlang, funksiyaning birinchi tur uzilish nuqtalaridagi sakrashini toping.

Yechilishi. 1) Berilgan funksiyaning $x=0$, $x=2$ nuqtalardagi chap va o'ng limitlarini topamiz:

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{1}{x-1}=-1, \qquad \lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}(x+1)^2=1;$$

$$\lim_{x\to 2-0} f(x)=\lim_{x\to 2-0}(x+1)^2=9, \qquad \lim_{x\to 2+0} f(x)=\lim_{x\to 2+0}(1-x)=-1.$$

Demak, berilgan funksiyaning uzilish nuqtalari $x=0$ va $x=2$ bo'lib, ular birinchi tur uzilish nuqtalari bo'lib hisoblanadi, uzilish nuqtalaridagi sakrashlari:

$$\Delta f(0)=|1+1|=2, \qquad \Delta f(2)=|-1-9|=|-10|=10.$$

Misolni MAPLE tizimidan foydalanib yechish:

> y: = piecewise (x < 0, 1/(x − 1), 0 < = x and x < = 2, (x + 1)^2, x > 2, 1 − x);

$$y:=\begin{cases}\frac{1}{x-1}, & x<0\\ (x+1)^2, & 0\le x \quad \text{and} \quad x\le 2\\ 1-x, & 2<x\end{cases}$$

> Limit (y, x = 0, left) = limit (y, x = 0, left); Limit (y, x = 0, right) =
limit (y, x = 0, right);

$$\lim_{x\to 0-}\begin{cases}\frac{1}{x-1}, & x<0\\ (x+1)^2, & 0\le x \quad \text{and} \quad x\le 2\\ 1-x, & 2<x\end{cases} = -1$$

$$\lim_{x\to 0+}\begin{cases}\frac{1}{x-1}, & x<0\\ (x+1)^2, & 0\le x \quad \text{and} \quad x\le 2\\ 1-x, & 2<x\end{cases} = 1$$

> Limit(y, $x=2$, left) = limit (y, $x=2$, left); Limit (y, $x=2$, right) =
limit(y, $x=2$, right);

$$\lim_{x\to 2-}\begin{cases}\frac{1}{x-1}, & x<0\\ (x+1)^2, & 0\le x \quad \text{and} \quad x\le 2\\ 1-x, & 2<x\end{cases} = 9$$

$$\lim_{x\to 2+}\begin{cases}\frac{1}{x-1}, & x<0\\ (x+1)^2, & 0\le x \quad \text{and} \quad x\le 2\\ 1-x, & 2<x\end{cases} = -1$$

2) ma'lumki,

$$|x+3| = \begin{cases} x+3, & x+3>0,\\ -(x+3), & x+3<0.\end{cases}$$

$x=-3$ nuqta berilgan funksiya uchun uzilish nuqtasi bo'ladi. $f(x)=\frac{|x+3|}{x+3}$ funksiyaning $x=-3$ nuqtadagi o'ng va chap limitlarini topamiz:

$$\lim_{x\to -3+0} f(x) = \lim_{x\to -3+0}\frac{(x+3)}{x+3} = 1, \qquad \lim_{x\to -3-0} f(x) = \lim_{x\to -3-0}\frac{-(x+3)}{x+3} = -1.$$

Demak, $x=-3$ nuqta berilgan funksiya uchun birinchi tur uzilish nuqta bo'lib, funksiyaning bu nuqtadagi sakrashi $\Delta f(-3)=2$.

Misolni MAPLE tizimidan foydalanib yechish:

> readlib(singular): singular(abs($x+3$)/($x+3$), x);

$\{x=-3\}$

> Limit(abs(x + 3)/(x + 3), x = −3, left) = limit(abs(x + 3)/(x + 3), x= −3, left);

$$\lim_{x \to -3-} \left(\frac{|x+3|}{x+3}\right) = -1$$

> Limit(abs(x + 3)/(x + 3), x = −3, right) = limit(abs(x + 3)/(x + 3), x = −3, right);

$$\lim_{x \to -3+} \left(\frac{|x+3|}{x+3}\right) = 1$$

3) $f(x) = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$ funksiya uchun $x = 2$, $x = -2$ nuqtalar ikkinchi tur uzilish nuqtalari bo'ladi, chunki

$$\lim_{x \to 2\pm 0} f(x) = \lim_{x \to 2\pm 0} \frac{1}{(x-2)(x+2)} = \pm\infty,$$

$$\lim_{x \to -2\pm 0} f(x) = \lim_{x \to -2\pm 0} \frac{1}{(x-2)(x+2)} = \mp\infty.$$

Misolni MAPLE tizimidan foydalanib yechish:

> readlib(singular):singular(1/(x^2 − 4), x);

{x=−2}, {x=2}

> Limit(1/(x^2 − 4), x = −2, left) = limit(1/(x^2 − 4), x = −2, left);

$$\lim_{x \to -2-} \left(\frac{1}{x^2-4}\right) = -\infty$$

> Limit(1/(x^2 − 4), x = −2, right) = limit(1/(x^2 − 4), x = −2, right);

$$\lim_{x \to -2+} \left(\frac{1}{x^2-4}\right) = \infty$$

> Limit(1/(x^2 − 4), x = 2, left) = limit(1/(x^2 − 4), x = −2, left);

$$\lim_{x \to 2-} \left(\frac{1}{x^2-4}\right) = -\infty$$

> Limit(1/(x^2 − 4), x = 2, right) = limit(1/(x^2 − 4), x = 2, right);

$$\lim_{x \to 2+} \left(\frac{1}{x^2-4}\right) = \infty.$$

4) ma'lumki,

$$|x-1| = \begin{cases} x-1, & x-1>0, \\ -(x-1), & x-1<0. \end{cases}$$

$f(x)=\frac{|x-1|}{x^2-x^3}$ funksiya uchun $x=0$ va $x=1$ nuqtalar uzilish nuqtalari bo'ladi. $\lim\limits_{x\to 0}\frac{|x-1|}{x^2(1-x)}=\infty$. Demak, $x=0$ nuqta ikkinchi tur uzilish nuqtasi bo'ladi. Endi $x=1$ nuqtada funksiyaning o'ng va chap limitlarini hisoblaymiz:

$$\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}\frac{|x-1|}{x^2-x^3}=\lim_{x\to 1+0}\frac{x-1}{x^2(1-x)}=-1,$$

$$\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}\frac{x-1}{x^2-x^3}=\lim_{x\to 1-0}\frac{-(x-1)}{x^2(1-x)}=1.$$

Shunday qilib, $x=1$ nuqta berilgan funksiya uchun birinchi tur uzilish nuqtasi bo'lib, funksiyaning bu nuqtadagi sakrashi $\Delta f(1)=2$.

Misolni MAPLE tizimidan foydalanib yechish:

> readlib(singular):singular(abs(x – 1)/(x^2 – x^3), x);

{x=1}, {x=0}

> Limit(abs(x – 1)/(x^2 – x^3), x = 0, left) = limit(abs(x – – 1)/(x^2 – x^3), x = 0, left);

$$\lim_{x\to 0-}\left(\frac{|x-1|}{x^2-x^3}\right)=\infty,$$

> Limit(abs(x – 1)/(x^2 – x^3), x = 0, right) = limit(abs(x – – 1)/(x^2 – x^3), x = 0,right);

$$\lim_{x\to 0+}\left(\frac{|x-1|}{x^2-x^3}\right)=\infty$$

> Limit(abs(x – 1)/(x^2 – x^3), x = 1, left) = limit(abs(x – – 1)/(x^2 – x^3), x = 1, left);

$$\lim_{x\to 1-}\left(\frac{|x-1|}{x^2-x^3}\right)=1$$

> Limit(abs(x – 1)/(x^2 – x^3), x = 1, 'right) = limit(abs(x – – 1)/(x^2 – x^3), x = 1, right);

$$\lim_{x\to 1+}\left(\frac{|x-1|}{x^2-x^3}\right)=-1$$

5) $f(x)=\frac{(x+1)^2-(x-1)^2}{x^2-x}$ funksiya uchun $x=0$ va $x=1$ uzilish nuqtalari bo'lib, bunda, ravshanki, $x=1$ nuqta ikkinchi tur, $x=0$ nuqta esa, yo'qotilishi mumkin bo'lgan uzilish nuqtasi bo'ladi, chunki

$$\lim_{x\to 1}\frac{(x+1)^2-(x-1)^2}{x(x-1)}=\infty,$$

$$\lim_{x\to 0}\frac{(x+1)^2-(x-1)^2}{x(x-1)}=\lim_{x\to 0}\frac{(1+x)^2-1+1-(x-1)^2}{x(x-1)}=$$

$$=\lim_{x\to 0}\frac{1}{x-1}\left[\frac{(1+x)^2-1}{x}+\frac{(x-1)^2-1}{-x}\right]=-4$$

ekanligini hisobga olsak, $x=0$ nuqta haqiqatdan ham yo'qotilishi mumkin bo'lgan uzilish nuqtasi bo'lib, undagi funksiyaning sakrashi $\Delta f(0)=0$ bo'ladi.

MAPLE tizimidan foydalanib misolni yechish:

> readlib(singular):singular(((x+1)^2-(x-1)^2)/(x^2-x), x);
{x=1}

> Limit(((x+1)^2-(x-1)^2)/(x^2-x), x=1, left)=
=limit(((x+1)^2-(x-1)^2)/(x^2-x), x=1, left);

$$\lim_{x\to 1-}\left(\frac{(x+1)^2-(x-1)^2}{x^2-x}\right)=-\infty$$

> Limit(((x+1)^2-(x-1)^2)/(x^2-x), x=1, right)=
=limit(((x+1)^2-(x-1)^2)/(x^2-x), x=1, right);

$$\lim_{x\to 1+}\left(\frac{(x+1)^2-(x-1)^2}{x^2-x}\right)=\infty.$$

> Limit(((x+1)^2-(x-1)^2)/(x^2-x), x=0, left)=
=limit(((x+1)^2-(x-1)^2)/(x^2-x), x=0, left);

$$\lim_{x\to 0-}\left(\frac{(x+1)^2-(x-1)^2}{x^2-x}\right)=-4.$$

> Limit(((x+1)^2-(x-1)^2)/(x^2-x), x=0, right)=
=limit(((x+1)^2-(x-1)^2)/(x^2-x), x=0, right);

$$\lim_{x\to 0+}\left(\frac{(x+1)^2-(x-1)^2}{x^2-x}\right)=-4.$$

8.3. Uzluksiz funksiyalarning xossalari. Nuqtada uzluksiz bo'lgan funksiyaning lokal xossalari. $f(x)$ funksiya X to'plamda aniqlangan bo'lsin. X to'plamdan biror $a\in X$ nuqta olib, bu nuqtaning shu X to'plamga tegishli bo'lgan yetarli kichik $U_\delta(a)$ atrofini qaraylik.

1°. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, u holda a nuqtaning yetarli kichik atrofida funksiya chegaralangan bo'ladi, ya'ni

$$\exists\delta>0\ \exists C>0:\ \forall x\in U_\delta(a)\ \to\ |f(x)|\le C.$$

2°. Agar $f(x)$ funksiya a nuqtada uzluksiz va $f(a)\neq 0$ bo'lsa, $f(a)$ son bilan a nuqtaning yetarli kichik atrofida $f(x)$ funksiyaning ishorasi bir xil bo'ladi, ya'ni

$$\exists\delta>0 : \forall x\in U_\delta(a) \to \text{sign}\, f(x)=\text{sign}\, f(a).$$

Natija. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lib, bu nuqtaning yetarli kichik atrofidan olingan x nuqtalarda ham musbat, ham manfiy ishorali qiymatlarni qabul qilaversa, funksiyaning a nuqtadagi qiymati nolga teng bo'ladi.

3°. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, a nuqtaning yetarli kichik atrofidan olingan x' va x'' nuqtalar uchun $|f(x')-f(x'')|<\varepsilon$ tengsizlik o'rinli bo'ladi, bunda $\forall\varepsilon>0$ son.

Funksiyaning nuqta atrofidagi xususiyatlariga uning lokal xususiyatlari deyiladi.

8.4. Uzluksiz funksiyalar ustida amallar. Uzluksiz funksiyalar uchun quyidagi tasdiqlar o'rinli:

8.2-teorema. *Agar $f(x)$ va $g(x)$ funksiyalar $X\subset R$ to'plamda aniqlangan bo'lib, ularning har biri $a\in X$ nuqtada uzluksiz, ya'ni*

$$\lim_{x\to a} f(x)=f(a),\ \lim_{x\to a} g(x)=g(a)$$

bo'lsa, $f(x)\pm g(x)$, $f(x)\cdot g(x)$, $\dfrac{f(x)}{g(x)}$ ($\forall x\in X$ lar uchun $g(x)\neq 0$) funksiyalar ham shu nuqtada uzluksiz. va

$$\lim_{x\to a}[f(x)\pm g(x)]=f(a)\pm g(a),$$

$$\lim_{x\to a} f(x)\cdot g(x)=f(a)\cdot g(a),$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}\cdot(g(a)\neq 0$$

tengliklar o'rinli bo'ladi.

8.2-eslatma. Ikkita funksiya yig'indisi, ayirmasi, ko'paytmasi va nisbati uzluksiz bo'lishidan, bu funksiyalardan har birining uzluksiz bo'lishi har doim ham kelib chiqavermaydi.

Masalan: 1) $f(x)=x\cos\frac{1}{x}$ funksiya $x=0$ nuqtada uzluksiz, lekin bu funksiyani hosil qiluvchi

$$f_1(x)=x,\quad f_2(x)=\cos\frac{1}{x}$$

funksiyalardan birinchisi $x=0$ nuqtada uzluksiz, ikkinchisi esa, bu nuqtada uzilishga ega.

2) $f(x)=[x]+\{x\}=x$. Berilgan funksiya R da uzluksiz, lekin bu funksiyani hosil qiluvchi $f_1(x)=[x]$, $f_2(x)=\{x\}$ ($[x]$ — x ning butun qismi, $\{x\}$ — x ning kasr qismi) funksiyalarning har biri $x=n$ $(n\in Z)$ nuqtada uzilishga ega.

8.18-misol. Ushbu $f(x)=5x^2-\sin^3 x+3^x$ funksiyaning $\forall a\in R$ nuqtada uzluksizligini ko'rsating.

Yechilishi. $\varphi(x)=x$, $g(x)=\sin x$, $h(x)=3^x$ funksiyalar R da uzluksiz. $f(x)$ funksiyani $f(x)=5x\cdot x+\sin x\cdot\sin x\cdot\sin x+3^x$ ko'rinishda yozamiz. U holda, uzluksiz funksiyalar ustidagi arifmetik amallarga ko'ra, $f(x)$ funksiyaning R da uzluksizligi kelib chiqadi, ya'ni $\lim\limits_{x\to a}(5x^2-\sin^3 x+3^x)=5a^2-\sin^3 a+3^a$.

8.19-misol. Ushbu $f(x)=\frac{x}{4+x^2}$ funksiyaning $\forall a\in R$ da uzluksizligini ko'rsating.

Yechilishi. $\forall x\in R$ lar uchun $4+x^2\neq 0$. 8.2-teoremaga asosan, berilgan funksiya $\forall x\in R$ da uzluksizdir, chunki $\lim\limits_{x\to a}x=a$, $\lim\limits_{x\to a}(4+x^2)=4+a^2$, ya'ni nisbat hosil qiluvchi funksiyalarning har biri ham uzluksizdir.

Demak, $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}\frac{x}{4+x^2}=\frac{a}{4+a^2}=f(a)$.

8.20-misol. Ushbu

$$1)\ f(x)=\frac{3x^5-10x+17}{x^4+4x^3+10x^2+12x+9};\quad 2)\ f(x)=\frac{5\cos^2 x+2\sin^3 x+3}{6\sin x-3}$$

funksiyalarni uzluksizlikka tekshiring.

Yechilishi. 1) Berilgan funksiya ikkita uzluksiz funksiyalarning nisbati shaklida tasvirlangan bo'lib, u faqat maxraj nolga aylanadigan nuqtalaridagina uzulishga ega bo'lishi mumkin, lekin bu holda maxraj

$$x^4+4x^3+10x^2+12x+9=(x^2+2x+3)^2.$$

$\forall x$ uchun $x^2+2x+3=(x+1)^2+2>0$. Demak, maxraj hech qanday nuqtada nolga aylanmaydi.

Shunday qilib, berilgan $f(x)$ funksiya ikkita uzluksiz funksiya nisbati sifatida R da uzluksiz.

2) $f(x)$ funksiya maxraj nolga teng bo'lgan, ya'ni $6\sin x-3=0$ yoki $\sin x=\frac{1}{2}$, $x=x_n=(-1)^n\frac{\pi}{6}+\pi n$ $(n=0,\ \pm1,\ \pm2,\ldots)$ nuqtalarda uzilishga ega.

Demak, berilgan $f(x)$ funksiya $x_n=(-1)^n\frac{\pi}{6}+\pi n(n=0,\pm1,\pm2,\ldots)$ nuqtalardan tashqari R da uzluksiz.

8.21-misol. Quyidagi funksiyalarni uzluksizlikka tekshiring va grafigini chizing:

$$1)\ y=\lim_{n\to\infty}\frac{1}{1+x^n},\quad x\geq0;\qquad 2)\ y=\lim_{n\to\infty}\frac{\ln(2^n+x^n)}{n},\ x\geq0$$

Yechilishi. 1) agar $x=0$ bo'lsa, $y=1$ bo'ladi, $0<x<1$ bo'lganda esa $\lim\limits_{n\to\infty}x^n=0$. Demak, $y=1$. Agar $x>1$ bo'lsa, u holda $\lim\limits_{n\to\infty}x^n=+\infty$. Bu holda $y=0$ bo'ladi. Agar $x=1$ bo'lsa, $y=\frac{1}{2}$ bo'ladi.

Shunday qilib, berilgan funksiyaning analitik ko'rinishi quyidagicha bo'ladi:

$$y=\begin{cases}1, & 0\leq x<1,\\ \frac{1}{2}, & x=1,\\ 0, & x>1.\end{cases}$$

Bundan, $x=1$ nuqta berilgan funksiya uchun birinchi tur uzilish nuqtasi ekanligini ko'rish qiyin emas. Bu funksiyaning grafigi 8.10-chizmada tasvirlangan.

2) agar $x=0$ va $x=1$ bo'lganda, $y=\ln2$ bo'ladi.

Agar $0<x<1$ bo'lganda, $y=\ln2$, $1<x<2$ bo'lganda esa,

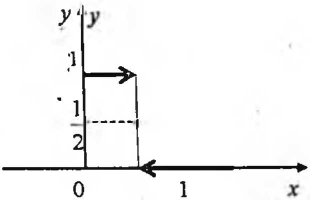
$$y=\lim_{n\to\infty}\frac{\ln(2^n+x^n)}{n}=\lim_{n\to\infty}\frac{n\ln2+\ln\left(1+\left(\frac{x}{2}\right)^n\right)}{n}=\ln2.$$

Agar $x=2$ da, $y=\ln2$ bo'ladi, agar $x>2$ bo'lganda esa, $y=\ln x$ bo'ladi.

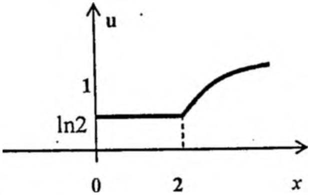
Shunday qilib, berilgan funksiyaning analitik ko'rinishi

$$y=\begin{cases}\ln 2, & 0\le x\le 2,\\ \ln x, & x>2\end{cases}$$

shaklda bo'ladi. $x=2$ nuqtada funksiyaning o'ng va chap limitlarini hisoblaymiz: $\lim\limits_{x\to 2+0} y=\ln 2$, $\lim\limits_{x\to 2-0} y=\ln 2$. Demak, $y(2-0)=y(2+0)=y(2)=\ln 2$ tenglik o'rinli bo'lganligi uchun, berilgan funksiya qaralayotgan sohada uzluksiz bo'ladi. Bu funksiyaning grafigi 8.11-chizmada tasvirlangan.



8.10-chizma.



8.11-chizma.

8.5. Murakkab funksiyaning uzluksizligi. $y=f(x)$ funksiya X to'plamda, $z=\varphi(y)$ funksiya esa Y to'plamda aniqlangan va $E(\varphi)\subseteq X$ bo'lsin, u holda ular yordamida $z=\varphi(f(x))$ murakkab funksiya tuzish mumkin bo'ladi.

8.3-teorema. $y=f(x)$ funksiya $a\in X$ nuqtada, $z=\varphi(y)$ funksiya esa a nuqtaga mos kelgan $y_a=f(a)$ nuqtada uzluksiz bo'lsa, $z=\varphi(f(x))$ murakkab funksiya a nuqtada uzluksiz bo'ladi.

8.22-misol. Ushbu

1) $y=\sin x^n$ $(n\in N)$; 2) $y=\sin(\log 4x)$; 3) $y=\sqrt{\frac{1}{2}-\sin^2 x}$

funksiyalarni uzluksizlikka tekshiring.

Yechilishi. 1) $u=x^n$ deb, $y=\sin u$ funksiyaga ega bo'lamiz. Ma'lumki, $y=\sin u$ funksiya, $\forall u$ uchun uzluksiz. $u=x^n$ funksiya esa darajali funksiya sifatida $\forall x\in R$ da uzluksiz. Demak, $y=\sin x^n$ murakkab funksiya 3-teoremaga asosan R da uzluksiz bo'ladi.

2) ma'lumki, $u=\log_3 4x$ funksiya $(0;\infty)$ da uzluksiz, $y=\sin u$ funksiya esa R da uzluksiz. Demak, murakkab funksiyalarning uzluk-

sizligi haqidagi 3-teoremaga ko'ra $y=\sin(\log_3 4x)$ murakkab funksiya $(0;\infty)$ oraliqda uzluksizdir.

3) $u=\sin x$ deb, $y=\sqrt{\frac{1}{2}-u^2}$ funksiyani hosil qilamiz. Bu funksiya $\frac{1}{2}-u^2\geq 0$, $u^2\leq\frac{1}{2}$ yoki $|u|\leq\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2}\leq u\leq\frac{\sqrt{2}}{2}$ da aniqlangan va uzluksiz.

$u=\sin x$ funksiya esa, R da uzluksiz.

Shunday qilib, $y=\sqrt{\frac{1}{2}-u^2}$ murakkab funksiya x ning $|\sin x|\leq\frac{\sqrt{2}}{2}$ tengsizlikni qanoatlantiradigan, ya'ni $\left\{\pi n-\frac{\pi}{4}\leq x\leq\frac{\pi}{4}+\pi n,\quad n\in Z\right\}$ qiymatlarida uzluksizdir.

8.23-misol. Ushbu

1) $y=\dfrac{1}{u^2-3u+2}$, $u=\dfrac{1}{x-2}$; 2) $y=u^2$, $u=\begin{cases}x-1, & x\geq 0,\\ x+1, & x<0;\end{cases}$

3) $y=\dfrac{1-u^2}{1+u^2}$, $u=\operatorname{tg}x$

murakkab funksiyalarning uzilish nuqtalarini toping va uzilish turlarini aniqlang.

Yechilishi. 1) $u=\frac{1}{x-2}$ funksiya $x=2$ nuqtada uzilishga ega, $y=\frac{1}{u^2-3u+2}$, funksiya esa, $u^2-3u+2=0$ tenglikni qanoatlantiradigan u ning qiymatlarida, ya'ni $u=1$ va $u=2$ nuqtalarda uzilishga ega. u ning bu qiymatlariga mos kelgan x ning qiymatlarini $1=\frac{1}{x-2}$, $2=\frac{1}{x-2}$ tenglamalarni yechish natijasida topamiz. Bulardan $x=3$; $x=\frac{5}{2}$.

Shunday qilib, berilgan murakkab funksiya $x=2$, $x=\frac{5}{2}$, $x=3$ nuqtalarda uzilishga ega. Endi uzilish nuqtalarining turini aniqlaymiz.

$$\lim_{x\to 2} y=\lim_{u\to\infty}\frac{1}{u^2-3u+2}=0$$

bo'lgani uchun $x=2$ yo'qotilishi mumkin bo'lgan uzilish nuqtasi bo'lar ekan.

$$\lim_{x\to\frac{5}{2}} y = \lim_{u\to 2} \frac{1}{u^2-3u+2} = \infty; \quad \lim_{x\to 3} y = \lim_{u\to 1} \frac{1}{u^2-3u+2} = \infty$$

bo'lgani uchun. $x=\frac{5}{2}$ va $x=\frac{3}{2}$ nuqtalar 2-tur uzilish nuqtlari bo'ladi.

2) $y=u^2$ funksiya u ning har qanday qiymatida uzluksiz.

$u=\begin{cases} x-1, & x\ge 0, \\ x+1, & x<0 \end{cases}$ funksiya uchun $x=0$ nuqta uzilish nuqtasi bo'lishi mumkin.

$u(0+0)=-1$, $u(0-0)=1$, $u(0)=-1$ bo'lgani uchun, $x=0$ nuqtada $u(x)$ funksiya o'ngdan uzluksiz, chapdan uzilishga ega. $x=0$ nuqta atrofida:

$$\lim_{x\to 0-0} y = \lim_{u\to 1} u^2 = 1, \quad \lim_{x\to 0+0} y = \lim_{u\to -1} u^2 = 1, \quad y(0)=y(-1)=1\,.$$

Demak, berilgan murakkab funksiya hamma joyda, ya'ni $\forall x\in R$ uchun uzluksizdir.

3) $u=\operatorname{tg}x$ funksiya $x=\frac{\pi}{2}+\pi n$ $(n=0, \pm 1, \pm 2, ...)$ nuqtalarda uzilishga ega.

$y=\frac{1-u^2}{1+u^2}$ funksiya har qanday u larda uzluksiz: $\lim_{x\to\frac{\pi}{2}} u = \lim_{x\to\frac{\pi}{2}} \operatorname{tg}x =$

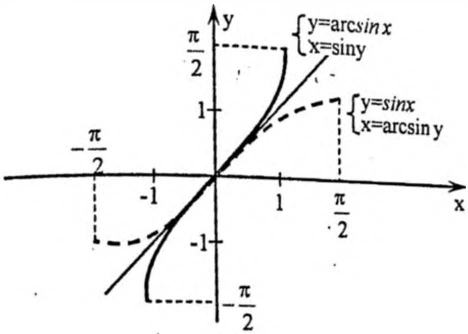
$=\pm\infty$, $\lim_{u\to\pm\infty} y = \lim_{u\to\pm\infty} \frac{1-u^2}{1+u^2} = -1$. Bu yerda, $\lim_{x\to\frac{\pi}{2}} y = \lim_{x\to\frac{\pi}{2}} \frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = -1.$

Demak, $x=\frac{\pi}{2}+\pi n$ $(n=0, \pm 1, \pm 2, ...)$ nuqtalar berilgan murakkab funksiya uchun yo'qotilishi mumkin bo'lgan uzilish nuqtalari bo'ladi.

8.6. MONOTON VA TESKARI FUNKSIYALARNING UZLUKSIZLIGI

8.4-teorema. *Agar $f(x)$ funksiya X oraliqda o'suvchi (kamayuvchi) bo'lib, uzilishga ega bo'lsa, uning uzilishi faqat birinchi tur uzilish bo'ladi.*

Agar $f(x)$ funksiya X oraliqda o'suvchi (kamayuvchi) bo'lib, uning qiymatlari Y oraliqni tutash to'ldirsa (ya'ni funksiya har bir $y\in Y$ qiymatni hech bo'lmaganda bir marta qabul qilsa), bu funksiya X da uzluksiz bo'ladi.



8.12-chizma.

8.5-teorema. *Uzluksiz $u=f(x)$ funksiya $(a;\ b)$ da teskari $x=g(y)$ funksiyaga ega bo'lishi uchun uning qat'iy monoton bo'lishi zarur va yetarlidir.*

8.24-misol. Ushbu $y=sinx$ funksiya uchun $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ kesmada teskari funksiya mavjudmi?

Yechilishi. Berilgan funksiya $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ kesmada uzluksiz va qat'iy o'suvchi. $u=\sin x$ funksiyaning qiymatlari to'plami $E(\sin x)=[-1;\ 1]$ kesmadan iborat. 5-teoremaga asosan, $[-1;\ 1]$ kesmada uzluksiz va o'suvchi teskari funksiya mavjud. Uning qiymatlari to'plami $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ kesmadan iborat. Berilgan funksiyaga teskari funksiya $x=\arcsin y$ ko'rinishda belgilanadi. $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ va $[-1;\ 1]$ kesmalarda mos ravishda to'g'ri va teskari funksiyalarning grafiklari (8.12-chizma) da berilgan.

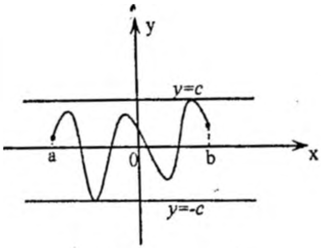
8.7. Kesmada uzluksiz bo'lgan funksiyalarning xossalari (global xossalar). $[a;\ b]$ kesmada aniqlangan va uzluksiz bo'lgan funksiyalarni qaraymiz, bunda a va b nuqtalardagi uzluksizliklar, mos ravishda, o'ngdan va chapdan uzluksizlik deb qaraladi.

1°. Veyershtrassning birinchi teoremasi. *Agar $f(x)$ funksiya $[a;\ b]$ kesmada aniqlangan va uzluksiz bo'lsa, $f(x)$ funksiya shu kesmada chegaralangan bo'ladi, ya'ni $\exists C>0$: $\forall x\in[a;\ b]\rightarrow |f(x)|\le C$ (8.13-chizma).*

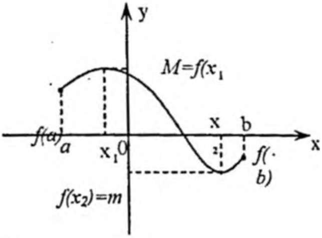
2°. Veyershtrassning ikkinchi teoremasi. *Agar $f(x)$ funksiya $[a;\ b]$ kesmada aniqlangan va uzluksiz bo'lsa, $f(x)$ funksiya shu kesmada* *o'zining aniq yuqori hamda aniq quyi chegaralariga erishadi, ya'ni [a; b] kesmada shunday x_1 va x_2 nuqtalar topiladiki,*

$$f(x_1) = \sup_{x\in[a,b]} \{f(x)\}, \quad f(x_1) = \inf_{x\in[a,b]} \{f(x)\}$$

tenglik o'rinli bo'ladi (8.14-chizma).



8.13-chizma.



8.13-chizma.

3°. Bolsano-Koshining birinchi teoremasi. *Agar f(x) funksiya [a; b] kesmada aniqlangan va uzluksiz bo'lib, kesmaning chetki nuqtalarida har xil ishorali qiymatlarga ega bo'lsa, shunday x_0 ($a<x_0<b$) nuqta topiladiki, unda f(x) funksiya nolga aylanadi: $f(x_0) = 0$ (8.15-chizma).*

4°. Bolsano-Koshining ikkinchi teoremasi. *Agar f(x) funksiya [a; b] kesmada aniqlangan va uzluksiz bo'lib, kesmaning chetki nuqtalarida $f(a)=A$, $f(b)=B$ qiymatlar qabul qilsa hamda $A \neq B$ bo'lsa, A va B sonlar orasida har qanday C son olinganda ham, a bilan b orasida shunday c nuqta topiladiki, $f(c) = C$ bo'ladi (8.16-chizma).*

8.25-misol. Ushbu $f(x) = \cos x + 2^x - x^4$ funksiyani [1; 5] kesmada chegaralanganlikka tekshiring.

Yechilishi. Berilgan funksiya [1; 5] kesmada uchta $f_1(x) = \cos x$, $f_2(x) = 2^x$, $f_3(x) = -x^4$ funksiyaning yig'indisi shaklida berilgan. Ravshanki, ularning har biri [1; 5] kesmada uzluksizdir. 8.2-teoremaga asosan berilgan funksiya ham [1; 5] kesmada uzluksiz. U holda, Beyershtrassning birinchi teoremasiga ko'ra berilgan funksiya [1; 5] kesmada chegaralangan.

8.3-eslatma. Beyershtrassning birinchi teoremasi (*a*, *b*) oraliq uchun har doim ham o'rinli emas. Masalan, $f(x) = \frac{1}{x}$ funksiya $x \in (0; 1)$ da uzluksiz, lekin bu oraliqda chegaralanmagan. Bu funksiya (0; 1) oraliqning ichki har bir nuqtasining kichik atrofida chegaralangan, lekin oraliqqa tegishli bo'lmagan nol nuqtaning atrofida chegaralanmagan.

8.4-eslatma. Uzilishga ega bo'lgan funksiya $[a; b]$ kesmaning har bir nuqtasida aniqlangan, lekin bu kesmada chegaralanmagan. Masalan,

$$f(x)=\begin{cases}\frac{1}{x}, 0<x\le 1 \text{ bo'lganda,}\\ 0,\ x=0 \text{ bo'lganda.}\end{cases}$$

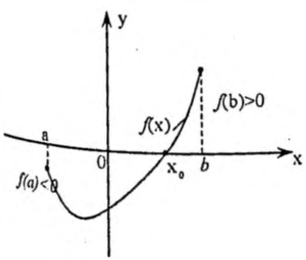
Shunday qilib, teoremadagi $f(x)$ funksiyaga qo'yilgan shartlarning har ikkalasi ham muhimdir.

8.26-misol. Ushbu $f(x)=x^3-3x$ funksiyaning $[-\sqrt{3}; \sqrt{3}]$ kesmada eng katta va eng kichik qiymatlari mavjudmi? (8.17-chizma).

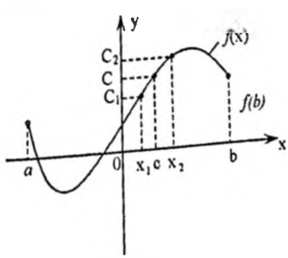
Yechilishi. Ravshanki, x^3 va $3x$ funksiyalarning har biri $[-\sqrt{3}; \sqrt{3}]$ kesmada uzluksiz. 8.2-teoremaga asosan berilgan $f(x)$ funksiya ham $[-\sqrt{3}; \sqrt{3}]$ kesmada uzluksiz. U holda Veyershtrassning ikkinchi teoremasiga ko'ra berilgan funksiya $[-\sqrt{3}; \sqrt{3}]$ kesmada aniq yuqori va aniq quyi chegaralariga erishadi: $\sup\limits_{x\in[-\sqrt{3};\sqrt{3}]}\{f(x)\}=f(-1)=2$, $\inf\limits_{x\in[-\sqrt{3};\sqrt{3}]}\{f(x)\}=f(1)=-2$.

8.5-eslatma. $[a; b]$ kesmada uzluksiz bo'lmagan funksiya uchun Veyershtrassning ikkinchi teoremasi o'rinli emas. Masalan,

$$f(x)=\begin{cases}x+2,\ -2\le x<0 \text{ bo'lganda.}\\ 0,\ x=0 \text{ bo'lganda,}\\ x-2,\ 0<x\le 2 \text{ bo'lganda (8.18-chizma)}\end{cases}$$



8.15-chizma



8.16-chizma

Yechilishi. Berilgan funksiya $\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ kesmada uzluksiz va kesmaning chetlarida har xil ishorali qiymatlar qabul qiladi: $f\left(-\frac{\pi}{2}\right)=-1$, $f\left(\frac{\pi}{2}\right)=1$ va

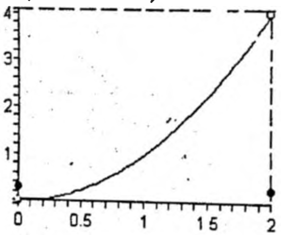
$-1\neq1$. Shartga ko'ra, $-1<\frac{1}{2}<1$. U holda, Bolsano-Koshining ikkinchi teoremasining shartlari bajarilayapti. Demak, shunday x_0 nuqta topiladiki, $f(x_0)=\frac{1}{2}$ bo'ladi, bunda $x_0=\frac{\pi}{6}\in\left[-\frac{\pi}{2};\ \frac{\pi}{2}\right]$ (8.22-chizma).

8.29-misol. Ushbu

$$f(x)=\begin{cases} x^2, & 0<x<2 \text{ bo'lganda,} \\ \frac{1}{3}, & x=0,\quad x=2 \text{ bo'lganda,} \end{cases}$$

funksiya [0; 2] segmentda o'zining aniq yuqori va aniq quyi chegarasiga erishadimi?

Yechilishi. Ravshanki, berilgan $f(x)$ funksiya [0; 2] segmentda chegaralangan. Bu oraliqda funksiyaning aniq yuqori chegarasi $M=4$, aniq quyi chegarasi $m=0$. Lekin, funksiya [0; 2]da o'zining aniq yuqori va aniq quyi chegarasiga erishmaydi, chunki [0; 2] segmentda funksiyaning qiymati 4 ga yoki 0 ga teng bo'ladigan nuqta mavjud emas (8.23-chizma).



8.23-chizma.

8.30-misol. [−2,2] segmentda ushbu

$$f(x)=\begin{cases} x^2+2, & -2\le x<0 \text{ bo'lganda,} \\ -(x^2+2), & 0\le x\le 2 \text{ bo'lganda} \end{cases}$$

funksiyani nolga aylantiradigan nuqta mavjudmi?

Yechilishi. Funksiyaning $[-2, 2]$ segmentning chetki nuqtalaridagi qiymatini hisoblaymiz:

$$f(-2)=6, \quad f(2)=-6.$$

Berilgan funksiya segmentning chetki nuqtalarida har xil ishorali qiymatlarni qabul qiladi, $[-2, 2]$ segmentning hech bir nuqtasida nolga aylanmaydi. Haqiqatdan ham, $\forall x \in [-2,2]$ uchun $(x^2+2)>0$ va $-(x^2+2)<0$ bo'ladi. Chunki, berilgan funksiya $[-2, 2]$ da uzluksiz emas, $x=0$ nuqtada uzilishga ega.

Mustaqil yechish uchun misol va masalalar

8.1. Ushbu 1) $f(x)=3x-2$, 2) $f(x)=x^3$ funksiyalar uchun uzluksizlikning «$\varepsilon-\delta$» ta'rifiga ko'ra $a=1$ nuqta uchun quyidagi jadvalni to'ldiring:

1)

ε	2	0,5	0,01	0,001	0,0001
δ					

2)

ε	2	0,5	0,01	0,001	0,0001
δ					

Koshi ta'rifidan foydalanib quyidagi funksiyalarning uzluksizligini ko'rsating:

8.2. $f(x)=x^2$. 8.3. $f(x)=\sqrt{x}$.

8.4. $f(x)=|x|$. 8.5. $f(x)=\operatorname{arctg} x$.

8.6. $f(x)=\begin{cases} x^2, & x-\text{ratsional son bo'lganda}, \\ -x^2, & x-\text{irratsional son bo'lganda}. \end{cases}$

8.7. $f(x)=\sqrt[3]{x}$. 8.8. $f(x)=2x-1$. 8.9. $f(x)=\frac{1}{x}$, $x \neq 0$.

8.10. $f(x)=x^2+2\sin x$. 8.11. $f(x)=\frac{2x-1}{x^2+2}$.

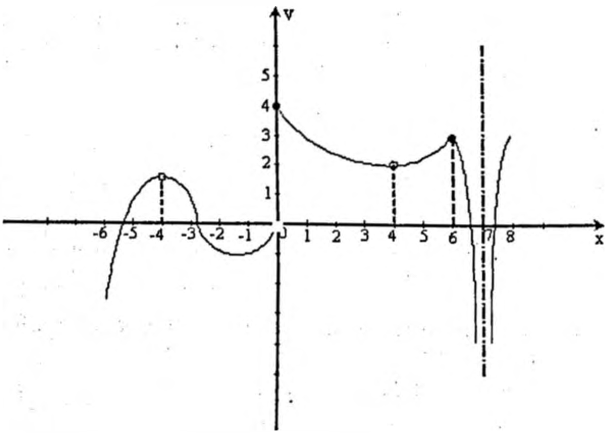
8.11. $f(x)=\cos^2 x$. 8.13. $f(x)=4x^2-3x+5$.

8.14. Agar $f(x)$ funksiya x_0 nuqtada uzluksiz bo'lsa, $|f(x)|$ funksiyaning shu nuqtada uzluksiz bo'lishini isbotlang.

8.15. $f(x)$ funksiya x_0 nuqtada uzluksizligining geometrik talqinini ifodalang.

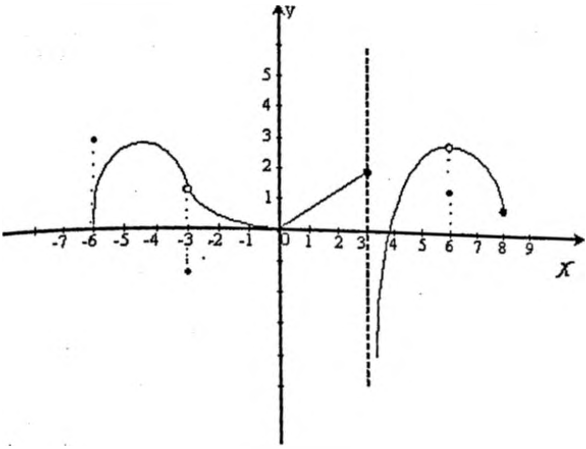
8.16. Agar $f(x)$ funksiya a nuqtada uzluksiz bo'lsa, $\varphi(x)=f(bx+c)(b\neq 0)$ funksiya $\frac{a-c}{b}$ nuqtada uzluksiz bo'lishini isbotlang.

8.17. 8.24-chizmada f funksiyaning grafigi berilgan. 1) qaysi nuqtalarda f funksiya uzilishga ega? 2) har bir uzilish nuqtasida f funksiya chapdan uzluksizmi, o'ngdanmi yoki har ikkala tomondan uzilishga egami, shuni aniqlang. 3) agar uzluksiz ega bo'lsa, f funksiya qaysi quqtalarda yo'qotishi mumkin bo'lgan uzilishga ega, qaysi nuqtalarda birinchi tur uzilishiga ega?



8.24-chizma.

8.18. 8.25-chizmada f funksiyaning grafigi berilgan. Funksiyaning uzluksizlik intervallarini toping.



8.25-chizma.

Berilgan funksiya ko'rsatilgan nuqtada uzluksizmi, yo'qmi, shuni aniqlang. Agar uzluksiz bo'lmasa, uzilishining turini aniqlang:

8.19. $f(x)=x^2-3x+1, \quad x_0=3.$ 8.20. $f(x)=\sqrt{x^2+4}, \quad x_0=2.$

8.21. $f(x)=\begin{cases} x^2+4, x<2, \\ x^3, \quad x\geq 2, \end{cases} \quad x_0=2.$

8.22. $f(x)=\begin{cases} x^2+9, \quad x<2, \\ 7, \quad x=2, \\ x^3, \quad x>3, \end{cases} \quad x_0=3.$

8.23. $f(x)=\begin{cases} -1, \quad x<0, \\ 0, \quad x=0, \\ 1, \quad x>0, \end{cases} \quad x_0=1.$

8.24. $f(x)=\begin{cases} \frac{x^2-1}{x+1}, \; x\neq -1, \\ -2, \quad x=-1, \end{cases} \quad x_0=-1.$

8.25. $f(x)=\begin{cases} -x^2, \quad x<0, \\ 1-\sqrt{x}, \quad x\geq 0, \end{cases} \quad x_0=0.$

8.26. $f(x)=x^2\operatorname{sign}x$ funksiyaning grafigini chizing. Agar funksiya grafigi uzilishga ega bo'lsa, uzilish nuqtalarida uzilishning turini aniqlang.

8.27. $f(x)=\frac{x^2-9}{x-3}$. 8.28. $f(x)=|x-1|$.

8.29. $f(x)=\begin{cases} x-1, & x<1, \\ 0, & x=1, \\ x^2, & x>1. \end{cases}$

8.30. $f(x)=\begin{cases} -1, & x<-1, \\ x^3, & -1\le x\le 1, \\ 1, & x>1. \end{cases}$ 8.31. $f(x)=\begin{cases} 1, & x\le 0, \\ x^2, & 0<x<1, \\ 1, & 1\le x<2, \\ x, & x\ge 2. \end{cases}$

8.32. $f(x)=\begin{cases} 2x+9, & x<-2, \\ x^2+1, & -2<x\le 1, \\ 3x-1, & 1<x<3, \\ x+6, & x>3. \end{cases}$

8.33. Quyidagi shartlarni qanoatlantiruvchi $f(x)$ funksiya grafigini chizing: 1) $D(f)=[-3;\ 3]$; 2) $f(-3)=f(-1)=1,\ f(2)=f(3)=2$; 3) $f(x)$ funksiya $x=-1$ nuqtada cheksiz uzilishga va $x=2$ nuqtada esa sakrash uzilishiga ega;
4) $f(x)$ funksiya $x=-1$ nuqtada o'ngdan, $x=2$ nuqtada chapdan uzluksizdir.

8.34—8.35 misollarda $f(x)$ funksiya R ning $x=1$ nuqtadan tashqari barcha nuqtalarida aniqlangan va uzluksiz. Agar mumkin bo'lsa, $f(1)$ qiymatni shunday tanlangki,natijada $f(x)$ funksiya butun R da uzluksiz bo'lsin:

8.34. $f(x)=\frac{x^2-1}{x-1}$. 8.35. $f(x)=\frac{x-1}{|x-1|}$.

8.36. Agar $f(x)=\begin{cases} x^2, & x<1 \\ Ax-3, & x\ge 1 \end{cases}$ funksiya $x=1$ nuqtada uzluksiz bo'lsa, u holda A ni toping.

8.37. $f(x)=\begin{cases} Ax-B, & x\le 1 \\ 3x, & 1<x<2 \\ Bx^2-A, & 2\le x \end{cases}$ uchun A va B larni shunday topingki, natijada $f(x)$ funksiya $x=1$ nuqtada uzluksiz, $x=2$ nuqtada esa, uzilishga ega bo'lsin.

8.38—8.39 misollarda $f(x)$ funksiyaning $x=5$ nuqtadagi qiymatini shunday aniqlangki, natijada funksiya uzluksiz bo'lsin:

8.38. $f(x)=\frac{\sqrt{x+4}-3}{x-5}$. 8.39. $f(x)=\frac{\sqrt{2x-1}-3}{x-5}$.

8.40—8.41 misollarda, berilgan funksiyaning qanday nuqtalarda uzluksiz ekanligini aniqlang.

8.40. $f(x)=\begin{cases} 1, & x \text{ ratsional bo'lganda,} \\ 0, & x \text{ irratsional bo'lganda.} \end{cases}$

8.41. $f(x)=\begin{cases} 2x, & x \text{ butun bo'lganda,} \\ x^2, & x \text{ qolgan hollarda.} \end{cases}$

8.42. $\lim\limits_{n\to 0} f(n+c)=f(c)$ shart $f(x)$ funksiyaning c nuqtada uzluksiz bo'lishi uchun zaruriy va yetarli shart ekanligini isbotlang.

8.43. $f(x)$ va $g(x)$ funksiyalar s nuqtada uzluksiz bo'lsin. Agar a) $f(c)>0$ bo'lsa, u holda $\exists\delta>0$ mavjud bo'lib, $\forall x\in(c-\delta,c+\delta)$ uchun $f(x)>0$ bo'lishni, b) $f(x)<g(c)$ bo'lsa, u holda $\exists\delta>0$ mavjud bo'lib, $\forall x\in(c-\delta,c+\delta)$ uchun $f(x)<g(c)$ bo'lishini isbotlang.

8.44. Agar: a) $|f(x)|$ uzluksiz bo'lsa, $f(x)$ ning uzluksiz bo'lmasligining ham mumkinligini isbotlang; b) hech qayerda uzluksiz bo'lmagan funksiyaga misol quring.

8.45. Faraz qilaylik, $\exists\delta>0$ mavjud bo'lib, $f(x)$ uchun

$$|f(x)-f(c)|\le B|x-c|, \quad (\forall x\in(c-\delta,c+\delta),\ \delta>0)$$

shart bajarilsin. U holda $f(x)$ fuknsiyaning $x=c$ da uzluksiz ekanligini isbotlang.

8.46. Faraz qilaylik, $f(x)$ funksiya uchun

$$|f(x)-f(t)|\le|x-t| \quad (\forall x,t\in(a,b))$$

shart bajarilsin. U holda $f(x)$ funksiyaning $(a;\ b)$ intervalda uzluksiz bo'lishini isbotlang.

8.47. Agar $\lim\limits_{n\to 0}\frac{f(c+n)-f(c)}{n}=4$ mavjud bo'lsa, u holda $f(x)$ funksiyaning $x=c$ nuqtada uzluksiz bo'lishini isbotlang.

8.48. Agar $f(x)$ funksiya $-\infty;+\infty$ da uzluksiz bo'lsa, u holda uni quyidagi ko'rinishda tasvirlash mumkinligini isbotlang: $f(x)=f(x)+f(x)$, bunda $f-(-\infty,+\infty)$ da toq funksiya.

8.49. Hech bir nuqtada uzluksiz bo'lmagan fuksiyaga misol keltiring.

8.50. Faqat: a) bitta $x_0 \in R$; b) ikkita $x_0, x_1 \in R$; d) n ta $x_1, x_2, ..., ..., x_n \in R$ nuqtalarda uzluksiz bo'lib, qolgan nuqtalarda uzluksiz bo'lmagan funksiyalarga misollar keltiring.

8.51. Uzilishga ega bo'lgan funksiyaning kvadrati ham uzilishga ega bo'ladimi?

8.52. Ushbu $f(x)=\begin{cases} x, & x-\text{ratsional son bo'lganda,} \\ 0, & x-\text{irratsional son bo'lganda,} \end{cases}$

funksiyaning $x=0$ nuqtada uzluksiz, qolgan nuqtalarda uzluksiz emasligini isbotlang.

8.53. Ixtiyoriy ko'phadning har bir nuqtada uzluksiz ekanligini isbotlang.

8.54. Ushbu $f(x)=\frac{P(x)}{Q(x)}$ (bu yerda $P(x)$, $Q(x)$ — nol bo'lmagan ko'phadlar bo'lib, $Q(x_0)\neq 0$) funksiyaning har bir $x_0 \in R$ nuqtada uzluksizligini isbotlang.

8.55. Agar: a) $f(x)$ funksiya biror x_0 nuqtada uzluksiz, $g(x)$ funksiya esa x_0 nuqtada uzilishga ega; b) $f(x)$ va $g(x)$ lar x_0 nuqtada uzilishga ega bo'lsa, u holda $f(x)\pm g(x)$, $f(x)\cdot g(x)$ funksiyalarning x_0 nuqtada uzluksizligi haqida nima deyish mumkin?

8.56. [0; 1] kesmaning har bir nuqtasida uzilishga ega bo'lgan funksiyaning kvadrati shu kesmada uzluksiz bo'ladigan funksiyaga misol keltiring.

8.57. Biror x_0 nuqtada $f(x)$ va $g(x)$ funksiyalar uzluksiz bo'lib, $\frac{f(x)}{g(x)}$ nisbati esa, x_0 nuqtada uzilishga ega bo'lgan funksiyaga misol keltiring.

Quyidagi funksiyalarning uzilish nuqtalarini toping, turlarini aniqlang, 1-tur uzilish nuqtalarida funksiyaning sakrashini hisoblang hamda grafigini chizing:

8.58. $f(x)=\begin{cases} x^2+5, & x<2, \\ x^3, & x\ge 2. \end{cases}$

8.59. $f(x)=\begin{cases} x^2+5, & x<2, \\ 10, & x=2, \\ x^3+1, & x<2. \end{cases}$

8.60. $f(x)=(\text{sign}x)^2$.

8.61. $f(x)=E(x)$ (butun funksiya).

8.62. $f(x)=\frac{|x|-x}{x^2}$.

8.63. $f(x)=\begin{cases} -x^2, & x<0, \\ 1-\sqrt{x}, & x\ge 0. \end{cases}$

8.64. $f(x)=\begin{cases} 2x+9, & x<-2, \\ x^2+1, & -2<x\le 1, \\ 3x-1, & 1<x<3, \\ x+6, & x\ge 3. \end{cases}$

8.65. $f(x)=\begin{cases} x+7, & x<-3, \\ |x-2|, & -3<x<-1, \\ x^2-2x, & -1<x<3, \\ 2x-3, & 3\le x. \end{cases}$

Quyidagi funksiyalarning uzilish nuqtalarini toping, ularning turlarini aniqlang va grafiklarini chizing:

8.66. $f(x)=\frac{|x+3|}{x+3}$.

8.67. $f(x)=\frac{1+x}{1+x^3}$.

8.68. $f(x)=\frac{|x-1|}{x^2-x^3}$.

8.69. $f(x)=\frac{x^2-4}{x^2-5x+6}$.

8.70. $f(x)=x-[x]$.

8.71. $f(x)=\frac{\frac{1}{x}-\frac{1}{x+1}}{\frac{1}{x-1}-\frac{1}{x}}$.

8.72. $f(x)=\begin{cases} x^2+2, & x\le 0, \\ x-1, & x>0. \end{cases}$

8.73. $f(x)=\frac{1}{\ln x}$.

8.74. $f(x)=\begin{cases} -\frac{1}{x}, & x<0, \\ 5x-x^2, & x\ge 0. \end{cases}$

8.75. $f(x)=e^{x+\frac{1}{x}}$.

8.76. $f(x)=\text{arctg}\,\frac{1}{x}$.

8.77. $f(x)=f(x)=\frac{1}{x^2-9}$.

8.78. $f(x)=\frac{1}{1-e^{\frac{x}{1-x}}}$.

8.79. $f(x)=\text{sign}(\cos x)$.

8.80. $f(x)=\frac{1}{1+3^{\frac{1}{x-4}}}$.

8.81. $f(x)=\begin{cases} x^2, & -\infty < x \le 0, \\ (x-1)^2, & 0 < x \le 2, \\ 5-x, & 2 < x < +\infty. \end{cases}$

8.82. $f(x)=\begin{cases} x+4, & x<-1, \\ x^2+2, & -1 \le x < 1, \\ 2x, & x \ge 1. \end{cases}$

8.83. $f(x)=\begin{cases} -1, & x<0, \\ \cos x, & 0 \le x \le \pi, \\ 1-x, & x>\pi. \end{cases}$

Berilgan funksiyalarni ko'rsatilgan nuqtalarda uzluksizlikka tekshiring:

8.84. $f(x)=2^{\frac{1}{x-5}}+1;\ x_1=5,\ x_2=6$.

8.85. $f(x)=\frac{(x-1)^2}{|x-1|},\ x_1=1,\ x_2=2.$

8.86. $f(x)=6^{\frac{1}{x-1}}-3;\ x_1=1,\ x_2=2$.

8.87. $f(x)=\frac{x-3}{x^2-9},\ x_1=3,\ x_2=-3.$

8.88. $f(x)=\frac{x+4}{x-3};\ x_1=3,\ x_2=4$.

8.89. $f(x)=\frac{x^2-4}{x-2},\ x_1=2,\ x_2=-2.$

8.90. $f(x)=\frac{2x}{x^2-1};\ x_1=-1,\ x_2=2$.

8.91. $f(x)=\frac{\sqrt{x+4}-3}{x-5},\ x_1=5,\ x_2=-3.$

8.92. $f(x)=\frac{x+3}{x+5},\ x_1=-5,\ x_2=-4$.

8.93. $f(x)=\frac{\sqrt{x+4}-3}{\sqrt{x-5}},\ x_1=5,\ x_2=12.$

8.94. $f(x)=\frac{\sqrt{x^2-7x+16}-\sqrt{6}}{(x-5)\sqrt{x+1}},\ x_1=5,\ x_2=-1.$

Quyidagi funksiyalarni uzluksizlikka tekshiring va grafigini chizing:

8.95. $y=\lim\limits_{n\to\infty}(1-x)^{2n}$, $|x|\le 1$.

8.96. $y=\lim\limits_{n\to\infty}\frac{x^n-x^{-n}}{x^n+x^{-n}}$, $x\ne 0$.

8.97. $y=\lim\limits_{n\to\infty}\frac{n^x-n^{-x}}{n^x+n^{-x}}$.

8.98. $y=\lim\limits_{n\to\infty}\sqrt[n]{1+x^n}$, $x\ge 0$.

8.99. $y=\lim\limits_{n\to\infty}[(x-1)\operatorname{arctg}x^n]$.

8.100. $y=\lim\limits_{n\to\infty}\sqrt[n]{(1+e^{n(x+1)})}$.

8.101. $y=\lim\limits_{n\to\infty}\frac{x^{n+2}}{\sqrt{2^{2n}+x^{2n}}}$, $x\ge 0$.

8.102. $y=\lim\limits_{n\to\infty}\sin^{2n}x$.

8.103. $y=\lim\limits_{n\to\infty}\frac{x+x^2e^{nx}}{1+e^{nx}}$.

8.104. $y=\lim\limits_{\lambda\to\infty}(1+x)\operatorname{th}\lambda x$.

Quyidagi funksiyalar a va b ning qanday qiymatlarida uzluksiz bo'ladi?

8.105. $f(x)=\begin{cases}(x-2)^3, & x\le 0,\\ ax^2+b, & 0<x<1,\\ 4\sqrt{x}, & x\ge 1.\end{cases}$

8.106. $f(x)=\begin{cases}ax^2+1, & x>0,\\ -x, & x\le 0.\end{cases}$

8.107. $f(x)=\begin{cases}\frac{(x-1)^2}{x^2-1}, & |x|\ne 1,\\ a, & x=-1,\\ b, & x=1.\end{cases}$

8.108. $f(x)=\begin{cases}\frac{2^x-1}{x}, & x\ne 0,\\ a, & x=0.\end{cases}$

8.109. $f(x)=\begin{cases}\frac{(1+x)^n-1}{x}, & x\ne 0,\\ a, & x=0.\end{cases}$

8.110. $f(x)=\begin{cases}\frac{9x}{\ln(1+3x)}, & x\ne 0,\\ a, & x=0.\end{cases}$

8.111. $f(x)=\begin{cases}\operatorname{ch}x, & x\le 0,\\ a(1-x)^4, & x>0\end{cases}$

8.112. $f(x)=\begin{cases}\frac{3+x}{27+x^3}, & x\ne -3,\\ a, & x=-3.\end{cases}$

8.113. $f(x)=\begin{cases}4x, & |x|\le 1,\\ x^2+ax+b, & |x|>1.\end{cases}$

Quyidagi $f[\varphi(x)]$ va $\varphi[f(x)]$ funksiyalarni uzluksizlikka tekshiring:

8.114. $f(x) = \operatorname{sign}x, \quad \varphi(x) = 4 + x^2.$

8.115. $f(x) = \operatorname{sign}(x-1), \quad \varphi(x) = \operatorname{sign}(x+1).$

8.116. $f(x) = \operatorname{sign}x, \quad \varphi(x) = x^3 - x.$

8.117. $f(x) = x(1-x^2), \quad \varphi(x) = \operatorname{sign}x.$

Quyidagi funksiyalarning x_0 dagi qiymatini shunday tanlash kerakki, funksiya shu nuqtada uzluksiz bo'lsin:

8.118. $f(x) = \frac{\sqrt{x+4}-2}{x}, \quad x_0 = 0.$

8.119. $f(x) = (x-\pi)\operatorname{ctg}x, \quad x_0 = \pi.$

8.120. $f(x) = \frac{x^2}{1-\cos x}, \quad x_0 = 0.$

8.121. $f(x) = \frac{1-\cos 2x}{1-\cos x}, \quad x_0 = 0.$

8.122. $f(x) = \frac{\operatorname{arctg}x}{\sin x}, \quad x_0 = 0.$

8.123. $f(x) = \frac{\cos 3x}{\sin 2x}, \quad x_0 = \frac{\pi}{2}.$

8.124. $f(x) = \operatorname{ctg}\frac{\pi}{1-x}, \quad x_0 = 0.$

8.125. $f(x) = \frac{\sin x}{x}, \quad x_0 = 0.$

8.126. $f(x) = \frac{x^3+1}{x^2-1}, \ x_0 = -1.$

8.127. $f(x) = \frac{\sin^2 x}{1-\cos x}, \ x_0 = 0.$

8.128. $f(x) = \frac{\sqrt{1+x}-1}{x}, \ x_0 = 0.$

8.129. $f(x) = \operatorname{tg}\frac{\pi}{2-x}, \quad x_0 = 0.$

8.130. $f(x) = \frac{4x^2-x}{3x}, \ x_0 = 0.$

8.131. $f(x) = \operatorname{arctg}\frac{1}{x}, \quad x_0 = 0.$

8.132. $f(x) = \frac{\sqrt{1+2x}-1}{\sin x}, \quad x_0 = 0.$

Quyidagi funksiyalarning berilgan kesmada chegaralanganligini ko'rsating:

8.133. $f(x) = \sin x \cos^2 x - \sqrt{x+6}, \quad x \in [0;\ 10].$

8.134. $f(x) = \operatorname{arctg}\frac{x^2+1}{2x} + 2^{\sin x} - x^2, \ x \in [1;\ 4].$

8.135. $f(x) = x(x-2)^2 \ln(4-x), \quad x \in [0;\ 3].$

8.136. $f(x) = \frac{x^2+1}{x^2+4}, \quad x \in [-4;\ 4].$

Quyidagi funksiyalarning berilgan kesmada eng katta va eng kichik qiymatlari mavjud bo'ladimi?

8.137. $f(x) = \sin x + 3^x, \ x \in [-1;\ 2].$

8.138. $f(x)=x^2-2,\quad x\in[-1;\ 2]$.

8.139. $f(x)=3,\quad x\in[-4;\ 4]$.

8.140. $f(x)=\begin{cases}-x^2+1, & -1\le x<0 \text{ bo‘lganda},\\ 0, & x=0 \text{ bo‘lganda},\\ x^2-1, & 0<x<1 \text{ bo‘lganda}\end{cases}$

funksiya berilgan sohada eng katta va eng kichik qiymatlarni qabul qiladimi?

Quyidagi tenglamalarning ko‘rsatilgan kesmada yechimga ega ekanligini ko‘rsating:

8.141. $x^3+2x+1=0,\ x\in[-1;\ 0]$.

8.142. $x^3-x^5-x+2=0,\quad x\in[0,5;\ 2]$.

8.143. $\sin^3 x-5\sin x+1=0,\quad x\in\left[0;\ \frac{\pi}{2}\right]$.

8.144. $\sin x-x+1=0,\quad x\in[0;\ \pi]$.

8.145. $2^x=1,\quad x\in[-0,2;\ 3]$.

8.146. $2^x=4x,\quad x\in[2;\ 5]$.

Quyidagi funksiyalarga teskari bo‘lgan funksiyalar mavjudmi?

8.147. $y=3x-6,\ x\in R$.

8.148. $y=3^{x-1},\ x\in R$.

8.149. $y=(x-2)^2,\ x\in(-\infty;2]$.

8.150. $y=\cos 2x,\ x\in\left[-\frac{\pi}{2};\ 0\right]$.

8.151. $y=\operatorname{tg}x,\ x\in\left[0;\ \frac{\pi}{4}\right]$.

8.152. $y=\sin x,\quad x\in\left[\frac{\pi}{2};\ \frac{3\pi}{2}\right]$.

8.153. $y=\ln(x-2),\quad x\in(2;\ +\infty)$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

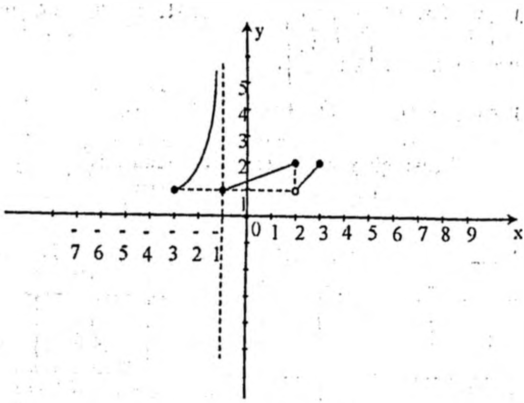
8.1. 1)

ε	2	0,5	0,01	0,001	0,0001
δ	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{300}$	$\frac{1}{3000}$	$\frac{1}{30000}$

2)

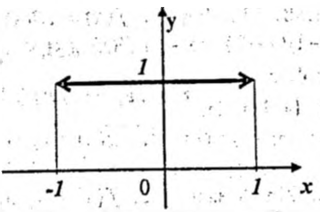
ε	2	0,5	0,01	0,001	0,0001
δ	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{700}$	$\frac{1}{7000}$	$\frac{1}{70000}$

8.17. 1) $x=-4$, $x=0$, $x=4$, $x=7$ uzilish nuqtalari; 2) $x=-4$ ikki tomnlama uzilish nuqtasi, $x=0$ da o'ngdan uzluksiz, $x=4$ da har ikki tomondan uzilish, $x=7$ har ikki tomondan uzilish; 3) $x=4$ yo'qotilishi mumkin bo'lgan uzilish nuqtasi, $x=0$ birinchi tur uzilish nuqtasi. **8.18.** $(-6;\ -3)$, $(-3;\ 3]$, $(3;\ 6)$, $(6;\ 8]$. **8.19.** Uzluksiz. **8.20.** Uzluksiz. **8.21.** Uzluksiz. **8.22.** Yo'qotilishi mumkin bo'lgan uzulish. **8.23.** Birinchi tur uzilish. **8.24.** Uzluksiz. **8.25.** Birinchi tur uzilish. **8.26.** Uzluksiz. **8.27.** $x=3$ da yo'qotilishi mumkin bo'lgan uzilish. **8.28.** Uzluksiz. **8.29.** Birinchi tur uzilish. **8.30.** Uzluksiz. **8.31.** $x=0$ va $x= =2$ da birinchi tur uzilish. **8.32.** $x=-2$ yo'qotilishi mumkin bo'lgan uzilish, $x=3$ da birinchi tur uzilish. **8.33.** 8.26-chizmada tasvirlangan. **8.34.** $f(1)=2$. **8.35.** Mumkin emas. **8.36.** $A=4$. **8.37.** $A-B=3$, $4-B=3$, $B\neq 3$. **8.38.** $f(5)=\frac{1}{6}$. **8.39.** $f(5)=\frac{1}{3}$. **8.40.** Hamma joyda uzilishga ega. **8.41.** $x=0$, $x=2$ va butun bo'lmagan hamma nuqtalarda. **8.44.** $f(x)=\begin{cases}1, & x \text{ ratsional bo'lganlda,}\\ -1, & x \text{ irratsional bo'lganda.}\end{cases}$

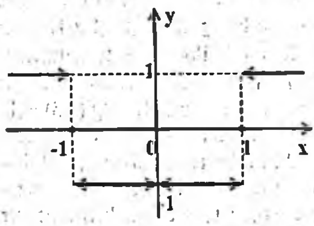


8.26-chizma.

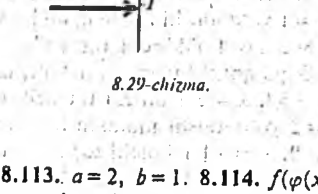
8.49. $D(x)$ — Dirixle funksiyasi. 8.50. Masalan: a) $f(x)=xD(x)$; b) $f(x)=x(x-4)D(x)$; d) $f(x)=(x-1)(x-2)...(x-n)\ D(x)$. 8.51. Yo'q. Masalan: $f(x)=\begin{cases}1, & x \text{ ratsional bo'lganlda,} \\ -1, & x \text{ irratsional bo'lganda.}\end{cases}$ 8.55. a) $f(x)\pm g(x)$ funksiyalar uzilishga ega bo'ladi. Masalan: 1) $f(x)=x^2$, $g(x)=\text{sign}x$; 2) $f(x)=1+x$, $g(x)=\text{sign}x$; b) $f(x)\cdot g(x)$ funksiya uzluksiz ham, uzilishga ega bo'lishi ham mumkin. Masalan: 1) $f(x)=\text{sign}x$, $g(x)=-\text{sign}x$; 2) $f(x)=\text{sign}x$, $g(x)=2\text{sign}x$. 8.56. $D(x)$ — Dirixli funksiyasi. 8.57. Masalan: $f(x)=x+x_0$, $g(x)=x^2-x_0^2$. 8.58. $x=2$ birinchi tur uzilish, $\Delta f(2)=-1$. 8.59. $x=2$ birinchi tur uzilish, $\Delta f(2)=1$. 8.60. $x=0$ yo'qotilishi mumkin bo'lgan nuqta. 8.61. $x=n$ birinchi tur uzilish, $\Delta f(n)=1$. 8.62. $x=0$ ikkinchi tur uzilish. 8.63. $x=0$ birinchi tur uzilish, $\Delta f(0)=1$. 8.64. $x=3$ birinchi tur uzilish, $\Delta f(3)=1$. 8.65. $x=-3$ birinchi tur uzilish, $\Delta f(-3)=1$. 8.66. $x=-3$ birinchi tur uzilish nuqtasi. 8.67. $x=-1$ da yo'qotilishi mumkin bo'lgan uzilish. 8.68. $x=0$ ikkinchi tur uzilish, $x=1$ birinchi tur uzilish. 8.69. $x=2$ da yo'qotilishi mumkin bo'lgan uzilish, $x=3$ ikkinchi tur uzilish. 8.70. $x=0, \pm1, \pm2, ...,$ birinchi tur uzilish nuqtalari. 8.71. $x=0$, $x=1$ yo'qotilishi mumkin bo'lgan uzilish, $x=-1$ ikkinchi tur uzilish. 8.72. $x=0$ birinchi tur uzilish. 8.73. $x=0$ da yo'qotilishi mumkin bo'lgan nuqta, $x=1$ ikkinchi tur uzilish. 8.74. $x=0$ ikkinchi tur uzilish. 8.75. $x=0$ ikkinchi tur uzilish. 8.76. $x=0$ birinchi tur uzilish. 8.77. $x=3$, $x=-3$ ikkinchi tur uzilish. 8.78. $x=0$, $x=1$ ikkinchi tur uzilish. 8.79. $x_n=\frac{\pi}{2}+\pi n$, $n\in Z$, birinchi tur uzilish nuqtalari. 8.80. $x=1$ birinchi tur uzilish. 8.81. $x=0$, $x=2$ birinchi tur uzilish. 8.82. $x=1$ birinchi tur uzilish. 8.83. $x=0$, $x=\pi$ birinchi tur uzilish. 8.84. $x_1=5$ ikkinchi tur uzilish, $x_2=6$ nuqtada uzluksiz. 8.85. $x_1=1$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=2$ nuqtada uzluksiz. 8.86. $x_1=1$ ikkinchi tur uzilish, $x_2=2$ nuqtada uzluksiz. 8.87. $x_1=3$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-3$ ikkinchi tur uzilish. 8.88. $x_1=3$ ikkinchi tur uzilish, $x_2=4$ nuqtada uzluksiz. 8.89. $x_1=2$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-2$ nuqtada uzluksiz. 8.90. $x_1=-1$ ikkinchi tur uzilish, $x_2=2$ nuqtada uzluksiz. 8.91. $x_1=5$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-3$ nuqtada uzluksiz. 8.92. $x_1=-5$ ikkinchi tur uzilish, $x_2=-4$ nuqtada uzluksiz. 8.93. $x_1=5$ ikkinchi tur uzilish, $x_2=12$ nuqtada uzluksiz. 8.94. $x_1=5$ yo'qotilishi mumkin bo'lgan nuqta, $x_2=-1$ ikkinchi tur uzilish. 8.95. $x=-1$, $x=1$ yo'qotilishi mum-



8.27-chizma.

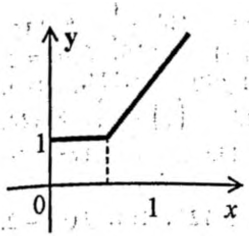


8.28-chizma.

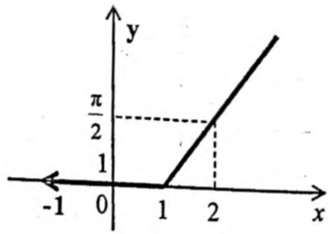


8.29-chizma.

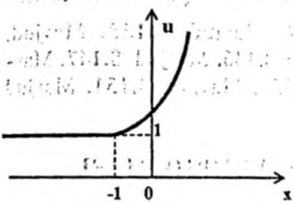
kin bo'lgan uzilish nuqtalari. Bu funksiyaning grafigi 8.27-chizmada tasvirlangan. **8.96.** $x=-1$, $x=1$ da birinchi tur uzilish, $x=0$ da esa, yo'qotilishi mumkin bo'lgan uzilish. Bu funksiyaning grafigi 8.28-chizmada tasvirlangan. **8.97.** $x=0$ birinchi tur uzilish nuqtasi. Bu funksiyaning grafigi 8.29-chizmada tasvirlangan. **8.98.** Uzluksiz. Berilgan funksiyaning grafigi 8.30-chizmada tasvirlangan. **8.99.** Uzluksiz. Berilgan funksiyaning grafigi 8.31-chizmada tasvirlangan. **8.100.** Uzluksiz. Bu funksiyaning grafigi 8.32-chizmada tasvirlangan. **8.101.** $x=2$ yo'qotilishi mumkin bo'lgan uzilish nuqtasi. Bu funksiyaning grafigi 8.33-chizmada tasvirlangan. **8.102.** $x=\frac{\pi}{2}+\pi n$, $n\in Z$ yo'qotilishi mumkin bo'lgan uzilish. Bu funksiyaning grafigi 8.34-chizmada tasvirlangan: **8.103.** Uzluksiz. Bu funksiyaning grafigi 8.35-chizmada tasvirlangan. **8.104.** $x=0$ birinchi tur uzilish. Bu funksiyaning grafigi 8.36-chizmada tasvirlangan. **8.105.** $a=12$, $b=8$. **8.106.** Shunday a son mavjud emas. **8.107.** Shunday a va b sonlar mavjud emas. **8.108.** $a=\ln 2$. **8.109.** $a=n$, $n\in N$. **8.110.** $a=3$. **8.111.** $a=1$. **8.112.** $a=\frac{1}{27}$. **8.113.** $a=2$, $b=1$. **8.114.** $f(\varphi(x))$ uzluksiz, $\varphi(f(x))$ funksiya $x=0$ nuqtada uzluksiz. **8.115.** $f(\varphi(x))$ funksiya $x=-1$ nuqtada uzluksiz, $\varphi(f(x))$ funksiya $x=1$ nuqtada uzluksiz. **8.116.** $f(\varphi(x))$ funksiya $x=0$, $x=\pm 1$ nuqtalarda uzluksiz $\varphi(f(x))$ uzluksiz. **8.117.** $f(\varphi(x))$ uzluksiz, $\varphi(f(x))$ funksiya $x=0$, $x=\pm 1$ nuqtalarda uzluksiz.



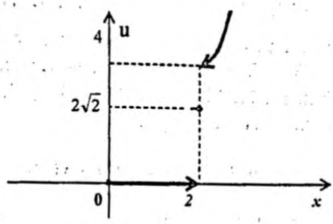
8.30-chizma.



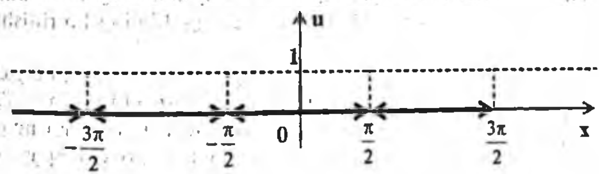
8.31-chizma.



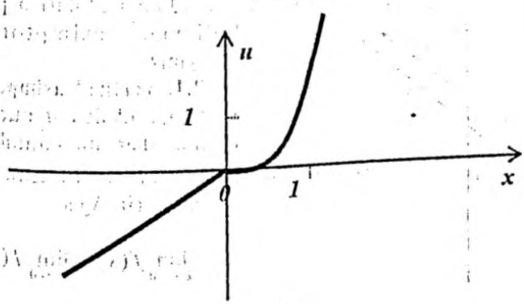
8.32-chizma.



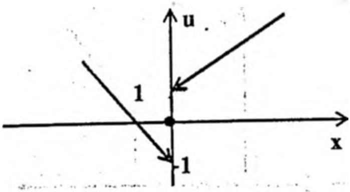
8.33-chizma.



8.34-chizma.



8.35-chizma.



8.36-chizma.

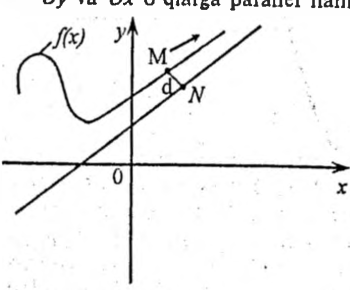
8.118. $f(0)=\frac{1}{4}$. **8.119.** $f(\pi)=1$. **8.120.** $f(0)=2$. **8.121.** $f(0)=4$. **8.122.** $f(0)=1$. **8.123.** $f\left(\frac{\pi}{2}\right)=-\frac{3}{2}$. **8.124.** Mumkin emas. **8.125.** Ha, $f(0)=1$. **8.126.** Ha, $f(-1)=\frac{-3}{2}$. **8.127.** Ha, $f(0)=2$. **8.128.** Ha, $f(0)=\frac{1}{2}$. **8.129.** Mumkin emas. **8.130.** Ha, $f(0)=-\frac{1}{3}$. **8.131.** Mumkin emas. **8.132.** Ha, $f(0)=1$. **8.137.** Mavjud. **8.138.** Mavjud. **8.139.** Mavjud emas. **8.140.** Mavjud emas. **8.146.** Mavjud. **8.147.** Mavjud. **8.148.** Mavjud. **8.149.** Mavjud. **8.150.** Mavjud. **8.151.** Mavjud emas. **8.152.** Mavjud emas.

9-§. FUNKSIYA GRAFIGINING ASIMPTOTALARI

Funksiyani tekshirish jarayonida uning grafigi koordinatalar boshidan cheksiz uzoqlashganda, yoki boshqacha aytganda, uning o'zgaruvchan nuqtasi cheksizlikka intilganda grafikning ko'rinishini bilib olish muhim.

9.1-ta'rif. Agar o'zgaruvchi $M(x, y)$ nuqta funksiya grafigi bo'yicha koordinatalar boshidan cheksiz uzoqlashganda, $y=f(x)$ funksiya grafigidagi o'zgaruvchi $M(x, y)$ nuqtadan to'g'ri chiziqdagi $N(x_1, y_1)$ nuqtagacha bo'lgan $d=MN$ masofa nolga intilsa, bu to'g'ri chiziq $y=f(x)$ funksiya *grafigining asimptotasi* deyiladi (9.1-chizma).

Oy va *Ox* o'qlarga parallel hamda koordinata o'qlariga parallel bo'lmagan asimptotalarni qaraymiz.



9.1-chizma.

9.1. Vertikal asimptotalar. $y=f(x)$ funksiya a nuqtaning biror $\varepsilon>0$ atrofida aniqlangan, ya'ni $x\in U_\varepsilon(a)$ bo'lsin.

9.2-ta'rif. Agar

$$\lim_{x\to a-0} f(x), \quad \lim_{x\to a+0} f(x)$$

lardan biri yoki ularning ikkalasi ham cheksiz bo'lsa, $x=a$ to'g'ri chiziq $f(x)$ *funksiya grafigining vertikal yoki Oy o'qqa parallel asimptotasi* deyiladi (9.2-*a*, *b*, *d*, *e* chizmalar).

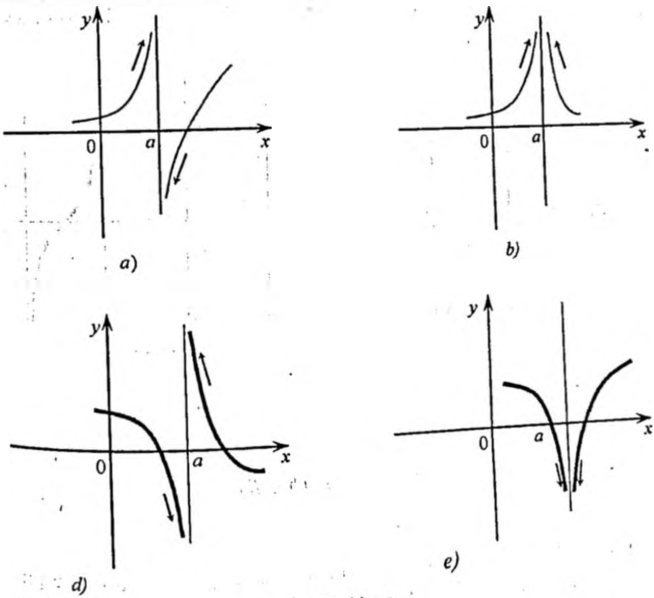
Demak, $y=f(x)$ funksiya grafigining vertikal asimptotalarini izlash uchun funksiyaning qiymatini cheksizlikka aylantiradigan (cheksiz uzilishga ega bo'lgan) $x=a$ nuqtani topish kerak ekan. Bunda $x=a$ to'g'ri chiziq vertikal asimptota bo'ladi.

9.1-eslatma. Umuman aytganda, $y=f(x)$ funksiyaning grafigi bir nechta vertikal asimptotalarga ega bo'lishi ham mumkin.

9.1-misol. Ushbu

$$f(x)=\frac{1}{x-2},\ x\in[-2;\ 3]$$

funksiya grafigining vertikal asimptotasini toping.



9.2-chizma.

Yechilishi. Berilgan funksiyaning maxraji $x=2$ nuqtada nolga aylanadi. $x \to 2 \pm 0$ da berilgan funksiyaning limitini hisoblaymiz:

$$\lim_{x \to 2-0} f(x) = \lim_{x \to 2-0} \frac{1}{x-2} = -\infty, \quad \lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} \frac{1}{x-2} = +\infty.$$

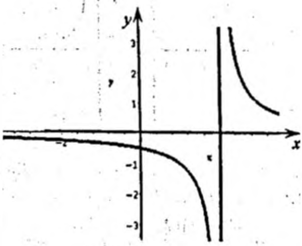
Demak, 9.2-ta'rifga ko'ra, berilgan funksiyaning grafigi uchun $x=2$ to'g'ri chiziq vertikal asimptota bo'ladi (9.3-chizma).

9.2-misol. Ushbu

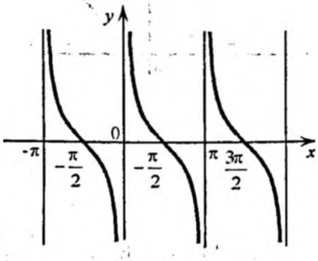
$$f(x) = \operatorname{ctg} x$$

funksiya grafigining vertikal asimptotalarini toping.

Yechilishi. Berilgan funksiya $x=\pi n$ $(n \in Z)$ nuqtalarda 2-tur uzilishga ega. $x \to \pi n \pm 0$ $(n \in Z)$da berilgan funksiyaning limiti $\pm\infty$ ga aylanadi. Shuning uchun, 9.2-ta'rifga asosan, funksiyaning grafigi cheksiz ko'p vertikal asimptotalarga ega (9.4-chizma): $x=0$, $x=\pm\pi n$, $x=\pm 2\pi$, ...



9.3-chizma.



9.4-chizma.

9.2. Gorizontal asimptotalar

9.3-ta'rif. Agar

$$\lim_{\substack{x \to +\infty \\ (x \to -\infty)}} f(x) = b \quad (b \in R)$$

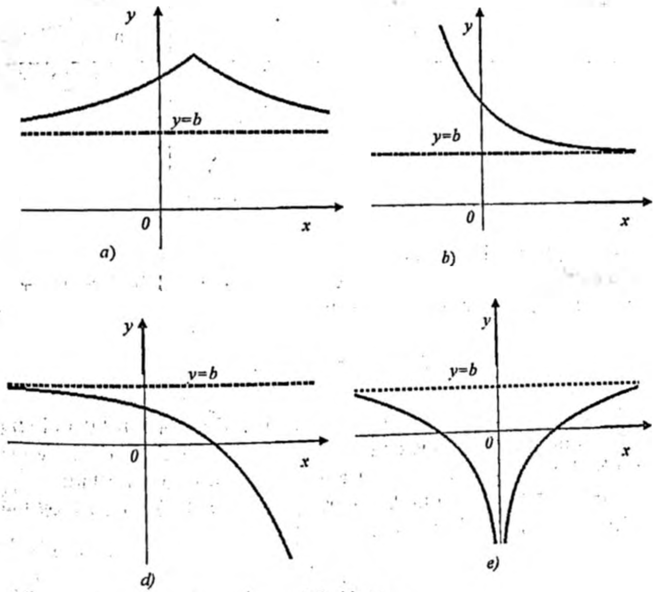
bo'lsa, $y=b$ to'g'ri chiziq $x \to +\infty$ $(x \to -\infty)$ da $y=f(x)$ funksiya grafigining *gorizontal yoki Ox o'qqa parallel asimptotasi* deyiladi (9.5-*a*), *b*), *d*), *e*) chizmalar).

9.3-misol. Ushbu $f(x)=\frac{x^2}{x^2+2}$ funksiya grafigining gorizontal asimptotasini toping.

Yechilishi. Berilgan funksiya R da aniqlangan. $x\to+\infty$ da berilgan funksiyaning limitini hisoblaymiz:

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^2}{x^2+2}=\lim_{x\to\pm\infty}\frac{1}{1+\frac{2}{x^2}}=1.$$

Demak, 9.3-ta'rifga ko'ra berilgan funksiyaning grafigi uchun $y=1$ to'g'ri chiziq gorizontal asimptota bo'ladi (9.6-chizma).



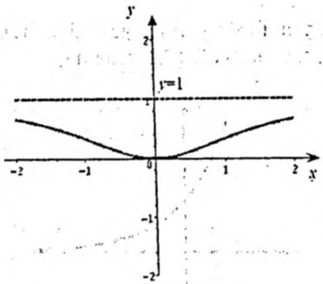
9.5-chizma.

9.4-misol. Ushbu $f(x)=\frac{1}{x}$ funksiya grafigining vertikal va gorizontal asimptotalarini toping.

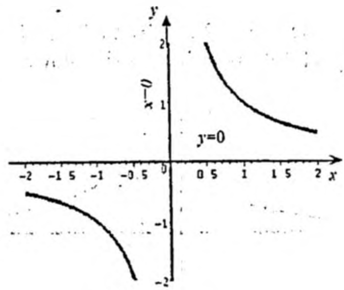
Yechilishi. Ravshanki, $\frac{1}{x}$ funksiyaning grafigi uchun $x=0$ va $y=0$ to'g'ri chiziqlar, mos ravishda, vertikal va gorizontal asimptotalar bo'ladi:

$$\lim_{x\to 0\pm 0} f(x) = \lim_{x\to 0\pm 0} \frac{1}{x} = \pm\infty,$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1}{x} = 0, \text{ (9.7-chizma).}$$



9.6-chizma.



9.7-chizma.

9.3. Og'ma asimptotalar

9.4-ta'rif. Shunday k va b chekli sonlar mavjud bo'lib, $x\to+\infty$ $(x\to-\infty)$ da $f(x)$ funksiya quyidagi

$$f(x) = kx + b + \alpha(x)$$

ko'rinishda ifodalansa (bunda $\lim\limits_{x\to\pm\infty}\alpha(x)=0$), $Y=kx+b$ to'g'ri chiziq $y=f(x)$ funksiya grafigining og'ma asimptotasi deyiladi. Xususiy holda $k=0$ bo'lsa, $Y=b$ to'g'ri chiziq gorizontal asimptota bo'ladi.

9.1-teorema. $y=f(x)$ funksiya grafigi $x\to\pm\infty$ da $Y=kx+b$ og'ma asimptotaga ega bo'lishi uchun

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=k, \quad \lim_{x\to\pm\infty}[f(x)-kx]=b \tag{9.1}$$

munosabatlar o'rinli bo'lishi zarur va yetarli.

(9.1) limitlarni hisoblashda quyidagi xususiy hollar bo'ladi:

1-hol. Argument x ning ishorasiga bog'liq bo'lmagan holda, ushbu

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{f(x)}{x}=k,$$
$$\lim_{x\to+\infty}[f(x)-kx]=\lim_{x\to-\infty}[f(x)-kx]=b$$

ikkala limit ham mavjud va chekli. Bu holda $Y=kx+b$ to'g'ri chiziq funksiya grafigining ikki tomonlama og'ma asimptotasi bo'ladi (quyida, 9.5-misolga qarang).

2-hol. Argument x ham musbat, ham manfiy ishorali cheksizlikka intilganda, ushbu

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k_1,\quad \lim_{x\to-\infty}\frac{f(x)}{x}=k_2,\quad \lim_{x\to+\infty}[f(x)-kx]=b_1,\quad \lim_{x\to-\infty}[f(x)-kx]=b_2$$

limitlar mavjud, lekin ular o'zaro har xil (hech bo'lmaganda $k_1\neq k_2$ yoki $b_1\neq b_2$). Bu holda $Y_1=k_1x+b_1$ va $Y_2=k_2x+b$ to'g'ri chiziqlar funksiya grafigining mos ravishda ikkita bir tomonli (o'ng va chap) og'ma asimptotalari bo'ladi (9.6-misolga qarang).

3-hol. Faqat $x\to+\infty$ da

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k,\quad \lim_{x\to+\infty}[f(x)-kx]=b$$

ikkala limit ham mavjud. Bu holda $Y=kx+b$ to'g'ri chiziq funksiya grafigining faqat o'ng og'ma asimptotasi bo'ladi (9.7-misolga qarang).

4-hol. Faqat $x\to-\infty$ da

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k,\quad \lim_{x\to-\infty}[f(x)-kx]=b$$

ikkala limit ham mavjud. Bu holda $Y=kx+b$ to'g'ri chiziq funksiya grafigining faqat chap og'ma asimptotasi bo'ladi.

Agar yuqoridagi hollarning barchasida $k=0$ bo'lsa, $Y=b$ to'g'ri chiziq gorizontal asimptota bo'ladi.

Funksiya grafigining asimptotalarga nisbatan joylashishini aniqlash uchun $x\to+\infty$, $x\to-\infty$ hollarning har birida $f(x)-(kx+b)$ ayirmaning ishorasi tekshiriladi.

Agar ayirmaning ishorasi musbat (manfiy) bo'lsa, funksiya grafigi asimtotadan yuqori (past)da joylashgan bo'ladi. Agar ayirma ishorasini o'zgartirsa, u holda asimptota funksiya grafigini kesadi.

9.5-misol. Ushbu

$$y=\sqrt[3]{x^3+1}$$

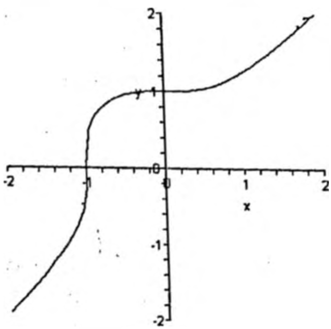
funksiya grafigining og'ma asimptotalarini toping.

Yechilishi. Og'ma asimptotalarni topamiz:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{\sqrt[3]{x^3+1}}{x}=1,$$

$$b=\lim_{x\to\pm\infty}[f(x)-kx]=\lim_{x\to\pm\infty}\left[\sqrt[3]{x^3+1}-x\right]=\lim_{x\to\pm\infty}\frac{x^3+1-x^3}{\sqrt[3]{(x^3+1)^2}+x\sqrt[3]{x^3+1}+x^2}=0$$

Demak, 9.1-teoremaga asosan, $y=x$ to'g'ri chiziq berilgan funksiyaning ikki tomonlama og'ma asimptotasi bo'ladi (9.8-chizma).



9.8-chizma.

9.6-misol. Ushbu

$$y=\sqrt{x^2+1}$$

funksiya grafigining og'ma asimptotalarini toping.

Yechilishi. Og'ma asimptotalarni topamiz:

$$k_1=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{x}=\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}=1,$$

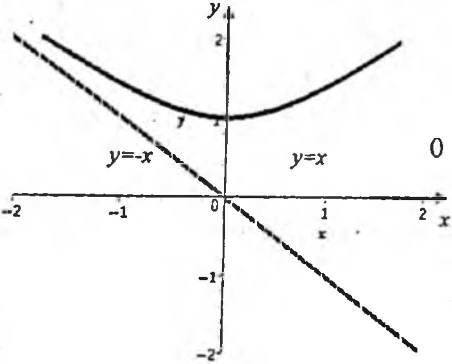
$$k_2=\lim_{x\to-\infty}\frac{y}{x}=\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}=\lim_{x\to\infty}\frac{|x|}{x}\sqrt{1+\frac{1}{x^2}}=-1,$$

$$b_1=\lim_{x\to+\infty}(y-k_1x)=\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}=0,$$

$$b_2=\lim_{x\to-\infty}(y-k_2x)=\lim_{x\to-\infty}\left(\sqrt{x^2+1}+x\right)=$$

$$=\lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}-x}=\lim_{x\to-\infty}\frac{1}{|x|\left(\sqrt{1+\frac{1}{x^2}}+1\right)}=0.$$

Demak, 9.1-teoremaga asosan, $Y_1 = x$ va $Y_2 = -x$ to'g'ri chiziqlar, mos ravishda, berilgan funksiyaning bir tomonli (o'ng va chap) og'ma asimptotalari bo'ladi (9.9-chizma). Berilgan funksiyani quyidagi ko'rinishda yozish mumkin: $y^2 - x^2 = 1$.



9.9-chizma.

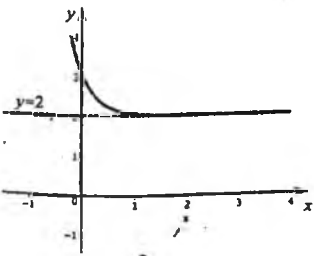
9.7-misol. Ushbu $y = e^{-3x} + 2$ funksiya grafigining og'ma asimptotalarini toping.

Yechilishi. Og'ma asimptotalarni topamiz:

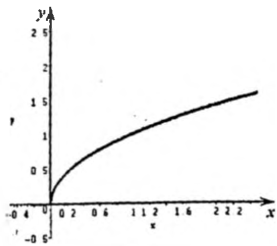
$$k_1 = \lim_{x\to+\infty} \frac{y}{x} = \lim_{x\to+\infty} \frac{e^{-3x}+2}{x} = 0, \quad k_2 = \lim_{x\to-\infty} \frac{y}{x} = \lim_{x\to-\infty} \frac{e^{-3x}+2}{x} = \infty,$$

$$b_1 = \lim_{x\to+\infty} (y - k_1 x) = \lim_{x\to+\infty} (e^{-3x} + 2) = 2.$$

Demak, 9.1-teoremaga ko'ra, $y = 2$ to'g'ri chiziq berilgan funksyaning faqat o'ng og'ma asimptotasi bo'ladi (9.10-chizma).



9.10-chizma.



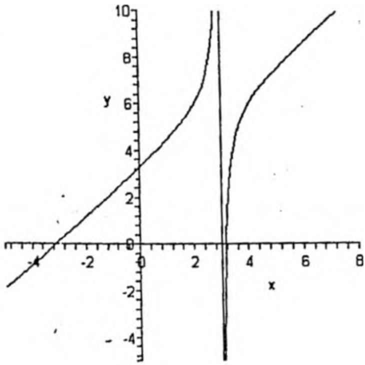
9.11-chizma.

9.2-eslatma. Berilgan $y=f(x)$ funksiya uchun faqat $\lim\limits_{x\to\infty}\frac{f(x)}{x}=k$ mavjud bo'lib, $\lim\limits_{x\to\infty}[f(x)-kx]$ mavjud bo'lmasa (yoki cheksiz bo'lsa) berilgan funksiya grafigi asimptotaga ega bo'lmaydi, lekin asimptotik yo'nalishga ega bo'ladi. Masalan, $y=\sqrt{x}$ parabola Ox o'qqa parallel bo'lgan asimptotik yo'nalishga ega, lekin gorizontal asimptotaga ega emas (9.11-chizma):

$$k=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{\sqrt{x}}{x}=0,\quad b=\lim_{x\to+\infty}(y-kx)=\lim_{x\to+\infty}\sqrt{x}=+\infty.$$

9.8-misol. Quyidagi $f(x)=\frac{x^2-10}{x-3}$ funksiya grafigi uchun $y=x+3$ to'g'ri chiziq og'ma asimptota bo'lishini ko'rsating.

Yechilishi. Berilgan funksiyaning ko'rinishini o'zgartirib, $f(x)=x+3-\frac{1}{x-3}$ ni hosil qilamiz. Bunda $x\to\pm\infty$ da $\alpha(x)=-\frac{1}{x-3}\to 0$ uchun $f(x)$ funksiyani $f(x)=x+3+\alpha(x)$ ko'rinishda ifodalash mumkin. Demak, 9.4-ta'rifga ko'ra, $y=x+3$ to'g'ri chiziq funksiya grafigining og'ma asimptotasi bo'ladi (9.12-chizma).



9.12-chizma.

9.9-misol. Ushbu $f(x)=\frac{x^2-2x}{x-1}$ funksiya grafigining asimptotalarini toping.

Yechilishi. (12.1) formuladan foydalanib, k va b larni topamiz:

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2-2x}{x(x-1)} = \lim_{x\to\pm\infty} \frac{1-\frac{2}{x}}{1-\frac{1}{x}} = 1, \quad k = 1,$$

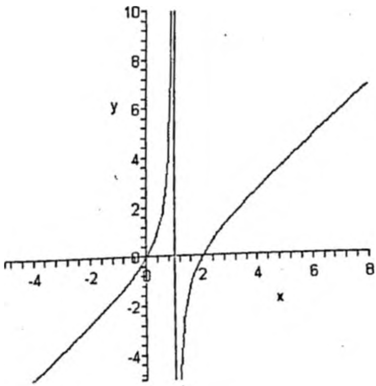
$$b = \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} \left(\frac{x^2-2x}{x-1} - x\right) = \lim_{x\to\pm\infty} \frac{-1}{1-\frac{1}{x}} = -1, \quad b = -1.$$

Demak, 9.1-teoremaga ko'ra, $y=x-1$ to'g'ri chiziq berilgan funksiya grafigining og'ma asimptotasi bo'ladi. $k\neq 0$ bo'lganligi uchun funksiya grafigi gorizontal asimptotaga ega emas.

Endi berilgan funksiyaning vertikal asimptotasini topamiz. Funksiya $x=1$ nuqtada 2-tur uzilishga ega. $x\to 1\pm 0$ da berilgan funksiyaning limitini hisoblaymiz:

$$\lim_{x\to 1\pm 0} f(x) = \lim_{x\to 1\pm 0} \frac{x^2-2x}{x-1} = \pm\infty.$$

Demak, 9.2-ta'rifga ko'ra, berilgan funksiyaning grafigi uchun $x=1$ to'g'ri chiziq vertikal asimptota bo'ladi (9.13-chizma).



9.13-chizma.

9.3-eslatma. $Y=kx+b$ asimptotalar bilan bir qatorda murakkab ko'rinishdagi asimptotalar ham qaraladi. Agar $f(x)$ funksiyani ushbu

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0+\alpha(x)$$

ko'rinishda yozish mumkin bo'lib, unda $\lim\limits_{x\to\infty}\alpha(x)=0$ bo'lsa, u holda

$$Y=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0 \qquad (12.2)$$

ko'phad bilan aniqlangan n-tartibli parabola $x\to+\infty$ da $y=f(x)$ funksiya grafigining asimptotasi deyiladi.

9.2-teorema. *$y=f(x)$ funksiyaning grafigi $x\to+\infty$ da (12.2) asimptotaga ega bo'lishi uchun $n+1$ ta limit qiymatlarning mavjud bo'lishi zarur va yetarli:*

$$\lim_{x\to+\infty}\frac{f(x)}{x^n}=a_n,\quad \lim_{x\to+\infty}\frac{f(x)-a_n x^n}{x^{n-1}}=a_{n-1},\ \ldots,$$

$$\lim_{x\to+\infty}\frac{f(x)-(a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2 x^2)}{x}=a_1,$$

$$\lim_{x\to+\infty}[f(x)-(a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x)]=a_0.$$

9.10-misol. Ushbu

$$f(x)=\frac{x^3+2x-3}{x-3}$$

funksiya grafigining asimptotasini toping.

Yechilishi. Berilgan funksiyani ushbu $f(x)=x^2+3x+11+\frac{30}{x-3}$ ko'rinishda tasvirlaymiz, bunda $\lim\limits_{x\to\infty}\alpha(x)=\lim\limits_{x\to\infty}\frac{30}{x-3}=0.$

Demak, $Y=x^2+3x+11$ chiziq ta'rifga ko'ra berilgan funksiya grafigining asimptotasi bo'ladi. Ravshanki, $x=3$ to'g'ri chiziq berilgan funksiya grafigining vertikal asimptotasi bo'ladi.

9.11-misol. Ushbu

$$f(x)=3^{\frac{1}{x}}$$

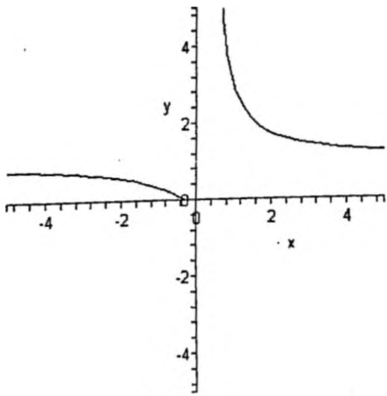
funksiyaning asimtotalarini toping va funksiya grafigining asimtotalarga nisbatan qanday joylashganligini aniqlang.

Yechilishi. Berilgan $f(x)=3^{\frac{1}{x}}$ funksiya $R\setminus\{0\}$ to'plamda aniqlangan. Funksiyaning $x\to\pm\infty$ va $x\to\pm0$ dagi limitlarini topamiz:

$$\lim_{x\to\pm\infty}3^{\frac{1}{x}}=1,\quad \lim_{x\to+0}3^{\frac{1}{x}}=+\infty,\quad \lim_{x\to-0}3^{\frac{1}{x}}=0.$$

Demak, $x\to\pm\infty$ da $y=1$ to'g'ri chiziq funksiya grafigi uchun gorizontal asimptota bo'ladi. $x\to+0$ da $x=0$ vertikal asimptota bo'ladi.

Ravshanki, $x>0$ da $3^{\frac{1}{x}}-1>0$, $3^{\frac{1}{x}}>1$, $x<0$ bo'lganda esa $3^{\frac{1}{x}}-1<0$, $3^{\frac{1}{x}}<1$ bo'lgani uchun, funksiya grafigi $x>0$ bo'lganda $y=1$ asimtotadan yuqorida, $x<0$ bo'lganda esa pastda joylashadi (9.14-chizma).



9.14-chizma.

Mustaqil ishlash uchun misollar

Quyidagi funksiya grafiklarining asimptotalarini toping:

9.1. $y = x + \frac{1}{x}$. 9.2. $y = \frac{x^2-6x+4}{3x-2}$. 9.3. $y = 10 + \frac{3}{x}$.

9.4. $y = \frac{3x^2-7}{2x+1}$. 9.5. $y = e^{\frac{1}{x}}$. 9.6. $y = \frac{21-x^2}{7x+9}$.

9.7. $y = \sqrt{x} + \sqrt{5-x}$. 9.8. $y = \frac{2x^2-1}{\sqrt{x^2-2}}$. 9.9. $y = \frac{x^3}{2(x+1)^2}$.

9.10. $y = \frac{x^3-2x^2-3x+2}{1-x^2}$. 9.11. $y = \frac{x^3}{6(3-x)^2}$. 9.12. $y = |x+2|\, e^{-\frac{1}{x}}$.

9.13. $y=x^x$. 9.14. $y = \frac{x^2+1}{x^2-1}$. 9.15. $y = \frac{x^2+16}{\sqrt{9x^2-8}}$.

9.16. $y = \frac{x-1}{\sqrt{x(3-x)}}$. 9.17. $y = \frac{(x+1)^2}{x-2}$. 9.18. $y = \frac{x^2-2x}{x-1}$.

9.19. $y = \frac{6(x^2-4)}{3x^2+8}$. 9.20. $y=\operatorname{tg}x$. 9.21. $y = \frac{2x^3-2x+1}{1+x^2}$.

9.22. $y = 4x + \frac{1}{x+3}$. 9.23. $y = \frac{2x^3-3x^2-2x+1}{1-3x^2}$.

9.24. $y = \frac{x^3-28}{x-3}$.

Mustaqil yechish uchun berilgan misollarning javoblari

9.1. $x=0$ — vertikal asimptota, $y=x$ — og'ma asimptota. 9.2. $x=\frac{2}{3}$ — vertikal asimptota, $y=\frac{1}{3}x-\frac{20}{9}$ — og'ma asimptota. 9.3. $y=10$ — gorizontal asimptota, $x=0$ — vertikal asimptota. 9.4. $x=-\frac{1}{2}$ — vertikal asimptota, $y=\frac{3}{2}x-\frac{3}{4}$ — og'ma asimptota. 9.5. $y=1$ — gorizontal asimptota, $x=0$ — vertikal asimptota. 9.6. $x=-\frac{9}{7}$ — vertikal asimptota, $y=-\frac{1}{7}x+\frac{9}{49}$ — og'ma asimptota. 9.7. Asimptota yo'q. 9.8. $x=\pm\sqrt{2}$ — vertikal asimptotalar, $y=-2x-1$, $y=2x-1$ — og'ma asimptotalar. 9.9. $x=-1$ — vertikal asimptota. 9.10. $x=\pm 1$ — vertikal asimptotalar, $y=-x+2$ — og'ma asimptota. 9.12. $x=3$ — vertikal asimptota, $y=\frac{1}{6}x+1$ — to'g'ri chiziq og'ma asimptota. 9.12. $y=-x-1$ va $y=x+1$ — og'ma asimptotalar. 9.13. Asimptota yo'q. 9.14. $x=1$, $x=-1$ — vertikal asimptota. 9.15. $x=-\frac{2\sqrt{2}}{3}$, $x=\frac{2\sqrt{2}}{3}$

– vertikal asimptotalar, $y=-\frac{1}{3}x$, $y=\frac{1}{3}x$ — og'ma asimptotalar. **9.16.** $x=0$, $x=3$ — vertikal asimptota. **9.17.** $x=2$ — vertikal asimptota, $y=x+4$ — og'ma asimptota **9.18.** $x=1$ — vertikal asimptota, $y=x-1$ — og'ma asimptota. **9.19.** $y=2$ — og'ma asimptota. **9.20.** $y=\frac{\pi}{2}+$ $+\pi n$, $n\in Z$ vertikal asimptotalar. **9.21.** $y=2x$ – og'ma asimptota. **9.22.** $y=4x$ — og'ma asimptota, $x=-3$ — vertikal asimptota. **9.23.** $x=-\frac{1}{\sqrt{3}}$, $x=\frac{1}{\sqrt{3}}$ — vertikal asimptotalar, $y=-\frac{2}{3}x+1$ — og'ma asimptota. **9.24.** $Y=x^2+x+1$ chiziq asimptotasi, $x=3$ — vertikal asimptota.

10-§. FUNKSIYANING TEKIS UZLUKSIZLIGI

10.1. Funksiyaning tekis va uzluksizligi uzluksizlik moduli. $f(x)$ funksiya X to'plamda aniqlangan bo'lsin. X to'plamning har bir nuqtasi uning limit nuqtasi bo'lsin.

10.1-ta'rif. $\forall\varepsilon>0$ son uchun shunday $\delta=\delta(\varepsilon)>0$ son topilib, X to'plamning $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x'$, x'' $(x',x''\in X)$ nuqtalari uchun

$$|f(x')-f(x'')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda *tekis uzluksiz* deyiladi.

Bu ta'rifni qisqacha,

$$\forall\varepsilon>0,\ \exists\delta(\varepsilon)>0,\ \forall x',\ x''\in X,\ |x'-x''|<\delta\Rightarrow|f(x')-f(x'')|<\varepsilon$$

ko'rinishda ifodalash ham mumkin.

10.1-eslatma. $f(x)$ funksiyaning tekis uzluksizligi ta'rifidagi $\delta>0$ son $\varepsilon>0$ songa bog'liq bo'lib, qaralayotgan x nuqtalarga bog'liq emas.

$f(x)$ funksiyaning X to'plamdagi tekis uzluksizligi inkorini, qisqacha, quyidagicha ta'riflash mumkin:

$$\exists\varepsilon>0,\ \forall\delta(\varepsilon)>0,\ \exists x',x''\in X,\ |x'-x''|<\delta\Rightarrow|f(x')-f(x'')|\geq\varepsilon.$$

10.1-teorema (Kantor teoremasi). *Agar $f(x)$ funksiya $[a;\ b]$ segmentda aniqlangan va uzluksiz bo'lsa, u shu segmentda tekis uzluksiz bo'ladi.*

$f(x)$ funksiya X to'plamda aniqlangan bo'lib, δ ixtiyoriy musbat son bo'lsin.

10.2-ta'rif. Ushbu

$$\omega(f;\delta)=\sup_{\substack{|x'-x''|<\delta \\ \forall x',\, x''\in X}}\{|\, f(x')-f(x'')\,|\}$$

ifodaga $f(x)$ funksiyaning X to'plamdagi *uzluksizlik moduli* deyiladi.

$f(x)$ funksiya X to'plamda aniqlangan bo'lsin.

10.3-ta'rif. Agar $\exists K>0$, α $(0<\alpha<1)$ sonlar mavjud bo'lib, $\forall x_1, x_2 \in X$ lar uchun

$$|\, f(x_1)-f(x_2)\,|\le K\,|\, x_1-x_2\,|^{\alpha} \qquad (10.1)$$

shart bajarilsa, u holda $f(x)$ funksiya X to'plamda *Gyolder shartini* qanoatlantiradi deyladi ($\alpha=1$ bo'lsa, u holda (10.1) shart *Lipshis sharti* deyiladi).

10.2-teorema. *$f(x)$ funksiya X to'plamda tekis uzluksiz bo'lishi uchun* $\lim\limits_{\delta\to 0+0}\omega(f;\delta)=0$ *shartining bajarlishi zarur va yetarli.*

Funksiyaning uzluksizlik moduli quyidagi xossalarga ega:

1°. $\omega(f;\delta)$ har doim $\omega(f;\delta)\ge 0$.

2°. $\omega(f;\delta)$ $\delta>0$ bo'yicha o'suvchi funksiya bo'ladi.

3°. Ixtiyoriy λ — musbat son uchun $\omega(f,\lambda\delta)\le(1+\lambda)\omega(f,\delta)$ tengsizlik o'rinli.

4°. Ixtiyoriy musbat son δ_1, δ_2 lar uchun

$$\omega(f,\delta_1+\delta_2)\le\omega(f,\delta_1)+\omega(f,\delta_2)$$

tengsizlik o'rinli.

10.1-misol. Ushbu

$$f(x)=x+\sin x$$

funksiyaning R da tekis uzluksizligini isbotlang.

Yechilishi. a) *«δ»ni topish.* $\forall\varepsilon>0$ son berilgan bo'lsin. Berilgan $\forall\varepsilon>0$ ga ko'ra shunday x ga bog'liq bo'lmagan $\delta=\delta(\varepsilon)>0$ ni topish kerakki, $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x',\ x''\in R$ lar uchun $|f(x')-f(x'')|<\varepsilon$ tengsizlik bajarilsin. Buning uchun $|f(x')-f(x'')|$, $|x'-x''|$ ifodalar o'rtasidagi bog'lanishni topamiz:

$$|f(x')-f(x'')|=|x'+\sin x'-x''-\sin x''|=$$
$$|x'-x''-(\sin x''-\sin x')|\le$$
$$\le|x'-x''|+2\left|\sin\frac{x'-x''}{2}\right|\left|\cos\frac{x'+x''}{2}\right|\le$$
$$\le 2|x'-x''|<\varepsilon$$

bo'lishi uchun $\delta=\frac{\varepsilon}{2}$ deb olish yetarli.

b) *δ ning «ishlash»ini* ko'rsatamiz.

$|x'-x''|<\frac{\varepsilon}{2}$ bo'lsin. Bundan, $\frac{|x'-x''|}{2}\ge\left|\sin\frac{x'-x''}{2}\right|$, $1\ge\left|\cos\frac{x'+x''}{2}\right|$ tengsizliklarni e'tiborga olgan holda,

$$\varepsilon>2|x'-x''|=|x'-x''+2\frac{|x'-x''|}{2}\cdot 1\ge$$
$$\ge|x'-x''|+2\left|\sin\frac{x'-x''}{2}\right|\left|\cos\frac{|x'+x''|}{2}\right|=$$
$$=|x'-x''|+|\sin x'-\sin x''|\ge|x'+\sin x'-x''-\sin x''|=$$
$$=|f(x')-f(x'')|$$

tengsizlikka ega bo'lamiz. Demak, berilgan ε bo'yicha topilgan δ tekis uzluksizlik ta'rifini qanoatlantiradi.

10.2-misol. Ushbu

$$f(x)=e^x\sin\frac{1}{x}$$

funksiyani $X=(0;\ 1)$ da tekis uzluksizlikka tekshiring.

Yechilishi. $x_n'=\frac{2}{(4n-1)\pi}$, $x_n''=\frac{2}{(4n+1)\pi}$ nuqtalarni olamiz.

$$|x_n' - x_n''| = \left|\frac{2}{(4n-1)\pi} - \frac{2}{(4n+1)\pi}\right| = \frac{2}{\pi}\left|\frac{1}{4n-1} - \frac{1}{4n+1}\right| = \frac{1}{\frac{\pi}{4}(16n^2-1)} \xrightarrow[n\to\infty]{} 0$$

$$|f(x_n') - f(x_n'')| = \left|e^{\frac{2}{\pi(4n-1)}}\sin(4n-1)\frac{\pi}{2} - e^{\frac{2}{\pi(4n+1)}}\sin(4n+1)\frac{\pi}{2}\right| =$$

$$= \left|-e^{\frac{2}{\pi(4n-1)}} - e^{\frac{2}{\pi(4n+1)}}\right| = \left|e^{\frac{2}{\pi(4n-1)}} + e^{\frac{2}{\pi(4n+1)}}\right| \geq 2 = \varepsilon.$$

Demak, berilgan funksiya (0; 1) da tekis uzluksiz emas.

10.3-misol. Ushbu

$$f(x) = \sin x^2$$

funksiyaning $X = R$ da tekis uzluksiz emasligini ko'rsating.

Yechilishi. Ravshanki, berilgan funksiya R da chegaralangan va uzluksiz, lekin uning R da tekis uzluksiz emasligini ko'rsatamiz: $\forall\delta > 0$ berilgan bo'lsin, R dan $\forall x_n' = \sqrt{n\pi}$, $x_n'' = \sqrt{n\pi + \frac{\pi}{2}}$ nuqtalarni olamiz. Unda,

$$|x_n' - x_n''| = \left|\sqrt{n\pi} - \sqrt{n\pi + \frac{\pi}{2}}\right| = \frac{\frac{\pi}{2}}{\sqrt{n\pi} + \sqrt{n\pi + \frac{\pi}{2}}} \xrightarrow[n\to\infty]{} 0.$$

Demak, bu ayirma n ning o'sishi bilan $\forall\delta > 0$ sondan kichik bo'lib boraveradi, lekin

$$|f(x_n') - f(x_n'')| = \left|\sin n\pi - \sin\left(n\pi + \frac{\pi}{2}\right)\right| = 1 > \varepsilon \quad (\forall\varepsilon < 1).$$

Shunday qilib, $f(x) = \sin x^2$ funksiya R da tekis uzluksiz emas ekan.

10.4-misol. Ushbu

1) $f(x) = 2x - 1$, $X = R$; 2) $f(x) = \sqrt[3]{x}$, $X = [0,2]$;

3) $f(x) = \cos\frac{1}{x}$, $X = (0,1)$;

4) $f(x) = x^2$; a) $X = (1;\ 1)$, b) $X = R$;

5) $f(x)=\frac{x}{2-x^2}$, $X=[-1,1]$

funksiyalarni ko'rsatilgan X to'plamda tekis uzluksizlikka tekshiring.

Yechilishi. 1) $\forall\varepsilon>0$ son berilgan bo'lsin. Bu berilgan $\varepsilon>0$ songa ko'ra x ga bog'liq bo'lmagan shunday $\delta=\delta(\varepsilon)$ son topishimiz kerakki, $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x', x''\in R$ lar uchun $|f(x')-f(x'')|<\varepsilon$ tengsizlik bajarilsin. Unda

$$|f(x')-f(x'')|=|2x'-1-2x''+1|=2|x'-x''|<\varepsilon$$

bo'lishi uchun $\delta=\frac{\varepsilon}{2}$ deb olish yetarli.

Demak, 10.1-ta'rifga asosan berilgan funksiya R da tekis uzluksiz bo'ladi.

2) Ravshanki, $f(x)=\sqrt[3]{x}$ funksiya $[0;\ 2]$ segmentda uzluksiz. U vaqtda, Kantor teoremasiga asosan berilgan funksiya $[0;\ 2]$ segmentda tekis uzluksiz bo'ladi.

3) $f(x)=\cos\frac{1}{x}$ funksiya $(0;\ 1)$da uzluksiz, lekin tekis uzluksiz emas. Haqiqatdan ham, $\forall\delta>0$ son berilgan bo'lsin.

Agar $\varepsilon\in(0;2)$, $x_n'=\frac{1}{2\pi n}$, $x_n''=\frac{1}{(2n+1)\pi}$ deb olsak, unda, bu

$$|x_n'-x_n''|=\left|\frac{1}{2\pi n}-\frac{1}{(2n+1)\pi}\right|=\frac{1}{2n(2n+1)\pi}$$

ayirma, n ning o'sishi bilan istalgancha kichik $\delta=\delta(\varepsilon)>0$ sondan kichik bo'lib boraveradi.

$$|f(x_n')-f(x_n'')|=|\cos 2\pi n-\cos(2n+1)\pi|=2>\varepsilon \qquad (0<\varepsilon<2).$$

Demak, $f(x)=\cos\frac{1}{x}$ funksiya $(0;\ 1)$da tekis uzluksiz emas ekan.

4) a) $\forall\varepsilon>0$ son berilgan bo'lsin. Unda berilgan ε ga ko'ra, shunday $\delta=\delta(\varepsilon)>0$ sonni topishimiz kerakki, $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x',x''\in(-1;1)$ lar uchun

$$|f(x')-f(x'')|=|(x')^2-(x'')^2|<\varepsilon \qquad (10.2)$$

tengsizlik bajarilsin. Buning uchun avvalo

$$|(x')^2-(x'')^2|,\ |x'-x''|$$

munosabatlar orasidagi bog'lanishni topamiz:

$$|f(x')-f(x'')|=|x'-x''||x'+x''|<2|x'-x''|.$$

(10.2) tengsizlik bajarilishi uchun $\delta=\frac{\varepsilon}{2}$ deb olish yetarli. Endi, berilgan ε ga ko'ra δ ning to'g'ri topilganligini tekshirib ko'ramiz: $|x'-x''|<\frac{\varepsilon}{2}$ bo'lsin. Bundan, $2|x'-x''|<\varepsilon$ $x',x''\in(-1,1)$ ekanligini inobatga olgan holda, keyingi tengsizlikdan

$$\varepsilon>|x'-x''||x'+x''|=|(x')^2-(x'')^2|$$

munosabatni hosil qilamiz.

Demak, $\delta=\frac{\varepsilon}{2}$ to'g'ri topilgan ekan, ya'ni berilgan funksiya $(-1;\ 1)$ da tekis uzluksiz.

b) $f(x)=x^2$ funksiya R da tekis uzluksiz emas. Haqiqatdan ham, $\forall\varepsilon<2$, $x'_n=n+\frac{1}{n}$, $x''_n=n$ deb olsak, unda $|x'_n-x''_n|=$ $=\frac{1}{n}\xrightarrow[n\to\infty]{}0$,

$$|f(x'_n)-f(x''_n)|=\left|\left(n+\frac{1}{n}\right)^2-n^2\right|=\left|n^2+2+\frac{1}{n^2}-n^2\right|=2+\frac{1}{n^2}>2>\varepsilon.$$

. Demak, tekis uzluksizlikning inkoriga asosan, $f(x)=x^2$ funksiya R da tekis uzluksiz emas.

5) Ravshanki, $f(x)=\frac{x}{2-x^2}$ funksiya $[-1;\ 1]$ segmentda uzluksiz. U holda Kantor teoremasiga asosan, $[-1;\ 1]$ segmentda tekis uzluksiz bo'ladi.

10.5-misol. Ushbu

1) $f(x)=3x^2+x+2,\quad X=[0,\ 1];$

2) $f(x)=x^2$, a) $X=[-a,a]$ $(a>0)$, b) $X=(-\infty,\infty)$;

3) $f(x)=\cos\frac{1}{x}$, $X=(0,\infty)$; 4) $f(x)=\sqrt[3]{x}$, $X=[0,2]$;

funksiyalarning uzluksizlik modulini toping va ularni tekis uzluksizlikka tekshiring.

Yechilishi. 1) $\forall\delta>0$ berilgan bo'lsin. $X=[0,1]$ to'plamdan ixtiyoriy x', x'' nuqtalarni olamiz va aniqlik uchun $x'>x''$ deb, ya'ni $x'=x''+\Delta x$, $0<\Delta x<\delta$ qaraymiz. U holda $\forall x''>0$ uchun

$$|f(x')-f(x'')|=|(3x'^2+x'+2)-(3x''^2+x''+2)|=$$
$$=|3(x''+\Delta x)^2+(x''+\Delta x)+2-(3x''^2+x''+2)|=$$
$$=|6x''\Delta x+3\Delta x^2+\Delta x|\le 6\delta+3\delta^2+\delta=7\delta+3\delta^2.$$

Bundan, $\omega(f,\delta)=\sup\limits_{|x'-x''|\le\delta}|f(x')-f(x'')|\le 7\delta+3\delta^2$. Lekin, $x'=$ $=x''+\delta$ nuqtalar uchun $|x'-x''|=\delta$ va

$$|f(x')-f(x'')|=|(3x'^2+x'+2)-(3x''^2+x''+2)|=$$
$$=|3x'^2+x'-(3x''^2+x'')|=7\delta+3\delta^2$$

bo'lishini e'tiborga olsak, unda bu tenglik va yuqoridagi tengsizlikdan

$$\omega(f,\delta)=7\delta+3\delta^2,\ \delta>0.$$

Bundan esa, $\lim\limits_{\delta\to0}\omega(f,\delta)=\lim\limits_{\delta\to0}(7\delta+3\delta^2)=0.$

Demak, berilgan $f(x)=3x^2+x+2$ funksiya $[0;1]$da tekis uzluksiz.

2) a) $\forall\delta>0$ berilgan. $|x'-x''|\le\delta$ tengsizlikni qanoatlantiruvchi $\forall x'$, $x''\in[-a,a]$ nuqtalarni olamiz.

$$|f(x')-f(x'')|=|x'^2-x''^2|=|x'+x''|\cdot|x'-x''|\le 2a\delta.$$

Bundan, $\omega(f,\delta)=\omega(x^2,\delta)\le 2a\delta$ bo'ladi. Ammo, $x'=a$, $x''=$ $=a-\delta$ $(0<\delta<2a)$ nuqtalar uchun

$$|f(x')-f(x'')|=|x'^2-x''^2|=a^2-(a-\delta)^2=|2a\delta-\delta^2|=2a\delta-\delta^2$$

bo'lganligi sababli, $\omega(x^2,\delta)=2a\delta-\delta^2$ bo'ladi. Bundan, $\lim_{\delta\to 0}\omega(x^2,\delta)=$ $=\lim_{\delta\to 0}(2a\delta-\delta^2)=0$.

Demak, berilgan $f(x)=x^2$ funksiya $[-a;\,a]$ $(a>0)$ segmentda tekis uzluksiz.

b) $X=(-\infty,\infty)$ to'plamdan $\forall x'=n+\frac{1}{n}$, $x''=n$ nuqtalarni olaylik:

$$|f(x')-f(x'')|=\left|\left(n+\frac{1}{n}\right)^2-n^2\right|=2+\frac{1}{n^2}$$

Bundan, $\forall\delta>0$ uchun $\omega(x^2,\delta)=3$, $\lim_{\delta\to 0}2\omega(x^2,\delta)=3\neq 0$ bo'lgani uchun $f(x)=x^2$ funksiya $X=(-\infty,\infty)$ da tekis uzluksiz emas.

3) $\forall\delta>0$ son berilgan bo'lsin.

Ravshanki, $\forall x'$, $x''\in(0,\infty)$ nuqtalar uchun:

$$|f(x')-f(x'')|=\left|\cos\frac{1}{x'}-\cos\frac{1}{x''}\right|\leq 2\,.$$

Xususiy holda $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi x' va x'' lar uchun ham

$$\left|\cos\frac{1}{x'}-\cos\frac{1}{x''}\right|\leq 2$$

tengsizlik bajariladi. Demak,

$$\omega\left(\cos\frac{1}{x},\delta\right)\leq 2\,.$$

Ammo $(0,\infty)$ to'plamdan $x'_n=\frac{1}{2\pi n}$, $x''_n=\frac{1}{(2n+1)\pi}$ nuqtalar olinsa, ravshanki, n ni yetarlicha katta qilib olish natijasida x'_n va x''_n nuqtalar uchun $|x'-x''|<\delta$ tengsizlikka erishish mumkin. U holda

$$\left|\cos\frac{1}{x'}-\cos\frac{1}{x''}\right|=|\cos 2\pi n-\cos(0)\pi|=2\,.$$

Bundan, $\omega\left(\cos\frac{1}{x},\delta\right)=2$, lekin, $\lim\limits_{\delta\to 0}\omega\left(\cos\frac{1}{x},\delta\right)=2\neq 0$ bo'lganligi uchun, berilgan $f(x)=\cos\frac{1}{x}$ funksiya $(0;\infty)$ da tekis uzluksiz emas.

4) $\forall\delta>0$ sonni tanlab, $\forall x',\ x''\in[0,\ 2]$ nuqtalarni olamiz. Aniqlik uchun $x'>x''$ deb, ya'ni $x'=x''+\Delta x$, $0<\Delta x\le\delta<2$ deb olamiz. Unda, $\forall x''>0$ uchun

$$|f(x')-f(x'')|=\left|\sqrt[3]{x'}-\sqrt[3]{x''}\right|=\sqrt[3]{x''+\Delta x}-\sqrt[3]{x''}\le\sqrt[3]{x''+\delta}-\sqrt[3]{x''}=$$

$$=\frac{\delta}{\sqrt[3]{x''^2}+\sqrt[3]{x''(x''+\delta)}+\sqrt[3]{(x''+\delta)^2}}\le\sqrt[3]{\delta}$$

bo'ladi. Bu tengsizlikdan $\omega\left(\sqrt[3]{x},\delta\right)\le\sqrt[3]{\delta}$. Ammo, $x'=x''+\delta$ bo'lganda,

$$\lim_{x''\to 0}\left(\sqrt[3]{x''+\delta}-\sqrt[3]{x''}\right)=\sqrt[3]{\delta}$$

bo'lgani uchun, $\omega\left(\sqrt[3]{x},\delta\right)\ge\sqrt[3]{\delta}$. Bundan va yuqoridagi tengsizlikdan $\forall\delta>0$ uchun $\omega\left(\sqrt[3]{x},\delta\right)=\sqrt[3]{\delta}$. $\lim\limits_{\delta\to 0}\omega\left(\sqrt[3]{x},\delta\right)=0$ bo'lganligi sababli, berilgan $f(x)=\sqrt[3]{x}$ funksiya [0; 2]da tekis uzluksiz.

Mustaqil yechish uchun misol va masalalar

10.1. Quyidagi berilgan $f(x)$ funksiyaning X to'plamda teks uzluksiz ekanligini isbotlang:

1) $f(x)=3x-1,\ X=R$; 2) $f(x)=\frac{x}{3-x^2},\ X=[-1;\ 1]$;

3) $f(x)=\sin x^2,\ X=[-3;\ 3]$;

4) $f(x)=\frac{\sin x}{x},\ X=(0;\ \pi)$; 5) $f(x)=\sin x,\ X=R$;

6) $f(x)=\frac{1}{x},\ X=[a;+\infty),\ a>0$;

7) $f(x)=\operatorname{arctg}x,\ X=(-\infty;+\infty)$;

8) $f(x)=\sqrt{x},\ X=(0;\ +\infty)$;

9) $f(x)=\frac{x^6-1}{\sqrt{1-x^4}},\ X=(-1;\ 1)$;

10) $f(x)=\sin\left(\frac{1}{x}\right),\ X=[0,01;\ +\infty)$.

10.2. $f(x)$ funksiya tekis uzluksiz bo'ladigan oraliqda berilgan $\varepsilon>0$ uchun $\delta=\delta(\varepsilon)$ni toping:

1) $f(x)=4x+5, \quad (-\infty<x<\infty)$;

2) $f(x)=x^2-3x-2, \quad (-1<x<5)$;

3) $f(x)=\frac{1}{x^2} \quad (0{,}1\le x\le 1)$;

4) $f(x)=2\sin x-\cos x, \quad (-\infty<x<+\infty)$;

5) $f(x)=x\cos\frac{1}{x}, \quad (x\ne 0)$ va $f(x)=0, \quad (0\le x\le \pi)$.

10.3. Quyida berilgan $f(x)$ funksiyaning X to'plamda tekis uzluksiz emasligini isbotlang:

1) $f(x)=\cos\left(\frac{1}{x}\right), \quad x=(0;\,1)$; 2) $f(x)=\sin x^2, \quad X=R$;

3) $f(x)=e^x, \quad X=R$; 4) $f(x)=\operatorname{ctg}x, \quad X=(0;\,1)$;

5) $f(x)=\ln x, \quad X=(0;\,1)$;

6) $f(x)=e^x\cos\frac{1}{x}, \quad X=(0;\,1)$;

7) $f(x)=x\sin x, \; X=(0;\,\infty)$; 8) $f(x)=\sin\left(\frac{1}{x}\right), \quad X=(0;\,1)$.

10.4. Quyidagi berilgan $f(x)$ funksiyani X to'plamda tekis uzluksizlikka tekshiring:

1) $f(x)=e^{-\arcsin x}, \; X=[-1;\,1]$; 2) $f(x)=\sqrt[3]{x}, \quad X=R$;

3) $f(x)=\begin{cases}1-x^2, & -1\le x\le 0,\\ 1+x, & 0<x\le 1,\end{cases} \; X=[-1;\,1]$;

4) $f(x)=\frac{1}{x}, \quad X=[0{,}1;\,1]$;

5) $f(x)=\cos x, \quad X=(0;\,1)$;

6) $f(x)=x^2$: $\quad a)\; X=[-e,e]$; $\quad b)\; X=R$.

10.5. Quyida berilgan $f(x)$ funksiyaning X to'plamada uzluksizlik modullarini toping va ularni uzluksizlik moduli yordamida tekis uzluksizlikka tekshiring:

1) $f(x)=2x-1, \quad X=R$;

2) $f(x)=\frac{1}{x}, \quad X=[a;+\infty), \quad a>0$;

3) $f(x)=x^{\varepsilon}+1, \quad \varepsilon>0, \quad X=[0;\,1]$;

4) $f(x)=\sin\frac{1}{x}, \quad X=(0;+\infty)$;

5) $f(x)=\frac{1}{x^2}, \quad X=(0;\,1)$.

10.6. Agar $\delta_1 < \delta_2$ bo'lsa, $\omega(f;\delta_1) \leq \omega(f;\delta_2)$ tengsizlikni isbotlang.

10.7. Agar X to'plam chegaralangan, X to'plamda $f(x)$ funksiya chegaralanmagan bo'lsa, u holda $\forall\delta > 0$ uchun $\omega(f;\delta) = +\infty$ ekanligini isbotlang.

10.8. Agar X — oraliq (interval) bo'lsa, u holda $\forall\delta_1 > 0$, $\delta_2 > 0$ uchun $\omega(f;\ \delta_1 + \delta_2) \leq \omega(f;\ \delta_1) + \omega(f;\ \delta_2)$ ekanligini isbotlang.

10.9. Agar $f(x)$ va $g(x)$ funksiyalar X to'plamda tekis uzluksiz bo'lsa, u holda $\forall\lambda, \beta \in R$ uchun $\lambda f(x) + \beta g(x)$ funksiya ham X to'plamda tekis uzluksiz bo'lishini isbotlang.

10.10. Agar $f(x)$ va $g(x)$ funksiyalar $[a;+\infty)$ da chegaralangan va tekis uzluksiz bo'lsa, u holda $f(x)\cdot g(x)$ funksiyaning $[a;+\infty)$ da tekis uzluksiz ekanligini isbotlang.

10.11. $[a;+\infty)$ oraliqda har biri tekis uzluksiz, ko'paytmasi esa tekis uzluksiz bo'lmaydigan funksiyalarga misol keltiring.

10.12. Agar $f(x)$ va $g(x)$ funksiyalar X chegaralangan to'plamda tekis uzluksiz bo'lsa, u holda ularning ko'paytmasini X to'plamda tekis uzluksiz bo'lishini isbotlang.

10.13. Ushbu a) $X = (0;\ 1)$; b) $X = [0;\ +\infty)$ to'plamlarda uzluksiz, lekin tekis uzluksiz bo'lmagan funksiyalarga misollar keltiring.

10.14. $X = [0;\ +\infty)$ to'plamda chegaralanmagan, lekin unda tekis uzluksiz bo'lgan funksiyalarga misollar keltiring.

10.15. $(0;\ 1)$ oraliqda chegaralangan, uzluksiz bo'lgan, ammo berilgan oraliqda tekis uzluksiz bo'lmaydigan funksiyalarga misol keltiring.

10.16. Ko'rsatilgan oraliqlarda $\sqrt{x}$ funksiya tekis uzluksiz bo'ladimi?

1) $[0;a]$ $(a > 0)$; 2) $(0;\ a)$, $(a > 0)$;

3) $[4;+\infty)$; 4) $[0,\ +\infty)$; 5) $(0;\ \infty)$.

10.17. Uzluksiz davriy funksiyaning tekis uzluksiz ekanligini isbotlang.

10.18. Gelder shartini qanoatlantiruvchi funksiyalarning tekis uzluksiz bo'lishini isbotlang.

10.19. $(a;\ b)$ oraliqda tekis uzluksiz bo'lgan funksiyaning shu oraliqda uzluksiz ekanligini isbotlang.

10.20. Ushbu

1) $f(x)=x^2$ $(0\le x\le 1)$, 2) $f(x)=\sqrt{x}$ $(0\le x\le a)$,

3) $f(x)=\sin x+\cos x$ $(0\le x\le 2\pi)$

funksiyalarning uzluksizlik modullari uchun $\omega(f,\delta)\le L\delta$ ko'rinishdagi tengsizliklarni isbotlang.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

10.2. 1) $\delta=\frac{\varepsilon}{4}$; 2) $\delta=\frac{\varepsilon}{4}$; 3) $\delta=\frac{0{,}0001\varepsilon}{3}$; 4) $\delta=\frac{\varepsilon}{3}$; 5) $\delta=\min\left\{\frac{\varepsilon}{[illegible]},\frac{\varepsilon^2}{[illegible]}\right\}$. **10.4.** 2) $\omega(f,\delta)=2\delta$, tekis uzluksiz; 3) $\forall\delta>0$ uchun $\omega(f,\delta)=+\infty$, tekis uzluksiz emas; 5) $\omega(f,\delta)=2\delta-\delta^2$; [illegible] tekis uzluksiz emas; 6) $\forall\delta>0$ uchun $\omega(f,\delta)=+\infty$, tekis uzluksiz emas. **10.13.** Masalan, $f(x)=\sin\frac{1}{x}$, $g(x)=\frac{1}{x}$. [illegible] $f(x)=x\sin x$, $g(x)=\sin x^2$. **10.14.** Masalan, $f(x)=\sqrt{x}$ [illegible]. **10.15.** Masalan, $y=\sin\frac{1}{x}$. **10.16.** 1) ha; 2) ha; 3) ha; 4) ha; 5) ha.

III BOB
DIFFERENSIAL HISOB VA UNING TATBIQLARI

11-§. FUNKSIYANING HOSILASI VA DIFFERENSIALI

11.1. Funksiya hosilasining ta'rifi. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lsin. Bu oraliqdan x_0 nuqta olib, unga $x_0+\Delta x\in(a,b)$ bo'lgan Δx ($\Delta x>0$ yoki $\Delta x<0$) orttirma beraylik. U holda $f(x)$ funksiya ham x_0 nuqtada

$$\Delta y=\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$$

orttirmaga ega bo'ladi. Ushbu

$$\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\quad(\Delta x\neq 0)$$

nisbatni qaraymiz. Ravshanki, bu nisbat Δx ning funksiyasi bo'lib, u Δx ning noldan farqli qiymatlarida, jumladan, nol nuqtaning yetarli kichik $U_\delta(0)$ $(\delta>0)$ atrofida aniqlangan. Bunda $\Delta x=0$ nuqta $U_\delta(0)$ to'plamning limit nuqtasidir.

11.1-ta'rif. Agar $\Delta x\to 0$ da $\frac{\Delta y}{\Delta x}$ nisbatning

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

limiti mavjud va chekli bo'lsa, bu limit $f(x)$ funksiyaning x_0 nuqtadagi hosilasi deyiladi va

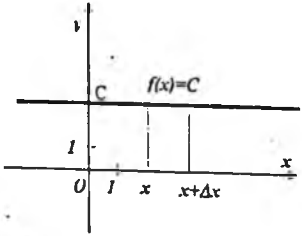
$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\tag{11.1}$$

kabi belgilanadi.

Agar $x_0+\Delta x=x$ deb olinsa, unda $\Delta x=x-x_0$ va $\Delta x\to 0$ da $x\to x_0$, natijada (11.1) ning ko'rinishi

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\tag{11.2}$$

shaklda bo'ladi. Hosila quyidagi $y'(x_0)$, y'_x (Lagranj), $\frac{dy}{dx}$, $\frac{df}{dx}$ (Leybnits), Dy, Df (Koshi) belgilar yordamida ham yoziladi.



11.1-chizma.

11.1-misol. Ushbu $f(x)=C$ funksiyaning hosilasini ta'rifdan foydalanib toping, bunda C-biror o'zgarmas son (11.1-*chizma*).

Yechilishi. Erkli o'zgaruvchining ikkita turli x va $x+\Delta x$ qiymatini olsak, bu qiymatlarda funksiyaning qiymatlari bir xil bo'ladi, ya'ni

$$f(x)=C,\quad f(x+\Delta x)=C.$$

Shuning uchun $\Delta y=f(x+\Delta x)-f(x)=C-C=0$, demak, $\frac{\Delta y}{\Delta x}=\frac{0}{\Delta x}=0$.

Shunday qilib,

$$y'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0,$$

ya'ni o'zgarmas sonning hosilasi nolga teng.

11.2-misol. Ushbu $y=\operatorname{tg}x$ funksiyaning hosilasini ta'rifdan foydalanib toping.

Yechilishi. Ma'lumki, $y=\operatorname{tg}x$ funksiya R ning $x=\frac{\pi}{2}+k\pi$ $(k\in Z)$ nuqtalardan tashqari nuqtalarida aniqlangan. $\forall x\in D(y)$ nuqtani olib, unga $x+\Delta x\in D(y)$ bo'ladigan Δx $(\Delta x<0$ yoki $\Delta x>0)$ orttirma beraylik. Bunda argumentning Δx orttirmasiga mos ravishda, berilgan funksiya ham

$$\Delta y = \operatorname{tg}(x+\Delta x) - \operatorname{tg} x = \frac{\sin(\Delta x)}{\cos(x+\Delta x)\cos x} \qquad (11.3)$$

orttirma oladi. (11.3) ning ikkala tomonini Δx ga bo'lib,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\cos(x+\Delta x)\cdot\cos x}\cdot\frac{\sin\Delta x}{\Delta x}$$

nisbatni hosil qilamiz va uning $\Delta x \to 0$ dagi limitini hisoblaymiz:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)\cdot\cos x}\cdot\frac{\sin\Delta x}{\Delta x} = \frac{1}{\cos^2 x}.$$

Demak, $y' = (tgx)' = \dfrac{1}{\cos^2 x}$ $\quad x \neq \dfrac{\pi}{2} + k\pi,\ k \in Z.$

11.3-misol. Ushbu $f(x) = (x-1)^2(x+3)$ funksiyaning $x_0 = 1$ nuqtadagi hosilasini ta'rifdan foydalanib, toping.

Yechilishi. (11.2) formulaga asosan: $f(1) = 0$ va

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(x-1)^2(x+3)}{x-1} = 0$$

bo'ladi. Demak, $f'(1) = 0$.

11.4-misol. Ushbu 1) $y = \sqrt{x},\ x_0 = 0$ va $\Delta x = 0{,}01$; 2) $y = 2x^2 + 3x + 5,\ x_0 = 0$ va $\Delta x = 0{,}001$; 3) $y = \dfrac{2}{x^2+x-4},\ x_0 = 1$ va $\Delta x = 0{,}2$ funksiyalar uchun x_0 va Δx larning ko'rsatilgan qiymatlarida Δy va $\dfrac{\Delta y}{\Delta x}$ larni toping.

Yechilishi. Δy va $\dfrac{\Delta y}{\Delta x}$ larni tuzib, hisoblaymiz:

1) $\Delta y = \sqrt{x_0 + \Delta x} - \sqrt{x_0} = \sqrt{0{,}01} = 0{,}1;$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{0{,}1}{0{,}01} = 10.$

2) $\Delta y = 2\cdot(0{,}001)^2 + 3\cdot 0{,}001 + 5 - 5 = 0{,}003002;$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{0{,}003002}{0{,}001} = 3{,}002.$

3) $\Delta y = \dfrac{1}{(1+0{,}2)^2 + (1+0{,}2) - 4} + \dfrac{1}{2} = \dfrac{1}{1{,}44 + 1{,}2 - 4} + \dfrac{1}{2} =$

$= -\dfrac{1}{1{,}36} + \dfrac{1}{2} = -\dfrac{1}{21};$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{-\frac{1}{21}}{\frac{1}{5}} = -\dfrac{5}{21}.$

Berilgan nuqtada chekli hosilaga ega bo'lgan funksiya, shu nuqtada albatta uzluksiz bo'ladi, ya'ni funksiyaning nuqtada chekli hosilaga ega bo'lishi uchun, uning shu nuqtada uzluksiz bo'lishi zaruriy shart bo'ladi, lekin yetarli shart bo'la olmaydi, masalan,

$$f(x)=|x|=\begin{cases} x, & x\geq 0, \\ -x, & x<0 \end{cases}$$

funksiya $x=0$ nuqtada uzluksiz, lekin bu nuqtada hosilaga ega emas.

11.2- ta'rif. Agar $\Delta x\to 0+0$ $(\Delta x\to 0-0)$ da $\frac{\Delta y}{\Delta x}$ ning chekli limiti

$$\lim_{\Delta x\to 0+0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$
$$\left(\lim_{\Delta x\to 0-0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0-0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right) \tag{11.4}$$

mavjud bo'lsa, bu limit $f(x)$ funksiyaning x_0 nuqtadagi o'ng (chap) hosilasi deb ataladi va $f'(x_0+0)$ $(f'(x_0-0))$ kabi belgilanadi.

Odatda, funksiyaning o'ng va chap hosilalari bir tomonli hosilalar deb ham aytiladi.

11.5-misol. Ushbu $y=|x-2|$ funksiyaning $x=2$ nuqtadagi o'ng va chap hosilalarini toping.

Yechilishi. (11.4) formuladan:

$$\lim_{\Delta x\to 0+0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{f(2+\Delta x)-f(2)}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{\Delta x}{\Delta x}=1,$$

$$\lim_{\Delta x\to 0-0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0-0}\frac{f(2+\Delta x)-f(2)}{\Delta x}=\lim_{\Delta x\to 0-0}\frac{|\Delta x|}{\Delta x}=\lim_{\Delta x\to 0-0}\frac{-\Delta x}{\Delta x}=-1.$$

Demak, $f(x)=|x-2|$ funksiyaning $x=2$ nuqtadagi o'ng hosilasi $f'(2+0)=1$, chap hosilasi esa $f'(2-0)=-1$ ekan.

11.1-eslatma. Agar $f(x)$ funksiya x_0 nuqtada hosilaga ega bo'lsa, u shu nuqtada bir tomonli $f'(x_0+0)$, $f'(x_0-0)$ hosilalarga ham ega bo'lib, $f'(x_0+0)=f'(x_0-0)=f'(x_0)$ tengliklar o'rinli.

11.2-eslatma. Agar $f(x)$ funksiya x_0 nuqtaning biror $U(x_0)$ atrofida uzluksiz, x_0 nuqtada bir tomonli $f'(x_0+0)$ va $f'(x_0-0)$ hosilalarga ega bo'lib, $f'(x_0+0)=f'(x_0-0)$ bo'lsa, $f(x)$ funksiya shu nuqtada $f'(x_0)$ hosilaga ega bo'ladi va $f'(x_0)=f'(x_0+0)=f'(x_0-0)$ tengliklar o'rinli.

11.3-eslatma. Agar $\Delta x \to 0$ da $\frac{\Delta y}{\Delta x}$ nisbat aniq ishorali cheksiz limitga ega, ya'ni

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\pm\infty \qquad (11.5)$$

bo'lsa, u ham $f(x)$ funksiyaning x_0 nuqtadagi hosilasi deb yuritiladi. Bunday hosila cheksiz hosila deb ataladi.

11.6-misol. Ushbu $y=\sqrt[3]{x}$ funksiyaning $x=0$ nuqtadagi hosilasini toping.

Yechilishi. Ta'rifga ko'ra:

$$\lim_{\Delta x\to 0+0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{\sqrt[3]{\Delta x}}{\Delta x}=\lim_{\Delta x\to 0+0}\frac{1}{\sqrt[3]{\Delta x^2}}=+\infty$$

bo'ladi. Demak, $f'(0)=+\infty$.

11.7-misol. Ushbu $f(x)=\sqrt[3]{(x-1)^2}$ funksiyaning $x=1$ nuqtadagi hosilasini toping.

Yechilishi. Berilgan funksiyaning $x=1$ nuqtadagi orttirmasini topamiz:

$$\Delta f(1)=\sqrt[3]{(1+\Delta x-1)^2}-\sqrt[3]{(1-1)^2}=(\Delta x)^{\frac{2}{3}}.$$

Endi $\frac{\Delta f}{\Delta x}$ nisbatni topamiz: $\frac{\Delta f}{\Delta x}=\frac{(\Delta x)^{\frac{2}{3}}}{\Delta x}=\frac{1}{(\Delta x)^{\frac{1}{3}}}$. 11.2-ta'rifga ko'ra:

$$\lim_{\Delta x\to +0}\frac{\Delta f}{\Delta x}=+\infty,\qquad \lim_{\Delta x\to -0}\frac{\Delta f}{\Delta x}=-\infty$$

bo'ladi. Demak, $f'_{+}(0)=+\infty$, $f'_{-}(0)=-\infty$.

11.8-misol. Ushbu $y=|\ln x|$ funksiyaning $x_0=1$ nuqtada hosilaga ega emasligini isbotlang.

Yechilishi. Berilgan funksiyaning $x_0=1$ nuqtadagi orttirmasi Δy ni topamiz:

$$\Delta y=\Delta f=|\ln(1+\Delta x)|-|\ln 1|=|\ln(1+\Delta x)|,$$

$$\Delta y=|\ln(1+\Delta x)|=\begin{cases}\ln(1+\Delta x), & \Delta x\geq 0\\ -\ln(1+\Delta x), & \Delta x<0.\end{cases}$$

Bundan:

$$\frac{\Delta y}{\Delta x}=\begin{cases}\dfrac{\ln(1+\Delta x)}{\Delta x}, & \Delta x>0,\\ -\dfrac{\ln(1+\Delta x)}{\Delta x}, & \Delta x<0,\end{cases}$$

$$\lim_{\Delta x\to 0+0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0+0}\ln(1+\Delta x)^{\frac{1}{\Delta x}}=1,\quad \lim_{\Delta x\to 0-0}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\to 0-0}\ln(1+\Delta x)^{\frac{1}{\Delta x}}=-1.$$

Demak, berilgan funksiyaning $x_0=1$ nuqtadagi o'ng va chap hosilalari mavjud va ular bir-biriga teng emas. Shuning uchun, $y=|\ln x|$ funksiya $x_0=1$ nuqtada hosilaga ega emas.

11.9-misol. Ushbu

1) $f(x)=(x-a)(x-b)^2(x-c)^3$; 2) $f(x)=x+(x-2)\arccos\sqrt{\dfrac{x}{x+2}}$

funksiyalarning, mos ravishda, 1) $f'(a), f'(b), f'(c)$; 2) $f'(2)$ hosilalarini toping.

Yechilishi. Hosilaning ta'rifiga ko'ra:

$$1)\ f'(a)=\lim_{\Delta x\to 0}\frac{(a+\Delta x-a)(a+\Delta x-b)^2(a+\Delta x-c)^3-0}{\Delta x}=(a-b)^2(a-c)^3;$$

$$f'(b)=\lim_{\Delta x\to 0}\frac{(b+\Delta x-a)(b+\Delta x-b)^2(b+\Delta x-c)^3-0}{\Delta x}=0;$$

$$f'(c)=\lim_{\Delta x\to 0}\frac{(c+\Delta x-a)(c+\Delta x-b)^2(c+\Delta x-c)^3-0}{\Delta x}=0;$$

$$2)\ f'(2)=\lim_{\Delta x\to 0}\frac{2+\Delta x+\Delta x\arccos\sqrt{\dfrac{2+\Delta x}{2+\Delta x+2}}-2}{\Delta x}=1+\arccos\sqrt{\frac{1}{2}}=1+\frac{\pi}{4}.$$

11.2. Hosilaning geometrik ma'nosi. $f(x)$ funksiya (a,b) oraliqda aniqlangan va uzluksiz bo'lsin. Uning grafigi uzluksiz biror Γ chiziqdan iborat bo'lib, u 11.2- chizmada tasvirlanganidek bo'lsin deb faraz qilaylik. Γ chiziqda $M_0(x_0, f(x_0))$ nuqtani olib, bu nuqtadan urinma o'tkazish masalasini qaraymiz. Γ chiziqda M_0 nuqtadan farqli $M(x_0+\Delta x, f(x_0+\Delta x))$ $(\Delta x \neq 0)$ nuqtani olib, bu nuqtalar orqali l kesuvchi o'tkazamiz. Bu kesuvchi x ning o'sish tomoniga qarab yo'nalgan bo'lsin deb faraz qilamiz. l kesuvchining Ox o'qning musbat yo'nalishi bilan tashkil qilgan burchagini φ deb belgilaymiz. φ burchak Δx ga bog'liq bo'ladi: $\varphi = \varphi(\Delta x)$.

11.3-ta'rif. Agar l kesuvchining $M(x_0+\Delta x, f(x_0+\Delta x))$ $(\Delta x \neq 0)$ nuqta Γ chiziq bo'ylab $M_0(x_0, f(x_0))$ nuqtaga intilganda ($\Delta x \to 0$ dagi) limit holati mavjud bo'lsa, u holda kesuvchining bu limit holatiga Γ chiziqqa $M_0(x_0, f(x_0))$ *nuqtada o'tkazilgan urinma* deyiladi.

Ravshanki, 11.2- chizmadan:

$$tg\varphi(\Delta x) = \frac{NM}{M_0N} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}.$$

Ma'lumki, $M_0(x_0, f(x_0))$ nuqtadan o'tuvchi to'g'ri chiziq $M_0(x_0, f(x_0))$ nuqtaning koordinatalari hamda bu to'g'ri chiziqning burchak koeffitsiyenti orqali to'liq aniqlanadi.

Agar $\Delta x \to 0$ sa, u holda $\Delta y \to 0$ va M nuqta Γ chiziq bo'ylab $M_0(x_0, f(x_0))$ nuqtaga intiladi, ya'ni l kesuvchining holati T urinma holatga o'tadi, ya'ni $\lim\limits_{\Delta x \to 0} tg\varphi(\Delta x) = tg\alpha$, bunda α burchak T urinmaning Ox o'q bilan tashkil qilgan burchagi. Ikkinchi tomondan,

$$\lim_{\Delta x \to 0} tg\varphi(\Delta x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = f'(x_0).$$

Shunday qilib, $f'(x_0) = tg\alpha$.

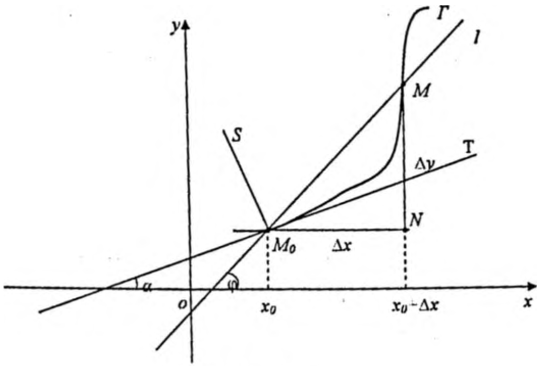
Demak, agar $f(x)$ funksiya $x_0 \in (a;b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lsa, u holda $f(x)$ funksiyaning grafigiga $M_0(x_0, f(x_0))$ nuqtada o'tkazilgan urinma mavjud bo'ladi (11.2-*chizma*). Ma'lumki, funksiyaning x_0 nuqtadagi $f'(x_0)$ hosilasi, shu urinmaning burchak koeffitsiyentini ifodalaydi. M_0T urinma chiziq tenglamasi

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{11.4}$$

bo'lib, bunda $f'(x_0) = \operatorname{tg}\alpha$, egri chiziqning $M_0(x_0, f(x_0))$ nuqtasiga o'tkazilgan M_0S normalning tenglamasi esa

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0) \tag{11.5}$$

ko'rinishda bo'ladi.

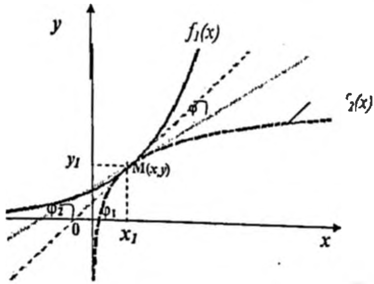


11.2-chizma.

$y_1 = f_1(x)$ va $y_2 = f_2(x)$ funksiyalar grafiklarining $M(x_1, y_1)$ kesishish nuqtasida o'tkazilgan urinmalar orasidagi φ burchak berilgan ikki egri chiziq orasidagi burchak bo'ladi va

$$\operatorname{tg}\varphi = \operatorname{tg}(\varphi_2 - \varphi_1) = \frac{f_2'(x_1) - f_1'(x_1)}{1 + f_1'(x_1) f_2'(x_1)} \tag{11.6}$$

formuladan topiladi (11.3-*chizma*).

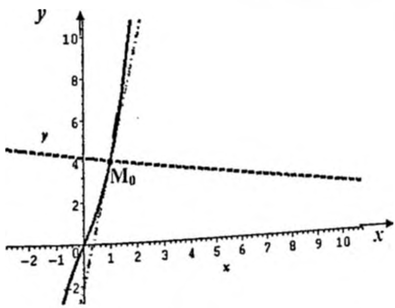


11.3-chizma.

11.10-misol. Ushbu $y = x^3 + 3x$ funksiya grafigiga $M_0(1;4)$ nuqtada o'tkazilgan urinma va normal tenglamalarini toping.

Yechilishi. Urinma to'g'ri chiziqning burchak koeffitsiyentini topish uchun avvalo berilgan funksiyaning hosilasini topamiz: $y' = 3x^2 + 3$. Bu hosilaning $x = 1$ nuqtadagi qiymati urinma to'g'ri chiziqning burchak koeffitsiyentini ifodalaydi, ya'ni $f'(1) = 3 \cdot 1 + 3 = 6$. Shunday qilib, (11.4) va (11.5) formulalarga asosan urinma va normal chiziq tenglamalari, mos ravishda, quyidagi ko'rinishlarda bo'ladi (11.4-*chizma*):

$y - 4 = 6(x - 1)$ yoki $y = 6x - 2$; $y - 4 = -\frac{1}{6}(x - 1)$ yoki $y = -\frac{1}{6}x + 4\frac{1}{6}$.



11.4-chizma.

MAPLE tizimidan foydalanib misolni yechish:

```
> V:=diff(x^3+3*x-y(x),x);
```

$$V := 3x^2 + 3 - \left(\frac{d}{dx}y(x)\right)$$

```
> W:=solve(V=0,diff(y(x),x));
```

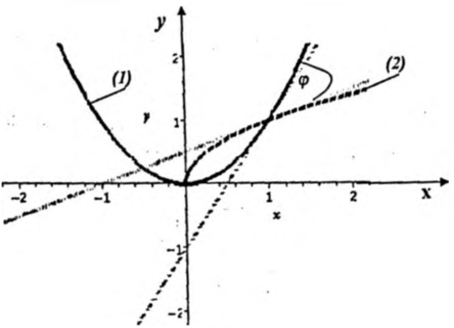
$$W := 3x^2 + 3$$

```
> subs(x=1,y=4,W);
```

11.11-misol. Ushbu $f(x) = x^2$ va $f(x) = \sqrt{x}$ funksiyalar grafiklarining $M(1,1)$ kesishish nuqtasida o'tkazilgan urinmalar orasidagi burchakni toping.

Yechilishi. Berilgan funksiyalarning $x = 1$ nuqtadagi hosilalarini topamiz:

$$f'(x) = 2x, \qquad f'(1) = 2, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(1) = \frac{1}{2}.$$



11.5-chizma.

(1) - $y=x^2$, (2) - $y=\sqrt{x}$

(11.6) formulaga ko'ra $tg\varphi = \dfrac{2-\frac{1}{2}}{1+2\cdot\frac{1}{2}} = \dfrac{3}{4}$ bo'ladi. Bu yerdan $\varphi = \operatorname{arctg}\dfrac{3}{4}$ (11.5-*chizma*).

MAPLE tizimidan foydalanib misolni yechish:

```
> solve(x^2=sqrt(x),x);
```

$1, 0$

```
> a:=diff(x^2,x);b:=diff(sqrt(x),x);
```

$a := 2x$

$b := \frac{1}{2\sqrt{x}}$

```
> arctan(subs(x=1,(a-b)/(1+a*b)));
```

$\text{arctg}\frac{3}{4}$.

11.12-misol. Ushbu $y = x^3 + 3x + 4$ chiziqqa o'tkazilgan urinma:

1) $y = 6x$ to'g'ri chiziqqa parallel bo'ladigan; 2) $y = -\frac{x}{12}$ to'g'ri chiziqqa perpendikulyar bo'ladigan nuqtalarni toping.

Yechilishi. Ma'lumki, berilgan $y = x^3 + 3x + 4$ chiziqqa o'tkazilgan urinmaning urinish nuqtasidagi burchak koeffitsiyenti $y' = 3x^2 + 3$ ga teng bo'ladi.

1) paralellik shartiga ko'ra: $3x^2 + 3 = 6$, bundan $x^2 = 1,\ x_1 = 1,\ x_2 = -1$. Izlanayotgan nuqtalar: $M_1(1; 8), M_2(-1; 0)$.

2) perpendikulyarlik shartiga ko'ra: $3x^2 + 3 = 12$, bundan $x_1 = -\sqrt{3}$, $x_2 = \sqrt{3}$. Izlanayotgan nuqtalar: $M_1\ (-\sqrt{3}\ ; -6\sqrt{3} + 4)$, $M_2\ (\sqrt{3}\ ; 6\sqrt{3} + 4)$

11.13-misol. Tenglamasi $x = 2\cos t - \cos 2t,\ y = 2\sin t - \sin 2t$ ko'rinishda bo'lgan $y = y(x)$ chiziqqa: 1) $t = \frac{\pi}{2}$; 2) $t = \pi$; 3) $t = \frac{3\pi}{2}$ nuqtalarda o'tkazilgan urinma chiziqlarning tenglamasini tuzing.

Yechilishi. $t = \frac{\pi}{2}$ bo'lganda, $x = 1, y = 2$; $t = \pi$ bo'lganda, $x = -3, y = 0$; $t = \frac{3\pi}{2}$ bo'lganda, $x = 1, y = -2$ bo'ladi. Endi berilgan $y = y(x)$ funksiyaning $y_x'(x)$ hosilasini topamiz:

$$y'_x=\frac{y'_t}{x'_t}=\frac{\sin\frac{3t}{2}\cdot\sin\frac{t}{2}}{\cos\frac{3t}{2}\cdot\sin\frac{t}{2}}=\operatorname{tg}\frac{3t}{2},$$

bundan, $y'_x\left(\frac{\pi}{2}\right)=-1;\ y'_x\left(\frac{3\pi}{2}\right)=1;\ t=\pi$ bo'lganda esa, $y'_x(x)$ funksiya aniqlanmagan.

Demak, 1) va 3) hollarda urinma to'g'ri chiziqlar tenglamalari, mos ravishda, $y-2=-(x-1),\ \ y+2=x-1$ ko'rinishda bo'ladi.

11.3. Hosilaning fizik ma'nosi. Moddiy nuqtaning to'g'ri chiziqli harakati $s=f(t)$ tenglama bilan ifodalangan bo'lsin, bunda t vaqt, s shu vaqt ichida o'tilgan yo'l (masofa). $s=f(t)$ funksiyaning t_0 nuqtadagi hosilasi $s=f(t)$ qonun bilan harakat qilayotgan moddiy nuqtaning t_0 momentdagi oniy tezligini bildiradi, ya'ni

$$v=f'(t_0)=\lim_{\Delta t\to 0}\frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}.$$

Moddiy nuqtaning berilgan $t=t_0$ momentdagi a tezlanishi esa v tezlikdan t vaqt bo'yicha olingan hosilaning $t=t_0$ dagi qiymatiga tengdir, ya'ni

$$a=v'(t_0)=\lim_{\Delta t\to 0}\frac{f'(t_0+\Delta t)-f'(t_0)}{\Delta t}$$

11.14-misol. Moddiy nuqta $s(t)=\frac{2}{3}t^3+\frac{1}{2}t^2-2t$ qonun bo'yicha to'g'ri chiziq bo'ylab harakat qiladi. Uning $t=4$ momentdagi tezligini toping.

Yechilishi. Moddiy nuqtaning istalgan t vaqtdagi harakat tezligini topamiz:

$$v(t)=s'(t)=2t^2+t-2.$$

Moddiy nuqtaning $t=4$ momentdagi harakat tezligini hisoblaymiz:

$$v(t)\big|_{t=4}=2\cdot 4^2+4-2=34(^m/_s).$$

11.4. Hosilani hisoblashning sodda qoidalari.

1. O'zgarmas sonning hosilasi nolga teng: $(C)'=0$ (bunda C- o'zgarmas son).

2. O'zgarmas sonni hosila ishorasidan tashqariga chiqarish mumkin: $(Cf(x))'=C\cdot(f(x))'$ (bunda C-o'zgarmas son).

3. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilalarga ega bo'lsa, $f(x)\pm g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x)\pm g(x)]'=f'(x)\pm g'(x) \tag{11.7}$$

formula bo'yicha topiladi.

4. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lsa, $f(x)\cdot g(x)$ funksiya ham x nuqtada hosilaga ega va u

$$[f(x)\cdot g(x)]'=f'(x)\cdot g(x)+f(x)\cdot g'(x) \tag{11.8}$$

formula bo'yicha topiladi.

5. Agar $f(x)$ va $g(x)$ funksiyalarning har biri $x\in(a,b)$ nuqtada $f'(x)$ va $g'(x)$ hosilaga ega bo'lib, $g(x)\neq 0$ bo'lsa, $\frac{f(x)}{g(x)}$ funksiya ham x nuqtada hosilaga ega va u

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)} \tag{11.9}$$

formula bo'yicha topiladi.

6. Agar $f(x)$ funksiya $x_0\in(a,b)$ nuqtada $f'(x_0)\neq 0$ hosilaga ega bo'lsa, bu funksiyaga teskari $x=f^{-1}(y)$ funksiya x_0 nuqtaga mos kelgan y_0 $(y_0=f(x_0))$ nuqtada hosilaga ega va

$$[f^{-1}(y)]'_{y=y_0}=\frac{1}{f'(x_0)} \tag{11.10}$$

tenglik o'rinli.

7. Agar $u=f(x)$ funksiya $x_0\in(a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lib, $y=F(u)$ funksiya esa x_0 nuqtaga mos kelgan u_0 $(u_0=f(x_0))$ nuqtada $F'(u_0)$ hosilaga ega bo'lsa, $\Phi(x)=F(f(x))$ murakkab funksiya ham x_0 nuqtada hosilaga ega va

$$\Phi'(x_0)=[F(f(x))]'\big|_{x=x_0}=F'(u_0)\cdot f'(x_0) \tag{11.11}$$

formula o'rinli.

8. Oshkormas, $F(x,y)=0$ funksiya uchun $y'(x)=-\frac{F'_x}{F'_y}$ formula o'rinli.

9. Parametrik tenglamasi bilan berilgan $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}, (t_0\le t\le t_1)$

funksiyaning hosilasi quyidagi

$$y'_x=\frac{\psi'(t)}{\varphi'(t)} \quad \text{yoki} \quad y'_x=\frac{y'_t(t)}{x'_t(t)} \tag{11.12}$$

formula bo'yicha topiladi.

11.5. Asosiy elementar funksiyalarning hosilalari jadvali

1. $y=C.$ $\quad y'=0.$
2. $y=x^\alpha.$ $\quad y'=\alpha\cdot x^{\alpha-1},\ \alpha\in R.$
3. $y=\sqrt{x}.$ $\quad y'=\frac{1}{2\sqrt{x}}.$
4. $y=e^x.$ $\quad y'=e^x.$
5. $y=a^x\ a>0,\ a\ne 1..$ $\quad y'=a^x\ln a.$
6. $y=\ln x.$ $\quad y'=\frac{1}{x}.$
7. $y=\log_a x,\ a>0,\ a\ne 1.$ $\quad y'=\frac{\log_a e}{x}=\frac{1}{x\ln a}.$
8. $y=\lg x.$ $\quad y'=\frac{1}{x}\lg e=\frac{1}{x\ln 10}.$
9. $y=\sin x.$ $\quad y'=\cos x.$

10. $y=\cos x$. $\quad y'=-\sin x$.

11. $y=\operatorname{tg} x$. $\quad y'=\frac{1}{\cos^2 x}=\sec^2 x$.

12. $y=\operatorname{ctg} x$. $\quad y'=-\frac{1}{\sin^2 x}=-\operatorname{cosec}^2 x$.

13. $y=\arcsin x$. $\quad y'=\frac{1}{\sqrt{1-x^2}}$.

14. $y=\arccos x$. $\quad y'=-\frac{1}{\sqrt{1-x^2}}$.

15. $y=\operatorname{arctg} x$. $\quad y'=\frac{1}{1+x^2}$.

16. $y=\operatorname{arcctg} x$. $\quad y'=-\frac{1}{1+x^2}$.

17. $y=\sec x$. $\quad y'=\frac{\sin x}{\cos^2 x}=\sin x\cdot sex^2 x$.

18. $y=\operatorname{cosec} x$. $\quad y'=-\frac{\cos x}{\sin^2 x}=-\cos x\cdot \operatorname{cosec}^2 x$.

19. $y=\operatorname{sh} x$. $\quad y'=\operatorname{ch} x$.

20. $y=\operatorname{ch} x$. $\quad y'=\operatorname{sh} x$.

21. $y=\operatorname{th} x$. $\quad y'=\frac{1}{\operatorname{ch}^2 x}$.

22. $y=\operatorname{cth} x$. $\quad y'=\frac{-1}{\operatorname{sh}^2 x}$.

23. $y=\operatorname{Arsh} x$. $\quad y'=\frac{1}{\sqrt{1+x^2}}$.

24. $y=\operatorname{Arch} x$. $\quad y'=\frac{1}{\sqrt{x^2-1}}$.

25. $y=\operatorname{Arth} x$. $\quad y'=\frac{1}{1-x^2}$.

26. $y=\operatorname{Arcth} x$. $\quad y'=\frac{-1}{1-x^2}$.

27. $y=x^x$. $\quad y'=x^x(1+\ln x)$.

11.15-misol. Ushbu

1) $y=2x^5-5x^3+7x+6$, 2) $y=\sqrt[3]{x}+\frac{1}{\sqrt{x}}-0{,}2x^5$;

3) $y=\frac{3x^3-2x^2+3}{x^2-x+1}$; 4) $y=\frac{x^2+\sqrt{x}}{x-2\sqrt[3]{x}}$;

5) $y=\frac{\sin\varphi+\cos\varphi}{1-\cos\varphi}$; 6) $y=3e^x-\ln x$; 7) $y=e^x(\operatorname{tg}x+\operatorname{ctg}x)$;

8) $y=\operatorname{arctg}x+x^2+\operatorname{arcctg}x$; 9) $y=e^x\log_2 x$; 10) $y=\operatorname{sh}x\operatorname{ch}x$

funksiyalarning hosilalarini toping.

Yechilishi. Hosilani hisoblashning sodda qoidalari va elementar funksiyalarning hosilalari jadvaliga asosan topamiz:

1) yig'indi (ayirma)ning hosilasini ifodalaydigan (11.7) formula va darajali funksiya hosilasi formulasidan foydalanamiz:

$$y'=(2x^5-5x^3+7x+6)'=(2x^5)'-(5x^3)'+(7x)'+(6)'=10x^4-15x^2+7;$$

MAPLE tizimidan foydalanib misolni yechish:

```
> Diff((2*x^5)-(5*x^3)+7*x+6,x)=

diff((2*x^5)-(5*x^3)+7*x+6,x);
```

$$\frac{d}{dx}(2x^5-5x^3+7x+6)=10x^4-15x^2+7$$

2) Avvalo (11.7) formuladan va so'ngra darajali funksiya hosilasini topish formulasidan foydalanamiz:

$$\left(\sqrt[3]{x}+\frac{1}{\sqrt{x}}-0{,}2x^5\right)'=\left(x^{1/3}+x^{-1/2}-0{,}2x^5\right)'=\frac{1}{3}x^{-\frac{2}{3}}-\frac{1}{2}x^{-\frac{3}{2}}-0{,}2\cdot5x^4=$$

$$=\frac{1}{3\sqrt[3]{x^2}}-\frac{1}{2x\sqrt{x}}-x^4;$$

MAPLE tizimidan foydalanib misolni yechish:

```
> Diff(x^(1/3)+1/sqrt(x)-0.2*x^5,x)=

diff(x^(1/3)+1/sqrt(x)-0.2*x^5,x);
```

$$\frac{d}{dx}\left(x^{(1/3)}+\frac{1}{\sqrt{x}}-0.2x^5\right)=\frac{1}{3x^{(2/3)}}-\frac{1}{2x^{(3/2)}}-1.0x^4$$

3) ikki funksiyaning nisbati (kasrning) hosilasini ifodalaydigan (11.9) formula va yuqorida foydalangan formulalarni qo'llaymiz:

$$y' = \frac{(3x^3-2x^2+3)'(x^2-x+1)-(3x^3-2x^2+3)(x^2-x+1)'}{(x^2-x+1)^2} =$$

$$= \frac{(9x^2-4x)(x^2-x+1)-(2x-1)\cdot(3x^3-2x^2+3)}{(x^2-x+1)^2} = \frac{3x^4-6x^3+11x^2-10x+3}{(x^2-x+1)^2};$$

MAPLE tizimidan foydalanib misolni yechish:

Diff(((3*x^3)-(2*x^2)+3)/((x^2)-

x+1),x)=diff(((3*x^3)-(2*x^2)+3)/((x^2)-x+1),x);

$$\frac{d}{dx}\frac{3x^3-2x^2+3}{x^2-x+1} = \frac{9x^2-4x}{x^2-x+1} - \frac{(3x^3-2x^2+3)(2x-1)}{(x^2-x+1)^2}$$

4) ikki funksiya nisbati (kasrning) hosilasini ifodalaydigan (11.9) formula, shuningdek, yig'indi (ayirma) ning hosilasini topish (11.7) formula hamda jadvaldagi ildizning hosilasi formulasidan foydalanamiz:

$$y' = \frac{\left(x^2+\sqrt{x}\right)'\cdot\left(x-2\sqrt[3]{x}\right)-\left(x-2\sqrt[3]{x}\right)'\cdot\left(x^2+\sqrt{x}\right)}{\left(x-2\sqrt[3]{x}\right)^2} =$$

$$= \frac{\left(2x+\frac{1}{2\sqrt{x}}\right)\left(x-2\sqrt[3]{x}\right)-\left(1-\frac{2}{3}\cdot x^{-\frac{2}{3}}\right)\left(x^2+\sqrt{x}\right)}{\left(x-2\sqrt[3]{x}\right)^2} = \frac{x^2-\frac{10}{3}x\sqrt[3]{x}-\frac{1}{2}\sqrt{x}-\frac{1}{3\sqrt[6]{x}}}{\left(x-2\sqrt[3]{x}\right)^2};$$

MAPLE tizimidan foydalanib misolni yechish:

> diff(((x^2)+sqrt(x))/(x-2*x^(1/3)),x);

$$\frac{2x+\frac{1}{2\sqrt{x}}}{x-2x^{(1/3)}} - \frac{\left(x^2+\sqrt{x}\right)\left(1-\frac{2}{3x^{(2/3)}}\right)}{\left(x-2x^{(1/3)}\right)^2}$$

5) Ikki funksiya nisbati (kasrning) hosilasini ifodalaydigan (11.9) formula, yig'indi (ayirma) ning hosilasi uchun keltirilgan (11.7) formula hamda jadvaldagi sinus va kosinus funksiyalarning hosilalari formulalaridan foydalanamiz:

$$y' = \frac{(\sin\varphi+\cos\varphi)'(1-\cos\varphi)-(1-\cos\varphi)'(\sin\varphi+\cos\varphi)}{(1-\cos\varphi)^2} =$$

$$= \frac{(\cos\varphi-\sin\varphi)(1-\cos\varphi)-(\sin\varphi+\cos\varphi)\sin\varphi}{(1-\cos\varphi)^2} = \frac{\cos\varphi-\sin\varphi-1}{(1-\cos\varphi)^2};$$

MAPLE tizimidan foydalanib misolni yechish:

> diff((sin(φ)+cos(φ))/(1-cos(φ)),φx);

$$\frac{\cos(\varphi)-\sin(\varphi)}{1-\cos(\varphi)} - \frac{(\sin(\varphi)+\cos(\varphi))\sin(\varphi)}{(1-\cos(\varphi))^2}$$

6) Avvalo (11.7) formuladan, so'ngra jadvaldagi ko'rsatkichli va logarifmik funksiyalar hosilalari formulalaridan foydalanamiz:

$$y' = (3e^x - \ln x)' = 3e^x - \frac{1}{x};$$

MAPLE tizimidan foydalanib misolni yechish:

> diff((3*ex'(x)-ln(x)),x);

$$3e^x - \frac{1}{x}$$

7) Ko'paytmaning hosilasi to'g'risidagi (11.8) formula va jadvaldagi ko'rsatkichli va trigonometrik funksiyalar hosilalari formulalaridan foydalanamiz:

$$y' = \left((e^x(\operatorname{tg}x+\operatorname{ctg}x)\right)' = e^x(\operatorname{tg}x+\operatorname{ctg}x) + e^x\left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}\right) = \frac{2e^x(\sin 2x - 2\cos 2x)}{\sin^2 2x}.$$

MAPLE tizimidan foydalanib misolni yechish:

> diff(ex'(x)*(tan(x)+cot(x)),x);

$$e^x(\tan(x)+\cot(x)) + e^x(\tan(x)^2 - \cot(x)^2)$$

8) Yig'indining hosilasini ifodalaydigan (11.7) formula, so'ngra jadvaldagi teskari trigonometrik funksiyalar va darajali funksiya hosilalari formulalaridan foydalanamiz:

$$y' = (\operatorname{arctg}x + x^2 + \operatorname{arcctg}x)' = \frac{1}{1+x^2} + 2x - \frac{1}{1+x^2} = 2x;$$

MAPLE tizimidan foydalanib misolni yechish:

> Diff(arctan(x)+x^2+arccot(x),x)=

diff(arctan(x)+x^2+arccot(x),x);

$$\frac{d}{dx}\left(\arctan(x)+x^2+\operatorname{arccot}(x)\right)=2\,x$$

9) Ko'paytmaning hosilasi to'g'risidagi (11.8) formula va jadvaldagi ko'rsatkichli va logarifmik funksiya hosilalari formulalaridan foydalanamiz

$$y'=(e^x\cdot\log_2 x)'=e^x\cdot\log_2 x+e^x\cdot\frac{1}{x\ln 2}=e^x\left(\log_2 x+\frac{1}{x\ln 2}\right);$$

MAPLE tizimidan foydalanib misolni yechish:

```
> diff(ex'(x)*log[2](x),x);
```

$$\frac{e^x\ln(x)}{\ln(2)}+\frac{e^x}{x\ln(2)}$$

10) Ko'paytmaning hosilasi formulasi (11.8) va jadvaldagi giperbolik funksiyalar hosilalari formulalaridan foydalanamiz:

$$y'=(\operatorname{sh}x\cdot\operatorname{ch}x)'=\operatorname{ch}^2x+\operatorname{sh}^2x.$$

MAPLE tizimidan foydalanib misolni yechish:

```
> Diff(sinh(x)*cosh(x),x)=diff(sinh(x)*cosh(x),x);
```

$$\frac{d}{dx}\left(\sinh(x)\cosh(x)\right)=\cosh(x)^2+\sinh(x)^2$$

11.16-misol. Murakkab funksiyaning hosilasini topish qoidasiga asosan, ushbu

1) $y=3^{\sin 2x}$; 2) $y=\sqrt{1+x^3}$; 3) $y=\ln\cos(x^2+1)$;

4) $y=\arccos\sqrt{1-x^2}$; 5) $y=\ln\sqrt{\dfrac{2^{3x}}{1+\sin 5x}}$; 6) $y=(3+\sin x)^x$

funksiyalarning hosilalarini toping.

Yechilishi. 1) Murakkab funksiyaning hosilasini topish qoidasini, ya'ni (11.11) formulani ikki marta qo'llab, topamiz.

$$y'=(3^{\sin 2x})'=3^{\sin 2x}\ln 3\cdot(\sin 2x)'=3^{\sin 2x}\ln 3\cos 2x\cdot 2=2\cdot\ln 3\; 3^{\sin 2x}\cos 2x;$$

MAPLE tizimidan foydalanib misolni yechish:

```
> diff(3^sin(2*x),x);
```

$$2\,3^{\sin(2\,x)}\cos(2\,x)\ln(3)$$

2) berilgan funksiya $z=1+x^3$ va $y=\sqrt{z}$ funksiyalar kompozitsiyasidan iborat bo'lib, $z_x=3x^2$, $y_z'=\frac{1}{2\sqrt{z}}$. Unda murakkab funksiyaning hosilasini topish formulasi (11.11) ga asosan, $y_x'=\frac{1}{2\sqrt{z}}\cdot 3x^2=\frac{3}{2}\cdot\frac{x^2}{\sqrt{1+x^3}}$.

MAPLE tizimidan foydalanib misolni yechish:

```
> diff(sqrt(1+x^3),x);
```

$$\frac{3x^2}{2\sqrt{1+x^3}}$$

3) murakkab funksiyaning hosilasini topish qoidasini ikki marta qo'llab, $y'=\frac{1}{\cos(x^2+1)}(\cos(x^2+1))'=-\frac{\sin(x^2+1)}{\cos(x^2+1}\cdot(x^2+1)'=-tg(x^2+1)\cdot 2x$ bo'lishini topamiz.

MAPLE tizimidan foydalanib misolni yechish:

```
> diff(ln(cos(x^2+1)),x);
```

$$-\frac{2\sin(x^2+1)\,x}{\cos(x^2+1)}$$

4) Murakkab funksiyaning hosilasini topish qoidasini uch marta qo'llab, $y'=-\frac{1}{\sqrt{1-(1-x^2)}}\cdot(\sqrt{1-x^2})'=-\frac{1}{|x|}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(1-x^2)'=\frac{x}{|x|\sqrt{1-x^2}}$

bo'lishini topamiz.

MAPLE tizimidan foydalanib misolni yechish:

```
> diff(arccos(sqrt(1-x^2)),x);
```

$$\frac{x}{\sqrt{1-x^2}\sqrt{x^2}}$$

5) bu holda berilgan funksiyani soddalashtirish maqsadga muvofiq bo'ladi:

$$y=\frac{1}{2}\left(\ln 2^{3x}-\ln(1+\sin 5x)\right)=\frac{3x}{2}\ln 2-\frac{1}{2}\ln(1+\sin 5x)$$

Endi berilgan funksiyadan hosila olish osonlashadi:

$$y'=\frac{3}{2}\cdot\ln 2-\frac{1}{2}\cdot(1+\sin 5x)'=\frac{3}{2}\ln 2-\frac{1}{2(1+\sin 5x)}\cdot\cos 5x\cdot(5x)'=\frac{3}{2}\ln 2-\frac{5\cos 5x}{2\,(1+\sin 5x)}$$

MAPLE tizimidan foydalanib misolni yechish:

> Diff(1.5*x*ln(2)-0.5*ln(1+sin(5*x)),x)=diff(1.5*x*ln(2)-0.5*ln(1+sin(5*x)),x);

$$\frac{d}{dx}(1.5\,x\ln(2)-0.5\ln(1+\sin(5\,x)))=1.5\ln(2)-\frac{2.5\cos(5\,x)}{1+\sin(5\,x)}$$

6) daraja-ko'rsatkichli funksiyaning ta'rifiga asosan berilgan funksiyani quyidagi ko'rinishda yozib olamiz: $y=e^{x\ln(3+\sin x)}$. Endi berilgan funksiyadan hosila olamiz:

$$y'=e^{x\ln(3+\sin x)}(x\ln(3+\sin x))'=e^{x\ln(3+\sin x)}\left(\ln(3+\sin x)+\frac{x}{3+\sin x}(3+\sin x)'\right)=$$

$$=e^{x\ln(3+\sin x)}\left(\ln(3+\sin x)+\frac{x\cos x}{3+\sin x}\right)=(3+\sin x)^x\left(\ln(3+\sin x)+\frac{x\cos x}{3+\sin x}\right)$$

MAPLE tizimidan foydalanib misolni yechish:

> diff((3+sin(x))^x,x);

$$(3+\sin(x))^x\left(\ln(3+\sin(x))+\frac{x\cos(x)}{3+\sin(x)}\right).$$

11.17-misol. Ushbu

1) $y=x+\frac{1}{5}x^5,\quad y=0,\quad y=\frac{6}{5};$

2) $y=0{,}1x+e^{0{,}1x},\quad y=1;\qquad$ 3) $y=2x^2-x^4,\quad x>1,\quad y=0$

funksiyalarga teskari bo'lgan funksiyalarning ko'rsatilgan nuqtadagi hosilasini toping.

Yechilishi. 1) berilgan funksiya $\forall x\in R$, qat'iy o'suvchi, $y'_x=1+x^4$ funksiya R ning hech bir nuqtasida nolga teng emas. Shuning uchun, bu funksiyaga teskari funksiya mavjud va uning hosilasi $x'_y=\frac{1}{y'_x}=\frac{1}{1+x^4}$. $y=0$ bo'lganda, $0=x\left(1+\frac{1}{5}x^4\right),\ x=0$ bo'ladi va $x_y(0)=1$. $y=\frac{6}{5}$ bo'lganda esa, $6=5x+x^5$, $x^5+5x-6=0$, $x=1$ va $x'_y\left(\frac{6}{5}\right)=\frac{1}{2}$ bo'ladi.

2) $y=0.1x+e^{0.1x}$ funksiya R da uzluksiz, qat'iy monoton va $y'_x=0,1+0,1e^{0,1x}$, R dagi biror nuqtada ham nolga aylanmaydi. Demak, berilgan funksiyaga teskari funksiya mavjud va uning hosilasi

$$x'_y=\frac{1}{y_x}=\frac{1}{0,1+0,1e^{0,1x}}=\frac{10}{1+e^{0,1x}}$$

$y=1$ bo'lganda, $1=0,1x+e^{0,1x}$, $x=0$ va $x'_y(1)=5$ bo'ladi.

3) $y=2x^2-x^4$ funksiya R da uzluksiz, $x>1$ bo'lganda $y'=4x(1-x^2)<0$ va nolga aylanmaydi. Berilgan funksiya $x>1$ da qat'iy kamayuvchi.

Shunday qilib, $x>1$ da $y=2x^2-x^4$ funksiyaga teskari funksiya mavjud va uning hosilasi $x'_y=\frac{1}{y'_x}=\frac{1}{4x(1-x^2)}$ ga teng. $y=0$ bo'lganda $x=\sqrt{2}$ bo'ladi.

Demak, $y'_x(0)=\frac{1}{4\cdot\sqrt{2}(1-2)}=-\frac{\sqrt{2}}{8}$.

11.18-misol. Ushbu $3y^3+2y=x$ tenglamadan topiladigan bir qiymatli $y=y(x)$ funksiyaning mavjudligini ko'rsating va uning hosilasi y'_x ni toping.

Yechilishi. Faraz qilaylik, berilgan tenglama ikkita haqiqiy $y_1(x)$ va $y_2(x)$ yechimlarga ega bo'lsin, ya'ni $3y_1^3+2y_1=x$, $3y_2^3+2y_2=x$. Bundan, $3(y_1^3-y_2^3)+2(y_1-y_2)=0$ yoki $(3y_1^2+3y_1y_2+3y_2^2+2)(y_1-y_2)=0$, bunda, $\forall x$ uchun $3y_1^2+3y_1y_2+3y_2^2+2>0$ bo'lganligidan, $y_1-y_2=0$, $y_1=y_2$.

Shunday qilib, berilgan tenglama yagona haqiqiy yechimga ega ekan. Haqiqiy yagona yechimning mavjudligini hosila yordamida ham topish mumkin. $x'_y=9y^2+2>0$ bo'lib, u hech bir nuqtada nolga aylanmaydi, ya'ni $x(y)$ funksiya qat'iy o'suvchi funksiya. Shuning uchun $x(y)$ funksiyaga teskari, yagona, monoton o'suvchi, differensiallanuvchi $y=y(x)$ funksiya mavjud bo'ladi va uning hosilasi (11.10) ga asosan, $y'_x=\frac{1}{x'_y}=\frac{1}{2+9y^2}$.

11.19-misol. Ushbu

1) $y^5+y^3+y-x=0$; 2) $(2a-x)y^2=x^3,\ y<0$;

3) $6xy+8y^2-12x-26y+11=0,\ y<2,\ x_0=\frac{11}{12}$

oshkormas shaklda berilgan $y=y(x)$ funksiyalarning hosilalari y'_x ni, 3)-da esa, $y'_x(x_0)$ ni toping.

Yechilishi. Agar biror oraliqda differensiallanuvchi $y=y(x)$ funksiya $F(x,y)=0$ tenglamani qanoatlantirsa, u holda uning hosilasi

$$\frac{d}{dx}F(x,y)=0 \qquad (*)$$

tenglamadan topiladi. 1) holda (*) tenglamaning ko'rinishi

$$\frac{d}{dx}(y^5+y^3+y-x)=0$$

shaklda bo'ladi. Bundan: $5y^4\cdot y_x'+3y^2\cdot y_x'+y_x'-1=0$,

$y_x'=\frac{1}{5y^4+3y^2+1}$.

MAPLE tizimidan foydalanib misolni yechish:

> z:=diff(y(x)^5+y(x)^3+y(x)-x,x);

$$z:=5y(x)^4\left(\frac{d}{dx}y(x)\right)+3y(x)^2\left(\frac{d}{dx}y(x)\right)+\left(\frac{d}{dx}y(x)\right)-1$$

> Q:=solve(z=0,diff(y(x),x));

$$Q:=\frac{1}{5\,y(x)^4+3\,y(x)^2+1}$$

2) holda (*) tenglamaning ko'rinishi $\frac{d}{dx}((2a-x)y^2-x^3)=0$ shaklda bo'ladi. Bu tenglamani differensiallash natijasida, ushbu

$$-y^2+(2a-x)\cdot 2y\cdot y_x'-3x^2=0$$

tenglamani hosil qilamiz. Bundan, $x\neq 2a$ shartda $y_x'=\frac{3x^2+y^2}{2y(2a-x)}$ bo'lishini topamiz. $y^2=\frac{x^3}{2a-x}$ munosabatdan, shartga ko'ra, $y<0$ bo'lgani uchun,

$y=-\sqrt{\frac{x^3}{2a-x}}$ ekanligini e'tiborga olgan holda, $0<x<2a$ shartda $y'_x=-\frac{(3a-x)y}{(2a-x)x}$ ekanligini topamiz.

MAPLE tizimidan foydalanib misolni yechish:

```
> z:=diff((2*a-x)*y(x)^2-x^3,x);
```

$$z:=-y(x)^2+2(2a-x)y(x)\left(\frac{d}{dx}y(x)\right)-3x^2$$

```
> Q:=solve(z=0,diff(y(x),x));
```

$$Q:=-\frac{-y(x)^2-3x^2}{4\,y(x)\,a-2\,y(x)\,x}$$

3) holda (*) tenglama

$$\frac{d}{dx}(6xy+8y^2-12x-26y+11)=0$$

shaklda bo'ladi. Bundan, $6y+6xy'_x+16y\cdot y'_x-12-26y'_x=0$, $y'_x=\frac{3(2-y)}{3x+8y-13}$

ekanligini topamiz. Shartga ko'ra, $y<2,\ x_0=\frac{11}{12}$ bo'lgani uchun, berilgan

tenglamadan $8y^2+\frac{11}{2}y-26y=0$ yoki $y\cdot\left(8y-\frac{41}{2}\right)=0$ bo'ladi, bundan

$y_1=0,\quad y_2=\frac{41}{16}>2$.

Demak, $y<2,\ x_0=\frac{11}{12}$ dagi hosila $y'_x\left(\frac{11}{12}\right)=-\frac{24}{41}$ ekan.

MAPLE tizimidan foydalanib misolni yechish:

```
> z:=diff(6*x*y(x)+8*y(x)^2-12*x-26*y(x)+11,x);
```

$$z:=6\left(\frac{d}{dx}y(x)\right)+6y(x)+16y(x)\left(\frac{d}{dx}y(x)\right)-12-26\left(\frac{d}{dx}y(x)\right)$$

```
> Q:=solve(z=0,diff(y(x),x));
```

$$Q:=-\frac{3\,y(x)-6}{3x+8\,y(x)-13}$$

11.20-misol. Parametrik shaklda berilgan ushbu

1) $x=\sin^2 t,\quad y=\cos^2 t,\quad 0<t<\frac{\pi}{2}$; 2) $x=e^{-t},\quad y=t^3,\quad -\infty<t<\infty$;

3) $x=a(t-\sin t),\quad y=a(1-\cos t),\quad -\infty<t<\infty$

$y=y(x)$ funksiyalarning y'_x hosilalarini toping.

Yechilishi. 1) $x(t), y(t)$ funksiyalar $\forall t$ larda differensiyalanuvchi hamda $0<t<\frac{\pi}{2}$ da $x'_t=2\sin t\cos t=\sin 2t\neq 0$. U holda, berilgan $y=y(x)$ funksiyaning hosilasi (11.12) formulaga asosan, $y'_x=\frac{y'_t}{x'_t}=\frac{-2\cos t\cdot\sin t}{2\sin t\cos t}=-1$, $0<x<1$.

MAPLE tizimidan foydalanib misolni yechish:

```
> s:=diff((cos(t))^2,t)/diff((sin(t))^2,t);
```

$$s:=-1$$

2) $x(t), y(t)$ funksiyalar $\forall t$ uchun differensiyalanuvchi va $-\infty<t<\infty$ da $x'_t=-e^{-t}\neq 0$. Shunday qilib, berilgan $y=y(x)$ funksiyaning hosilasi:

$$y'_x=-\frac{3t^2}{e^{-t}}=-3t^2e^t.$$

MAPLE tizimidan foydalanib misolni yechish:

```
s:=diff(t^3,t)/diff(ex'(-t),t);
```

$$s:=-\frac{3t^2}{e^{(-t)}}$$

3) $x(t), y(t)$ funksiyalar ham, ko'rsatilgan oraliqda differensiallanuvchi va $x'_t=a(1-\cos t)$, $y'_t=a\sin t$, $t\neq 2k\pi$, bo'lganda

$$y'_x=\frac{a\sin t}{a(1-\cos t)}=\operatorname{ctg}\frac{t}{2}.$$

MAPLE tizimidan foydalanib misolni yechish:

```
s:=diff(a*(1-cos(t)),t)/diff(a*(t-sin(t)),t);
```

$$s:=\frac{\sin(t)}{1-\cos(t)}$$

11.21-misol. Tenglamasi ushbu

1) $r=a\varphi$, $\frac{4\pi}{3}<\varphi<2\pi$, $x_0=0$; 2) $r=e^{\varphi}$, $-\frac{\pi}{6}<\varphi<\frac{\pi}{6}$, $x_0=1$

ko'rinishda berilgan (bunda r va φ lar $(x;y)$ nuqtaning qutb koordinatalari) $y=y(x)$ funksiyalarning $y'(x_0)$ hosilalarini toping.

Yechilishi. 1) $r=a\varphi$ funksiyaning ifodasini parametrik shaklga keltiramiz: $x=r\cos\varphi=a\varphi\cos\varphi,\ y=r\sin\varphi=a\varphi\sin\varphi$. Bundan,

$$x'_\varphi=a\cos\varphi-a\varphi\sin\varphi,\ y'_\varphi=a\sin\varphi+a\varphi\cos\varphi,\ y'_x=\frac{\sin\varphi+\varphi\cos\varphi}{\cos\varphi-\varphi\sin\varphi}.$$

Shartga ko'ra, $\frac{4\pi}{3}<\varphi<2\pi,\ x_0=0$ bo'lgani uchun $0=a\cdot\varphi\cos\varphi$,

bundan $\varphi=\frac{3\pi}{2}$. Demak, $y'_x(0)=\frac{-1}{\frac{3\pi}{2}}=-\frac{2}{3\pi}$.

2) Berilgan $r=e^\varphi$ funksiyaning tenglamasini parametrik shaklga keltiramiz: $x=r\cos\varphi=e^\varphi\cos\varphi,\ y=r\sin\varphi=e^\varphi\sin\varphi$. Bundan,

$$x'_\varphi=e^\varphi(\cos\varphi-\sin\varphi),\ y'_\varphi=e^\varphi(\sin\varphi+\cos\varphi),\quad y'_x(x)=\frac{\sin\varphi+\cos\varphi}{\cos\varphi-\sin\varphi}.$$

Shartga ko'ra, $-\frac{\pi}{6}<\varphi<\frac{\pi}{6},\ x_0=1$ bo'lgani uchun $1=e^\varphi\cos\varphi,\ \varphi=0$ bo'ladi.

Demak, $y'_x(1)=1$.

11.6. Funksiyaning differensiali. $y=f(x)$ funksiya $(a;b)$ oraliqda berilgan bo'lsin. $x_0\in(a,b)$ nuqtani olib, unga $\Delta x(\Delta x>0$ ёки $\Delta x<0)$ orttirma beramiz $(x_0+\Delta x\in(a,b))$. Natijada berilgan funksiya ham shu nuqtada orttirma oladi va u $\Delta y=f(x_0+\Delta x)-f(x_0)$ kabi ifodalanadi.

11.4-ta'rif. Agar $y=f(x)$ funksiyaning $x_0\in(a,b)$ nuqtadagi Δy orttirmasi ushbu

$$\Delta y=\Delta f(x_0)=A\cdot\Delta x+\alpha\cdot\Delta x \tag{11.15}$$

(bunda A – Δx ga bog'liq bo'lmagan o'zgarmas son, $\alpha=\alpha(\Delta x)$ bo'lib, $\Delta x\to0$ da $\alpha(\Delta x)\to0$) ko'rinishda ifodalansa, funksiya x_0 nuqtada *differensiallanuvchi* deyiladi.

(11.15) munosabatni quyidagicha

$$\Delta y = A\cdot\Delta x + o(\Delta x) \qquad (11.16)$$

yozish ham mumkin.

11.5- ta'rif. $f(x)$ funksiya Δy orttirmasining Δx ga nisbatan chiziqli bosh qismi $A\Delta x$ funksiyaning *differensiali* deyiladi va $dy = df(x_0)$ kabi belgilanadi.

Demak, $dy = df(x_0) = A\Delta x$, $\Delta x = dx$ ekanligini e'tiborga olsak (erkli o'zgaruvchi x ning Δx orttirmasini uning differensiali dx bilan almashtirish mumkin), $dy = Adx$ bo'ladi.

(11.15) formulani ushbu

$$y(x_0+\Delta x) = y(x_0) + dy(x_0) + \alpha(\Delta x)\Delta x$$

ko'rinishda ham yozish mumkin. Agar $dy(x_0)\neq 0$ bo'lsa, funksiyaning $x_0+\Delta x$ nuqtadagi qiymatini taqribiy

$$y(x_0+\Delta x)\approx y(x_0) + dy(x_0) \qquad (11.17)$$

formula bilan hisoblash mumkin

11.22-misol. Ushbu $f(x) = x^3 - 2x^2 + 3$ funksiyaning x_0 $(\forall x_0 \in R)$ nuqtada differensiallanuvchiligini ko'rsating.

Yechilishi. Berilgan funksiyaning x_0 nuqtadagi orttirmasini topamiz:

$$\Delta f(x_0) = f(x_0+\Delta x) + f(x_0) = (x_0+\Delta x)^3 - 2(x_0+\Delta x)^2 + 3 - (x_0^3 - 2x_0^2 + 3) =$$
$$= (3x_0^2 - 4x_0)\Delta x + (3x_0\Delta x + (\Delta x)^2 - 2\Delta x)\Delta x.$$

Agar $A = 3x_0^2 - 4x_0$, $\alpha(\Delta x) = 3x_0\Delta x + (\Delta x)^2 - 2\Delta x$ deyilsa, $\Delta f(x_0) = A\Delta x + \alpha(\Delta x)\Delta x$ bo'lishi kelib chiqadi. Bu esa berilgan funksiyaning x_0 nuqtada differensiallanuvchi ekanligini bildiradi.

11.23-misol. Ushbu

$$f(x) = x\sin\frac{1}{x}, \qquad f(0) = 0$$

funksiya $x = 0$ nuqtada differensiallanuvchi bo'ladimi?

Yechilishi. Berilgan funksiyaning $x=0$ nuqtadagi orttirmasini topamiz:

$$\Delta f(0)=f(0+\Delta x)-f(0)=\Delta x\sin\frac{1}{\Delta x}.$$

Bu tenglikdan ko'rinadiki, berilgan funksiyaning $x=0$ nuqtadagi $\Delta f(0)$ orttirmasini (11.15) ko'rinishda ifodalab bo'lmaydi. Demak, funksiya $x=0$ nuqtada differensiallanuvchi bo'lmaydi.

11.1-teorema. *$f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lishi uchun uning shu nuqtada chekli $f'(x)$ hosilaga ega bo'lishi zarur va yetarli.*

Agar $y=f(x)$ funksiya differensiallanuvchi bo'lsa,

$$dy=f'(x_0)dx \tag{11.18}$$

ekanligini ko'rish qiyin emas. Ma'lumki, differensiallanuvchi funksiyalar uchun dy bilan dx lar proportsional o'zgarib, $f'(x)$ proportsionallik koeffitsiyentini ifodalaydi.

Ixtiyoriy differensiallanuvchi u va v funksiyalar uchun ushbu

$$d(\alpha u\pm\beta v)=\alpha du\pm\beta dv,\ \alpha,\beta\in R;\ \ d(u\cdot v)=vdu+udv;\ \ d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2},\ v\neq 0 \tag{11.19}$$

tengliklar o'rinli bo'ladi.

11.24-misol. $y=2x^2-x+1$ funksiyaning $x=1$ nuqtadagi differensialini toping.

Yechilishi. 1-usul. Berilgan funksiyaning $x=1$ nuqtadagi orttirmasini topamiz:

$$\Delta f(1)=f(1+\Delta x)+f(1)=2(1+\Delta x)^2-(1+\Delta x)+1-2=3\Delta x+2(\Delta x)^2.$$

Funksiyaning orttirmasi (11.15) shaklda tasvirlandi, bu holda $A=3,\ \alpha(\Delta x)=2(\Delta x)^2$.

Shunday qilib, $dy|_{x=1}=3dx$

2-usul. $y=2x^2-x+1$ funksiyaning $x=1$ nuqtadagi hosilasini topamiz: $y'(x)=4x-1$, $y'(1)=3$ (11.18) formulaga binoan, $dy|_{x=1}=3dx$.

11.25–misol. Ushbu 1) $\sin 31^\circ$; 2) $\sqrt[4]{17}$ ifodalarning taqribiy qiymatlarini toping.

Yechilishi. Berilgan ifodalarning taqribiy qiymatlarini hisoblashda (11.17) formuladan foydalanamiz.

1) $x_0=\frac{\pi}{6}$, $\Delta x=\frac{\pi}{180}$ deb olib, $y=\sin x$ funksiyani va uning hosilasining $x_0=\frac{\pi}{6}$ nuqtadagi qiymatlarini hisoblaymiz:

$$y\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2};\quad y'\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2};\quad \sin 31^\circ=\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\pi}{180}\approx 0{,}501$$

2) $x_0=16$, $\Delta x=1$ deb olib, (11.17) formula bo'yicha

$$\sqrt[4]{17}\approx\sqrt[4]{16}+\left(\sqrt[4]{x}\right)'\big|_{x=16}\cdot 1=2+\frac{1}{4\sqrt[4]{16^3}}=2\frac{1}{32}\ .$$

Funksiya differensialining (11.18) ifodasidan foydalanib, elementar funksiyalarning differensiallari jadvalini keltiramiz:

1. $d(x^\mu)=\mu x^{\mu-1}dx \quad (x>0)$.
2. $d(a^x)=a^x\ln a\,dx \quad (a>0,\ a\neq 1)$.
3. $d(\log_a x)=\frac{1}{x}\log_a e \quad (x>0,\ a>0,\ a\neq 1)$.
4. $d(\sin x)=\cos x\,dx$.
5. $d(\cos x)=-\sin x\,dx$.
6. $d(\mathrm{tg}\,x)=\frac{1}{\cos^2 x}dx,\quad x\neq\frac{\pi}{2}+k\pi$, k=0, ±1, ±2,. . .
7. $d(\mathrm{stg}\,x)=-\frac{1}{\sin^2 x}dx,\quad x\neq k\pi$, k=0, ±1, ±2,. . .
8. $d(\ln x)=\frac{1}{x}dx$.
9. $d(e^x)=e^x dx$.

10. $d(\arcsin x)=\frac{1}{\sqrt{1-x^2}}dx, \quad -1<x<1.$

11. $d(\arccos x)=-\frac{dx}{\sqrt{1-x^2}}, \quad -1<x<1.$

12. $d(\text{arctg}\, x)=\frac{dx}{1+x^2}.$

13. $d(\text{arcctg}\, x)=-\frac{dx}{1+x^2}.$

11.26-misol. Ushbu

$1)\ y=2\sqrt{x^3}(3\ln x-2); \qquad 2)\ y=\arccos e^x;$

$3)\ y=\frac{\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{\frac{1-x}{1+x}}; \quad 4)\ y=\frac{x^2\cdot 2^x}{x^x}, \quad x_1=1, \quad x_2=2$

funksiyalarning differensiyalarini toping.

Yechilishi. (11.19) formulalardan foydalanib, berilgan funksiyalarning differensiallarini hisoblaymiz:

$$1)\ dy=d\left(2\sqrt{x^3}(3\ln x-2)\right)=2\sqrt{x^3}\,d(3\ln x-2)+(3\ln x-2)d\left(2\sqrt{x^3}\right)=$$
$$=2\sqrt{x^3}\cdot\frac{3}{x}dx+(3\ln x-2)\cdot\frac{3x^2}{\sqrt{x^3}}dx=9\sqrt{x}\cdot\ln x\,dx;$$

$$2)\ dy=d(\arccos e^x)=-\frac{1}{\sqrt{1-e^{2x}}}\cdot e^x\cdot dx;$$

$$3)\ dy=d\left(\frac{\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{\frac{1-x}{1+x}}\right)=d\left(\frac{\arcsin x}{\sqrt{1-x^2}}\right)+d\left(\ln\sqrt{\frac{1-x}{1+x}}\right)=$$
$$=\frac{\left(\sqrt{1-x^2}+x\arcsin x\right)}{(1-x^2)^{3/2}}dx-\frac{dx}{\frac{1-x}{1+x}\cdot(1+x)^2}=\frac{x\arcsin x}{(1-x^2)\sqrt{1-x^2}}dx;$$

$$4)\ dy=d\left(\frac{x^2\cdot 2^x}{x^x}\right)=\frac{x^x d(x^2\cdot 2^x)-x^2\cdot 2^x d(x^x)}{(x^x)^2}=\frac{2x\cdot 2^x+x^2\cdot 2^x(\ln 2-\ln x-1)}{x^x}dx.$$

Bundan $dy|_{x=1}=(2+\ln 4)dx, \quad dy|_{x=2}=0.$

11.27-misol. MAPLE tizimidan foydalanib, ushbu $y=\sqrt{1-x^2}\arcsin x$ funksiyaning birinchi tartibli differensialini toping.

Yechilishi:

> X:=subs(D(arcsin(x))=diff(arcsin(x),x)*D(x),D(sqrt(1-x^2)*arcsin(x)));

$$X := -\frac{D(x)\, x \arcsin(x)}{\sqrt{1-x^2}} + D(x)$$

Mustaqil yechish uchun misol va masalalar

11.1. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning $f'(x)$ hosilalarini toping.

1) $f(x)=x^3-2x$. 2) $f(x)=2^x\cos x$. 3) $f(x)=\cos 5x$.

4) $f(x)=\operatorname{tg} x+2x$. 5) $f(x)=\sqrt[3]{x}$. 6) $f(x)=\ln(4x-1)$.

7) $f(x)=\arccos 2x$. 8) $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0, \\ 0, & x=0. \end{cases}$ 9) $f(x)=e^{-x}+e^x$.

11.2. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning $f'(x_0)$ hosilalarini toping:

1) $f(x)=(x-4)^3(x+3)$, $f'(4)$. 2) $f(x)=\dfrac{(x-2)^3\ln x}{\sin x}$, $f'(2)$.

3) $f(x)=\operatorname{ctg} x+2x$, $f'\left(\dfrac{\pi}{4}\right)$. 4) $f(x)=\sqrt[3]{x-4}$, $f'(4)$.

5) $f(x)=\sqrt[3]{x^4}$, $f'(0)$. 6) $f(x)=\dfrac{1}{x^2}$, $f'(1)$

7) $f(x)=\sqrt{\dfrac{1-x^2}{1+x^2}}$, $f'(0)$. 8) $f(x)=\dfrac{x^2}{\sqrt{x^2+4}}$, $f'(0)$.

9) $f(x)=\dfrac{\sqrt{x+1}}{x-1}$, $f'(2)$. 10) $f(x)=3|x+1|$, $f'(-2)$

11.3. Hosila ta'rifidan foydalanib, quyidagi funksiyalarning hosilalari mavjudligini tekshiring:

1) $f(x)=|\ln x|$, $x_0=1$. 2) $f(x)=|(x-1)(x-2)|$, $x_0=1$, $x_0=2$.

3) $f(x)=|\sin x|$, $x_0=\pi$. 4) $f(x)=|\cos x|$. 5) $f(x)=|\pi^2-x^2|\cdot\sin^2 x$.

Quyidagi funksiyalarning hosilalarini toping:

11.82. $y=\sqrt{x}+\ln x-\dfrac{1}{\sqrt{x}}$. **11.83.** $y=\dfrac{x^4}{4}\left[(\ln x)^2-\dfrac{1}{2}\ln x+\dfrac{1}{8}\right]$

11.84. $y=\operatorname{ctg}\pi x+\dfrac{\cos\pi x}{2\sin^3\pi x}$. **11.85.** $y=e^{\sqrt[3]{x}}\left(\sqrt[3]{x^2}-2\sqrt[3]{x^2}+2\right)$

11.86. $y=\dfrac{1}{\sqrt{2}}\ln\left(\sqrt{2}\cdot\operatorname{tg}x+\sqrt{1+2\operatorname{tg}^2x}\right)$. **11.87.** $y=\dfrac{1}{2a}\left[\ln\dfrac{\sqrt{a^2+x^2}}{a+x}-\dfrac{a}{a+x}\right]$

11.88. $y=+\dfrac{\sin x}{2\cos^2 x}-\dfrac{1}{2}\ln\operatorname{tg}\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$. **11.89.** $y=\ln\operatorname{tg}\dfrac{x}{2}-\operatorname{ctg}x\cdot\ln(1+\sin x)-x$

11.90. $y=\dfrac{x^2\ln x}{a+bx^2}-\dfrac{1}{b}\ln\sqrt{a+bx^2}$.

11.91. $y=\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\dfrac{1}{2\sqrt{2}}\operatorname{arctg}\dfrac{x\sqrt{2}}{1-x^2}$.

11.92. $y=\dfrac{2}{3}\operatorname{arctg}\left(\dfrac{5}{3}\operatorname{arctg}\dfrac{x}{2}+\dfrac{4}{3}\right)$

11.93. $y=\dfrac{x}{\sqrt{1-x^2}}\arcsin x+\ln\sqrt{1-x^2}$

11.94. $y=\sqrt{x}-\sqrt{1-x}\cdot\arcsin\sqrt{x}$.

11.95. $y=\dfrac{3x+2}{4x^2}\sqrt{x-1}+\dfrac{3}{4}\operatorname{arctg}\sqrt{x-1}$.

11.96. $y=\arcsin(\sin x-\cos x)+\ln\left(\sin x+\cos x+\sqrt{\sin 2x}\right)$

11.97. $y=2\operatorname{arctg}\dfrac{\sqrt{2\operatorname{tg}x}}{1-\operatorname{tg}x}+\ln\dfrac{1+\sqrt{2\operatorname{tg}x}+\operatorname{tg}x}{1-\sqrt{2\operatorname{tg}x}+\operatorname{tg}x}$.

11.98. $y=\ln\dfrac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-2\operatorname{arctg}\dfrac{\sqrt[4]{1+x^4}}{x}$,

11.99. $y=x^{\sin x}$. **11.100.** $y=A\cdot e^{-k^2x}\sin(\omega x+\alpha)$.

11.101. $y=\sqrt[m+n]{(1-x)^m(1+x)^n}$.

11.102. $y=\dfrac{1}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$. **11.103.** $y=\dfrac{x^2+4}{x\cdot\sqrt{4+\left(\dfrac{x^2-4}{2x}\right)^2}}$.

11.104. $y=\dfrac{1}{(1+x^2)\sqrt{1+x^2}}$. **11.105.** $y=\log_2^3(2x+3)^2$.

11.106. $y=\sin\ln|x|$. **11.107.** $y=\sin[\sin(\sin x)]$.

11.108. $y=2^{\operatorname{tg}\frac{1}{x}}$. **11.109.** $y=\sin\left[\cos^2\left(\operatorname{tg}^3 x\right)\right]$.

11.110. $y=x^{a^a}+a^{x^a}+a^{a^x}$ $(a>0)$. **11.111.** $y=\ln\left(\ln^2\left(\ln^3 x\right)\right)$.

11.112. $y=\dfrac{1}{2}\operatorname{th}x\dfrac{\sqrt{2}}{8}\ln\dfrac{1+\sqrt{2}\operatorname{th}x}{1-\sqrt{2}\operatorname{th}x}$. **11.113.** $y=\sin\cos^2 x\cdot\cos\sin^2 x$.

11.114. $y=\cos^n x\cdot\cos nx$. **11.115.** $y=\sin(\arcsin x)$.

11.116. $y=\cos(2\arccos x)$. **11.117.** $y=\log_2(\log_3(\log_5 x))$.

11.118. $y=e^{e^x}+x^{e^x}$.

11.119. $y=\dfrac{1}{2a}\ln\dfrac{x-a}{x+a}$, $(a<0)$. **11.120.** $y=\arcsin(\sin x)$.

11.121. $y=\dfrac{\arcsin x}{\sqrt{1-x^2}}+\dfrac{1}{2}\ln\dfrac{1-x}{1+x}$. **11.122.** $y=(\sin x)^{\cos x}$.

11.123. $y=\dfrac{\operatorname{ch}x^2}{\operatorname{sh}^2 x^2}-\ln\operatorname{cth}\dfrac{x^2}{2}$. **11.124.** $y=\ln\dfrac{\sqrt{3}-\sqrt{2}\cos x}{\sqrt{3}+\sqrt{2}\cos x}$.

11.125. $y=\dfrac{1}{\sin\alpha}\ln\dfrac{1+x}{1-x}-\operatorname{ctg}\alpha\cdot\ln\dfrac{1+x\cos\alpha}{1-x\cos\alpha}$.

11.126. $y=\operatorname{th}x+\dfrac{\sqrt{2}}{4}\ln\dfrac{1+\sqrt{2}\operatorname{th}x}{1-\sqrt{2}\operatorname{th}x}$.

11.127. $y=x-\ln\sqrt{1+e^{2x}}+e^{-x}\cdot\operatorname{arcctg}e^x$.

11.128. $y=2^{x^x}$. **11.129.** $y=x^{x^3}$. **11.130.** $y=x^{e^x}$.

11.131. $y=x^{x^x}$. **11.132.** $y=x^{\frac{2}{\ln x}}-2x^{\log_x e}e^{1+\ln x}+e^{1+\frac{2}{\log_x e}}$

11.133. $y=\left(\dfrac{a}{b}\right)^x\cdot\left(\dfrac{b}{x}\right)^a\cdot\left(\dfrac{x}{a}\right)^b$ $(a>0,\ b>0)$. **11.134.** $y=e^x+e^{e^x}+e^{e^{e^x}}$.

11.135. $y=\dfrac{1}{2\sqrt{ab}}\ln\dfrac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}$ $(a>0,\ b>0)$

11.136. $y=-\frac{\cos x}{2\sin^2 x}+\ln\sqrt{\frac{1+\cos x}{\sin x}}$.

11.137. $y=\ln\frac{b+a\cos x+\sqrt{b^2-a^2}\sin x}{a+b\cos x}$.

11.138. $y=\ln\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]$.

11.139. $y=x[\sin(\ln x)-\cos(\ln x)]$

11.140. $y=x\arcsin\sqrt{\frac{x}{1+x}}+\operatorname{arctg}\sqrt{x}-\sqrt{x}$. **11.141.** $y=\arcsin(\sin x-\cos x)$.

11.142. $y=\operatorname{arctg}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$

11.143. $y=\frac{a}{b}x+\frac{2\sqrt{a^2-b^2}}{a}\operatorname{arctg}\left(\sqrt{\frac{a-b}{a+b}}\,th\frac{x}{2}\right)$ $(0\le|b|<a)$

11.144. $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$. **11.145.** $y=\frac{(\ln x)^x}{x^{\ln x}}$.

11.146. Quyidagi funksiyalarning ko'rsatilgan nuqtalardagi o'ng va chap hosilalarini toping.

1) $f(x)=|x+3|$, $f'(-3+0)$, $f'(-3-0)$.

2) $y=f(x)=|x^2-5x+6|$, $x=2$, $x=3$. 3) $f(x)=|2^x-2|$, $x=1$.

4) $f(x)=\sqrt[3]{\sin\pi x}$, $x=k$, $k\in Z$. 5) $f(x)=\arccos\left(\frac{1}{x}\right)$, $x=-1$, $x=1$.

11.147. Quyidagi funksiyalarning $x=0$ nuqtadagi o'ng va chap hosilalarini toping:

1) $f(x)=\begin{cases}x, & x\le 0,\\ \sqrt[3]{x^4}\ln x, & x>0;\end{cases}$ 2) $f(x)=\begin{cases}1+e^{1/x}, & x<0,\\ \sqrt{1+\sqrt[3]{x^4}}, & x\ge 0.\end{cases}$

11.148. Funksiyalarning uzilish nuqtalaridagi o'ng va chap hosilalarini toping.

1) $f(x)=\sqrt{\frac{x^2+x^3}{x}}$; 2) $f(x)=\begin{cases}\frac{1}{1+e^{1/x}}, & x\ne 0,\\ 0, & x=0;\end{cases}$

3) $f(x)=arctg\frac{1+x}{1-x}$;   4) $f(x)=(1-x^2)signx$.

11.149. $f(x)=|x-x_0|\cdot\varphi(x)$ funksiya uchun $f'_-(x_0)$ va $f'_+(x_0)$ larni toping, bunda $\varphi(x)$-berilgan x_0 nuqtada uzluksiz funksiya.

11.150. $y=\frac{1}{2}x^2-\ln x$ funksiya grafigiga abssissasi $x_0=2$ bo'lgan nuqtada o'tkazilgan urinmaning burchak koeffitsiyentini toping.

11.151. $y=x^3-3x+2$ parabolaga abssissasi $x_0=2$ bo'lgan nuqtada o'tkazilgan urinmaning burchak koeffitsiyentini toping.

11.152. $y=4\sin\frac{x}{3}$ funksiya grafigining $M(\frac{3\pi}{2},4)$ nuqtasidan o'tkazilgan urinma tenglamasini yozing.

11.153. $y=x^2+1$ egri chiziqqa o'tkazilgan urinma $y=2x+3$ to'g'ri chiziqqa parallel. Urinish nuqtasining ordinatasini toping.

11.154. $y=x^2-2x+1$ egri chiziqdagi qanday nuqtada unga o'tkazilgan urinma $y=-4(x+1)$ to'g'ri chiziqqa parallel bo'ladi?

11.155. $y=\frac{x}{1-x}$ funksiya grafigiga abssissasi $x_0=3$ bo'lgan nuqtadan o'tkazilgan urinmaning Ox o'qi bilan tashkil etgan burchagi α bo'lsa, $tg2\alpha$ ni toping.

11.156. $y=\frac{x+2}{x-2}$ funksiya grafigiga qanday nuqtalarda o'tkazilgan urinma, Ox o'qining musbat yo'nalishi bilan 135^0 li burchak tashkil etadi?

11.157. $y=\sqrt[3]{x}$ funksiyaning grafigi qanday nuqtada abssissa o'qiga 30^0 li burchak ostida joylashgan bo'ladi?

11.158. $y=x^3+2x-1$ funksiya grafigiga qanday nuqtada o'tkazilgan urinma, $x+y=0$ to'g'ri chiziqqa perpendikulyar bo'ladi?

11.159. $y=x^4$ va $y^4=x$ funksiyalarning grafiklari qaysi nuqtalarda, qanday burchak ostida kesishishlarini aniqlang.

11.160. $y=\ln x$ chiziq Ox o'qni qanday burchak ostida kesadi?

11.161. $y=\sin x$ chiziq (sinusoida) Ox o'qni qanday burchak ostida kesadi?

11.162. a ning qanday qiymatida $y=a^x$ chiziq Oy o'qni 45° li burchak ostida kesadi?

11.163. Ushbu

1) $y=\sin x\sqrt{3}$; 2) $y=\dfrac{x}{1+x^2}$; 3) $y=\dfrac{x}{\sqrt{3+x^2}}$

funksiyalar Oy o'qni qanday burchak ostida kesadi?

11.164. $x=a\cos t,\ y=b\sin t$ ellipsga o'tkazilgan urinmaning Ox o'q bilan tashkil qilgan burchagini toping.

11.165. $(2;1)$ nuqtada $x=t^2-3t+4,\ y=t^2-4t+4$ chiziqqa o'tkazilgan urinmani toping.

11.166. $x=2t^3-9t^2+12t-1,\ y=t^3+t+1$ chiziqqa qanday nuqtada o'tkazilgan urinma Oy o'qqa parallel bo'lali.

11.167. $x=2t-t^2,\ y=3t+t^3$ chiziqqa: 1) $t=-1$; 2) $t=1$; 3) $t=\sqrt{2}$ nuqtalarda o'tkazilgan urinma to'g'ri chiziqlar tenglamasini yozing.

11.168. Quyidagi funksiyalarning grafiklari qaysi nuqtalarda qanday burchak ostida kesishishlarini aniqlang:

1) $f_1(x)=x-x^3,\quad f_2(x)=5x$.

2) $f_1(x)=\sqrt{2}\sin x,\quad f_2(x)=\sqrt{2}\cos x$. 3) $f_1(x)=\dfrac{1}{x},\quad f_2(x)=\sqrt{x}$.

4) $f_1(x)=\ln x,\ f_2(x)=\dfrac{x^2}{2e}$. 5) $f_1(x)=x^2-4x+4,\ f_2(x)=-x^2+6x-4$.

6) $f_1(x)=x^3,\quad f_2(x)=\dfrac{1}{x^2}$. 7) $f_1(x)=4x^2+2x-8,\ f_2(x)=-x^3-x+10$.

11.169. Qanday nuqtalarda quyidagi $y=y(x)$ funksiyalar grafigiga o'tkazilgan urinmalar berilgan to'g'ri chiziqlarga parallel bo'ladi?

1) $y=x^2-7x+3, \quad 5x+y-3=0$. 2) $y=\frac{1}{3}\sin 3x+\frac{\sqrt{3}}{3}\cos 3x, \quad y=-x$

3) $y=x^3-3x+5, \quad y=-2x$. 4) $y=\ln(4x-1), y=x$.

5) $y=x^2, \; y=2x+5$. 6) $y=(3-x^2)e^x \; x=0$. 7) $y=|x-5|\cdot(x-3)^3, \quad x=0$.

11.170. Qanday nuqtalarda quyidagi $y=y(x)$ funksiyalar grafigiga o'tkazilgan urinmalar berilgan to'g'ri chiziqlarga perpendikulyar bo'ladi?

1) $y=\ln x, \quad 2y+x+1=0$. 2) $y=x^3-3x+5, \quad y=-\frac{x}{9}$.

3) $y=-\sqrt{2x^3}, \quad 4x-3y+2=0$. 4) $y=\sin x, \; x-10=0$.

5) $y=tgx, \quad x+y=0$. 6) $y=x^2, \; y=2x+5$

11.171. $y=f(x)$ funksiya grafigiga berilgan nuqtada o'tkazilgan urinma tenglamasini yozing:

1) $y=\mathrm{arctg}\,2x, x=0$. 2) $y=e^x, x=1$. 3) $y=|x-1|\sqrt[3]{x+2}, \; x=6$.

4) $y=\sqrt{5-x^2}, \; x=1$. 5) $y=4\,\mathrm{ctg}\,x-\frac{\cos x}{\sin^2 x}, \quad x=\frac{\pi}{2}$.

6) $x^2+y^2-2x+6y=0, \quad y>-3, \; x=0$.

7) $x=a(t-\sin t), \; y=a(1-\cos t), \; t=t_0 \neq 2n\pi, \; n\in Z$.

8) $x=te^t, \; y=te^{-t}, \; t>-1, t=t_0>-1$.

11.172. $s=2\sin 3t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilayotgan nuqtaning $t=\frac{\pi}{9}$ paytdagi tezligini toping.

11.173. $s=\sin^2 t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilayotgan nuqtaning $t=\frac{\pi}{6}$ paytdagi tezligini toping.

11.174. $s=e^t+\cos t+5t$ qonuniyat bo'yicha harakatlanayotgan moddiy nuqtaning $t=0$ dagi tezligini toping.

11.175. Ikki moddiy nuqta, mos ravishda, $s_1 = 2,5t^2 - 6t + 1$ va $s_2 = 0,5t^2 + 2t - 3$ qonuniyatlar bo'yicha harakatlanmoqda. Qaysi vaqtda birinchi nuqtaning tezligi ikkinchisinikidan 3 marta katta bo'ladi?

11.176. Moddiy nuqta $s = \ln t + \frac{1}{16}t$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilmoqda. Harakat boshlangandan qancha vaqt o'tgach, nuqtaning tezligi $\frac{1}{8}$ м/с bo'ladi?

11.177. Massasi $m = 1,5$ bo'lgan jism $s(t) = t^2 + t + 1$ qonuniyat bo'yicha to'g'ri chiziqli harakat qilmoqda. Jismning harakati boshlangandan 5 sekund vaqt o'tgandagi kinetik energiyasini toping (m massa kilogrammlarda, s yo'l — metrlarda berilgan).

11.178. Abssissalar o'qi bo'ylab ikkita nuqta, mos ravishda, $x = 100 + 5t$ va $x = \frac{t^2}{2}$ qonuniyatlar bo'yicha harakat qilmoqda. Bu nuqtalar uchrashish paytida (momentida) bir-biridan qanday tezlikda uzoqlashadi? (x metrlar bilan o'lchanadi, t — sekundlar bilan).

11.179. G'ildirak shunday aylanadiki, uning burilish burchagi vaqtning kvadratiga proporsionaldir. Birinchi aylanish 8 sekund vaqt davomida amalga oshirildi. Harakat boshlangandan 64 sekund vaqt o'tgandagi burchak tezligini toping.

11.180. Ko'rsatilgan nuqtalarda quyidagi funksiyalarga teskari bo'lgan funksiyalarning hosilalarini toping:

1) $y = 2x - \frac{\cos x}{2}$, $y = -\frac{1}{2}$. 2) $y = 2x^2 - x^4$, $0 < x < 1$, $y = \frac{3}{4}$.

11.181. Parametrik shaklda berilgan quyidagi $y = y(x)$ funksiyalarning y'_x hosilalarini toping:

1) $x = e^{-t}$, $y = t^3$, $-\infty < t < +\infty$. 2) $x = a\cos t$, $y = b\sin t$, $0 < t < \pi$.

3) $x = \ln(1 + t^2)$, $y = t - \operatorname{arctg} t$. 4) $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

5) $x = \dfrac{a\sin t}{1+b\cos t}$, $y = \dfrac{c\cos t}{1+b\cos t}$.   6) $x = a\cos^3 t, y = a\sin^3 t$.

7) $x = a\,\mathrm{ch}\,t$, $y = b\,\mathrm{sh}\,t$.   8) $x = \arcsin\dfrac{t}{\sqrt{1+t^2}}$, $y = \arccos\dfrac{1}{\sqrt{1+t^2}}$

11.182. Oshkormas shaklda berilgan quiydagi $y = y(x)$ funksiyalarning y'_x hosilalarini toping:

1) $x + \sqrt{xy} + y = a$.   2) $e^x \cdot \sin y - e^{-y}\cos x = 0$. 3) $y^2 = 2px, y > 0$.

4) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $y > 0$. 5) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $y > 0$.

6) $e^y + xy = e$, $x = 0, y = 1$. 7) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.   8) $\mathrm{arctg}\dfrac{y}{x} = \ln\sqrt{x^2+y^2}$.

9) $x^2 + y^2 - 6x + 10y + 2 = 0$, $y > -5$, $x_0 = 0$.

10) $x^2 - 4xy + 4y^2 + 4x - 3y - 7 = 0$, $x < 2y - 1$.

11.183-misol. Ushbu

1) $r = a(1+\cos\varphi)$, $\varphi \in (0; \frac{2\pi}{3})$.

2) $r = a\sqrt{\cos 2\varphi}$, $0 < \varphi < \frac{\pi}{4}$, $x_0 = a\dfrac{\sqrt{6}}{4}$.

3) $r = a\sqrt{\cos 2\varphi}$, $0 \le \varphi \le \frac{\pi}{4}$, $f'_+(0) = ?$, $f'_-(0) = ?$;

4) $r = a(1+\cos\varphi)$. 5) $r = ae^{m\varphi}$.

ko'rinishda berilgan $y = y(x)$ funksiyalarning y'_x hosilalarini toping (bunda r va φ lar $(x; y)$ nuqtaning qutb koordinatalari).

Quyidagi funksiyalarning differensialini toping.

11.184. $y = \ln x + x^2$.   **11.185.** $y = e^{3x} + \sqrt{x}$.   **11.286.** $y = \cos^2 x + 3$.

11.187. $y = \mathrm{tg}\,4x + \dfrac{2}{x}$.   **11.188.** $y = \log_3 x + \sin 5x$

11.189. Quyidagi funksiyalarning differensialini toping.

1) $y = \ln\ln\left(\dfrac{x}{2}\right)$.   2) $y = \cos\dfrac{1}{\log_2 x}$.   3) $y = e^{\frac{\sqrt{1-x}}{1+x}}$.   4) $y = x^{x^2}$.

5) $y=\operatorname{arctg}\frac{\ln x}{x}$, $x_0=\frac{1}{e}$, $x_0=e$.

6) $y=\frac{x^2\cdot 2x}{x^2}$, $x_0=1$, $x_0=2$. 7) $y=\arcsin^2 x+\operatorname{arctg}^3 x$.

11.190. Funksiyaning orttirmasini uning differensiali bilan almashtirib, quyidagi $y=f(x)$ funksiyalarning ko'rsatilgan nuqtadagi taqribiy qiymatlarini toping.

1) $y=\sqrt[3]{x}$, $a)\, x=65$; $b)\, x=125{,}1324$. 2) $y=\sqrt[4]{x}$, $a)\, x=90$; $b)\, x=15{,}8$.

3) $y=tgx$, $x=44°50'$. 4) $y=\sqrt{\frac{2-x}{2+x}}$, $x=0{,}15$.

11.191. Funksiyaning orttirmasini uning differensiali bilan almashtirib, quyidagi ifodalarning taqribiy qiymatlarini toping.

1) $\sqrt[3]{1{,}02}$. 2) $\sin 29°$. 3) $\cos 151°$. 4) $\lg 11$.

11.192. Oshkormas yoki parametrik shaklda berilgan $y=y(x)$ funksiyalarning differensiallarini quyidagi berilgan nuqtalarda toping.

1) $y^3-y=6x^2$, $(1;2)$. 2) $y^5+x^4=xy^2$, (x_0,y_0). 3) $xy-\sqrt[3]{xy^2+6}=0$, $(2;1)$.

4) $4xy^3+\ln\sqrt{\frac{x}{x+y}}=0$, $(1;0)$. 5) $x=(t-1)^2(t-2)$, $y=(t-1)^2\cdot(t-3)$, $(4;0)$.

6) $x=\frac{e^t}{t}$, $y=(t-1)^2 e^t$, $\left(-\frac{2}{\sqrt{e}};\frac{9}{4\sqrt{e}}\right)$.

11.193. Quyidagi $y=f(x)$ funksiyalarni berilgan nuqtalarda differensiallanuvchilikka tekshiring.

1) $y=\sqrt{x^3}$, $x_0=0$. 2) $y=\sqrt{1-x}$, $x_0=\frac{1}{2}$. 3) $y=3x^3+4x^2+5x-2$, $\forall x_0\in R$.

4) $y=x\cos\frac{1}{x}$, $x_0=0$. 5) $y=\begin{cases} x\cdot\sin\frac{1}{x^2}, & x\neq 0,\\ 0, & x=0,\end{cases}$ $x_0=0$.

6) $y=x\sqrt{\ln(1+x^2)}$, $x_0=0$. 7) $y=\arcsin\frac{1-x^2}{1+x^2}$, $x_0=1$; $x_0=-1$; $x_0=0$.

8) $y=\begin{cases}\operatorname{arctg}x, & x\geq 0,\\ x^2+x, & x<0,\end{cases}$ $\forall x_0\in R$. 9) $y=3^{\pi^{\frac{1}{x}}}$, $x_0=\frac{1}{\pi}$.

10) $y=\ln(1+\sin^2 x)-2\sin x\cdot \operatorname{arctg}\sin x,\quad x_0=\frac{\pi}{2}.$

11) $y=\begin{cases}(x-2)\,\operatorname{arctg}\dfrac{1}{x-2}, & x\neq 2,\\ 0, & x=2,\end{cases}\quad x_0=2.$ 12) $y=\left(\frac{\sin x}{x}\right)^x,\quad x_0=\frac{\pi}{2}.$

13) $y=\sqrt{1-e^{-x^2}},\quad x_0=0.$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

11.1. 1) $3x^2-2x$. 2) $2^x(\ln 2\cos x-\sin x)$. 3) $-5\sin 5x$. 4) $\frac{1}{\cos^2 x}+2$. 5) $\frac{1}{3\sqrt[3]{x^2}}$.

6) $\frac{4}{4x-1}$. 7) $-\frac{2}{\sqrt{1-4x^2}}$. 8) $\begin{cases}2x\sin\frac{1}{x}+\cos\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$ 9) $-e^{-x}+e^{x}$.

11.2. 1) 0. 2) 0. 3) 0. 4) ∞. 5) 0. 6) -2. 7) 0. 8) 0. 9) $-\frac{1}{\sqrt{3}}$. 10) -3.

11.3. 1) mavjud emas. 2) mavjud emas. 3) mavjud emas.

4) $x=\frac{2k-1}{2}\pi,\ k\in Z$ nuqtada hosilaga ega emas. 5). Hamma joyda hosilaga ega. 6) $f'(0)=0$. 7) Mavjud emas. 8) Mavjud emas. 9) $f'(0)=0$. 10) $f'(0)=0$.

11.4. $6(x+1)$ **11.5.** $\frac{1}{\sqrt{x}}+\frac{3}{x^4}$. **11.6.** $-\frac{6x^2}{(1+x^3)^3}$.

11.7. $60(1+2x)^{29}$. **11.8.** $16\left(7x^2-\frac{4}{x}+6\right)^7\left(7x+\frac{2}{x^2}\right)$. **11.9.** $12\left(t^3-\frac{1}{t^3}+3\right)^3\left(t^2+\frac{1}{t^4}\right)$.

11.10. $-\frac{x}{\sqrt{1-x^2}}$. **11.11.** $\frac{3-x}{2(1-x)^{3/2}}$. **11.12.** $\frac{t^2(3-t)}{(1-t)^3}$. **11.13.** $12x^{-5}+5x^{-6}$.

11.14. $\frac{2(1-x^2)}{(x^2-x+1)^2}$. **11.15.** $\frac{7}{8}x^{-\frac{1}{8}}$. **11.16.** $\frac{4}{3\sqrt[3]{2x+1}}$.

11.17. $\frac{12-6x-6x^2+2x^3+5x^4-3x^5}{(1-x)^3}$. **11.18.** $-\frac{(1-x)^{p-1}[(p+q)+(p-q)x]}{(1+x)^{q+1}}$.

11.19. $x^2\cos x$. **11.20.** $\cos x\left(3\sin^2 x-2\operatorname{cosec}^3 x\right)$.

11.21. $4\sin x\cdot\sec^5 x$. **11.22.** $\dfrac{x-0{,}5\sin 2x}{x^2\cos^2 x}$.

11.23. $-\sin^3 x$. **11.24.** $\sin x\cos x\left(6-5\sin^3 x\right)$. **11.25.** $-\dfrac{1}{x^2}\cos\dfrac{1}{x}$.

11.26. $\cos(\sin x)\cos x$. **11.27.** $-6\sin 8x\cos 4x$. **11.28.** $4\sin 2x\left(1+\sin^2 x\right)^3$.

11.29. $\dfrac{2x}{5\sin^2\sqrt[5]{1+x^2}\,\sqrt[5]{\left(1+x^2\right)^4}}$. **11.30.** $\dfrac{1}{2\sqrt{1+\operatorname{tg}x}}\cdot\dfrac{1}{\cos^2 x}$.

11.31. $-3\sin 3x\sin(2\cos 3x)$.

11.32. $\dfrac{\cos x+\cos 2x}{(1+\cos x)^2}$. **11.33.** $\dfrac{1-\sin x+x\cdot\cos x}{(1-\sin x)^2}$.

11.34. $-0{,}64(2\cos(8x+5)-3\sin 0{,}8x)(2\sin(8x+5)+0{,}3\cos 0{,}8x)$.

11.35. $a(a\cos x+b\sin x)^{n-1}(-a\sin x+b\cos x)$.

11.36. $-4\cos 8x,\quad x\neq\dfrac{\pi}{8}+\dfrac{\pi}{4}k,\ x\neq\dfrac{\pi}{4}+\dfrac{\pi}{2}k,\ x\neq\dfrac{\pi}{2}k,\ k\in Z$.

11.37. $3x^2\log_4 x+\dfrac{x^2}{2\ln 2}$. **11.38.** $\ln x+1$. **11.39.** $\dfrac{1}{2\sqrt{\ln x}}\cdot\dfrac{1}{x}$.

11.40. $-\dfrac{4x}{1-2x^2}$. **11.41.** $\dfrac{2(x-2)}{\left(x^2-4x\right)\ln 2}$.

11.42. $5\ln^4(\sin x)\operatorname{ctg}x$. **11.43.** $\dfrac{2}{\arccos 2x}\dfrac{1}{\sqrt{1-4x^2}}$.

11.44. $n(1+\ln\sin x)^{n-1}\operatorname{ctg}x$. **11.45.** $\dfrac{1+x^2-2x^2\ln x}{x\left(1+x^2\right)}$.

11.46. $\dfrac{x}{\operatorname{arctg}\sqrt{1+x^2}}\cdot\dfrac{1}{\left(2+x^2\right)\sqrt{1+x^2}}$. **11.47.** $\dfrac{6\ln^2\operatorname{tg}x}{\sin 2x}$.

11.48. $\dfrac{6x^2-5\sin 5x}{\cos 5x+2x^3}$. **11.49.** $3^x\ln 3$. **11.50.** $2^{1+\sin 2x}\cdot\cos 2x\cdot\ln 2$.

11.51. $-4\cdot 5^{\cos 4x}\cdot\sin 4x\cdot\ln 5$. **11.52.** $\dfrac{2\left(x-x^2\ln 2\right)}{4^x}$. **11.53.** $e^x(\cos x-\sin x)$.

11.54. $4x^3-2\cdot 5^{2x}\ln 5$. **11.55.** $\dfrac{2e^x}{\left(1-e^x\right)^2}$.

11.56. $[2x-5+\left(x^2-5x+6\right)\ln 6]6^x$. **11.57.** $3\cdot 10^{3x-4}\cdot\ln 10$. **11.58.** $-4e^{-4x}$.

11.59. $\dfrac{2}{\sqrt{1-4x^2}}e^{\arcsin 2x}$. **11.60.** $a^x x^a\ln a+a^{1+x}x^{a-1}$. **11.61.** $e^{\sqrt{x+1}}\dfrac{1}{2\sqrt{x+1}}$.

11.62. $\dfrac{4}{\left(e^x+e^{-x}\right)^2}$. **11.63.** $e^{ax}(a\cos bx-b\sin bx)$. **11.64.** $\dfrac{1}{2}\operatorname{sh}2x$.

11.65. $\operatorname{th}x$. **11.66.** $2\operatorname{sh}2x$. **11.67.** $3\operatorname{th}^2 x\cdot\dfrac{1}{\operatorname{ch}^2 x}$. **11.68.** $x(2\operatorname{sh}x+x\operatorname{ch}x)$

11.69. $\dfrac{2}{\operatorname{ch}^2 x\cdot(1-\operatorname{th}x)^2}$. **11.70.** $\dfrac{1}{9\operatorname{ch}^2\dfrac{x}{3}}\cdot\left(1-\operatorname{th}\dfrac{x}{3}\right)$. **11.71.** $-\dfrac{4}{\operatorname{sh}^2 4x}$.

11.72. $5\cdot 3^{\operatorname{ch}5x}\ln 3\operatorname{sh}5x$. **11.73.** $\arcsin x+\dfrac{x}{\sqrt{1-x^2}}$. **11.74.** $-\dfrac{2\arccos x}{\sqrt{1-x^2}}$.

11.75. $\dfrac{1}{2\sqrt{x}}\operatorname{arctg}x+\dfrac{\sqrt{x}}{1+x^2}$. **11.76.** $-\dfrac{3x^3}{\sqrt{x^2-9}}\operatorname{sign}x$.

11.77. $\dfrac{2x\operatorname{arctg}x+2x^3\operatorname{arctg}x-x^2}{\left(1+x^2\right)\operatorname{arctg}^2 x}$. **11.78.** $-\dfrac{x+\sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}}$.

11.79. $\dfrac{x+\sqrt{1-x^2}}{2\sqrt{1-x^2}\left(1-x\sqrt{1-x^2}\right)}$. **11.80.** $\dfrac{x\arccos x-\sqrt{1-x^2}}{\sqrt{\left(1-x^2\right)^3}}$. **11.81.** $\arccos x$.

11.82. $\dfrac{x+2\sqrt{x}+1}{2x\sqrt{x}}$. **11.83.** $x^3\ln^2 x$. **11.84.** $-\dfrac{3\pi}{2\sin^4\pi x}$.

11.85. $\dfrac{1}{3}e^{\sqrt[3]{x}}$. **11.86.** $\dfrac{1}{\sqrt{1-\sin^4 x}}$.

11.87. $\dfrac{x^2}{\left(x^2+a^2\right)(x+a)^2}$. **11.88.** $\dfrac{1}{\cos^3 x}$. **11.89.** $\dfrac{\ln(1+\sin x)}{\sin^2 x}$.

11.90. $\dfrac{2ax\ln x}{\left(a+bx^2\right)^2}$. **11.91.** $\dfrac{1}{1+x^4}$. **11.92.** $\dfrac{1}{5+4\sin x}$.

11.93. $\dfrac{\arcsin x}{\left(1-x^2\right)\sqrt{1-x^2}}$. **11.94.** $\dfrac{\arcsin\sqrt{x}}{2\sqrt{1-x}}$. **11.95.** $\dfrac{1}{x^3\sqrt{x-1}}$.

11.96. $\sqrt{2\operatorname{ctg}x}$. **11.97.** $2\sqrt{2\operatorname{ctg}x}$. **11.98.** $\dfrac{4}{\sqrt[4]{1+x^4}}$.

11.99. $x^{\sin x}\left(\dfrac{\sin x}{x}+\cos x\cdot\sin x\right)$.

11.100. $Ae^{-k^2x}\left(\omega\cos(\varpi x+a)-k^2\sin(\omega x+a)\right)$.

11.101. $\frac{n-m-(n+m)x}{(n+m)\sqrt[m+n]{(1-x)^{n}(1+x)^{m}}}$. **11.102.** $-\frac{1}{\sqrt{(1+x^2)^3}}$.

11.103. $0,\ x\neq 0$. **11.104.** $\frac{-3x}{\sqrt{(1+x^2)^5}}$.

11.105. $\frac{12}{\ln 2}\frac{\log_2^2(2x+3)^2}{2x+3}$. **11.106.** $\frac{\cos\ln|x|}{x}$.

11.107. $\cos x\cdot\cos(\sin x)\cdot\cos[\sin(\sin x)]$. **11.108.** $-\frac{1}{x^2}2^{\operatorname{tg}\frac{1}{x}}\cdot\sec^2\frac{1}{x}\cdot\ln 2$.

11.109. $-3\operatorname{tg}^2 x\cdot\sec^2 x\cdot\sin(2\cdot\operatorname{tg}^3 x)\cdot\cos[\cos^2(\operatorname{tg}^3 x)]$.

11.110. $a^a\cdot x^{a^a-1}+ax^{a-1}a^{x^a}\ln a+a^x\cdot a^{a^x}\cdot\ln^2 a$. **11.111.** $\frac{6}{x\ln x\ln(\ln^2 x)}(x>e)$.

11.112. $\frac{1}{1-\operatorname{sh}^4 x}$. **11.113.** $-\sin 2x\cdot\cos(\cos 2x)$.

11.114. $-n\cos^{n-1}x\cdot\sin(n+1)x$. **11.115.** $1,\ |x|<1$. **11.116.** $4x,\ |x|<1$.

11.117. $\frac{1}{\ln 2}\frac{1}{x\ln x\ln\log_5 x}$, $x>5$. **11.118.** $e^x\left(e^{e^x}+x^{e^x}\left(\frac{1}{x}+\ln x\right)\right)$.

11.119. $\frac{1}{x^2-a^2}$. **11.120.** $\operatorname{sgn}(\cos x)$, $\left(x\neq\frac{2k-1}{k}\pi,\ k\in Z\right)$.

11.121. $\frac{x\arcsin x}{\sqrt{(1-x^2)^3}}$. **11.122.** $(\sin x)^{1+\cos x}(\operatorname{ctg}^2 x-\ln(\sin x))$,

$(2k\pi<x<(2k+1)\pi,\ k\in Z)$. **11.123.** $-\frac{4x}{\operatorname{sh}^3 x^2}$.

11.124. $\frac{2\sqrt{6}\sin x}{3-2\cos^2 x}$. **11.125.** $\frac{2\sin\alpha}{(1-x^2)(1-x^2\cos^2\alpha)}$, $|x|<1$. **11.126.** $\frac{2}{1-\operatorname{sh}^4 x}$.

11.127. $-\frac{\operatorname{arctg} e^x}{e^x}$. **11.128.** $(\ln 2)2^{x^x}\cdot x^x(1+\ln x)$. **11.129.** $x^{1+x^2}(1+2\ln x)$.

11.130. $e^x\cdot x^{e^x}\left(\frac{1}{x}+\ln x\right)$. **11.131.** $x^{x^x}\cdot x^{x-1}(x\ln^2 x+x\ln x+1)$.

11.132. $2e(x-e)$, $x>0,\ x\neq 1$. **11.133.** $\left(\frac{a}{b}\right)^x\cdot\left(\frac{b}{x}\right)^a\cdot\left(\frac{x}{a}\right)^b\left(\ln\frac{a}{b}-\frac{a-b}{x}\right)$, $(x>0)$.

11.134. $e^x\left[1+e^{e^x}\left(1+e^{e^{e^x}}\right)\right]$. **11.135.** $\frac{1}{a-bx^2}$, $\left(|x|<\sqrt{\frac{a}{b}}\right)$.

11.136. $\frac{\cos^2 x}{\sin^3 x}$ $(0 < x - 2k\pi < \pi,\ k \in Z)$. **11.137.** $\frac{\sqrt{b^2 - a^2}}{a + b\cos x}$.

11.138. $-\frac{1 + x + \frac{1}{x} + \ln\frac{1}{x}}{\left(1 + x\ln\frac{1}{x}\right)\left[1 + x\ln\left(\frac{1}{x} + \ln\frac{1}{x}\right)\right]}$. **11.139.** $2\sin(\ln x)$ $(x > 0)$.

11.140. $\arcsin\sqrt{\frac{x}{1+x}}$ $(x \ge 0)$. **11.141.** $\frac{\sin x + \cos x}{\sqrt{\sin 2x}}$, $\left(0 < x - k\pi < \frac{\pi}{2},\ k \in Z\right)$.

11.142. 1, $\left(x \ne \frac{\pi}{4} + k\pi,\ k \in Z\right)$. **11.143.** $\frac{a + b\,\mathrm{ch}\,x}{b + a\,\mathrm{ch}\,x}$.

11.144. $(\sin x)^{1+\cos x}\left(\mathrm{ctg}^2 x - \ln\sin x\right) - \cos x^{1+\sin x}\left(\mathrm{tg}^2 x - \ln\cos x\right)$, $\left(0 < x - 2k\pi < \frac{\pi}{2},\ k \in Z\right)$.

11.145. $\frac{(\ln x)^{x-1}}{x^{\ln x + 1}}\left[x - 2\ln^2 x + x\ln x \cdot \ln(\ln x)\right]$ $(x > 1)$.

11.146. 1) $f_+'(-3+0) = 1$, $f_-'(-3-0) = -1$. 2) $f_-'(2) = f_-'(3) = -1$, $f_+'(2) = f_+'(3) = 1$.

3) $f_+'(1) = \ln 4$, $f_-'(1) = -\ln 4$. 4) $f'(2k) = +\infty$, $f'(2k-1) = -\infty$, $k \in Z$.

4) $f(x) = \sqrt[3]{\sin \pi x}$, $x = k$, $k \in Z$. 5) $f_-'(-1) = f_+'(1) = +\infty$, $f_+'(-1)$ va $f_-'(1)$

mavjud emas. **11.147.** 1) $f_-'(0)=1$, $f_+'(0)=0$. 2) $f'(0)=0$.

11.148. 1) $f_-'(0) = -\frac{1}{2}$, $f_+'(0) = \frac{1}{2}$. 2) $f_-'(0) = -\infty$, $f_+'(0) = 0$.

3) $f_-'(1) = f_+'(1) = \frac{1}{2}$. 4) $f_+'(0) = +\infty$, $f_-'(0) = +\infty$.

11.149. $f_+'(x_0) = \varphi(x_0)$, $f_-'(x_0) = -\varphi(x_0)$ **11.150.** k=3/2. **11.151.** k = 1.

11.152. $y = 4$. **11.153.** $y = 2$. **11.154.** (-1; 4). **11.155.** $\frac{8}{15}$.

11.156. (4;3), (0;-1). **11.157.** $\left(\frac{1}{\sqrt[4]{27}}; \frac{1}{\sqrt[4]{3}}\right)$. **11.158.** (-1;0), (1,-2).

11.159. $\varphi = \mathrm{arctg}\frac{3}{8}$. **11.160.** 45°.

11.161. $x = 2\pi n$ da 45°, $x = (2n+1)\pi$ da -45° $(n \in Z)$. **11.162.** $a = e$.

11.163. 1) 30°. 2) 45°. 3) 60°. **11.164.** $\mathrm{tg}\,\alpha = -\frac{b}{a}\mathrm{ctg}\,t$.

11.165. $y = 2x - 3$. **11.166.** (4;3) va (3;7).

11.167. 1) va 3) hollarda, mos ravishda, $y+2=0$ va $y-\sqrt{2}=\frac{3}{2}(1+\sqrt{2})(x-2\sqrt{2}+2)$; 2) holda $t=1$ nuqtada $\frac{3(1-t^2)}{2(1-t)}$ funksiya aniqlanmagan.

11.168. 1) $(0;0)$, $\varphi=\text{arctg}(2/3)$. 2) $\left(\frac{\pi}{4}+\pi k;\ (-1)^k\right), (k\in Z), \varphi=\frac{\pi}{2}$.

3) $(1;1)$, $\varphi=\text{arctg}3$. 4) $(\sqrt{e};1/2)$, $\varphi=0$. 5) $(1;1)$ va $(4;4)$, $\varphi=\text{arctg}\frac{6}{7}$.

6) $(1;1)$, $\varphi=\frac{5\pi}{4}$. 7) $(3;34)$, $\varphi=0$.

11.169. 1) $(1;-3)$. 2) $\left(\frac{1}{3}(\frac{\pi}{3}-2k\pi);\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{3}(-\frac{2\pi}{3}-2m\pi);-\frac{1}{\sqrt{3}}\right)$, $k,m\in Z$.

3) $M_1\left(-\frac{1}{\sqrt{3}};\ 5+8\frac{\sqrt{3}}{9}\right)$, $M_2\left(\frac{1}{\sqrt{3}};\ 5+8\frac{\sqrt{3}}{9}\right)$. 4) $\left(\frac{5}{4};\ \ln 4\right)$. 5) $(1;1)$. 6) $(1;2e)$.

7) $(3;0)$; $\left(\frac{9}{2};\frac{27}{16}\right)$.

11.170. 1) $\left(\frac{1}{2};-\ln 2\right)$. 2) $M_1(-2;3)$, $M_2(2;7)$. 3) $\left(\frac{1}{8};-\frac{1}{16}\right)$.

4) $\left(\frac{\pi}{2}+k\pi;1\right)$, $(k\in Z)$. 5) $(k\pi;0)$, $(k\in Z)$. 6) $\left(-\frac{1}{4};\frac{1}{16}\right)$.

11.171. 1) $2x-y=0$. 2) $ex-y=0$.

3) $29x-12y-54=0$. 4) $x+2y-5=0$. 5) $y=-3x+\frac{3\pi}{2}$. 6) $x-3y=0$.

7) $y=\left(\text{ctg}\left(\frac{t_0}{2}\right)\right)x+2a-at_0\,\text{ctg}(t_0/2)$. 8) $(1-t_0)e^{-t_0}x-(1+t_0)e^{t_0}y+2t_0^2=0$.

11.172. $3\sqrt{3}$ (m/s). **11.173.** $\frac{\sqrt{3}}{2}$(m/s). **11.174.** 6 (m/s). **11.175.** 6 (s).

11.176. 16 (s). **11.177.** 90,75 Joul . **11.178.** $15\frac{\text{m}}{\text{s}}$. **11.179.** $4\pi\cdot$rad/s.

11.180. 1) $x'\left(-\frac{1}{2}\right)=\frac{1}{2}$. 2) $x'\left(\frac{3}{4}\right)=\frac{\sqrt{2}}{2}$. **11.181.** 1) $y'_x=-3t^2e^t$.

2) $y'_x=-\frac{b}{a}\text{ctg}\,t$. 3) $y'_x=\frac{t}{2}$. 4) $y'_x=\text{ctg}\frac{t}{2}$ $(t\ne 2k\pi, k\in Z)$. 5) $y'_x=\frac{-c\sin t}{a(b+\cos t)}$.

6) $y'_x=-\text{tg}\,t$ $\left(t\ne\frac{2k+1}{2}\pi,\ k\in Z\right)$. 7) $y'_x=\frac{b}{a}\text{cth}\,t$ $(|t|>0)$ 8) $y'_x=\text{sign}\,t$ $(0<|t|<+\infty)$

11.182. 1) $y'_x=\frac{2a-2x-y}{x+2y-2a}$. 2) $y'_x=\frac{e^x\sin y+e^{-y}\sin x}{e^x\cos y+e^{-y}\cos x}$. 3) $y'_x=\frac{p}{y}$.

4) $y'_x=-\sqrt[3]{\frac{y}{x}}$, $|x|<a$. 5) $y'_x=\frac{b^2x}{a^2y}$, $|x|>a$. 6) $y'_x(0)=-\frac{1}{e}$. 7) $y'_x=-\frac{b^2x}{a^2y}$.

8) $y'_x=\frac{x+y}{x-y}$. 9) $y'_x(0)=\frac{1}{\sqrt{3}}$. 10) $y'_x=\frac{4y-2x-4}{8y-4x-3}$.

11.183. 1) $y'_x=-\text{ctg}\frac{3\varphi}{2}$. 2) $y'\left(a\frac{\sqrt{6}}{4}\right)=1$. 3) $y'_+(0)=1$, $y'_-(0)=-\infty$.

4) $y'_x=-\text{ctg}\frac{3\varphi}{2}, \varphi\ne 0$. 5) $y'_x=\text{tg}\left(\varphi+\text{arctg}\frac{1}{m}\right)$.

11.184. $\left(\frac{1}{x}+2x\right)dx$. **11.185.** $\left(3e^{3x}+\frac{1}{2\sqrt{x}}\right)dx$. **11.286.** $-\sin 2x\cdot dx$.

11.187. $\left(\frac{4}{\cos^2 4x}-\frac{2}{x^2}\right)dx$. **11.188.** $\left(\frac{1}{x\ln 3}+5\cos 5x\right)dx$.

11.189. 1) $\frac{dx}{x\ln\frac{x}{2}}$, $x>2$. 2) $\frac{\sin\left(\frac{1}{\log_2 x}\right)}{\left(x\log_2^2 x\right)\ln 2}dx$. 3) $-\frac{1}{\sqrt{1+x}\sqrt{1-x^2}}e^{\sqrt{\frac{1-x}{1+x}}}dx$.

4) $x^{1+x^2}(1+2\ln x)dx$. 5) $\frac{2e^2dx}{e^2+1}$; 0. 6) $(2+\ln 4)dx$; 0. 7) $\left(\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{3\,\text{arctg}^2 x}{1+x^2}\right)dx$.

11.190. 1) *a*) 4,0208; *b*) 5,00177. 2) *a*) 3,083; *b*) 1,9938. 3) 0,9942. 4) 0,925.

11.191. 1) 1,007. 2) 0,4849. 3) $-0{,}8748$. 4) 1,043.

11.192. 1) $\frac{12}{11}dx$. 2) $\frac{y_0^3-4x_0^3}{5y_0^4-2x_0y_0}dx$. 3) $-\frac{11}{20}dx$. 4) 0. 5) $\frac{1}{2}dx$. 6) $\frac{1}{8}dx$.

11.193. 1) differensiallanuvchi; 2) differensiallanuvchi;

3) differensiallanuvchi; 4) differensiallanuvchi emas;

5) differensiallanuvchi emas; 6) differensiallanuvchi;

7) $x_0 = 1$, $x_0 = -1$ da differensiallanuvchi, $x_0 = 0$ da esa differensiallanuvchi emas; 8) sonlar o'qining hamma joyida;

9) differensiallanuvchi; 10) differensiallanuvchi;

11) differensiallanuvchi emas; 12) differensiallanuvchi;

13) differensiallanuvchi emas.

12-§. FUNKSIYANING YUQORI TARTIBLI HOSILASI VA DIFFERENSIALI

12.1. Funksiyaning yuqori tartibli hosilasi. $y = f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin.

12.1-ta'rif. Agar $f(x)$ funksiya (a,b) oraliqning har bir $x \in (a,b)$ nuqtasida $f'(x)$ hosilaga ega bo'lib, bu $f'(x)$ funksiya ham $x_0 \in (a,b)$ nuqtada hosilaga ega bo'lsa, uni $f(x)$ funksiyaning x_0 nuqtadagi *ikkinchi tartibli hosilasi* deb ataladi va $y''|_{x=x_0}$, $f''(x_0)$, $\left(\frac{d^2y}{dx^2}\right)\Big|_{x=x_0}$ kabi belgilardan biri orqali yoziladi.

$f(x)$ funksiya (a,b) oraliqning har bir $x \in (a,b)$ nuqtasida $(n-1)$ tartibli $f^{(n-1)}(x)$ hosilaga ega bo'lsin. Bu $f^{(n-1)}(x)$ funksiyaning $x_0 \in (a,b)$ nuqtadagi hosilasi (agar u mavjud bo'lsa) $f(x)$ funksiyaning x_0 nuqtadagi *n-tartibli hosilasi* deb ataladi va u $y^{(n)}(x_0)$, $f^{(n)}(x_0)$, $\frac{d^n y(x)}{dx^n}\Big|_{x=x_0}$ belgilardan biri orqali yoziladi. Odatda $f(x)$ funksiyaning $f''(x), f'''(x), \ldots$ hosilalari uning *yuqori tartibli hosilalari* deyiladi.

Agar $s = s(t)$ - to'g'ri chiziq bo'ylab harakat qilayotgan material nuqtaning harakat qonunini ifodalasa, u holda $s''(t)$ - shu nuqtaning t vaqt ichidagi tezlanishini ifodalaydi. Demak, ikkinchi tartibli hosilaning fizik ma'nosi – material nuqtaning tezlanishidan iborat ekan.

12.1-eslatma. $f(x)$ funksiyaning biror $x \in (a,b)$ nuqtadagi $f'(x)$ hosilasi mavjudligidan uning shu nuqtadagi yuqori tartibli hosilalarga ega bo'lishi har doim ham kelib chiqavermaydi. Masalan, $f(x) = \sqrt{x^5}$ funksiya $x \geq 0$ da, jumladan, $x = 0$ nuqtada ham, $f'(x) = \frac{5}{2}x^{3/2}$, $f''(x) = \frac{15}{4}x^{1/2}$

hosilalarga ega, lekin, bu funksiya $x=0$ nuqtada chekli uchinchi tartibli hosilaga ega emas.

$f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda aniqlangan bo'lib, ular $x\in(a,b)$ nuqtada n- tartibli $f^{(n)}(x)$, $g^{(n)}(x)$ hosilalarga ega bo'lsin. (Buni quyidagicha tushunish lozim: $f(x)$ va $g(x)$ funksiyalar x nuqtani o'z ichiga olgan $(\alpha,\beta)\subset(a,b)$ oraliqda $f', f'',...,f^{(n-1)}$ hamda $g', g'',...,g^{(n-1)}$ hosilalarga ega bo'lib, x nuqtada esa, $f^{(n)}(x)$, $g^{(n)}(x)$ hosilalarga ega). U holda

1) $[Cf(x)]^{(n)}=Cf^{(n)}(x)$, $C=const$;

2) $(f(x)\pm g(x))^{(n)}= f^{(n)}(x)\pm g^{(n)}(x)$;

3) $(f(x)\cdot g(x))^{(n)}= f^{(n)}(x)\cdot g(x)+C_n^1 f^{(n-1)}(x)\cdot g'(x) + C_n^2 f^{(n-2)}(x)\cdot g''(x)$ $+...+$

$$+C_n^k f^{(n-k)}(x)g^{(k)}(x)+...+f(x)g^{(n)}(x), \qquad (12.1)$$

bunda $C_n^k=\frac{n(n-1)...(n-k+1)}{k!}$. (12.1) formula *Leybnits formulasi* deyiladi.

Yuqori tartibli hosilalarni topishda quyidagi, asosiy elementar funksiyalarning n- tartibli hosilalarini topish formulalari muhim ahamiyatga ega:

1. $y=x^m$ $\qquad y^{(n)}=m(m-1)(m-2)...(m-n+1)x^{m-n}$.

Agar m butun son va $n>m$ bo'lsa, $y^{(n)}=0$ bo'ladi. Xususiy holda, $m=-1$ bo'lsa, $y=\frac{1}{x}$ funksiyaning n- tartibli hosilasi $y^{(n)}=\frac{(-1)^n m!}{x^{n+1}}$ bo'ladi.

2. $y=\ln x$, $\qquad y^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^n}$.

3. $y=\log_a x$, $\qquad y^{(n)}=(-1)^{n-1}\frac{(n-1)!}{\ln a}\frac{1}{x^n}$.

4. $y=e^{\lambda x}$, $\qquad y^{(n)}=\lambda^n e^{\lambda x}$.

5. $y=a^{bx}$, $\quad y^{(n)}=b^n a^x \ln^n a$.

6. $y=\sin bx$, $\quad y^{(n)}=b^n \sin(bx+n\frac{\pi}{2})$.

7. $y=\cos bx$, $\quad y^{(n)}=b^n \cos(bx+n\frac{\pi}{2})$

8. $y=(ax+b)^{\alpha}$, $\quad y^{(n)}=((ax+b)^{\alpha})^{(n)}=a^n\alpha(\alpha-1)\cdots(\alpha-n+1)(ax+b)^{\alpha-n}$.

12.1-misol. Ushbu

1). $y=\ln(x^2-3x+2)$; 2). $y=3^{5x}$;

3). $y=\dfrac{ax+b}{cx+d}$; 4). $y=x^3\sin x$; 5). $y=\operatorname{arctg} x$, $y^{(n)}(0)$

funksiyalarning n- tartibli hosilalarini toping.

Yechilishi. 1) $y=\dfrac{2x-3}{x^2-3x+2}$. Hisoblashlarni soddalashtirish uchun oxirgi funksiyani quyidagicha shakl almashtiramiz:

$$y'=\frac{2x-3}{x^2-3x+2}=\frac{x-2+x-1}{(x-1)(x-2)}=\frac{1}{x-1}+\frac{1}{x-2}=(x-1)^{-1}+(x-2)^{-1}.$$

Bundan 8) formulada $\alpha=-1$, $a=1$, $b=-1$, $b=-2$ deb olib,

$$\left(\frac{1}{x-1}\right)^{(n)}=(-1)(-2).....(-1-n+1)(x-1)^{-1-n},$$

$$\left(\frac{1}{x-2}\right)^{(n)}=(-1)(-2).....(-1-n+1)(x-2)^{-1-n},$$

bo'lishini topamiz. Demak,

$$y^{(n)}=(\ln(x^2-3x+2))^{(n)}=(-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n}+\frac{1}{(x+2)^n}\right].$$

2) $y'=3^{5x}\cdot\ln 3\cdot 5$, $\quad y''=(3^{5x}\cdot 5\cdot\ln 3)'=3^{5x}\cdot 5^2\ln^2 3$,

$y^{(3)}=3^{5x}5^3\cdot\ln^3 3$ va hokazo $y^{(n)}=3^{5x}\cdot 5^n\cdot\ln^n 3$ deb yozish mumkin. Bu formulaning to'g'riligini matematik induksiya usuli bilan ko'rsatamiz: Bu formula $n=1$ da o'rinli. Faraz qilaylik, $n=k$ da ham o'rinli bo'lsin, ya'ni $y^{(k)}=3^{5x}\cdot 5^k\cdot\ln^k 3$. U holda, $(y^{(k)})'=(3^{5x}\cdot 5^k\cdot\ln^k 3)'=3^{5x}\cdot 5^{k+1}\ln^{k+1}3$.

Demak, formula $n=k+1$ bo'lganda ham o'rinli ekan. Bundan, $\forall n$ uchun o'rinli ekanligi kelib chiqadi. Shu bilan biz, 5) formulaning ham to'g'riligini isbotladik.

3) Hisoblashni soddalashtirish maqsadida berilgan funksiyaning shaklini quyidagicha o'zgartiramiz:

$$y=\frac{ax+b}{cx+d}=\frac{a}{c}+\frac{bc-ad}{c(cx+d)}=\frac{a}{c}+\frac{bc-ad}{c}(cx-d)^{-1}.$$

Bundan 8) formulani e'tiborga olgan holda, $(x=-1,\ a=c,\ b=-d)$,

$$y^{(n)}=\frac{(-1)^n n!c^{n-1}}{(cx+d)^{n+1}}\cdot(bc-ad)$$

bo'lishini topamiz.

4) $u=\sin x,\ v=x^3$ deb olib, Leybnis (12.1) formulasidan foydalanib:

$$(x^3\sin x)^{(n)}=C_n^0x^3(\sin x)^{(n)}+C_n^1(x^3)'(\sin x)^{(n-1)}+$$
$$+C_n^2(x^3)''(\sin x)^{n-2}+C_n^3(x^3)'''(\sin x)^{n-3}$$

topamiz. Qolgan hadlar nolga teng bo'ladi, chunki $(x^3)^{(k)}=0$ $k>3$. sinx funksiyaning $n, n-1, n-2, n-3$ tartibli hosilalarini topishda 6) formuladan $b=1$ bo'lgan holda foydalanamiz:

$$(\sin x)^{(n)}=\sin(x+n\cdot\frac{\pi}{2}),\quad (\sin x)^{(n-1)}=\sin(x+(n-1)\cdot\frac{\pi}{2})=-\cos(x+\frac{n\pi}{2}),$$

$$(\sin x)^{(n-2)}=\sin(x+(n-2)\frac{\pi}{2})=-\sin(x+\frac{n\pi}{2}),\quad (\sin x)^{(n-3)}=\sin(x+(n-3)\cdot\frac{\pi}{2})=$$
$$=\cos(x+\frac{n\pi}{2}).$$

Shunday qilib,

$$y^{(n)}=(x^3\cdot\sin x)^{(n)}=x^3\cdot\sin(x+\frac{n\pi}{2})-3x^2n\cos(x+\frac{n\pi}{2})+$$
$$+\frac{6x\cdot n(n-1)}{2!}\sin(x+\frac{n\pi}{2})+\frac{n(n-1)(n-2)}{3!}\cdot 6\cos(x+\frac{n\pi}{2})=$$
$$=(x^3-3xn(n-1))\sin(x+\frac{n\pi}{2})+(n(n-1)(n-2)-3nx^2)\cos(x+\frac{n\pi}{2}).$$

5) $y' = \frac{1}{1+x^2}$, bundan $(1+x^2)y' = 1$. Keyingi tenglikning ikkala tomonidan $n-1$ tartibli hosila olamiz. Chap tomonning hosilasini topishda $u = y'$, $v = (1+x^2)$ deb olib, Leybnis formulasini qo'llaymiz;

$$y^{(n)}(1+x^2) + 2(n-1)xy^{(n-1)} + (n-1)(n-2)y^{(n-2)} = 0.$$

Bundan $x = 0$ bo'lganda, ushbu $y^{(n)}(0) = -(n-1)(n-2)y^{(n-2)}(0)$ rekurrent formulaga ega bo'lamiz. n juft $(n = 2k)$ bo'lganda, $y^{(2k)}(0) = 0$ n toq $(n = 2k+1)$ bo'lganda esa $y^{(2k+1)}(0) = -(2k)(2k-1)y^{(2k-1)}(0) = ...$

$... = (-1)^k (2k)! y'(0) = (-1)^k (2k)!$, chunki $y'(0) = 1$.

12.2-misol. Ushbu $y = x + x^3$, $x \in R$ funksiyaga teskari bo'lgan funksiyaning ikkinchi tartibli hosilasini toping.

Yechilishi. Berilgan funksiya hamma joyda uzluksiz,

$$y'_x = 1 + 3x^2 > 0, \qquad y'_x = 1 + 3x^2 \neq 0, \qquad \forall x \in R$$

bo'lgani uchun unga teskari funksiya mavjud va uning hosilasi

$$x'_y = \frac{1}{y'_x} = \frac{1}{1+3x^2}.$$

Bu tenglikning ikkala tomonidan y bo'yicha hosila olamiz:

$$x''_{yy} = \left(\frac{1}{1+3x^2}\right)' \cdot x'_y = \frac{-6x}{(1+3x^2)^3}.$$

12.3-misol. $y = f(x)$ funksiya ushbu

$x = x(t)$, $y = y(t)$, $t \in (a,b)$

parametrik tenglama bilan berilgan bo'lib, $x(t)$ va $y(t)$ funksiyalar $\forall t \in (a,b)$ uchun ikki marta differensiallanuvchi va $x'(t) \neq 0$ bo'lsin. U holda y''_{xx} ni toping.

Yechilishi. Ma'lumki, berilgan funksiyaning birinchi tartibli hosilasi

$$y'_x = \frac{y'_t}{x'_t}$$

formula orqali topiladi. Bu tenglikning ikkala tomonidan x bo'yicha hosila olamiz:

$$y''_{xx}=\left(\frac{y'_t}{x'_t}\right)'_t\cdot t'_x=\left(\frac{y'_t}{x'_t}\right)'_t\cdot\frac{1}{x'_t},$$

$$y''_{xx}=\frac{x'_t\cdot y''_{tt}-y'_t\cdot x''_{tt}}{(x'_t)^3}. \qquad (12.2)$$

12.4-misol. Quyidagi: 1). $x=t^3,\ \ y=t^2$, 2). $x=a(t-\sin t),\ y=a(1-\cos t)$ parametrik tenglamalar berilgan $y(x)$ funksiyalarning y''_{xx} hosilalarini toping.

Yechilishi. Berilgan funksiyalarning y''_{xx} hosilalari (12.2) formula orqali topiladi:

1) $x'_t=3t^2,\ x''_{tt}=6t;\ \ y'_t=2t,\ y''_{tt}=2,\ y''_{xx}=\dfrac{x'_t\cdot y''_{tt}-y'_t\cdot x''_{tt}}{(x'_t)^3}=\dfrac{6t^2-12t^2}{27\cdot t^6}=-\dfrac{2}{9t^4}.$

2) $x'_t=a(1-\cos t),\ \ x''_{tt}=a\sin t;\ \ y'_t=a\sin t,\ \ y''_{tt}=a\cos t,\ \ y''_{xx}=-\dfrac{1}{4a\sin^4\frac{t}{2}}(t\neq 2\pi n,\ n\in Z).$

12.5-misol. Parametrik shaklda berilgan ushbu

$$x=\frac{2t-t^2}{t-1},\qquad y=\frac{t^2}{t-1}.$$

$y(x)$ funksiyaning $(0;4)$ nuqtadagi y''_{xx} hosilasini toping.

Yechilishi. Berilgan $y(x)$ funksiyaning y''_{xx} hosilasini (12.2) formula orqali topamiz:

$$x'_t=\frac{t^2-2t+2}{(t-1)^2},\quad x''_{tt}=\frac{2}{(t-1)^3};\quad y'_t=\frac{t^2-2t}{(t-1)^2},\quad y''_{tt}=\frac{2}{(t-1)^3}.$$

Shartga ko'ra $x=0,\ y=4$ bo'lgani uchun

$$\begin{cases}2t-t^2=0,\\ 4t-4-t^2=0\end{cases}\Rightarrow\begin{cases}t(2-t)=0,\\ 4t-4-t^2=0,\end{cases}$$

bunda $t=2$ yechim bo'ladi. Endi $x'_t, x''_{tt}, y'_t, y''_{tt}$ larning $t=2$ dagi qiymatini topamiz:

$$x_t'(2)=-2,\ x_t''(2)=2;\ y_t'(2)=0,\ y_t''(2)=2.$$

Topilgan qiymatlarni (12.2) formulaga qo'yib, $y''_{xx}\big|_{(0,4)}=\frac{1}{2}$ ekanligini topamiz.

12.6-misol. Oshkormas shaklda berilgan ushbu $x^2+y^2=25$ $y(x)$ funksiyaning $(3;4)$ nuqtadagi y'_x, y''_{xx} hosilalarini toping.

Yechilishi. Berilgan funksiyaning y'_x hosilasi ushbu

$$\frac{d}{dx}(x^2+y^2-25)=0$$

tenglamadan topiladi: $2x+2yy'_x=0$. Bu tenglamadan:

$$y'_x=-\frac{x}{y}. \tag{12.3}$$

(12.3) ning ikkala tomonida x bo'yicha hosila olamiz: $y''_{xx}=-\frac{y-xy'_x}{y^2}$,

(12.3) ni e'tiborga olgan holda,

$$y''_{xx}=-\frac{y-\frac{x^2}{y}}{y^2}=-\frac{y^2+x^2}{y^3}=-\frac{25}{y^3}. \tag{12.4}$$

munosabatni hosil qilamiz.

Shartga ko'ra $x=3,\ y=4$ bo'lgani uchun (12.3), (12.4) lardan

$$y'_x\big|_{(3,4)}=-\frac{3}{4},\ y''_{xx}\big|_{(3,4)}=-\frac{25}{64}.$$

12.7-misol. Ushbu $y=|x|^2$ funksiyaning $x=0$ nuqtada nechanchi tartibgacha hosilaga ega ekanligini aniqlang.

Yechilishi. $x\neq 0$ bo'lsa, $y'_x=\begin{cases}2x, & x>0 \text{ bo'lganda,}\\ -2x, & x<0 \text{ bo'lganda.}\end{cases}$

$x=0$ nuqtada hosilaning ta'rifiga ko'ra,

$$y'(0)=\lim_{\Delta x\to 0}\frac{y(0+\Delta x)-y(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|^2}{\Delta x}=0.$$

Demak, berilgan funksiya $\forall x$ uchun birinchi tartibli hosilaga ega:

$$y'(x)=2|x|.$$

Ma'lumki, $|x|$ funksiya $x=0$ nuqtada differensiallanuvchi emas. Shuning uchun $y=|x|^2$ funksiya $x=0$ nuqtada faqat birinchi tartibli hosilaga ega bo'lib, ikkinchi tartibli hosilaga ega emas.

12.8-misol. Ikkita moddiy nuqta berilgan bo'lib, ulardan biri $s_1(t)=t^3+\frac{t^2}{2}+t+\frac{1}{2}$ qonun bo'yicha, ikkinchisi esa $s_2(t)=\frac{2}{3}t^3+3t^2-5t$ (s_1, s_2- metrda, t-sekundda o'lchanadi) qonun bo'yicha harakatlanadi. Bu moddiy nuqtalarning tezliklari o'zaro teng bo'lgan nuqtadagi tezlanishini toping.

Yechilishi. Ma'lumki, harakatlanayotgan moddiy nuqtalarning tezliklari, mos ravishda, $v_1(t)=s_1'(t)$, $v_2(t)=s_2'(t)$ formulalar bilan topiladi:

$$v_1(t)=s_1'(t)=3t^2+t+1,\quad v_2(t)=s_2'(t)=2t^2+6t-5.$$

Shartga ko'ra, $3t^2+t+1=2t^2+6t-5$, bundan $t^2-5t+6=0$, $t_1=2$, $t_2=3$.

1) $t=2$ sekundda birinchi va ikkinchi moddiy nuqtalarning tezlanishlari:

$$a_1(t)\big|_{t=2}=v_1'(t)\big|_{t=2}=(6t+1)\big|_{t=2}=13\frac{m}{sek^2},\ a_2(t)\big|_{t=2}=v_2'(t)\big|_{t=2}=(4t+6)\big|_{t=2}=14\frac{m}{sek^2}$$

2) $t=3$ sekundda:

$$a_1(t)\big|_{t=3}=v_1'(t)\big|_{t=3}=(6t+1)\big|_{t=3}=19\frac{m}{sek^2},\ a_2(t)\big|_{t=3}=v_2'(t)\big|_{t=3}=(4t+6)\big|_{t=3}=18\frac{m}{sek^2}.$$

12.2. Funksiyaning yuqori tartibli differensiali. $y=f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin. Agar $f(x)$ funksiya $x\in(a,b)$ nuqtada chekli $f'(x)$ hosilaga ega bo'lsa, funksiyaning differensiali ushbu $dy=f'(x)dx$ formula bilan hisoblanishini 11-§ ning 6-bandida ko'rdik, bunda dx

miqdor $f(x)$ funksiya argumenti x ning ixtiyoriy Δx orttirmasini ifodalaydi.

Faraz qilaylik, $f(x)$ funksiya $x\in(a,b)$ nuqtada ikkinchi tartibli hosilaga ega bo'lsin. U holda, belgilangan dx lar uchun funksiyaning differensiali dy faqat x ning funksiyasi bo'ladi va uning differensialini hisoblash mumkin, bunda ham $dx=\Delta x$ deb olinadi.

12.2-ta'rif. $f(x)$ funksiya differensiali dy ning $x\in(a,b)$ nuqtadagi differensialiga berilgan $f(x)$ funksiyaning *ikkinchi tartibli differensiali* deb ataladi va u d^2y ёки $d^2f(x)$ kabi belgilanadi, ya'ni $d^2y=d(dy)$ ёки $d^2f(x)=d(df(x))$.

Differensiallash qoidasidan foydalanib,

$$d^2y=d(dy)=d(y'd(x))=d(y')dx=y''(dx)^2$$

bo'lishini topamiz

Shunday qilib, funksiyaning ikkinchi tartibli differensiali uning ikkinchi tartibli hosilasi orqali quyidagicha yoziladi:

$$d^2y=y''dx^2, \qquad (12.5)$$

bunda $dx^2=dx\cdot dx=(dx)^2$. Xuddi shunday, x — erkli o'zgaruvchi bo'lganda, funksiyaning n- tartibli differensiali ta'rifini berish mumkin.

$f(x)$ funksiya $x\in(a,b)$ nuqtada n-tartibli $f^{(n)}(x)$ hosilaga ega bo'lsin. Funksiyaning $(n-1)$ tartibli differensiali $d^{n-1}y$ dan olingan differensial, berilgan $f(x)$ funksiyaning $x\in(a,b)$ nuqtadagi n- *tartibli differensiali* deb ataladi va u d^ny ёки $d^nf(x)$ kabi belgilanadi, ya'ni

$$d^ny=d(d^{n-1}y) \text{ ёки } d^nf(x)=d(d^{n-1}f(x)), \quad d^ny=y^{(n)}dx^n. \qquad (12.6)$$

Erkli o'zgaruvchi x ning n - tartibli differensiali $n>1$ da, ta'rif bo'yicha $d^nx=0$ deb olinadi. $f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda berilgan bo'lib, ular $x\in(a,b)$ nuqtada differensialga ega bo'lsin. U holda quyidagi

1) $d^n[Cf(x)] = Cd^n f(x),\ C = const;$

2) $d^n[f(x) \pm g(x)] = d^n f(x) \pm d^n g(x),$

3) $d^n[f(x)\cdot g(x)] = d^n[f(x)]\cdot g(x) + C_n^1 d^{n-1}[f(x)]\cdot d[g(x)] + \ldots +$
$+ C_n^k d^{n-k}[f(x)]\cdot d^k[g(x)] + \ldots + f(x)\cdot d^n[g(x)],$

formulalar o'rinli bo'ladi, bunda $C_n^k = \dfrac{n(n-1)..(n-k+1)}{k!}$.

$u = f(x)$ funksiya (a,b) oraliqda, $y = F(u)$ funksiya esa, (c,d) oraliqda berilgan bo'lib, ular yordamida $y = F(f(x))$ murakkab funksiya tuzilgan bo'lsin. $u = f(x)$ funksiya $x \in (a,b)$ nuqtada $f'(x)$, $F(u)$ funksiya esa, mos $u \in (c,d)$ nuqtada $F'(u)$ hosilaga ega deb, $y = F(f(x))$ funksiyaning differensialini hisoblaymiz:

$$dy = F'(f(x))\cdot f'(x)dx = F'(f(x))\cdot df(x).$$

12.2.-eslatma. (12.5) va (12.6) formulalar $n > 1$ bo'lganda faqat x - erkli o'zgaruvchi bo'lgan holda o'rinli.

x erksiz o'zgaruvchi bo'lgan holda, ya'ni $y = y(x(t))$ murakkab funksiya uchun (12.5) formula ushbu:

$$d^2 y = d(dy) = d(y'_x dx) = d(y'_x)\cdot dx + y'_x d(dx) = y''_{xx} dx^2 + y'_x d^2 x \qquad (12.7)$$

ko'rinishda bo'ladi.

Agar x erkli o'zgaruvchi bo'lsa, $d^2x = 0$ bo'ladi va (12.5) formula (12.7) formulaga teng bo'ladi.

12.9-misol. x ni erkli o'zgaruvchi deb, ushbu $y = xe^{x^2}$ funksiyaning uchinchi tartibli differensialini toping.

Yechilishi. 1-usul. Ikkinchi tartibli differensialning ta'rifiga ko'ra:

$$d^2y=d(dy)=d(xde^{x^2}+e^{x^2}dx)=d(2x^2e^{x^2}dx+e^{x^2}dx)=$$
$$=2d(x^2e^{x^2})\cdot dx+d(e^{x^2})\cdot dx=$$
$$=2\left[d(e^{x^2})\cdot x^2+e^{x^2}d(x^2)\right]dx+d(e^{x^2})dx= \qquad (12.8)$$
$$=4e^{x^2}\cdot x^3dx^2+4xe^{x^2}dx^2+2xe^{x^2}dx^2=$$
$$=(6e^{x^2}x+4e^{x^2}\cdot x^3)dx^2=2e^{x^2}(3x+2x^3)dx^2.$$

Uchinchi tartibli differensialning ta'rifiga ko'ra:

$$d(d^2y)=d\left[2e^{x^2}(3x+2x^3)dx^2\right]=2d\left[e^{x^2}(3x+2x^3)dx^2\right]=$$
$$=2\left[e^{x^2}d(3x+2x^3)+(3x+2x^3)d(e^{x^2})\right]dx^2=$$
$$=2\left[e^{x^2}(3+6x^2)+(3x+2x^3)2xe^{x^2}\right]dx^3=2e^{x^2}(3+12x^2+4x^4)dx^3.$$

2-usul. Berilgan funksiyaning uchinchi tartibli hosilasini topamiz: (12.8) ni e'tiborga olib,

$$y'''=(xe^{x^2})'''=\left[2e^{x^2}3x+2x^3\right]'=$$
$$=2\left[(e^{x^2})'(3x+2x^3)+e^{x^2}(3x+2x^3)'\right]=2e^{x^2}(3+12x^2+4x^4).$$

(12.6) formulaga asosan $(n=3)$, $d^3y=2e^{x^2}(3+12x^2+4x^4)dx^3$ bo'lishini topamiz.

MAPLE tizimidan foydalanib misolni yechish:

> Diff(x*exp(x^2),x$3)=diff(x*exp(x^2),x$3);

$$\frac{d^3}{dx^3}\left(x\cdot e^{x^2}\right)=6e^{(x^2)}+24x^2e^{(x^2)}+8x^4e^{(x^2)}.$$

12.10-misol. Quyidagi: *a)* x biror erkli o'zgaruvchining funksiyasi; *b)* x erkli o'zgaruvchi bo'lgan hollarda, $y=e^{\sqrt{x}}$ funksiyaning ikkinchi tartibli differensialini toping.

Yechilishi. a) 1-usul. Ikkinchi tartibli differensialning ta'rifiga ko'ra:

$$d^2y=d\left(d\left(e^{\sqrt{x}}\right)\right)=d\left(\frac{1}{2\sqrt{x}}e^{\sqrt{x}}\right)dx=\frac{e^{\sqrt{x}}}{2\sqrt{x}}d^2x+\frac{e^{\sqrt{x}}}{4x}\left(1-\frac{1}{\sqrt{x}}\right)dx^2.$$

2-usul. Berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$y'_x = \frac{e^{\sqrt{x}}}{2\sqrt{x}},\ y''_{xx} = \frac{e^{\sqrt{x}}}{2x}\left(\frac{1}{2} - \frac{2}{3\sqrt{x}}\right).$$

(12.7) formulaga asosan:

$$d^2y = \frac{e^{\sqrt{x}}}{2\sqrt{x}}d^2x + \frac{e^{\sqrt{x}}}{4x}\left(1 - \frac{1}{\sqrt{x}}\right)dx^2.$$

b) Bu holda $d^2x = 0$, demak, unda $d^2y = \frac{e^{\sqrt{x}}}{4x}\left(1 - \frac{1}{\sqrt{x}}\right)dx^2$ bo'ladi.

12.11-misol. Ushbu $y = \sqrt{1-x^2}\arcsin x$ funksiyaning ikkinchi tartibli differensialini toping.

Yechilishi. Ikkinchi tartibli differensialni topish uchun (12.5) munosabatdan foydalanamiz:

$$y' = \frac{-x}{\sqrt{1-x^2}}\arcsin x + 1,$$

$$d^2y = y''dx^2 = \left(-\frac{x}{1-x^2} - \frac{1}{\sqrt{(1-x^2)^3}}\arcsin x\right)dx^2$$

$$y'' = -\frac{x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{(1-x^2)}}\arcsin x,$$

MAPLE tizimidan foydalanib misolni yechish:

```
> X:=subs(D(arcsin(x))=diff(arcsin(x),x)*D(x),
D(sqrt(1-x^2)*arcsin(x)));
```

$$X := -\frac{D(x)\,x\arcsin(x)}{\sqrt{1-x^2}} + D(x)$$

```
> F:=subs(D(D(X))=0,D(arcsin(x))=
diff(arcsin(x),x)*D(x),D(X));
```

$$F := -\frac{D(x)^2\arcsin(x)}{\sqrt{1-x^2}} - \frac{D(x)^2x^2\arcsin(x)}{(1-x^2)^{(3/2)}} - \frac{D(x)^2x}{1-x^2}$$

```
> simplify(F);
```

$$-\frac{(x\sqrt{1-x^2} + \arcsin(x))\,D(x)^2}{(1-x^2)^{(3/2)}}$$

12.12-misol. Ushbu $y=\sin 5x\cdot\cos 2x$ funksiyaning n- tartibli differensialini toping.

Yechilishi. Berilgan funksiyani $y=\frac{1}{2}[\sin 7x+\sin 3x]$ ko'rinishda ifodalaymiz. Bu funksiyaning n-tartibli differensialini topish uchun $y^{(n)}=(\sin bx)^{(n)}=b^n\sin(bx+n\frac{\pi}{2})$ va (12.6) formulalardan foydalanamiz:

$$d^n y=y^{(n)}dx^n=\frac{1}{2}\left[7^n\sin\left(7x+\frac{n\pi}{2}\right)+3^n\sin\left(3x+\frac{n\pi}{2}\right)\right]dx^n.$$

12.13-misol. Ushbu $y=\frac{3x+2}{x^2-2x+5}$ funksiyaning $x=0$ nuqtadagi ikkinchi tartibli differensialini toping.

Yechilishi. Berilgan funksiyaning ikkinchi tartibli differensialini (12.6) formula bo'yicha $(n=2)$ topamiz:

$$y'_x=\frac{-3x^2-4x+19}{(x^2-2x+5)^2},\quad y''_{xx}=\frac{6x^3+12x^2-114x+56}{(x^2-2x+5)^3}.$$

(12.6) formulaga asosan: $d^2y\big|_{x=0}=y''_{xx}\big|_{x=0}dx^2=\frac{56}{125}dx^2.$

Mustaqil yechish uchun misol va masalalar

Quyidagi funksiyalarning ko'rsatilgan tartibdagi hosilalarini toping.

12.1. $y=x(2x-1)^2(x+3)^3$, $y^{(7)}=?$ **12.2.** $y=\frac{1}{1+x^3}$, $y''=?$

12.3. $y=1+10x+\frac{1}{x^{98}}$, $y''=?$ **12.4.** $y=\cos^2 x$, $y''=?$

12.5. $y=(1+x^2)\text{arctg}\,x$, $y''=?$ **12.6.** $y=\text{arctg}(x+\sqrt{x^2+1})$, $y''=?$

12.7. $y=\sqrt{1-x^2}\arcsin x$, $y''=?$ **12.8.** $y=\ln(x+\sqrt{x^2+1})$, $y''=?$

12.9. $y=\sqrt[3]{x^3}$, $y'''=?$ **12.10.** $y=x^5\ln x$, $y'''=?$

12.11. $y=x^2\cos 3x$, $y^{IV}=?$ **12.12.** $y=x^2\sin 2x$, $y^{(50)}=?$

12.13. $y=\frac{e^x}{x}$, $y^{(10)}=?$ **12.14.** $y=\sin x\sin 2x\sin 3x$, $y^{(10)}=?$

12.15. $y=\sin^2 x\ln x$, $y^{(6)}=?$ **12.16.** $y=x^2e^{2x}$, $y^{(50)}=?$

12.17. $y=a^{3x},\quad y^{(n)}=?$

12.18. $y=\dfrac{1+x}{1-x},\quad y^{(n)}=?$

12.19. $y=\sin ax+\cos bx,\quad y^{(n)}=?$

12.20. $y=\dfrac{\alpha x+\beta}{\gamma x+\delta},\quad y^{(n)}=?$

12.21. $y=\dfrac{x}{a+bx},\quad y^{(n)}=?$

12.22. $y=\dfrac{1}{x(x+1)},\quad y^{(n)}=?$

12.23. $y=\sin x\cos x,\quad y^{(n)}=?$

12.24. $y=\sin 5x\cos 2x,\quad y^{(n)}=?$

12.25. $y=\log_a x,\quad y^{(n)}=?$

12.26. $y=\sin^4 x+\cos^4 x,\quad y^{(n)}=?$

12.27. $y=\sin 3x\cos^2 x,\quad y^{(n)}=?$

Quyidagi funksiyalarning berilgan nuqtalardagi ko‘rsatilgan tartibdagi hosilalarini toping.

12.28. $y=x^4-4x^3+4,\quad y^{(n)}(1)=?$

12.29. $y=\dfrac{x^3}{(x-1)^4},\quad y^{(n)}(5)=?$

12.30. $y=\dfrac{\arcsin x}{\sqrt{1+x^2}},\quad y^{(n)}(0)=?$

12.31. $y=\mathrm{arctg}\, x,\quad y^{(n)}(1)=?$

12.32. $y=e^{\sqrt{x}},\quad y^{(n)}(4)=?$

12.33. $y=\dfrac{1+x}{\sqrt{1-x}},\quad y^{(100)}(x)=?$

12.34. $y=\dfrac{1}{1-x},\quad y^{(n)}(x)=?$

Quyidagi funksiyalarning ko‘rsatilgan tenglamalarni qanoatlantirishini isbotlang.

12.35. $y=e^x\sin x,\quad y''-2y'+2y=0;$

12.36. $y=e^{-x}\sin x,\quad y''+2y'+2y=0;$

12.37. $y=c_1\cos x+c_2\sin x,\quad y''+y=0$ (c_1 va c_2 - ixtiyoriy o‘zgarmas sonlar);

12.38. $y=c_1 chx+c_2 shx,\quad y''-y=0$ (c_1 va c_2 - ixtiyoriy o‘zgarmas sonlar);

12.39. $y=A\sin(\omega t+\omega_0)+B\cos(\omega t+\omega_0),\quad \dfrac{d^2y}{dt^2}+\omega^2 y=0$ (A, B, ω, ω_0 - ixtiyoriy o‘zgarmas sonlar);

12.40. $y=\sin(n\arcsin x),\quad (1-x^2)y''-xy'+n^2y=0;$

12.41. $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$, $y'' - (\lambda_1 + \lambda_2) y' + \lambda_1 \lambda_2 y = 0$ (c_1, c_2 - ixtiyoriy o'zgarmas sonlar, λ_1, λ_2 - o'zgarmas);

12.42. $y = \cos(m \ln x)$, $x^2 y^{(n+2)} + (2n+1) x y^{(n+1)} + (n^2 + m^2) y^{(n)} = 0$;

Berilgan $y = y(x)$ funksiyalarning quyidagi tenglamalarni qanoatlantirishini aniqlang:

12.43. $y = A \cos ax + B \sin ax$, $y'' + a^2 y = 0$.

12.44. $y = Ae^x + Be^{-x} - \dfrac{1}{x}$, $y'' - y = \dfrac{x^2 - 2}{x^3}$.

12.45. $y = 1 + \cos e^x + \sin e^x$, $y'' - y' + e^{2x} y = 0$.

Quyidagi parametrik shaklda berilgan $y = y(x)$ funksiyalarning ko'rsatilgan tartibdagi hosilalarini toping:

12.46. $x = at^2$, $y = bt^3$; $x''_{yy} = ?$

12.47. $x = t^3 + 3t + 1$, $y = t^3 - 3t + 1$; $y''_{xx} = ?$

12.48. $x = e^{\alpha t} \cos \beta t$, $y = e^{\alpha t} \sin \beta t$; $y''_{xx} = ?$

12.49. $x = a(\cos t - \ln ctg \dfrac{t}{2})$, $y = a \sin t$; $y''_{xx} = ?$

12.50. $x = t\,ch\,t - sh\,t$, $y = t\,sh\,t - ch\,t$; $y''_{xx} = ?$

12.51. $x = a \cos^3 t$, $y = a \sin^3 t$; $y'''_{xxx} = ?$

12.52. $x = a(1 + \cos t)\cos t$, $y = a(1 - \cos t)\sin t$; $y''_{x^2} = ?$

12.53. $x = \ln t$, $y = t^2 - 1$; $y'_{x^2} = ?$

12.54. $x = 2t \cos t + (t^2 - 2)\sin t$, $y = 2t \sin t - (t^2 - 2)\cos t$; $y''_{x^2} = ?$

12.55. $x = 2^{\cos^2 t}$, $y = 2^{\sin^2 t}$; $y''_{x^2} = ?$

Quyidagi parametrik shaklda berilgan $y = y(x)$ funksiyalarning berilgan nuqtada ko'rsatilgan tartibdagi hosilalarini toping:

12.56. $x = \ln(1 + \sin \varphi)$, $y = \ln(1 - \cos 2\varphi)$; $(\ln(\dfrac{3}{2}), \ln(\dfrac{1}{2}))$; $y''_{xx} = ?$

12.57. $x = ch\,t \sin t + sh\,t \cos t$, $y = ch\,t \cos t - sh\,t \sin t$; $(0,1)$; $y''_{xx} = ?$

Parametrik shaklda berilgan $y=y(x)$ funksiyalarning berilgan tenglamalarni qanoatlantirishini isbotlang:

12.58. $x=e^t\sin t,\quad y=e^t\cos t;\quad y''(x+y)^2=2(xy'-y).$

12.59. $x=\sin t,\quad y=\sin kt;\quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+k^2y=0.$

12.60. $x=\sin t,\quad y=Ae^{\sqrt{2}t}+Be^{-\sqrt{2}t};\quad (1-x^2)y''-xy'-2y=0,\ -\frac{\pi}{2}<t<\frac{\pi}{2},\ A$ va B - ixtiyoriy o‘zgarmas sonlar.

Quyidagi oshkormas shaklda berilgan $y=y(x)$ funksiyalarning x bo‘yicha ko‘rsatilgan tartibdagi hosilalarini toping:

12.61. $y^2=2px,\ y''_{x^2}=?$

12.62. $e^{x-y}=x+y,\ y''_{x^2}=?$

12.63. $\operatorname{arctg} y-y+x=0,\ y''_{x^2}=?$

12.64. $e^x-e^y=y-x,\ y''_{x^2}=?$

12.65. $x^2+5xy+y^2-2x+y-6=0,$ $(1,1)$ nuqtadagi $y''_{x^2}=?$

Quyidagi funksiyalarning $x=0$ nuqtada nechanchi tartibli hosilalarga ega ekanligini aniqlang va mavjud hosilalarning bu nuqtadagi qiymatini hisoblang:

12.66. $y=\begin{cases}1-\cos x, & x<0 \text{ bo'lganda},\\ \ln(1+x)-x, & x\geq 0 \text{ bo'lganda}.\end{cases}$

12.67. $y=\begin{cases}shx-x, & x<0 \text{ bo'lganda},\\ x-\sin x, & x\geq 0 \text{ bo'lganda}.\end{cases}$

12.68. $y=\begin{cases}shx, & x<0 \text{ bo'lganda},\\ \sin x\, chx, & x\geq 0 \text{ bo'lganda}.\end{cases}$

12.69. $y=\begin{cases}x^{10}, & x - \text{rasional son bo'lganda},\\ -x^{10}, & x - \text{irrasional son bo'lganda}.\end{cases}$

12.70. $y=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq 0 \text{ bo'lganda},\\ 0, & x=0 \text{ bo'lganda}.\end{cases}$

x ni erkli o‘zgaruvchi deb, quyidagi $y=y(x)$ funksiyalarning ko‘rsatilgan tartibdagi differensiallarini toping:

12.71. $y=(x+1)^3(x-1)^2,\ d^2y=?$

12.72. $y=(x^3+2x^2+x+3)e^{-2x},\ d^2y=?$

12.73. $y=\sin^2 x,\ d^4y=?$

12.74. $y=x\cos 2x,\ d^{12}y=?$

12.75. $y=\operatorname{arctg}\left(\frac{b}{a}tgx\right);\ d^2y=?$

12.76. $y=\cos x\, ch x;\ d^4y=?$

12.77. $y=\sqrt{\ln^2 x-4};\ d^2y=?$

12.78. $y=3^{-x^2};\ d^3y=?$

Agar du, d^2u, dv, dv^2 lar mavjud bo'lsa, quyidagi $y=y(x)$ funksiyalar uchun d^2y ni toping.

12.79. $y=\sqrt{u^2+v^2};\ d^2y=?$

12.80. $y=u^v;\ d^2y=?$

12.81. $y=\frac{2u+v}{u};\ d^2y=?$

12.82. $y=u\ln v;\ d^2y=?$

Quyidagi $y=y(x)$ funksiyalarning berilgan nuqtadagi ko'rsatilgan tartibdagi differensiallarini toping:

12.83. $y=xe^{x^2};\ d^2y\big|_{x=1}=?$

12.84. $y=\cos^2 x;\ d^3y\Big|_{x=\frac{\pi}{4}}=?$

12.85. $y=x\sqrt[3]{(x-5)^2};\ d^2y\big|_{x=-3}=?$

12.86. $y=\frac{1}{ax+b};\ d^ny\big|_{x=0}=?$

12.87. $y=\left(\sqrt{x^2-1}+\sqrt{x^2-1}\right)^7;\ d^{16}y\big|_{x=1}=?$

12.88. $y=\sin x\sin 2x\sin 3x;\ d^{10}y\Big|_{x=\frac{\pi}{6}}=?$

Oshkormas shaklda berilgan $y=y(x)$ funksiyalarning $(x_0;y_0)$ nuqtadagi ikkinchi tartibli differensialini toping:

12.89. $x^2+2xy+y^2-4x+2y-2=0,\ (1;1).$

12.90. $3(y-x+1)+arctg(y/x)=0,\ (1;0).$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

12.1. 0. **12.2.** $\frac{6x(2x^3-1)}{(x^3+1)^3}$. **12.3.** $9702/x^{100}$. **12.4.** $-2\cos 2x$. **12.5.** $\frac{2x}{1+x^2}+2arctgx$. **12.6.** $-x(1+x^2)^{-2}$. **12.7.** $-\frac{\arcsin x+x\sqrt{1-x^2}}{\sqrt{(1-x^2)^3}}$.

12.8. $-x(x^2+1)^{-3/2}$. **12.9.** $\frac{42}{125}x^{-\frac{13}{5}}$. **12.10.** $x^2(60\ln x+47)$.

12.11. $27(3x^2\cos 3x+8x\sin 3x-4\cos 3x)$.

12.12. $2^{50}(-x^2\sin 2x+50x\cos 2x+\frac{1225}{2}\sin 2x)$.

12.13. $\frac{e^x}{x}-10\frac{e^x}{x^2}+90\frac{e^x}{x^3}-720\frac{e^x}{x^4}+5040\frac{e^x}{x^5}-30240\frac{e^x}{x^6}+151200\frac{e^x}{x^7}-604800\frac{e^x}{x^8}+$
$+1814400\frac{e^x}{x^9}-3628800\frac{e^x}{x^{10}}+3628800\frac{e^x}{x^{11}}$. **12.14.**
$-2^9\sin 2x-2^{18}\sin 4x+2^8 3^{10}\sin 6x$.

12.15. $-60/x^6+\left(\frac{144}{x^5}-\frac{160}{x^3}-\frac{96}{x}\right)\sin 2x+\left(\frac{60}{x^6}-\frac{180}{x^4}+\frac{120}{x^2}+32\ln x\right)\cos 2x$.

12.16. $2^{49}e^{2x}(2x^2+10x+1225)$. **12.17.** $27a^{3x}\ln^3 a$. **12.18.** $2(n!)(1-x)^{-n-1}$.

12.19. $a^n\sin(ax+n\frac{\pi}{2})+b^n\cos(bx+n\frac{\pi}{2})$.

12.20. $(-1)^{n-1}n!(\alpha\delta-\beta\gamma)\gamma^{n-1}(\gamma x+\delta)^{-n-1}$.

12.21. $(-1)^{n-1}n!ab^{n-1}(a+bx)^{-n-1}$. **12.22.** $(-1)^n n![x^{-n-1}-(x+1)^{-n-1}]$.

12.23. $2^{n-1}\sin(2x+n\frac{\pi}{2})$. **12.24.** $\frac{1}{2}[7^n\sin(7x+n\frac{\pi}{2})+3^n\sin(3x+n\frac{\pi}{2})]$.

12.25. $(-1)^{n-1}\frac{(n-1)!}{x^n\ln a}$. **12.26.** $4^{n-1}\cos(4x+n\frac{\pi}{2})$.

12.27. $\frac{1}{4}\sin(x+n\frac{\pi}{2})+\frac{3^n}{2}\sin(3x+n\frac{\pi}{2})+\frac{5^n}{4}\sin(5x+n\frac{\pi}{2})$.

12.28. 360. **12.29.** $\frac{625}{1024}$. **12.30.** 0. **12.31.** $-\frac{1}{2}$. **12.32.** $e^{2/32}$.

12.33. $\frac{1\cdot 3\cdot 5\cdot\ldots\cdot 197(399-x)}{2^{100}(1-x)^{201/2}}$. **12.34.** $\frac{5!}{(1-x)^6}$. **12.46.** $-\frac{2a}{9b^2t^4}$.

12.47. $\frac{4t}{3(t^2+1)^3}$. **12.48.** $\frac{(\alpha^2+\beta^2)\beta e^{-\alpha t}}{(\alpha\cos\beta t-\beta\sin\beta t)^3}$. **12.49.** $\frac{\sin t(1+3\sin^2 t)}{a^2\cos^7 t}$.

12.50. $-1/(tsh^3 t)$. **12.51.** $\frac{5\cos^2 t-4}{9a^2\sin^3 t\cos^7 t}$. **12.52.** $\frac{3}{16a^2}\frac{\cos 2t}{\left(\sin\frac{3t}{2}\right)^3\left(\cos\frac{t}{2}\right)^3}$.

12.53. $4t^3$. **12.54.** $\frac{1}{(at+b)\cos^3 t}$. **12.55.** $2^{3\sin^2 t-1}$. **12.56.** -12.

12.57. $-\frac{1}{2}$. **12.61.** $-p^2/y^3$. **12.62.** $4(x+y)/(x+y+1)^3$.

12.63. $-\frac{2(1+y^2)}{y^5}$. **12.64.** $(e^x-e^y)(1-e^{x+y})/(1+e^y)^3$.

12.65. $\frac{111}{256}$. **12.66.** $y'(0)=0$, $y''(0)$ mavjud emas.

12.67. $y'(0)=0$, $y''(0)=0$, $y'''(0)=1$, $y^{(IV)}(0)=0$, $y^{(V)}(0)$ mavjud emas. **12.68.** $y'(0)=1$, $y''(0)=0$, $y'''(0)$ mavjud emas. **12.69.** $y'(0)=0$, $y''(0)$ mavjud emas. **12.70.** $y^{(n)}(0)=0, n\in N$. **12.71.** $4(x+1)(5x^2-2x-1)dx^2$.

12.72. $2(2x^3-2x^2-3x+6)e^{-2x}dx^2$. **12.73.** $-8\cos 2x dx^4$.

12.74. $4096(6\sin 2x+x\cos 2x)dx^{12}$.

12.75. $\frac{ab(a^2-b^2)\sin 2x}{(a^2\cos^2 x+b^2\sin^2 x)}dx^2$. **12.76.** $17\cos x ch x dx^8$.

12.77. $\frac{4\ln x-4-\ln^3 x}{x^2\sqrt{(\ln^2 x-4)^3}}dx^2$. **12.78.** $4\ln 3\cdot 3^{-x^2}x[2x-\ln 3(2x^2\ln 3-1)]dx^3$.

12.79. $\frac{(u^2-v^2)(ud^2u+vd^2v)+(vdu-udv)^2}{(u^2-v^2)^{3/2}}$.

12.80. $u^v\left(\frac{v}{u}d^2u+\ln u d^2v+\frac{v(v-1)}{u^2}du^2+\frac{2(v\ln u+1)}{u}dudv+\ln^2 u dv^2\right)$

12.81. $\frac{1}{u^3}\left(u^2d^2v-uvd^2u-2ududv+2vdu^2\right)$

12.82. $\ln v d^2u+\frac{2}{v}dudv+\frac{u}{v}d^2v-\frac{u}{v^2}dv^2$.

12.83. $10edx^2$. **12.84.** $4dx^3$. **12.85.** $-\frac{5}{8}dx^2$.

12.86. $\frac{(-1)^n a^n n!}{b^{n+1}}dx^n$. **12.87.** $\frac{(27)!!\sqrt{2}}{2^{25}}dx^{16}$

12.88. $-2^7\cdot 1025\sqrt{3}dx^{10}$. **12.89.** $-\frac{1}{3}dx^2$. **12.90.** $\frac{3}{8}dx^2$.

13-§. DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI

13.1. Nuqtada funksiyaning o'sishi (kamayishi). Funksiyaning lokal ekstremum qiymatlari. $y = f(x)$ funksiya biror belgilangan c nuqtaning atrofida aniqlangan bo'lsin.

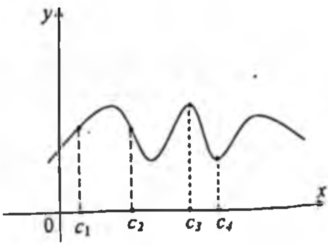
13.1-ta'rif. Agar c nuqtaning shunday $U_\delta(c)\,(\delta>0)$ atrofi mavjud bo'lib, $x<c$ bo'lganda $f(x)<f(c)$ tengsizlik, $x>c$ bo'lganda esa $f(x)>f(c)$ tengsizlik o'rinli bo'lsa, $y=f(x)$ funksiya c *nuqtada o'sadi* deyiladi.

13.2-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $x<c$ bo'lganda, $f(x)>f(c)$ tengsizlik, $x>c$ bo'lganda esa $f(x)<f(c)$ tengsizlik bajarilsa, $y=f(x)$ funksiya c *nuqtada kamayadi* deyiladi.

13.3-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $f(c)$ qiymat funksiyaning $U_\delta(c)$ atrofdagi qiymatlari ichida eng kattasi (eng kichigi) bo'lsa, $y=f(x)$ funksiya c *nuqtada lokal maksimum (lokal minimum) ga ega* deyiladi.

Odatda funksiyaning c nuqtadagi lokal maksimum va lokal minimum qiymatlari birgalikda lokal ekstremum qiymati deb yuritiladi.

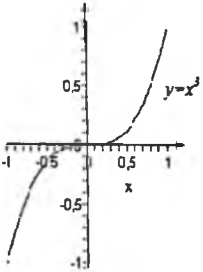
13.1-chizmada funksiyaning c_1 nuqtada o'suvchi, c_2 nuqtada kamayuvchi, c_3 nuqtada lokal maksimumga, c_4 nuqtada esa lokal minimumga ega ekanligi tasvirlangan (13.1- chizma).



13.1-chizma

13.1-teorema (Funksiyaning nuqtada o'suvchi, kamayuvchi bo'lishining yetarli sharti). *Agar $y=f(x)$ funksiya c nuqtada differensiallanuvchi bo'lib, uning bu nuqtadagi $f'(c)$ hosilasi musbat (manfiy) bo'lsa, u holda bu funksiya c nuqtada o'suvchi (kamayuvchi) bo'ladi.*

13.1-eslatma. $y=f(x)$ funksiyaning c nuqtada o'suvchi (kamayuvchi) bo'lishi uchun uning shu nuqtadagi $f'(c)$ hosilasining musbat (manfiy) bo'lishi zaruriy shart bo'la olmaydi. Masalan, $y=x^3$ funksiya $x=0$ nuqtada o'suvchi, lekin uning $x=0$ nuqtadagi hosilasi $f'(0)=0$ (13.2-chizma).



13.2-chizma

13.2-eslatma. Agar funksiya x_0 nuqtada o'suvchi bo'lsa, uning x_0 nuqtaning biror atrofida o'suvchi bo'lishi shart emas. Masalan,

$$f(x)=\begin{cases} 0, & x=0, \\ \dfrac{x}{2}-x^2\sin\dfrac{1}{x}, & x\neq 0. \end{cases}$$

funksiya berilgan bo'lsa, hosilaning ta'rifiga asosan:

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{\dfrac{x}{2}-x^2\sin\dfrac{1}{x}}{x}=\frac{1}{2}>0.$$

Demak, $f(x)$ funksiya $x=0$ nuqtada o'suvchi, lekin, bu funksiya monoton emas, chunki

$$f'(x)=\frac{1}{2}-2x\sin\frac{1}{x}+\cos\frac{1}{x}$$

funksiya $x=0$ nuqtaning ixtiyoriy kichik atrofida musbat qiymatni ham, manfiy qiymatni ham qabul qiladi: $x_k=\dfrac{1}{k\pi}$ $(k=1,2,...)$ bo'lib, k juft

13-§. DIFFERENSIAL HISOBNING ASOSIY TEOREMALARI

13.1. Nuqtada funksiyaning o'sishi (kamayishi). Funksiyaning lokal ekstremum qiymatlari. $y=f(x)$ funksiya biror belgilangan c nuqtaning atrofida aniqlangan bo'lsin.

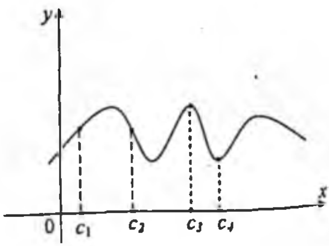
13.1-ta'rif. Agar c nuqtaning shunday $U_\delta(c)\,(\delta>0)$ atrofi mavjud bo'lib, $x<c$ bo'lganda $f(x)<f(c)$ tengsizlik, $x>c$ bo'lganda esa $f(x)>f(c)$ tengsizlik o'rinli bo'lsa, $y=f(x)$ funksiya c *nuqtada o'sadi* deyiladi.

13.2-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $x<c$ bo'lganda, $f(x)>f(c)$ tengsizlik, $x>c$ bo'lganda esa $f(x)<f(c)$ tengsizlik bajarilsa, $y=f(x)$ funksiya c *nuqtada kamayadi* deyiladi.

13.3-ta'rif. Agar c nuqtaning shunday $U_\delta(c)$ atrofi mavjud bo'lib, $f(c)$ qiymat funksiyaning $U_\delta(c)$ atrofdagi qiymatlari ichida eng kattasi (eng kichigi) bo'lsa, $y=f(x)$ funksiya c *nuqtada lokal maksimum (lokal minimum) ga ega* deyiladi.

Odatda funksiyaning c nuqtadagi lokal maksimum va lokal minimum qiymatlari birgalikda lokal ekstremum qiymati deb yuritiladi.

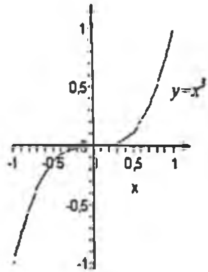
13.1-chizmada funksiyaning c_1 nuqtada o'suvchi, c_2 nuqtada kamayuvchi, c_3 nuqtada lokal maksimumga, c_4 nuqtada esa lokal minimumga ega ekanligi tasvirlangan (13.1- chizma).



13.1-chizma

13.1-teorema (Funksiyaning nuqtada o'suvchi, kamayuvchi bo'lishining yetarli sharti). *Agar $y=f(x)$ funksiya c nuqtada differensiallanuvchi bo'lib, uning bu nuqtadagi $f'(c)$ hosilasi musbat (manfiy) bo'lsa, u holda bu funksiya c nuqtada o'suvchi (kamayuvchi) bo'ladi.*

13.1-eslatma. $y=f(x)$ funksiyaning c nuqtada o'suvchi (kamayuvchi) bo'lishi uchun uning shu nuqtadagi $f'(c)$ hosilasining musbat (manfiy) bo'lishi zaruriy shart bo'la olmaydi. Masalan, $y=x^3$ funksiya $x=0$ nuqtada o'suvchi, lekin uning $x=0$ nuqtadagi hosilasi $f'(0)=0$ (13.2-chizma).



13.2-chizma

13.2-eslatma. Agar funksiya x_0 nuqtada o'suvchi bo'lsa, uning x_0 nuqtaning biror atrofida o'suvchi bo'lishi shart emas. Masalan,

$$f(x)=\begin{cases} 0, & x=0, \\ \frac{x}{2}-x^2\sin\frac{1}{x}, & x\neq 0. \end{cases}$$

funksiya berilgan bo'lsa, hosilaning ta'rifiga asosan:

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{\frac{x}{2}-x^2\sin\frac{1}{x}}{x}=\frac{1}{2}>0.$$

Demak, $f(x)$ funksiya $x=0$ nuqtada o'suvchi, lekin, bu funksiya monoton emas, chunki

$$f'(x)=\frac{1}{2}-2x\sin\frac{1}{x}+\cos\frac{1}{x}$$

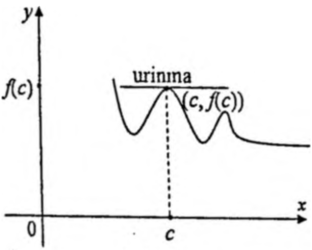
funksiya $x=0$ nuqtaning ixtiyoriy kichik atrofida musbat qiymatni ham, manfiy qiymatni ham qabul qiladi: $x_k=\frac{1}{k\pi}$ $(k=1,2,...)$ bo'lib, k juft bo'lganda $f'(x)$ ning qiymati $\frac{3}{2}$ ga, k toq son bo'lganda esa $-\frac{1}{2}$ ga teng bo'ladi.

13.2. Ferma, Roll, Lagranj va Koshi teoremalari

13.2-teorema (Ferma). *$y=f(x)$ funksiya biror X oraliqda aniqlangan bo'lib, bu oraliqning ichki c nuqtasida o'zining eng katta (eng kichik) qiymatiga erishsin. Agar c nuqtada funksiya chekli $f'(x)$ hosilaga ega bo'lsa, u holda*

$$f'(c)=0$$

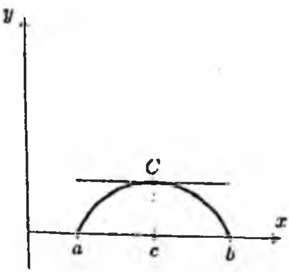
bo'ladi.



13.3-chizma

Ferma teoremasi sodda geometrik ma'noga ega. $y=f(x)$ funksiya Ferma teoremasining shartlarini qanoatlantirganda, $f(x)$ funksiyaning grafigidagi $(c, f(c))$ nuqtaga o'tkazilgan urinma Ox o'qqa parallel bo'ladi (13.3-chizma).

Ferma teoremasining fizik ma'nosi quyidagicha: to'g'ri chiziq bo'ylab harakat qilayotgan zarrachaning qaytish momenti tezligi nolga teng bo'ladi.



13.4-chizma.

13.3-eslatma. $f'(c)=0$ shart funksiyaning c nuqtada lokal ekstremumga ega bo'lishi uchun yetarli shart bo'la olmaydi.

Masalan, $f(x)=x^3$ funksiyaning hosilasi $x=0$ nuqtada $f'(0)=0$ bo'lsa-da, funksiya $x=0$ nuqtada o'sadi.

13.3-teorema (Roll). *$y=f(x)$ funksiya $[a,b]$ da aniqlangan bo'lib: 1) uzluksiz; 2) aqalli (a,b) da chekli hosilaga ega; 3) $[a,b]$ ning chetlarida o'zaro teng $(f(a)=f(b))$ qiymatlarni qabul qilsa, u holda, kamida bitta shunday c $(a<c<b)$ nuqta topiladiki, $f'(c)=0$ bo'ladi.*

Roll teoremasining geometrik ma'nosi quyidagicha: $y=f(x)$ funksiya Roll teoremasining hamma shartlarini qanoatlantirganda, bu funksiyaning grafigida shunday $(c;f(c))$ nuqta topiladiki, bu nuqtada funksiya grafigiga o'tkazilgan urinma Ox o'qqa parallel bo'ladi (13.4-chizma).

13.4-eslatma. Roll teoremasida $y=f(x)$ funksiyadan uning $[a,b]$ segmentda uzluksizligi, segmentning ichki nuqtalarida esa, uning differensiallanuvchiligi talab qilingan edi. Funksiya segmentning ichki nuqtalarida differensiallanuvchiligidan uning shu nuqtalarda uzluksizligi kelib chiqadi, shuning uchun Rolli teoremasidagi funksiyaga qo'yilgan 1) shartning o'rniga, $f(x)$ ning a nuqtadan o'ngdan, b nuqtada esa chapdan uzluksiz bo'lishini talab qilish yetarli.

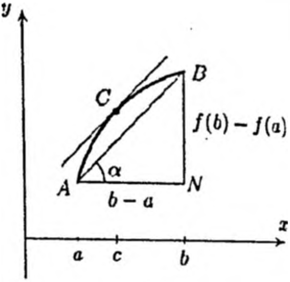
13.5-eslatma. Roll teoremasining barcha shartlari muhim. Agar teoremadagi $y=f(x)$ funksiyaga qo'yilgan shartlarning birortasi bajarilmasa, teoremaning tasdig'i o'rinli bo'lmasligi mumkin.

Masalan, $f(x)=1-\sqrt[3]{x^2}$ funksiya $[-1;1]$ segmentda uzluksiz bo'lib, bu funksiya uchun $f(-1)=f(1)=0$. Lekin, bu funksiyaning hosilasi $(-1;1)$ intervalning birorta nuqtasida ham nolga aylanmaydi. Bunga sabab, berilgan funksiya $(-1;1)$ intervalning $x=0$ nuqtadan tashqari qolgan hamma nuqtalarida hosilaga ega. $x=0$ nuqta ichki nuqta bo'lganligi uchun Roll teoremasining 3) sharti bajarilmaydi. Shuning uchun berilgan funksiyaga Roll teoremasini qo'llab bo'lmaydi.

13.4-teorema (Lagranj). *Agar $f(x)$ funksiya: 1) $[a,b]$ kesmada aniqlangan va uzluksiz; 2) aqalli (a,b) oraliqda chekli hosilaga ega bo'lsa, u holda shunday c $(a<c<b)$ nuqta topiladiki, bu nuqtada*

$$\frac{f(b)-f(a)}{b-a}=f'(c) \qquad (13.1)$$

tenglik o'rinli bo'ladi.



13.5-chizma.

Lagranj teoremasining geometrik ma'nosi quyidagicha: faraz qilaylik, $f(x)$ funksiya Lagranj teoremasining barcha shartlarini qanoatlantirsin. $f(x)$ funksiya grafigining $A(a,f(a))$, $B(b,f(b))$ nuqtalarini to'g'ri chiziq bilan tutashtiramiz. $\frac{f(b)-f(a)}{b-a}=\frac{NB}{AN}$ ifoda AB kesuvchining burchak koeffitsiyentini ifoda qoladi. $f'(c)$- esa $f(x)$ funksiya grafigining $C(c,f(c))$ nuqtasidan o'tkazilgan urinmaning burchak koeffitsiyentidir, ya'ni $\operatorname{tg}\alpha=f'(c)$. Shunday c $(a<c<b)$ nuqta topiladiki, $f(x)$ funksiya grafigiga $C(c,f(c))$ nuqtada o'tkazilgan urinma AB to'g'ri chiziqqa parallel bo'ladi (13.4-chizma). (13.1) formulani boshqacha ham yozish mumkin: $\forall x_0\in[a;b]$ nuqtani olib, unga ixtiyoriy Δx orttirma beramiz $(x_0+\Delta x\in[a;b])$. $[x_0,x_0+\Delta]$ segment uchun (13.1) Lagranj formulasini yozamiz:

$$f(x_0+\Delta x)-f(x_0)=\Delta x\cdot f'(c), \qquad (13.2)$$

bunda $\forall c\in(x_0,x_0+\Delta x)$. $c=x_0+\theta\Delta x,\ \ 0<\theta<1$ deb belgilasak,

$$f(x_0+\Delta x)-f(x_0)=\Delta x\cdot f'(x_0+\theta\Delta x) \qquad (13.3)$$

Odatda, (13.2) yoki (13.3) formula chekli orttirmalar haqidagi *Lagranj formulasi deb yuritiladi.*

13.6-eslatma. Agar (13.1) formulada $f(a)=f(b)$ deb olinsa, u holda $f'(c)=0$ $(a<c<b)$ bo'lib, Lagranj teoremasidan Roll teoremasining kelib chiqishini ko'ramiz.

Lagranj teoremasidan quyidagi natijalar kelib chiqadi.

13.1-natija. Agar $f(x)$ funksiya $(a;b)$ oraliqda differensiallanuvchi va bu oraliqda $f'(c)=0$ bo'lsa, u holda bu oraliqda $f(x)$ funksiya o'zgarmas bo'ladi.

13.2-natija. $f(x)$ va $g(x)$ funksiyalar biror $(a;b)$ oraliqda uzluksiz, bu oraliqda differensiallanuvchi bo'lib, $f'(x)=g'(x)$, $x\in(a;b)$ bo'lsa, u holda bu funksiyalarning biri ikkinchisidan o'zgarmas songa farq qiladi, ya'ni $f(x)=g(x)+c$.

13.3-natija. $f(x)$ funksiya biror x_0 nuqtaning $U_\delta(x_0)$ atrofida uzluksiz va $U_\delta(x_0)$ atrofda differensiallanuvchi bo'lsin. Agar chekli $\lim\limits_{x\to x_0} f'(x)=A$ mavjud bo'lsa, u holda $f(x)$ funksiya x_0 nuqtada differensiallanuvchi deyiladi va $f'(x_0)=A$ bo'ladi.

13.5-teorema (Koshi). *$f(x)$ va $g(x)$ funksiyalar:*

1) $[a,b]$ segmentda aniqlangan va uzluksiz bo'lsin;

2) $[a,b]$da chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo'lib, $\forall x\in(a,b)$ uchun $g'(x)\neq 0$ bo'lsin.

U holda shunday $c(a<c<b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

(13.4)

tenglik o'rinli bo'ladi.

13.7-eslatma. (13.1) Lagranj formulasi (13.4) Koshi formulasidan ($g'(x)=x$ bo'lganda) kelib chiqadi.

13.8-eslatma. (13.4) formulada $b > a$ deb olish shart emas.

13.1-misol. Ushbu $f(x) = 3x^2 - 1$ funksiya $[1; 2]$ segmentda Ferma teoremasining hamma shartlarini qanoatlantiradimi?

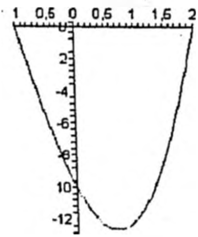
Yechilishi. Berilgan funksiya $[1; 2]$ segmentda monoton o'sadi. Demak, u $x = 1$ da eng kichik qiymatini, $x = 2$ da esa eng katta qiymatini qabul qiladi, lekin bu nuqtalar $[1; 2]$ segmentning ichki nuqtasi emas. Shunday qilib $f(x) = 3x^2 - 1$ funksiya uchun Ferma teoremasini qo'llab bo'lmaydi, ya'ni

$f'(1) = f'(2) = 0$ deb aytish noto'g'ri bo'ladi, chunki $f'(1) = 6$, $f'(2) = 12$.

13.2-misol. Quyidagi funksiyalar uchun ko'rsatilgan oraliqda Roll teoremasining shartlarini tekshiring:

a) $f(x) = x^3 + 4x^2 - 7x - 10,\ x \in [-1; 2]$; *b*) $f(x) = \begin{cases} x, & x \in [0; 1), \\ 0, & x = 1; \end{cases}$

c) $f(x) = |x - 1|,\ x \in [0; 2]$; *d*) $f(x) = x,\ x \in [0; 1]$.



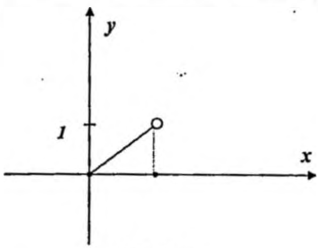
13.6-chizma.

Yechilishi. *a*) 1) $[-1; 2]$ kesmada $f(x) = x^3 + 4x^2 - 7x - 10$ funksiya aniqlangan va uzluksiz; 2) $(-1; 2)$ oraliqda $f'(x) = x^2 + 8x - 7$ chekli hosila mavjud; 3) oraliqning chetki nuqtalarida funksiya o'zaro teng $f(-1) = f(2)$ qiymatlarni qabul qiladi (13.6-chizma).

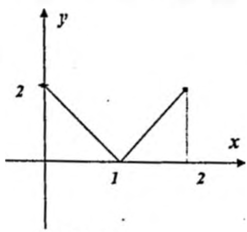
Demak, berilgan $f(x) = x^3 + 4x^2 - 7x - 10$ funksiya $[-1; 2]$ oraliqda Roll teoremasining hamma shartlarini qanoatlantiradi. U holda, $f'(x) = 0$ tenglamani qanoatlantiradigan $x_1 = -\frac{4 + \sqrt{47}}{3}$, $x_2 = -\frac{-4 + \sqrt{47}}{3}$ nuqtalar mavjud (13.5-chizma).

b) Ravshanki, berilgan funksiya Roll teoremasidagi 2) va 3) shartlarni qanoatlantiradi, lekin 1) shart bajarilmaydi. Funksiya qaralayotgan kesmada uzluksiz emas, $x=1$ nuqtada u uzilishga ega, chunki $\lim_{x\to 1-0} f(x)=1$, ammo $f(1)=0$. Demak, $(0;1)$ oraliqda $f'(c)=0$ bo'ladigan $x=c$ nuqta mavjud emas (13.7-chizma).

c) $[0;2]$ kesmada $f(x)=|x-1|$ funksiya Roll teoremasining 1) va 3) shartlarini qanoatlantiradi, lekin 2) shartini qanoatlantirmaydi, berilgan



13.7-chizma.



13.8-chizma

funksiya $x=1$ nuqtada differensiallanuvchi emas (13.8-chizma).

Demak, (0;2) oraliqda $f'(c)=0$ tenglamani qanoatlantiradigan c nuqta mavjud emas.

d) $[0;1]$ kesmada $f(x)=x$ funksiya Roll teoremasining 1) va 2) shartlarini qanoatlantiradi, 3) shartini qanoatlantirmaydi, chunki $f(0)=0$, $f(1)=1$. Demak, $(0;1)$ oraliqda tenglamani qanoatlantiradigan c nuqta mavjud emas.

13.3-misol. $[-2;0]$ kesmada $f(x)=2x^2-7$ funksiya Lagranj teoremasining hamma shartlarini qanoatlantiradimi? Agar $(f(b)-f(a)=f'(c)(b-a))$ qanoatlantirsa Lagranj formulasida qatnashadigan c nuqtani toping.

Yechilishi. Berilgan $f(x)=2x^2-7$ funksiya $[-2;0]$ kesmada uzluksiz va $(-2;0)$ da chekli $f'(x)=4x$ hosilaga ega. U holda izlanayotgan c nuqta

$$f'(c)=4c=\frac{f(0)-f(-2)}{0-(-2)}=\frac{-7-1}{2}=-4,$$

bundan $c=-1$.

13.4-misol. Ushbu

1) $\ln(1+x)<x \quad (x>0)$,

2) $|\arg tgx_2-\arg tgx_1|\le|x_2-x_1|, \quad x_1,x_2\in(-\infty,\infty)$

tengsizliklarni isbotlang.

Yechilishi. **1)** $[0;x]$ kesmada berilgan $f(x)=\ln(1+x)$ funksiya Lagranj teoremasining hamma shartlarini qanoatlantiradi, u holda teoremaning tasdig'iga ko'ra shunday $c(0<c<x)$ nuqta topiladiki,

$$f(x)-f(0)=\ln(1+x)=\frac{1}{1+c}x<x$$

bo'ladi, chunki $\frac{1}{1+c}<1$.

2) $[x_1,x_2]$kesmada berilgan $f(x)=\operatorname{arctg}x$ funksiya Lagranj teoremasining hamma shartlarini qanoatlantiradi, u holda Lagranj formulasiga asosan:

$$|\operatorname{arctg}x_2-\operatorname{arctg}x_1|=\left|\frac{1}{1+c^2}(x_1-x_2)\right|\le|x_2-x_1|,$$

chunki $0<\frac{1}{1+c^2}\le1$

13.5-misol. Ushbu

$$f(x)=x^2-2x+3, \quad g(x)=x^3-7x^2+20x-5$$

funksiyalar $[1;4]$ kesmada Koshi teoremasining shartlarini qanoatlantiradimi? Agar qanoatlantirsa, (13.4) Koshi formulasida qatnashgan c nuqtani toping.

Yechilishi. Berilgan $f(x)$ va $g(x)$ funksiyalar hamma joyda uzluksiz, jumladan, $[1,4]$ da ham uzluksiz. Berilgan funksiyalar mos ravishda $(1;4)$ da chekli

$$f'(x)=2x-2,\ g'(x)=3x^2-14x+20$$

hosilalarga ega bo'lib, x ning hech bir haqiqiy qiymatida $g'(x)$ funksiya nolga aylanmaydi. U holda (13.4) formulaga asosan:

$$\frac{f(4)-f(1)}{g(4)-g(1)}=\frac{f'(c)}{g'(c)},\quad \frac{11-2}{27-9}=\frac{2c-2}{3c^2-14c+20}\quad (1<x<4),$$

bu tenglamani c ga nisbatan yechib, $c_1=2$ va $c_2=4$ larni topamiz. Bulardan $c_1=2$ nuqta ichki nuqta bo'ladi.

Mustaqil yechish uchun misol va masalalar

13.1. Ushbu $f(x)=2x^2-1$ funksiya uchun $[1,2]$ kesmada Ferma teoremasining shartlari bajariladimi?

13.2. Ushbu $f(x)=5\sqrt{2x+1}-x$ funksiya uchun $[4;40]$ kesmada Ferma teoremasining shartlari bajariladimi?

13.3. Ushbu $f(x)=x\ln 5-x\ln x$ funksiya uchun $\left[\frac{5}{3};2{,}5\right]$ kesmada Ferma teoremasining shartlari bajariladimi?

13.4. Ushbu $f(x)=\ln\sin x$ funksiya uchun $\left[\frac{\pi}{6};\frac{5\pi}{6}\right]$ kesmada Roll teoremasining shartlari bajariladimi?

13.5. Ushbu

1) $f(x)=\sqrt[3]{x^2-3x+2}$, $[1,2]$; 2) $f(x)=4^{\sin x}$, $[0;\pi]$

funksiyalar ko'rsatilgan kesmada Roll teoremasining shartlarini qanoatlantirishini tekshiring.

13.6. Ushbu $f(x)=\sin x$ funksiya uchun $[1;2]$ kesmada Roll teoremasining shartlari bajariladimi?

13.7. Quyidagi 1) $f(x)=x^3-x$, $x\in[0;1]$; 2) $f(x)=\sin 2x$, $x\in[0;2\pi]$

funksiyalar ko'rsatilgan oraliqda Roll teoremasining shartlarini qanoatlantirishini ko'rsating va $f'(c)=0$ ni qanoatlantiruvchi c sonlarni toping.

13.8. Quyidagi

1) $f(x)=x^2$ $x\in[1;2]$; 2) $f(x)=x^3$, $x\in[1;3]$; 3) $f(x)=\sqrt{1-x^2}$ $x\in[0;1]$

funksiyalar uchun ko'rsatilgan oraliqda o'rta qiymat haqidagi teoremaning shartlarini tekshiring va teorema tasdig'ini qanoatlantiruvchi barcha c sonlarni toping.

13.9. Ushbu $f(x)=3x^2-5$ funksiya $[-2;0]$ kesmada Lagranj teoremasining shartlarini qanoatlantiradimi? Agar qanoatlantirsa, $f(b)-f(a)=f'(c)(b-a)$ Lagranj formulasidagi c nuqtani toping.

13.10. $f(x)=\ln x$ funksiyaga $[1;e]$ kesmada Lagranj formulasini qo'llang va unda qatnashadigan c ning qiymatini toping.

13.11. $f(x)=\sin 3x$ funksiya uchun $[x_1;x_2]$ kesmada Lagranj formulasini yozing.

13.12. $f(x)=\arcsin(2x)$ funksiya uchun $[x_0;x_0+\Delta x]$ kesmada Lagranj formulasini yozing.

13.13. $f(x)=x^n$ funksiyaning $[0;a]$ kesmada $(n>0;\ a>0)$ Lagranj teoremasining shartlarini qanoatlantirishini ko'rsating.

13.14. $y=|x|$ funksiya uchun $[0;a]$ kesmada Roll teoremasi o'rinli emasligini ko'rsating.

13.15. Agar $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_{n-1}x=0$ tenglama $x=x_0$ musbat ildizga ega bo'lsa, $na_0x^{n-1}+(n-1)a_1x^{n-2}+...+a_{n-1}=0$ tenglama ham musbat, x_0 dan kichik ildizga ega bo'lishini isbotlang.

13.16. $x^3-3x+c=0$ tenglamaning $(0;1)$ oraliqda ikkita har xil ildizga ega bo'lmasligini isbotlang.

13.17. $y=x^3$ chiziqda shunday nuqtani topingki, unga o'tkazilgan urinma $A(-1;1)$ va $B(2;8)$ nuqtalarni birlashtiruvchi vatarga parallel bo'lsin.

Lagranj teoremasidan foydalanib, quyidagi tengsizliklarni isbotlang:

13.18. $e^x>ex, \quad x>1.$

13.19. $\frac{a-b}{a}\le\ln\frac{a}{b}\le\frac{a-b}{b}, \quad 0<b\le a.$

13.20. $e^x>1+x, \quad x\in R.$

13.21. $|\sin x-\sin y|\le|x-y|.$

13.22. $nb^{n-1}(a-b)<a^n-b^n<n\cdot a^{n-1}(a-b), \quad n>1, \quad a>b.$

13.23. $\frac{\alpha-\beta}{\cos^2\beta}\le \mathrm{tg}\alpha-\mathrm{tg}\beta\le\frac{\alpha-\beta}{\cos^2\alpha}, \quad 0<\beta\le\alpha<\frac{\pi}{2}.$

13.24. Ushbu $f(x)=e^x, \quad g(x)=\frac{x^2}{1+x^2}$ funksiyalar uchun $[-3;3]$ kesmada Koshi teoremasi o'rinlimi?

13.25. Ushbu $f(x)=x^2$ va $g(x)=x^3$ funksiyalar uchun $[-1;1]$ kesmada Koshi teoremasi o'rinlimi?

13.26. Funksiyaning o'zgarmaslik alomatidan foydalanib, elementar matematikadan ma'lum bo'lgan quyidagi formulalarni isbot qiling:

1) $\arcsin x + \arccos x = \dfrac{\pi}{2}$; 2) $\sin^2 x = \dfrac{1-\cos 2x}{2}$;

3) $\arccos\dfrac{1-x^2}{1+x^2} = 2\,\mathrm{arctg}\, x,\quad 0 < x < \infty$; 4) $\arcsin\dfrac{2x}{1+x^2} = \begin{cases} \pi - 2tgx, & x \geq 1, \\ 2\arg tgx, & -1 \leq x \leq x, \\ -\pi - 2arctgx, & x \leq 1. \end{cases}$

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

13.1. Bajarilmaydi. **13.2.** Bajariladi. **13.3.** Bajariladi.

13.4. Bajariladi. **13.6.** Bajariladi. **13.7.** 1) $c = \dfrac{\sqrt{3}}{3}$; 2) $c = \dfrac{\pi}{4}; \dfrac{3\pi}{4}; \dfrac{5\pi}{4}; \dfrac{7\pi}{4}$.

13.8. 1) $c = \dfrac{3}{2}$; 2) $c = \dfrac{1}{4}\sqrt{39}$; 3) $c = \dfrac{1}{4}\sqrt{2}$. **13.9.** $c = 1$. **13.10.** $c = e - 1$.

13.11. $\sin(3x_2) - \sin(3x_1) = 3(x_2 - x_1)\cos(3c)$, $(x_1 < c < x_2)$.

13.12. $\arcsin[2(x_0 - \Delta x)] - \arcsin 2x_0 = \dfrac{2\Delta x}{\sqrt{1-4c^2}}$, $x_0 < c < x_0 + \Delta x$. **13.17.** $M(1;1)$. **13.24.** O'rinli emas. **13.25.** O'rinli emas.

14-§. LOPITAL QOIDALARI

Funksiyalarning limitini hisoblash jarayonida $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ ko'rinishidagi aniqmasliklarni ochish vaqtida, ba'zan qiyinchiliklarga duch kelinadi. Agar berilgan funksiyalarning hosilalari mavjud bo'lsa, ulardan foydalanganda berilgan aniqmasliklarni ochish yengillashadi. Odatda, hosilalardan foydalanib aniqmasliklarni ochish *Lopital qoidalari* deb ataladi.

14.1. Lopitalning birinchi qoidasi $\left(\frac{0}{0}\right)$. Agar $x\to a$ da $f(x)\to 0$, $g(x)\to 0$ bo'lsa, $\frac{f(x)}{g(x)}$ ning $x\to a$ dagi limiti $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikni ifodalaydi. Ba'zi hollarda, $x\to a$ da $\frac{f(x)}{g(x)}$ nisbatning limitini topishga qaraganda $\frac{f'(x)}{g'(x)}$ nisbatning limitini topish yengil bo'ladi.

14.1-teorema (Lopitalning birinchi qoidasi). $f(x)$ *va* $g(x)$ *funksiyalar* (a,b) *da aniqlangan, uzluksiz bo'lib, ular quyidagi shartlarni qanoatlantirsin:*

1) $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=0$;

2) (a,b) *da chekli* $f'(x)$ *va* $g'(x)$ *hosilalar mavjud va* $\forall x\in(a,b),\ g'(x)\neq 0$;

3) $\exists\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=A$ *(A-chekli yoki cheksiz) bo'lsin.*

U holda, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ *ham mavjud va*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A \tag{14.1}$$

tenglik o'rinli.

14.1-eslatma. 14.1-teoremaning 3) sharti bajarilmaganda ham, $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ mavjud bo'lishi mumkin. Masalan, $f(x)=x^2\sin\frac{1}{x}$, $g(x)=2^x-1$ bo'lsin. Bu funksiyalar uchun 14.1-teoremaning shartlarini tekshiramiz:

1) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x^2\cdot\sin\frac{1}{x}=0;$ $\lim\limits_{x\to 0} g(x)=\lim\limits_{x\to 0}(2^x-1)=0;$

2) $f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$, $g'(x)=2^x\cdot\ln 2$ bo'lib, $x\to 0$ da

$$\frac{f'(x)}{g'(x)}=\frac{2x\cdot\sin\frac{1}{x}-\cos\frac{1}{x}}{2^x\cdot\ln 2}$$

nisbatning limiti mavjud emas, chunki $\lim\limits_{x\to 0}\cos\frac{1}{x}$ limit mavjud emas.

Lekin, $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{2^x-1}=\lim\limits_{x\to 0}\frac{x\cdot\sin\frac{1}{x}}{\frac{2^x-1}{x}}=0.$

14.2-eslatma. 14.1-teoremada $f'(x)$ va $g'(x)$ hosilalarning $x=a$ nuqtada uzluksizligi talab qilinsa, u holda $g'(a)\neq 0$ shartda (14.1) formulani quyidagicha yozish mumkin:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

14.3-eslatma. 14.1- teorema $a=+\infty$ yoki $a=-\infty$ bo'lgan hol uchun ham o'rinli, ya'ni $f(x)$ va $g(x)$ funksiyalar $c<x<\infty$ da aniqlangan va shu oraliqda differensiallanuvchi bo'lib, quyidagi shartlarni qanoatlantirsin:

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty} g(x)=0,\qquad g'(x)\neq 0,\ \forall x\in(c;+\infty).$$

U holda, $\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}$ mavjud bo'lsa, $\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)}$ ham mavjud bo'ladi va

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$$

tenglik o'rinli bo'ladi.

14.4-eslatma. Agar $f(x)$ va $g(x)$ funksiyalar 14.1-teoremaning barcha shartlarini qanoatlantirsa, Lopital qoidasini takroriy qo'llash mumkin, ya'ni

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}.$$

14.2. Lopitalning ikkinchi qoidasi $\left(\frac{\infty}{\infty}\right)$. Agar $x\to a$ da $f(x)\to\infty$, $g(x)\to\infty$ bo'lsa, $\frac{f(x)}{g(x)}$ ning $x\to a$ dagi limiti $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmaslikni ifodalaydi. Ba'zi hollarda, bunday aniqmasliklarni ochishda ham $f(x)$ va $g(x)$ funksiyalarning hosilalaridan foydalanish maqsadga muvofiq bo'ladi.

14.2-teorema (Lopitalning ikkinchi qoidasi). *$f(x)$ va $g(x)$ funksiyalar (a, b) da quyidagi shartlarni qanoatlantirsin:*

1) $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=\infty$;

2) *(a, b) da chekli $f'(x)$ va $g'(x)$ hosilalar mavjud va* $\forall x\in(a, b)$, $g'(x)\neq 0$;

3) $\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$ *(A-chekli yoki cheksiz), u holda*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$$

tenglik o'rinli.

14.5-eslatma. Agar $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=\infty$ bo'lsa, $f(x)\cdot g(x)$ ifoda $(0\cdot\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu ko'rinishdagi aniqmaslikni ochishda, uni

$$f(x)\cdot g(x)=\frac{f(x)}{\frac{1}{g(x)}}=\frac{g(x)}{\frac{1}{f(x)}}$$

kabi yozish orqali $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltirilib, Lopital qoidalari qo'llaniladi.

14.6-eslatma. Agar $\lim\limits_{x\to a} f(x)=+\infty$, $\lim\limits_{x\to a} g(x)=+\infty$ bo'lsa, $f(x)-g(x)$ ifoda $(\infty-\infty)$ ko'rinishdagi aniqmaslikni ifoda qiladi, uni ham quyidagi

$$f(x)-g(x)=\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)}\cdot\frac{1}{g(x)}}$$

kabi yozish bilan $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltiriladi va Lopital qoidalari qo'llaniladi.

14.7-eslatma. Agar $x\to a$da $f(x)$ funksiya $1, 0$ va ∞ga, $g(x)$ funksiya esa mos ravishda, $\infty, 0$ va 0 ga intilganda $(f(x))^{g(x)}$ – daraja – ko'rsatkichli ifoda $(1^{\infty}),(0^{0}),(\infty^{0})$ ko'rinishdagi aniqmasliklarni ifoda qiladi. Bu ko'rinishdagi aniqmasliklarni ochish uchun, avvalo berilgan ifoda logarifmlanadi:

$$\ln y = g(x)\cdot\ln f(x),$$

bu ifoda, $x\to a$da $(0\cdot\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi, ya'ni yuqorida o'rganilgan holga keltiriladi.

Shunday qilib: 1) $(0\cdot\infty)$ yoki $(\infty-\infty)$ ko'rinishdagi aniqmasliklar algebraik almashtirishlar natijasida $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltiriladi va ularga Lopital qoidalari qo'llaniladi.

2) $(1^{\infty}),(0^{0}),(\infty^{0})$ ko'rinishdagi aniqmasliklar logarifmlash yoki $(f(x))^{g(x)}=e^{g(x)\ln(f(x))}$ shakl o'zgartirishlar orqali $(0\cdot\infty)$ ko'rinishdagi aniqmaslikka keltiriladi, so'ngra uni $\left(\frac{0}{0}\right)$ yoki $\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmasliklarga keltirilib, Lopital qoidalari qo'llaniladi.

14.1-misol. $\lim\limits_{x\to 2}\dfrac{\ln(x^2-3)}{x^2+3x-10}$ limitni hisoblang.

Yechilishi. Ravshanki, $f(x)=\ln(x^2-3)$, $g(x)=x^2+3x-10$ funksiyalar $x\to 2$ da $f(x)=\ln(x^2-3)\to 0$, $g(x)=x^2+3x-10\to 0$. $x=2$ nuqtaning $x=\pm\sqrt{3}$ nuqtalarni o'z ichida saqlamaydigan ixtiyoriy kichik atrofida $f'(x)=\dfrac{2x}{x^2-3}$, $g'(x)=2x+3$ hosilalar mavjud va $g'(x)=2x+3\neq 0$ $\left(x>-\dfrac{3}{2}\right)$,

$$\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\lim_{x\to 2}\frac{\frac{2x}{x^2-3}}{2x+3}= =\lim_{x\to 2}\frac{2x}{(x^2-3)(2x+3)}=\frac{4}{7}$$ mavjud.

Demak, berilgan limitni hisoblashga Lopitalning birinchi qoidasini qo'llash mumkin:

$$\lim_{x\to 2}\frac{\ln(x^2-3)}{x^2+3x-10}=\lim_{x\to 2}\frac{2x}{(x^2-3)(2x+3)}=\frac{4}{7}.$$

MAPLE tizimidan foydalanib misolni yechish:

> Limit(ln(x^2-3)/(x^2+3*x-10),x=2)=

limit(ln(x^2-3)/(x^2+3*x-10),x=2);

$$\lim_{x\to 2}\left(\frac{\ln(x^2-3)}{x^2+3x-10}\right)=\frac{4}{7}.$$

14.2-misol. $\lim\limits_{x\to+\infty}\dfrac{x^a}{e^{\alpha x}}$ $(a>0,\alpha>0)$ limitni hisoblang.

Yechilishi. Bu holda, $f(x)=x^a, g(x)=e^{\alpha x}$ bo'lib, ular 14.2-teoremaning barcha shartlarini qanoatlantiradi. Shuning uchun, Lopitalning ikkinchi qoidasiga ko'ra,

$$\lim_{x\to+\infty}\frac{x^a}{e^{\alpha x}}=\lim_{x\to+\infty}\frac{\alpha x^{a-1}}{a e^{\alpha x}}=0,$$

chunki, $x\to+\infty$ da asosi birdan katta ko'rsatkichli funksiya darajali funksiyaga qaraganda tezroq o'sadi. Bu limitni Lopital qoidasini k $(k=[\alpha]+1,\ \alpha-k<0)$ marta qo'llab ham topish mumkin:

$$\lim_{x\to+\infty}\frac{x^a}{e^{\alpha x}}=\lim_{x\to+\infty}\frac{\alpha x^{a-1}}{a e^{\alpha x}}=\ldots=\lim_{x\to+\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)x^{\alpha-k}}{a^k e^{\alpha x}}=0.$$

14.3-misol. $\lim\limits_{x\to 1}\dfrac{x^x - x}{\ln x - x + 1}$ limitni hisoblang.

Yechilishi. Bu holda, $f(x) = x^x - x$, $g(x) = \ln x - x + 1$ bo‘lib, ular 14.1-teoremaning barcha shartlarini qanoatlantiradi, jumladan, $x = 1$ nuqtaning ixtiyoriy kichik atrofida

$$f'(x) = x^x(\ln x + 1) - 1, \quad g'(x) = \frac{1}{x} - 1$$

hosilalar mavjud bo‘lib, $g'(x) = \frac{1}{x} - 1 \neq 0$ $(x \neq 1)$. Lekin, $f'(x)$ va $g'(x)$ funksiyalar ham o‘z navbatida $x = 1$ nuqtaning kichik atrofida 14.1-teoremaning barcha shartlarini qanoatlantiradi. Shuning uchun, berilgan limitni hisoblashga Lopitalning birinchi qoidasini ikki marta qo‘llaymiz:

$$\lim_{x\to 1}\frac{x^x - x}{\ln x - x + 1} = \lim_{x\to 1}\frac{x^{x-1}(\ln x + 1) - x}{1 - x} = \lim_{x\to 1}\left[1 - x^{x-1}(\ln x + 1)\left(1 + \frac{1}{x} + \ln x\right) - x^x\right] = -2.$$

MAPLE tizimidan foydalanib misolni yechish:

> Limit(((x)^x-x)/(ln(x)-x+1),x=1)=

limit(((x)^x-x)/(ln(x)-x+1),x=1);

$$\lim_{x\to 1}\left(\frac{x^x - x}{\ln(x) - x + 1}\right) = -2.$$

14.4-misol. $\lim\limits_{x\to+\infty}\dfrac{\ln^n x}{x^a}$ $(a > 0,\ n > 0)$ limitni hisoblang.

Yechilishi. $\ln x = t,\ x = e^t$ almashtirishni olib, $\lim\limits_{x\to+\infty}\dfrac{\ln^n x}{x^a} = \lim\limits_{t\to+\infty}\dfrac{t^n}{e^{at}} = 0$ bo‘lishini topamiz (14.2-misolga qarang).

14.5-misol. $\lim\limits_{x\to 0+0} x^\alpha \ln^\beta\left(\dfrac{1}{x}\right)$ $(\alpha > 0,\ \beta > 0)$ limitni hisoblang.

Yechilishi. Bu holda $\lim\limits_{x\to 0+0} f(x) = \lim\limits_{x\to 0+0} x^\alpha = 0$, $\lim\limits_{x\to 0+0} g(x) = \lim\limits_{x\to 0+0} \ln^\beta\left(\dfrac{1}{x}\right) = +\infty$.

Demak, berilgan ifodaning $x \to 0+0$ dagi limiti $(0\cdot\infty)$ ko‘rinishdagi aniqmaslikni ifodalaydi. Bu aniqmaslikni ochish uchun, uni $\left(\frac{0}{0}\right)$ yoki

$\left(\frac{\infty}{\infty}\right)$ ko'rinishdagi aniqmaslikka keltirib,. Lopitalning birinchi qoidasini qo'llashda, soddalik uchun, $\ln\frac{1}{x}=t,\ x=e^{-t}$ almashtirishni bajarib, berilgan ifodaning limiti

$$\lim_{x\to+0+0} x^{\alpha}\cdot\ln^{\beta}\left(\frac{1}{x}\right)=\lim_{t\to+\infty}\frac{t^{\beta}}{e^{\alpha t}}=0$$

topamiz (14.2-misolga qarang).

14.6-misol. $\lim_{x\to0}[\ln(1+\sin^2 x)\cdot ctg(\ln^2(1+x))]$ limitni hisoblang.

Yechilishi. $\lim_{x\to0} f(x)=\lim_{x\to0}\ln(1+\sin^2 x)=0,\ \ \lim_{x\to0} ctg\ln^2(1+x)=\infty.$ Berilgan ifodaning limiti $(0\cdot\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu aniqmaslikni ochish uchun berilgan ifodaning shaklini o'zgartirib, uni $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltirib, keyin Lopitalning birinchi qoidasini takroriy qo'llaymiz:

$$\lim_{x\to0}[\ln(1+\sin^2 x)\cdot ctg\ln^2(1+x)]\overset{(0\cdot\infty)}{=}=\lim_{x\to0}\frac{\ln(1+\sin^2 x)}{tg\ln^2(1+x)}\overset{\left(\frac{0}{0}\right)}{=}$$

$$=\lim_{x\to0}\frac{\frac{1}{1+\sin^2 x}\sin 2x}{2[1+tg^2\ln^2(1+x)]\cdot\ln(1+x)\cdot\frac{1}{1+x}}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{\sin 2x}{2\cdot\ln(1+x)}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{\cos 2x}{\frac{1}{1+x}}=1.$$

14.7-misol. $\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ limitni hisoblang.

Yechilishi. Berilgan funksiya $x=0$ nuqtada $(\infty-\infty)$ ko'rinishdagi aniqmaslikni ifodalaydi. Bu aniqmaslikni $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltirib, so'ngra Lopitalning birinchi qoidasini takroriy qo'llab,

$$\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)\overset{(\infty-\infty)}{=}\lim_{x\to0}\frac{x-\sin x}{x\cdot\sin x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=0$$

bo'lishini topamiz.

14.8-misol. Ushbu

$$1)\ \lim_{x\to 0+0}(1+x)^{\ln x};\quad 2)\ \lim_{x\to 0}(\mathrm{ctg}x)^{\sin x};\quad 3)\ \lim_{x\to 1-0}[\arcsin(\ln x)]^{\mathrm{tg}x}$$

limitlarni hisoblang

Yechilishi. 1) $y=(1+x)^{\ln x}$ funksiya $x=0$ nuqtada (1^{∞}) ko'rinishdagi aniqmaslikni ifodalaydi. Bu aniqmaslikni ochish uchun avvalo, berilgan funksiyani logarifmlab, $(0\cdot\infty)$ ko'rinishdagi aniqmaslikka keltiramiz:

$$\lim_{x\to 0+0}\ln y=\lim_{x\to +0}\ln x\ln(1+x).$$

Buni shakl almashtirish natijasida $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltiramiz va Lopitalning birinchi qoidasini qo'llaymiz:

$$\lim_{x\to 0+0}\ln y=\lim_{x\to 0+0}\frac{\ln(1+x)}{\frac{1}{\ln x}}^{\left(\frac{0}{0}\right)}=-\lim_{x\to 0+0}\frac{\frac{1}{1+x}}{\frac{1}{x}\cdot\frac{1}{\ln^2 x}}=-\lim_{x\to 0+0}\frac{x\cdot\ln^2 x}{1+x}^{(0\cdot\infty)}==\lim_{x\to 0+0}\frac{1}{1+x}\cdot x\ln^2 x=$$

$$=\lim_{x\to 0+0}\frac{\ln^2 x}{\frac{1}{x}}^{\left(\frac{\infty}{\infty}\right)}=\lim_{x\to 0+0}\frac{2\cdot\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}}=-2\lim_{x\to 0+0}x\ln x\overset{(0\cdot\infty)}{=}-2\lim_{x\to 0+0}\frac{\ln x}{\frac{1}{x}}^{\left(\frac{\infty}{\infty}\right)}=2\lim_{x\to 0+0}x^2=0.$$

Demak, $\lim\limits_{x\to 0+0} y=e^0=1$.

MAPLE tizimidan foydalanib misolni yechish:

> Limit((1+x)^(ln(x)),x=0,right)=
limit((1+x)^(ln(x)),x=0,right);

$$\lim_{x\to 0+}(1+x)^{\ln(x)}=1.$$

2) Berilgan $y=(\mathrm{ctg}\,x)^{\sin x}$ funksiya $x=0$ nuqtada (∞^0) ko'rinishdagi aniqmaslikni ifodalaydi. $y=(\mathrm{ctg}\,x)^{\sin x}=e^{\sin x\cdot\ln(\mathrm{ctg}\,x)}$ ayniyatdan foydalanib,

$$\lim_{x\to 0}(\mathrm{ctg}\,x)^{\sin x}=\lim_{x\to 0}e^{\sin x\ln\mathrm{ctg}\,x}=e^{\lim\limits_{x\to 0}\sin x\ln\mathrm{ctg}\,x}. \qquad (*)$$

bo'lishini topamiz. So'ngra, (*) ning o'ng tomonidagi ko'rsatkichli funksiyaning limitini hisoblaymiz:

$$\lim_{x\to 0}\sin x\ln\mathrm{ctg}\,x=\lim_{x\to 0}\frac{\ln\mathrm{ctg}\,x}{\frac{1}{\sin x}}=\lim_{x\to 0}\frac{\frac{1}{\mathrm{ctg}\,x}\cdot\left(-\frac{1}{\sin^2 x}\right)}{-\frac{1}{\sin^2 x}\cdot\cos x}=\lim_{x\to 0}\frac{1}{\mathrm{ctg}\,x\cdot\cos x}=\lim_{x\to 0}\frac{\sin x}{\cos^2 x}=0.$$

Buni hisobga olsak, (*) dan

$$\lim_{x\to 0}(\operatorname{ctg} x)^{\sin x} = e^0 = 1$$

bo‘ladi.

MAPLE tizimidan foydalanib misolni yechish:

> Limit((ctan(x))^(sin(x)),x=0)=

limit((ctan(x))^(sin(x)),x=0);

$\lim\limits_{x\to 0}(c\tan(x)^{\sin(x)}) = 1.$

3) Bu holda berilgan funksiya $x=0$ nuqtada (0^0) ko‘rinishdagi aniqmaslikni ifodalaydi. Endi

$$[\arcsin x]^{\operatorname{tg} x} = e^{(\operatorname{tg} x \cdot \ln(\arcsin x))}$$

ayniyatga asosan, $\lim\limits_{x\to 0+0}(\arcsin x)^{\operatorname{tg} x} = e^{\lim\limits_{x\to 0+0} \operatorname{tg} x \ln(\arcsin x)}$ bo‘ladi. Oxirgi tenglikning o‘ng tomonidagi ko‘rsatkichli funksiyaning limitini hisoblaymiz:

$$\lim_{x\to 0+0} \operatorname{tg} x \cdot \ln(\arcsin x) \overset{(0\cdot\infty)}{=} \lim_{x\to 0+0} \frac{\ln(\arcsin x)}{\frac{1}{\operatorname{tg} x}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to 0+0} \frac{\frac{1}{\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\operatorname{tg}^2 x}\cdot\frac{1}{\cos^2 x}} =$$

$$= -\lim_{x\to 0+0} \frac{1}{\sqrt{1-x^2}} \cdot \lim_{x\to 0+0} \frac{\sin^2 x}{\arcsin x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0+0} \frac{\sin 2x}{\frac{1}{\sqrt{1-x^2}}} = 0.$$

Shunday qilib, $\lim\limits_{x\to 0+0} (\arcsin x)^{\sin x} = e^0 = 1.$

MAPLE tizimidan foydalanib misolni yechish:

> Limit((arcsin((x)))^tan(x),x=0,right)=

limit((arcsin((x)))^tan(x),x=0,right);

$\lim\limits_{x\to 0+}[\arcsin(x)]^{\tan(x)} = 1.$

14.9-misol. Ushbu

$$1)\ \lim_{x\to 0}\frac{x^2\cdot\sin\frac{1}{x}}{\sin x}; \qquad 2)\ \lim_{x\to\frac{\pi}{2}}\frac{\operatorname{tg} x}{\sec x}$$

limitlarni hisoblashga Lopital qoidalarini qo‘llash mumkinmi?

Yechilishi. 1) Bu holda, $f(x)=x^2\sin\frac{1}{x}$, $g(x)=\sin x$. $\frac{f(x)}{g(x)}$ nisbatning $x\to 0$dagi limiti mavjud va u 0 ga teng. Haqiqatdan ham,

$$\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot x\sin\left(\frac{1}{x}\right)=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}\cdot\lim_{x\to 0}x\cdot\sin\frac{1}{x}=1\cdot 0=0.$$

$\frac{f'(x)}{g'(x)}$ning $x\to 0$ dagi limiti mavjud emas:

$$\lim_{x\to 0}\frac{2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)}{\cos x}=\frac{\lim\limits_{x\to 0}2x\sin\left(\frac{1}{x}\right)-\lim\limits_{x\to 0}\cos\left(\frac{1}{x}\right)}{\lim\limits_{x\to 0}\cos x}=0-\lim_{x\to 0}\cos\frac{1}{x},$$

bunda $\lim\limits_{x\to 0}\cos\frac{1}{x}$ mavjud emas. Demak, bu holda Lopital qoidasini qo'llash mumkin emas.

MAPLE tizimidan foydalanib misolni yechish:

> Limit((x^2)*sin(1/x)/sin(x),x=0)=

limit((x^2)*sin(1/x)/sin(x),x=0);

$$\lim_{x\to 0}\left(\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin(x)}\right)=0$$

2) Bu holda $f(x)=\mathrm{tg}\,x$, $g(x)=\sec x$ bo'lib, $\frac{f(x)}{g(x)}$ nisbatning $x\to\frac{\pi}{2}$ dagi limiti mavjud:

$\lim\limits_{x\to\frac{\pi}{2}}\frac{\mathrm{tg}\,x}{\sec x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\sin x\cos x}{\cos x}=\lim\limits_{x\to\frac{\pi}{2}}\sin x=1.$ Lekin Lopital qoidasini qo'llash bilan maqsadga erishib bo'lmaydi:

$$\lim_{x\to\frac{\pi}{2}}\frac{\mathrm{tg}\,x}{\sec x}=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{\sec x\cdot\mathrm{tg}\,x}=\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\mathrm{tg}\,x}=\lim_{x\to\frac{\pi}{2}}\frac{\mathrm{tg}\,x}{\sec x}=\ldots$$

Mustaqil yechish uchun misol va masalalar

Lopital qoidalaridan foydalanib, quyidagi funksiyalarning limitini hisoblang

14.1. $\lim\limits_{x\to1}\dfrac{3x^2+5x-8}{4x^2+3x-7}$.

14.2. $\lim\limits_{x\to4}\dfrac{\ln(x^2-15)}{3x^2-10x-8}$.

14.3. $\lim\limits_{x\to0}\dfrac{e^{ax}-e^{-2ax}}{\ln(1+x)}$.

14.4. $\lim\limits_{x\to-1}\dfrac{\sqrt[3]{1+2x}+1}{\sqrt{2+x}+x}$.

14.5. $\lim\limits_{x\to0}\dfrac{\ln\sin 2x}{\ln\sin x}$.

14.6. $\lim\limits_{x\to0}\dfrac{\ln x}{\ln\sin x}$.

14.7. $\lim\limits_{x\to0}\dfrac{e^{1/x^2}-1}{\operatorname{arctg}x^2-\pi}$.

14.8. $\lim\limits_{x\to0}\dfrac{\ln\cos 2x}{\ln\cos 5x}$.

14.9. $\lim\limits_{x\to1}\dfrac{a^{\ln x}-x}{\ln x}$.

14.10. $\lim\limits_{x\to1}\dfrac{\operatorname{tg}x-x}{x-\sin x}$.

14.11. $\lim\limits_{x\to0}\dfrac{x\cdot\operatorname{ctg}x-1}{x^2}$.

14.12. $\lim\limits_{x\to0}\dfrac{\sin ax-\sin bx}{\operatorname{sh}ax-\operatorname{sh}bx}$.

14.13. $\lim\limits_{x\to0}\dfrac{3\operatorname{tg}4x-12\operatorname{tg}x}{3\sin 4x-12\sin x}$.

14.14. $\lim\limits_{x\to1}\dfrac{x^\alpha-1}{x^\beta-1}$, $\beta\neq0$.

14.15. $\lim\limits_{x\to+\infty}\dfrac{x^3}{e^x}$.

14.16. $\lim\limits_{x\to+\infty}\dfrac{x^n}{e^{ax}}$, $a>0$.

14.17. $\lim\limits_{x\to0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$.

14.18. $\lim\limits_{x\to1}\dfrac{1-4\sin^2\left(\frac{\pi x}{6}\right)}{1-x^2}$.

14.19. $\lim\limits_{x\to0}\dfrac{a^x-b^x}{x\sqrt{1-x^2}}$.

14.20. $\lim\limits_{x\to+\infty}\dfrac{\pi-2\operatorname{arctg}x}{\ln\left(1+\frac{1}{x}\right)}$.

14.21. $\lim\limits_{x\to0}\dfrac{a^x-b^x}{c^x-d^x}$.

14.22. $\lim\limits_{x\to a}\dfrac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$.

14.23. $\lim\limits_{x\to0}\dfrac{1-\cos x^2}{x^2\cdot\sin x^2}$.

14.24. $\lim\limits_{x\to0}\dfrac{\arcsin 2x-2\arcsin x}{x^3}$.

14.25. $\lim\limits_{x\to0}\dfrac{a^x-a^{\sin x}}{x^3}$.

14.26. $\lim\limits_{x\to a}\dfrac{x^a-a^x}{x^a-a^a}$.

14.27. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{\operatorname{tg}^2 x}$.   14.28. $\lim\limits_{x\to 0}\dfrac{\ln\left(\frac{2}{\pi}\arccos x\right)}{\ln(1+x)}$.

14.29. $\lim\limits_{x\to 1}\dfrac{x^x-1}{\ln x}$.   14.30. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\sin x\cos x}$.

14.31. $\lim\limits_{x\to +\infty}\left(x^n\cdot e^{-x}\right)$ 14.32. $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\dfrac{\pi x}{2}$.

14.33. $\lim\limits_{x\to a}\arcsin(x-a)\cdot\operatorname{ctg}(x-a)$.

14.34. $\lim\limits_{x\to\infty}\left[x\cdot\sin\dfrac{\alpha}{x}\right]$.   14.35. $\lim\limits_{x\to +0}x^\alpha\ln^\beta\left(\dfrac{1}{x}\right)$, $\alpha>0,\ \beta>0$.

14.36. $\lim\limits_{\varphi\to\alpha}\left[\left(\alpha^2-\varphi^2\right)\operatorname{tg}\dfrac{\pi\varphi}{2\alpha}\right]$. 14.37. $\lim\limits_{x\to\infty}\left(\alpha^{\frac{1}{x}}-1\right)x$, $(\alpha>0)$. 14.38. $\lim\limits_{x\to 0+0}\left(x^x-1\right)\ln x$.

14.39. $\lim\limits_{x\to\infty}x^\alpha a^x$, $a>0$, $a\neq 1$.   14.40. $\lim\limits_{x\to\infty}x\ln\left(\dfrac{2}{\pi}\operatorname{arctg}x\right)$.

14.41. $\lim\limits_{x\to 1}\left[\dfrac{1}{\ln x}-\dfrac{x}{\ln x}\right]$.   14.42. $\lim\limits_{x\to 0}\left(\operatorname{ctg}x-\dfrac{1}{x}\right)$. 14.43. $\lim\limits_{x\to\infty}\left[x\left(e^{\frac{1}{x}}-1\right)\right]$.

14.44. $\lim\limits_{x\to 0}\left[\dfrac{\ln(1+x)^{1+x}}{x^2}-\dfrac{1}{x}\right]$.   14.45. $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\dfrac{1}{x}\right)\right]$.

14.46. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\operatorname{ctg}^2 x\right)$ 14.47. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)$.

14.48. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$ 14.49. $\lim\limits_{x\to 1}\left(\dfrac{\alpha}{1-x^\alpha}-\dfrac{\beta}{1-x^\beta}\right)$. 14.50. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$.

14.51. $\lim\limits_{x\to 0}\left(\operatorname{ctg}x-\dfrac{1}{x}\right)$. 14.52. $\lim\limits_{x\to 0}\left[\dfrac{1}{\ln\left(x+\sqrt{1+x^2}\right)}-\dfrac{1}{\ln(1+x)}\right]$.

14.53. $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\operatorname{tg}x}$. 14.54. $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}$. 14.55. $\lim\limits_{x\to 0+0}|\ln x|^{2x}$. 14.56. $\lim\limits_{x\to 0}x^{\frac{1}{\ln\left(e^x-1\right)}}$.

14.57. $\lim\limits_{x\to\frac{\pi}{2}}(\operatorname{tg}x)^{2x-\pi}$. 14.58. $\lim\limits_{x\to 0}\left[\dfrac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$. 14.59. $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\operatorname{ch}x}\right)^{\frac{1}{x^2}}$.

14.60. $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$. 14.61. $\lim\limits_{x\to\infty}\left(\dfrac{2}{\pi}\operatorname{arctg}x\right)^x$. 14.62. $\lim\limits_{x\to 0}(1+x)^{\ln x}$.

14.63. $\lim\limits_{x\to\frac{\pi}{2}-0}(\operatorname{tg}x)^{\cos x}$. **14.64.** $\lim\limits_{x\to 0}(3x^2+3x)^{\frac{1}{x}}$. **14.65.** $\lim\limits_{x\to+0+0}x^{\frac{1}{\ln \operatorname{sh} x}}$.

14.66. $\lim\limits_{x\to+0-0}x^{x^x-1}$. **14.67.** $\lim\limits_{x\to 0+0}\left(\frac{1}{x}\right)^{\operatorname{tg}x}$. **14.68.** $\lim\limits_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{1/x}$.

14.69. $\lim\limits_{x\to+0+0}(\arcsin x)^{\operatorname{tg}x}$. **14.70.** $\lim\limits_{x\to 0}\left(x^{x^x}-1\right)$. **14.71** $\lim\limits_{x\to 0}\frac{\ln \operatorname{ch}x}{\sqrt[4]{\operatorname{ch}x}-\sqrt[3]{\operatorname{ch}x}}$.

14.72 $\lim\limits_{x\to+0-0}\frac{x^{\ln x}}{(\ln x)^x}$. **14.73.** $\lim\limits_{x\to+0+0}\left[\left(\sqrt[3]{x^3+x^2+x+1}-\sqrt{x^2+x+1}\right)\frac{\ln\left(e^x+x\right)}{x}\right]$

14.74. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\arcsin x}\right)$ **14.75.** $\lim\limits_{x\to 0}\left[\frac{\ln(1+x)^{1-x}}{x^2}-\frac{1}{x}\right]$

14.76. $\lim\limits_{x\to 0}\left(\frac{5}{2+\sqrt{9+x}}\right)^{\frac{1}{\sin x}}$. **14.77.** $\lim\limits_{x\to 0+0}\frac{e^{1/x^2}-1}{2\operatorname{arctg}x^2-\pi}$.

14.78. $\lim\limits_{x\to 0}\frac{sh^2 x}{\ln(ch3x)}$. **14.79.** $\lim\limits_{x\to 1}\left(\frac{\pi}{2}\operatorname{tg}\frac{\pi x}{2}-\frac{1}{1-x}\right)$

14.80. $\lim\limits_{x\to+\infty}\left[(x+\alpha)^{1+\frac{1}{x}}-x^{1+\frac{1}{x+\alpha}}\right]$ **14.81.** $\lim\limits_{x\to 0}\frac{\ln(1+x)+\frac{x^2}{2}-\sin x}{\operatorname{arctg}^3 x}$.

14.82. $\lim\limits_{x\to 0}\left(\frac{1}{\operatorname{tg}x}-\frac{1}{e^x-1}\right)$ **14.83.** $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{x}{\operatorname{ctg}x}-\frac{\pi}{2\cos x}\right)$

14.84. $\lim\limits_{x\to 0}\left[\frac{\pi x-1}{2x^2}+\frac{\pi}{x(e^{2\pi x}-1)}\right]$ **14.85.** $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{\operatorname{tg}\frac{\pi x}{2a}}$.

14.86. $\lim\limits_{x\to 0}\left(2\sqrt{x}+x\right)^{\frac{1}{\ln x}}$. **14.87.** $\lim\limits_{x\to\infty}\left(\operatorname{ctg}\frac{\pi x}{1+x}\right)^{\frac{1}{x}}$. **14.88.** $\lim\limits_{x\to\infty}\left(x+2^x\right)^{\frac{1}{x}}$.

14.89. Quyidagi limitlarni Lopital qoidasi bo'yicha hisoblash mumkin emasligini ko'rsating va ularning limitini hisoblang:

1). $\lim\limits_{x\to 0}\frac{x^2\sin(1/x)}{\sin^2 x}$; 2). $\lim\limits_{x\to\infty}\frac{x+\cos x}{x-\cos x}$.

14.90. $\lim\limits_{x\to\infty}\frac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ limitni hisoblashga Lopital qoidasini qo'llash mumkinmi, agar limit mavjud bo'lsa, uni hisoblang.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

14.1. 1. **14.2.** $\frac{4}{7}$. **14.3.** $3a$. **14.4.** $\frac{4}{9}$. **14. 5.** 1. **14.6.** 1. **14.7.** 0. **14.8.** $\frac{4}{25}$. **14.9.** $\ln a-1$. **14.10.** 2. **14.11.** $-\frac{1}{3}$. **14.12.** 1. **14.13.** -2. **14.14.** $\frac{\alpha}{\beta}$. **14.15.** 0. **14.16.** 0. **7.17.** 2. **14.18.** $\frac{\pi\sqrt{3}}{6}$. **14.19.** $\ln\frac{a}{b}$. **14.20.** 2. **14.21.** $\frac{\ln\frac{a}{b}}{\ln\frac{c}{d}}$. **14.22.** $\cos a$. **14.23.** $\frac{1}{2}$. **14.24.** 1. **14.25.** $\frac{1}{6}\ln a$. **14.26.** $1-\ln a$. **14.27.** $-\frac{1}{2}$. **14.28.** $-\frac{2}{\pi}$. **14.29.** 1. **14.30.** 2. **14.31.** 0. **14.32.** $\frac{2}{\pi}$. **14.33.** 1. **14.34.** a. **14.35.** 0. **14.36.** $\frac{4a^2}{\pi}$. **14.37.** $\ln a$. **14.38.** 0. **14.39.** 0, agar $0<a<1$, α -ixtiyoriy bo‘lganda, $+\infty$, agar $a>1$, α - ixtiyoriy bo‘lganda .

14.40. $-\frac{2}{\pi}$. **14.41.** -1. **14.42.** 0. **14.43.** 1. **14.44.** $\frac{1}{2}$. **14.45.** $\frac{1}{2}$. **14.46.** $\frac{2}{3}$. **14.47.** $-\frac{1}{3}$. **14.48.** 0. **14.49.** $\frac{\alpha-\beta}{2}$. **14.50.** $\frac{1}{2}$. **14.51.** 0. **14.52.** $-\frac{1}{2}$. **14.53.** 1. **14.54.** e^{-1}. **14.55.** 1. **14.56.** e. **14.57.** 1. **14.58.** $e^{-\frac{1}{2}}$. **14.59.** e^{-1}. **14.60.** e^{2}. **14.61.** $e^{-\frac{2}{\pi}}$. **14.62.** 1. **14.63.** 1. **14.64.** 3. **14.65.** e. **14.66.** 1. **14.67.** 1. **14.68.** $e^{-\frac{2}{\pi}}$. **14.69.** 1. **14.70.** -1. **14.71.** $\frac{mn}{n-m}$. **14.72.** 0. **14.73.** $-\frac{1}{6}$. **14.74.** 0. **14.75.** $\frac{1}{2}$. **14.76.** $e^{-\frac{1}{30}}$. **14.77.** $-\frac{1}{2}$. **14.78.** $\frac{2}{9}$. **14.79.** 0. **14.80.** a. **17. 81.** $\frac{1}{2}$. **14.82.** $\frac{1}{2}$. **14.83.** -1. **14.84.** $\frac{\pi^2}{6}$. **14.85.** $e^{-\frac{2}{\pi}}$. **14.86.** $\sqrt{e}$. **14.87.** 1. **14.88.** 2. **14.90.** Lopital qoidasini qo‘llash mumkin emas, limit mavjud emas.

15-§. TEYLOR FORMULASI

15.1. Teylor teoremasi. Tabiatda ko'pgina masalalar funksiyaning berilgan nuqtadagi qiymatini topishga bog'liq bo'ladi. Funksiya murakkab bo'lgan hollarda funksiyaning berilgan nuqtadagi qiymatini hisoblash har doim ham yengil bo'lavermaydi. Bunday hollarda, nuqtadagi qiymatini hisoblash noqulay bo'lgan funksiyani, o'ziga qaraganda sodda va hisoblash uchun qulay bo'lgan funksiyaga yaqinlashtirish-almashtirishga to'g'ri keladi. Berilgan $f(x)$ funksiyani biror g(x) funksiyaga yaqinlashtirish-almashtirishda quyidagi ikki momentni e'tiborga olish muhimdir:

1) $f(x)$ ga yaqinlashadigan g(x) funksiyaning tanlab olinishi va uning tuzilishi (soddaligi, hisoblash uchun qulayligi);

2) $f(x)$ ni g(x) ga yaqinlashtirishdagi qo'yilgan xatolikni aniqlash va uni hisoblash.

Odatda yaqinlashadigan funksiya sifatida butun ratsional $P_n(x)$ ko'phad olinadi.

1885-yilda buyuk nemis matematigi K.Veyershtrass $[a,b]$ kesmada uzluksiz bo'lgan $f(x)$ funksiyani $P_n(x)$ ko'phad bilan yaqinlashtirish mumkinligi haqidagi teoremani isbot qiladi, lekin bu teorema $R_n(x)=f(x)-P_n(x)$ ayirmani baholashni va uning nolga intilish tartibini aniqlab bermaydi. Keyingi yillardagi ilmiy izlanishlar $R_n(x)$ ning nolga intilish tartibi yaqinlashtiriladigan $f(x)$ funksiyaning hosilalarga ega bo'lishiga bog'liq ekanligini ko'rsatdi.

$f(x)$ funksiya biror x_0 nuqtaning atrofida yuqori tartibli hosilalarga ega bo'lsa, bu hosilalardan foydalanib, avvalo $P_n(x)$ ko'phadni tuzish va $f(x)$ funksiyani bu ko'phad bilan yaqinlashtirish masalasini qarash mumkin bo'ladi. Bu masalani yechishda Teylor formulasi muhim rol o'ynaydi.

15.1-teorema (Teylor). *$f(x)$ funksiya biror a nuqtaning atrofida $n+1$ tartibgacha hosilalarga ega, x - funksiya argumentining a nuqtaning atrofidagi ixtiyoriy qiymati, r – ixtiyoriy musbat son bo'lsin. U holda a va x nuqtalar orasida shunday ξ nuqta topiladiki, bu nuqtada*

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+...+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_{n+1}(x) \quad (15.1)$$

formula o'rinli bo'ladi, bunda

$$R_{n+1}(x)=\left(\frac{x-a}{x-\xi}\right)^p\cdot\frac{(x-\xi)^{n+1}}{n!\,p}\cdot f^{(n+1)}(\xi). \qquad (15.2)$$

(15.1) formulaga *Teylor formulasi* deyiladi, $R_{n+1}(x)$ ifoda esa, *Teylor formulasining qoldiq hadi* deyiladi. Odatda, (15.2) ko'rinishdagi qoldiq had *umumiy yoki Shlyomilh-Rosh ko'rinishdagi qoldiq had* deb ham yuritiladi.

Teylor formulasidan kengroq foydalanish maqsadida, uning qoldiq hadining umumiy ko'rinishi (15.2) dan xususiy holda kelib chiqadigan quyidagi turli xil ko'rinishlarini keltiramiz:

$$R_{n+1}(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],\quad 0<\theta<1,$$

$$R_{n+1}(x)=\frac{(x-a)^{n+1}\cdot(1-\theta)^n}{n!}\cdot f^{(n+1)}[a+\theta(x-a)],\ 0<\theta<1,$$

$$R_{n+1}(x)=o\left((x-a)^n\right) \qquad (15.3)$$

Qoldiq hadning bu ko'rinishlari, mos ravishda, qoldiq hadning *Lagranj, Koshi va Peano ko'rinishlari* deyiladi.

15.1-eslatma. Peano ko'rinishidagi qoldiq hadni keltirib chiqarishda 15.1-teoremadagi $f(x)$ funksiyaga nisbatan qo'yilgan shartni «yengillashtirish» mumkin, ya'ni, $f(x)$ funksiya a nuqtaning atrofida

$f'(x), f''(x), \ldots, f^{(n)}(x)$ hosilalarga ega bo'lib, $f^{(n)}(x)$ hosila esa a nuqtada uzluksiz bo'lsin, degan shart yetarli.

Yechilayotgan masalaning xususiyatiga qarab, u yoki bu ko'rinishdagi qoldiq hadli Teylor formulasidan foydalaniladi. Masalan, a nuqta atrofidagi $x (x \neq a)$ nuqtalarda $f(x)$ funksiyaning qiymatlarini hisoblash kerak bo'lganda, Koshi yoki Lagranj ko'rinishidagi qoldiq hadli Teylor formulasidan foydalanilgan ma'qul. Agar $x \to a$ da qoldiq hadning nolga intilish tartibini bilish lozim bo'lsa, u holda Peano ko'rinishidagi qoldiq hadli Teylor formulasidan foydalanish qulay bo'ladi.

$f(x)$ funksiyaning (15.1) Teylor formulasida $a=0$ deb olinsa, ushbu

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + R_{n+1}(x) \qquad (*)$$

formula hosil bo'ladi. Odatda bu formula *Makloren formulasi* deyiladi. Bu holda qoldiq had $R_{n+1}(x)$ quyidagicha:

1) *Lagranj ko'rinishida*: $R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x), \ (0 < \theta < 1)$;

2) *Koshi ko'rinishida*: $R_{n+1}(x) = \frac{x^{n+1}(1-\theta)^n}{n!} f^{(n+1)}(\theta x), \ (0 < \theta < 1)$;

3) *Peano ko'rinishida*: $R_{n+1}(x) = o(x^n)$

kabi yozilishi mumkin.

Ko'p hollarda, (15.1) Teylor formulasi quyidagi ko'rinishda ham yoziladi: (15.1) formulada $a = x_0$, $x - a = \Delta x$, qoldiq had Lagranj ko'rinishida olinsa,

$$f(x_0 + \Delta x) - f(x_0) = \frac{f'(x_0)}{1!}\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(\Delta x)^n + \frac{f^{(n+1)}(x_0 + \theta\Delta x)}{(n+1)!} \cdot (\Delta x)^{n+1} \quad (0 < \theta < 1) \qquad (15.4)$$

formula hosil bo'ladi. (15.4) Teylor formulasi chekli orttirmalar haqidagi Lagranj formulasining umumlashmasi bo'lib hisoblanadi (16-§ ga q.). Xususiy holda, (15.4) da $n = 0$ deb olsak Lagranj formulasi kelib chiqadi.

Ixtiyoriy funksiya uchun Lagranj ko'rinishidagi qoldiq hadli Makloren formulasini qaraymiz va uning qoldiq hadini baholaymiz.

Agar shunday o'zgarmas M son mavjud bo'lib, argument x ning $x_0=0$ nuqta atrofidagi barcha qiymatlarida hamda $\forall n \in N$ uchun

$$\left|f^{(n)}(x)\right| \le M$$

tengsizlik o'rinli bo'lsa, u holda

$$\left|R_{n+1}(x)\right| = \left|\frac{f^{(n+1)}(\theta x) x^{n+1}}{(n+1)!}\right| \le M \cdot \frac{|x|^{n+1}}{(n+1)!}$$

tengsizlik o'rinli bo'ladi. x ning har bir belgilangan qiymatida

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

bo'lishini e'tiborga olsak, u holda n ning yetarli katta qiymatlarida $R_{n+1}(x)$ ning yetarli kichik bo'lishiga ishonch hosil qilamiz. Bu holda $x_0 = 0$ nuqtaning atrofida $f(x)$ funksiyani

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

ko'phad bilan taqribiy almashtirish mumkin bo'ladi:

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n.$$

$f(x)$ funksiya $x_0 = 0$ nuqtaning atrofida istalgan tartibdagi hosilaga ega (cheksiz differensiallanuvchi) bo'lganda:

a) agar $f(x)$ – juft bo'lsa, u holda $\forall n \in N$ uchun

$$f(x) = \sum_{k=0}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} + R_{2n+1}(x);$$

b) agar $f(x)$ – toq bo'lsa, u holda $\forall n \in N$ uchun

$$f(x) = \sum_{k=0}^{n} \frac{f^{(2k+1)}}{(2k+1)!} x^{2k+1} + R_{2n+2}(x)$$

bo'ladi.

15.2. Elementar funksiyalar uchun Makloren formulasi

$f(x)=e^x$, $f(x)=\sin x$, $f(x)=\cos x$, $f(x)=shx$, $f(x)=chx$,
$f(x)=(1+x)^m$, $f(x)=\ln(1+x)$

funksiyalar uchun Peano ko'rinishidagi qoldiq hadli Makloren formulasi quyidagicha bo'ladi:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+o(x^n), \tag{15.5}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-....+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+o(x^{2n+2}), \tag{15.6}$$

$$shx=x+\frac{x^3}{3!}+\frac{x^5}{5!}+....+\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2}), \tag{15.7}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+....+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n+1}), \tag{15.8}$$

$$chx=1+\frac{x^2}{2!}+\frac{x^4}{4!}+....+\frac{x^{2n}}{(2n)!}+o(x^{2n+1}),$$

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+....+\frac{m(m-1)....(m-(n-1))}{n!}x^n+o(x^n), \tag{15.9}$$

yoki

$$(1+x)^m=\sum_{k=0}^{n}C_\alpha^k x^k+0(x^n),\ C_\alpha^0=1, C_\alpha^k=\frac{\alpha(\alpha-1)....(\alpha-(k-1))}{k!},\ k=1,2,...$$

Xususiy holda,

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n), \tag{15.10}$$

$$\frac{1}{1-x}=\sum_{k=0}^{n}x^k+o(x^n),$$

$$\ln(1+x)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}x^k+o(x^n) \tag{15.11}$$

$$\ln(1-x)=-\sum_{k=1}^{n}\frac{x^k}{k}+o(x^n). \tag{15.12}$$

$$tgx=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+o(x^{11}).$$

$$a^x = 1 + (\ln a)x + \frac{1}{2!}(\ln a)^2 x^2 + \ldots + \frac{1}{n!}(\ln a)^n x^n + o\left(x^{n+1}\right), \quad a > 0, \ a \neq 1.$$

Agar

$$f(x) = \sum_{k=0}^{n} a_k (x - x_0)^k + o((x - x_0)^n), \qquad g(x) = \sum_{k=0}^{n} b_k (x - x_0)^k + o((x - x_0)^n)$$

bo'lsa,

$$f(x) + g(x) = \sum_{k=0}^{n} (a_k + b_k) x^k + o((x - x_0)^n), \tag{15.13}$$

$$f(x) \cdot g(x) = \sum_{k=0}^{n} C_k (x - x_0)^k + o((x - x_0)^n), \tag{15.14}$$

bunda $\quad C_k = \sum_{p=0}^{k} a_p b_{k-p}.$

15.1-misol. Ushbu $f(x) = arctgx$ funksiyani $o\left(x^n\right)$ hadgacha Makloren formulasiga yoying.

Yechilishi. Ma'lumki,

$$f^{(n)}(x) = \frac{(n-1)!}{(1+x^2)^{n/2}} \cdot \sin\left[n(arctg\, x + \frac{\pi}{2})\right],$$

$$f^{(n)}(0) = \begin{cases} 0, & n \text{ juft bo'lganda}, \\ (-1)^{\frac{n-1}{2}} (n-1)!, & n \text{ toq bo'lganda}. \end{cases}$$

Demak, Peano ko'rinishidagi qoldiq hadli (*) Makloren formulasiga asosan,

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{\frac{n-1}{2}} \frac{x^n}{n} + o(x^n) \tag{15.15}$$

15.2-misol. Ushbu

$$f(x) = \sqrt{1 - 2x + x^3} - \sqrt[3]{1 - 3x + x^2}$$

funksiyani $o\left(x^3\right)$ hadgacha Makloren formulasiga yoying.

Yechilishi. $f_1(x) = \sqrt{1 - 2x + x^3}$, $f_2(x) = \sqrt[3]{1 - 3x + x^2}$ deb belgilab, (15.9) formulaga asosan:

1) $m=\frac{1}{2}$ bo'lgan holda,

$$f_1(x)=[1-(2x-x^3)]^{\frac{1}{2}}=1-\frac{1}{2}(2x-x^3)-\frac{1}{8}(2x-x^3)^2-\frac{1}{16}(2x-x^3)^3+o(x^3),$$

2) $m=\frac{1}{3}$ bo'lgan holda,

$$f_2(x)=\sqrt[3]{1-3x+x^2}=[1-(3x-x^2)]^{\frac{1}{3}}=1-\frac{1}{3}(3x-x^2)-\frac{1}{9}(3x-x^2)^2-\frac{5}{81}(3x-x^2)^3+o(x^3)$$

ifodalarga ega bo'lamiz.

Shunday qilib, (18.3) formulaga asosan

$$f(x)=f_1(x)-f_2(x)=\frac{1}{6}x^2+x^3+o(x^3).$$

MAPLE tizimidan foydalanib misolni yechish:

> Order:=6;

$$Order := 6$$

> series(sqrt(1-2*x+x^3)-surd(1-3*x+x^2,3),x=0);

$$\frac{1}{6}x^2+x^3+\frac{119}{72}x^4+\frac{239}{72}x^5+O(x^6)$$

15.3-misol. Ushbu $f(x)=e^{2x+3x^2}$ funksiyani $o(x^3)$ gacha Makloren formulasiga yoying.

Yechilishi. (15.5) formulaga asosan:

$$e^{2x+3x^2}=1+\frac{(2x+3x^2)}{1!}+\frac{(2x+3x^2)^2}{2!}+\frac{(2x+3x^2)^3}{3!}+o(x^3)=1+2x+5x^2+\frac{22}{3}x^3+o(x^3)$$

ni hosil qilamiz.

MAPLE tizimidan foydalanib misolni yechish:

> Order:=6;

$$Order := 6$$

> series(ex' (2*x+3*x^2),x=0);

$$1+2x+5x^2+\frac{22}{3}x^3+\frac{67}{6}x^4+\frac{199}{15}x^5+O(x^6)$$

15.4-misol. Ushbu

1) $f(x)=\frac{1}{3x+4}$; 2) $f(x)=\ln(2+3x)$; 3) $f(x)=\frac{1}{\sqrt[3]{1-x}}$

funksiyalarni $o(x^n)$ hadgacha Makloren formulasiga yoying.

Yechilishi. 1) Berilgan $\frac{1}{3x+4}$ funksiyani $\frac{1}{3x+4}=\frac{1}{3\left(1+\frac{4}{3}x\right)}$ kabi tasvirlab, (15.10) formulani e'tiborga olgan holda,

$$\frac{1}{3x+4}=\frac{1}{3}\sum_{k=0}^{n}(-1)^k\left(\frac{4}{3}\right)^k\cdot x^k+o(x^n)=\sum_{k=0}^{n}(-1)^k\cdot\frac{4^k}{3^{k+1}}\cdot x^k+o(x^n)$$

yoyilmaga ega bo'lamiz.

2) Ushbu $\ln(2+3x)=\ln 2+\ln\left(1+\frac{3}{2}x\right)$ tenglikdan hamda (15.11) formuladan

$$\ln(2+3x)=\ln 2+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left(\frac{3}{2}\right)^k\cdot x^k+o(x^n)$$

yoyilmaga ega bo'lamiz.

3) Berilgan funksiyani $\frac{1}{\sqrt[3]{1-x}}=(1+(-1)x)^{-\frac{1}{3}}$ kabi tasvirlab, (15.9) formuladan, $m=-\frac{1}{3}$ bo'lganda $\frac{1}{\sqrt[3]{1-x}}=1+\sum_{k=1}^{n}C_{-\frac{1}{3}}^{k}\cdot x^k+o(x^n)$ yoyilmani topamiz, bunda

$$C_{-\frac{1}{3}}^{k}=\frac{\left(-\frac{1}{3}\right)\cdot(-1/3-1)\ldots(-1/3-(k-1))}{k!}=(-1)^k\cdot\frac{(3k-2)!}{3^k\cdot k!}$$

15.5-misol. Ushbu

1) $f(x)=(x+2)e^{\frac{1}{2}x}$; 2) $f(x)=e^{3x}\cdot\ln(1-x)$, $n=4$

funksiyalarni $o(x^n)$ aniqlikda Makloren formulasiga yoying.

Yechilishi. **1)** Avvalo, berilgan $f(x)=(x+2)e^{\frac{1}{2}x}$ funksiyani $f(x)=xe^{\frac{x}{2}}+2e^{\frac{x}{2}}$ ko‘rinishda yozib, so‘ngra (15.5) va (15.13) formulalardan foydalanib, hamda

$$f(x)=(x+2)e^{\frac{1}{2}}=\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)^k\cdot x^k}{k!}+o(x^{n-1})+2\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)^k\cdot x^k}{k!}+o(x^n)=\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)^k\cdot x^{k+1}}{k!}+$$

$$+\sum_{k=0}^{n}\frac{2\left(\frac{1}{2}\right)^k\cdot x^k}{k!}+o(x^n),\quad \sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)^k\cdot x^{k+1}}{k!}=\sum_{k=1}^{n}\frac{\left(\frac{1}{2}\right)^{k-1}}{(k-1)!}x^k;$$

$$\sum_{k=0}^{n}\frac{2\cdot\left(\frac{1}{2}\right)^k x^k}{k!}=2+\sum_{k=1}^{n}\frac{2\cdot\left(\frac{1}{2}\right)^k x^k}{k!}.$$

ekanligini e’tiborga olib,

$$f(x)=2+\sum_{k=1}^{n}\left(\frac{\left(\frac{1}{2}\right)^{k-1}}{(k-1)!}+\frac{2\cdot\left(\frac{1}{2}\right)^k}{k!}\right)x^k+o(x^n)=2+\sum_{k=1}^{n}\frac{\left(\frac{1}{2}\right)^{k-1}}{k!}(k+1)x^k+o(x^n)=$$

$$=\sum_{k=0}^{n}\frac{1}{2^{k-1}k!}(k+1)x^k+o(x^n).$$

yoyilmani hosil qilamiz.

2) (15.5), (15.12) va (15.14) yoyilmalarni e’tiborga olib,

$$f(x)=e^{3x}\cdot\ln(1-x)=\left(1+3x+\frac{9}{2!}x^2+\frac{9}{2}x^3+o(x^3)\right)\cdot(-1)\left(x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)+o(x^4)=$$

$$=-x-\left(\frac{1}{2}+3\right)x^2-\left(\frac{1}{3!}+\frac{3}{2}+\frac{9}{2}\right)x^3-\left(\frac{1}{4!}+\frac{1}{2!}+\frac{9}{2!}\cdot\frac{1}{2!}+\frac{9}{2}\right)x^4+o(x^4)=$$

$$=-\left(x+\frac{7}{2}x^2+\frac{37}{6}x^3+\frac{175}{24}x^4\right)+o(x^4).$$

yoyilmaga ega bo‘lamiz.

15.6-misol. Ushbu

$$f(x)=\cos x\cdot\sin^2 x$$

funksiyani $o(x^{2n+1})$ hadgacha Makloren formulasiga yoying.

Yechilishi. Berilgan funksiyaning shaklini quyidagicha o'zgartiramiz: $\cos x\sin^2 x=\cos x(1-\cos^2 x)=\cos x-\cos^3 x$. So'ngra, $\cos 3x=4\cos^3 x-3\cos x$, munosabatdan $\cos^3 x=\frac{1}{4}\cos 3x+\frac{3}{4}\cos x$ kabi yozish mumkin. Keyingi tenglikni e'tiborga olsak,

$$\cos x\sin^2 x=\cos x-\frac{1}{4}\cos 3x-\frac{3}{4}\cos x=\frac{1}{4}\cos x-\frac{1}{4}\cos 3x$$

bo'ladi. So'ngra, (15.8) va (15.14) formulalarga binoan,

$$\cos x\sin^2 x=\frac{1}{4}\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}-\frac{1}{4}\sum_{k=0}^{n}(-1)^k\cdot\frac{3^{2k}\cdot x^{2k}}{(2k)!}+o(x^{2n+1})=$$

$$=\sum_{k=0}^{n}\frac{1}{4}\frac{(-1)^k}{(2k)!}\cdot(3^{2k}-1)x^{2k}+o(x^{2n+1}).$$

yoyilmani hosil qilamiz.

15.7-misol. [0; 1] segmentda aniqlangan $f(x)=\ln(1+x)$ funksiyani x ning darajalari bo'yicha Makloren formulasiga yoying. Yoyilmaning birinchi o'nta hadigacha yo'l qo'yilgan xatoni baholang.

Yechilishi. Ma'lumki, $f^{(n)}(x)=(\ln(1+x))^{(n)}=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$

Bundan, $f(0)=0,\ f^{(n)}(0)=(-1)^{n-1}(n-1)!$. Buni e'tiborga olib, $\ln(1+x)$ funksiyaning Makloren formulasini yozamiz:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+\frac{x^9}{9}+R_{10}(x).$$

Qoldiq had $R_{10}(x)$ ni baholashda uning Lagranj ko'rinishidan foydalanamiz:

$$R_{10}(x)=\frac{f^{(10)}(\xi)}{10!}x^{10}=-\frac{9!}{10!(1+\xi)^{10}}\cdot x^{10}=-\frac{x^{10}}{10(1+\xi)^{10}}\qquad(0<\xi<x)$$

$0\le x\le 1,\ \ \xi>0$ larni e'tiborga olgan holda, $R_{10}(x)$ ni absolyut qiymati bo'yicha baholaymiz:

$$|R_{10}(x)| = \left|\frac{-x^{10}}{10(1+\xi)^{10}}\right| < \frac{1}{10}.$$

15.8-misol. x ning qanday qiymatida

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

taqribiy formulada yo'l qo'yilgan xato 0,0005 dan kichik bo'ladi.

Yechilishi. $(\sin x)^{(7)} = -\cos x$ bo'lgani uchun,

$$|R_7(x)| = \left|\frac{-\cos\theta x}{7!}x^7\right| \le \frac{|x|^7}{7!}.$$

Shartga ko'ra, $\frac{|x|^7}{7!} < 0{,}0005$. Demak, $|x| < 0{,}821$ bo'lganda.

15.9-misol. Makloren formulasidan foydalanib, ushbu 1) $\sin 20^0$, 2) $\sqrt[4]{83}$ miqdorlarni $o(x^5)$ hadgacha taqribiy hisoblang.

Yechilishi. 1) (15.15) formulaga asoslanib,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

munosabatni olamiz va undan quyidagi taqribiy formulaga ega bo'lamiz:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

bundan

$$\sin 20^0 = \sin\frac{\pi}{9} \approx \frac{\pi}{9} - \frac{1}{6}\cdot\frac{\pi^3}{729} + \frac{\pi^5}{120\cdot 9^5} \approx 0.3420213308.$$

MAPLE tizimidan foydalanib misolni yechish:

```
> C:=taylor(sin(x), x='i/6,5);
```

$$C := \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{1}{12}\sqrt{3}\left(x-\frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x-\frac{\pi}{6}\right)^4 + o\left(x-\frac{\pi}{6}\right)^5$$

```
> V:=subs(x='i/9,C);convert(evalf(V),' olynom);
```

$$V := \frac{1}{2}+\frac{1}{36}\sqrt{3}\pi-\frac{1}{1296}\pi^2+\frac{1}{69984}\sqrt{3}\pi^3+\frac{1}{5038848}\pi^4+o\left(-\frac{1}{1889568}\pi^5\right)$$

0.3420213308

2) (15.9) formuladan

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+o(x^3)$$

ni hosil qilamiz. Bundan esa,

$$(1+x)^m \approx 1+mx+\frac{m(m-1)}{2!}x^2$$

taqribiy formulaga ega bo'lamiz. Bu formuladan foydalanib, $\sqrt[4]{83}$ miqdorni taqribiy hisoblaymiz:

$$\sqrt[4]{83}=\sqrt[4]{81+2}=3\left(1+\frac{2}{81}\right)^{\frac{1}{4}}\approx 3\left(1+\frac{1}{4}\frac{2}{81}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{2}{81}\right)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-2\right)}{3!}\left(\frac{3}{81}\right)^3\right)=$$

$$=3\left(1+\frac{1}{162}-\frac{3\cdot 4}{4\cdot 4\cdot 2!\,81\cdot 81}+\frac{3\cdot 7\cdot 2^3}{4^3\cdot 6\cdot 81^3}\right)=3\left(\frac{163}{162}-\frac{1}{8\cdot 27\cdot 81}+\frac{7}{16\cdot 81^3}\right)=$$

$$=\left(\frac{163}{54}-\frac{1}{8\cdot 9\cdot 81}+\frac{7}{16\cdot 27\cdot 81^2}\right)\approx 3{,}018349479.$$

Shunday qilib, $\sqrt[4]{83}\approx 3{,}018342$.

MAPLE tizimidan foydalanib misolni yechish:

> C:=taylor(3*surd(1+x,4), x=2/81,4);

$$C := \frac{1}{27}83^{(1/4)}81^{(3/4)}+\frac{3}{332}83^{(1/4)}81^{(3/4)}\left(x-\frac{2}{81}\right)+\frac{729}{220448}83^{(1/4)}81^{(3/4)}\left(x-\frac{2}{81}\right)^2+$$
$$+\frac{137781}{73188736}83^{(1/4)}81^{(3/4)}\left(x-\frac{2}{81}\right)^3+O\left(\left(x-\frac{2}{81}\right)^4\right)$$

> V:=subs(x=2/81,C);convert(evalf(V),' olynom);

$$V := \frac{1}{27}83^{(1/4)}81^{(3/4)}$$

3.018349479

15.3. Teylor formulasidan foydalanib limitlarni hisoblash

Ushbu $\lim\limits_{x\to0}\frac{f(x)}{g(x)}$ limitni topish talab qilingan bo'lsin, bunda $f(0)=g(0)=0$. $f(x)$ va $g(x)$ lar Makloren formulasiga yoyiladigan funksiyalar bo'lsin. Bu funksiyalarning Makloren formulasidagi yoyilmalarining noldan farqli birinchi hadi bilan chegaralanamiz:

$$f(x)=ax^n+o(x^n),\quad a\neq0,$$
$$g(x)=bx^m+o(x^m),\quad b\neq0.$$

Agar $m=n$ bo'lsa,

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{ax^n+o(x^n)}{bx^m+o(x^m)}=\frac{a}{b}$$

bo'ladi.

Agar $n>m$ bo'lsa,

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{ax^n+o(x^n)}{bx^m+o(x^m)}=0 \qquad (2)$$

bo'ladi.

Agar $n<m$ bo'lsa,

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{ax^n+o(x^n)}{bx^m+o(x^m)}=\infty \qquad (3)$$

bo'ladi.

15.10-misol. Ushbu

$$\lim_{x\to0}\frac{\operatorname{tg}x-\sin x}{x^3}$$

limitni toping.

Yechilishi. $\operatorname{tg}x=x+\frac{x^3}{3}+o(x^4)$,

$$\sin x=x-\frac{x^3}{6}+o(x^4)$$

yoyilmalardan foydalanib, berilgan limitni topamiz:

$$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}=\lim_{x\to 0}\frac{\frac{1}{2}x^3+o(x^4)}{x^3}=\lim_{x\to 0}\left[\frac{1}{2}+\frac{o(x^3)}{x^3}\right]=\frac{1}{2}.$$

15.11-misol. Ushbu

$$\lim_{x\to 0}\frac{\sqrt[5]{1+2x}-1}{\sqrt[4]{1+x}-\sqrt{1-x}}$$

limitni toping.

Yechilishi. $x\to 0$ da

$$\sqrt[4]{1+x}=1+\frac{1}{4}x+o(x),$$

$$\sqrt{1-x}=1-\frac{1}{2}x+o(x)$$

yoyilmalardan foydalanib, berilgan kasr maxrajining $x\to 0$ dagi Makloren formulasiga yoyilmasini topamiz:

$$x\to 0 \text{ da} \quad \sqrt[4]{1+x}-\sqrt{1-x}=\frac{3}{4}x+o(x).$$

Bu yoyilmani e'tiborga olib, berilgan kasrning suratini $o(x)$ gacha Makloren formulasiga yoyamiz:

$$\sqrt[5]{1+2x}-1=1+\frac{1}{5}2x+o(x)-1=\frac{2}{5}x+o(x).$$

Shunday qilib,

$$\lim_{x\to 0}\frac{\sqrt[5]{1+2x}-1}{\sqrt[4]{1+x}-\sqrt{1-x}}=\lim_{x\to 0}\frac{\frac{2}{5}x+o(x)}{\frac{3}{4}x+o(x)}=\frac{8}{15}.$$

15.12-misol. Ushbu

$$\lim_{x\to 0}\frac{\sqrt[3]{1-x^2}-x\operatorname{ctg}x}{x\sin x}$$

limitni toping.

Yechilishi. (15.6), (15.8) va (15.9) formulalardan foydalanib, kasrning suratida x ning uchinchi darajasigacha bo'lgan hadlarni qoldirib, berilgan limitni hisoblaymiz:

$$\lim_{x\to0}\frac{\sqrt[3]{1-x^2}-x\operatorname{ctg}x}{x\sin x}=\lim_{x\to0}\frac{\sin x\cdot\sqrt[3]{1-x^2}-x\cos x}{x\sin^2 x}=\lim_{x\to0}\frac{\sin x\cdot\sqrt[3]{1-x^2}-x\cos x}{x^3}=$$

$$=\lim_{x\to0}\left[\left(x-\frac{x^3}{3!}+o(x^2)\right)\left(1-\frac{1}{3}x^2+o(x^3)\right)-x\left(1-\frac{x^2}{2!}+o(x^3)\right)\right]\cdot x^{-3}=$$

$$=\lim_{x\to0}\frac{x-\frac{1}{2}x^3-\frac{x^3}{3!}-x+\frac{x^3}{2!}+o(x^3)}{x^3}=0$$

Demak,

$$\lim_{x\to0}\frac{\sqrt[3]{1-x^2}-x\cos x}{x\sin x}=0.$$

15.13-misol. Ushbu

$$\lim_{x\to0}\frac{e^{-x}-\sqrt{1+x^2}-\arcsin x}{\operatorname{sh}(x-x^2)-\ln\sqrt{1+2x}}$$

limitni toping.

Yechilishi. $x\to0$ da (15.7) va (15.11) yoyilmalardan foydalanib,

$$\operatorname{sh}(x-x^2)-\ln\sqrt{1+2x}=x-x^2+\frac{(x-x^2)^3}{3!}+o(x^3)-\frac{1}{2}(2x-2x^2+o(x^3))=\frac{x^3}{3!}+o(x^3).$$

yoyilmaga ega bo'lamiz. Shunga ko'ra, berilgan ifodaning suratini $x\to0$ da x ning darajalari bo'yicha $o(x^3)$ hadgacha Makloren formulasiga yoyamiz. $x\to0$ da ushbu

$$\sin x=x-\frac{x^3}{6}+o(x^3),\qquad e^t=1+\frac{t}{1!}+\frac{t^2}{2!}+o(t^3),$$

$$(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}x^2+o(x^3),\quad \arcsin x=x+\frac{x^3}{6}+o(x^3)$$

yoyilmalardan foydalanib, $x\to 0$ da suratning yoyilmasini topamiz:

$$e^{\sin x}-\sqrt{1+x^2}-\arcsin x=-\frac{x^3}{3}+o(x^3).$$

Shunday qilib, berilgan kasr $x\to 0$ da $\dfrac{-\frac{x^3}{3}+o(x^3)}{\frac{x^3}{3!}+o(x^3)}$ shaklida tasvirlanadi.

Bundan esa, izlanayotgan limitning -2 ga tengligi kelib chiqadi.

Ko‘p hollarda, $\lim\limits_{x\to 0} f(x)^{g(x)}$ $(f(x)>0, \lim\limits_{x\to 0} g(x)=\infty)$ ko‘rinishdagi limitlarni hisoblashda ham Teylor formulasi qo‘llaniladi. Faraz qilaylik, xususiy holda $x_0=0$ da $f(x)$ va $g(x)$ funksiyalar

$$f(x)=1+ax^k+o(x^k),\quad g(x)=1/(bx^k+o(x^k))$$

kabi tasvirlangan bo‘lsin, bunda $a\neq 0$, $b\neq 0$, $k\in N$.

$$\lim_{x\to 0}(1+ax^k+o(x^k))^{\frac{1}{(ax^k+o(x^k))}}=e,\qquad \lim_{x\to 0}\frac{ax^k+o(x^k)}{bx^k+o(x^k)}=\frac{a}{b}$$

bo‘lgani uchun

$$\lim_{x\to 0} f(x)^{g(x)}=\lim_{x\to 0}(1+ax^k+o(x^k))^{\frac{1}{(bx^k+o(x^k))}}=e^{\frac{a}{b}}. \qquad (15.16)$$

bo‘ladi.

Agar $x\to 0$ da

$$f(x)=\frac{1+ax^k+o(x^k)}{1+cx^k+o(x^k)},\quad g(x)=\frac{1}{bx^k+o(x^k)},$$

$a\neq 0$, $c\neq 0$, $b\neq 0$, $k\in N$ bo‘lsa,

$$\lim_{x\to 0} f(x)^{g(x)}=e^{\frac{a-c}{b}} \qquad (15.17)$$

bo‘ladi.

$x\to 0$ da $(f(x))^{g(x)}$ funksiyaning limitini topishda, avvalo $g(x)$ va $\ln f(x)$ funksiyalarni Maklorcn formulasiga yoyib, so'ngra $\lim\limits_{x\to 0} g(x)\ln f(x)$ ni topib, berilgan ifodaning limitini topish ham mumkin.

15.14-misol. Ushbu

$$\lim_{x\to 0}\left(e^{\operatorname{tg} x}+\ln(1-x)\right)^{\operatorname{ctg} x^3}$$

limitni toping.

Yechilishi. $x\to 0$ da $\operatorname{ctg} x^3=\frac{1}{\operatorname{tg} x^3}=\frac{1}{x^3+o(x^3)}$ bo'lgani uchun, $f(x)=e^{\operatorname{tg} x}+\ln(1-x)$ funksiyani $o(x^3)$ hadgacha Makloren formulasiga yoyish kerak bo'ladi.

$$\operatorname{tg} x=x+\frac{x^3}{3}+o(x^3),\qquad \ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}+o(x^3),$$

$$t\to 0 \text{ da } \quad e^t=1+\frac{t}{1!}+\frac{t^2}{2!}+o(x^3)$$

yoyilmalarni e'tiborga olgan holda,

$$e^{\operatorname{tg} x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3),$$

$$e^{\operatorname{tg} x}+\ln(1-x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-x-\frac{x^2}{2}-\frac{x^3}{3}+o(x^3)=-\frac{1}{6}x^3+o(x^3)$$

yoyilmani topamiz.

Shunday qilib, (15.16) formulaga asosan,

$$\lim_{x\to 0}\left(e^{\operatorname{tg} x}+\ln(1-x)\right)^{\operatorname{ctg} x^3}=\lim_{x\to 0}\left(-\frac{1}{6}x^3+o(x^3)\right)^{\frac{1}{x^3+o(x^3)}}=e^{-\frac{1}{6}}$$

bo'ladi.

15.15-misol. Ushbu

$$\lim_{x\to 0}\left(\frac{3\sin x}{2x+\operatorname{tg} x}\right)^{\frac{1}{1-\cos x}}$$

limitni toping.

Yechilishi. $x \to 0$ da ushbu

$$1-\cos x=\frac{x^2}{2}+o(x^2), \quad \sin x=x-\frac{1}{6}x^3+o(x^3),$$
$$tgx=x+\frac{1}{3}x^3+o(x^3)$$

yoyilmalardan foydalanib,

$$\left(\frac{3\sin x}{2x+\operatorname{tg}x}\right)^{\frac{1}{1-\cos x}}=\left(\frac{3x-\frac{x^3}{2}+o(x^3)}{3x+\frac{1}{3}x^3+o(x^3)}\right)^{\frac{1}{\frac{x^2}{2}+o(x^2)}}=\left(\frac{1-\frac{1}{6}x^3+o(x^3)}{1+\frac{1}{9}x^3+o(x^3)}\right)^{\frac{1}{\frac{x^2}{2}+o(x^2)}}$$

ni hosil qilamiz. Bundan, (15.17) formulaga asosan, izlanayotgan limit

$\lim\limits_{x\to 0}\left(\frac{3\sin x}{2x+tgx}\right)^{\frac{1}{1-\cos x}}=e^{\left(-\frac{1}{6}-\frac{1}{9}\right)2}=e^{-\frac{5}{9}}$ bo'ladi.

15.16-misol. Ushbu

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}-\operatorname{ctg}x\right)$$

limitni toping.

Yechilishi. Berilgan ifodani

$$\frac{1}{x}\left(\frac{1}{x}-\operatorname{ctg}x\right)=\frac{\sin x-x\cos x}{x^2\cdot\sin x}$$

ko'rinishda tasvirlab, $x \to 0$ da

$$\sin x=x-\frac{x^3}{6}+o(x^3), \quad \cos x=1-\frac{x^2}{2!}+o(x^3), \quad x^2\sin x=x^3+o(x^3)$$

yoyilmalarni e'tiborga olib,

$$\frac{1}{x}\left(\frac{1}{x}-ctgx\right)=\frac{x-\frac{x^3}{6}-x+\frac{x^3}{2!}+o(x^3)}{x^3+o(x^3)}=\frac{\frac{1}{3}x^3+o(x^3)}{x^3+o(x^3)}$$

yoyilmaga ega bo'lamiz. Bundan izlanayotgan limitning $\frac{1}{3}$ ga tengligi kelib chiqadi.

Quyidagi ko'phadlarni $x-x_0$ ning manfiy bo'lmagan darajalari bo'yicha Teylor formulasiga yoying:

15.1. $P(x)=1+3x+5x^2-2x^3,\quad x_0=-1.$

15.2. $P(x)=x^4-4x^3+7x^2-5x+3,\quad x_0=2.$

15.3. $P(x)=x^9+3x^4+2x-2,\quad x_0=1.$

Quyidagi funksiyalarni x ning manfiy bo'lmagan darajalari bo'yicha ko'rsatilgan tartibgacha Makloren formulasiga yoying:

15.4. $f(x)=\dfrac{(1+x)^{100}}{(1-2x)^{40}\cdot(1+2x)^{60}},\quad o(x^2)$ hadgacha.

15.5. $f(x)=e^{\sqrt{1+2x}},\quad o(x^2)$ hadgacha

15.6. $f(x)=xe^x,\quad o(x^3)$ hadgacha.

15.7. $f(x)=\dfrac{e^x+e^{-x}}{2},\quad o(x^4)$ hadgacha.

15.8. $f(x)=\sqrt{1-2x+x^3}-\sqrt[3]{1-3x+x^3},\quad o(x^3)$ hadgacha.

15.9. $f(x)=\ln(1+\arcsin x),\quad o(x^3)$ hadgacha.

15.10. $f(x)=\dfrac{5x^4-11}{x^6-x^3-2},\quad o(x^4)$ hadgacha.

15.11. $f(x)=\ln\sqrt{\dfrac{e-x^3}{1-ex^3}},\quad o(x^4)$ hadgacha.

15.12. $f(x)=\sqrt[3]{\sin x^3},\quad o(x^{13})$ hadgacha.

15.13. $f(x)=\ln\cos x,\quad o(x^6)$ hadgacha.

15.14. $f(x)=\sin(\sin x),\quad o(x^3)$ hadgacha.

15.15. $f(x)=tgx,\quad o(x^5)$ hadgacha.

15.16. $f(x)=\sin x\cdot\sin 3x,\quad o(x^5)$ hadgacha.

15.17. $f(x)=\sin^2 x\cdot\cos^4 x,\quad o(x^7)$ hadgacha.

15.18. $f(x)=\cos^6 x+\sin^6 x, \quad o(x^5)$ hadgacha.

Quyidagi funksiyalarni Makloren formulasiga $o(x^4)$ hadgacha yoying:

15.19. $f(x)=e^{2x+1}$.

15.20. $f(x)=5^{2-x}$.

15.21. $f(x)=\ln(e^x+3)$.

15.22. $f(x)=\dfrac{1}{2x+5}$.

15.23. $f(x)=\ln\dfrac{2-3x}{3+2x}$.

15.24. $f(x)=\ln(x^2+3x+2)$.

15.25. $f(x)=\dfrac{x}{\sqrt[3]{9-6x+x^2}}$.

15.26. $f(x)=\dfrac{x^2+1}{2x-3}$.

15.27. $f(x)=\dfrac{3x^2+5x-5}{x^2+x-2}$

Quyidagi funksiyalarni Makloren formulasiga $o(x^{2n})$ hadgacha yoying:

15.28. $f(x)=x\,\text{ch}\,5x$.

15.29. $f(x)=\text{ch}\,x\cdot\text{ch}\,2x$.

15.30. $f(x)=\sin^3 x\cos x$

Quyidagi funksiyalarni Makloren formulasiga $o(x^{2n+1})$hadgacha yoying:

15.31. $f(x)=\text{sh}\,x\cdot\text{sh}\,5x$.

15.32. $f(x)=\cos^4 x+\sin^4 x$.

15.33. $f(x)=\dfrac{1}{\sqrt{x^2+2}+\sqrt{2-x^2}}$.

Quyidagi funksiyalarni Teylor formulasiga x_0 nuqtaning atrofida $o\left((x-x_0)^n\right)$ hadgacha yoying:

15.34. $f(x)=(x^2-1)e^{2x}$, $x_0=-1$.

15.35. $f(x)=\ln(x^2-7x+12)$, $x_0=1$.

15.36. $f(x)=\frac{2x+1}{x-1}\ln x$, $x_0=1$.

15.37. $f(x)=\frac{x+7}{x(2x+7)}$, $x_0=-2$.

15.38. $f(x)=\frac{x^2-4x+5}{x^2-5x+6}$, $x_0=1$.

Quyidagi funksiyalarni Teylor formulasiga x_0 nuqtaning atrofida $o\left((x-x_0)^{2n}\right)$ hadgacha yoying:

15.39. $f(x)=(x+3)e^{3x^2+18x}$, $x_0=-3$.

15.40. $f(x)=\frac{x^2-2x+1}{\sqrt[3]{x(2-x)}}$, $x_0=+1$.

15.41. $f(x)=\frac{x-1}{3x^2-6x+5}$, $x_0=1$.

Quyidagi funksiyalarni Teylor formulasiga x_0 nuqtaning atrofida $o\left((x-x_0)^{2n+1}\right)$ hadgacha yoying:

15.42. $f(x)=x(x-2)2^{x^2-2x-1}$, $x_0=1$.

15.43. $f(x)=\left(x+\frac{\pi}{4}\right)(\sin x+\cos x)$, $x_0=-\frac{\pi}{4}$.

15.44. $f(x)=\log_5\sqrt[3]{\frac{2x-1}{3-2x}}$, $x_0=1$.

15.45. $f(x)=\frac{1-4x+4x^2}{\sqrt{x}+\sqrt{1-x}}$, $x_0=\frac{1}{2}$.

15.46. Ushbu $f(x)=2^{x^3-3x^2+3x}$ funksiyani Teylor formulasiga $x_0=1$ nuqtaning atrofida $o\left((x-x_0)^{3n}\right)$ hadgacha yoying.

15.47. Ushbu $f(x)=\dfrac{x-2}{\sqrt[3]{(x-4)(x^2-2x+4)}}$ funksiyani Teylor formulasiga $x_0=2$ nuqtaning atrofida $o\left((x-2)^{3n+1}\right)$ hadgacha yoying.

15.48. Ushbu $f(x)=\sin(3x^2+6x+4)$ funksiyani Teylor formulasiga $x_0=-1$ nuqtaning atrofida $o\left((x+1)^{4n}\right)$ hadgacha yoying.

15.49. Ushbu $f(x)=\cos\left(\dfrac{8}{\pi}x^2-4x+\pi\right)$ funksiyani Teylor formulasiga $x_0=\dfrac{\pi}{4}$ nuqta atrofida $o\left(\left(x-\dfrac{\pi}{4}\right)^{4n-3}\right)$ hadgacha yoying.

15.50. Teylor formulasidan foydalanib, ushbu ifodaning taqribiy qiymatini toping.

1) $\sqrt[5]{250}$; 2) $\sqrt[12]{4000}$; 3) $\operatorname{arctg}0{,}8$; 4) $\sin 36°$; 5) $(1{,}2)^{1,1}$.

15.51. Quyidagi miqdorlarni Teylor formulasidan foydalanib, ko'rsatilgan aniqlikgacha taqribiy hisoblang:

1) e sonini 10^{-9} aniqlikgacha; 2) $\cos 10°$ sonini 10^{-3} aniqlikgacha;

3) $\sqrt[3]{30}$ sonini 10^{-4} aniqlikgacha; 4) $\lg 11$ sonini 10^{-5} aniqlikgacha;

Teylor formulasidan foydalanib, quyidagi limitlarni toping.

15.52. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-\sqrt{1+2x}}{x^2}$. **15.53.** $\lim\limits_{x\to 0}\dfrac{e^x-(1+x)}{x}$.

15.54. $\lim\limits_{x\to 0}\dfrac{\cos x-1+\dfrac{x^2}{2}}{x^4}$. **15.55.** $\lim\limits_{x\to+\infty}x^2\left(e^{-\frac{1}{x^2}}-1\right)$.

15.56. $\lim\limits_{y\to 0}\dfrac{1-\cos y-\dfrac{y^2}{2}}{y^4}$. **15.57.** $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}$.

15.58. $\lim\limits_{t\to\infty}(t+1)\cdot\sin\dfrac{1}{t+1}$. **15.59.** $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-x^2/2}}{x^4}$.

15.60. $\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{1-\cos x}$. **15.61.** $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}$, $(m\neq n, m\neq 0, n\neq 0)$.

15.62. $$\lim_{x\to 0}\frac{\ln(1+x)-x(1+x)^{a}}{x^2}.$$

15.63. $$\lim_{x\to+\infty}\left(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}\right).$$

15.64. $$\lim_{x\to 0}\frac{ch3x+\cos 3x-2}{x^4}.$$

15.65. $$\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt[3]{1+x}-2\sqrt[4]{1-x}}{x}.$$

15.66. $$\lim_{x\to 0}\frac{\operatorname{arctg}x-\arcsin x}{x^2}.$$

15.67. $$\lim_{x\to 0}\frac{\operatorname{arctg}x-\arcsin x}{\operatorname{tg}x-\sin x}.$$

15.68. $$\lim_{x\to+\infty}\left[\left(x^3-x^2+\frac{x}{2}\right)e^{1/x}-\sqrt{x^6+1}\right].$$

15.69. $$\lim_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right].$$

15.70. $$\lim_{x\to 0}\frac{3\cos x+\arcsin x-3\sqrt[3]{1+x}}{\ln(1-x^2)}.$$

15.71. $$\lim_{x\to 0}\frac{\sqrt[3]{1-x^2}-x\operatorname{ctg}x}{x\sin x}.$$

15.72. $$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right).$$

15.73. $$\lim_{x\to 0}\frac{\sin(\sin x)-x\sqrt[3]{1-x^2}}{x^5}.$$

15.74. $$\lim_{x\to 1}\frac{\sin(\sin\pi x)}{\ln(1+\ln x)}.$$

15.75. $$\lim_{x\to 0}\frac{\operatorname{sh}(\operatorname{tg}x)-x}{x^3}.$$

15.76. $$\lim_{x\to 0}\frac{\ln(1+x^2)-2\sin x+2x\cos x^2}{\operatorname{arctg}x^3}.$$

15.77. $$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\operatorname{tg}x}\right).$$

15.78. $$\lim_{x\to 0}\frac{e^{\sin x}-\sqrt{1+x^2}-x\cos x}{\ln^3(1-x)}.$$

15.79. $$\lim_{x\to 0}\frac{e^{\operatorname{arctg}x}+\ln(1-x)-1}{2-\sqrt{4+x^2}}.$$

15.80. $$\lim_{x\to 0}\frac{\sqrt{1+2x^3}-\cos x^4}{\operatorname{tg}x-x}.$$

15.81. $$\lim_{x\to 0}\frac{e^{\sin x}+\ln(1-x)-1}{\arcsin x-\sin x}.$$

15.82. $$\lim_{x\to 0}\frac{\sqrt[3]{1+3x}-e^{\sin x}+\frac{3}{2}x^2}{\arcsin x-\operatorname{tg}x}.$$

15.83. $$\lim_{x\to 0}\frac{e^{\operatorname{tg}(x/2)}-\sqrt{1+\sin x}-\frac{x^2}{4}}{\arccos x-\operatorname{arcctg}x}.$$

15.84. $$\lim_{x\to 0}\frac{\sqrt[3]{1+x^3}-x\operatorname{ctg}x-\frac{1}{2}x^2}{x\cos x-\sin x}.$$

15.85. $$\lim_{x\to 0}\frac{\ln\left(1+x-\frac{1}{6}x^3\right)-\operatorname{sh}x+\frac{2}{3}x^2}{\sin 2x-2x\cos x}.$$

15.86. $\lim\limits_{x\to 0} \dfrac{e^x - \sqrt{1+2x+2x^2}}{x + \mathrm{tg}\,x - \sin 2x}$.

15.87. $\lim\limits_{x\to 0} \dfrac{\sin(\mathrm{arctg}\,x) - \mathrm{tg}\,x}{e^{\mathrm{sh}\,x} - (1+2x)^{1/2} - x^2}$.

15.88. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+2x} - e^{\mathrm{tg}\,x} + 6x^3 + x^2}{\ln(1+x) - \mathrm{arctg}\,x + \dfrac{x^2}{2}}$.

15.89. $\lim\limits_{x\to 0} \dfrac{x + \mathrm{ch}\,x - e^{\arcsin x}}{\mathrm{tg}\,x + \sqrt[3]{1-3x} - 2\cos x + 1}$.

15.90. $\lim\limits_{x\to 0} \left(\dfrac{\cos x}{\mathrm{ch}\,3x}\right)^{\frac{1}{x^2}}$.

15.91. $\lim\limits_{x\to 0} \left(\dfrac{\mathrm{tg}\,x}{\mathrm{arctg}\,x}\right)^{\frac{1}{x^2}}$.

15.92. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} e^{\frac{x}{1+x}} - \dfrac{1}{\sin x}\right)^{1/\mathrm{arctg}\,x}$.

15.93. $\lim\limits_{x\to 0} \left(\dfrac{\mathrm{ch}\,x - \cos x}{2\sqrt{1+2x} - 2\sqrt[3]{1+3x}}\right)^{\frac{1}{x}}$.

15.94. $\lim\limits_{x\to 0} \left(\mathrm{tg}\left(\dfrac{x}{3}\right) + 2 - \sqrt[3]{1+x}\right)^{\mathrm{ctg}^2 x}$.

15.95. $\lim\limits_{x\to 0} \left(\dfrac{2\cos x + x}{2\sqrt{1+x}}\right)^{1/x^2}$.

15.96. $\lim\limits_{x\to 0} \left(\dfrac{\sqrt{\cos x}}{e^x - \ln(1+x)}\right)^{1/x^2}$.

15.97. $\lim\limits_{x\to 0} \left(\dfrac{x^2 - (\mathrm{arctg}\,x)^2}{x^2 \cdot \sin \dfrac{2}{3} x^2}\right)^{1/x^2}$.

15.98. $\lim\limits_{x\to 0} \left(\dfrac{\mathrm{sh}(x + \sin x)}{\sin x + \mathrm{arc}\ln x}\right)^{\mathrm{ctg}^2 x}$

15.99. $\lim\limits_{x\to 0}\left(\dfrac{\ln(1+x)}{x}+\dfrac{x}{\ln(e^2-xe^2)}\right)^{1/x^2}$.

15.100. $\lim\limits_{x\to 0}\left(\dfrac{2x}{x-2}+\ln(e+xe^{x+1})\right)^{1/x^2}$.

15.101. $\lim\limits_{x\to 0}\left(1+\dfrac{1}{2}\ln\dfrac{1+x}{1-x}-\operatorname{arctg}x\right)^{\frac{1}{\arcsin x^3}}$.

15.102. $\lim\limits_{x\to 0}\left(\dfrac{3}{2}x^2+\sqrt[3]{1+3\sin x}+\ln(1-x)\right)^{\frac{1}{sh^3 x}}$.

15.103. $\lim\limits_{x\to 0}\left(\sqrt{1-2x+3x^2}+x(1-shx)\right)^{ctg^3 x}$.

15.104. $\lim\limits_{x\to 0}\left(1+th(xe^x)+\dfrac{1}{2}\ln(1-2x)\right)^{\frac{1}{x^3}}$.

15.105. $\lim\limits_{x\to 0}\left(e^{\sin x}-\dfrac{x^2}{2}-x\cos x\right)^{\frac{1}{\ln^3(1-\frac{x}{2})}}$.

15.106. $\lim\limits_{x\to 0}\left(e^{x-x^2}-x\sqrt[3]{1-\dfrac{3}{2}x}\right)^{\frac{1}{tgx-x}}$.

Mustaqil yechish uchun misol va masalalarning javoblari

15.1. $5-13(x+1)+11(x+1)^2-2(x+1)^3$.

15.2. $5+7(x-2)+7(x-2)^2+4(x-2)^3+(x-2)^4$.

15.3. $2-3(x-1)+(x-1)^2+15(x-1)^3+25(x-1)^4+\dfrac{7}{3}(x-1)^5+7(x-1)^6+(x-1)^7$.

15.4. $1+60x+195x^2+o(x^2)$. **15.5.** $e+ex+o(x^2)$.

15.6. $x+\dfrac{x^2}{1!}+\dfrac{x^3}{2!}+o(x^3)$. **15.7.** $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+o(x^4)$.

15.8. $\dfrac{1}{6}x^2+x^3+o(x^3)$. **15.9.** $x-\dfrac{x^2}{2}+\dfrac{x^3}{2}+(x^3)$.

15.10. $\frac{11}{2}-\frac{11}{4}x^3+\frac{13}{8}x^6+o(x^6).$

15.11. $\frac{1}{2}+x^3\,\mathrm{sh}\,1+o(x^4).$

15.12. $x-\frac{x^7}{18}-\frac{x^{13}}{3240}+o(x^{13}).$

15.13. $-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}+o(x^6).$

15.14. $x-\frac{x^3}{3}+o(x^3).$

15.15 $x+\frac{x^3}{3}+\frac{2x^5}{15}+o(x^5).$

15.16. $3x^2-5x^4+o(x^5).$

15.17. $x^2-\frac{7}{3}x^4+\frac{107}{45}+o(x^6).$

15.18. $1-3x^2+4x^4+o(x^5)$

15.19. $\sum_{k=1}^{n} e\cdot\frac{2^k}{k!}x^k+o(x^n).$

15.20. $\sum_{k=0}^{n}(-1)^k\cdot\frac{25\cdot(\ln 5)^k}{k!}x^k+o(x^n).$

15.21. $\ln 3+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\left(\frac{e}{3}\right)^k x^k+o(x^n).$

15.22. $\sum_{k=1}^{n}(-1)^k\cdot\frac{2^k}{5^{k+1}}x^k+o(x^n).$

15.23. $\ln\frac{2}{3}+\sum_{k=1}^{n}\frac{(-4)^k-9^k}{k\cdot 6^k}x^k+o(x^n).$

15.24. $\ln 2+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}(1+2^{-k})x^k+o(x^n).$

15.25. $\sum_{k=1}^{n}3^{\frac{1}{3}-k}(-1)^{k-1}C^{k-1}_{-2/3}\cdot x^k+o(x^n).$

15.26. $-\frac{1}{3}-\frac{2}{9}x-\frac{13}{12}\sum_{k=2}^{n}\left(\frac{2}{3}\right)^k x^k+o(x^n).$

15.27. $\frac{5}{2}+\sum_{k=1}^{n}\left((-1)^k 2^{-(k+1)}-1\right)x^k+o(x^n).$

15.28. $\sum_{k=0}^{n-1}\frac{5^{2k}}{(2k)!}x^{2k+1}+o(x^{2n}).$

15.29. $\sum_{k=0}^{n-1}\frac{3^{2k+1}-1}{2(2k+1)!}x^{2k+1}+o(x^{2n}).$

15.30. $\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)!}x^{2k-1}(1-2^{2k})x^{2k+1}+o(x^{2n}).$

15.31. $\sum_{k=0}^{n}\frac{6^{2k}-4^{2k}}{2\cdot(2k)!}x^{2k}+o(x^{2n+1}).$

15.32. $1+\sum\limits_{k=1}^{n}(-1)^k\frac{4^{2k}}{(2k)!}x^{2k}+o(x^{2n+1})$.

15.33. $\sum\limits_{k=0}^{n}C_{1/2}^{k+1}\cdot 2^{-\left(k+\frac{3}{2}\right)}(1+(-1)^k)x^{2k}+o(x^{2n+1})$.

15.34. $\sum\limits_{k=1}^{n}e^{-2}\frac{2^{k-2}(k-5)}{(k-1)!}(x+1)^k+o\left((x+1)^n\right)$.

15.35. $\ln 6-\sum\limits_{k=1}^{n}\frac{2^{-k}+3^{-k}}{k}(x-1)^k+o\left((x-1)^n\right)$.

15.36. $3+\sum\limits_{k=1}^{n}\frac{(-1)^k(k-2)}{k(k+1)}(x-1)^k+o\left((x-1)^n\right)$.

15.37. $\sum\limits_{k=0}^{n}\left(\frac{(-1)^{k+1}\cdot 2k}{3^{k-1}}-\frac{1}{2^{k+1}}\right)(x+2)^k+o\left((x+2)^n\right)$

15.38. $1+\sum\limits_{k=1}^{n}(1-2^{-k})(x-1)^k+o\left((x-1)^n\right)$.

15.39. $\sum\limits_{k=0}^{n-1}e^{-27}\cdot\frac{3^k}{k!}(x+3)^{2k+1}+o\left((x+3)^{2n}\right)$.

15.40. $(x-1)^2\sum\limits_{k=1}^{n-1}\frac{1\cdot 4\cdot 7\ldots(3k-2)}{3^k k!}(x-1)^{2k+2}+o\left((x-1)^{2n}\right)$.

15.41. $\sum\limits_{k=0}^{n-1}(-1)^k\frac{3^k}{2^{k+1}}(x-1)^{2k+1}+o\left((x-1)^{2n}\right)$.

15.42. $-\frac{1}{4}+\sum\limits_{k=1}^{n}\frac{(\ln 2)^{k-1}}{4\cdot k!}(k-\ln 2)(x-1)^{2k}+o\left((x-1)^{2n+1}\right)$.

15.43. $\sum\limits_{k=1}^{n}\frac{\sqrt{2}(-1)^{k+1}}{(2k-1)!}\left(x+\frac{\pi}{4}\right)^{2k}+o\left((x+\frac{\pi}{4})^{2n+1}\right)$.

15.44. $\frac{2}{3\ln 5}\sum\limits_{k=1}^{n+1}\frac{2^{2k-1}}{2k-1}(x-1)^{2k-1}+o\left((x-1)^{2n+1}\right)$.

15.45. $\sum\limits_{k=0}^{n-1}C_{1/2}^{2k+1}2^{2k+\frac{3}{2}}\left(x-\frac{1}{2}\right)^{2k+2}+o\left(\left(x-\frac{1}{2}\right)^{2n+1}\right)$.

15.46. $\sum_{k=0}^{n} \frac{2(\ln 2)^k}{k!}(x-1)^{3k} + o\left((x-1)^{3n}\right)$.

15.47. $\sum_{k=0}^{n} \frac{(-1)^{k+1} C_{-1/3}^{k}}{2^{3k-1}}(x-2)^{3k-1} + o\left((x-2)^{3n+1}\right)$.

15.48. $\sum_{k=0}^{n-1} \frac{(-1)^k 3^{2k+1} \cos 1}{(2k+1)!}(x+1)^{4k-2} + \sum_{k=0}^{n} \frac{(-1)^k 3^{2k} \cdot \sin 1}{(2k)!}(x+1)^{4k} + o\left((x+1)^{4n}\right)$.

15.49. $\sum_{k=0}^{n} (-1)^{2k+1}\left(\frac{8}{\pi}\right)^{2k+1} \frac{\left(x-\frac{\pi}{4}\right)^{4k+2} 3^{2k-1} \cos 1}{(2k+1)!} + o\left(\left(x-\frac{\pi}{4}\right)^{4n+3}\right)$.

15.50. 1) $\sqrt[5]{250} \approx 3{,}017$; 2) $\sqrt[12]{4000} \approx 1{,}9961$; 3) $\operatorname{arctg} 0{,}8 \approx 0{,}6747409422$; 4) $\sin 36^\circ \approx 0{,}5877852524$; 5) $(1{,}2)^{1,1} \approx 1{,}222079251$ **15.51.** 1) 2.718281828; 2) 0.9848077530; 3) 3.107232506; 4) 1.041392685. **15.52.** $-\frac{1}{2}$

15.53. $\frac{1}{2}$ **15.54.** $\frac{1}{24}$ **15.55.** -1 **15.56.** $\frac{1}{4}$. **15.57.** $-\frac{1}{2}$.

15.58. 0. **15.59.** $-\frac{1}{12}$. **15.60.** 2. **15.61.** $\frac{1}{m}-\frac{1}{n}$.

15.62. $-\frac{1}{2}-\alpha$. **15.63.** $\frac{1}{3}$ **15.64.** $\frac{27}{4}$. **15.65.** $\frac{4}{3}$. **15.66.** 0.

15.67. -1. **15.68.** $\frac{1}{6}$. **15.69.** $\frac{1}{2}$. **15.70.** $\frac{7}{6}$. **15.71.** 0.

15.72. 0. **15.73.** $\frac{1}{3}$. **15.74.** $-\pi$. **15.75.** $\frac{1}{2}$. **15.76.** $\frac{4}{3}$.

15.77. $\frac{1}{3}$. **15.78.** $-\frac{1}{2}$. **15.79.** 2. **15.80.** 3. **15.81.** -1.

15.82. -10. **15.83.** $-\frac{1}{6}$. **15.84.** -1. **15.85.** -1. **15.86.** $\frac{2}{5}$.

15.87. 5. **15.88.** 9. **15.89.** $\frac{1}{4}$. **15.90.** e^{-5}. **15.91.** $e^{2/3}$.

15.92. $e^{-\frac{7}{3}}$. **15.93.** $e^{\frac{7}{3}}$. **15.94.** $e^{1/9}$. **15.95.** $e^{-\frac{3}{8}}$.

15.96. $e^{-\frac{5}{4}}$. **15.97.** $e^{-\frac{23}{30}}$. **15.98.** $e^{\frac{7}{12}}$. **15.99.** $e^{\frac{7}{3}}$. **15.100.** $e^{-\frac{5}{12}}$.

15.101. $e^{\frac{2}{3}}$. **15.102.** $e^{\frac{7}{6}}$. **15.103.** e. **15.104.** $e^{-\frac{7}{6}}$.

15.105. e^{-4}. **15.106.** $e^{-\frac{7}{4}}$.

16-§. FUNKSIYANI HOSILA YORDAMIDA TEKSHIRISH

16.1. Funksiyaning monotonlik oraliqlarini aniqlash. Kamaymovchi (o'smovchi), o'suvchi (kamayuvchi) funksiyalarning ta'riflarini yana bir eslab o'tamiz.

1^0. $x_1 < x_2$ shartni qanoatlantiradigan $\forall x_1, x_2 \in (a,b)$ uchun $f(x_1) \le f(x_2)$, $(f(x_1) \ge f(x_2))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya $(a;b)$ *oraliqda kamaymovchi (o'smovchi)* deyiladi.

2^0. $x_1 < x_2$ shartni qanoatlantiradigan $\forall x_1, x_2 \in (a,b)$ uchun $f(x_1) < f(x_2)$, $(f(x_1) > f(x_2))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya $(a;b)$ *oraliqda o'suvchi (kamayuvchi)* deyiladi. Ba'zan, o'suvchi (kamayuvchi) o'rniga *qat'iy o'suvchi (qat'iy kamayuvchi)* deb ham yuritiladi.

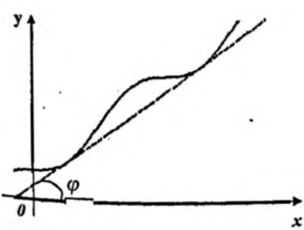
16.1-teorema. *$f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lsin. Funksiyaning shu oraliqda kamaymovchi (o'smovchi) bo'lishi uchun*

$$f'(x) \ge 0 \quad (f'(x) \le 0), \quad x \in (a,b)$$

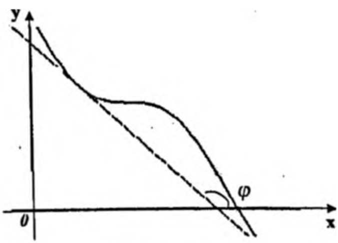
tengsizlikning bajarilishi zarur va yetarli.

16.2-teorema. *$f(x)$ funksiya (a,b) oraliqda chekli hosilaga ega bo'lib, $f'(x) > 0$ $(f'(x) < 0)$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya (a,b) oraliqda qat'iy o'suvchi(qat'iy kamayuvchi) bo'ladi.*

16.3-teorema. *$f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega*



16.1-chizma.



16.2-chizma.

bo'lsin. Bu funksiyaning (a,b) oraliqda o'zgarmas bo'lishi uchun

$$f'(x)=0,\ \ x\in(a,b)$$

bo'lishi zarur va yetarli.

Natija. Agar $f(x)$ va $g(x)$ funksiyalar (a,b) oraliqda aniqlangan bo'lib, chekli $f'(x)$ va $g'(x)$ hosilalarga ega bo'lib, $\forall x\in(a;b)$ da $f'(x)=g'(x)$ bo'lsa, (a,b) da bu funksiyalarning biri ikkinchisidan o'zgarmas songa farq qiladi, ya'ni $f(x)=g(x)+C$, $\forall x\in(a;b)$.

Masalan, $\operatorname{arctg}x$ va $\arcsin\frac{x}{\sqrt{1+x^2}}$ funksiyalar $-\infty<x<+\infty$ da aniqlangan, chekli hosilalarga ega bo'lib, ularning hosilalari bir-biriga teng. Haqiqatdan ham:

$$\frac{d}{dx}\left(\arcsin\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\cdot\frac{\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}}{1+x^2}=\frac{1}{1+x^2},\quad \frac{d}{dx}(\operatorname{arctg}x)=\frac{1}{1+x^2}$$

Yuqoridagi natijaga asosan, $\operatorname{arctg}x=\arcsin\frac{x}{\sqrt{1+x^2}}+C$. Bunda o'zgarmas C sonni aniqlash uchun tenglikning ikkala tomoniga $x=0$ qiymatni qo'yib, $C=0$ ekanligini topamiz. Shunday qilib, $-\infty<x<\infty$ da $\operatorname{arctg}x=\arcsin\frac{x}{\sqrt{1+x^2}}$ tenglikni hosil qilamiz.

16.2-teoremaning geometrik ma'nosi quyidagicha:

1) $f'(x)>0$ $(tg\alpha>0)$ shart funksiya grafigining har bir nuqtasiga o'tkazilgan urinma abssissalar o'qining musbat yo'nalishi bilan o'tkir burchak tashkil qilishini (16.1-chizma);

2) $f'(x)<0$ shart esa, o'tmas burchak tashkil qilishini (16.2-chizma) anglatadi.

16.1-eslatma. $f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lib, bu funksiyaning (a,b) oraliqda qat'iy o'suvchi (qat'iy kamayuchi) bo'lishidan, $f'(x)$ ning $\forall x\in(a,b)$ da musbat (manfiy) bo'lishi har doim ham kelib chiqavermaydi. Masalan, ma'lumki, $f(x)=x^3$ funksiya R da qat'iy o'suvchi, lekin uning $y'=3x^2$ hosilasi hamma joyda musbat emas, $x=0$ da esa nolga aylanadi.

Shunday qilib, funksiya hosilasining (a,b) oraliqda musbat (manfiy) bo'lishi funksiyaning qat'iy monoton bo'lishi uchun zaruriy shart bo'la olmaydi.

Funksiyani monotonlikka tekshirganda avvalo uning hosilasini topish (u mavjud bo'lgan joyda) kerak, so'ngra hosila musbat (manfiy) bo'ladigan oraliqlarini aniqlash kerak. Hosilasi musbat (manfiy) bo'lgan oraliqlarda funksiya monoton o'suvchi (kamayuvchi) bo'ladi.

16.1-misol. Ushbu

$$1)\ f(x)=x^3+2x-5;\quad 2)\ f(x)=\frac{2x}{1+x^2};$$

$$3)\ f(x)=\cos\frac{\pi}{x};\quad 4)\ f(x)=x+|\sin 2x|$$

funksiyalarning monotonlik oraliqlarini aniqlang.

Yechilishi. **1)** Berilgan funksiya $\forall x\in R$ uchun aniqlangan bo'lib u differensiallanuvchi: $f'(x)=3x^2+2>0,\ \forall x\in R$. Demak, berilgan funksiya R da o'suvchi.

2) $f(x)=\frac{2x}{1+x^2}$ funksiya $\forall x\in R$ da aniqlangan va u differensiyalanuvchi: $f'(x)=\frac{2(1-x^2)}{(1+x^2)^2}$. Bundan $(-\infty,-1)\cup(1,+\infty)$ oraliqlarda $f'(x)<0$, $(-1;1)$ oraliqda esa $f'(x)>0$ bo'lishi kelib chiqadi. Demak, funksiya $(-\infty,-1)$ va $(1,+\infty)$ oraliqlarda qat'iy kamayuvchi, $(-1;1)$ oraliqda qat'iy o'suvchi.

3) $f(x)=\cos\frac{\pi}{x}$ funksiya R ning $x=0$ nuqtadan tashqari barcha nuqtalarida aniqlangan va differensiallanuvchi: $f'(x)=\frac{\pi}{x^2}\cdot\sin\frac{\pi}{x}$. Juft

funksiya bo'lganligi uchun $x\ge 0$ holni qaraymiz. $f'(x)=\frac{\pi}{x^2}\cdot\sin\frac{\pi}{x}>0$ ning ishorasi $\sin\frac{\pi}{x}$ ning ishorasi kabi bo'ladi. Bundan $0<\frac{\pi}{x}<\pi$ yoki $2k\pi<\frac{\pi}{x}<(2k+1)\pi,$ $k\in N$ da $\sin\frac{\pi}{x}>0$, bu yerdan $x>1$ yoki $\frac{1}{2k+1}<x<\frac{1}{2k},\ k\in N.$

Shunday qilib, $(1;+\infty)$ yoki $\left(\frac{1}{2k+1};\frac{1}{2k}\right),\ k\in N$ oraliqlarda funksiya qat'iy o'suvchi.

Xuddi shunday $\left(\frac{1}{2k},\frac{1}{2k-1}\right), k\in N$ oraliqlarda $f'(x)<0$ bo'ladi. Shuning uchun berilgan funksiya qat'iy kamayuvchi.

Agar $x<0$ bo'lsa, u holda berilgan funksiyaning juftligini e'tiborga olib $\left(-\frac{1}{2k},-\frac{1}{2k+1}\right), k\in N$ oraliqlarda funksiya qat'iy o'suvchi; $(-\infty;-1)$ va $\left(-\frac{1}{2k+1},-\frac{1}{2k}\right), k\in N$ oraliqlarda esa, qat'iy kamayuvchi.

4) Berilgan funksiya R da aniqlangan va differensiallanuvchi.

Ravshanki,

$$f(x)=\begin{cases} x+\sin 2x, & k\pi<x<\frac{\pi}{2}+k\pi,\ k\in Z\ ; \\ x-\sin 2x, & \pi\left(k+\frac{1}{2}\right)<x<\pi+k\pi, k\in Z\ ; \\ \frac{k\pi}{2}, & x=\frac{k\pi}{2}, k\in Z\ . \end{cases}$$

$$f'(x)=\begin{cases} 1+2\cos 2x, & k\pi<x<\frac{\pi}{2}+k\pi,\ \ k\in Z\ ; \\ 1-2\cos 2x, & \frac{\pi}{2}+k\pi<x<\pi+k\pi, k\in Z\ . \end{cases}$$

Bundan,

$$\begin{cases} 1+2\cos 2x>0; \\ k\pi<x<\frac{\pi}{2}+k\pi; \end{cases} \qquad \begin{cases} 1-2\cos 2x>0; \\ \frac{\pi}{2}+k\pi<x<\pi+k\pi \end{cases}$$

sistemalarni yechib, $k\pi<x<\frac{\pi}{3}+k\pi$, $k\in Z$ va $\pi\left(k+\frac{1}{2}\right)<x<\frac{\pi}{3}+\pi\left(k+\frac{1}{2}\right)$, $k\in Z$ lar uchun $f'(x)>0$ bo'lishini topamiz. Yuqoridagi hosil bo'lgan tengsizliklarni birlashtirsak, $\frac{k\pi}{2}<x<\frac{\pi}{3}+\frac{k\pi}{2}$, $k\in Z$ tengsizlik hosil bo'ladi.

Shunday qilib, $\left(\frac{k\pi}{2},\frac{\pi}{3}+\frac{k\pi}{2}\right)$, $k\in Z$ oraliqda $f'(x)>0$, $\left(\frac{k\pi}{2}+\frac{\pi}{3},\frac{k\pi}{2}+\frac{\pi}{2}\right)$, $k\in Z$ oraliqda $f'(x)<0$ bo'ladi.

Demak, funksiya $\left(\frac{k\pi}{2},\frac{\pi}{3}+\frac{k\pi}{2}\right)$, $k\in Z$ oraliqda o'suvchi, $\left(\frac{k\pi}{2}+\frac{\pi}{3},\frac{k\pi}{2}+\frac{\pi}{2}\right)$, $k\in Z$ oraliqda kamayuvchi bo'ladi.

16.2-misol. Ushbu $f(x)=x^2\ln x$ funksiyani monotonlikka tekshiring.

Yechilishi. Berilgan funksiya $(0;+\infty)$ oraliqda aniqlangan. Uning hosilasi $f'(x)=2x\ln x+x$ bo'ladi. Endi 16.1-teoremaga ko'ra,

$$f'(x)\geq 0, \text{ ya'ni } 2x\ln x+x\geq 0$$

yoki

$$f'(x)\leq 0, \text{ ya'ni } 2x\ln x+x\leq 0$$

bo'ladigan nuqtalar to'plamini topamiz:

a) $x(2\ln x+1)\geq 0 \Rightarrow\ 2\ln x+1\geq 0 \Rightarrow\ x\geq e^{-\frac{1}{2}}$, $[e^{-\frac{1}{2}},+\infty)$,

b) $x(2\ln x+1)\leq 0 \Leftrightarrow 2\ln x+1\leq 0 \Rightarrow\ x\leq e^{-\frac{1}{2}}$, $\left[0;e^{-\frac{1}{2}}\right]$

Bundan berilgan funksiya uchun $[e^{-\frac{1}{2}};+\infty)$ da $f'(x)\geq 0$, $\left[0;e^{-\frac{1}{2}}\right]$ da $f'(x)\leq 0$ bo'lishini olamiz. Demak, berilgan funksiya $\left[0;e^{-\frac{1}{2}}\right]$ da kamayuvchi, $[e^{-\frac{1}{2}};+\infty)$ da esa o'suvchi ekan.

16.3-misol. Ushbu funksiyani monotonlikka tekshiring:

$$f(x)=x^3-\frac{1}{x}$$

Yechilishi. Berilgan funksiya uchun $D(f)=(-\infty;0)\cup(0;+\infty)$. Uning hosilasi $f'(x)=3x^2+\frac{1}{x^2}$ bo'ladi. Ravshanki, $(-\infty;0)$ va $(0;+\infty)$ oraliqlarda $f'(x)>0$ bo'ladi.

Demak, 16.2-teoremaga ko'ra, berilgan funksiya $(-\infty;0)$ va $(0;+\infty)$ oraliqlarda qat'iy o'suvchidir.

16.2. Funksiyaning ekstremum qiymatlari. $f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lib, $x_0\in(a,b)$ bo'lsin. Funksiyaning ekstremum qiymatlari tushunchasini yana bir marta eslatib o'tamiz.

16.1-ta'rif. Agar $x_0\in(a,b)$ nuqtaning shunday $U_\delta(x_0)\subset(a;b)$ atrofi mavjud bo'lib, $\forall x\in U_\delta(x_0)$ uchun

$$f(x)\leq f(x_0)\quad (f(x)\geq f(x_0))$$

tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada lokal maksimumga (lokal minimumga) ega deyiladi. $f(x_0)$ qiymat esa, $f(x)$ funksiyaning $U_\delta(x_0)$ atrofdagi *lokal maksimumi (lokal minimumi)* deyiladi.

16.2-ta'rif. Agar $x_0\in(a,b)$ nuqtaning shunday $U_\delta(x_0)\subset(a;b)$ atrofi mavjud bo'lib, $\forall x\in U_\delta(x_0)$ uchun $f(x)<f(x_0)$ $(f(x)>f(x_0))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada qat'iy maksimumga (qat'iy minimumga) ega deyiladi. $f(x_0)$ qiymat esa $f(x)$ funksiyaning $U_\delta(x_0)$ atrofdagi *qat'iy lokal maksimumi (qat'iy lokal minimumi)* deyiladi.

Funksiyaning maksimumi va minimumi umumiy nom bilan uning *ekstremumi* deyiladi.

Yuqorida keltirilgan ta'riflardagi x_0 nuqta $f(x)$ funksiyaning *lokal maksimum (lokal minimum), qat'iy maksimum (qat'iy minimum) nuqtasi* deb yuritiladi. Funksiyaning maksimum va minimum nuqtalari uning *ekstremum nuqtalari* deyiladi.

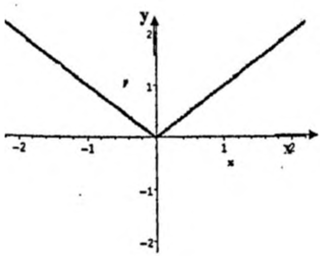
Funksiyaning $U_\delta(x_0)$ atrofdagi lokal maksimum (lokal minimum) qiymatlari $f(x_0)=\max\limits_{x\in U_\delta(x_0)}\{f(x)\}$ $\left(f(x_0)=\min\limits_{x\in U_\delta(x_0)}\{f(x)\}\right)$ kabi belgilanadi.

16.4-teorema (Funksiya ekstremumga ega bo'lishining zaruriy sharti). *Agar $f(x)$ funksiya x_0 $(x_0\in(a,b))$ nuqtada hosilaga ega bo'lib, u shu nuqtada ekstremumga ega bo'lsa, $f'(x_0)=0$ bo'ladi.*

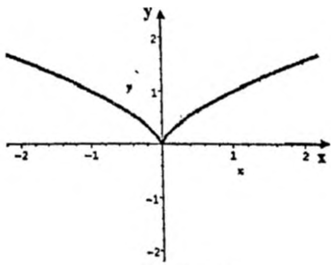
Bu shart funksiya ekstremumga ega bo'lishi uchun yetarli shart bo'la olmaydi. Masalan, $y=x^3$ funksiyaning $x=0$ nuqtadagi hosilasi nolga teng, ya'ni $y'(0)=0$, lekin funksiya bu nuqtada ekstremumga ega emas.

Odatda funksiyaning hosilasi nolga aylanadigan nuqtalar statsionar (turg'un, kritik) nuqtalar deb ataladi.

x_0 nuqtada funksiya hosilaga ega bo'lmasa ham ekstremumga ega bo'lishi mumkin. Masalan, 1) $f(x)=|x|$ funksiyaning $x=0$ nuqtadagi



16.3-chizma.



16.4-chizma.

hosilasi mavjud emas, lekin funksiya $x=0$ nuqtada minimumga ega (16.3-chizma). 2) $f(x)=x^{\frac{2}{3}}$ funksiyaning $x=0$ nuqtadagi hosilasi cheksiz, lekin funksiya $x=0$ nuqtada minimumga ega (16.4-chizma).

Demak, funksiyaning hosilasi cheksiz yoki hosilasi mavjud bo'lmagan nuqtalarda ham ekstremum mavjud bo'lishi mumkin ekan.

Shunday qilib, $f(x)$ funksiyaga ekstremum qiymat beruvchi nuqtalarni funksiyaning statsionar nuqtalari, funksiyaning hosilasi mavjud bo'lmagan nuqtalar, funksiyaning hosilasi cheksiz bo'lgan nuqtalar orasidan izlash kerak ekan. Odatda bunday nuqtalar ekstremumga shubhali nuqtalar deb ataladi.

a) **Ekstremum mavjud bo'lishining birinchi yetarli sharti.** $x_0 \in (a,b)$ nuqtaning $U_\delta^-(x_0)=\{x\in R:\ x_0-\delta<x<x_0;\ \delta>0\}$,

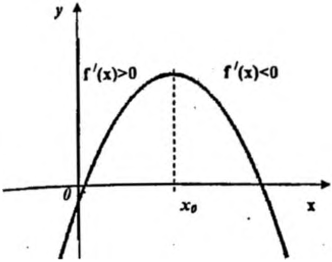
$U_\delta^+(x_0)=\{x\in R:\ x_0<x<x_0+\delta;\ \delta>0\}$ chap va o'ng atroflarini qaraymiz.

Faraz qilaylik, $y=f(x)$ funksiya x_0 nuqtada uzluksiz bo'lib, $U_\delta(x_0)\subset(a;b)$ da chekli $f'(x)$ hosilaga ega bo'lsin (x_0 nuqtada hosila mavjud bo'lmasligi ham mumkin).

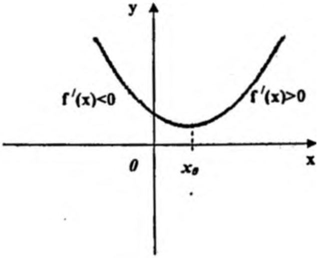
1. Agar $\forall x\in U_\delta^-(x_0)$ uchun, $f'(x)>0$, $\forall x\in U_\delta^+(x_0)$ uchun $f'(x)<0$ bo'lsa,

ya'ni $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini «+» dan «-» ga o'zgartirsa, $f(x)$ funksiya x_0 nuqtada lokal maksimumga erishadi (16.5-chizma).

2. Agar $\forall x \in U_\delta^-(x_0)$ uchun $f'(x)<0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x)>0$



16.5-chizma.



16.6-chizma

ya'ni $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini «-» dan «+» ga o'zgartirsa x_0 nuqtada minimumga erishadi (16.6-chizma)

Agar $\forall x \in U_\delta^-(x_0)$ uchun $f'(x)>0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x)>0$ yoki $\forall x \in U_\delta^-(x_0)$ uchun $f'(x)<0$, $\forall x \in U_\delta^+(x_0)$ uchun $f'(x)<0$ bo'lsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.

$y=f(x)$ funksiyaga ekstremum qiymat beruvchi nuqtalarni ***birinchi tartibli hosila yordamida topish qoidasi:***

1. $f'(x)$ hosila topiladi.

2. $y=f(x)$ funksiyaning kritik nuqtalari, ya'ni $f'(x)$ hosila nolga aylanadigan yoki uzilishga ega bo'lgan nuqtalar topiladi.

3. Topilgan kritik nuqtalar $f(x)$ funksiyaning aniqlanish sohasini oraliqlarga ajratadi, shu oraliqlarda $f'(x)$ hosilaning ishorasi tekshiriladi.

4. Funksiyaning ekstremum nuqtalardagi qiymatlari hisoblanadi.

16.4-misol. $f(x)=\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x+3$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiya $\forall x \in R$ uchun aniqlangan va differensiallanuvchi.

1. $f'(x) = x^3 - 6x^2 + 11x - 6$.

2. $f'(x) = 0 \Rightarrow x^3 - 6x^2 + 11x - 6 = 0 \Rightarrow (x-1)(x-2)(x-3) = 0$,

bundan kritik nuqtalar $x_1 = 1, x_2 = 2, x_3 = 3$ ekanligini topamiz.

3. Oraliqlar usuli yordamida quyidagi jadvalni tuzamiz:

x	$(-\infty;1)$	$(1;2)$	$(2;3)$	$(3;+\infty)$
$f'(x)$	−	+	−	+
$f(x)$	↘	↗	↘	↗

4. Ekstremum mavjud bo'lishining birinchi yetarli shartiga asosan

$f_{min}(1) = \frac{3}{4}$; $f_{max}(2) = 1$, $f_{min}(3) = \frac{3}{4}$ bo'lishini topamiz.

16.5-misol. $f(x) = \dfrac{x^3 + 2x^2}{(x-1)^2}$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiya $x = 1$ nuqtadan tashqari $\forall x \in R$ da aniqlangan va differensiallanuvchi

1. $f'(x) = \dfrac{(3x^2 + 4x)(x-1)^2 - 2(x-1)(x^3 + 2x^2)}{(x-1)^4} = \dfrac{x(x+1)(x-4)}{(x-1)^3}$

2. $f'(x) = 0 \Rightarrow \dfrac{x(x+1)(x-4)}{(x-1)^3} = 0 \Rightarrow x(x+1)(x-4) = 0$.

Bundan, $x_1 = 0, x_2 = -1, x_3 = 4$ kritik nuqtalar ekanligini aniqlaymiz.

3. Oraliqlar usuli yordamida quyidagi jadvalni tuzamiz:

x	$(-\infty;-1)$	$(-1;0)$	$(0;1)$	$(1;4)$	$(4;\infty)$
$f'(x)$	+	−	+	−	+
$f(x)$	↗	↘	↗	↘	↗

4. Ekstremum mavjud bo‘lishining birinchi yetarli shartiga asosan,

$$f_{max}(-1)=\frac{1}{4};\quad f_{max}(0)=0,\quad f_{min}(4)=\frac{32}{3}$$

bo‘ladi.

16.6-misol. Tenglamasi parametrik shaklida berilgan ushbu

$$x=\frac{1}{t(t+1)},\qquad y=\frac{(t+1)^2}{t},\quad t>0$$

funksiyani ekstremumga tekshiring.

Yechilishi. $x(t)$ va $y(t)$ funksiyalar t parametrning $t>0$ qiymatlarida differensiallanuvchi:

$$x_t'=-\frac{2t+1}{t^2(t+1)^2},\quad t>0 \text{ da } \quad x_t'\neq 0. \qquad y_t'=\frac{t^2-1}{t^2}.$$

1. Ma’lumki, bu holda berilgan funksiyaning $y_x'=f'(x)$ hosilasi $f'(x)=\frac{y_t'}{x_t'}$ formula bo‘yicha topiladi: $f'(x)=\frac{(t+1)^2(t^2-1)}{2t+1},\quad t>0.$

2. $f'(x)=\frac{(t+1)^3(t-1)}{2t+1}$ hosila $t=1$ nuqtada nolga aylanadi.

Demak, berilgan funksiya bitta kritik nuqtaga ega: $t=1$ da $x=\frac{1}{2}$.

Agar x, $x=\frac{1}{2}$ nuqtaning chap tomonida bo‘lsa t parametr, $t=1$ nuqtaning chap tomonida bo‘ladi, t ning bu qiymatlarida $f'(x)<0$, $x=\frac{1}{2}$ nuqtaning o‘ng atrofida hosila $f'(x)>0$ bo‘ladi.

Shuning uchun $x=\frac{1}{2}$ nuqtada funksiya minimumga ega va u $f_{min}\left(\frac{1}{2}\right)=4$ bo‘ladi.

16.7-misol. Ushbu $f(x)=5\sqrt[5]{x^4}-x^4$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiya R da aniqlangan va uzluksiz. Funksiyaning hosilasi $f'(x)=4\left(\frac{1}{\sqrt[3]{x}}-x^3\right)$ ga teng. $f'(x)=0$ tenglamaning ildizlarini topamiz: $x=\pm1$. Hosila $x=0$ nuqtada cheksizlikka aylanadi, ya'ni bu nuqtada chekli hosila mavjud emas.

Demak, funksiyaga ekstremum beradigan nuqtalarni $x_1=-1, x_2=0, x_3=1$ nuqtalar orasidan izlash kerak.

Oraliqlar usuli yordamida quyidagi jadvalni tuzamiz:

x	$(-\infty;-1)$	$(-1;0)$	0	$(0;1)$	$(1;\infty)$
$f'(x)$	+	–	*mavjud emas*	+	–
$f(x)$	↗	↘	$f_{min}(0)=0$	↗	↘

Bu jadvaldan, $f'(x)$ hosilaning $x_1=-1$ nuqtadan o'tishda o'z ishorasini «+» dan «-» ga, $x_3=1$ nuqtadan o'tishda ham «+» dan «-» ga o'zgartiradi degan xulosaga kelamiz. Ravshanki, berilgan funksiya $x_1=-1$, $x_3=1$ nuqtalarda uzluksiz. Demak, berilgan funksiya $x_1=-1$, $x_3=1$ nuqtalarda maksimumga ega va uning maksimum qiymatlari: $f_{max}(-1)=4$, $f_{max}(1)=4$; nihoyat, $x=0$ nuqtada berilgan funksiyaning minimumga ega bo'lishi yuqoridagidek ko'rsatiladi va uning $x=0$ nuqtadagi minimumi qiymati $f_{min}(0)=0$ bo'ladi.

16.8-misol. $f(x)=\sqrt[3]{x^2}(x-5)$ funksiyani ekstremumga tekshiring.

Yechilishi. 1. $f'(x)=\frac{2}{3\sqrt[3]{x}}(x-5)+\sqrt[3]{x^2}=\frac{5}{3}\frac{x-2}{\sqrt[3]{x}}$.

2. $x=0$ (bu nuqtada hosila uzilishga ega) va $x=2$ (bu nuqtada hosila nolga aylanadi) nuqtalar kritik nuqtalardir.

3. Quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;2)$	2	$(2;\infty)$
$f'(x)$	+	Mavjud emas	-	0	+
$f(x)$	↗	$f_{max}(0)=0$	↘	$f_{min}(2)\approx -4,8$	↗

4. $f_{max}(0)=0,\ f_{min}(2)=-3\sqrt[3]{4}\approx -4,8.$

b) **Ekstremum mavjud bo'lishining ikkinchi yetarli sharti.** x_0 nuqta $f(x)$ funksiyaning statsionar nuqtasi, ya'ni $f'(x_0)=0$ bo'lsin. Agar $f(x)$ funksiyaning x_0 nuqtadagi ikkinchi tartibli hosilasi mavjud bo'lib, $f''(x_0)<0$ $(f''(x_0)>0)$ bo'lsa, $f(x)$ funksiya x_0 nuqtada maksimumga (minimumga) erishadi.

$y=f(x)$ funksiyaga ekstremum qiymat beruvchi nuqtalarni *ikkinchi tartibli hosila yordamida topish qoidasi:*

1. $f'(x)$ hosila topiladi.

2. Berilgan funksiyaning kritik nuqtalari, ya'ni $f'(x)=0$ bo'ladigan nuqtalar topiladi.

3. Ikkinchi tartibli hosila $f''(x)$ topiladi.

4. Ikkinchi tartibli hosilaning ishorasi har bir kritik nuqtada tekshiriladi. Bunda agar ikkinchi tartibli hosila manfiy bo'lsa, u holda funksiya tekshirilayotgan nuqtada maksimumga, musbat bo'lsa, minimumga ega bo'ladi. Agar ikkinchi tartibli hosila nolga teng bo'lsa, u holda funksiyaning ekstremumini birinchi yetarli shart bo'yicha tekshirish yoki yuqori tartibli hosilalardan foydalanib tekshirishga to'g'ri keladi (c) bandga qarang).

5. Funksiyaning ekstremum nuqtalardagi qiymatlari hisoblanadi.

16.9-misol. $f(x)=x^3-9x^2+24x-12$ funksiyani ikkinchi tartibli hosila yordamida ekstremumga tekshiring.

Yechilishi. Berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$f'(x)=3x^2-18x+24=3(x^2-6x+8)\Rightarrow f''(x)=6(x-3).$$

$3(x^2-6x+8)=0$ tenglamadan $x_1=2,\ x_2=4$ statsionar nuqtalarni topamiz. Topilgan ikkinchi tartibli hosilalarning har bir statsionar nuqtalardagi ishoralarini aniqlaymiz: $f''(2)=-6<0,\ f''(4)=6>0.$

Demak, $x_1=2$ nuqtada funksiya maksimumga $f_{max}(2)=8$, $x_2=4$ nuqtada funksiya minimumga $f_{min}(4)=4$ ega bo'ladi.

16.10-misol. $f(x)=\dfrac{x^3+2x^2}{(x-1)^2}$ funksiyani ikkinchi tartibli hosila yordamida ekstremumga tekshiring.

Yechilishi. Berilgan funksiya R ning $x=1$ nuqtadan tashqari barcha nuqtalarida aniqlangan va differensiallanuvchi. Funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$f'(x)=\frac{x(x^2-3x-4)}{(x-1)^3},\quad f''(x)=\frac{14x+4}{(x-1)^4}.$$

$x(x^2-3x-4)=0$ tenglamadan $x_1=-1,\ x_2=0,\ x_3=4$ statsionar nuqtalarni topamiz. Ikkinchi tartibli hosilaning har bir statsionar nuqtadagi ishorasini aniqlaymiz:

$$f''(-1)=-\frac{5}{8}<0\qquad f''(0)=4>0,\qquad f''(4)=\frac{20}{27}>0.$$

Shunday qilib, funksiya, ekstremum mavjud bo'lishining ikkinchi yetarli shartiga ko'ra, $x_1=-1$ nuqtada maksimumga: $f_{max}(-1)=\frac{1}{4}$; $x_2=0$

nuqtada minimumga: $f_{min}(0)=0$; $x_3=4$ nuqtada esa, minimumga: $f_{min}(4)=\frac{32}{3}$ ega bo'lar ekan.

16.11-misol. $f(x)=(x-2)^4$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz: $f'(x)=4(x-2)^3$, $f''(x)=12(x-2)^2$.

$4(x-2)^3=0$ tenglamadan $x=2$ statsionar nuqta ekanligini topamiz. Ikkinchi tartibli hosila $x=2$ nuqtada nolga aylanadi, shuning uchun ekstremumning birinchi yetarli sharti bo'yicha berilgan funksiyani ekstremumga tekshiramiz.

Ravshanki, $\forall x\in U_\delta^-(2)$, ya'ni $x<2$ uchun $f'(x)=4(x-2)^3<0$, $\forall x\in U_\delta^+(2)$, ya'ni $x>2$ uchun $f'(x)=4(x-2)^3>0$ $\Rightarrow$ $x>2$ bo'ladi.

Demak, berilgan funksiya, ekstremumning birinchi yetarli shartiga ko'ra, $x=2$ nuqtada minimumga erishadi: $f_{min}(2)=f(2)=0$.

16.12-misol. $f(x)=2\sin x+\cos 2x$ funksiyani ikkinchi tartibli hosila yordamida ekstremumga tekshiring.

Yechilishi. Berilgan funksiya davriy funksiya bo'lgani uchun uni ekstremumga tekshirishni $[0,2\pi]$ kesmada olib boramiz. Funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$f'(x)=2\cos x-2\sin 2x=2\cos x(1-2\sin x);\; f''(x)=-2\sin x-4\cos 2x.$$

$2\cos x(1-2\sin x)=0$ tenglamadan funksiyaning $[0;2\pi]$ kesmadagi statsionar nuqtalarini topamiz: $x_1=\frac{\pi}{6}$, $x_2=\frac{\pi}{2}$, $x_3=\frac{5\pi}{6}$, $x_4=\frac{3\pi}{2}$.

Ikkinchi tartibli hosilaning har bir statsionar nuqtalardagi ishoralarini topamiz:

$$f''\left(\frac{\pi}{6}\right)=-3<0,\quad f''\left(\frac{\pi}{2}\right)=2>0,\quad f''\left(\frac{5\pi}{6}\right)=-3<0,\quad f''\left(\frac{3\pi}{2}\right)=6>0.$$

Shunday qilib, funksiya ekstremum mavjud bo‘lishining ikkinchi yetarli shartga ko‘ra, $x_1=\frac{\pi}{6}$ nuqtada maksimumga: $f_{max}\left(\frac{\pi}{6}\right)=\frac{3}{2}$; $x_2=\frac{\pi}{2}$ nuqtada minimumga: $f_{min}\left(\frac{\pi}{2}\right)=1$, $x_3=\frac{5\pi}{6}$ nuqtada minimumga: $f_{max}\left(\frac{5\pi}{6}\right)=\frac{3}{2}$; $x_4=\frac{3\pi}{2}$ nuqtada esa, yana minimumga: $f_{min}\left(\frac{3\pi}{2}\right)=-3$; ega bo‘lar ekan.

c) Ekstremum mavjud bo‘lishining uchinchi yetarli sharti. $f(x)$ funksiyaning $x_0\in(a,b)$ nuqtada $f'(x_0), f''(x_0),\dots,f^{(n)}(x_0)$ hosilalari mavjud bo‘lib, biror $n>2$ son uchun $f'(x_0)=f''(x_0)=\dots=$ $=f^{(n-1)}(x_0)=0$, $f^{(n)}(x_0)\neq 0$ bo‘lsin.

Agar: *a*) n juft son bo‘lib ($n=2m$, $m\in N$), $f^{(n)}(x_0)=f^{(2m)}(x_0)<0$ tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya x_0 nuqtada maksimumga; $f^{(n)}(x_0)=f^{(2m)}(x_0)>0$ tengsizlik o‘rinli bo‘lsa, $f(x)$ funksiya x_0 nuqtada minimumga ega bo‘ladi.

b) n toq son bo‘lsa ($n=2m+1$, $m\in N$), $f(x)$ funksiya ekstremumga ega bo‘lmaydi.

16.13-misol. Ushbu $f(x)=(x-c)^n$ funksiyani ekstremumga tekshiring.

Yechilishi. Ravshanki, $f'(c)=f^{(2)}(c)=\dots=f^{(n-1)}(c)=0$, $f^{(n)}(c)=n!>0$ bo‘ladi. Ekstremumning uchinchi yetarli shartiga ko‘ra, n juft bo‘lganda funksiya $x=c$ nuqtada minimumga ega bo‘ladi, n toq bo‘lganda esa ekstremumga ega bo‘lmaydi.

16.14-misol. $f(x)=\cos x-1+\frac{x^2}{2}$ funksiyani ekstremumga tekshiring.

Yechilishi. Berilgan funksiya uchun $f'(x)=-\sin x+x$ bo‘lib, $f'(x)$ hosila $x=0$ nuqtada nolga aylanadi. $x=0$ statsionar nuqtada funksiyaning ikkinchi, uchinchi va to‘rtinchi tartibli hosilalarini topamiz:

$$f''(x)=-\cos x+1;\ f''(0)=0;\ f'''(x)=\sin x;\ f'''(0)=0;$$
$$f^{(4)}(x)=\cos x;\ f^{(4)}(0)=1\neq 0.$$

Shunday qilib, $x=0$ statsionar nuqtada to'rtinchi tartibli, ya'ni juft tartibli, hosila noldan farqli bo'lib, $f^{(4)}(0)=1>0$ bo'lganligi uchun funksiya $x=0$ nuqtada minimumga ega, va $f_{min}(0)=0$.

16.3. Funksiyaning eng katta va eng kichik qiymatlarini topish. $f(x)$ funksiya $[a;b]$ segmentda aniqlangan va uzluksiz bo'lsin. Veyershtrassning ikkinchi teoremasiga ko'ra, funksiyaning $[a;b]$ da eng katta hamda eng kichik qiymatlari mavjud va funksiya bu qiymatlarga segmentning nuqtalarida erishadi.

Funksiyaning eng katta qiymati quyidagicha topiladi:

1) $f(x)$ funksiyaning $(a;b)$ oraliqdagi maksimum qiymatlari topiladi. $f(x)$ funksiyaning $(a;b)$ dagi hamma maksimum qiymatlaridan iborat to'plam $\{\max f(x)\}$ bo'lsin.

2) $f(x)$ funksiyaning $[a;b]$ segmentning chegarasidagi, ya'ni $x=a,\ x=b$ nuqtalardagi $f(a)$ va $f(b)$ qiymatlari hisoblanadi. So'ngra $\{\max f(x)\}$ to'plamning barcha elementlari bilan $f(a)$ va $f(b)$ lar taqqoslanadi. Bu qiymatlar ichida eng kattasi $f(x)$ funksiyaning $[a;b]$ dagi eng katta qiymati bo'ladi. Xuddi shu usulda funksiyaning eng kichik qiymati ham topiladi.

Biror oraliqda uzluksiz bo'lgan funksiyaning eng katta va eng kichik qiymatlarini topish uchun:

1) bu oraliqda funksiyaning tegishli statsionar nuqtalarini topish, bu topilgan statsionar nuqtalarni ekstremumga tekshirish va funksiyaning bu nuqtalardagi qiymatlarini hisoblash;

2) funksiyaning oraliqning chetki nuqtalaridagi qiymatlarini topish ;

3) topilgan qiymatlarni funksiyaning oraliqning ichidagi nuqtalaridagi ekstremum qiymatlari bilan solishtirish kerak; bu qiymatlarning eng

kichigi va eng kattasi, mos ravishda, funksiyaning qaralayotgan oraliqdagi eng kichik va eng katta qiymatlari bo'ladi.

16.15-misol. Ushbu $f(x)=x^2-4x+6$ funksiyaning $[-3;10]$ oraliqda eng katta va eng kichik qiymatlarini toping.

Yechilishi. **1)** $f'(x)=2x-4$; $f'(x)=2x-4=0$; bunda $x=2\in[-3;10]$ statsionar nuqta bo'ladi. $f'(x)$ hosila $x=2$ nuqtadan o'tishda o'z ishorasini «-» dan «+» ga o'zgartiradi.

Demak, funksiya ekstremumi mavjud bo'lishining birinchi yetarli shartiga ko'ra, $x=2$ nuqtada minimum qiymatga ega bo'ladi: $f_{\min}(2)=2$.

2) $f(-3)=27$, $f(10)=66$.

3) Shunday qilib, funksiyaning $[-3;10]$ oraliqdagi eng kichik qiymati 2 ga teng bo'lib, funksiya unga oraliqning ichki nuqtasida erishadi, eng katta qiymati 66 ga teng bo'lib, funksiya unga oraliqning o'ng chetida erishadi: $f_{eng\ kichik}=f(2)=2$, $f_{eng\ katta}=f(10)=66$.

16.16-misol. Ushbu $f(x)=\sqrt{5-4x}$ funksiyaning $[-1;1]$ oraliqda eng katta va eng kichik qiymatlarini toping.

Yechilishi. Berilgan funksiya $[-1;1]$ oraliqda noldan farqli hosilaga ega: $f'(x)=-\frac{2}{\sqrt{5-4x}}$. Bundan, funksiya eng katta va eng kichik qiymatiga oraliqning chetki nuqtalarida erishadi: $f(-1)=3$, $f(1)=1$.

Shunday qilib, berilgan funksiyaning $[-1;1]$ oraliqdagi eng katta qiymati 3 ga, eng kichik qiymati esa 1 ga teng:

$$f_{eng\ katta}=f(-1)=3,\quad f_{eng\ kichik}=f(1)=1.$$

16.17-misol. $f(x)=\arccos x^2$ funksiyaning $\left[-\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right]$ oraliqdagi eng katta va eng kichik qiymatlarini toping.

Yechilishi. Berilgan funksiya qaralayotgan oraliqda $f'(x)=-\frac{2x}{\sqrt{1-x^4}}$

hosilaga ega. $-\frac{2x}{\sqrt{1-x^4}}=0$ tenglamadan $x=0$ statsionar nuqtani topamiz.

$f'(x)$ hosila $x=0$ nuqtadan o'tishda o'z ishorasini «+» dan «-» ga o'zgartiradi.

Demak, berilgan funksiya $x=0$ nuqtada maksimum qiymatga erishadi: $f_{max}(0)=\frac{\pi}{2}$. Funksiyaning oraliqning chetki nuqtalaridagi qiymatlari $f(\pm\sqrt{2}/2)=\frac{\pi}{3}$.

Shunday qilib, funksiyaning $\left[-\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right]$ oraliqdagi eng katta qiymati $f(0)=\frac{\pi}{2}$ ga, eng kichik qiymati esa $f(\pm\sqrt{2}/2)=\frac{\pi}{3}$ ga teng.

16.18-misol. Konserva bankasi radiusi r va balandligi h bo'lgan silindrdan iborat. r va h lar orasidagi munosabat qanday bo'lganda to'la sirti o'zgarmas bo'lgan konserva bankasi eng katta hajmga ega bo'ladi?

Yechilishi. Konserva bankasining to'la sirtini S bilan belgilaymiz. Ma'lumki,

$$S=2\pi r^2+2\pi rh,\ h=const, \quad (1)$$

bundan $h=\frac{S}{2\pi r}-r$. Konserva bankasining hajmi $V=\pi r^2h=\frac{S}{2}r-\pi r^3$.

Demak, masala $V(r)=\frac{S}{2}r-\pi r^3$ funksiyaning eng katta qiymatini topishga keltirildi. Shuning uchun bu funksiyani maksimumga tekshiramiz. $V'(r)=\frac{S}{2}-3\pi r^2$, $r>$o ekanligini e'tiborga olib,

$$r=\sqrt{\frac{S}{6\pi}} \quad (2)$$

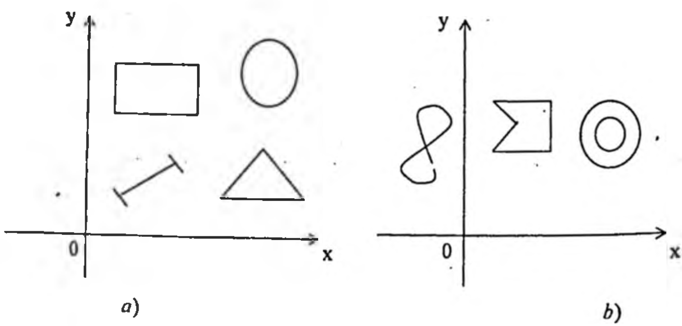
bo'lishini topamiz.

Funksiya ekstremumga ega bo'lsa, u faqat $r=\sqrt{\frac{S}{6\pi}}$ nuqtada ega bo'lishi mumkin. $r<\sqrt{\frac{S}{6\pi}}$ bo'lganda $V'(r)=3\pi\left(\frac{S}{6\pi}-r^2\right)>0$ bo'ladi. $r>\sqrt{\frac{S}{6\pi}}$ bo'lganda esa, $V'(r)<0$ bo'ladi.

Demak, funksiya ekstremumga ega bo'lishining birinchi yetarli shartiga asosan, $V(r)$ funksiya $r=\sqrt{\frac{S}{6\pi}}$ nuqtada maksimumga erishadi. Endi konserva bankasi, eng katta hajmga ega bo'lishi uchun r bilan h orasida qanday bog'lanish borligini aniqlaymiz.(1) bilan (2) dan $\frac{h}{r}=2$, $h=2r$. Demak, eng katta hajmga ega bo'lgan konserva bankasini yasashda uning balandligini diametrga teng qilib olish yetarli.

16.4. Funksiya grafigining qavariqligi va botiqligi. Funksiya grafigining egilish (bukilish) nuqtalarini topishda qavariq to'plam va qavariq funksiya tushunchasi muhim ahamiyatga ega. Shuning uchun biz avvalo qavariq to'plam va qavariq funksiya tushunchalarini beramiz.

Bo'sh bo'lmagan G $(G\subset R^2)$ to'plam berilgan bo'lsin.



16.7-chizma.

16.3-ta'rif. Agar G to'plamning ixtiyoriy ikki nuqtasi bilan birga ularni tutashtiruvchi kesma ham shu to'plamga qarashli bo'lsa, G – *qavariq to'plam* deyiladi. Boshqacha aytganda, agar $\forall x, y \in G$ va barcha $\lambda \in [0;1]$ uchun $\lambda x + (1-\lambda) y \in G$ bo'lsa, G – *qavariq to'plam* deyiladi.

16.7-a)-chizmadagi to'g'ri to'rtburchak, doira, uchburchaklar qavariq to'plamlarga misol bo'la oladi, 16.7-b)-chizmadagi shakllar esa, qavariq bo'lmagan to'plamlardir. Yagona elementli va bo'sh to'plamlarni ham qavariq to'plam deb qarash mumkin.

16.4-ta'rif. Agar ixtiyoriy $x, y \in G$, $x \neq y$ nuqtalar va barcha $\lambda \in (0,1)$ sonlar uchun $\lambda x + (1-\lambda) y$ – G to'plamning ichki nuqtasi bo'lsa, G – *qat'iy qavariq to'plam* deyiladi. Masalan, doira qat'iy qavariq to'plam bo'ladi, parallepiped esa qat'iy qavariq to'plam emas.

$f(x)$ funksiya G – qavariq to'plamda berilgan bo'lsin.

16.5-ta'rif. Agar $\forall x, y \in G$ va $\forall \lambda \in [0;1]$ uchun

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

tengsizlik o'rinli bo'lsa, qavariq G to'plamda aniqlangan va chekli $f(x)$ funksiya *qavariq* deyiladi (16.7-a)-chizma).

16.6-ta'rif. Agar $\forall x, y \in G$, $x \neq y$ nuqtalar va $\forall \lambda \in (0;1)$ uchun

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$$

qat'iy tengsizlik bajarilsa, $f(x)$ funksiya G to'plamda *qat'iy qavariq* deyiladi.

16.7-ta'rif. Agar $g(x) = -f(x)$ funksiya qavariq $G \subset R^2$ to'plamda qavariq bo'lsa, $f(x)$ funksiya G to'plamda *botiq* deyiladi (16.8-b)-chizma).

Misollar. 1) $f(x) = |x|$, $x \in R$, funksiya qavariqdir, chunki $\forall x_1, x_2 \in R$, $x_1 \neq x_2$ nuqtalar va $\forall \lambda \in [0;1]$ uchun $|\lambda x_1 + (1-\lambda) x_2| \leq \lambda |x_1| + (1-\lambda)|x_2|$ tengsizlik o'rinli bo'ladi.

2) $f(x)=e^x$ funksiya R da qat'iy qavariq funksiyadir, chunki $\forall x_1, x_2 \in R,\ x_1 \neq x_2$ nuqtalar va $\forall \lambda \in (0,1)$ uchun

$$e^{\lambda x_1+(1-\lambda)x_2} < \lambda e^{x_1}+(1-\lambda)e^{x_2}$$

tengsizlik bajariladi.

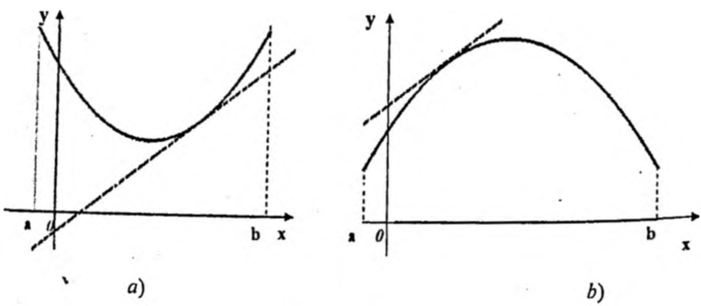
3) $f(x)=-e^x$ funksiya R da botiq funksiya bo'ladi.

Amaliyotda, ko'pincha, funksiya grafigi qavariqligi (botiqligi)ni tekshirishda yuqoridagi keltirilgan ta'riflarga ekvivalient bo'lgan quyidagi ta'riflardan foydalaniladi.

$y=f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin. Agar $y=f(x)$ funksiyaning grafigi (a,b) oraliqning ixtiyoriy nuqtasidan o'tkazilgan urinmadan yuqorida (pastda) yotsa, bu funksiyaning grafigi *qavariq (botiq)* deyiladi (16.7-a), 16.8-b) chizmalar).

Hosila yordamida funksiya grafigining qavariqligi va botiqligini tekshirish mumkin.

$y=f(x)$ funksiya (a,b) oraliqda chekli $f'(x)$ hosilaga ega bo'lsin.



16.8-chizma.

16.5-teorema. *$f(x)$ funksiyaning grafigi (a,b) oraliqda qavariq (qat'iy qavariq) bo'lishi uchun, uning $f'(x)$ hosilasining shu oraliqda kamayuvchi (qat'iy kamayuvchi) bo'lishi zarur va yetarli.*

16.6-teorema. *$f(x)$ funksiyaning (a,b) oraliqda botiq (qat'iy botiq) bo'lishi uchun, uning $f'(x)$ hosilasining shu oraliqda o'suvchi (qat'iy o'suvchi) bo'lishi zarur va yetarli.*

$y=f(x)$ funksiya (a,b) oraliqda ikkinchi tartibli hosilaga ega bo'lsin.

16.7-teorema. *$f(x)$ funksiyaning grafigi (a,b) oraliqda qavariq (botiq) bo'lishi uchun $f''(x)\geq 0$ ($f''(x)\leq 0$) tengsizlikning o'rinli bo'lishi zarur va yetarli.*

16.19-misol. Ushbu $f(x)=x^4-2x^3+6x-4$ funksiya grafigining qavariqlik va botiqlik oraliqlarini toping.

Yechilishi. $f'(x)=4x^3-6x^2+6$, $f''(x)=12x^2-12x=12(x^2-x)=12x(x-1)$ larni topamiz. Ravshanki, $(-\infty;0)$ va $(1;\ \infty)$ oraliqlarda $f''(x)>0$ tengsizlik o'rinli, ya'ni bu oraliqlarda funksiyaning grafigi qavariq bo'ladi, $(0,1)$ oraliqda esa $f''(x)<0$ tengsizlik o'rinli, ya'ni bu oraliqda funksiyaning grafigi botiq bo'ladi.

16.5. Funksiya grafigining egilish nuqtalari. $f(x)$ funksiya x_0 nuqtaning biror $U_\delta(x_0)$ $(\delta>0)$ atrofida aniqlangan bo'lsin.

16.8-ta'rif. Agar $f(x)$ funksiya $U_\delta^-(x_0)$ oraliqda botiq (qavariq) bo'lib, $U_\delta^+(x_0)$ oraliqda esa qavariq (botiq) bo'lsa, ya'ni $f(x)$ funksiya x_0 nuqtadan o'tishda o'z qavariqligining yo'nalishini o'zgartirsa, u holda x_0 nuqta $f(x)$ funksiyaning *egilish nuqtasi* deyiladi, bu holda $(x_0, f(x_0))$ nuqta *$f(x)$ funksiya grafigining egilish nuqtasi* deyiladi.

Agar $x_0\in(a,b)$ - $f(x)$ funksiya grafigi egilish nuqtasining abssissasi bo'lsa, bu nuqtada ikkinchi tartibli hosila mavjud bo'lishi ham, bo'lmasligi ham mumkin.

Funksiyaning ikkinchi tartibli hosilasi nolga aylanadigan yoki mavjud bo'lmaydigan nuqtalar *II tur kritik nuqtalar* deyiladi. Bu nuqtalarda egilish mavjud bo'lishi ham, bo'lmasligi ham mumkin. Masalan, $x=0$ nuqta $y=x^3$ va $y=x^{\frac{1}{3}}$ funksiyalar uchun egilish nuqtasi bo'lib, $y=x^3$ funksiyaning $x=0$ nuqtada ikkinchi tartibli hosilasi mavjud, $y=x^{\frac{1}{3}}$ funksiyaning ikkinchi tartibli hosilasi esa, mavjud emas.

16.8-teorema (egilish nuqtasi bo'lishning zaruriy sharti). *Agar $M(x_0,f(x_0))$ nuqta $f(x)$ funksiya grafigining egilish nuqtasi bo'lib, $f(x)$ funksiya x_0 nuqtada ikkinchi tartibli hosilaga ega bo'lsa, u holda $f''(x_0)=0$ bo'ladi.*

Bu shart funksiya grafigi egilish nuqtasiga ega bo'lishi uchun yetarli shart bo'la olmaydi.

Masalan, $f(x)=x^4$ funksiyaning $f''(x)=12x^2$ hosilasi $x=0$ nuqtada nolga aylanadi, lekin funksiyaning grafigi $M(0;0)$ nuqtada egilishga ega emas.

16.9-teorema (egilish nuqtasi bo'lishning birinchi yetarli sharti). *$f(x)$ funksiya x_0 nuqtaning biror atrofida ikkinchi tartibli hosilaga ega va $f''(x_0)=0$ bo'lsin. U holda, ko'rsatilgan atrofda $f''(x_0)$ ikkinchi tartibli hosila x_0 nuqtaning chap va o'ng atrofida har xil ishoraga ega bo'lsa, u holda $M(x_0,f(x_0))$ nuqta $f(x)$ funksiya grafigining egilish nuqtasi bo'ladi.*

16.10-teorema (egilish nuqtasi bo'lishning ikkinchi yetarli sharti). *Agar $f(x)$ funksiya x_0 nuqtada chekli uchinchi tartibli hosilaga ega va bu nuqtada $f''(x_0)=0$, $f'''(x_0)\neq 0$ shartlarni qanoatlantirsa, u holda $M(x_0,f(x_0))$ nuqta $f(x)$ funksiya grafigining egilish nuqtasi bo'ladi.*

16.11-teorema (egilish nuqtasi bo'lishning uchinchi yetarli sharti). *$n\geq 2$ – biror juft son bo'lsin. Agar $f(x)$ funksiya x_0 nuqtaning biror atrofida* *n tartibli hosilaga, x_0 nuqtaning o'zida esa $n+1$ tartibli hosilaga ega bo'lib, $f^{(1)}(x_0)=f^{(2)}(x_0)=...=f^{(n)}(x_0)=0$, $f^{(n+1)}(x_0)\neq 0$ shartlar bajarilsa, u holda $M(x_0,f(x_0))$ nuqta $f(x)$ funksiya grafigining egilish nuqtasi bo'ladi.*

$y=f(x)$ funksiya grafigining egilish nuqtalarini topish qoidasi:

1. Funksiyaning ikkinchi tartibli hosilasi $f''(x)$ topiladi;

2. $y=f(x)$ funksiyaning II tur kritik nuqtalari, ya'ni $f''(x)$ hosila nolga aylanadigan yoki uzilishga ega bo'lgan nuqtalar topiladi;

3. Topilgan kritik nuqtalar $f(x)$ funksiyaning aniqlanish sohasini oraliqlarga ajratadi. Bu oraliqlarda ikkinchi tartibli $f''(x)$ hosilaning ishorasi tekshiriladi. Agar bunda x_0 kritik nuqta qavariqlik va botiqlik oraliqlarini ajratib tursa, x_0 nuqta funksiya grafigining egilish nuqtasi abssissasidan iborat bo'ladi;

4. Funksiyaning egilish nuqtalaridagi qiymatlari hisoblanadi.

16.20-misol. $f(x)=6x^2-x^3$ funksiya grafigining egilish nuqtalarini toping.

Yechilishi. 1. $f'(x)=12x-3x^2$, $f''(x)=12-6x$.

2. $f''(x)=0 \Rightarrow 12-6x=0$, ya'ni $x=2$ yagona kritik nuqta.

3. $(-\infty;2)$ oraliqda $f''(x)>0$, $(2;+\infty)$ oraliqda esa, $f''(x)<0$ bo'lgani uchun 16.9-teoremaga ko'ra, $x=2$ nuqta funksiya grafigi egilish nuqtasining abssissasidir.

4. Bu nuqtaning ordinatasini topamiz: $f(2)=16$.

Shunday qilib, $(2;16)$ -funksiya grafigining egilish nuqtasi bo'lar ekan.

16.21-misol. $f(x)=3x^4-8x^3+6x^2+12$ funksiya grafigining qavariqlik, botiqlik oraliqlarini va egilish nuqtalarini toping.

Yechilishi. Berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$f'(x)=12x^3-24x^2+12x,\ f''(x)=36x^2-48x+12.$$

$36x^2-48x+12=0$ tenglamadan $x_1=1$ va $x_2=\frac{1}{3}$ ildizlarni topib, funksiyaning ikkinchi tartibli hosilasining 1 va 1/3 nuqtadagi qiymatlarini hisoblaymiz:

$$x_1=1,\quad x_2=\frac{1}{3},\quad f''(1)=0,\quad f''\left(\frac{1}{3}\right)=0.$$

Quyidagi jadvalni tuzamiz:

x	$-\infty<x<\frac{1}{3}$	$\frac{1}{3}<x<1$	$1<x<\infty$
$sign f''$	+	-	+
$f(x)$	∪ qavariq	∩ botiq	∪ qavariq

Shunday qilib, $\left(-\infty;\frac{1}{3}\right)$, $(1;\infty)$ oraliqlarda $f''(x)>0$, $\left(\frac{1}{3};1\right)$ oraliqda esa $f''(x)<0$ bo'ladi.

Demak, 16.7-teoremaga asosan, $\left(-\infty;\frac{1}{3}\right)$ va $(1;\infty)$ oraliqlarda funksiya grafigi qavariq, $\left(\frac{1}{3};1\right)$ oraliqda botiq. $x_1=1, x_2=\frac{1}{3}$ nuqtalardan o'tishda $f''(x)$ hosila o'z ishorasini o'zgartiradi. U holda, 16.9-teoremaga asosan, $(1;13)$, $\left(\frac{1}{3};\frac{335}{27}\right)$ nuqtalar berilgan funksiyaning grafigi uchun egilish nuqtalari bo'ladi.

16.22-misol. Ushbu $f(x)=x^4-6x^2+6x-1$ funksiya grafigining qavariqlik, botiqlik oraliqlarini va egilish nuqtalarini toping.

Yechilishi. $f(x)$ funksiyaning birinchi, ikkinchi va uchinchi tartibli hosilalarini topamiz:

$$f'(x)=4x^3-12x-6,\quad f''(x)=12x^2-12=12(x^2-1),\qquad f'''(x)=24x$$

$12(x^2-1)=0$ tenglamadan $x_1=-1$ va $x_2=1$. Bu nuqtalarda $f''(\pm1)=0$ va $f'''(\pm1)\neq0$. Quyidagi jadvalni tuzamiz:

x	$-\infty<x<-1$	$-1<x<1$	$1<x<\infty$
sign f''	+	-	+
$f(x)$	$\cup$ qavariq	$\cap$ botiq	$\cup$ qavariq

Bu jadvaldan, 16.7-teoremaga asosan, berilgan funksiyaning grafigi $(-\infty;-1)$, $(1;+\infty)$oraliqlarda qavariq, $(-1;1)$ oraliqda esa, botiqligiga ishonch hosil qilamiz.

$f''(\pm1)=0$ bo'lganligi, hamda $x=\pm1$ nuqtalarning chap va o'ng tomonlarida $f''(x)$ hosila o'z ishorasini o'zgartirgani uchun $x=\pm1$ nuqtalar funksiya grafigi egilish nuqtalarining abssissalari bo'ladi, ya'ni $(-1;0)$, $(1;-12)$ nuqtalar funksiya grafigining egilish nuqtalari bo'ladi.

$(-1;0)$, $(1;-12)$ nuqtalar funksiya grafigi uchun egilish nuqtalari ekanligini, boshqa usul, ya'ni 16.11-teorema orqali ham ko'rsatish mumkin.

Haqiqatan ham, $f''(\pm1)=0$, $f'''(x)=24x$, $f'''(\pm1)\neq0$ bo'lganligi uchun, 16.10-teoremaga asosan, $(-1;0)$, $(1;-12)$ nuqtalar funksiya grafigi uchun egilish nuqtalari bo'lar ekan.

16.23-misol. $f(x)=x\sin(\ln x)$ $(x>0)$ funksiya grafigining qavariqligi, botiqliq oraliqlarini va egilish nuqtalarini toping.

Yechilishi. Berilgan funksiyaning birinchi va ikkinchi tartibli hosilalarini topamiz:

$$f'(x)=\sin(\ln x)+x\cos(\ln x)\cdot\frac{1}{x}=\sin(\ln x)+\cos(\ln x),$$

$$f''(x)=\cos(\ln x)\cdot\frac{1}{x}-\sin(\ln x)\cdot\frac{1}{x}=\frac{\sqrt{2}}{x}\cos\left(\ln x+\frac{\pi}{4}\right).$$

$x_k = e^{\frac{\pi}{4}+2k\pi}$ $(k = 0,\pm1,\pm2,\ldots)$ nuqtalarda $f''(x) = 0$ bo'ladi.

$-\frac{3\pi}{4} + 2k\pi < \ln x < \frac{\pi}{4} + 2k\pi$ $\quad (k = 0,\pm1,\pm2,\ldots)$ oraliqda $f''(x) > 0$,

$\frac{\pi}{4} + 2k\pi < \ln x < \frac{5\pi}{4} + 2k\pi$ $\quad (k = 0,\pm1,\pm2,\ldots)$ oraliqda esa $f''(x) < 0$ bo'ladi.

Shunday qilib,

$$e^{-\frac{3\pi}{4}+2k\pi} < x < e^{\frac{\pi}{4}+2k\pi}, \quad (k = 0,\pm1,\pm2,\ldots)$$

oraliqda funksiya grafigi qavariq,

$$e^{\frac{\pi}{4}+2k\pi} < x < e^{\frac{5\pi}{4}+2k\pi}, \quad (k = 0,\pm1,\pm2,\ldots)$$

oraliqda botiq,

$$\left(e^{\frac{\pi}{4}+2k\pi};\ \frac{(-1)^k e^{\frac{\pi}{4}+2k\pi}}{\sqrt{2}} \right) \quad (k = 0,\pm1,\pm2,\ldots)$$

nuqtalar esa, funksiya grafigining egilish nuqtalari bo'ladi.

16.24-misol. Tenglamasi parametrik shaklda berilgan ushbu

$$x = t \cdot e^{t},\ y = t\ e^{-t},\ t > 0 \qquad (*)$$

funksiya grafigining egilish nuqtasini toping.

Yechilishi. Ma'lumki, (*) tenglama bilan berilgan $y = f(x)$ funksiyaning birinchi va ikkinchi tartibli hosilalari quyidagi formulalar bilan topiladi:

$$y'_x = \frac{y'_t}{x'_t}, \quad y''_{xx} = (y'_x)'_t \cdot \frac{1}{x'_t}, \ x'_t \neq 0.$$

Bunda $y'_t = e^{-t}(1-t),\ x'_t = e^{t}(1+t)$, $\quad (y'_x)'_t = -\frac{2e^{-2t}(2-t^2)}{(1+t)^2}$.

Shunday qilib, $y''_{xx} = -\frac{2e^{-2t}(2-t^2)}{(1+t)^3}$ ikkinchi tartibli hosila $t = \sqrt{2}$ da nulga aylanadi va y $t = \sqrt{2}$ nuqtadan o'tishdan o'z ishorasini o'zgartiradi. Parametrning bu qiymatida $y = f(x)$ funksiyaning grafigi egilish nuqtasiga ega bo'ladi. Parametrning $t=\sqrt{2}$ qiymatiga funksiya grafigining $\left(\sqrt{2}\cdot e^{\sqrt{2}};\ \sqrt{2}\cdot e^{-\sqrt{2}}\right)$ nuqtasi to'g'ri keladi.

Demak, $\left(\sqrt{2}\cdot e^{\sqrt{2}};\ \sqrt{2}\cdot e^{-\sqrt{2}}\right)$ nuqta funksiyaning grafigi uchun egilish nuqtasi bo'ladi.

16.25-misol. a, b, c koeffitsiyentlarning qanday qiymatlarida ushbu

$$f(x)=ax^4+bx^3+cx^2+dx+e$$

funksiyaning grafigi egilish nuqtasiga ega bo'ladi.

Yechilishi. Berilgan funksiyaning ikkinchi tartibli hosilasini topamiz: $f''(x)=12ax^2+6bx+2c$. Funksiyaning grafigi egilish nuqtasiga ega bo'lishi uchun $6ax^2+3bx+c=0$ tenglama ikkita har xil haqiqiy ildizga ega bo'lishi zarur va yetarli, yoki $3b^2-8ac>0$ shart bajarilishi kerak.

16.6. Qavariq funksiyalarning tengsizliklarni yechishda qo'llanilishi. Qavariq to'plamlar va qavariq funksiyalar tushunchasi klassik asosiy tengsizliklarni isbot qilishda muhim rol o'ynaydi. Asosiy klassik tengsizliklar ichida eng muhimi Yensen tengsizligi hisoblanadi. Qolgan klassik tengsizliklar, xususiy holda, ya'ni natija sifatida, Yensen tengsizligidan kelib chiqadi. Shuning uchun Yensen tengsizligini qavariq funksiya ta'rifidan foydalanib, batafsil isbot qilamiz.

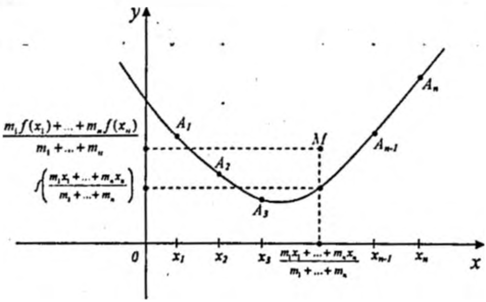
Teorema (Yensen tengsizligi). *$f(x)$ funksiya biror (a,b) intervalda qavariq funksiya bo'lsin. $x_1,x_2,\dots,x_n$ lar - (a,b) intervaldan olingan ixtiyoriy nuqtalar, $\lambda_1,\lambda_2,\dots,\lambda_n$ lar esa, ixtiyoriy musbat sonlar bo'lib, ularning yig'indisi 1 ga teng bo'lsin. U holda*

$$f(\lambda_1x_1+\dots+\lambda_nx_n)\le\lambda_1f(x_1)+\lambda_2f(x_2)+\dots+\lambda_nf(x_n) \qquad (1)$$

tengsizlik o'rinli. Odatda (1) tengsizlik - *Yensen tengsizligi* deyiladi.

Isboti. Shartga ko'ra $y=f(x)$ qavariq funksiya bo'lgani uchun uning grafigini 16.9-chizmada berilgan ko'rinishda tasvirlash mumkin. $y=f(x)$ funksiya grafigidan abssissalari $x_1x_2,\dots,x_n$ bo'lgan $A_1,A_2,\dots,A_n$ nuqtalarni olib, bu nuqtalarga massalari $m_1, m_2, ..., m_n$ bo'lgan yuklar joylashtirilgan, deb faraz qilamiz. Unda fizikadan ma'lumki, bu nuqtalardagi massalar markazining koordinatalari quyidagi ko'rinishda bo'ladi:

$$\left(\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; \frac{\sum_{i=1}^{n} m_i f(x_i)}{\sum_{i=1}^{n} m_i}\right).$$



16.9-chizma.

$A_1, A_2, ..., A_n$ nuqtalar qavariq funksiya grafigining yuqorisidagi qavariq to'plamda yotganligi uchun ular massalarining markazi ham shu qavariq to'plamda yotadi. Shuning uchun massalar markazi – M nuqtaning ordinatasi $y = f(x)$ funksiya grafigidagi abstsissasi $\frac{\sum_{i=1}^{n} m_i x}{\sum_{i=1}^{n} m_i}$ bo'lgan nuqtaning ordinatasi $f\left(\frac{\sum m_i x_i}{\sum m_i}\right)$ dan kichik bo'lmaydi, ya'ni

$$f\left(\frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n}\right) \le \frac{m_1 f(x_1) + m_2 f(x_2) + ... + m_n f(x_n)}{m_1 + m_2 + ... + m_n} \quad (2)$$

tengsizlik o'rinli bo'ladi. Bunda $m_1=\lambda_1,...,m_n=\lambda_n$ deb olsak, (1) Yensen tengsizligi isbot bo'ladi. (1) va (2) tengsizliklar o'zaro ekvivalent. Haqiqatdan ham, (1) tengsizlikni isbot qilish davomida (2) tengsizlikni isbot qildik, agar (1) da $\lambda_i=\frac{m_i}{m_1+...+m_n}$ $(i=\overline{1,n})$ deb olsak, natijada (2) tengsizlikni ham Yensen tengsizligi deb aytish mumkin.

Agar $f(x)$ botiq funksiya bo'lsa, u holda (1) va (2) Yensen tengsizliklaridagi tengsizlik ishorasi qarama-qarshisiga o'zgaradi. Bunga ishonch hosil qilish uchun $-f(x)$ qavariq funksiyani qarash yetarli.

16.25-misol. Ushbu

$$(a_1b_1+....+a_nb_n)^2\le(a_1^2+a_2^2+...+a_n^2)\cdot(b_1^2+b_2^2+...+b_n^2)$$

Koshi – Bunyakovskiy tengsizligini isbot qiling. Bunda a_i va b_i $(i=\overline{1,n})$ lar ixtiyoriy musbat sonlar.

Isboti. Ravshanki, $y=x^2$ qavariq funksiyadir. Shu sababga ko'ra, bu funksiya uchun Yensen tengsizligini qo'llash mumkin:

$$\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)^2\le\frac{m_1x_1^2+m_2x_2^2+....+m_nx_n^2}{m_1+m_2+...+m_n}\quad(m_i>0)$$

Bundan $(m_1x_1+....+m_nx_n)^2\le(m_1x_1^2+...+m_nx_n^2)(m_1+...+m_n)$ tengsizlik kelib chiqadi. Bu yerda $m_i=b_i^2$, $x_i=\frac{a_i}{b_i}$ deb olsak, Koshi – Bunyakovskiy tengsizligi isbot bo'ladi.

16.26-misol. O'rta arifmetik va o'rta geometrik miqdorlar haqida Koshi tengsizligini, ya'ni

$$\frac{x_1+x_2+...+x_n}{n}\ge\sqrt[n]{x_1...x_n}\quad(x_i>0,\ i=\overline{1,n})$$

tengsizlikni isbotlang.

Yechilishi. Bu tengsizlikning ikkala tomonini logarifmlash natijasida

$$\ln\left(\frac{1}{n}x_1+\frac{1}{n}x_2+\ldots+\frac{1}{n}x_n\right)\ge\frac{1}{n}\ln x_1+\frac{1}{n}\ln x_2+\ldots+\frac{1}{n}\ln x_n$$

tengsizlikni hosil qilamiz. Bu tengsizlik Yensen tengsizligini eslatadi, lekin tengsizlik qarama – qarshi ishorali, chunki $\ln x$ funksiya qavariq emas, lekin $-\ln x$ funksiya qavariq bo'ladi. Shuning uchun $\lambda_i=\frac{1}{n}\left(i=\overline{1,n}\right)$ deb olib, $-\ln x$ funksiya uchun Yensen tengsizligini qo'llasak, natijada Koshi tengsizligini hosil qilamiz.

16.27-misol. Ushbu

$$\sum_{i=1}^{n}a_ib_i\le\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1,\ (a_i,b_i>0)$$

tengsizlikni (Geldyor tengsizligini) isbotlang.

Yechilishi. Ravshanki, $x>0, p>1$ bo'lganda $y=x^p$ funksiya qavariq funksiya. Demak, bu funksiyaga (2) Yensen tengsizligini qo'llash mumkin.

$$\left(\frac{\sum_{i=1}^{n}m_ix_i}{\sum_{i=1}^{n}m_i}\right)^p\le\frac{\sum_{i=1}^{n}m_ix_i^p}{\sum_{i=1}^{n}m_i}$$

Bu tengsizlikdan, $\sum_{i=1}^{n}m_ix_i\le\left(\sum_{i=1}^{n}m_i\right)^{\frac{p-1}{p}}\left(\sum_{i=1}^{n}m_ix_i^p\right)^{\frac{1}{p}}$ Shartga ko'ra, $\frac{1}{p}+\frac{1}{q}=1$ bo'lgani uchun, $\frac{p-1}{p}=\frac{1}{q}$. Buni e'tiborga olgan holda, keyingi tengsizlikdan

$$\sum_{i=1}^{n}m_ix_i\le\left(\sum_{i=1}^{n}m_ix_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}m_i\right)^{\frac{1}{q}}$$

bo'lishi kelib chiqadi. Bu tengsizlikda $m_i=b_i^q$, $x_i=a_ib_i^{1-q}$ deb olinsa, u holda isbot qilinishi kerak bo'lgan Geldyor tengsizligi kelib chiqadi.

16.28-misol. Ushbu

$$\sqrt[n]{a_1a_2...a_n}+\sqrt[n]{b_1...b_n}\le\sqrt[n]{(a_1+b_1).....(a_n+b_n)}\quad (a_i,b_i>0)$$

Minkovskiy tengsizligini isbotlang.

Yechilishi. Tengsizlikning ikkala tomonini $\sqrt[n]{a_1a_2...a_n}$ ga bo'lamiz natijada

$$1+\left(\frac{b_1}{a_1}\right)^{\frac{1}{n}}...\left(\frac{b_n}{a_n}\right)^{\frac{1}{n}}\le\left(1+\frac{b_1}{a_1}\right)^{\frac{1}{n}}.....\left(1+\frac{b_n}{a_n}\right)^{\frac{1}{n}}$$

tengsizlikni hosil qilamiz. Bunda $x_i=\ln\frac{b_i}{a_i}$, $\frac{b_i}{a_i}=e^{x_i}$ deb tengsizlikni

quyidagi $1+e^{\sum_{i=1}^{n}\frac{1}{n}x_i}\le\prod\left(1+e^{x_i}\right)^{\frac{1}{n}}$ ko'rininshga keltiramiz. Tengsizlikning ikkala tomonini logarifmlash natijasida

$$\ln\left(1+e^{\sum_{i=1}^{n}\frac{1}{n}x_i}\right)\le\sum\frac{1}{n}\ln\left(1+e^{x_i}\right)$$

ni hosil qilamiz. $y=\ln\left(1+e^{x}\right)$ funksiya qavariq funksiya. Keyingi tengsizlik $y=\ln\left(1+e^{x}\right)$ uchun Yensen tengsizligini ifodalaydi.

16.7. Funksiyalarni to'liq tekshirish va ularning grafiklarini chizish

Biz III bobning yuqoridagi paragraflarida funksiyalarning o'zgarish xarakterini hosilalar yordamida o'rgandik. Funksiyaning o'zgarish xarakterini hosila yordamida o'rganish funksiya grafigini aniqroq yasashda muhim rol o'ynaydi.

Funksiyalarni to'liq tekshirish va ularning grafiklarini yasashni quyidagi sxema bo'yicha olib borish maqsadga muvofiq bo'ladi:

1. Funksiyaning aniqlanish sohasini topish.
2. Funksiyani uzluksizlikka tekshirish va uzilish nuqtalarini topish.
3. Funksiyaning juft, toqligi hamda davriyligini aniqlash.

4. Funksiya grafigining o'qlar bilan kesishish nuqtalarini topish.

5. Funksiyaning ishorasi saqlanadigan oraliqlarni aniqlash.

6. Funksiya grafigining asimptotalarini topish.

7. Funksiyaning monotonlik oraliqlarini topish va ekstremumga tekshirish.

8. Funksiya grafigining qavariqligi hamda botiqligini aniqlash, egilish nuqtalarini topish.

9. Funksiyaning grafigini chizish.

16.29-misol. $f(x)=x^3-3x^2+4$ funksiyani to'liq tekshiring va grafigini chizing.

Yechilishi. Berilgan funksiya R da aniqlangan, uzluksiz va differensiallanuvchi. Funksiyaning grafigi asimptotalarga ega emas,

$$\lim_{x\to-\infty} f(x)=-\infty, \quad \lim_{x\to+\infty} f(x)=+\infty.$$

Berilgan funksiyani ushbu $f(x)=(x+1)(x-2)^2$ ko'rinishda tasvirlaymiz. $(x+1)(x-2)^2=0$ tenglamadan $x_1=-1$, $x_2=2$ nuqtalar $f(x)=0$ tenglamaning ildizlari ekanligini topamiz.

Demak, funksiyaning grafigi $(-1;0)$, $(2;0)$ nuqtalarda abssissa o'qi bilan, $(0;4)$ nuqtada esa, ordinata o'qi bilan kesishadi.

Funksiyani ekstremumga tekshiramiz: $f'(x)=3x^2-6x=3x(x-2)$. Bundan birinchi tartibli hosila $x_1=0, x_2=2$ nuqtalarda nolga aylanishi kelib chiqadi, ya'ni bu nuqtalar-kritik nuqtalar bo'ladi. $x\in(-\infty;0)$ bo'lganda $f'(x)>0$, $x\in(0;2)$ bo'lganda $f'(x)<0$, $x\in(2;+\infty)$ bo'lganda esa, $f'(x)>0$ bo'ladi.

Shunday qilib, funksiyaning birinchi tartibli $f'(x)$ hosilasi $x_1=0$ nuqtadan o'tishda o'z ishorasini «+» dan «-» ga, $x_2=2$ nuqtadan o'tishda esa, «–» dan «+» ga o'zgartiradi. Demak, ekstremum mavjud bo'lishining birinchi yetarli shartiga asosan, funksiya $x_1=0$ nuqtada maksimum $\left(f_{max}(0)=4\right)$ ga, $x_2=2$ nuqtada esa minimum $\left(f_{min}(2)=0\right)$ ga cga bo'ladi. Funksiya grafigini chizishda qulaylik uchun, funksiya to'g'risida yuqorida olingan ma'lumotlar yordamida quyidagi jadvalni tuzamiz:

1-jadval.

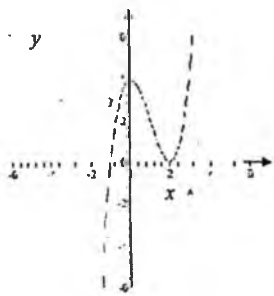
x	$(-\infty;0)$	0	$(0;2)$	2	$(2;+\infty)$
sign y'	+	0	-	0	+
$y=f(x)$ funksiyaning o'zgarishi	↗	$f_{max}(0)=4$	↘	$f_{min}(2)=$	↗

Funksiyaning ikkinchi tartibli hosilasini topamiz: $f''(x)=6(x-1)$. Funksiyaning ikkinchi tartibli hosilasi $x=1$ nuqtada nolga aylanadi, ya'ni $x=1$ nuqta II-tur kritik nuqta bo'ladi. $x\in(-\infty;1)$ da $f''(x)<0$, $x\in(1;+\infty)$da esa $f''(x)>0$. Dcmak, 16.7-teorcmaga asosan, $(-\infty;1)$ oraliqda funksiyaning grafigi botiq, $(1;+\infty)$oraliqda qavariq, $x=1$ nuqta funksiyaning grafigi cgilish nuqtasining abssissasi bo'lib, $(1;2)$ nuqta egilish nuqtasi bo'ladi.

2-jadval

x	$(-\infty;1)$	1	$(1;+\infty)$
sign f''	-	0	+
$y=f(x)$ funksiya grafigi qavariqligining yo'nalishi.	∩	(1;2)	∪

Yuqoridagi mulohazalar va 1-,-2-jadvallar yordamida funksiyaning grafigini (16.10-chizma) chizamiz.



16.10-chizma.

16.30-misol. $f(x)=\dfrac{(x-1)^2}{(x+1)^3}$ funksiyani to'liq tekshiring va grafigini chizing.

Yechilishi. Berilgan funksiya R ning $x=-1$ nuqtadan tashqari barcha nuqtalarida aniqlangan. Ravshanki, funksiya grafigi, mos ravishda, $(1;0)$ va $(0,1)$ nuqtalarda abssissa va ordinata o'qlarini kesadi. $x<-1$ da funksiya manfiy, $x>-1$ da esa, musbat. Funksiya $x=-1$ nuqtada II-tur uzilishga ega va $\lim\limits_{x\to-1-0}f(x)=-\infty$, $\lim\limits_{x\to-1+0}f(x)=+\infty$ bo'lgani uchun $x=-1$ to'g'ri chiziq funksiya grafigi uchun vertikal asimptota, $y=0$ to'g'ri chiziq esa, gorizontal asimptota.

Berilgan funksiya davriy ham emas, juft ham emas va toq ham emas.

Funksiyani ekstremumga tekshirish uchun uning birinchi tartibli hosilasini topamiz:

$$f'(x)=-\frac{x^2-6x+5}{(x+1)^4}=-\frac{(x-1)(x-5)}{(x+1)^4}.$$

Birinchi tartibli $f'(x)$ hosila $x_1=1$ va $x_2=5$ nuqtalarda nolga aylanadi, ya'ni bu nuqtalar kritik nuqtalar bo'ladi. $x\in(-\infty;1)$ da $f'(x)<0$, $x\in(1,5)$ da $f'(x)>0$, $x\in(5,+\infty)$ da esa $f'(x)<0$ bo'ladi. Demak,

$f'(x)$ hosila $x_1=1$ nuqtadan o'tishda o'z ishorasini «-» dan «+» ga o'zgartiradi, $x_2=5$ nuqtadan o'tishda esa, «+» dan «-» ga o'zgartiradi. Shunday qilib, funksiya $x_1=1$ nuqtada minimum $f_{min}(1)=0$ qiymatga, $x_2=5$ nuqtada esa, maksimum $f_{max}(5)=\frac{2}{27}$ qiymatga ega bo'ladi.

1-jadval

x	$(-\infty;1)$	1	(1;5)	5	$(5;\infty)$	x<-1	-1	x>-1
$sign f'$	-	0	+	0	-	-	∞	-
$y=f(x)$- funksiyaning o'zgarishi	↘	0	↗	$\frac{2}{27}$	↘	Manfiy	∞	Musbat

Funksiyaning qavariqlik, botiqlik oraliqlari va egilish nuqtalarini topamiz:

$$f''(x)=\frac{2(x^2-10x+13)}{(x+1)^5}.$$

Ikkinchi tartibli $f''(x)$ hosila $x_1=5+\sqrt{12}$, $x_2=5-\sqrt{12}$ nuqtalarda nolga aylanadi, ya'ni bu nuqtalar II-tur kritik nuqta bo'ladi. $x\in(-\infty;5-\sqrt{12})$ da $f''(x)>0, x\in(5-\sqrt{12};5+\sqrt{12})$ da $f''(x)<0, x\in(5+\sqrt{12};+\infty)$ da esa $f''(x)>0$.

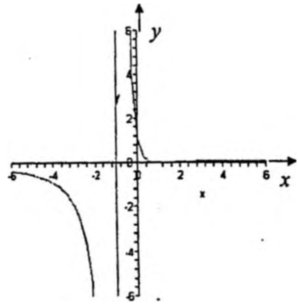
Demak, $(-\infty;5-\sqrt{12})$ oraliqda funksiya grafigi qavariq $(5-\sqrt{12};5+\sqrt{12})$ oraliqda botiq, $(5+\sqrt{12};+\infty)$ oraliqda qavariq.

Shunday qilib, $x_1=5+2\sqrt{3}, x_2=5-2\sqrt{3}$ nuqtalar funksiya grafigi egilish nuqtalarining abssissalari bo'lib, $(5+2\sqrt{3};\ 0,0659)$ va $(5-2\sqrt{3};\ 0,0196)$ nuqtalar egilish nuqtasi bo'ladi.

2-jadval

x	$(-\infty;5-2\sqrt{3})$	$5-2\sqrt{3}$	$(5-2\sqrt{3},5+2\sqrt{3})$	$5+2\sqrt{3}$	$(5+2\sqrt{3},+\infty)$
sign f''	+	0	-	0	+
$f(x)$ funksiya grafigi qavariqligining yo'nalishi.	∪	0,0196	∩	0,0659	∪

Yuqoridagi mulohazalar va 1-, 2- jadvallarga binoan berilgan funksiyaning grafigini (16.11-chizma) chizamiz.



16.11-chizma.

16.31-misol. Ushbu

$$y = f(x) = \frac{\sin x}{2 + \cos x}$$

funksiyani to'liq tekshiring va grafigini chizing.

Yechilishi. Berilgan funksiya R da aniqlangan uzluksiz, toq va 2π davrga ega bo'lgan davriy funksiya, koordinata boshi, simmetriya markazi bo'ladi. $x = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$da $y = 0$ bo'ladi. Ravshanki, uning ishorasi $\sin x$ ning ishorasi kabi bo'ladi, ya'ni $sign(y) = sign(\sin x)$. Funksiya davriy bo'lgani uchun tekshirishni $[0, 2\pi]$ da olib borish yetarli.

Funksiya grafigi asimptotaga ega emas. Funksiyaning birinchi tartibli hosilasini topamiz:

$$f'(x) = \frac{1 + 2\cos x}{(2 + \cos x)^2}$$

Birinchi tartibli $f'(x)$ hosila $x_1 = \frac{2\pi}{3}, x_2 \frac{4\pi}{3}$ nuqtalarda nolga aylanadi, ya'ni bu nuqtalar kritik nuqtalar bo'ladi.

$x\in\left[0;\frac{2\pi}{3}\right)$; $x\in\left(\frac{4\pi}{3};2\pi\right]$ larda $f'(x)>0$, $x\in\left(\frac{2\pi}{3};\frac{4\pi}{3}\right)$ larda $f'(x)<0$ bo‘ladi.

Shuning uchun $x_1=\frac{2\pi}{3}$ nuqtada maksimumga, ya’ni $f_{max}\left(\frac{2\pi}{3}\right)=\frac{1}{\sqrt{3}}$;

$x_1=\frac{4\pi}{3}$ nuqtada esa, minimum ga, ya’ni $f_{min}\left(\frac{4\pi}{3}\right)=-\frac{1}{\sqrt{3}}$ ga ega bo‘ladi.

1-jadval.

x	$\left[0;\frac{2\pi}{3}\right)$	$\frac{2\pi}{3}$	$\left(\frac{2\pi}{3};\frac{4\pi}{3}\right)$	$\frac{4\pi}{3}$	$\left(\frac{4\pi}{3};2\pi\right]$
sign f'	+	0	-	0	+
$y=f(x)$ – funksiyaning o‘zgarishi	↗	$\frac{1}{\sqrt{3}}$	↘	$-\frac{1}{\sqrt{3}}$	↗

Endi, funksiya grafigining qavariqlik, botiqlik oraliqlari va egilish nuqtalarini topamiz: buning uchun avvalo, funksiyaning ikkinchi tartibli hosilasini topamiz:

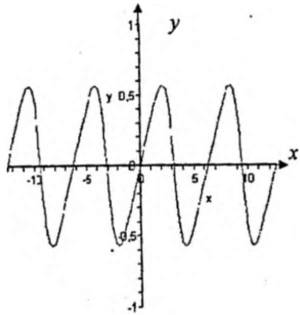
$$y''=f''(x)=\frac{2\sin x(\cos x-1)}{(2+\cos x)^2}.$$

$x\in(0;\pi)$ da $f''(x)<0$, $x\in(\pi;2\pi)$ da $f''(x)>0$ bo‘ladi.

Demak, $(\pi;0)$ oraliqda funksiya grafigi botiq, $(\pi;2\pi)$ oraliqda esa, qavariq bo‘lib, $(0;\pi)$ nuqta egilish nuqtasi bo‘ladi. 2-jadval.

X	$(0;\pi)$	π	$(\pi;2\pi)$
sign y''	-	0	+
Funksiya grafigi qavariqligining yo‘nalishi.	∩	0	∪

Yuqoridagi mulohaza va 1- , 2 – jadvallarga asoslanib, berilgan funksiyaning grafigi chiziladi (16.12- chizma).



16.12-chizma.

16.8. MAPLE tizimidan foydalanib funksiyalarni to'liq tekshirish va ularning grafiklarini yasash

16.32-misol. MAPLE tizimidan foydalanib, ushbu $f(x)=\dfrac{x^3}{6(3-x)^2}$ funksiyani to'liq tekshiring va uning grafigini chizing.

Yechilishi. 1 Funksiyaning aniqlanish sohasi: D(f)= $(-\infty;3) \cup (3;\infty)$.

2. MAPLE tizimidan foydalanib, berilgan funksiyani uzluksizlikka tekshiramiz:

```
> readlib(singular):singular((x^3)/(6*(3-x)^2),x);
```

$$\{x=3\},\ \{x=-\infty\},\ \{x=+\infty\}$$

```
> limit(x^3/(6*(3-x)^2),x=3,left)=limit(x^3/(6*(3-x)^2),x=3,left);
```

$$\lim_{x\to 3-0}\left(\frac{x^3}{6(3-x)^2}\right)=\infty$$

```
> limit(x^3/(6*(3-x)^2),x=3,right)=limit(x^3/(6*(3-
x)^2),x=3,right);
```

$$\lim_{x\to 3+0}\left(\frac{x^3}{6(3-x)^2}\right)=+\infty$$

Demak, MAPLE tizimidagi natijalariga ko'ra, $x=3$ funksiyaning 2-tur uzilish nuqtasi bo'ladi.

3. MAPLE tizimidan foydalanib, berilgan funksiyani davriylikka, juft va toqlikka tekshiramiz:

> evalb(y(x)=y(-x));

false

> evalb(y(x)=-y(-x));

false

> solve(y(x)=y(x+T),T);

$$0, -\frac{(2x^2+27-18x-3\sqrt{-27+12x})x}{2(9-6x+x^2)}, -\frac{(2x^2+27-18x+3\sqrt{-27+12x})x}{2(9-6x+x^2)}$$

Demak, funksiya davriy ham emas, juft ham emas; toq ham emas ekan.

4. MAPLE tizimidan foydalanib, funksiyaning koordinatalar o'qlari bilan kesishish nuqtalarini topamiz:

> y:=x->x^3/(6*(3-x)^2);

$$y := x \to \frac{x^3}{6(3-x)^2}$$

> solve(y(x)=0,x);

0, 0, 0

Shunday qilib, funksiya grafigi faqat bitta O(o,o) nuqtada koordinatalar o'qlari bilan kesishadi.

5. MAPLE tizimida funksiyaning ishorasi saqlanadigan intervallarni aniqlaymiz:

> solve(y(x)>0,x);

Re*alRange*(*Open*(0), *Open*(3)), Re*alRange*(*Open*(3), ∞)

> solve(y(x)<0,x);

Re*alRange*(−∞, *Open*(0))

MAPLE tizimidagi natijalariga ko'ra, quyidagi jadvalni tuzamiz:

X	(-∞;0)	0	(0;3)	3	(3; ∞)
signy	–	0	+	∞	+
grafikning	O*x* o'qi		O*x*		O*x* o'qi

joylanishi	ostida		o'qi ustida		ustida

6. MAPLE tizimidan foydalanib, berilgan funksiya grafigining asimptotalarini topamiz:

```
> alpha[1]:=limit(y(x)/x,x=+infinity);
```

$$a_1 := \frac{1}{6}$$

```
> alpha [2]:=limit(y(x)/x,x=-infinity);
```

$$a_2 := \frac{1}{6}$$

```
> a:=alpha[1];
```

$$a := \frac{1}{6}$$

```
> beta[1]:=limit(y(x)-a*x,x=+ infinity );
```

$$b_1 := 1$$

```
> beta[2]:= limit (y(x)-a*x,x=- infinity );
```

$$b_2 := 1$$

Demak, $y=\frac{1}{6}x+1$ - to'g'ri chiziq funksiya grafigining og'ma asimptotasi bo'ladi.

7. MAPLE tizimidan foydalanib, berilgan funksiyaning monotonlik oraliqlari va ekstremum qiymatlarini topamiz:

```
> solve (diff (y(x),x)>0,x);
```

$$RealRange(-\infty, Open(0)), RealRange(Open(0), Open(3)), RealRange(Open(9), \infty)$$

```
> solve (diff (y(x),x)<0,x);
```

$$RealRange(Open(3), Open(9))$$

```
> solve (diff (y(x),x)=0,x);
```

$$9, 0, 0$$

```
> eval(diff(y(x),x$2),x=0);
```

$$0$$

```
> eval(diff(y(x),x$2),x=9);
```

$$\frac{1}{16}$$

```
> eval(y(x),x=0);
```

$$0$$

```
> eval(y(x),x=9);
```

$$\frac{27}{8}$$

MAPLE tizimidagi natijalariga ko‘ra, quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;3)$	3	$(3;9)$	+	$(9;\infty)$
$sign(y')$	+	0	+	∞	-	0	+
funksiyaning o‘zgarishi	↗	0	↗	∞	↘	$\frac{27}{8}$	↗

$y_{min}=y(9)=\frac{27}{8}$. A(9, $\frac{27}{8}$)-berilgan funksiya grafigining minimum nuqtasi bo‘ladi.

8. MAPLE tizimidan foydalanib, berilgan funksiyaning qavariqlik va botiqlik oraliqlarini topamiz:

```
> solve (diff (y(x),,x$2)>0,x);
```

$RealRange(Open(0), Open(3)), RealRange(Open(3), \infty)$.

```
> solvc (diff (y(x),x$2)<0,x);
```

$RealRange(-\infty, Open(0))$

```
> solve (diff (y(x),x$2)=0,x);
```

$$0$$

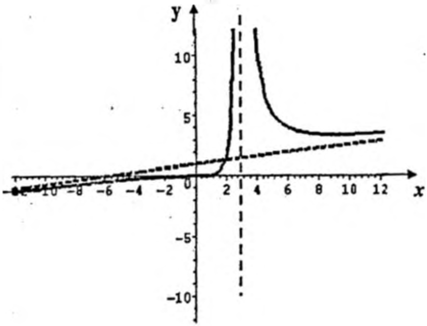
MAPLE tizimidagi natijalariga ko‘ra, quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;3)$	3	$(3;+\infty)$
$sign\ y''$	-	0	+	∞	+
Funksiya grafigining qavariqlik yo‘nalishi	∩ ↑	0	↓ ∪	∞	↓ ∪

9. MAPLE tizimidan foydalanib, berilgan funksiyaning grafigini chizamiz:

```
> plot([y(x),[3,t,t=-4..12]],x=-6..16,-
6..6,color=[red,black],thickness=3,linestyle=[1,3]);
```

Funksiyaning grafigi 16.13-chizmada tasvirlangan.



16.13-chizma

16.33-misol. MAPLE tizimidan foydalanib, ushbu

$$u = \frac{x^2}{1+x^2}$$

funksiyani to'la tekshiring va grafigini chizing.

Yechilishi. 1 Funksiyaning aniqlanish sohasi: $D(f) = (-\infty;\infty)$

2. MAPLE tizimidan foydalanib, berilgan funksiyani uzluksizlikka tekshiramiz:

```
> readlib(singular):singular((x^2)/(1+x^2),x);
false
```

Demak, berilgan funksiyaning uzilish nuqtasi yo'q.

3. MAPLE tizimidan foydalanib, berilgan funksiyani davriylikka, juft va toqlikka tekshiramiz:

> evalb(y(x)=y(-x));

true

> cvalb(y(x)=-y(-x));

false

> solve(y(x)=y(x+T),T);

0, -2 x

Dcmak, Funksiya davriy ham emas, juft funksiya.

4. MAPLE tizimidan foydalanib, funksiyaning koordinatalar o'qlari bilan kesishish nuqtalarini topamiz:

> y:=x->x^2/(1+x^2);

$$y := x \to \frac{x^2}{1+x^2}$$

> solve(y(x)=0,x);

0, 0

Shunday qilib, funksiya grafigi faqat bitta O(o,o) nuqtada koordinatalar o'qlari bilan kesishadi.

5. MAPLE tizimida funksiyaning ishorasi saqlanadigan intervallarni aniqlaymiz:

> solve(y(x)>0,x);
Re*alRange*(−∞, *Open*(0)), Re*alRange*(*Open*(0), ∞)
> solve(y(x)<0,x);
Yuqaridagi xulosaga ko'ra,quyidagi jadvalni tuzamiz:

x	(-∞;0)	0	(0; ∞)
signy	+	0	+
grafikning joylanishi	O*x* o'qi ustida		O*x* o'qi ustida

6. MAPLE tizimidan foydalanib, berilgan funksiya grafigining asimptotalarini topamiz:

> alpha[1]:=limit(y(x)/x,x=+infinity);

$\alpha_1 := 0$

```
> alpha[2]:=limit(y(x)/x,x=-infinity );
```

$$a_2 := 0$$

```
> a:=alpha[1];
```

$$a := 0$$

```
> beta[1]:=limit(y(x)-a*x,x=+ infinity );
```

$$b_1 := 1$$

```
> beta [2]:=limit(y(x)-a*x,x=- infinity );
```

$$b_2 := 1$$

Demak, $y=1$ - to'g'ri chiziq funksiya grafigining gorizontal asimptotaci bo'ladi.

7. MAPLE tizimidan foydalanib, berilgan funksiyaning monotonlik oraliqlari va ekstremum qiymatlarini topamiz:

```
> solve(diff(y(x),x)>0,x);
```

$$RealRange(Open(0), \infty)$$

```
> solve(diff(y(x),x)<0,x);
```

$$RealRange(-\infty, Open(0))$$

```
> solve(diff(y(x),x)=0,x);
```

$$0$$

Demak, MAPLE tizimidagi natijalariga ko'ra, quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;\ \infty)$
sign y'	-	0	+
funksiyaning o'zgarishi	↘	0	↗

$y_{min}=y(0)=0.$ A(0,0)-berilgan funksiya grafigining minimum nuqtasi bo'ladi.

8. MAPLE tizimidan foydalanib, berilgan funksiyaning qavariqlik va botiqlik oraliqlarini topamiz:

```
> solve(diff(y(x),x$2)>0,x);
```

$RealRange(Open(-\frac{1}{3}\sqrt{3}),(Open(\frac{1}{3}\sqrt{3}))$

> solve(diff(y(x),x$2)<0,x);

$RealRange(-\infty, Open(-\frac{1}{3}\sqrt{3})), RealRange(Open(\frac{1}{3}\sqrt{3}),\infty))$

> solve(diff(y(x),x$2)=0,x);

0

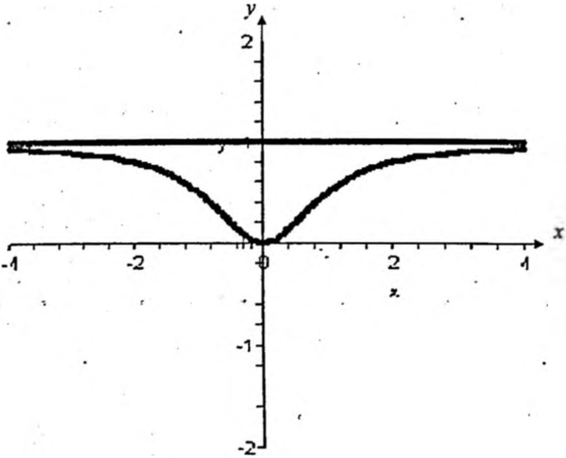
Demak, MAPLE tizimidagi natijalariga ko'ra, quyidagi jadvalni tuzamiz:

x	$(-\infty;-\frac{1}{3}\sqrt{3})$	$-\frac{1}{3}\sqrt{3}$	$(-\frac{1}{3}\sqrt{3};\frac{1}{3}\sqrt{3})$	$\frac{1}{3}\sqrt{3}$	$(\frac{1}{3}\sqrt{3};+\infty)$
sign y''	-	0	+	0	-
Funksiya grafiginig qavariqlik yo'nalishi	∩ ↑ ↑	0	↓ ∪		↓ ∪

9. MAPLE tizimidan foydalanib, berilgan funksiyaning grafigini chizamiz:

> plot([y(x),[3,t,t=-4..12]],x=-4..4,-2..2,color=[red,black],thickness=3,linestyle=[1,3]);

Berilgan funksiyaning grafigi 16.14-chizmada tasvirlangan.



16.14-chizma.

Mustaqil yechish uchun misol va masalalar

Quyidagi funksiyalarni monotonlikka tekshiring.

16.1. $y=3x-x^2$. **16.2.** $y=\dfrac{\sqrt{x}}{x+100}$ $(x\geq 0)$. **16.3.** $y=x+\sin x$.

16.4. $y=x^2-\ln x^2$. **16.5.** $y=x^2e^{-x}$. **16.6.** $y=\dfrac{1-x+x^2}{1+x+x^2}$. **16.7.** $y=2\sin x+\cos x, (0\leq x\leq 2\pi)$. **16.8.** $y=\ln\left(x+\sqrt{1+x^2}\right)$. **16.9.** Quyidagi funksiyalarning o‘suvchi va kamayuvchi bo‘lish oraliqlarini toping:

1) $y=\dfrac{\sin x+\cos x}{1+|\cos x|}$. 2) $y=(x-2)^5(2x+1)^4$.

3) $y=\sqrt[3]{(2x-a)(a-x)^2}$. 4) $y=\dfrac{2x}{1+2x}$.

5) $y=x-e^x$. 6) $y=x-2\sin x$ $(0\leq x\leq 2\pi)$.

7) $y=\frac{x}{\sqrt[3]{x^2-1}}$. 8) $y=\frac{\sqrt[3]{x}}{x+50}$.

16.10. Ushbu 1) $y=\frac{a^2-1}{3}x^3+(a-1)x^2+2x$; 2) $y=ax+3\sin x+4\cos x$

funksiyalar a ning qanday qiymatlarida o'suvchi bo'ladi.

Quyidagi funksiyalarni ekstremumga tekshiring.

16.11. $y=2+x-x^2$. **16.12.** $y=(x-1)^3$ **16.13.** $y=\frac{3}{4}x^4+x^3-9x^2+7$.

16.14. $y=x^4e^{-x^2}$. **16.15.** $y=2\sin x+\cos 2x$. **16.16.** $y=\frac{3x^2+4x+4}{x^2+x+1}$.

16.17. $y=\sin x+\frac{1}{2}\sin 2x$. **16.18.** $y=(x^2-2x)\ln x-\frac{3}{2}x^2+4x$

16.19. $y=\frac{1}{2}\left(x^2-\frac{1}{2}\right)\arcsin x+\frac{1}{4}x\sqrt{1-x^2}-\frac{\pi}{12}x^2$

16.20. $y=\text{arctg}\, x-\frac{1}{2}\ln(1+x^2)$ **16.21.** $y=|x-5|(x-3)^3$.

16.22. $y=(x+1)^5e^{-x}$. **16.23.** $y=\ln(x^2+1)-2\,\text{arctg}\, x$.

16.24. $y=\left(\frac{1}{2}-x\right)\cos x+\sin x-\frac{x^2-x}{4}\left(0\le x\le\frac{\pi}{2}\right)$ **16.25.** $y=ae^{px}+be^{-px}$.

16.26. $y=\begin{cases}1+x, & x\le 0,\\ x^{x^2\ln x}, & x>0\end{cases}$ **16.27.** $y=\begin{cases}1+x, & x\le 0,\\ x^3, & x>0\end{cases}$

16.28. $y=|x-1|\sqrt[3]{x+2}$ **16.29.** $y=\sin(x+1)-(\cos x)$, $x\in(0,\pi)$.

16.30. $y=\frac{1+|\cos x|}{2+\cos x+\sqrt{3}\sin x}$, $x\in(0,\pi)$

Quyidagi funksiyalarning ko'rsatilgan oraliqlarda eng katta va eng kichik qiymatlarini toping.

16.31. $y=2x^3-3x^{25}-12x+1$, $x\in[-2{,}2{,}5]$. **16.32.** $y=x+\sqrt{x}$, $x\in[0{,}4]$.

16.33. $y=x^3-3x^2+1$, $x\in[-1;4]$.

16.34. $y = \mathrm{arctg}\,x - \frac{1}{2}\ln x,\ x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$ **16.35.** $y = 2\sin x + \sin 2x,\ x \in \left[0; \frac{3}{2}\pi\right]$.

16.36. $y = x - 2\ln 2,\ x \in [1; e]$. **16.37.** $y = \begin{cases} 2x^2 + \frac{2}{x^2},\ x \in [-2; 0) \cup (0; 2], \\ 1,\ x = 0. \end{cases}$

16.38. $y = -\frac{1}{3}x^3 - \frac{1}{6}x$ funksiyaning $x \in [-1; 1]$ kesmadagi eng katta va eng kichik qiymatlari yig'indisini hisoblang.

Quyidagi funksiyalar grafigining qavariqlik va botiqlik oraliqlarini toping.

16.39. $y = x^4 + x^3 - 18x^2 + 24x - 12$. **16.40.** $y = x + x^{5/3}$.

16.41. $y = x + \sin x$. **16.42.** $y = 2 - |x^5 - 1|$

16.43. $y = 3x^4 - 4x^3 + 1$. **19.44.** $y = x^\alpha, \alpha > 1, x > 0$.

16.45. $y = x\ln x$. **16.46.** $y = 4\sqrt{(x-1)^5} + 20\sqrt{(x-1)^3}$ $(x \geq 1)$.

16.47. $y = \frac{10}{x}\ln\frac{10}{x}$. **16.48.** $y = e^{\mathrm{arctg}\,x}$. **16.49.** $y = \frac{\sqrt{x}}{x+1}$.

Quyidagi funksiyalar grafigining egilish nuqtalarini toping.

16.50. $y = x + 36x^2 - 2x^3 - x^4$. **16.51.** $y = 1 + x^2 - \frac{x^4}{2}$.

16.52. $y = 3x^4 - 8x^3 + 6x^2 - 12$. **16.53.** $y = \frac{x+1}{x^2+1}$.

16.54. $y = \frac{2x^2 - x - 4}{x^2 - 4x + 4}$. **16.55.** $y = \frac{\ln^2 x}{x}$ $(x > 0)$.

16.56. $y = e^{\sin x},\ x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. **16.57.** $y = e^{-2x}\sin^2 x$.

16.58. a parametrning qanday qiymatlarida $f(x) = ax^3 + e^x$ funksiya egilish nuqtasiga ega bo'ladi.

Quyidagi funksiyalarni to'liq tekshiring va ularning grafigini chizing.

16.59. $y = \frac{x^4}{(1+x)^3}$. **16.60.** $y = \sqrt[3]{x^2} - \sqrt[3]{x^2 - 4}$.

16.61. $y = x^2\ln(x+2)$. **16.62.** $y = x^3e^{-4x}$. **16.63.** $y = \frac{1}{3}\sqrt{(2x+1)^3} + 4\sqrt{x}$.

16.64. $y = \frac{x^2\sqrt{x^2-1}}{2x^2-1}$. **16.65.** $y = |x|\sqrt{1-x^2}$. **16.66.** $y = (x^2 - 2)e^{-2x}$.

16.67. $y=\frac{x^2+2x-3}{x}e^{1/x}$. **16.68.** $y=\frac{\ln x}{x}$. **16.69.** $y=\ln\left|\frac{x-1}{x+1}\right|+\frac{6}{x+1}$.

16.70. $y=\sin x-\sin^2 x$. **16.71.** $y=\sin x\sin 3x$.

16.72. $y=\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$. **16.73.** $y=\frac{3}{2}x-\arccos\frac{1}{x}$.

16.74. $y=e^{-\operatorname{arctg} x}$. **16.75.** $y=\left(1+\frac{1}{x}\right)^x$. **16.76.** $y=x^6-3x^4+3x^2-5$.

16.77. $y=x^2e^{1/x}$. **16.78.** $y=x^2e^{-x}$. **16.79.** $y=\frac{x^3}{x-1}$.

16.80. $y=\cos x+\frac{1}{2}\sin 2x$. **16.81.** $y=\ln x-x+1$. **16.82.** $y=\frac{x^2+x-1}{x^2-2x+1}$.

Mustaqil yechish uchun berilgan misol va masalalarning javoblari

16.1. $\left(-\infty;\frac{1}{2}\right)$ da funksiya o'suvchi, $\left(\frac{1}{2};\infty\right)$ da esa funksiya kamayuvchi.

16.2. $(0;100)$ da funksiya o'suvchi, $(100;\infty)$ da esa, funksiya kamayuvchi. **16.3.** R da funksiya o'suvchi. **16.4.** $(-\infty;-1)\cup(0;1)$ da funksiya kamayuvchi, $(-1;0)\cup(1;\infty)$ da esa, funksiya o'suvchi.

16.5. $(-\infty;0)\cup(2;\infty)$ da funksiya kamayuvchi, $(0;2)$ da funksiya o'suvchi. **16.6.** $(-\infty;-1)\cup(1;+\infty)$ da funksiya o'suvchi, $(-1;1)$ da esa, funksiya kamayuvchi. **16.7.** $\left(0;\frac{\pi}{6}\right)\cup\left(\frac{\pi}{2},\frac{5\pi}{6}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$ da funksiya o'suvchi, $\left(\frac{\pi}{6},\frac{\pi}{2}\right)\cup\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$ da esa, funksiya kamayuvchi. **16.8.** O'suvchi. **16.9.** 1) $\left(-\frac{\pi}{2}+2k\pi;\frac{\pi}{2}+2\pi k\right), k\in Z$ da funksiya o'suvchi, $\left(\frac{\pi}{2}+2k\pi;\frac{3\pi}{2}+2k\pi\right), k\in Z$ da esa funksiya kamayuvchi. 2) $\left(-\infty;-\frac{1}{2}\right)$ da o'suvchi; $\left(-\frac{1}{2},\frac{11}{18}\right)$ da kamayuvchi, $\left(\frac{11}{18};\infty\right)$ da o'suvchi. 3) $\left(-\infty;-\frac{2}{3}\right)$da o'suvchi; $\left(\frac{2}{3}a,a\right)$ da kamayuvchi, $(a;\infty)$ da o'suvchi. 4) $(-\infty;-1)$da kamayuvchi; $(-1,1)$ da o'suvchi; $(1;\infty)$ da kamayuvchi. 5)

$(-\infty;0)$ da o'suvchi; $(0,\infty)$ da kamayuvchi. 6) $\left(-\infty;\frac{\pi}{3}\right)$ kamayuvchi; $\left(\frac{\pi}{3},\frac{5\pi}{3}\right)$ da o'suvchi ; $\left(\frac{5\pi}{3},2\pi\right)$da kamayuvchi. 7) $\left(-\infty;-\sqrt{3}\right)$, $\left(\sqrt{3};\infty\right)$ larda o'suvchi; $\left(-\sqrt{3};-1\right)$, $(-1,1)$, $\left(1;\sqrt{3}\right)$ larda kamayuvchi. 8) $(-\infty;-50)\cup(-50;25)$ da funksiya o'suvchi, $(25;+\infty)$ da esa, funksiya kamayuvchi. **16.10.** 1) $a\le 3$, $a\ge 1$; 2) $a\ge 5$.**16.11.** $y_{max}\left(\frac{1}{2}\right)=2\frac{1}{4}$. **16.12.** Ekstremumga ega emas.

16.13. $y_{min}(-2)=-9$, $y_{min}(3)=-40{,}5$, $y_{max}(0)=7$.

16.14. $y_{max}\left(\pm\sqrt{2}\right)=4e^{-2}$, $y_{min}(0)=0$.

16.15. $y_{max}\left(\frac{\pi}{6}\right)=\frac{3}{2}$, $y_{max}\left(\frac{5\pi}{6}\right)=\frac{3}{2}$, $y_{min}\left(\frac{\pi}{2}\right)=1$, $y_{max}\left(\frac{\pi}{2}\right)=1$, $y_{min}\left(\frac{3\pi}{2}\right)=-3$.

16.16. $y_{max}(0)=4$, $y_{min}(-2)=\frac{2}{3}$. **16.17.** $y_{min}\left(2\pi k-\frac{\pi}{3}\right)=-\frac{3\sqrt{3}}{4}, k\in Z$,

$y_{max}\left(2\pi k+\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{4}, k\in Z.$ **16.18.** $y_{max}(1)=2\frac{1}{2}$, $y_{min}(e)=\frac{e(4-e)}{2}$.

16.19. $y_{max}(0)=0$, $y_{min}\left(\frac{1}{2}\right)=\frac{3\sqrt{3}-2\pi}{48}$. **16.20.** $y_{max}(1)=\frac{\pi}{4}-\frac{1}{2}\ln 2\approx 0{,}439$.

16.21. $y_{max}\left(\frac{9}{2}\right)=\frac{27}{16}$, $y_{min}(5)=0$. **16.22.** $y_{max}(4)=5^5\cdot e^{-4}$.

16.23. $y_{min}(1)=\ln 2-\frac{\pi}{2}$.

16.24. $y_{max}\left(\frac{1}{2}\right)=\sin\frac{1}{2}+\frac{1}{16}$, $y_{min}\left(\frac{\pi}{6}\right)=\frac{36\sqrt{3}-12\pi\sqrt{3}+72-\pi^2+6\pi}{144}$.

16.25. Agar $ab\le 0$ bo'lsa, ekstremum yo'q; agar $ab>0$va $a>0$bo'lsa, $y_{min}\left(\frac{1}{2p}\ln\frac{b}{a}\right)=2\sqrt{ab}$; agar $ab>0$va $a<0$bo'lsa, $y_{max}\left(\frac{1}{2p}\ln\frac{b}{a}\right)=-2\sqrt{ab}$.

16.26. $y_{max}\left(\frac{1}{e}\right)=e^{\frac{1}{e^2}}$, $y_{min}(1)=1$. **16.27.** $y_{max}(0)=1$, $y_{min}\left(\frac{1}{\sqrt[3]{e}}\right)=e^{-\frac{1}{3e}}$.

16.28. $y_{max}\left(-\frac{5}{4}\right)=\frac{9\cdot\sqrt[3]{6}}{8}$, $y_{min}(1)=0$. **16.29.** $y_{max}\left(\frac{\pi}{2}\right)=\cos 1$.

16.30. $y_{max}\left(\frac{\pi}{2}\right)=2-\sqrt{3}$. **16.31.** $y_{eng\ katta}(-1)=8$, $y_{eng\ kichik}(2)=19$.

16.32. $y_{eng\ katta}(4)=6$, $y_{eng\ kichik}(0)=0$.

16.33. $y_{eng\ katta}(4)=17$, $y_{eng\ kichik}(2)=y_{eng\ kichik}(-1)=-3$.

16.34. $y_{eng\ katta}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}+0{,}25\cdot\ln 3,\quad y_{eng\ kichik}(\sqrt{3})=\frac{\pi}{6}-0{,}25\cdot\ln 3.$

16.35. $y_{eng\ katta}\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2},\quad y_{eng\ kichik}\left(\frac{3\pi}{2}\right)=-2.$

16.36. $y_{eng\ katta}(1)=1,\ y_{eng\ kichik}(2)=2(1-\ln 2).$

16.37.Eng kattasi yo'q, $y_{eng\ kichik}(0)=1$.

16.38. 0. 16.39. $(-\infty;-2)\cup\left(\frac{3}{2};\infty\right)$ da qavariq; $\left(-2;\frac{3}{2}\right)$da botiq.

16.40. $(-\infty;0)$da botiq; $(0;\infty)$ da qavariq. **16.41.** $(2\pi k,(2k+1)\pi),k\in Z$ da botiq; $((2k+1)\pi,(2k+2)\pi),k\in Z$ da qavariq. **16.42.** $(-\infty;0)\cup(1;\infty)$ da botiq; $(0;1)$ da qavariq. **16.43.** $(-\infty;0)\cup\left(\frac{2}{3};+\infty\right)$ da qavariq; $\left(0;\frac{2}{3}\right)$ da botiq. **16.44.** Qavariq. **16.45.** Qavariq.

16.46. Hamma joyda qavariq. **16.47.** $(0;10e\sqrt{e})$ da botiq, $(10e\sqrt{e};+\infty)$ da esa qavariq. **16.48.** $(-\infty;0{,}5)$ da qavariq. **16.49.** $\left(0;\frac{2+\sqrt{3}}{\sqrt{3}}\right)$ da botiq, $\left(\frac{2+\sqrt{3}}{\sqrt{3}};+\infty\right)$ da esa qavariq. **16.50.** $(-3;294)$.

16.51. $\left(\frac{1}{\sqrt{3}};\frac{23}{18}\right),\left(-\frac{1}{\sqrt{3}};\frac{23}{18}\right)$. **16.52.** $\left(\frac{1}{3};12\frac{11}{27}\right),(1;13)$. **16.53.** $\left(-2-\sqrt{3};\frac{-\sqrt{3}-1}{4}\right),\left(-2+\sqrt{3};\frac{\sqrt{3}+1}{4}\right),(1;1)$. **16.54.** $\left(\frac{8}{7};-\frac{31}{9}\right)$.

16.55. $\left(e^{\frac{3-\sqrt{5}}{2}};\left(\frac{3-\sqrt{5}}{2}\right)^2\cdot e^{\frac{\sqrt{5}-3}{2}}\right),\left(e^{\frac{3+\sqrt{5}}{2}};\left(\frac{3+\sqrt{5}}{2}\right)^2\cdot e^{\frac{-\sqrt{5}-3}{2}}\right)$. **16.56.** Abssissasi $x=\arcsin\frac{\sqrt{5}-1}{2}$ bo'lgan nuqtada.

16.57. $\left((6k+(-1)^k)\frac{\pi}{12};\ \frac{2-\sqrt{3}}{4}e^{-(6k+(-1)^k)\frac{\pi}{6}}\right),k\in Z.$ **16.58.** $a\in\left(-\infty;-\frac{e}{6}\right),\ a\in(0;+\infty).$

19.59. Funksiyaning aniqlanish sohasi: $(-\infty;-1)\cup(-1;+\infty)$. $x=-1$ - vertikal asimptota, $y=x-3$ og'ma asimptota. $y_{min}(0)=0,\ y_{max}(-4)=-\frac{256}{27}$. $\left(-6;-\frac{3296}{125}\right)$ va $\left(2;\frac{16}{27}\right)$nuqtalar egilish nuqtalari (16.15-chizma). **16.60.** R da aniqlangan, juft funksiya. Grafik Oy o'qiga nisbatan simmetrik, $y=0$ -

gorizontal asimptota. $y_{min}(0)=\sqrt[3]{4}$, $y_{max}(\pm\sqrt{2})=2\sqrt[3]{2}$. $(2;\sqrt[3]{4})$, $(-2;\sqrt[3]{4})$- egilish nuqtalari (16.16-chizma). **16.61.** Funksiya $(-2;+\infty)$ oraliqda aniqlangan. $x=-2$ vertikal asimptota. $y_{min}(0)=0$, $y_{max}(-0{,}73)\approx 0{,}12$. $(-0{,}37;0{,}075)$ - egilish nuqtasi (16.17-chizma).**16.62.** Funksiya R da aniqlangan, $x\to+\infty$ da $y=0$ - gorizontal asimptota. $y_{max}\left(\frac{3}{4}\right)=\left(\frac{3}{4e}\right)^3$. Egilish nuqtalari: $(0;0)$, $\left(\frac{3-\sqrt{3}}{4};\left(\frac{3-\sqrt{3}}{4}\right)^3 e^{\sqrt{3}-3}\right)$, $\left(\frac{3+\sqrt{3}}{4};\left(\frac{3+\sqrt{3}}{4}\right)^3 e^{-\sqrt{3}-3}\right)$ (16.18-chizma). **16.63.** Funksiya $x\geq 0$da aniqlangan, ordinata o'qi bilan esa $\left(0;\frac{1}{3}\right)$ nuqtada kesishadi; funksiya qa'tiy o'suvchi; $\left(\frac{\sqrt{5}+1}{2};\approx 8\right)$- egilish nuqtasi (16.19-chizma). **16.64.** Funksiya $|x|\geq 1$da aniqlangan; ordinata o'qiga nisbatan simmetrik; ordinata o'qi bilan kesishish nuqtalari: $(1;0)$, $(-1;0)$; $x\to+\infty$ da $y=\frac{x}{2}$ va $x\to-\infty$da $y=-\frac{x}{2}$ asimptotalari; $(-\infty;-1)$ da kamayuvchi $(1;+\infty)$da o'suvchi (16.20-chizma). **16.65.** Funksiya $|x|\leq 1$da aniqlangan; ordinata o'qiga nisbatan simmetrik; o'qlar bilan kesishish nuqtalari: $(-1,0);(0,0),(1,0)$; $y_{min}(0)=0$, $y_{max}\left(\pm\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$ (16.21-chizma). **16.66.** Funksiya R da aniqlangan, koordinatalar o'qlari bilan kesishish nuqtalari: $(-\sqrt{2};0)$, $(\sqrt{2};0)$, $(0;-2)$; $x\to+\infty$ da $y=0$ asimptota; $y_{min}(-1)\approx-7{,}4$, $y_{max}(2)\approx 0{,}04$; funksiya egilish nuqtalarining abssissalari: $x=1-\sqrt{10}/2\approx-0{,}6$, $x=1+\sqrt{10}/2\approx 2{,}6$. (16.22-chizma). **16.67.** Funksiya $x=0$ dan tashqari R da aniqlangan; abssissa o'qi bilan kesishish nuqtalari: $(-3;0)$ va $(1;0)$; $x\to 0+$ da $y=x+3$ va $x=0$ asimptotalar; $y(-0)=0$, $y'(-0)=0$; $y_{max}(-1)=\frac{4}{e}$; funksiyaning egilish nuqtalari abssissalari: $x=-5\pm\sqrt{22}$ (16.23-chizma). **16.68.** Funksiyaning aniqlanish sohasi: $x>0$; abssissa o'qi bilan kesishish nuqtasi: $(1;0)$; asimptotalari: $x\to+\infty$ da $y=0$ va $x\to 0+$ da $x=0$, $y_{max}(e)=\frac{1}{e}$; egilish nuqtasi: $\left(e^{\frac{3}{2}};1{,}5e^{-\frac{3}{2}}\right)$ (16.24-chizma).

16.69. Funksiyaning aniqlanish sohasi: $(-\infty;-1)\cup(-1;1)\cup(1;+\infty)$; asimptotalari: $x=-1$, $x=1$, $y=0$, koordinatalar o'qlari bilan kesishish nuqtalari: $(\approx 0{,}9;0)$, $(\approx 1{,}2;0)$, $(0;6)$, $y_{max}(2)=2-\ln 3$, egilish nuqtalari: $(0{,}5;4-\ln 3)$,

$(3,1;\ 5-\ln 2)$ (16.25-chizma). **16.70.** Funksiya R da aniqlangan; 2π davrga ega bo'lgan davriy funksiya; funksiyaning nollari: $y(0)=y\left(\frac{\pi}{2}\right)=y(\pi)=0$; $y_{max}\left(\frac{\pi}{6}\right)=\frac{1}{4}$, $y_{max}\left(\frac{5\pi}{6}\right)=\frac{1}{4}$, $y_{min}\left(\frac{\pi}{2}\right)=0$, $y_{min}\left(\frac{3}{2}\pi\right)=-2$; funksiyaning egilish nuqtalari abssissalari:

$$x=\arcsin\frac{1+\sqrt{33}}{8},\ x=\pi-\arcsin\frac{1+\sqrt{33}}{8},\ x=\pi+\arcsin\frac{\sqrt{33}-1}{8},\ x=2\pi-\arcsin\frac{\sqrt{33}-1}{8}$$

(16.26-chizma). **16.71.** Funksiya R da aniqlangan; π davrga ega bo'lgan davriy funksiya: funksiyaning grafigi ordinata o'qiga simmetrik: $y(0)=y\left(\frac{\pi}{3}\right)=y\left(-\frac{\pi}{3}\right)=0$; $y_{min}(0)=0$, $y_{min}\left(\pm\frac{\pi}{2}\right)=-1$, $y_{max}(\pm\arccos(1/4))=9/16$;

funksiyaning egilish nuqtalari abssissalari: $x=\pm\frac{1}{2}\arccos\frac{\sqrt{129}+1}{16}$,

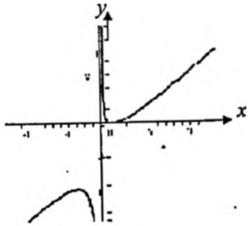
$x=\pm\frac{1}{2}\left(\pi-\arccos\frac{\sqrt{129}-1}{16}\right)$, (16.27-chizma). **16.72.** Funksiya R da aniqlangan; 2π davrga ega bo'lgan davriy funksiya: funksiyaning grafigi koordinatalar boshiga nisbatan simmetrik; funksiyaning nollari: $y(0)=y(\pi)=0$; $[0;\pi]$ dagi maksimumlari $y_{max}\left(\frac{\pi}{4}\right)=(3+4\sqrt{2})/6$, $y_{max}\left(\frac{3\pi}{4}\right)=\sqrt{2}-1/2$, minimumi: $y_{min}\left(\frac{2\pi}{3}\right)=\sqrt{3}/4$; Funksiyaning egilish nuqtalarining abssissalari: $x=0, x=\pi$, $x=\arcsin\frac{\sqrt{7}-1}{6}$, $x=\pi-\arcsin\frac{\sqrt{7}-1}{6}$ (16.28-chizma). **16.73.** Funksiyaning aniqlanish sohasi: $(-\infty;-1]\cup[1;+\infty)$; $(0;-\frac{\pi}{2})$ simmetriya markazi; $x\to+\infty$ da $y=\frac{3x-\pi}{2}$ asimptota; $y_{max}\left(-\frac{2\sqrt{3}}{3}\right)=-\frac{6\sqrt{3}+5\pi}{6}\approx-4{,}4$; $y_{min}\left(-\frac{2\sqrt{3}}{3}\right)=\frac{6\sqrt{3}-\pi}{6}\approx1{,}2$; $(-\infty;-1)$ da botiq (16.29-chizma). **16.74.** Funksiyaning R da aniqlangan, asimptotalari: $x\to-\infty$ da $y=e^{\frac{\pi}{2}}$, $x\to+\infty$ da $y=e^{-\frac{\pi}{2}}$; kamayuvchi funksiya; egilish nuqtasi-$\left(-\frac{1}{2};e^{\operatorname{arctg}0,5}\right)$. (16.30-chizma). **16.75.** Funksiyaning aniqlanish sohasi: $(-\infty;-1)\cup(0;+\infty)$; asimptotalari: $x\to\pm\infty$ da $y=e$, $y\to+\infty$

da $x=-1$; $(-\infty;-1)$ da botiq, $(0;+\infty)$ da esa qavariq. (16.31-chizma). **16.76.** Aniqlanish sohasi: R; juft funksiya; asimptotaga ega emas; $(-\infty;0)$ da kamayadi, $(0;\infty)$ da o'sadi; $y_{min}(0)=-5$; egilish nuqtalari: $\left(\frac{1}{\sqrt{5}};-4,51\right)$, $(1;-4)$, $\left(-\frac{1}{\sqrt{5}};-4,51\right)$, $(-1;-4)$; $(-\infty;-1)\cup\left(-\frac{1}{\sqrt{5}};\frac{1}{\sqrt{5}}\right)\cup(1;\infty)$da qavariq; $\left(-1;-\frac{1}{\sqrt{5}}\right)\cup\left(\frac{1}{\sqrt{5}};1\right)$ da botiq; 16.32-chizma. **16.77.** Aniqlanish sohasi: $(-\infty;0)\cup(0;\infty)$; $x=0$ funksiyaning ikkinchi tur uzilish nuqtasi; asimptota: $x=0$; $y_{min}\left(\frac{1}{2}\right)\approx 1,87$; $(-\infty;0)\cup(0;\infty)$ da qavariq; 16.33-chizma.

16.78. Aniqlanish sohasi: R; koordinatalar o'qlari bilan kesishish nuqtasi: $O(0;0)$; asimptota: $x\to\infty$ da $y=0$; $y_{min}(0)=0$, $y_{max}(2)=4e^{-2}\approx 0,54$; egilish nuqtalari: $(\sqrt{3}-1;0,3)$, $(\sqrt{3}+1,0,47)$, $(\sqrt{3}-1;\sqrt{3}+1)$ da botiq; $(-\infty;\sqrt{3}-1)$ da qavariq; 16.34-chizma. **16.79.** Aniqlanish sohasi: $(-\infty;1)\cup(1;\infty)$; $x=1$ funksiyaning ikkinchi tur uzilish nuqtasi; koordinatalar o'qlari bilan kesishish nuqtasi: $O(0;0)$; asimptota: $x=1$; $y_{min}\left(\frac{3}{2}\right)=\frac{27}{4}$; egilish nuqtalari $O(0;0)$; $(-\infty;0)\cup(1;\infty)$ da qavariq; $(0;1)$ da botiq; 16.35-chizma. **16.80.** Aniqlanish sohasi: R; davriy funksiya: $T=2\pi$; koordinatalar o'qlari bilan kesishish nuqtalari: $(0;1)$, $\left(\frac{\pi}{2};0\right)$, $\left(\frac{3\pi}{2};0\right)$; asimptotaga ega emas; $y_{max}\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{4}$, $y_{min}\left(\frac{5\pi}{6}\right)=-\frac{3\sqrt{3}}{4}$; egilish nuqtalari: $\left(\frac{\pi}{2};0\right)$, $\left(\pi+\arcsin\frac{1}{4};\frac{-3\sqrt{15}}{16}\right)$, $\left(\frac{3\pi}{2};0\right)$, $\left(2\pi-\arcsin\frac{1}{4};\frac{3\sqrt{15}}{16}\right)$; 16.36-chizma. **16.81.** Aniqlanish sohasi: $(0;\infty)$; koordinatalar o'qlari bilan kesishish nuqtalari: $(1;0)$; asimptota: $x=0$; $y_{min}(1)=0$; $(0;\infty)$ da botiq; 16.37-chizma.

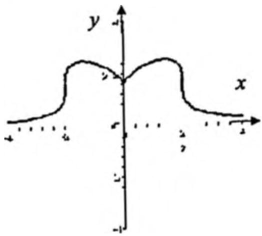
16.82. Funksiya $x=1$ dan tashqari butun R da aniqlangan; koordinatalar o'qlari bilan kesishish nuqtalari: $\left(\frac{-\sqrt{5}-1}{2};\ 0\right)$, $\left(\frac{\sqrt{5}-1}{2};\ 0\right)$, $(0;-1)$; asimptotalari: $y=1$ va $x=1$ to'g'ri chiziqlar; $y_{min}\left(\frac{1}{3}\right)=-\frac{5}{4}$; nuqta egilish nuqtasi $(0;-1)$ (16.38-chizma).

16.59.



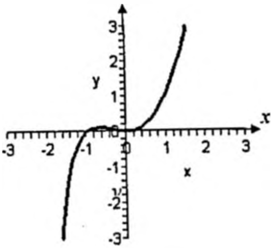
16.15-chizma.

16.60.



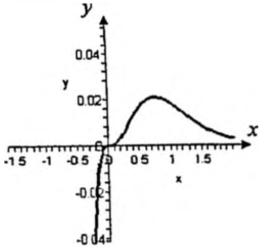
16.16-chizma.

16.61.



16.17-chizma .

16.62.



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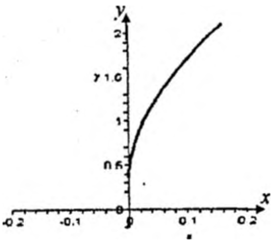
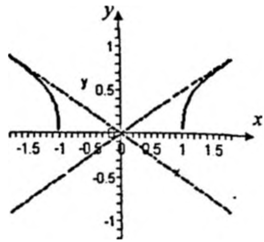
16.63.



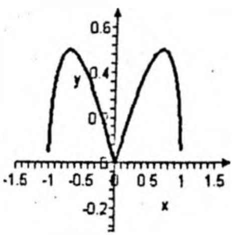
16.19-chizma.

16.64.



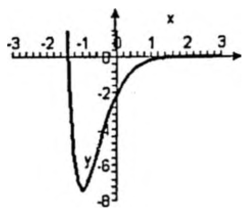
16.20-chizma.

16.65.



16.21-chizma.

16.66.



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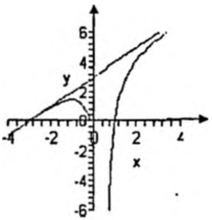
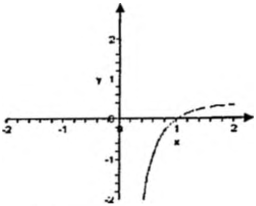
16.67.



16.23-chizma.

16.68.



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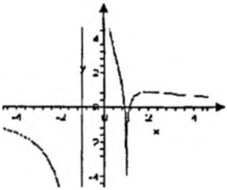
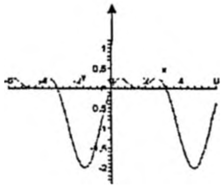
16.69.



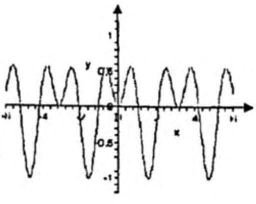
16.25-chizma

16.70.



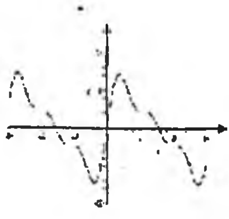
16.26-chizma

16.71.



16.27-chizma

16.72.



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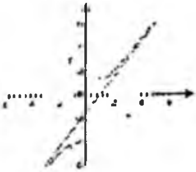
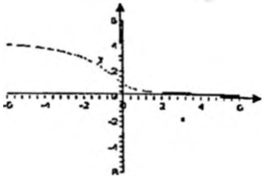
16.73.



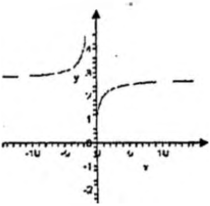
16.29-chizma

16.74.



16.30-chizma.

16.75.



16.31-chizma

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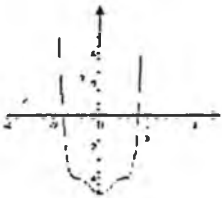
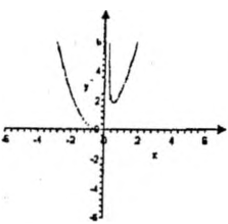


16.32-chizma

16.77.



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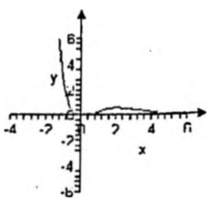
16.78.



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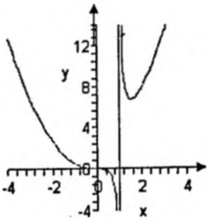
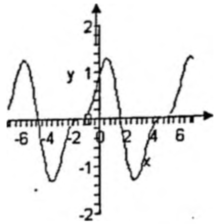
16.79.



16.35-chizma

16.80.



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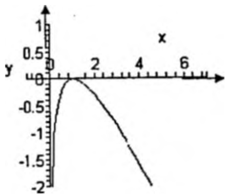
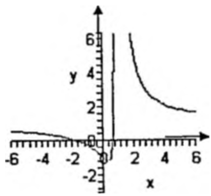
16.81.



16.37-chizma.

16.82.



16.38-chizma

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MUNDARIJA

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MATEMATIK ANALIZDAN MISOL VA MASALALAR

1-QISM

o'quv qo'llanma sifatida tavsiya etilgan

Toshkent — «TURON-IQBOL» — 2012
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Tel.: 244-25-58. Faks: 244-20-19.

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Kompyuterda sahifalovchi *E. Muratov*

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Terishga berildi 15.06.2012. Bosishga ruxsat etildi 08.12.2012.
«Times» garniturasi. Ofset usulida chop etildi. Qog'oz bichimi $60\times90^1/_{16}$.
Shartli bosma tabog'i 30,0. Nashr tabog'i 30,7. Adadi 210 nusxa.
Buyurtma № 1119.

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100200. Toshkent sh., Radial tor ko'chasi, 10-uy.