O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

ALISHER NAVOIY NOMIDAGI
SAMARQAND DAVLAT UNIVERSITETI

FUNKSIONAL ANALIZDAN MASALALAR TO‘PLAMI

III qism
CHIZIQLI OPERATORLAR

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Mas'ul muharrir:
Lakaev Saidaxmad Norjigitovich, Fizika-matematika fanlari doktori

Tuzuvchilar:
Abdullayev Janikul Ibragimovich, Fizika-matematika fanlari doktori
G'anixo'jayev Rasul Nabiyevich, Fizika-matematika fanlari doktori
Ikromov Isroil Akramovich, Fizika-matematika fanlari doktori

Taqrizchilar:
Imomkulov Sevdiyor Akramovich, Fizika-matematika fanlari doktori
Niyozov Iqbol Ergashovich, Fizika-matematika fanlari nomzodi

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Mundarija

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Ushbu uslubiy qo'llanma funksional analiz fanidan namunaviy o'quv dasturga moslab tuzilgan masalalar to'plamidir. U universitetlarning matematika va mexanika bakalavriyat yo'nalishlari bo'yicha ta'lim olayotgan talabalari uchun mo'ljallangan. Bundan tashqari masalalar to'plamidan matematik tahlil va matematik fizika mutaxassisliklari bo'yicha ta'lim olayotgan magistrlar hamda katta ilmiy xodim – izlanuvchilar foydalanishlari mumkin. Masalalar to'plami funksional analizning asosiy boblari chiziqli operatorlar nazariyasi va chiziqli integral tenglamalarni o'z ichiga olgan. U nisbatan soddaroq misollardan tashkil topgan bo'lib, o'quvchilarni misol yechishga rag'batlantiradi.

Uslubiy qo'llanmaning asosiy maqsadi bo'lg'usi mutaxassislarni funksional analizning asosiy tushunchalari va usullari bilan tanishtirish va funksional masalalarni yechishda ko'nikmalar hosil qilishdan iborat. Qo'llanma talabalarni funksional analizga oid masalalarni yechishga o'rgatadi hamda ularda yetarli darajada texnik mahorat hosil qiladi. Ushbu to'plam O'zMU va SamDUda "Funksional analiz" fanidan ma'ruza va amaliy mashg'ulotlar olib boruvchi professor-o'qituvchilarning ko'p yillik ish tajribalari asosida tuzilgan.

Uslubiy qo'llanma II bob, 6 paragrafdan iborat. Har bir paragraf boshida qisqacha nazariy material berilgan. Har bobdan so'ng talabalar o'z bilimlarini tekshirishlari uchun test savollari javoblari bilan berilgan.

Mualliflar uslubiy qo'llanmani yaxshilashda bergan foydali maslahatlari uchun mas'ul muharrir va taqrizchilarga o'z minnatdorchiliklarini bildiradi.

Masalalar to'plami birinchi marta chop qilinayotgani uchun xato va kamchiliklar bo'lishi mumkin. Xato va kamchiliklar haqidagi fikrlaringizni jabdullaev@mail.ru elektron manziliga jo'natishlaringizni so'raymiz.

I bob. Chiziqli operatorlar

Bu bobda normalangan fazolarda chegaralangan chiziqli operatorlar, ularning normasini topish, chiziqli operatorlarning teskarisi mavjud yoki mavjud emasligini tekshirish, agar teskari operator mavjud bo'lsa, uni aniqlash, chiziqli operatorlarga qo'shma operatorlarni aniqlash (Banax fazolarida Banax bo'yicha qo'shma operatorni, Hilbert fazolarida Hilbert qo'shmasini), chiziqli operatorlarnmg xos qiymatlari, xos vektorlari, spektri va rezolventasini aniqlashga doir masalalar jamlangan.

Bobning $1-\S$ da operatorlarning chiziqli chegaralanganligini tekshirib, ularning normasini topishga doir mashqlar bor. Chiziqli operatorning aniqlanish sohasini ko'rsatib, uning chegaralanmaganligini yoki uzluksiz emasligini ko'rsatishga doir mashqlar ham shu paragrafdan joy olgan. Chiziqli operatorlar ketma-ketligining yaqinlashishiarini tekshirishga doir mashqlar ham shu paragrafga kiritilgan. $2-\S$ da berilgan chiziqli operatorga teskari operator mavjud ekanligini ko'rsatib, uning teskarisini topishga doir mashqlar keltirilgan. Bundan tashqari bu paragrafda teskari operatorning mavjud emasligini ko'rsatishga doir mashqlar ham bor. $3-\S$ da esa, berilgan chiziqli operatorga mos Banax yoki Hilbert qo'shmasini topishga doir mashqlar keltirilgan. Oxirgi $4-\S$ da esa chiziqli operatorning xos qiymatlari, xos vektorlari, spektri va rezolventasini hamda spektral yoyilmasini topishga doir mashqlar jamlangan.

. 1-§. Chiziqli uzluksiz operatorlar

Bu paragrafda biz normalangan fazolarda aniqlangan chiziqli operatorlarni qaraymiz. Chiziqli normalangan fazolarni X, Y va Z bilan, chiziqli operatorlarni esa A, B va C harflari bilan belgilaymiz.

1.1-ta'rif. *X chiziqli normalangan fazodan olingan har bir x elementga Y fazoning yagona y elementini mos qo'yuvchi $Ax = y$ akslantirish operator deyiladi.*

Umuman, A operator X ning hamma yerida aniqlangan bo'lishi shart emas. Bu holda Ax mavjud va $Ax \in Y$ bo'lgan barcha $x \in X$ lar to'plami A operatorning *aniqlanish sohasi* deyiladi va u $D(A)$ bilan belgilanadi, ya'ni:

$$D(A) = \{ x \in X \ : \ Ax \ \text{ mavjud va } \ Ax \in Y \}.$$

1.2-ta'rif. *Agar ixtiyoriy $x, y \in D(A)$ elementlar va ixtiyoriy $\alpha, \beta \in \mathbb{C}$ sonlar uchun $\alpha x + \beta y \in D(A)$ bo'lib,*

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

tenglik o'rinli bo'lsa, A ga chiziqli operator deyiladi.

1.3-ta'rif. *Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $\delta = \delta(\varepsilon) > 0$ mavjud bo'lib, $\|x - x_0\| < \delta$ tengsizlikni qanoatlantiruvchi barcha $x \in D(A)$ lar uchun $\|Ax - Ax_0\| < \varepsilon$ tengsizlik bajarilsa, A operator $x = x_0$ nuqtada uzluksiz deyiladi. Agar A operator ixtiyoriy $x \in D(A)$ nuqtada uzluksiz bo'lsa, A ga uzluksiz operator deyiladi.*

1.4-ta'rif. *$Ax = \theta$ tenglikni qanoatlantiruvchi barcha $x \in X$ lar to'plami A operatorning yadrosi deyiladi va u $Ker A$ bilan belgilanadi.*

1.5-ta'rif. *Biror $x \in D(A)$ uchun $y = Ax$ tenglik bajariladigan barcha $y \in Y$ lar to'plami A operatorning qiymatlar sohasi yoki tasviri deyiladi va u $Im A$ yoki $R(A)$ bilan belgilanadi.*

Matematik simvollar yordamida operator yadrosi va qiymatlar sohasini quyidagicha yozish mumkin:

$$Ker A = \{ x \in D(A) : \ Ax = \theta \},$$

$$R(A) := Im A = \{ y \in Y : \text{ biror } x \in D(A) \text{ uchun } y = Ax \}.$$

1.6-ta'rif. *Agar shunday $C > 0$ son mavjud bo'lib, barcha $x \in D(A)$ lar uchun*

$$\|Ax\| \leq C \cdot \|x\| \tag{1.1}$$

tengsizlik bajarilsa, A chegaralangan operator deyiladi.

1.7-ta'rif. (1.1) *tengsizlikni qanoatlantiruvchi C sonlar to'plamining aniq quyi chegarasi A operatorning normasi deyiladi va u $\|A\|$ bilan belgilanadi, ya'ni $\|A\| = \inf C$.*

1.1-teorema. *$A : X \to Y$ chiziqli chegaralangan operatorning normasi $\|A\|$ uchun quyidagi tenglik o'rinli*

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}. \qquad (1.2)$$

Chiziqli operatorlarning uzluksizligi va chegaralanganligi orasida quyidagi bog'lanish mavjud.

1.2-teorema. *A chiziqli operator chegaralangan bo'lishi uchun uning uzluksiz bo'lishi zarur va yetarli.*

X chiziqli normalangan fazoni Y chiziqli normalangan fazoga akslantiruvchi chiziqli chegaralangan operatorlar to'plamini $L(X, Y)$ bilan belgilaymiz. Xususan, $X = Y$ bo'lsa $L(X, X) = L(X)$. $L(X, \mathbb{C})$ bilan X ga qo'shma fazo belgilanadi. Hilbert fazolarini H bilan belgilaymiz.

1.8-ta'rif. *$A : X \to Y$ va $B : X \to Y$ chiziqli operatorlarning yig'indisi deb, $x \in D(A) \cap D(B)$ elementga $y = Ax + Bx \in Y$ elementni mos qo'yuvchi $C = A + B$ operatorga aytiladi.*

Agar $A, B \in L(X, Y)$ bo'lsa, u holda C ham chiziqli chegaralangan operator bo'ladi va quyidagi tengsizlik o'rinli

$$\|C\| = \|A + B\| \leq \|A\| + \|B\|.$$

1.9-ta'rif. *A chiziqli operatorning α songa ko'paytmasi x elementga αAx elementni mos qo'yuvchi operator sifatida aniqlanadi, ya'ni*

$$(\alpha A)(x) = \alpha Ax.$$

$L(X, Y)$ to'plamda kiritilgan operatorlarni qo'shish va operatorni songa ko'paytirish amallari chiziqli fazo ta'rifidagi 1-8 shartlarni qanoatlantiradi. Demak, $L(X, Y)$ to'plam operatorlarni qo'shish va operatorni songa ko'paytirish amallariga nisbatan chiziqli fazo tashkil qiladi. Bu fazoda aniqlangan $p : L(X, Y) \to \mathbb{R}$, $p(A) = \|A\|$ funksional (1.59-misolga qarang) normaning barcha shartlarini qanoatlantiradi. Demak, $L(X, Y)$ chiziqli normalangan fazodir. Bu fazoning to'laligi haqida quyidagi teorema o'rinli.

1.3-teorema. *Agar Y to'la normalangan fazo bo'lsa, u holda $L(X, Y)$ ham to'la normalangan fazo ya'ni Banax fazosi bo'ladi.*

1.10-ta'rif. *$A : X \to Y$ va $B : Y \to Z$ chiziqli operatorlar berilgan bo'lib, $R(A) \subset D(B)$ bo'lsin. B va A operatorlarning ko'paytmasi deganda, har bir $x \in D(A)$ ga Z fazoning $z = B(Ax)$ elementini mos qo'yuvchi $C = BA : X \to Z$ operatorga aytiladi.*

Agar A va B lar chiziqli chegaralangan operatorlar bo'lsa, u holda $C = BA$ ham chiziqli chegaralangan operator bo'ladi va

$$\|C\| = \|BA\| \leq \|B\| \cdot \|A\| \tag{1.3}$$

tengsizlik o'rinli.

1.11-ta'rif. *Agar $\{A_n\} \subset L(X, Y)$ operatorlar ketma-ketligi uchun shunday $A \in L(X, Y)$ operator mavjud bo'lib, $\lim_{n\to\infty} \|A_n - A\| = 0$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga norma bo'yicha yoki tekis yaqinlashadi deyiladi va $A_n \Longrightarrow A$ $(A_n \overset{u}{\Longrightarrow} A)$ shaklda belgilanadi.*

1.12-ta'rif. *Agar ixtiyoriy $x \in X$ uchun $\lim_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchli yoki nuqtali yaqinlashadi deyiladi va $A_n \longrightarrow A$ $(A_n \overset{s}{\longrightarrow} A)$ shaklda belgilanadi.*

1.13-ta'rif. *Agar ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim_{n\to\infty} |f(A_n x) - f(Ax)| = 0$ bo'lsa, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchsiz yoki kuchsiz ma'noda yaqinlashuvchi deyiladi va $A_n \longrightarrow A$ $(A_n \overset{w}{\longrightarrow} A)$ shaklda belgilanadi.*

Bu ta'rifni Hilbert fazosida quyidagicha bayon qilish mumkin.

1.14-ta'rif. *Agar ixtiyoriy $x, y \in H$ uchun $\lim_{n\to\infty} (A_n x, y) = (Ax, y)$*

bo'lsa, $\{A_n\} \subset L(H)$ operatorlar ketma-ketligi A operatorga kuchsiz yaqinlashuvchi deyiladi.

1.1. X – ixtiyoriy chiziqli normalangan fazo bo'lsin.

$$Ix = x, \quad x \in X$$

akslantirish *birlik operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring. Agar chegaralangan bo'lsa, uning normasini toping.

Yechish. Bu operatorning chiziqliligi va uzluksizligi quyidagi tengliklardan bevosita kelib chiqadi:

$$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha Ix + \beta Iy, \quad \|I(x - x_0)\| = \|x - x_0\|.$$

1.2-teoremadan uning chegaralangan ekanligi kelib chiqadi. 1.1-teoremaga ko'ra, $\|I\| = \sup\limits_{\|x\|=1} \|Ix\| = \sup\limits_{\|x\|=1} \|x\| = 1$ ekanligini olamiz. Qo'shimcha qilib aytishimiz mumkinki, uning aniqlanish sohasi, qiymatlar sohasi va yadrosi uchun quyidagilar o'rinli:

$$D(I) = X, \quad R(I) = X, \quad Ker I = \{\theta\}.$$

1.2. X va Y ixtiyoriy chiziqli normalangan fazolar bo'lsin.

$$\Theta : X \to Y, \ \Theta x = \theta$$

operator *nol operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring. Chegaralangan ekanligini ko'rsatib, normasini toping. Aniqlanish sohasi, qiymatlar sohasi va yadrosi haqida ma'lumot bering.

Yechish. Nol operatorning chiziqliligi va uzluksizligi bevosita ta'rifdan kelib chiqadi. Endi nol operatorning chegaralangan ekanligini ko'rsatib, uning normasini topamiz. Istalgan $x \in X$ uchun $\|\Theta x\| = \|\theta\| = 0$ tenglik o'rinli. Bundan $\|\Theta\| = \sup\limits_{\|x\|=1} \|\Theta x\| = \sup\limits_{\|x\|=1} \|\theta\| = 0$ ekanligi kelib chiqadi. Nol operator $L(X, Y)$ chiziqli normalangan fazoning nol elementi bo'ladi. Uning aniqlanish sohasi, qiymatlar sohasi va yadrosi uchun quyidagilar o'rinli:

$$D(\Theta) = X, \quad R(\Theta) = \{\theta\}, \quad Ker\,\Theta = X.$$

1.3. Aniqlanish sohasi $D(A) = C^{(1)}[a, b] \subset C[a, b]$ bo'lgan va $C[a, b]$ fazoni o'zini-o'ziga akslantiruvchi

$$A : C[a, b] \to C[a, b], \quad (Af)(x) = f'(x)$$

operatorni qaraymiz. Bu operator *differensial operator* deyiladi. Uni chiziqlilik va uzluksizlikka tekshiring.

Yechish. Differensial operatorni chiziqli ekanligini ko'rsatamiz. Buning uchun ixtiyoriy $f, g \in D(A)$ elementlarning chiziqli kombinatsiyasi bo'lgan $\alpha f + \beta g$ elementga A operatorning ta'sirini qaraymiz:

$$(A(\alpha f + \beta g))(x) = (\alpha f(x) + \beta g(x))' =$$

$$= \alpha f'(x) + \beta g'(x) = \alpha\,(Af)(x) + \beta\,(Ag)(x).$$

Biz bu yerda yig'indining hosilasi hosilalar yig'indisiga tengligidan, hamda o'zgarmas sonni hosila belgisi ostidan chiqarish mumkinligidan foydalandik. Demak, A operator chiziqli ekan. Uni nol nuqtada uzluksizlikka tekshiramiz. Ma'lumki, $A\theta = \theta$, bu yerda θ – $C[a, b]$ fazoning nol elementi, ya'ni $\theta(x) = 0$. Endi nolga yaqinlashuvchi $f_n \in D(A)$ ketma-ketlikni tanlaymiz. Umumiylikni buzmagan holda $a = 0$, $b = 1$ deymiz.

$$f_n(x) = \frac{x^{n+1}}{n+1}, \quad \lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \max_{0\le x\le 1} \left|\frac{x^{n+1}}{n+1}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Ikkinchi tomondan,

$$(Af_n)(x) = x^n, \quad \lim_{n\to\infty} \|Af_n - A\theta\| = \lim_{n\to\infty} \max_{0\le x\le 1} |x^n| = \lim_{n\to\infty} 1 = 1 \ne 0.$$

Demak, A operator nol nuqtada uzluksiz emas ekan. Uning chiziqliligidan, differensial operator aniqlanish sohasining barcha nuqtalarida uzilishga ega bo'ladi. 1.2-teoremaga ko'ra differensial operator chegaralanmagan operator bo'ladi. A ning qiymatlar sohasi va yadrosi uchun quyidagilar o'rinli:

$$R(A) = C[a,b], \quad Ker A = \{const\}.$$

1.4. $C[a, b]$ fazoni o'zini-o'ziga akslantiruvchi B operatorni quyidagicha aniqlaymiz:

$$(Bf)(x) = \int_a^b K(x,t) f(t)\, dt. \tag{1.4}$$

B ga *integral operator* deyiladi. Bu yerda $K : [a, b]\times[a, b] \to \mathbb{R}$ – uzluksiz funksiya, u integral operatorning *o'zagi (yadrosi)* deyiladi. B operatorni chiziqlilik va uzluksizlikka tekshiring.

Yechish. Ma'lumki, ixtiyoriy $f \in C[a, b]$ uchun $K(x, t)\, f(t)$ funksiya x va t ning uzluksiz funksiyasidir. Matematik analiz kursidan ma'lumki,

$$\int_a^b K(x, t)\, f(t)\, dt$$

integral parametr $x \in [a, b]$ ning uzluksiz funksiyasi bo'ladi. Bulardan B operatorning aniqlanish sohasi $D(B)$ uchun $D(B) = C[a, b]$ tenglik o'rinli ekanligi kelib chiqadi. Integral operatorning chiziqli ekanligi integrallash amalining chiziqlilik xossasidan kelib chiqadi, ya'ni ixtiyoriy $f, g \in C[a, b]$ va $\alpha, \beta \in \mathbb{C}$ lar uchun

$$(B(\alpha f + \beta g))(x) = \int_a^b K(x,t)\,(\alpha f(t) + \beta g(t))\, dt =$$

$$= \alpha \int_a^b K(x,t) f(t)\, dt + \beta \int_a^b K(x,t) g(t)\, dt = \alpha (Bf)(x) + \beta (Bg)(x)$$

tengliklar o'rinli. Endi B operatorning uzluksiz ekanligini ko'rsatamiz. $f_0 \in C[a, b]$ ixtiyoriy tayinlangan element va $\{f_n\} \subset C[a, b]$ unga yaqinlashuvchi ixtiyoriy ketma-ketlik bo'lsin. U holda

$$\|Bf_n - Bf_0\| = \max_{a\le x\le b}\left|\int_a^b K(x,t)\,(f_n(t) - f_0(t))\,dt\right| \le$$

$$\le \max_{a\le t\le b}|f_n(t) - f_0(t)|\max_{a\le x\le b}\int_a^b |K(x,t)|\,dt = C\cdot\|f_n - f_0\|. \quad (1.5)$$

Bu yerda

$$C = \max_{a\le x\le b}\int_a^b |K(x,t)|\,dt.$$

C ning chekli ekanligi $[a, b]$ kesmada uzluksiz funksiyaning chegaralangan ekanligidan kelib chiqadi. Agar (1.5) tengsizlikda $n\to\infty$ da limitga o'tsak,

$$0 \le \lim_{n\to\infty}\|Bf_n - Bf_0\| \le C\cdot\lim_{n\to\infty}\|f_n - f_0\| = 0$$

ekanligini olamiz. Demak,

$$\lim_{n\to\infty}\|B f_n - B f_0\| = 0.$$

Shunday qilib, B integral operator ixtiyoriy nuqtada uzluksiz ekan. 1.2-teoremaga ko'ra, u chegaralangan operator bo'ladi. B integral operatorning qiymatlar sohasi va yadrosi uning o'zagi K ning berilishiga bog'liq. Masalan, $K(x,t)\equiv 1$ bo'lsa, B operatorning qiymatlar sohasi ImB o'zgarmas funksiyalardan iborat, ya'ni $ImB = \{f\in C[a,b] : f(t) = const\}$ uning yadrosi $KerB$ o'zgarmasga ortogonal funksiyalardan iborat, ya'ni

$$KerB = \left\{f\in C[a,\ b] : \int_a^b f(t)\,dt = 0\right\}. \qquad \square$$

Quyidagi belgilashlarni kiritamiz. Faraz qilaylik, $G\subset\mathbb{R}^m$ biror soha bo'lsin. $C(G,\mathbb{R}^n)$ bilan $u(p) = (u_1(p), u_2(p),\ldots,u_n(p))\in\mathbb{R}^n$, $u_j\in C(G)$, $p\in G$, $j = 1,2,\ldots,n$ vektor funksiyalar to'plamini belgilaymiz. Xuddi shunday $L_2(G,\mathbb{R}^n)$ bilan $u(p) = (u_1(p), u_2(p),\ldots,u_n(p))\in\mathbb{R}^n$,

$u_j \in L_2(G)$, $j = 1, 2, \ldots, n$ vektor funksiyalar to'plamini belgilaymiz. Bu ikkala fazoda ham skalyar ko'paytma bir xilda kiritiladi, ya'ni

$$(u, v) = \sum_{k=1}^{n} \int_G u_k(p)\, v_k(p)\, dp.$$

1.5. Har bir $u \in C(G)$, $G \subset \mathbb{R}^3$ funksiyaga, uning $\ell(\cos\alpha, \cos\beta, \cos\gamma)$ yo'nalish bo'yicha olingan hosilasini mos qo'yib, natijada

$$A : C(G) \to C(G), \ (Au)(p) = \cos\alpha \frac{\partial u(p)}{\partial x} + \cos\beta \frac{\partial u(p)}{\partial y} + \cos\gamma \frac{\partial u(p)}{\partial z}$$

operatorga ega bo'lamiz. Bu operator chiziqli, uzluksiz bo'ladimi? Agar bu operatorni $A : C^{(1)}(G) \to C(G)$ akslantirish sifatida qarasak u uzluksiz bo'ladimi?

1.6. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning divergensiyasini mos qo'yib, natijada

$$A : C^{(1)}(G) \to C(G), \ (Au)(p) = \frac{\partial u(p)}{\partial x} + \frac{\partial u(p)}{\partial y} + \frac{\partial u(p)}{\partial z}$$

operatorga ega bo'lamiz. Bu operator chiziqli, uzluksiz bo'ladimi?

1.7. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning gradiyentini mos qo'yib, natijada

$$A : C^{(1)}(G) \to C(G, \mathbb{R}^3), \ (Au)(p) = \left(\frac{\partial u(p)}{\partial x}, \frac{\partial u(p)}{\partial y}, \frac{\partial u(p)}{\partial z} \right)$$

operatorga ega bo'lamiz. Bu operatorning tabiiy aniqlanish sohasini toping. U chiziqli, uzluksiz bo'ladimi?

1.8. Har bir $u \in C^{(1)}(G)$, $G \subset \mathbb{R}^3$ funksiyaga uning gradiyenti uzunligini mos qo'yib, natijada $A : C^{(1)}(G) \to C(G)$,

$$(Au)(p) = \sqrt{\left(\frac{\partial u(p)}{\partial x}\right)^2 + \left(\frac{\partial u(p)}{\partial y}\right)^2 + \left(\frac{\partial u(p)}{\partial z}\right)^2}$$

operatorga ega bo'lamiz. Bu operator chiziqli bo'ladimi?

1.9. Har bir $u \in C(G, \mathbb{R}^3)$, $u(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ vektor funksiyaga uning uyurmasi (rotori) ni mos qo'yib, natijada

$$A : C(G, \mathbb{R}^3) \to C(G, \mathbb{R}^3),$$

$$(Au)(p) = \left(\frac{\partial R(p)}{\partial y} - \frac{\partial Q(p)}{\partial z}, \ \frac{\partial P(p)}{\partial z} - \frac{\partial R(p)}{\partial x}, \ \frac{\partial Q(p)}{\partial x} - \frac{\partial P(p)}{\partial y} \right)$$

operatorga ega bo'lamiz. Bu operatorning tabiiy aniqlanish sohasini toping. U chiziqli, uzluksiz bo'ladimi?

1.10. Laplas (kinetik energiya) operatori $\Delta : C(\mathbb{R}^3) \to C(\mathbb{R}^3)$,

$$(\Delta u)(x, y, z) = \frac{\partial^2 u(x, y, z)}{\partial x^2} + \frac{\partial^2 u(x, y, z)}{\partial y^2} + \frac{\partial^2 u(x, y, z)}{\partial z^2}$$

ni chiziqli, uzluksizlikka tekshiring. Dastlab uning tabiiy aniqlanish sohasini toping.

1.11. Quyidagi operatorning chiziqli chegaralanganligini ko'rsatib, normasini toping:

$$A : C^{(2)}[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = \frac{d^2 x(t)}{dt^2} := x''(t).$$

Yechish. Berilgan operatorning aniqlanish sohasi $D(A) = C^{(2)}[0,\ 1]$. Operator chiziqli, chunki ixtiyoriy $x,\ y \in C^{(2)}[0,\ 1]$ va $\alpha,\ \beta$ sonlar uchun

$$[A(\alpha x + \beta y)]\ (t) = (\alpha x + \beta y)''\ (t) = \alpha x''(t) + \beta y''(t) =$$

$$= \alpha\ (Ax)\ (t) + \beta\ (Ay)\ (t) = (\alpha Ax + \beta Ay)\ (t)$$

tengliklar o'rinli. Endi operatorning chegaralanganligini ko'rsatamiz:

$$\|Ax\| = \max_{0 \le t \le 1} |(Ax)\ (t)| = \max_{0 \le t \le 1} |x''(t)| \le \|x\|,$$

ya'ni ixtiyoriy $x \in C^{(2)}[0,\ 1]$ uchun

$$\|Ax\| \le \|x\|. \tag{1.6}$$

(1.6) tengsizlikdan $\|A\| \le 1$ ni olamiz. A operator normasini 1.1-teorema yordamida topamiz. Quyidagi funksiyalar ketma-ketligini qaraymiz:

$$x_n(t) = e^{-nt} \in D(A) = C^{(2)}[0,\ 1].$$

U holda $\|x_n\|$ va $\|Ax_n\|$ lar uchun

$$\|x_n\| = \max_{0\le t\le 1}\left|e^{-nt}\right| + \max_{0\le t\le 1}\left|-ne^{-nt}\right| + \max_{0\le t\le 1}\left|n^2e^{-nt}\right| = 1 + n + n^2,$$

$$\|Ax_n\| = \max_{0\le t\le 1}\left|n^2e^{-nt}\right| = n^2$$

tengliklar o'rinli bo'ladi. (1.2) tenlikka ko'ra,

$$\|A\| = \sup_{x\ne 0}\frac{\|Ax\|}{\|x\|} \ge \sup_n \frac{\|Ax_n\|}{\|x_n\|} = \sup_n \frac{n^2}{1+n+n^2} = 1.$$

Olingan $\|A\| \le 1$ va $\|A\| \ge 1$ tengsizliklardan $\|A\| = 1$ ekanligi kelib chiqadi.

1.12. Quyidagi operatorning chiziqli chegaralanganligini ko'rsatib, normasini toping:

$$A : L_5[-1,\ 1] \to L_3[-1,\ 1],\quad (Ax)(t) = t^4\,x(t^3).$$

Yechish. Dastlab ixtiyoriy $x \in L_5[-1\,,\ 1]$ uchun $Ax \in L_3[-1\,,\ 1]$ ekanligini ko'rsatamiz. Buning uchun $t^3 = r$ almashtirishdan foydalanamiz:

$$\|Ax\|_3 = \left(\int_{-1}^{1}\left|t^4x(t^3)\right|^3 dt\right)^{\frac{1}{3}} = \left(\frac{1}{3}\int_{-1}^{1}|r|^{\frac{10}{3}}\,|x(r)|^3\,dr\right)^{\frac{1}{3}}.$$

Oxirgi integralni baholashda quyidagi umumlashgan Gyolder tengsizligidan foydalanamiz:

$$\|x\cdot y\|_s \le \|x\|_k \cdot \|y\|_r\,,$$

bu yerda $x \in L_k[a,\ b],\quad y \in L_r[a,\ b],\quad \frac{1}{k}+\frac{1}{r}=\frac{1}{s}$. Biz qarayotgan holda $k=\frac{15}{2},\ r=5,\ s=3$. Shuning uchun

$$\|Ax\|_3 = \left(\int_{-1}^{1}\left|\frac{1}{\sqrt[3]{3}}r^{\frac{10}{9}}x(r)\right|^3 dr\right)^{\frac{1}{3}} \le$$

$$\le \left(\int_{-1}^{1}\left|\frac{1}{\sqrt[3]{3}}r^{\frac{10}{9}}\right|^{\frac{15}{2}} dr\right)^{\frac{2}{15}} \left(\int_{-1}^{1}|x(r)|^{5} dr\right)^{\frac{1}{5}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}} \cdot \|x\|_5.$$

Demak, ixtiyoriy $x \in L_5[-1,\ 1]$ uchun

$$\|Ax\|_3 \le \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}} \cdot \|x\|_5 \tag{1.7}$$

tengsizlik o‘rinli. Shunday qilib, har bir $x \in L_5[-1,\ 1]$ uchun $Ax \in L_3[-1,\ 1]$. Demak, $D(A) = L_5[-1,\ 1]$. Endi A operatorning chiziqli ekanligini ko‘rsatamiz:

$$[A(\alpha x + \beta y)](t) = t^4(\alpha x + \beta y)(t^3) = \alpha t^4 x(t^3) + \beta t^4 y(t^3) =$$

$$= \alpha\,(Ax)\,(t) + \beta\,(Ay)\,(t) = [\alpha\,Ax + \beta\,Ay]\,(t)\,,\ \forall t \in [-1,\ 1],$$

ya'ni $\forall x, y \in L_5[-1,\ 1]$ va $\forall \alpha, \beta \in \mathbb{C}$ uchun $A(\alpha x + \beta y) = \alpha Ax + \beta Ax$. Operatorning chegaralanganligi (1.7) dan kelib chiqadi. Uning normasini topish uchun $x_0(t) = t^{\frac{5}{3}} \in L_5[-1\,,\ 1]$ elementni qaraymiz:

$$\|x_0\|_5 = \left(\int_{-1}^{1}|t|^{\frac{25}{3}}dt\right)^{\frac{1}{5}} = \left(\frac{3}{14}\right)^{\frac{1}{5}}, \qquad (Ax_0)\,(t) = t^9,$$

$$\|Ax_0\| = \left(\int_{-1}^{1}|t|^{27}dt\right)^{\frac{1}{3}} = \left(\frac{1}{14}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{1}{3}},$$

$$\frac{\|Ax_0\|_3}{\|x_0\|_5} = \frac{\frac{1}{\sqrt[3]{3}}(3/14)^{\frac{1}{3}}}{(3/14)^{\frac{1}{5}}} = \frac{1}{\sqrt[3]{3}}\left(\frac{3}{14}\right)^{\frac{2}{15}}. \tag{1.8}$$

(1.7) va (1.8) dan $\|A\| = \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{3}{14}\right)^{\frac{2}{15}}$ tenglikni hosil qilamiz. □

1.13-1.32-misollarda berilgan operatorlarning chiziqli, chegaralangan ekanligini ko'rsating, ularning normalarini toping.

1.13. $A : C[-2,\ 2] \to C[-2,\ 2]\ ,\ (Ax)(t) = te^t x(t)\ .$

1.14. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1}(ts + t^2 s^2)\, x(s)\, ds.$

1.15. $A : L_2[-1,\ 1] \to L_2[-1,\ 1]\ ,\ (Ax)(t) = (t^2 - t)\, x(t)\ .$

1.16. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(t) = t^3 x(t^{1/3})\, .$

1.17. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(t) = t^2 x\,(t^3)\ .$

1.18. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^{1}(t+s)^2\, x\,(s)\, ds\, .$

1.19. $A : L_3[-1,\ 1] \to L_3[-1,\ 1],\ (Ax)(t) = t\, x\ \left(\sqrt[3]{t}\right)\ .$

1.20. $A : L_3[-1,\ 1] \to L_3[-1,\ 1],\ (Ax)(t) = \sqrt[5]{1-t}\, x(t)\, .$

1.21. $A : L_5[0, 2] \to L_5[0, 2]\, ,\ (Ax)(t) = (t^2 - 2t + 1)\, x(t)\, .$

1.22. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\ (Af)(n) = f(n+1)\, .$

1.23. $A : \ell_2 \to \ell_2,\ Ax = \left(2x_1, \frac{3}{2}x_2, \ldots, \frac{n+1}{n}x_n, \ldots\right).$

1.24. $A : \ell_2 \to \ell_2,\ Ax = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots, \frac{1}{n}x_n, \ldots\right).$

1.25. $A : \ell_1 \to \ell_1,\ Ax = \left(2x_1, \left(1+\frac{1}{2}\right)^2 x_2, \ldots, \left(1+\frac{1}{n}\right)^n x_n, \ldots\right).$

1.26. $A : \ell_3 \to \ell_3,\ Ax = ((1+1)x_1, (1+1/2\,)x_2, \ldots, (1+1/n)\, x_n, \ldots).$

1.27. $A:\ell_5\to\ell_5$, $Ax=\left(\frac{x_1}{5},\frac{x_2}{5^2},\ldots,\frac{x_n}{5^n},\ldots\right)$.

1.28. $A:\ell_4\to\ell_4$, $Ax=\left(\sin\frac{\pi}{8}\cdot x_1,\sin\frac{2\pi}{8}\cdot x_2,\ldots,\sin\frac{n\pi}{8}\cdot x_n,\ldots\right)$.

1.29. $A:\ell_{5/2}\to\ell_{5/2}$, $Ax=\left(\frac{1}{2}x_1,\frac{1}{\sqrt{2}}x_2,\ldots,\frac{1}{\sqrt[n]{2}}x_n,\ldots\right)$.

1.30. $A:\ell_{5/4}\to\ell_{5/4}$, $Ax=\left(0,\frac{1}{2}x_2,\frac{2}{3}x_3,\ldots,(1-\frac{1}{n})x_n,\ldots\right)$.

1.31. $A:C[0,\ 4]\to M[0,\ 4]$. $(Ax)(t)=[t]\,x(t)$.

1.32. $B:AC_0[-1,\ 4]\to V_0[-1,\ 4]$. $(Bx)(t)=\operatorname{sign}t\,x(t)$.

1.15-misolning yechimi. *Chiziqliligi.* Ixtiyoriy $x\in L_2[-1,\ 1]$ uchun $Ax\in L_2[-1,\ 1]$ ekanligidan $D(A)=L_2[-1,\ 1]$ ekanligi kelib chiqadi.

$$(A(\alpha x+\beta y))(t)=(t^2-t)\,(\alpha x+\beta y)(t)=(t^2-t)\,(\alpha x(t)+\beta y(t))=$$

$$=\alpha(t^2-t)\,x(t)+\beta(t^2-t)\,y(t)=\alpha(A\,x)(t)+\beta(A\,y)(t)$$

tenglikdan esa berilgan operatorning chiziqli ekanligi kelib chiqadi.

Chegaralanganligi.

$$\|Ax\|^2=\int_{-1}^{1}|(t^2-t)\,x(t)|^2\,dt\le\max_{-1\le t\le1}|t^2-t|^2\int_{-1}^{1}|\,x(t)|^2\,dt=4\,\|x\|^2.$$

Bu yerdan berilgan operatorning chegaralanganligi va $\|A\|\le2$ ekanligi kelib chiqadi.

Normasi. Berilgan operatorning normasini topish uchun quyidagicha yo'l tutamiz. $B_n=[-1,\ -1+\frac{1}{n}]$ va $x_n(t)=\sqrt{n}\,\chi_{B_n}(t)$ deymiz. U holda

$$\|x_n\|^2=\int_{-1}^{1}|x_n(t)|^2\,dt=\int_{-1}^{-1+\frac{1}{n}}n\,dt=1\iff\|x_n\|=1.$$

Xuddi shunday $\|Ax_n\|^2$ ni hisoblaymiz:

$$\|Ax_n\|^2 = \int_{-1}^{1} |(t^2-t)\,x_n(t)|^2\,dt = \int_{B_n} |\sqrt{n}(t^2-t)|^2 dt = n \int_{-1}^{-1+\frac{1}{n}} (t^4-2t^3+t^2)dt.$$

Bu jadval integrali bo'lib, uning qiymati

$$4 - \frac{6}{n} + \frac{7}{3n^2} - \frac{3}{2n^3} + \frac{1}{5n^4} = \|Ax_n\|^2$$

dir. Endi

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \geq \sup_{n\geq 1} \|Ax_n\| = \sup_{n\geq 1} \sqrt{4 - \frac{6}{n} + \frac{7}{3n^2} - \frac{3}{2n^3} + \frac{1}{5n^4}} = 2$$

munosabatdan $\|A\| \geq 2$ ni olamiz. Yuqorida $\|A\| \leq 2$ tengsizlik ko'rsatilgan edi. Bulardan $\|A\| = 2$ kelib chiqadi. □

1.32-misolning yechimi. *Chiziqliligi* 1.15-misol kabi ko'rsatiladi. *Chegaralanganligi.* Berilgan operatorning chegaralangan ekanligini ko'rsatishda o'zgarishi chegaralangan funksiyalar uchun o'rinli bo'lgan

$$V_a^b[f \cdot g] \leq \sup_{x\in[a,\,b]} |f(x)| \cdot V_a^b[g] + \sup_{x\in[a,\,b]} |g(x)| \cdot V_a^b[f]$$

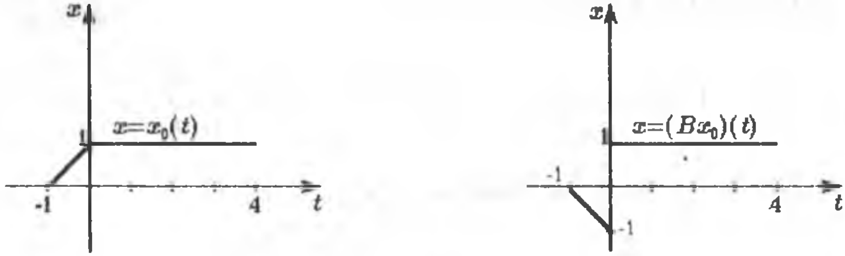
tengsizlikdan (I qism 9.50-misolga qarang) foydalanamiz:

$$\|Bx\| = \mathop{V}_{-1}^{4}[Bx] \leq \sup_{-1\leq t\leq 4} |\text{sign}\, t|\, \mathop{V}_{-1}^{4}[x] + \sup_{-1\leq t\leq 4} |x(\,t)|\, \mathop{V}_{-1}^{4}[\text{sign}] \leq 3\,\|x\|\,.$$

Bu yerda biz $\sup\limits_{-1\leq t\leq 4} |\text{sign}\, t| = 1.$ $\mathop{V}\limits_{-1}^{4}[\text{sign}] = 2$ va barcha $x \in AC_0[-1,\ 4]$ lar uchun o'rinli bo'lgan $\max\limits_{-1\leq t\leq 4} |x(\,t)| \leq \mathop{V}\limits_{-1}^{4}[x] = \|x\|$ tengsizlikdan foydalandik. Yuqoridagi tengsizlikdan $\|B\| \leq 3$ kelib chiqadi.

$$x_0(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ 1, & 0 \leq t \leq 4 \end{cases}, \qquad x_0 \in AC_0[-1,\ 4]$$

element uchun $\|x_0\| = 1$ va $\|Bx_0\| = 3$ tengliklar o'rinli (1.1-chizma). Bu yerdan $\|B\| \geq 3$ ekanligi kelib chiqadi. Demak, $\|B\| = 3$ ekan. □



1.1-chizma

1.33. X va Y chiziqli normalangan fazolar, $A : X \to Y$ chiziqli operator. A operator chegaralanmagan bo'lishi uchun $D(A)$ da $\|x_n\| = 1$ va $\lim\limits_{n\to\infty} \|Ax_n\| = \infty$ shartlarni qanoatlantiruvchi ketma-ketlikning mavjud bo'lishi zarur va yetarli. Isbotlang.

Endi chegaralanmagan operatorlarga misol keltiramiz.

1.34. $I : \ell_2 \to \ell_1$, $Ix = x$ operatorning chegaralanmagan ekanligini ko'rsating.

Yechish. Bu operatorning aniqlanish sohasi $D(A) = \ell_1$ fazodan iborat. Quyidagi ketma-ketlikni qaraymiz: $x_n = (1, 2, \dots, n, 0, 0, \dots) \in D(A)$. U holda

$$\|x_n\|_{\ell_2} = \sqrt{1^2 + 2^2 + \cdots + n^2} = \sqrt{\frac{n(n+1)(2n+1)}{6}},$$

$$\|Ix_n\|_{\ell_1} = \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Bulardan

$$\sup_{n\geq 1} \frac{\|Ix_n\|}{\|x_n\|} = \sup_{n\geq 1} \frac{n(n+1)}{2} \cdot \sqrt{\frac{6}{n(1+n)(2n+1)}} = \infty.$$

Bu esa $I : \ell_2 \to \ell_1$ operatorning chegaralanmagan ekanligini ko'rsatadi.

1.35-1.48-misollarda quyidagi savollarga javob bering. $X, Y-$ haqiqiy chiziqli normalangan fazolar, $A: X \to Y$ chiziqli operator.

1) A operatorning aniqlanish sohasi $D(A)$ butun X fazoga tengmi?

2) Berilgan operator uzluksizmi?

1.35. $A: C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = \dfrac{dx(t)}{dt}.$

1.36. $A: AC_0[0,\ 1] \to L_1[0,\ 1], \quad (Ax)(t) = \dfrac{dx(t)}{dt}.$

1.37. $A: C[0,\ 1] \to V_0[0,\ 1], \quad (Ax)(t) = t\,x(t).$

1.38. $A: AC_0[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t^{-0,1}\,x(t).$

1.39. $A: AC_0[0,\ 1] \to L_1[0,\ 1], \quad (Ax)(t) = t^{-0,3}\,x(t).$

1.40. $A: C^{(1)}[-1,\ 1] \to L_1[-1,\ 1], \quad (Ax) = \dfrac{d^2x(t)}{dt^2}.$

1.41. $A: L_1[0,\ 1] \to L_1[0,\ 1], \quad (Ax)(t) = x(t^3)\,.$

1.42. $A: L_1(\mathbb{R}_+) \to L_1(\mathbb{R}_+), \quad (Ax)(t) = t\,x(\sqrt{t}\,)$

1.43. $A: L_1(\mathbb{R}) \to L_2(\mathbb{R}), \quad (Ax)(t) = \dfrac{1}{1+|t|}x(t)\,.$

1.44. $A: L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+), \quad (Ax)(t) = \int\limits_0^{\infty} (ts+1)\,x\,(s)\,ds.$

1.45. $A: \ell_3 \to \ell_1, \quad Ax = (x_1, 2x_2, x_3, 2x_4 \ldots, x_{2n-1}, 2x_{2n}, \ldots)\,.$

1.46. $A: \ell_2 \to \ell_2, \quad Ax = \left(\operatorname{ctg} 1 \cdot x_1, \operatorname{ctg}\dfrac{1}{2} \cdot x_2, \ldots, \operatorname{ctg}\dfrac{1}{n} \cdot x_n, \ldots\right).$

1.47. $A: \ell_2 \to \ell_1, \quad Ax = \left(x_1, 2x_2, \dfrac{3}{2}x_3, \ldots, \dfrac{n}{n-1}x_n, \ldots\right).$

1.48. $A: \ell_2 \to \ell_1, \quad Ax = \left(x_1, \dfrac{x_2}{\sqrt{2}}, \ldots, \dfrac{x_n}{\sqrt{n}}, \ldots\right).$

1.36-misolning yechimi. $[a,\ b]$ kesmada absolyut uzluksiz F funksiyaning hosilasi $F'(x) = f(x)$ integrallanuvchidir (I qism 10.18-misol). Demak, operatorning aniqlanish sohasi $D(A) = AC_0[0,\ 1]$ ekan. I qism 9.76-misol tasdig'iga ko'ra, ixtiyoriy $x \in AC_0[0,\ 1]$ uchun

$$V_0^1[Ax] = \int_0^1 |x(t)|\, dt$$

tenglik o'rinli. Bu tenglikning chap tomoni $\|Ax\|$ ga o'ng tomoni $\|x\|$ ga teng. Demak, barcha $x \in AC_0[0,\ 1]$ uchun $\|Ax\| = \|x\|$ tenglik o'rinli. Bu yerdan A operatorning chegaralanganligi va $\|A\| = 1$ ekanligi kelib chiqadi. □

1.42-misolning yechimi. Agar $x_0(t) = (1+t^2)^{-1}$ desak, u holda $x_0 \in L_1(\mathbb{R}_+)$ bo'ladi, $(Ax_0)(t) = t\,(1+t)^{-1}$ bo'lib, u integrallanuvchi emas, ya'ni $Ax_0 \notin L_1(\mathbb{R}_+)$. Demak, $D(A) \neq L_1(\mathbb{R}_+)$. Agar $x_n(t) = \chi_{[n,\ n+1)}(t)$ desak, u holda $x_n \in D(A)$, $\|x_n\| = 1$ bo'lib.

$$\|Ax_n\| = \int_0^\infty t\,\chi_{[n,\ n+1)}(\sqrt{t})dt = \int_0^\infty 2\,s^3\,\chi_{[n,\ n+1)}(s)\,ds = 2\int_n^{n+1} s^3\,ds.$$

Bu integralning qiymati $\|Ax_n\| = 2n^3 + 3n^2 + 2n + 0,5$ ga teng. 1.33-misolga ko'ra, A ning chegaralanmagan operator ekanligi kelib chiqadi. 1.2-teoremaga ko'ra, u uzluksiz ham emas. □

1.49-1.54-misollarda quyidagi savollarga javob bering. $X, Y-$ kompleks chiziqli normalangan fazolar, $A : X \to Y$ chiziqli operator.

1) A operatorning aniqlanish sohasi $D(A)$ butun X fazoga tengmi?

2) Berilgan operator uzluksizmi?

1.49. $A : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$, $\quad (Ax)(t) = t\,x(t)$.

1.50. $A : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $\quad (Ax)(t) = \frac{1}{\sqrt{t}}x(t)$.

1.51. $A : \ell_1(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $\quad (Af)(n) = \sqrt{n}f(n)$.

1.52. $A:\ell_2(\mathbb{Z})\to\ell_1(\mathbb{Z}),\quad (Af)(n)=\dfrac{1}{\sqrt{n+0,1}}f(n)\,.$

1.53. $A:\ell_2\to\ell_3,\quad Ax=(x_1,\sqrt{2}\,x_2,\ldots,\sqrt{n}\,x_n,\ldots)\,.$

1.54. $A:\ell_1\to\ell_1,\quad Ax=(x_1,\ln 2\cdot x_2,\ldots,\ln n\cdot x_n,\ldots).$

1.54-misolning yechimi. Matematik analiz kursidan ma'lum bo'lgan quyidagi qatorlarni qaraymiz:

$$1)\quad 1+\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln^2 n},\qquad 2)\quad 1+\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n}.$$

Ulardan 1)-si yaqinlashuvchi, 2)-si esa uzoqlashuvchi. Bu tasdiq integral alomat yordamida ko'rsatiladi. Agar biz $x_0=\left(1,\dfrac{1}{2\ln^2 2},\ldots,\dfrac{1}{n\ln^2 n},\ldots\right)$ desak, u holda $x_0\in\ell_1$ bo'ladi, $Ax_0=\left(1,\dfrac{1}{2\cdot\ln 2},\ldots,\dfrac{1}{n\cdot\ln n},\ldots\right)$ bo'lib, uning hadlarining modullaridan tuzilgan qator (2-qator) uzoqlashuvchi, ya'ni $Ax_0\notin\ell_1$. Demak, $D(A)\neq\ell_1$. Agar

$$x_m=\left(1,\frac{1}{2\cdot\ln^2 2},\ldots,\frac{1}{m\cdot\ln^2 m},0,0,\ldots\right)$$

desak, u holda $x_m\in D(A)$ bo'lib, biror $C>0$ va barcha m larda $\|x_m\|\le C$ bo'ladi. Ammo $\lim\limits_{m\to\infty}\|Ax_m\|=\infty$. Ya'ni A chegaralanmagan operatordir. 1.2-teoremaga ko'ra, u uzluksiz ham emas. □

1.55. Shunday $A,\,B\in L(X)$ operatorlarga misol keltiringki, $AB\neq BA$ bo'lsin.

1.56. $A,\,B\in L(X,\,Y)$ noldan farqli operatorlar bo'lib, $R(A)\cap R(B)=0$ bo'lsa, A va B larning chiziqli erkli ekanligini isbotlang.

1.57. $A,\,B\in L(X,\,Y)$ va $R(A)=R(B)$, $KerA=KerB$ bo'lishidan $A=B$ ekanligi kelib chiqadimi?

1.58. X, Y lar normalangan fazolar, $U \subset X$ ochiq to'plam, $V \subset X$ yopiq to'plam hamda $A \in L(X, Y)$ bo'lsa. $A(U)$ ochiq. $A(V)$ esa yopiq to'plam bo'ladimi?

1.59. $p : L(X,Y) \to \mathbb{R}$, $p(A) = \|A\|$ funksional norma shartlarini qanoatlantirishini isbotlang.

1.60. $p : L(X,Y) \to \mathbb{R}$, $p(A) = \|A\|$ akslantirishning uzluksiz ekanligini isbotlang.

1.61. $L(\mathbb{R}^n, \mathbb{R}^m)$ fazoning o'lchamini toping.

1.62. X chiziqli normalangan fazo, X' uning qism fazosi bo'lsin. $M = \{A \in L(X) : Ker A = X'\}$ to'plam $L(X)$ ning qism fazosi bo'ladimi?

1.63. X chiziqli normalangan fazo, X' uning qism fazosi bo'lsin. $M = \{A \in L(X) : Ker A \supset X'\}$ to'plam $L(X)$ ning qism fazosi bo'ladimi?

1.64. X chiziqli normalangan fazo, $A \in L(X)$ ixtiyoriy element, $N_k = Ker\, A^k$, $k = 0, 1, 2, \ldots$ bo'lsin. Quyidagilarni isbotlang.

a) $N_0 \subset N_1 \subset \cdots \subset N_k \subset N_{k+1} \subset \cdots$ munosablar o'rinli.

b) faraz qilaylik, biror $m \in \mathbb{N}$ soni uchun $N_m = N_{m+1}$ bo'lsin. U holda barcha $p \in \mathbb{N}$ uchun $N_{m+p} = N_m$ tenglik o'rinli.

1.65. X chiziqli normalangan fazo, $A \in L(X)$ tayinlangan element bo'lsin. $AB = BA$ shartni qanoatlantiruvchi barcha $B \in L(X)$ lar to'plami $L(X)$ ning qism fazosi bo'ladimi?

1.66. X chiziqli normalangan fazo, $A \in L(X)$ tayinlangan element bo'lsin. $AB = 0$ shartni qanoatlantiruvchi barcha $B \in L(X)$ lar to'plami $L(X)$ ning qism fazosi bo'ladimi?

1.67. H Hilbert fazosi, $A_n \in L(H)$, $n \in \mathbb{N}$ va har bir $x, y \in H$ uchun $\sup\limits_{n \in \mathbb{N}} |(A_n x, y)| < \infty$ tengsizlik o'rinli bo'lsin. U holda $\sup\limits_{n \in \mathbb{N}} \|A_n\| < \infty$ tengsizlik ham o'rinli. Isbotlang.

1.68. X, Y lar Banax fazolari, $A_n \in L(X, Y)$ $(n \in \mathbb{N})$ va har bir $x \in X$ da $A_n x$ ketma-ketlik fundamental bo'lsin. U holda shunday $A \in L(X, Y)$ operator mavjud bo'lib, $\{A_n\}$ operatorlar ketma-ketligi A operatorga kuchli ma'noda yaqinlashadi. Isbotlang.

1.69. $C[-\pi, \pi]$ Banax fazosining $M = \{x \in C[-\pi, \pi] : \ x(-\pi) = x(\pi)\}$ qism fazosini qaraymiz va har bir $x \in M$ uchun

$$(A_n x)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(s)\, ds + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{n} \cos k(t-s) x(s) ds$$

deymiz. Quyidagilarni isbotlang.

a) Quyidagi tenglik o'rinli:

$$(A_n x)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(2n+1)(s-t)]}{\sin[(s-t)/2]}\, x(s)\, ds.$$

b) $A_n \in L(M)$ va quyidagi tenglik o'rinli

$$\|A_n\| = \frac{1}{2\pi} \max_{t \in [-\pi, \pi]} \int_{-\pi}^{\pi} \left| \frac{\sin[(2n+1)(s-t)]}{\sin[(s-t)/2]} \right| ds.$$

c) $\Phi \subset C[-\pi, \pi]$ - trigonometrik ko'phadlardan iborat qism fazo bo'lsin. Φ da A_n operatorlar ketma-ketligi birlik operatorga kuchli ma'noda yaqinlashadi.

1.70. $C[0, 1]$ fazoni o'zini-o'ziga akslantiruvchi A, A_n, B_n operatorlarni quyidagicha aniqlaymiz:

$$(Ax)(t) = \int_0^1 e^{st} x(s)\, ds, \quad (A_n x)(t) = \int_0^1 \left[\sum_{k=0}^{n} \frac{(s\,t)^k}{k!} \right] x(s)\, ds, \ n \in \mathbb{N},$$

$$(B_n x)(t) = \int_{1/n}^{1-1/n} e^{st} x(s) ds, \quad n \in \mathbb{N}.$$

A_n, B_n operatorlar A operatorga yaqinlashadimi? Yaqinlashish xarakterini (tekis, kuchli, kuchsiz) aniqlang.

1.71. $C[0,\ 1]$ ni o'zini-o'ziga akslantiruvchi A_n, $n \in \mathbb{N}$ operatorlarni

$$(A_n x)(t) = x(t^{1+1/n})$$

tenglik yordamida aniqlaymiz. Quyidagilarni isbotlang.

a) Har bir $n \in \mathbb{N}$ uchun $A_n \in L(C[0,\ 1])$;

b) $\{A_n\}$ ketma-ketlik birlik operatorga kuchli yaqinlashadi.

c) $\{A_n\}$ operatorlar ketma-ketligi birlik operatorga tekis yaqinlashmaydi.

1.72. X, Y lar Banax fazolari, $x_n, x \in X$, $x_n \to x$, $A_n, A \in L(X,Y)$. Agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo'lsa, u holda $\lim\limits_{n\to\infty} \|A_n x_n - Ax\| = 0$ munosabatni isbotlang.

1.73. X, Y, Z lar Banax fazolari, $A_n, A \in L(X,Y)$, $B_n, B \in L(Y,Z)$ bo'lib, A_n operatorlar ketma-ketligi A ga, B_n operatorlar ketma-ketligi B ga kuchli ma'noda yaqinlashsin. U holda $B_n \cdot A_n$ operatorlar ketma-ketligi $B \cdot A$ operatorga kuchli ma'noda yaqinlashadi. Isbotlang.

1.74. X, Y, Z lar Banax fazolari, $A_n, A \in L(X,Y)$, $B_n, B \in L(Y,Z)$ bo'lib, A_n operatorlar ketma-ketligi A ga B_n operatorlar ketma-ketligi B ga tekis (norma bo'yicha) yaqinlashsin. U holda $B_n \cdot A_n$ operatorlar ketma-ketligi $L(X,\ Z)$ fazoda $B \cdot A$ operatorga tekis yaqinlashadi.

1.75. Shunday X normalangan fazoga va $A,\ B \in L(X)$ operatorlarga misol keltiringki, $\|A \cdot B\| < \|A\| \cdot \|B\|$ bo'lsin.

1.76. X normalangan fazo va $A \in L(X)$, $B : X \to X$ chegaralanmagan operator bo'lsin. B ning aniqlanish sohasi $D(B)$ X ning hamma yerida zich bo'lsin. $A \cdot B$ va $B \cdot A$ larning chegaralangan, chegaralanmagan hollariga misollar keltiring.

1.77. H Hilbert fazosi, $L \subset H$ uning qism fazosi bo'lsin. $P : H \to H$, $Px = u$, $x = u+v$, $u \in L$, $v \in L^{\perp}$ operator L ga *ortogonal proyeksiyalash* operatori deyiladi. P ning chiziqli chegaralangan ekanligini ko'rsatib, normasini toping.

1.78. $L_2[-1, 1]$ Hilbert fazosida A, B operatorlarni quyidagicha aniqlaymiz:

$$(Ax)(t) = \frac{1}{2}[x(t) + x(-t)], \qquad (Bx)(t) = \frac{1}{2}[x(t) - x(-t)].$$

a) $R(A)$, $R(B)$ to'plamlarni tavsiflang. Ular $L_2[-1, 1]$ ning yopiq qism fazolari bo'ladimi?

b) A, B operatorlarning chiziqli chegaralangan ekanligini ko'psatib, normalarini toping.

c) A^2, B^2 operatorlarni toping. A va B operatorlar ortogonal proyeksiyalash operatorlari bo'ladimi?

d) $A \cdot B$ va $B \cdot A$ operatorlarni toping.

1.79. H Hilbert fazosi, L_1, $L_2 \subset H$ uning qism fazolari bo'lsin. P_1, P_2 lar mos ravishda L_1, L_2 larga ortogonal proyeksiyalash operatorlari bo'lsa, $\|P_1 - P_2\| \leq 1$ ekanligini isbotlang.

1.80. Agar $A \cdot B = 0$ bo'lsa, A va B operatorlar *ortogonal* deyiladi. H Hilbert fazosi, L_1. $L_2 \subset H$ uning qism fazolari, P_1, P_2 lar mos ravisbda L_1, L_2 larga ortogonal proyeksiyalash operatorlari bo'lsin. $P_1 \cdot P_2 = 0$ bo'lishi uchun L_1 va L_2 qism fazolar o'zaro ortogonal bo'lishi zarur va yetarli. Isbotlang.

2- §. Teskari operatorlar

Biz bu paragrafda o'zaro bir qiymatli chiziqli akslantirishlarni qaraymiz. X va Y lar Banax fazolari, A esa X ni Y ga akslantiruvchi chiziqli operator, $D(A)$ – uning aniqlanish sohasi, ImA esa uning qiymatlar sohasi bo'lsin.

2.1-ta'rif. *Agar ixtiyoriy $y \in ImA$ uchun $Ax = y$ tenglama yagona yechimga ega bo'lsa, u holda A teskarilanuvchan operator deyiladi.*

2.2-ta'rif. *Agar A teskarilanuvchan operator bo'lsa, u holda ixtiyoriy $y \in ImA$ ga $Ax = y$ tenglamaning yechimi bo'lgan yagona $x \in D(A)$ element mos keladi. Bu moslikni o'rnatuvchi operator A operatorga teskari operator deyiladi va A^{-1} bilan belgilanadi.*

Teskari operator ta'rifidan quyidagilar kelib chiqadi:

$$A^{-1} : Y \to X, \ \ D(A^{-1}) = ImA, \ ImA^{-1} = D(A).$$

Bundan tashqari teskari operator uchun

$$A^{-1}Ax = x, \ x \in D(A), \quad AA^{-1}y = y, \ y \in D(A^{-1}) \qquad (2.1)$$

tengliklar o'rinli.

$A : X \to X$ chiziqli operator bo'lsin. Agar biror $B \in L(X)$ operator uchun $BA = I$ bo'lsa, u holda B operator A operatorga *chap teskari operator* deyiladi. Xuddi shunday, $AC = I$ tenglik bajarilsa, C operator A ga *o'ng teskari operator* deyiladi. Adabiyotlarda A ga chap teskari operator A_l^{-1}, o'ng teskari operator esa A_r^{-1} orqali belgilanadi.

2.1-lemma. *Agar A operator uchun ham chap teskari, ham o'ng teskari operatorlar mavjud bo'lsa, u holda ular o'zaro teng.*

Isbot. A uchun A_l^{-1} chap teskari, A_r^{-1} o'ng teskari operatorlar bo'lsin, u holda

$$A_l^{-1} = A_l^{-1}I = A_l^{-1}(A\,A_r^{-1}) = (A_l^{-1}A)A_r^{-1} = IA_r^{-1} = A_r^{-1}. \ \square$$

Ma'lumki, agar A uchun bir vaqtda ham o'ng teskari, ham chap teskari operatorlar mavjud bo'lsa, u holda A teskarilanuvchan operator bo'ladi va $A^{-1} = A_l^{-1} = A_r^{-1}$ tenglik o'rinli.

2.1-teorema. *Chiziqli operatorga teskari operator chiziqlidir.*

2.2-teorema *(Teskari operator haqida Banax teoremasi). A operator X Banax fazosini Y Banax fazosiga biyektiv akslantiruvchi chiziqli chegaralangan operator bo'lsin. U holda A^{-1} operator mavjud va chegaralangan.*

Hozir biz chiziqli operator teskarilanuvchan bo'lishligining zarur va yetarli shartini keltiramiz.

2.3-teorema. *$A : X \to Y$ chiziqli operator teskarilanuvchan bo'lishi uchun $Ax = \theta$ tenglama faqat $x = \theta$ yechimga ega bo'lishi zarur va yetarli.*

Endi chegaralangan teskari operator mavjud bo'lishligining zarur va yetarli shartini keltiramiz.

2.4-teorema. *$Im A$ da A ga chegaralangan teskari operator mavjud bo'lishi uchun, shunday $m > 0$ son mavjud bo'lib, barcha $x \in D(A)$ larda*

$$\|Ax\| \geq m \|x\| \tag{2.2}$$

tengsizlikning bajarilishi zarur va yetarli.

Teskari operatorni topishda foydali bo'lgan quyidagi ikki teoremani keltiramiz.

2.5-teorema. *X – Banax fazosi va $A \in L(X)$. Agar $\|A\| < 1$ bo'lsa, u holda $I - A$ operator uchun chegaralangan teskari operator mavjud.*

2.2-lemma. *Agar $A, B \in L(X)$ bo'lib, $A^{-1}, B^{-1} \in L(X)$ bo'lsa, u holda AB operatorga chegaralangan teskari operator mavjud va $(AB)^{-1} = B^{-1}A^{-1}$ tenglik o'rinli.*

Lemmaning isboti $ABB^{-1}A^{-1} = I$, $B^{-1}A^{-1}AB = I$ tengliklardan hamda 2.1-lemmadan kelib chiqadi.

2-§. Teskari operatorlar

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Hozir biz chiziqli operator teskarilanuvchan bo'lishligining zarur va yetarli shartini keltiramiz.

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tengsizlikning bajarilishi zarur va yetarli.

Teskari operatorni topishda foydali bo'lgan quyidagi ikki teoremani keltiramiz.

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2.2-lemma. *Agar $A, B \in L(X)$ bo'lib, $A^{-1}, B^{-1} \in L(X)$ bo'lsa, u holda AB operatorga chegaralangan teskari operator mavjud va $(AB)^{-1} = B^{-1}A^{-1}$ tenglik o'rinli.*

Lemmaning isboti $ABB^{-1}A^{-1} = I$, $B^{-1}A^{-1}AB = I$ tengliklardan hamda 2.1-lemmadan kelib chiqadi.

2.6-teorema. $A \in L(X)$ *operatorga chegaralangan teskari operator mavjud bo'lsin. Agar* $A' : X \to X$ *operatorning normasi*

$$\|A'\| < \frac{1}{\|A^{-1}\|}$$

tengsizlikni qanoatlantirsa, u holda $B = A - A'$ *operatorga chegaralangan teskari operator mavjud.*

2.1. $A : \ell_2 \to \ell_2$, $Ax = (0, x_1, x_2, \ldots, x_{n+1}, \ldots)$ operatorga chap teskari operatorni toping. A *o'ngga siljitish operatori* deyiladi.

Yechish. $B : \ell_2 \to \ell_2$ bilan *chapga siljitish operatorini* belgilaymiz:

$$Bx = (x_2, x_3, \ldots, x_{n+1}, \ldots).$$

Endi BA operatorning $x \in \ell_2$ elementga ta'sirini qaraymiz.

$$BAx = B(Ax) = B(0, x_1, x_2, \ldots, x_{n-1}, \ldots) = (x_1, x_2, \ldots, x_n, \ldots) = Ix.$$

Demak, B operator A ga chap teskari operator ekan, ya'ni $B = A_l^{-1}$.
□

2.2. 2.1-misolda keltirilgan o'ngga siljitish operatori $A : \ell_2 \to \ell_2$ ga o'ng teskari operator mavjudmi?

Yechish. Faraz qilaylik, A ga o'ng teskari operator mavjud bo'lsin, u holda 2.1-lemmaga ko'ra $A_r^{-1} = B = A_l^{-1}$ bo'ladi, ya'ni

$$A_r^{-1}x = (x_2, x_3, \ldots, x_{n+1}, \ldots).$$

Endi AA_r^{-1} operatorning nolmas $x \in \ell_2$ elementga ta'sirini qaraymiz:

$$AA_r^{-1}x = A(A_r^{-1}x) = A(x_2, x_3, \ldots, x_{n+1}, \ldots) = (0, x_2, x_3, \ldots, x_n, \ldots) \neq Ix.$$

Demak, A uchun o'ng teskari operator mavjud emas ekan. □

2.3. $\ell_2(\mathbb{Z})$ Hilbert fazosida o'ngga siljitish operatori

$$A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Af)(n) = f(n-1)$$

ni qaraymiz. Uning uchun o'ng va chap teskari operatorlar mavjudligini isbotlang.

2.4. $L_2[0,\ 1]$ fazoda x ga ko'paytirish operatorini, ya'ni

$$A : L_2[0,\ 1] \to L_2[0,\ 1], \quad (Af)(x) = xf(x) \tag{2.3}$$

operatorni qaraymiz. Bu operator 2.3-teorema shartlarini qanoatlantiradimi? A teskarilanuvchan operator bo'ladimi?

2.5. (2.3) tenglik bilan aniqlangan operator 2.4-teorema shartlarini qanoatlantiradimi? A ga chegaralangan teskari operator mavjudmi?

2.6. Hilbert fazosi $L_2[-1,\ 1]$ ni o'zini-o'ziga akslantiruvchi

$$A : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (Af)(x) = \cos x\, f(x)$$

operatorni qaraymiz. A operator 2.4-teorema shartlarini qanoatlantiradimi? A ga chegaralangan teskari operator mavjudmi?

2.7. $A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(s) = \int\limits_0^1 e^{s+t} x(t)\, dt + x(s)$,

operator berilgan. Operator teskarilanuvchanmi? Agar teskarilanuvchan bo'lsa, teskari operatorni toping.

Yechish. Dastlab berilgan operatorning teskarilanuvchanligini tekshiramiz. 2.3-teoremaga ko'ra, A operator teskarilanuvchan bo'lishi uchun $Ax = 0$ tenglama faqat nol yechimga ega bo'lishi zarur va yetarli. $Ax = 0$ tenglamani qaraymiz, ya'ni

$$\int\limits_0^1 e^{s+t} x(t)\, dt + x(s) = 0 \quad \text{yoki} \quad x(s) = -c(x)\, e^s, \tag{2.4}$$

bu yerda

$$c(x) = \int_0^1 e^t x(t)\,dt. \tag{2.5}$$

Endi (2.4) ni (2.5) ga qo'ysak,

$$c(x) = -c(x)\int_0^1 e^{2t}dt = -\frac{1}{2}c(x)\left(e^2 - 1\right) \quad \text{yoki} \quad \frac{1}{2}\left(e^2 + 1\right) c(x) = 0$$

tenglikka ega bo'lamiz. Bundan $c(x) = 0$. (2.4) dan esa $x(s) \equiv 0$ ekanligiga ega bo'lamiz. Demak, $Ax = 0$ tenglama faqat $x = 0$ yechimga ega, shuning uchun A teskarilanuvchan operator. A^{-1} ni topish maqsadida ixtiyoriy $y \in C[0,\ 1]$ element uchun $Ax = y$ tenglamani, ya'ni

$$\int_0^1 e^{s+t}x(t)\,dt + x(s) = y(s) \qquad \text{yoki} \qquad x(s) = y(s) - c(x)\,e^s \tag{2.6}$$

tenglamani yechamiz. Bu yerda $c(x)$ (2.5) ko'rinishga ega. $x(s)$ uchun olingan (2.6) ifodani (2.5) ga qo'ysak,

$$c(x) = \int_0^1 e^t y(t)\,dt - c(x)\int_0^1 e^{2t}\,dt = \int_0^1 e^t y(t)\,dt - \frac{1}{2}c(x)\left(e^2 - 1\right)$$

ni olamiz. Bundan

$$c(x) = \frac{2}{e^2 + 1}\int_0^1 e^t y(t)\,dt \tag{2.7}$$

ni olamiz. $c(x)$ uchun olingan (2.7) ifodani (2.6) ga qo'ysak, $Ax = y$ tenglama yechimi quyidagi ko'rinishni oladi:

$$x(s) = y(s) - \frac{2}{e^2 + 1}\int_0^1 e^{s+t}y(t)\,dt.$$

Demak, har bir $y \in C[0,\ 1]$ elementga $Ax = y$ tenglama yechimini mos qo'yuvchi A^{-1} operator quyidagi formula yordamida aniqlanar ekan:

$$A^{-1} : C[0,\ 1] \to C[0,\ 1],\ (A^{-1}x)(s) = x(s) - \frac{2}{e^2+1}\int_0^1 e^{s+t}x(t)\,dt. \square$$

2.8-2.27-misollarda berilgan operatorlarning teskarilanuvchan ekanligini ko'rsating va ularga teskari operatorlarni toping.

2.8. $A : \mathbb{R}^3 \to \mathbb{R}^3,\quad Ax = (x_1 + x_3,\ x_1 + x_2, x_2 + x_3).$

2.9. $A : \mathbb{R}^3 \to \mathbb{R}^3,\quad Ax = (x_1, x_1 - x_2, x_2 - x_3)\,.$

2.10. $A : \mathbb{R}^4 \to \mathbb{R}^4,\quad Ax = (x_2, x_3, x_4, x_1)\,.$

2.11. $A : \mathbb{R}^5 \to \mathbb{R}^5,\quad Ax = (x_1 + x_2 + x_3, x_1 - 2x_2, x_3, x_4, x_5)\,.$

2.12. $A : \mathbb{R}^7 \to \mathbb{R}^7,\quad Ax = (x_1 - x_2, x_1 + x_2, x_2 + x_3, x_4, x_5, x_6, x_7)\,.$

2.13. $A : \ell_2 \to \ell_2,\ Ax = (x_1, x_2 + x_3, x_1 - x_2 + x_3, x_4, x_5, \dots)\,.$

2.14. $A : \ell_2 \to \ell_1,\quad Ax = \left(x_1, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n, \dots\right).$

2.15. $A : \ell_1 \to \ell_1,\ Ax = (x_1, x_2, x_2 + x_3, x_3 + x_4, x_3 + x_4 + x_5, x_6, x_7, \dots)\,.$

2.16. $A : m \to \ell_2,\quad Ax = \left(x_1, \frac{1}{\sqrt{2}}x_2, \dots, \frac{1}{\sqrt{n}}x_n, \dots\right).$

2.17. $A : \ell_2 \to \ell_2,\quad Ax = \left(x_1, \frac{1}{2}x_2, \frac{2}{3}x_3, \dots, \frac{n-1}{n}x_n, \dots\right).$

2.18. $A :\ \ell_1 \to C[0,\ 1],\ (Ax)(t) = \sum\limits_{k=1}^{\infty} x_k \sin 2\pi k t.$

2.19. $A : m \to C[0,\ 1],\ (Ax)(t) = \sum\limits_{k=1}^{\infty} \frac{x_k}{k^2}\cos 2\pi kt.$

2.20. $A : C_0^{(1)}[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x'(t)\,.$

2.21. $A : C[0,\ 1] \to C[0,\ 1]$, $(Ax)(t) = \int\limits_0^t s\,x(s)\,ds.$

2.22. $A : C[0,\ 1] \to C[0,\ 1]$, $(Ax)(t) = (t+2)x(t) + \int\limits_0^1 s x(s)\,ds.$

2.23. $A : C[0,\ 1] \to C[0,\ 1]$, $(Ax)(t) = (1+t)\,x(t) + \int\limits_0^1 tsx(s)\,ds.$

2.24. $A : C[0,\ \pi] \to C[0,\ \pi]$, $(Ax)(t) = (\sin t + 1)x(t)$.

2.25. $A : C[0,\ 2] \to C[0,\ 2]$, $(Ax)(t) = (t+1)x(t) + x(1)\,t + x(0)$.

2.26. $A : C[0,\ 1] \to C[0,\ 1]$, $(Ax)(t) = t^2 x(t) + x(1)$.

2.27. $A : C[0,\ \pi] \to C[0,\ \pi]$, $(Ax)(t) = x(t) + \int\limits_0^\pi \sin t \cdot \sin s\, x(s)\,ds.$

2.9-misolning yechimi. Ma'lumki.

$$Ax = 0 \iff (x_1, x_1 - x_2, x_2 - x_3) = (0, 0, 0)$$

tenglama yagona $x = 0$ yechimga ega. 2.3-teoremaga ko'ra, A^{-1} operator mavjud. Teskari operatorni topish maqsadida

$$Ax = y \iff (x_1, x_1 - x_2, x_2 - x_3) = (y_1, y_2, y_3)$$

tenglamadan $x = (x_1, x_2, x_3)$ ni topamiz. Bu tenglama

$$\begin{cases} x_1 = y_1 \\ x_1 - x_2 = y_2 \\ x_2 - x_3 = y_3 \end{cases}$$

sistemaga teng kuchli. Uning yechimi $x_1 = y_1$, $x_2 = y_1 - y_2$, $x_3 = y_1 - y_2 - y_3$ dan iborat. Shunday qilib, teskari operator

$$A^{-1}y = (y_1, y_1 - y_2, y_1 - y_2 - y_3)$$

tenglik bilan aniqlanar ekan. □

2.28. Quyidagi operatorning teskarilanuvchan emasligini ko'rsating:

$$A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = x(0)\ t + x(1)\ t^2. \qquad (2.8)$$

Yechish. 2.3-teoremaga ko'ra, A chiziqli operator teskarilanuvchan bo'lishi uchun $Ax = 0$ tenglama faqat $x = 0$ yechimga ega bo'lishi zarur va yetarli. (2.8) formula bilan berilgan operator uchun $x_0(t) = t(t-1) \neq 0$ funksiyani olsak, $x_0(0) = x_0(1) = 0$ bo'lganligi uchun

$$(Ax_0)(t) = x_0(0)\, t + x_0(1)\, t^2 = 0, \quad \forall t \in [0,\ 1].$$

Demak, $Ax = 0$ tenglama nolmas yechimga ega. Shunday ekan 2.3-teoremaga ko'ra, A teskarilanuvchan operator emas. □

2.29-2.48-misollarda keltirilgan operatorlarning teskarilanuvchan emasligini ko'rsating.

2.29. $A : \mathbb{R}^4 \to \mathbb{R}^4, \quad Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_1 + x_4)$.

2.30. $A : \ell_2 \to \ell_2, \quad Ax = (x_1, 0, x_3, 0, x_5, 0, x_7, x_8, x_9, \ldots)$.

2.31. $A : \ell_1 \to \ell_1, \quad Ax = (0, x_2, 0, x_4, 0, x_6, x_7, x_8, x_9, \ldots)$.

2.32. $A : \ell_1 \to \ell_1, \quad Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_1 + x_4, x_5, x_6, \ldots)$.

2.33. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Af)(n) = \chi_{\{-1,0,1\}}(n) f(n)$.

2.34. $A : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}), \quad (Af)(n) = (1 - \chi_{\mathbb{N}}(n)) f(n)$.

2.35. $A : m \to m, \quad Ax = (x_1, x_2, 0, 0, 0, x_6, x_7, x_8, \ldots)$.

2.36. $A : AC[0,\ 1] \to L_1[0,\ 1], \quad (Ax)(t) = x'(t)$.

2.37. $A : C^{(2)}[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = x''(t)$.

2.38. $A : C[-1,\ 1] \to C[-1,\ 1], \quad (Ax)(t) = \int_{-1}^{1} ts\, x(s)\, ds$.

2.39. $A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = x(0) \cdot (t^2 + 1) + x(1)(t^2 + 3t + 2)$.

2.40. $A : C^{(1)}[0, 1] \to C[0, 1]$, $(Ax)(t) = x(0) + x'(t)$.

2.41. $A : C[0, 1] \to C[0, 1]$, $(Ax)(t) = (t^2 + t + 1)\int\limits_0^1 s\,x(s)\,ds$.

2.42. $A : C[0, 2] \to C[0, 2]$, $(Ax)(t) = x(0) + x(1)\,t + x(2)\,t^2$.

2.43. $A : C^{(1)}[0, 1] \to C^{(1)}[0, 1]$, $(Ax)(t) = \int\limits_0^t x'(s)\,ds + [x(0) - x(1)]\,t$.

2.44. $A : C^{(1)}[0, 1] \to C[0, 1]$, $(Ax)(t) = x'(t) + 2x(t)$.

2.45. $A : C[0, \pi] \to C[0, \pi]$, $(Ax)(t) = (\sin t + \cos t)x(0) - \cos 2t\, x(\pi)$.

2.46. $A : C^{(1)}[0, 1] \to C[0, 1]$, $(Ax)(t) = x'(t) - \dfrac{x(0)}{t^2 + 1}$.

2.47. $A : L_2[-1, 1] \to L_2[-1, 1]$, $(Ax)(t) = \int\limits_{-1}^1 t^2(s^2 + 1)\,x(s)\,ds$.

2.48. $A : L_2[-1, 1] \to L_2[-1, 1]$, $(Ax)(t) = \cos t \int\limits_{-1}^1 \sin s\, x(s)\,ds$.

2.31-misolning yechimi. Yuqorida keltirilgan 2.29-2.48-misollarni 2.3-teoremadan foydalanib yechish qulay. $e_1 = (1, 0, 0, \ldots, 0, \ldots)$ noldan farqli element uchun $Ae_1 = 0$ tenglik o'rinli. 2.3-teoremaga ko'ra, berilgan operatorga teskari operator mavjud emas. □

2.38-misolning yechimi. Bu misolni yechishda ham 2.3-teoremadan foydalanamiz. $x_0(s) \equiv 1$ noldan farqli element uchun

$$(Ax_0)(t) = \int\limits_{-1}^{1} ts\,ds = t\,\frac{s^2}{2}\Big|_{-1}^{1} = \frac{t}{2}(1^2 - (-1)^2) = 0$$

tenglik o'rinli. Demak, $Ax = 0$ tenglama nolmas $x_0 \in C[-1, 1]$ yechimga ega. 2.3-teoremaga ko'ra, A ga teskari operator mavjud emas. □

2.49. X, Y chiziqli normalangan fazolar, $A : X \to Y$ teskarilanuvchan chiziqli operator bo'lsin. U holda $x_1, x_2, \ldots, x_n \in D(A)$ va $Ax_1, Ax_2,$

$\dots, Ax_n$ elementlar sistemasi bir vaqtda yo chiziqli bog'langan, yo chiziqli bog'lanmagan bo'ladi. Isbotlang.

2.50. X chiziqli normalangan fazo, $A : X \to X$ chiziqli operator bo'lsin. Agar biror $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ uchun $I + \lambda_1 \cdot A + \lambda_2 \cdot A^2 + \dots + + \lambda_n \cdot A^n = 0$ shart bajarilsa A ga teskari operator mavjudligini isbotlang.

2.51. X chiziqli normalangan fazo, $A, B : X \to X$ chiziqli operatorlar bo'lib, $D(A) = D(B) = X$, hamda $(AB)^{-1}$, $(BA)^{-1}$ operatorlar mavjud bo'lsin. A va B larga teskari operatorlar mavjudmi?

2.52. X chiziqli normalangan fazo, $A : X \to X$ chiziqli operator va $D(A)$ da $\|x_n\| = 1$ va $\lim\limits_{n\to\infty} \|A x_n\| = 0$ shartni qanoatlantiruvchi ketma-ketlik mavjud bo'lsin. U holda A ga chegaralangan teskari operator mavjud emasligini isbotlang.

2.53. $C^{(1)}[0, 1]$ Banax fazosi, $L = \{x \in C^{(1)}[0, 1] : x(0) = 0\}$ uning qism fazosi va $A : L \to C[0, 1]$ chiziqli operatorni

$$(A x)(t) = \frac{dx(t)}{dt} + u(t)\, x(t), \quad u \in C[0, 1]$$

tenglik bilan aniqlaymiz. A operatorning chegaralangan teskarisi mavjudligini isbotlang.

2.54. $A : C^{(1)}[0, 1] \to C[0, 1]$, $(A x)(t) = \dfrac{dx(t)}{dt}$ chiziqli operatorga o'ng teskari operator mavjud, chap teskari operator mavjud emas. Isbotlang.

2.55. $A : C[0, 1] \to C[0, 1]$ operatorni

$$(A x)(t) = \int_0^t x(s)\, ds$$

tenglik bilan aniqlaymiz. Uning qiymatlar sohasi $R(A)$ qanday shartlarni qanoatlantiruvchi funksiyalardan iborat? $R(A)$ da A ga teskari operator mavjudmi? Agar mavjud bo'lsa, u chegaralanganmi?

2.56. $A : C[0, 1] \to C[0, 1]$, $(Ax)(t) = \int\limits_0^t x(s)\,ds + x(t)$ operatorni qaraymiz:

a) $Ker A = \{\theta\}$ tenglikni isbotlang.

b) A ga chegaralangan teskari operator mavjudligini isbotlang. A^{-1} ni toping.

2.57. $A : C[0, \pi] \to C[0, \pi]$, $(Ax)(t) = x(t) + \int\limits_0^\pi \cos(t-s)\,x(s)\,ds$ operatorga teskari operator mavjudligini ko'rsating va uni toping.

2.58. $A : C[0, 1] \to C[0, 1]$ operatorni quyidagicha aniqlaymiz:

$$(Ax)(t) = \frac{d^2x(t)}{dt^2} + x(t), \ \ D(A) = \{x \in C^{(2)}[0, 1] : \ x(0) = x'(0) = 0\}.$$

a) A chegaralanmagan operator. Isbotlang.

b) A ga chegaralangan teskari operator mavjudligini isbotlang va uni toping.

2.59. H Hilbert fazo, $A \in L(H)$, $R(A) = H$ va A ga chegaralangan o'ng teskari A_r^{-1} operator mavjud bo'lsin. U holda A ga chegaralangan teskari operator mavjud. Isbotlang.

3-§. Qo'shma operatorlar

Bu paragrafda bizning asosiy maqsadimiz Banax yoki Hilbert fazolarida aniqlangan operatorlarga qo'shma operatorlarni topish va ularning asosiy xossalarini o'rganishdir. Bundan tashqari biz musbat, normal, unitar, izometrik va giponormal operator tushunchalarini kiritamiz va ularga doir misollar qaraymiz.

X va Y – chiziqli normalangan fazolar, $A : X \to Y$ chiziqli chegaralangan operator bo'lsin. $f : X \to \mathbb{C}$ funksionalning x nuqtadagi qiymatini (f, x) deb belgilaymiz.

3.1-ta'rif. *Agar biror $A^* : Y^* \to X^*$ chiziqli chegaralangan operator va barcha $x \in X$, $g \in Y^*$ lar uchun*

$$(g,\, Ax) = (A^*g,\, x)$$

tenglik o'rinli bo'lsa, A^* operator A ga qo'shma operator deyiladi.

Hilbert fazosida qo'shma operator quyidagicha ta'riflanadi.

3.2-ta'rif. *H Hilbert fazosi va $A \in L(H)$ operator berilgan bo'lsin. Agar biror $A^* : H \to H$ operator va ixtiyoriy $x, y \in H$ lar uchun*

$$(Ax, y) = (x, A^*y)$$

tenglik o'rinli bo'lsa, A^* operator A ga qo'shma operator deyiladi.

3.3-ta'rif. *Agar $A = A^*$ bo'lsa, $A \in L(H)$ o'z-o'ziga qo'shma operator deyiladi.*

3.4-ta'rif. *$A : H \to H$ chiziqli operator va $H_0 \subset H$ qism fazo berilgan bo'lsin. Agar ixtiyoriy $x \in H_0$ uchun $Ax \in H_0$ bo'lsa, u holda H_0 qism fazo A operatorga nisbatan invariant qism fazo deyiladi.*

Endi Hilbert fazosida chegaralanmagan $A : H \to H$ chiziqli operator berilgan va uning aniqlanish sohasining yopig'i $\overline{D(A)} = H$ bo'lsin. $y \in H$ shunday elementki, biror $y^* \in H$ va barcha $x \in D(A)$ lar uchun $(Ax, y) = (x, y^*)$ tenglik o'rinli bo'lsin. $D(A)$ ning H da zichligidan $y^* \in H$ element bir qiymatli aniqlanadi. Bu $y^* = A^*y$ moslikni o'rnatuvchi $A^* : H \to H$ operator *A ga qo'shma operator* deyiladi.

3.5-ta'rif. *Agar $A : H \to H$ va $A_1 : H \to H$ operatorlar uchun $D(A_1) \subset D(A)$ bo'lib $Ax = A_1x$, $x \in D(A_1)$ bo'lsa, u holda A operator A_1 operatorning davomi deyiladi, A_1 esa A operatorning $D(A_1)$ dagi qismi deyiladi, bu holat $A_1 \subset A$ shaklda yoziladi.*

A operatorning grafigi deb $Gr(A) = \{(x, Ax) : x \in D(A)\} \subset H \times H$ to'plamga aytiladi. Agar A operatorning grafigi $Gr(A)$ yopiq bo'lsa, A chiziqli operator *yopiq* deyiladi.

3.6-ta'rif. *Agar $A \subset A^*$ bo'lib, $\overline{D(A)} = H$ bo'lsa, u holda $A : H \to H$ chiziqli operator simmetrik deyiladi. Agar $A = A^*$ bo'lsa, u holda A chiziqli operator o'z-o'ziga qo'shma deyiladi.*

3.7-ta'rif. *O'z-o'ziga qo'shma A operator uchun barcha $x \in D(A)$ larda $(Ax, x) \geq 0$ bo'lsa, A ga musbat operator deyiladi va bu $A \geq 0$ shaklda yoziladi.*

O'z-o'ziga qo'shma A va B operatorlar uchun $A \geq B$ yozuvi $A - B \geq 0$ ekanligini anglatadi.

3.8-ta'rif. *Agar $A \geq 0$ operator uchun shunday $B \geq 0$ operator mavjud bo'lib, $B^2 = A$ bo'lsa, B operator A operatorning musbat kvadrat ildizi deyiladi va $B = A^{\frac{1}{2}}$ shaklda belgilanadi.*

Hilbert fazolarida aniqlangan o'z-o'ziga qo'shma operatorlarning muhim sinfi bo'lgan proyeksiyalash operatorlariga ta'rif beramiz. H Hilbert fazosi L uning biror qism fazosi bo'lsin. U holda har bir $x \in H$ element yagona usul bilan quyidagicha tasvirlanadi,

$$x = y + z, \quad \text{bu yerda} \quad y \in L, \ \ z \in L^{\perp}.$$

3.9-ta'rif. *Har bir $x \in H$ ga $Px = y$ ni mos qo'yib, H ning hamma yerida aniqlangan P operatorni hosil qilamiz. Uning qiymatlar sohasi L bo'ladi. Shuni ta'kidlash kerakki, agar $x \in L$ bo'lsa $x = y$ va $z = 0$ bo'ladi. Bu operator proyeksiyalash operatori yoki L ning ustiga ortogonal proyeksiyalash operatori deyiladi. Zarurat bo'lgan holda P_L ko'rinishida ham belgilanadi.*

3.10-ta'rif. *Agar ikkita P_1 va P_2 proyeksiyalash operatorlari uchun $P_1 P_2 = 0$ bo'lsa, ular o'zaro ortogonal deyiladi.*

$P_1 P_2 = 0$ shart $P_2 P_1 = 0$ shartga teng kuchli, chunki $(P_1 P_2)^* = P_2 P_1 = 0$ bo'ladi va teskarisi ham o'rinli.

3.11-ta'rif. *Agar P_1 va P_2 proyeksiyalash operatorlari uchun $P_1 P_2 = P_2$ bo'lsa, P_2 proyeksiyalash operatori P_1 proyeksiyalash operatorining qismi deyiladi.*

Bu ta'rifdan bevosita kelib chiqadiki, P_{L_2} proyeksiyalash operatori P_{L_1} proyeksiyalash operatorining qismi bo'lishi uchun L_2 qism fazo L_1 qism fazoning qismi bo'lishi zarur va yetarli.

3.12-ta'rif. *Agar $AB = BA$ bo'lsa, A va B operatorlar o'rin almashinuvchi operatorlar deyiladi. $[A, B] = AB - BA$ operatorga A va B operatorlarning kommutatori deyiladi.*

Demak, o'rin almashinuvchi A va B operatorlarning kommutatori nolga teng bo'ladi va aksincha.

3.13-ta'rif. *Agar $[A, A^*] = 0$ bo'lsa, A ga normal operator deyiladi.*

3.14-ta'rif. *Agar barcha $x \in H$ lar uchun $\|A^*x\| \leq \|Ax\|$ bo'lsa, A ga giponormal operator deyiladi.*

3.15-ta'rif. *Agar $AA^* = A^*A = I$ bo'lsa, A unitar, agar barcha $x \in H$ uchun $\|Ax\| = \|x\|$ bo'lsa, A izometrik operator deyiladi. Agar $A : H \to H$ $(H = L \oplus L^\perp)$ operator L da izometrik bo'lib, barcha $x \in L^\perp$ lar uchun $Ax = 0$ bo'lsa, A qisman izometrik operator deyiladi.*

Unitar va izometrik operator ta'riflarini, H_1 Hilbert fazosini H_2 Hilbert fazosiga akslantiruvchi $U : H_1 \to H_2$ operatorlar uchun ham keltirish mumkin.

3.16-ta'rif. *Agar $U : H_1 \to H_2$ biyektiv akslantirish bo'lib, barcha $x \in H$ uchun $\|Ux\| = \|x\|$ bo'lsa, U ga unitar operator deyiladi. Agar $U : H_1 \to H_2$ inyektiv akslantirish bo'lib, barcha $x \in H$ uchun $\|Ux\| = \|x\|$ bo'lsa, U ga izometrik operator deyiladi.*

3.17-ta'rif. *Agar shunday C teskarilanuvchan operator mavjud bo'lib, $A = C^{-1}BC$ bo'lsa, A va B operatorlar o'xshash deyiladi.*

3.1. $T \in L(\ell_1)$ o'ngga siljitish operatori, ya'ni

$$Tx = T(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots), \quad x \in \ell_1$$

bo'lsin. T ga qo'shma T^* operatorni toping.

Yechish. Ma'lumki, $T \in L(X, Y)$ operatorning Banax qo'shmasi hamma $x \in X$ va $f \in Y^*$ lar uchun

$$(T^* f)(x) = f(Tx) \tag{3.1}$$

tenglikni qanoatlantiruvchi va Y^* fazoni X^* fazoga akslantiruvchi T^* operatordan iborat. Bizga ma'lumki, $\ell_1^* \cong m$, boshqacha aytganda har qanday $f \in \ell_1^*$ uchun shunday yagona $y \in m$ mavjudki,

$$f(x) = \sum_{k=1}^{\infty} x_k y_k, \quad y = (y_1, y_2, \ldots, y_n, \ldots) \in m, \tag{3.2}$$

tenglik ixtiyoriy $x \in \ell_1$ lar uchun o'rinli bo'ladi. Xuddi shuningdek, shunday $\xi \in m$ mavjudki,

$$(T^* f)(x) = \sum_{k=1}^{\infty} x_k \xi_k, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) \in m, \tag{3.3}$$

tenglik ixtiyoriy $x \in \ell_1$ lar uchun bajariladi. (3.2) va (3.3) tengliklarni hisobga olsak, berilgan T operator uchun (3.1) shart quyidagi ko'rinishga keladi:

$$\sum_{k=1}^{\infty} x_k \xi_k = \sum_{k=2}^{\infty} x_{k-1} y_k = \sum_{k=1}^{\infty} x_k y_{k+1}. \tag{3.4}$$

Bu tenglik barcha $x \in \ell_1$ lar uchun bajariladi. Xususiy holda, $e_k \in \ell_1$, $k = 1, 2, 3, \ldots$ elementlar uchun (3.4) tenglik $\xi_k = y_{k+1}$, $k = 1, 2, 3, \ldots$ tengliklarga aylanadi. Shunday qilib, $T^* : m \to m$ operator

$$T^* y = T^*(y_1, y_2, \ldots, y_n, \ldots) = (y_2, y_3, \ldots, y_{n+1}, \ldots)$$

formula bilan aniqlanar ekan. □

Ma'lumki, agar $T \in L(X, Y)$ bo'lsa, $T^* \in L(Y^*, X^*)$ bo'ladi va

$$\|T^*\| = \|T\| \tag{3.5}$$

tenglik o'rinli. Qaralayotgan misolda bu tasdiqning bajarilishini tekshirib ko'ramiz. T^* operatorning chiziqli ekanligi uning aniqlanishidan ko'rinib turibdi. (3.5) tenglik ham bajariladi. Haqiqatan ham,

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|\leq 1} \left(\sum_{k=1}^{\infty} |x_k|\right) = 1,$$

$$\|T^*\| = \sup_{\|y\|\leq 1} \|T^*y\| = \sup_{\|y\|\leq 1} \sup_{2\leq k<\infty} |y_k| = 1.$$

3.2-3.11-misollarda Banax fazosida berilgan $T \in L(X, Y)$ operatorga qo'shma T^* operatorni toping.

3.2. $T : \mathbb{C}^4 \to \mathbb{C}^3,\ Tx = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3 + \lambda_4 x_4)$.

3.3. $T : \mathbb{C}^4 \to \mathbb{C}^4,\ Tx = (0, x_1, 2x_2, (2+i)x_3)$.

3.4. $T : c_0 \to c_0,\ Tx = (\lambda_2 x_2, \lambda_3 x_3, \ldots, \lambda_n x_n, \ldots),\ |\lambda_n| \leq 5,\ \forall n \in \mathbb{N}$.

3.5. $T : \ell_1 \to c_0,\ Tx = (0, \mu_1 x_1, \mu_2 x_2, \ldots, \mu_n x_n, \ldots),\ |\mu_n| \leq 1,\ \forall n \in \mathbb{N}$.

3.6. $T : c_0 \to \ell_1,\quad Tx = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$.

3.7. $T : \ell_1 \to \ell_1,\quad Tx = (\underbrace{0, \ldots, 0}_{n-1}, x_1, 0, 0, \ldots)$.

3.8. $T : \ell_1 \to c_0,\ Tx = (e^i x_1, e^{2i} x_2, \ldots, e^{in} x_n, \ldots)$.

3.9. $T : \ell_1 \to \ell_2,\ Tx = \left(0, \frac{1}{2}x_1, \frac{2}{3}x_2, \ldots, \frac{n-1}{n}x_{n-1}, \ldots\right)$.

3.10. $T : m \to m,\quad Tx = (0, 0, x_1, 2x_2, \ldots, 50x_{50}, 0, 0, \ldots)$.

3.11. $T : \ell_3 \to \ell_3,\quad Tx = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots),\ |\lambda_n| \leq 2,\ n \in \mathbb{N}$.

3.12. $\ell_2 = \left\{x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{k=1}^{\infty} |x_k|^2 < \infty\right\}$ kompleks Hilbert fazosida $(Tx)_n = x_{n+1}$, ya'ni $Tx = (x_2, x_3, \ldots, x_{n+1}, \ldots)$ operator berilgan bo'lsin. T^* operatorni toping.

Yechish. ℓ_2 fazo quyidagi $(x,y) = \sum_{k=1}^{\infty} x_k \overline{y_k}$ skalyar ko'paytmaga nisbatan Hilbert fazosi bo'ladi. Shuning uchun 3.2-ta'rifdan foydalanamiz.

$$(Tx, y) = \sum_{k=1}^{\infty} (Tx)_k \overline{y_k} = \sum_{k=1}^{\infty} x_{k+1} \overline{y_k} =$$

$$= \sum_{k=2}^{\infty} x_k \overline{y_{k-1}} = (x, T^* y) = \sum_{k=1}^{\infty} x_k \overline{(T^* y)_k}.$$

Bu tenglik barcha $x \in \ell_2$ lar uchun o'rinli. Bundan esa T^* operatorning

$$(T^* y)_1 = 0, \quad (T^* y)_k = y_{k-1}, \quad k = 2, 3, \ldots,$$

yoki

$$T^* y = T^* (y_1, y_2, \ldots, y_n, \ldots) = (0, y_1, y_2, \ldots, y_n, \ldots)$$

formula bilan aniqlanishini ko'ramiz.

$$(Tx)_n = x_{n+1} = x_{n-1} = (T^* x)_n, \quad n = 1, 2, 3, \ldots$$

tenglik ℓ_2 fazoning faqat nol vektori uchun bajariladi. Shu sababli T operator o'z-o'ziga qo'shma operator bo'la olmaydi. □

3.13-3.18-misollarda kompleks Hilbert fazosida berilgan $T \in L(H)$ operatorga qo'shma T^* operatorni toping.

3.13. $T : \ell_2 \to \ell_2,\ Tx = (2x_1, i x_2, (1+i)x_3, 0, 0, \ldots)$.

3.14. $T : \ell_2 \to \ell_2,\ Tx = (\lambda_1 x_1, \ldots, \lambda_n x_n, \ldots),\ \lambda = \{\lambda_n\} \in \ell_2$.

3.15. $T : \ell_2 \to \ell_2,\ Tx = (x_1 + x_3, x_2 + x_4, \ldots, x_n + x_{n+2}, \ldots)$.

3.16. $T : \ell_2 \to \ell_2,\ Tx = (x_1 + 2x_2 + x_3, \ldots, x_n + 2x_{n+1} + x_{n+2}, \ldots)$.

3.17. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\ (Tf)(n) = f(n-1) + f(n+1)$.

3.18. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\ (Tf)(n) = f(n-1) - 2f(n) + f(n+1)$.

3.19. $K(s,t)$ funksiya kvadrati $[a,\ b]\times[a,\ b]$ da integrallanuvchi bo'lsin. Quyidagilarni isbotlang.

a) $T: L_2[a,\ b]\to L_2[a,\ b],\ (Tx)(t)=\int\limits_a^b K(t,s)f(s)ds$ operatorga qo'shma operator

$$T^*: L_2[a,\ b]\to L_2[a,\ b],\ (T^*x)(t)=\int\limits_a^b \overline{K(s,t)}\, f(s)ds$$

formula bilan aniqlanadi.

b) $T=T^*$ bo'lishi uchun deyarli barcha $s,t\in[a,\ b]$ larda $K(t,s)=\overline{K(s,t)}$ tenglikning bajarilishi zarur va yetarli.

3.20. Kompleks Hilbert fazosida berilgan operatorlarning qo'shmasi quyidagi xossalarga ega. Isbotlang.

a) $(A+B)^*=A^*+B^*$. b) $(\alpha A)^*=\bar{\alpha}A^*$. c) $(AB)^*=B^*A^*$.

3.21. $L_2[a,\ b]$ kompleks Hilbert fazosida, uzluksiz u funksiyaga ko'paytirish operatori, ya'ni

$$(Tx)(t)=u(t)\,x(t),\quad x\in L_2[a,\ b]$$

operatorni qaraymiz. T ga qo'shma operatorni toping.

Yechish. 3.2-ta'rifga ko'ra $T\in L(H)$ operatorning Hilbert qo'shmasi barcha $x,y\in H$ lar uchun

$$(Tx,y)=(x,T^*y) \tag{3.6}$$

tenglikni qanoatlantiruvchi $T^*\in L(H)$ operatordan iborat. $L_2[a,\ b]$ fazo

$$(x,y)=\int\limits_a^b x(t)\overline{y(t)}\,dt \tag{3.7}$$

skalyar ko'paytmaga nisbatan Hilbert fazosi bo'ladi. Shunday ekan misolda berilgan T operator uchun (3.6) tenglik

$$\int_a^b u(t)\, x(t)\, \overline{y(t)}\, dt = \int_a^b x(t)\, \overline{(T^*y)(t)}\, dt$$

ko'rinishda bo'ladi. Bu tenglikni quyidagicha ham yozish mumkin:

$$\int_a^b x(t)\, \overline{\overline{u(t)}\, y(t)}\, dt = \int_a^b x(t)\, \overline{(T^*y)(t)}\, dt. \tag{3.8}$$

Agar $z(t) = \overline{u(t)}\, y(t)$ belgilashni kiritsak, (3.7) ga ko'ra (3.8) tenglik $(x, z) = (x, T^*y)$ yoki $(x, T^*y) - (x, z) = (x, T^*y - z) = 0$ ko'rinishga keladi. Oxirgi tenglik barcha $x \in L_2[a, b]$ lar uchun o'rinli bo'ladi. Xususiy holda, $x = T^*y - z$ elementni olsak, $(T^*y - z, T^*y - z) = 0$ tenglik hosil bo'ladi. Skalyar ko'paytmaning ta'rifiga asosan oxirgi tenglik o'rinli bo'lishi uchun $T^*y - z = 0$, ya'ni $T^*y = z$ bo'lishi kerak. Shunday qilib, qo'shma $T^* : L_2[a, b] \to L_2[a, b]$ operator

$$(T^*y)(t) = \overline{u(t)}\, y(t), \quad y \in L_2[a, b]$$

formula yordamida aniqlanar ekan. □

Ma'lumki, $T^* = T$ bo'lsa, T operator o'z-o'ziga qo'shma operator deyiladi. Shuning uchun 3.21-misoldagi T operator u funksiya faqat haqiqiy qiymatlar qabul qilgandagina (ya'ni, $\overline{u(t)} \equiv u(t)$ bo'lganda) o'z-o'ziga qo'shma operator bo'ladi.

3.22-3.28-misollarda kompleks Hilbert fazosida berilgan $T \in L(H)$ operatorga qo'shma T^* operatorni toping.

3.22. $(Tx)(t) = \int_0^1 [t\, s + i \cos(t + s)]\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

3.23. $(Tx)(t) = \int_0^1 (t^2 + t + s)\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

3.24. $(Tx)(t) = \int\limits_0^t s\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

3.25. $(Tx)(t) = (\cos t + i \sin t)\, x(t) + \int\limits_{-1}^{1} (ts - it^2 s^2)\, x(s)\, ds, \quad x \in L_2[-1,\ 1].$

3.26. $(Tx)(t) = (t + it^2)\, x(t) + \int\limits_0^1 (t + is)\, x(s)\, ds, \quad x \in L_2[0,\ 1].$

3.27. $(Tx)(t) = x(t + h), \ h > 0, \quad x \in L_2(\mathbb{R}).$

3.28. $(Tx)(t) = u(t)\, x(t + h), \ h > 0, \ u \in M(\mathbb{R}), \ x \in L_2(\mathbb{R}).$

3.26-misolning yechimi. Berilgan operatorni $T = A + B$ yig'indi shaklda yozamiz. Bu yerda

$$(Ax)(t) = (t + it^2)x(t), \quad (Bx)(t) = \int\limits_0^1 (t + is)x(s)\, ds, \quad x \in L_2[0,\ 1].$$

3.21-misol tasdig'iga ko'ra, A operatorga qo'shma operator

$$(A^*x)(t) = (t - it^2)x(t), \quad x \in L_2[0,\ 1]$$

formula yordamida, 3.19-misolga ko'ra, B operatorga qo'shma operator

$$(B^*x)(t) = \int\limits_0^1 (s - it)x(s)\, ds, \quad x \in L_2[0,\ 1]$$

formula yordamida aniqlanadi. 3.20-misolning a) tasdig'iga ko'ra, ularning yig'indisi bo'lgan $T = A + B$ operatorga qo'shma bo'lgan T^* operator

$$(T^*x)(t) = ((A^* + B^*)x)(t) = (t - it^2)x(t) + \int\limits_0^1 (s - it)x(s)\, ds, \quad x \in L_2[0,\ 1]$$

tenglik bilan aniqlanadi. □

3.29-3.32-misollarda berilgan $T \in L(H)$ operatorning o'z-o'ziga qo'shma bo'lish shartini toping.

3.29. $T : \ell_2 \to \ell_2,\ Tx = (\mu_1 x_1, \ldots, \mu_n x_n, \ldots),\ \mu_n \in \mathbb{C},\ |\mu_n| \le 1,\ \forall n \in \mathbb{N}.$

3.30. $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}),\quad (Tf)(n) = v(n)f(n),\quad |v(n)| \le M,\quad \forall n \in \mathbb{N}.$

3.31. $T : L_2[a,b] \to L_2[a,b],\ (Tx)(t) = \int\limits_a^b K(t,s)\,x(s)\,ds,\ K \in L_2([a,b]^2)$

3.32. $T : L_2[a,b] \to L_2[a,b],\ (Tx)(t) = (u(t) + iv(t))x(t),\ u, v \in C[a,b].$

3.30-misolning yechimi. 3.21-misoldagidek ko'rsatish mumkinki, $\ell_2(\mathbb{Z})$ fazoda ham $(Tf)(n) = v(n)f(n)$ ko'paytirish operatoriga qo'shma operator $(T^*f)(n) = \overline{v(n)}\,f(n)$ tenglik bilan aniqlanadi. Demak, $T = T^*$ bo'lishi uchun, barcha $n \in \mathbb{Z}$ larda $v(n) = \overline{v(n)}$ tenglikning bajarilishi zarur va yetarli. □

Quyidagi 3.33-3.37-misollarda $\alpha,\ \beta \in \mathbb{C}$ parametrlarni shunday tanlangki, natijada $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lsin.

3.33. $(Tx)(t) = \int\limits_{-\pi}^{\pi} [\alpha \sin(s-t) + \beta \cos(s-t)]\,x(s)\,ds,\quad x \in L_2[-\pi, \pi].$

3.34. $(Tx)(t) = [\alpha\, u(t) + i\beta\, v(t)]x(t),\quad x \in L_2[a,b],\ u, v \in C[a,b].$

3.35. $(Tx)(t) = \int\limits_{-\pi}^{\pi} \exp\{\alpha\, ts + i\,\beta t^2 s^2\}\,x(s)ds,\quad x \in L_2[-\pi, \pi].$

3.36. $T = \alpha A + \beta A^*,\quad A \in L(H).$

3.37. $T = A + \alpha\beta A^*,\quad A \in L(H).$

3.35-misolning yechimi. 3.14-misolning b) bandiga ko'ra $T = T^*$ bo'lishi uchun

$$K(t,s) = \exp\{\alpha\, ts + i\,\beta t^2 s^2\} = \overline{\exp\{\alpha\, st + i\,\beta s^2 t^2\}} =$$

$$= \exp\{\overline{\alpha}\, ts - i\overline{\beta}\, t^2 s^2\} = \overline{K(s,t)}$$

tenglikning bajarilishi zarur va yetarli. Bu tenglik $\alpha = \overline{\alpha}$ va $\beta = -\overline{\beta}$ tengliklarga teng kuchli. Bu yerdan α haqiqiy, β sof mavhum son ekanligi, ya'ni $Im\alpha = 0$ va $Re\beta = 0$ shartlar kelib chiqadi. □

3.38-3.55-misollarda keltirilgan tasdiqlarni isbotlang.

3.38. $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lishi uchun bichiziqli $f(x, y) = (Tx, y)$ funksionalning simmetrik bo'lishi zarur va yetarli.

3.39. $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lishi uchun $f(x, x) = \varphi(x) = (Tx, x)$ kvadratik formaning barcha $x \in H$ larda haqiqiy bo'lishi zarur va yetarli.

3.40. Agar A va B operatorlar o'xshash bo'lsa, u holda ixtiyoriy $n \in \mathbb{N}$ uchun A^n va B^n lar ham o'xshash bo'ladi.

3.41. Agar A va B operatorlar o'xshash bo'lsa, u holda A^* va B^* lar ham o'xshash bo'ladi.

3.42. Agar A va B operatorlardan hech bo'magandа biri teskarilanuvchan bo'lsa, u holda AB va BA lar o'xshash bo'ladi.

3.43. Ixtiyoriy $T \in L(H)$ uchun $W(T) = \{(Tx, x) : \|x\| = 1\}$ to'plam $\mathbb{C}$ da qavariq to'plam bo'ladi. $W(T)$ to'plam T *operatorning sonli aksi* deyiladi.

3.44. $w(T) = \sup\{|(Tx, x)| : \|x\| = 1\}$ son T *operatorning sonli radiusi* deyiladi. Quyidagi $\|T\| \leq 2\, w(T)$ tengsizlik o'rinli.

3.45. Agar $T \in L(H)$ o'z-o'ziga qo'shma operator bo'lsa, u holda $\|T\| = w(T)$ tenglik o'rinli.

3.46. O'z-o'ziga qo'shma A va B operatorlarning ko'paytmasi AB o'z-o'ziga qo'shma bo'lishi uchun $[A, B] = 0$ bo'lishi zarur va yetarli.

3.47. Agar o'z-o'ziga qo'shma A operatorga teskari operator mavjud bo'lsa, u holda A^{-1} ham o'z-o'ziga qo'shma bo'ladi.

3.48. Ixtiyoriy $T \in L(H)$ operator yagona usulda $T = A + i\,B$ ko'rinishda tasvirlanadi. Bu yerda A va B lar o'z-o'ziga qo'shma operatorlar.

3.49. Agar A o'z-o'ziga qo'shma operator bo'lsa, u holda $I + iA$ ga chegaralangan teskari operator mavjud.

3.50. Agar $A \geq 0$ bo'lsa, u holda barcha $n \in \mathbb{N}$ da $A^n \geq 0$ bo'ladi.

3.51. Agar A o'z-o'ziga qo'shma bo'lsa, u holda barcha $n \in \mathbb{N}$ da $A^{2n} \geq 0$ bo'ladi.

3.52. Agar $[A, B] = 0$ va $A \geq 0$, $B \geq 0$ bo'lsa, u holda $AB \geq 0$ bo'ladi.

3.53. $u : H \to H$ izometrik operator uchun $u^*u = I$, $uu^* = I - P$ tengliklar o'rinli. Bu yerda P ortogonal proyeksiyalash operatori.

3.54. Ortogonal proyeksiyalash operatori $P : H \to H$ o'z-o'zoga qo'shma operator bo'ladi.

3.55. Ortogonal proyeksiyalash operatori $P : H \to H$ musbat operator bo'ladi.

3.50-misolning yechimi. $A \geq 0$ ekanligidan barcha $x \in H$ larda $(Ax, x) = (x, Ax) \geq 0$ ekanligi ma'lim. Bu yerdan barcha $x \in H$ larda $(A^{2n+1}x, x) = (AA^n x, A^n x) \geq 0$ ekanligi kelib chiqadi, ya'ni A^{2n+1} musbat operator. $(A^{2n}x, x) = (A^n x, A^n x) = \|A^n x\|^2 \geq 0$. Demak, barcha $n \in \mathbb{N}$ larda $A^n \geq 0$ ekan. □

3.56-3.63-misollarda berilgan $U \in L(H_1, H_2)$ yoki $F \in L(H_1, H_2)$ akslantirishning unitar operator ekanligini isbotlang.

3.56. Istiyoriy $s \in \mathbb{Z}$ uchun $U_s : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(U_s f)(n) = f(n + s)$.

3.57. Barcha $s \in \mathbb{Z}^n$ uchun $U_s : \ell_2(\mathbb{Z}^n) \to \ell_2(\mathbb{Z}^n)$, $(U_s f)(n) = f(n + s)$.

3.58. Furye akslantirishi

$$F : \ell_2(\mathbb{Z}) \to L_2[-\pi, \pi], \quad (Ff)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \exp\{inx\} f(n).$$

3.59. Furye akslantirishi

$$F : \ell_2(\mathbb{Z}^n) \to L_2\left([-\pi,\ \pi]^n\right), \quad (Ff)(x) = \frac{1}{(2\pi)^{n/2}} \sum_{n\in\mathbb{Z}^n} \exp\{i(n,x)\} f(n).$$

3.60. Furye akslantirishi

$$F : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \quad (Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{ixy\} f(y)dy.$$

3.61. Furye akslantirishi

$$F : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n), \quad (Ff)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\{i(x,y)\} f(y)dy.$$

3.62. $U : \ell_2 \to \ell_2, \quad Ux = \left(e^{i}x_1, e^{2i}x_2, \ldots, e^{ni}x_n, \ldots\right).$

3.63. Ixtiyoriy $A \in L(H)$, $A = A^*$ uchun $U = e^{iA}$ unitar operator bo'ladi.

3.64. $U : \ell_2 \to \ell_2$, $Ux = (0, e^{i}x_1, e^{2i}x_2, \ldots, e^{ni}x_n, \ldots)$ izometrik, lekin unitar emasligini isbotlang.

3.65. $U : \ell_2 \to \ell_2$, $Ux = (x_2, x_3, \ldots, x_{n+1}, \ldots)$ ning qisman izometrik operator ekanligini isbotlang.

3.66. Shunday A va B o'z-o'ziga qo'shma operatorlarga misol keltiringki, $A \geq B$ va $A \leq B$ munosabatlarning hech biri bajarilmasin.

3.67. Shunday $0 \leq A \leq B$ operatorlarga misol keltiringki. $A^2 \leq B^2$ tengsizlik bajarilmasin.

3.68. Agar A va B lar o'z-o'ziga qo'shma chegaralangan operatorlar bo'lib, $[A, B] = 0$ bo'lsa, $T = A + iB$ normal operator bo'ladi va aksincha.

3.69. $U_s : \ell_2(\mathbb{Z}^n) \to \ell_2(\mathbb{Z}^n)$, $(U_s f)(n) = f(n + s)$ operatorlar oilasini ko'paytirish amaliga nisbatan Abel gruppasi bo'lishini isbotlang.

3.70-3.74-misollarda berilgan operatorlarning giponormal ekanligini isbotlang.

3.70. Ixtiyoriy unitar operator.

3.71. Ixtiyoriy izometrik operator.

3.72. Ixtiyoriy normal operator.

3.73. Ixtiyoriy ortogonal proyeksiyalash operatori.

3.74. Ixtiyoriy o'z-o'ziga qo'shma operator.

3.71-misolning isboti. Bizga U izometrik operator berilgan bo'lsin. Izometrik operator ta'rifiga ko'ra $\|Ux\| = \|x\|$ va $\|U^*x\| \leq \|x\|$ tengliklar ixtiyoriy $x \in H$ uchun o'rinli. Bu yerdan $\|U^*x\| \leq \|Ux\|$ tengsizlik kelib chiqadi. Ya'ni U giponormal operator bo'ladi. □

3.75. Giponormal operatorga qo'shma operator giponormal bo'lmasligi mumkin. Misol keltiring.

3.76. $T : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$, $(Tx)(t) = e^{it}x(t)$ operatorning sonli aksi $W(T) = \{(Tx, x) : \|x\| = 1\}$ to'plamni va T operatorning sonli radiusi $w(T) = \sup\{|(Tx, x)| : \|x\| = 1\}$ sonlarni toping.

3.77. $T : L_2[-1, 1] \to L_2[-1, 1]$, $(Tx)(t) = t\,x(t)$ operatorning sonli aksi $W(T) = \{(Tx, x) : \|x\| = 1\}$ to'plamni va T operatorning sonli radiusi $w(T) = \sup\{|(Tx, x)| : \|x\| = 1\}$ sonlarni toping.

3.78-3.86-misollarda keltirilgan tasdiqlarni isbotlang.

3.78. Agar T normal operator bo'lsa, u holda T^* ham normal operator bo'ladi.

3.79. $A : L(H) \to L(H)$, $A(T) = T^*$ operator additiv va qo'shma bir jinsli bo'ladi.

3.80. Nolmas $L \subset H$ qism fazoning ustiga ortogonal proyeksiyalash operatori P o'z-o'ziga qo'shma, normasi birga teng bo'lgan va $P^2 = P$ shartni qanoatlantiruvchi operator bo'ladi.

3.81. $P^2 = P$ shartni qanoatlantiruvchi o'z-o'ziga qo'shma P chiziqli operator, biror $L \subset H$ qism fazoga ortogonal proyeksiyalash operatori bo'ladi.

3.82. Faraz qilaylik, P_1 va P_2 lar mos ravishda L_1 va L_2 qism fazolarga ortogonal proyeksiyalash operatorlari bo'lsin. P_1 va P_2 proyeksiyalash operatorlari o'zaro ortogonal bo'lishi uchun ularga mos L_1 va L_2 qism fazolar o'zaro ortogonal bo'lishi zarur va yetarli.

3.83. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining yig'indisi proyeksiyalash operatori bo'lishi uchun, bu operatorlar o'zaro ortogonal bo'lishi zarur va yetarli. Agar bu shart bajarilgan bo'lsa, u holda $P_{L_1} + P_{L_2} = P_{L_1 \oplus L_2}$ tenglik o'rinli.

3.84. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining ko'paytmasi proyeksiyalash operatori bo'lishi uchun, bu operatorlar o'rin almashinuvchi (kommutativ) bo'lishi zarur va yetarli. Agar bu shart bajarilgan bo'lsa, u holda $P_{L_1} P_{L_2} = P_{L_2} P_{L_1} = P_{L_1 \cap L_2}$ tenglik o'rinli.

3.85. Ikkita P_{L_1} va P_{L_2} proyeksiyalash operatorlarining ayirmasi proyeksiyalash operatori bo'lishi uchun, P_{L_2} ning P_{L_1} uchun qism bo'lishi zarur va yetarli. Agar bu shart bajarilgan bo'lsa, u holda $P_{L_1} - P_{L_2} = P_{L_1 \ominus L_2}$ tenglik o'rinli.

3.86. Ortogonal proyeksiyalash operatori $P : H \to H$ musbat operator bo'ladi.

4-§. Chiziqli operator spektri

Chiziqli operator nazariyasida eng muhim tushunchalardan biri bu spektr tushunchasidir. Spektrning asosida esa xos qiymat tushunchasi yotadi. Spektr haqida masala qaralayotganda X fazoni o'zini-o'ziga aks-lantiruvchi chiziqli A operatorlar nazarda tutiladi. Shunday qilib $A : X \to X$ chiziqli operatorning xos qiymati ta'rifiga kelamiz.

4.1-ta'rif. *Agar biror $\lambda \in \mathbb{C}$ soni uchun $(A - \lambda I)x = 0$ tenglama nolmas ($x \neq 0$) yechimga ega bo'lsa, λ soni A operatorning xos qiymati deyiladi, nolmas yechim x esa xos vektor deyiladi.* $\dim Ker(A-\lambda I) = n$ *soni λ xos qiymatning karraligi deyiladi. Agar $n = 1$ bo'lsa, λ soni A operatorning oddiy xos qiymati, $n \geq 2$ bo'lsa, λ soni A operatorning karrali xos qiymati, $n = \infty$ bo'lsa, λ soni A operatorning cheksiz karrali xos qiymati deyiladi.*

4.2-ta'rif. *Agar $\lambda \in \mathbb{C}$ kompleks soni uchun $A - \lambda I$ ga teskari operator mavjud bo'lib, u X ning hamma yerida aniqlangan bo'lsa, λ soni A operatorning regulyar nuqtasi deyiladi,*

$$R_\lambda(A) = (A - \lambda I)^{-1}$$

operator esa A operatorning λ nuqtadagi rezolventasi deyiladi. Barcha regulyar nuqtalar to'plami $\rho(A)$ orqali belgilanadi.

4.3-ta'rif. *A operatorning regulyar bo'lmagan nuqtalari uning spektri deyiladi, ya'ni $\sigma(A) = \mathbb{C}\backslash\rho(A)$ to'plamga A operatorning spektri deyiladi.*

Demak, $\lambda \in \mathbb{C}$ soni A operatorning spektriga qarashli bo'lsa, u holda yo $A - \lambda I$ ga teskari operator mavjud emas, yo mavjud bo'lganda ham, u butun X fazoda aniqlanmagan bo'ladi. Agar $\lambda \in \mathbb{C}$ soni A operatorning xos qiymati bo'lsa, $(A - \lambda I)x = 0$ tenglama nolmas yechimga esa, ikkinchidan bu bir jinsli tenglama nol yechimga ham ega. Demak, 2.3-teoremaga ko'ra, $A - \lambda I$ ga teskari operator mavjud emas. Shunday qilib operatorning xos qiymatlari, uning spektriga qarashli bo'ladi.

Chekli o'lchamli fazolarda berilgan chiziqli operatorlarning spektri aniq tavsiflanadi. Unga qisqacha to'xtalamiz. Faraz qilaylik, $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operator berilgan bo'lsin. Ma'lumki, har bir $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operatorga $\{a_{ij}\}$ – $n \times n$ matritsa mos keladi va aksincha. Bu $A-\lambda I$ matritsa determinanti $det(A-\lambda I)$, parametr λ ning $n-$darajali ko'phadi bo'ladi.

4.1-teorema. *$A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operatorning spektri chekli sondagi chekli karrali xos qiymatlardan iborat. Bu xos qiymatlar $det(A - \lambda I)$ ko'phadning nollari bo'ladi va aksincha.*

Agar $A : X \to X$ chiziqli operator bo'lib, dim $X = \infty$ bo'lsa, uning spektri ixtiyoriy tabiatli yopiq to'plam bo'lishi mumkin. Odatda spektr quyidagi qismlarga ajratiladi.

4.4-ta'rif. *a) A operatorning barcha chekli karrali yakkalangan xos qiymatlari to'plami nuqtali spektr deyiladi va $\sigma_{pp}(A)$ bilan belgilanadi.*

b) Agar $\lambda \in \sigma(A)$ xos qiymat bo'lmasa va $\overline{Im(A - \lambda I)} \neq X$, ya'ni $A - \lambda I$ operatorning qiymatlar sohasi X ning hamma yerida zich emas. Bunday λ lar to'plami A operatorning qoldiq spektri deyiladi va $\sigma_{qol}(A)$ bilan belgilanadi.

c) Agar $\lambda \in \mathbb{C}$ xos qiymatlar to'plamining limitik nuqtasi bo'lsa, yoki $\lambda \in \mathbb{C}$ operatorning cheksiz karrali xos qiymati bo'lsa, yoki $\lambda \in \sigma(A)$ uchun $\overline{Im(A - \lambda I)} = X$ bo'lsa, bunday λ lar A operatorning muhim spektriga qarashli deyiladi. Operatorning muhim spektri $\sigma_{ess}(A)$ bilan belgilanadi.

Asosan o'z-o'ziga qo'shma operatorlarning spektri o'rganiladi. Endi o'z-o'ziga qo'shma operatorlar uchun muhim spektr ta'rifini keltiramiz.

4.5-ta'rif. *Agar biror $\lambda \in \mathbb{C}$ soni uchun nolga kuchsiz yaqinlashuvchi $\{f_n\} \subset H$ birlik vektorlar ketma-ketligi mavjud bo'lib,*

$$\lim_{n\to\infty} \|(A - \lambda I)f_n\| = 0$$

bo'lsa, λ soni $A = A^*$ operatorning muhim spektriga qarashli deyiladi.

Chegaralangan operatorlarning spektri haqida quyidagi tasdiq o'rinli.

4.2-teorema. *Agar $A \in L(X)$ bo'lsa, u holda $\sigma(A)$ chegaralangan yopiq to'plam bo'ladi.*

4.6-ta'rif. *Agar $A \in L(H, H_1)$ va $ImA \subset H_1$ qism fazo chekli o'lchamli bo'lsa, u holda A operator chekli o'lchamli operator deyiladi.*

4.7-ta'rif. *Agar $A : H \to H_1$ operator H dagi har qanday chegaralangan to'plamni H_1 dagi nisbiy kompakt to'plamga akslantirsa, u holda A kompakt yoki to'la uzluksiz operator deyiladi.*

4.8-ta'rif. *Agar biror U unitar operator uchun $B = UAU^{-1} = UAU^*$ tenglik o'rinli bo'lsa, u holda B operator A operatorga unitar ekvivalent deyiladi.*

4.9-ta'rif. *Agar $P \in L(H)$ uchun $P^* = P$ va $P^2 = P$ bo'lsa, P ga proyektor yoki proyeksiyalash operatori deyiladi.*

4.3-teorema. *Agar $A \in L(X)$ va $|\lambda| > \|A\|$ bo'lsa, u holda λ regulyar nuqta bo'ladi.*

Bu teoremadan chegaralan operatorning spektri markazi koordinatalar boshida, radiusi $r = \|A\|$ bo'lgan yopiq doirada saqlanishi kelib chiqadi.

4.4-teorema. *$A \in L(H)$ o'z-o'ziga qo'shma operator bo'lsin, u holda.*

a) $\sigma_{qol}(A)$ – bo'sh to'plam,

b) $\sigma(A)$ to'plam $\mathbb{R}$ ning qismi, ya'ni $\sigma(A) \subset \mathbb{R}$,

c) A operatorning har xil xos qiymatlariga mos keluvchi xos vektorlari o'zaro ortogonaldir.

O'z-o'ziga qo'shma $A \in L(H)$ operator uchun (Ax, x) barcha $x \in H$ larda haqiqiy (3.39-misol) bo'ladi. Quyidagi belgilashlarni kiritamiz:

$$\inf_{\|x\|=1} (Ax, x) = m, \qquad \sup_{\|x\|=1} (Ax, x) = M.$$

m soni o'z-o'ziga qo'shma $A \in L(H)$ operatorning *quyi chegarasi*, M

esa uning *yuqori chegarasi* deyiladi.

4.5-teorema. *O'z-o'ziga qo'shma $A \in L(H)$ operatorning spektri $\sigma(A) \subset [m, M]$ bo'ladi.*

4.6-teorema (spektral teorema). *A H Hilbert fazosida aniqlangan chegaralangan, o'z-o'ziga qo'shma operator bo'lib, m uning quyi chegarasi, M esa uning yuqori chegarasi bo'lsin. U holda quyidagi shartlarni qanoatlantiruvchi proyektorlar oilasi mavjud:*

1) agar $\lambda < m$ bo'lsa, $E_\lambda = 0$ va $M < \lambda$ bo'lsa, $E_\lambda = I$;

2) ixtiyoriy $\lambda \in \mathbb{R}$ uchun $\lim\limits_{\lambda \to \lambda_0 + 0} E_\lambda = E_{\lambda_0}$;

3) ixtiyoriy $\lambda < \mu$ uchun $E_\lambda \leq E_\mu$;

4) $A = \int\limits_{-\infty}^{\infty} \lambda \, dE_\lambda \Longleftrightarrow \forall x, y \in H$ uchun $(Ax, y) = \int\limits_{-\infty}^{\infty} \lambda \, d(E_\lambda x, y)$ tenglik o'rinli. E_λ larga A operatorning spektral proyektorlari deyiladi.

Dastlab chekli o'lchamli fazolarda berilgan operatorlarning spektriga oid misollar qaraymiz.

4.1. $A : \mathbb{C}^3 \to \mathbb{C}^3$, $Ax = (x_1 + x_2, x_2 - x_1, 2x_3)$ operatorning xos qiymatlari va xos vektorlarini toping.

Yechish. 4.1-teoremaga ko'ra, bu operatorning spektri faqat xos qiymatlardan iborat. Shuning xos qiymatga nisbatan tenglama, ya'ni

$$Ax = \lambda x \Longleftrightarrow (x_1 + x_2, x_2 - x_1, 2x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \tag{4.1}$$

tenglamani qaraymiz. Agar (4.1) tenglama biror $\lambda \in \mathbb{C}$ da nolmas yechimga ega bo'lsa, $\lambda \in \mathbb{C}$ ning bu qiymati xos son bo'ladi. (4.1) tenglama quyidagi sistemaga teng kuchli

$$\begin{cases} x_1 + x_2 = \lambda x_1 \\ x_2 - x_1 = \lambda x_2 \\ 2x_3 = \lambda x_3 \end{cases} \Longleftrightarrow \begin{cases} (1-\lambda)x_1 + x_2 = 0 \\ -x_1 + (1-\lambda)x_2 = 0 \\ (2-\lambda)x_3 = 0. \end{cases} \tag{4.2}$$

Bu sistema nolmas yechimga ega bo'lishi uchun, uning determinanti

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-2\lambda+\lambda^2)$$

nolga teng bo'lishi zarur va yetarli. Uning nollari $\lambda_1 = 2$, $\lambda_2 = 1 + i$, $\lambda_3 = 1 - i$ lardir. Endi bu xos qiymatlarga mos xos vektorlarni topamiz. Masalan, $\lambda = 2$ bo'lsin. λ ning bu qiymatida (4.2) sistema $x^{(1)} = (0, 0, 1)$ nolmas yechimga ega. Xuddi shunday $\lambda_2 = 1 + i$, $\lambda_3 = 1 - i$ xos sonlariga mos xos vektorlar $x^{(2)} = (1, i, 0)$, $x^{(3)} = (1, -i, 0)$ bo'ladi. □

4.2-4.7-misollarda berilgan operatorlarning xos qiymatlari va xos vektorlarini toping. O'z-o'ziga qo'shma bo'lgan holda (4.2 va 4.5-misollarda) xos qiymatlarning haqiqiyligini va har xil xos qiymatlarga mos xos vektorlarni ortogonalligini tekshiring.

4.2. $A : \mathbb{C}^3 \to \mathbb{C}^3$, $Ax = (x_1 + x_2, x_2 + x_1, 3x_3)$.

4.3. $A : \mathbb{C}^3 \to \mathbb{C}^3$, $Ax = (x_1, x_2 + x_3, x_3 - x_2)$.

4.4. $A : \mathbb{C}^3 \to \mathbb{C}^3$, $Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1)$.

4.5. $A : \mathbb{C}^4 \to \mathbb{C}^4$, $Ax = (x_1, 2x_2, 3x_3, 4x_4)$.

4.6. $A : \mathbb{C}^4 \to \mathbb{C}^4$, $Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1, 5x_4)$.

4.7. $A : \mathbb{C}^4 \to \mathbb{C}^4$, $Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_1)$.

4.8. Hilbert fazosi $L_2[a, b]$ da

$$(Ax)(t) = u(t) \int_a^b u(s)\, x(s) ds, \quad x \in L_2[a, b]$$

formula vositasida aniqlangan A operatorning xos qiymatlari va xos funksiyalarini toping. Bu yerda $u : [a, b] \to \mathbb{R}$ berilgan uzluksiz funksiya.

Yechish. Ta'rifga ko'ra, nol bo'lmagan biror $x \in L_2[a, b]$ funksiya va $\lambda \in \mathbb{C}$ soni uchun

$$(Ax)(t) = \lambda x(t) \tag{4.3}$$

tenglik bajarilsa, x funksiya A operatorning xos funksiyasi, λ son esa unga mos keluvchi xos qiymat deyiladi. Qaralayotgan operator uchun (4.3) tenglik quyidagi ko'rinishga ega bo'ladi:

$$u(t) \int_a^b u(s)\, x(s) ds = \lambda\, x(t). \tag{4.4}$$

Bu yerda $x \neq 0$. Faraz qilaylik, $\lambda \neq 0$ bo'lsin. U holda $x \neq 0$ bo'lgani uchun

$$\alpha_x = \int_a^b u(s) x(s) ds \neq 0 \tag{4.5}$$

tengsizlik bajarildi. (4.4) tenglikda (4.5) ni e'tiborga olsak,

$$x(t) = \lambda^{-1} \alpha_x\, u(t)$$

tenglikni olamiz. (4.5) tenglikka x funksiyaning bu ifodasini qo'yib,

$$\alpha_x = \alpha_x\, \lambda^{-1} \int_a^b u^2(t) dt \quad \text{yoki} \quad \alpha_x \left(\lambda - \int_a^b u^2(t)\, dt \right) = 0$$

tenglikka kelamiz. Bunda $\alpha_x \neq 0$ bo'lgani uchun $\lambda = \int_a^b u^2(t) dt$ son A operatorning xos qiymati va $x(t) = u(t)$ esa unga mos xos vektor ekanligi kelib chiqadi. Yana shuni ta'kidlash kerakki, agar

$$\int_a^b u(t) x(t) dt = 0 \tag{4.6}$$

shartni qanoatlantiruvchi nolmas x funksiya mavjud bo'lsa, u holda $\lambda = 0$ soni uchun ham (4.3) tenglik bajariladi. Albatta (4.6) shartni qanoatlantiruvchi nolmas x funksiya mavjud. Demak, A operator ikkita $\lambda = 0$ va $\mu = \int\limits_a^b u^2(t)dt$ xos qiymatlarga ega. (4.6) shartni qanoatlantiruvchi funksiyalar $\lambda = 0$ xos qiymatga mos keluvchi xos funksiyalar bo'ladi. □

4.9-4.15-misollarda berilgan operatorlarning xos qiymatlari va xos funksiyalarini toping. 4.11-4.15-misollarda keltirilgan operatorlarning o'z-o'ziga qo'shma bo'lishini ko'rsating. Ular uchun 4.4-teoremaning b) va c) tasdiqlarining bajarilishi tekshiring.

4.9. $(Ax)(t) = t\int\limits_{-1}^{1} s\,x(s)ds, \quad x \in C[-1,\ 1].$

4.10. $(Ax)(t) = \int\limits_{-1}^{1}(1+ts)\,x(s)ds, \quad x \in C[-1,\ 1].$

4.11. $(Ax)(t) = \int\limits_{-\pi}^{\pi}(1+\sin t\,\sin s)\,x(s)ds, \quad x \in L_2[-\pi,\ \pi].$

4.12. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos(t-s)\,x(s)ds, \quad x \in L_2[-\pi,\ \pi].$

4.13. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\sin(t+s)\,x(s)ds, \quad x \in L_2[-\pi,\ \pi].$

4.14. $(Ax)(t) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}(1+\cos(t-s))\,x(s)ds, \quad x \in L_2[-\pi,\ \pi].$

4.15. $(Ax)(t) = x(t)+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}(\cos(t-s)-\sin t\,\sin s)\,x(s)ds, \quad x \in L_2[-\pi,\ \pi].$

4.16. $L_2[a,b]$ Hilbert fazosida erkin o'zgaruvchi x ga ko'paytirish operatori, ya'ni

$$A: L_2[a,b] \to L_2[a,b], \quad (Af)(x) = xf(x)$$

operatorni qaraymiz. Uning nuqtali, qoldiq va muhim spektrini toping.

Yechish. 3.21-misol natijasi va $u(x) = x = \overline{x} = \overline{u(x)}$ tenglikka ko'ra, berilgan operator o'z-o'ziga qo'shma, ya'ni $A = A^*$ dir. 4.4-teoremaning a) tasdig'iga ko'ra, $\sigma_{qol}(A) = \emptyset$. Ma'lumki,

$$(Af)(x) = \lambda f(x) \quad \text{ya'ni} \quad (x - \lambda)f(x) = 0 \tag{4.7}$$

tenglama ixtiyoriy $\lambda \in \mathbb{C}$ uchun yagona nol yechimga ega. Demak, A operator xos qiymatlarga ega emas, ya'ni $\sigma_{pp}(A) = \emptyset$. (4.7) tenglama faqat nol yechimga ega ekanligidan 2.3-teoremaga ko'ra, $(A - \lambda I)f(x) = g(x)$ tenglamaning ixtiyoriy $g \in Im\, A$ da yagona yechimga ega ekanligi kelib chiqadi. Ko'rsatish mumkinki $A - \lambda I$ operatorga teskari operator

$$(A - \lambda I)^{-1}g(x) = \frac{g(x)}{x - \lambda} \tag{4.8}$$

formula bilan aniqlanadi. Agar $\lambda \notin [a,\ b]$ bo'lsa, u holda $x - \lambda \neq 0$, natijada $(A - \lambda I)^{-1}$ operator $L_2[a,\ b]$ fazoning hamma yerida aniqlangan. 4.2-ta'rifga ko'ra, $\lambda \notin [a,\ b]$ regulyar nuqta, ya'ni $\sigma(A) \subset [a,\ b]$. Lekin (4.8) formula bilan aniqlangan teskari operator $\lambda \in [a,\ b]$ bo'lganda $L_2[a,\ b]$ fazoning hamma yerida aniqlanmagan. Demak, $[a,\ b] \subset \sigma(A)$. Bulardan, $\sigma(A) = [a,\ b]$. Endi A operatorning spektridagi ixtiyoriy nuqta uning muhim spektriga qarashli ekanligini ko'rsatamiz. Ixtiyoriy $\lambda \in [a,\ b)$ uchun

$$f_n(x) = \sqrt{n(n+1)}\,\chi_{A_n}(x), \quad A_n = \left[\lambda + \frac{1}{n+1},\ \lambda + \frac{1}{n}\right) \tag{4.9}$$

deymiz. Ma'lum nomerdan boshlab $\lambda + \dfrac{1}{n} < b$ bo'ladi va bunday nomerlar uchun $\|f_n\| = 1$ tenglik o'rinli. Bundan tashqari har xil n va m larda $A_n \cap A_m = \emptyset$ bo'lgani uchun $(f_n, f_m) = 0$ tenglik o'rinli, ya'ni $\{f_n\}$ ortonormal sistema ekan. Ma'lumki, (II qism, 11.75-misol) ixtiyoriy ortonormal sistema nolga kuchsiz ma'noda yaqinlashadi, shuning uchun $\{f_n\}$ ketma-ketlik ham nolga kuchsiz ma'noda yaqinlashadi. Endi

$\|(A-\lambda I)f_n\|$ norma kvadratini hisoblaymiz:

$$\|(A-\lambda I)f_n\|^2 = n(n+1)\int\limits_{\lambda+\frac{1}{n+1}}^{\lambda+\frac{1}{n}}(t-\lambda)^2dt = \frac{3n^2+3n+1}{3n^2(n+1)^2}\to 0,\, n\to\infty.$$

4.5-ta'rifga ko'ra, $\lambda\in[a,b)$ son A operatorning muhim spektriga qarashli ekan. Agar $\lambda=b$ bo'lsa, u holda nolga kuchsiz yaqinlashuvchi

$$g_n(x)=\sqrt{n(n+1)}\,\chi_{B_n}(x),\quad B_n=\left[b-\frac{1}{n},\, b-\frac{1}{n-1}\right),\quad (g_n, g_m)=\delta_{nm}$$

ketma-ketlik uchun $\lim\limits_{n\to\infty}\|(A-bI)g_n\|=0$ shart bajariladi. Bu yerdan, $\lambda=b$ nuqta ham A operatorning muhim spektriga qarashli ekanligi kelib chiqadi. Shunday qilib, A operatorning spektri faqat muhim spektrdan iborat bo'lib, u $[a,\, b]$ kesma bilan ustma-ust tushadi. Xulosa

$$\sigma_{qol}(A)=\sigma_{pp}(A)=\emptyset,\quad \sigma_{ess}(A)=\sigma(A)=[a,\, b].$$ □

4.17. 4.16-misolda qaralgan A operatorni $C[a,\, b]$ Banax fazosida qaraymiz, ya'ni

$$A: C[a,\, b]\to C[a,\, b],\quad (Af)(x)=xf(x)$$

operatorni qaraymiz. Uning nuqtali va qoldiq spektrini toping.

Yechish. Ma'lumki, ((4.7) ga qarang)

$$(Af)(x)=\lambda f(x)\quad \text{ya'ni}\quad (x-\lambda)f(x)=0,\quad f\in C[a,\, b] \qquad (4.10)$$

tenglama ixtiyoriy $\lambda\in\mathbb{C}$ uchun yagona nol yechimga ega. Demak, A operator xos qiymatlarga ega emas, ya'ni $\sigma_{pp}(A)=\emptyset$. (4.10) tenglama faqat nol yechimga ega ekanligidan 2.3-teoremaga ko'ra, $A-\lambda I$ operatorga teskari operator mavjud va u (4.8) formula bilan aniqlanadi. Xuddi 4.16-misoldagi kabi ko'rsatishimiz mumkinki, $\sigma(A)=[a,\, b]$ tenglik o'rinli. Haqiqatan ham, agar $\lambda\notin[a,\, b]$ bo'lsa, u holda (4.8) ning o'ng tomoni ixtiyoriy $g \in C[a, b]$ da uzluksiz funksiya bo'ladai, ya'ni $D((A - \lambda I)^{-1}) = C[a, b]$. 4.2-ta'rifga ko'ra λ regulyar nuqta, ya'ni $\sigma(A) \subset [a, b]$. Agar $\lambda \in [a, b]$ bo'lsa, u holda (4.8) formula bilan aniqlangan $(A-\lambda I)^{-1}$ operator $C[a, b]$ fazoning hamma yerida aniqlanmagan, bundan $[a, b] \subset \sigma(A)$. Bulardan, $\sigma(A) = [a, b]$ ekanligi kelib chiqadi. Endi $\sigma(A) = \sigma_{qol}(A)$ ekanligini ko'rsatamiz. Ixtiyoriy $\lambda \in [a, b]$ uchun $A - \lambda I$ operatorning qiymatlar sohasi

$$Im(A - \lambda I) = \{g \in C[a, b] : g(x) = (x - \lambda)f(x)\}$$

$C[a, b]$ fazoda zich emas. Haqiqatan ham, $Im(A-\lambda I)$ chiziqli ko'pxillilikdagi ixtiyoriy g uchun $g(\lambda) = 0$ shart bajariladi. Agar biz $f_0(x) \equiv 1$ desak, u holda ixtiyoriy $g \in Im(A - \lambda I)$ uchun

$$\|g - f_0\| = \max_{x\in[a,b]} |g(x) - f_0(x)| \geq |g(\lambda) - f_0(\lambda)| = 1$$

tengsizlik o'rinli. Demak, $A-\lambda I$ operatorning qiymatlar sohasi $Im(A-\lambda I)$ dan $f_0(x) \equiv 1$ elementga yaqinlashuvchi ketma-ketlik ajratish mumkin emas, ya'ni $\overline{Im(A - \lambda I)} \neq C[a, b]$. 4.4-ta'rifga ko'ra, barcha $\lambda \in [a, b]$ lar uchun $\lambda \in \sigma_{qol}(A)$ munosabat o'rinli. Bundan $\sigma(A) \subset \sigma_{qol}(A)$ kelib chiqadi. Teskari munosabat $\sigma(A) \supset \sigma_{qol}(A)$ doim o'rinli. Demak, $\sigma(A) = \sigma_{qol}(A) = [a, b]$. □

4.16 va 4.17-misollarda bir xil qonuniyat bo'yicha ta'sir qiluvchi A operator har xil $L_2[a, b]$ va $C[a, b]$ fazolarda qaralgan. Har ikki holda ham A operatorning spektri $[a, b]$ kesma bilan ustma-ust tushgan, lekin spektrning qismlarida (strukturasida) o'zgarish bo'ldi. Birinchi holda (4.16-misolda) $\sigma_{qol}(A) = \emptyset$ edi, ikkinchi holda $\sigma_{qol}(A) = [a, b]$.

4.18-4.24-misollarda berilgan operatorlarning spektri (nuqtali, qoldiq va muhim) va rezolventasini toping.

4.18. $(Ax)(t) = (t^2 + 1)x(t), \quad x \in C[-1, 1]$.

4.19. $(Ax)(t) = t\,x(t) + x(0)\,t^2, \quad x \in C[-1, 1]$.

4.20. $(Ax)(t) = \sin t\, x(t) + x(0)\cos t, \quad x \in C[-\pi,\ \pi]$.

4.21. $(Ax)(t) = (t^2 + 1)\, x(t), \quad x \in L_2[0,\ \infty)$.

4.22. $(Ax)(t) = (1 - \cos t)\, x(t), \quad x \in L_2[-\pi,\ \pi]$.

4.23. $(Ax)(t) = (1 - 4\cos t + 3\sin t)x(t), \quad x \in L_2[-\pi,\ \pi]$.

4.24. $(Ax)(t) = t^2\, x(t), \quad x \in L_2(\mathbb{R})$

4.25. Kompleks Hilbert fazosi $L_2[0,\ 1]$ da aniqlangan

$$(Ax)(t) = t\,x(t) + \int_0^1 t\,s\,x(s)\,ds, \quad x \in L_2[0,1]$$

operatorning spektri va rezolventasini toping.

Yechish. A operatorni 4.16 va 4.8-misollarda spektri va xos qiymati o'rganilgan B va C operatorlarning yig'indisi ko'rinishida tasvirlash mumkin:

$$(Bx)(t) = t\,x(t), \quad (Cx)(t) = \int_0^1 t\,s\,x(s)\,ds, \quad x \in L_2[0,1].$$

3.21-misolda $L_2[a,\ b]$ fazoda ko'paytirish operatorining o'z-o'ziga qo'shmalik shartlari topilgan. $u(t) = t = \overline{t} = \overline{u(t)}$ tenglikdan B operatorning o'z-o'ziga qo'shma ekanligi kelib chiqadi. 3.19-misolda $L_2[a,\ b]$ fazoda integral operatorining o'z-o'ziga qo'shmalik shartlari keltirilgan. $K(t,s) = t \cdot s = \overline{s \cdot t} = \overline{K(s,t)}$ tenglikdan C operatorning o'z-o'ziga qo'shma ekanligi kelib chiqadi. $A^* = (B + C)^* = B^* + C^* = B + C = A$ tenglik A operatorning o'z-o'ziga qo'shma ekanligini bildiradi. 4.4-teoremaning b) qismiga ko'ra, A operatorning spektri $\mathbb{R}$ ning qism to'plami bo'ladi. Xususan, uning xos qiymatlari ham haqiqiy bo'ladi. A operator spektrini ikki qismga ajratib topamiz: a) qismida uning xos qiymatlarini, b) qismida esa uning muhim spektrini topamiz.

a) A operatorning xos qiymatlarini topish uchun quyidagi tasdiqdan foydalanamiz.

4.1-tasdiq. $\lambda \in \mathbb{R}\backslash[0,\ 1]$ soni A operatorning xos qiymati bo'lishi uchun

$$\Delta(\lambda) := 1 + \int_0^1 \frac{s^2}{s-\lambda} ds = 0$$

tenglikning bajarilishini zarur va yetarli.

Isbot. *Zaruriyligi.* Aytaylik, $\lambda \in \mathbb{R}\backslash[0,\ 1]$ soni A operatorlarning xos qiymati bo'lsin, ya'ni biror nolmas $x \in L_2[0,\ 1]$ element uchun

$$(Ax)(t) = \lambda x(t) \Longleftrightarrow tx(t) + \int_0^1 tsx(s)ds = \lambda x(t)$$

tenglik bajarilsin. U holda

$$(t-\lambda)x(t) + t \cdot \alpha_x = 0, \tag{4.11}$$

bunda

$$\alpha_x = \int_0^1 sx(s)ds. \tag{4.12}$$

Agar $\alpha_x = 0$ bo'lsa, (4.11) tenglik $(t-\lambda)x(t) = 0$ tenglikka aylanadi. Bundan x ning nol ekanligiga kelamiz. Farazimizga ko'ra x xos vektor, ya'ni $x \neq 0$. Shunday ekan, $\alpha_x \neq 0$. Yuqoridagi (4.11) tenglikdan

$$x(t) = -\frac{\alpha_x \cdot t}{t-\lambda}$$

ni topamiz va buni (4.12) ga qo'yib,

$$\alpha_x = -\alpha_x \int_0^1 \frac{s^2}{s-\lambda} ds \qquad \Longleftrightarrow \qquad \alpha_x \left(1 + \int_0^1 \frac{s^2}{s-\lambda} ds\right) = 0$$

tenglikka ega bo'lamiz. $\alpha_x \neq 0$ bo'lganligi uchun

$$1 + \int_0^1 \frac{s^2}{s-\lambda} ds = 0 \qquad \text{ya'ni} \qquad \Delta(\lambda) = 0$$

tenglikni hosil qilamiz.

Yetarliligi. Aytaylik, $\lambda \in \mathbb{R}\setminus[0,\ 1]$ soni uchun $\Delta(\lambda) = 0$ tenglik bajarilsin. U holda $x(t) = t\,(t-\lambda)^{-1}$ funksiyani olsak,

$$(A-\lambda I)x(t) = \frac{(t-\lambda)t}{t-\lambda} + t\int_0^1 \frac{s\cdot s}{s-\lambda}\,ds = t\left(1+\int_0^1 \frac{s^2 ds}{s-\lambda}\right) = t\Delta(\lambda) = 0$$

tenglik bajariladi. Bundan λ soni A operator uchun xos qiymat bo'lishi va $x(t) = t\,(t-\lambda)^{-1}$ unga mos xos funksiya bo'lishi kelib chiqadi. □

A operatorning $[0,\ 1]$ kesmadan tashqaridagi xos qiymatlarini topamiz. Barcha manfiy λ lar uchun

$$\Delta(\lambda) = 1 + \int_0^1 \frac{s^2}{s-\lambda}\,ds \geq 1$$

tengsizlik o'rinli. 4.1-tasdiqqa ko'ra, A operatorning manfiy xos qiymatlari yo'q. Endi A operatorning 1 dan katta xos qiymatlari mavjudmi degan savolga javob beramiz. Aytaylik, $\lambda > 1$ bo'lsin. U holda

$$\Delta'(\lambda) = \int_0^1 \frac{s^2\,ds}{(s-\lambda)^2} > 0, \qquad \lim_{\lambda\to 1}\Delta(\lambda) = -\infty, \qquad \lim_{\lambda\to+\infty}\Delta(\lambda) = 1$$

munosabatlar o'rinli bo'ladi. Hosilaning musbatligidan $\Delta(\cdot)$ ning $(1,\infty)$ intervalda o'suvchi ekanligi kelib chiqadi. Limitik munosabatlardan biror ε va N uchun $\Delta(1+\varepsilon)\,\Delta(N) < 0$ tengsizlik kelib chiqadi. Ya'ni $\Delta(\cdot)$ funksiya $[1+\varepsilon,\ N]$ kesmaning chetki nuqtalarida har xil ishorali qiymatlar qabul qiladi. Bolsana-Koshi teoremasiga ko'ra, shunday $\lambda_0 \in (1+\varepsilon,\ N)$ nuqta mavjudki, $\Delta(\lambda_0) = 0$ bo'ladi. $\Delta(\cdot)$ funksiyaning qat'iy monotonligidan $\lambda_0 \in (1,\infty)$ nuqtaning yagonaligi kelib chiqadi. Shunday qilib, $[0,\ 1]$ kesmadan tashqarida A operatorning yagona xos qiymati bor ekan. Agar $\lambda \in \mathbb{C}\setminus[0,\ 1]$ soni A operatorning xos qiymati bo'lmasa, $A - \lambda I$ operator teskarilanuvchan bo'ladi. Rezolventa uchun tenglama

$$(A-\lambda I)x(t)=y(t) \quad \Longleftrightarrow \quad (t-\lambda)x(t)+t\int\limits_0^1 sx(s)ds=y(t)$$

dan (4.12) ni e'tiborga olgan holda $x(t)$ ni topamiz:

$$(t-\lambda)x(t)+t\cdot\alpha_x=y(t),$$

bundan

$$x(t)=\frac{y(t)}{t-\lambda}-\alpha_x\frac{t}{t-\lambda}. \tag{4.13}$$

$x(t)$ uchun olingan (4.13) ifodani (4.12) ga qo'ysak, α_x uchun

$$\alpha_x=\int\limits_0^1\frac{sy(s)\,ds}{s-\lambda}-\alpha_x\int\limits_0^1\frac{s^2\,ds}{s-\lambda}\Longleftrightarrow\alpha_x\left(1+\int\limits_0^1\frac{s^2ds}{s-\lambda}\right)=\int\limits_0^1\frac{sy(s)\,ds}{s-\lambda}$$

tenglamaga ega bo'lamiz. Bu yerdan $\Delta(\lambda)\neq 0$ bo'lganligi uchun

$$\alpha_x=\frac{1}{\Delta(\lambda)}\int\limits_0^1\frac{sy(s)}{s-\lambda}ds$$

ni hosil qilamiz. Demak,

$$x(t)=\frac{y(t)}{t-\lambda}-\frac{t}{t-\lambda}\frac{1}{\Delta(\lambda)}\int\limits_0^1\frac{sy(s)}{s-\lambda}ds.$$

Shunday qilib, A operatorning rezolventasi quyidagi formula bilan aniqlanadi:

$$R_\lambda(A)y(t)=\frac{y(t)}{t-\lambda}-\frac{t}{t-\lambda}\frac{1}{\Delta(\lambda)}\int\limits_0^1\frac{sy(s)}{s-\lambda}ds, \qquad y\in L_2[0,\ 1].$$

Olingan natijalardan ko'rinib turibdiki, agar $\lambda\in\mathbb{C}\setminus([0,\ 1]\cup\{\lambda_0\})$ bo'lsa, u holda $D(R_\lambda(A))=L_2[0,\ 1]$, ya'ni λ regulyar nuqta. Bu yerdan $\rho(A)\supset\mathbb{C}\setminus([0,\ 1]\cup\{\lambda_0\})$ ekanligi kelib chiqadi.

b) 4.16-misol xulosasidan kelib chiqadiki, $(Bx)(t) = tx(t)$, $x \in L_2[0, 1]$ operatorning spektri faqat muhim spektrdan iborat bo'lib, u $[0, 1]$ kesma bilan ustma-ust tushadi. Hozir biz ko'rsatamizki, $A = B + C$ operatorning ham muhim spektri $[0, 1]$ kesma bilan ustma-ust tushadi. Aytaylik, $\lambda \in [0, 1)$ va $\{f_n\}$ – (4.9) tenglik bilan aniqlanuvchi nolga kuchsiz yaqinlashuvchi ketma-ketlik bo'lsin. Ko'rsatamizki,

$$\lim_{n\to\infty} \|(A - \lambda I)f_n\| = \lim_{n\to\infty} \|(B - \lambda I)f_n + Cf_n\| = 0 \tag{4.14}$$

bo'ladi. $\|(A - \lambda I)f_n\| \le \|(B - \lambda I)f_n\| + \|Cf_n\|$ tengsizlikni inobatga olsak, faqat $\lim_{n\to\infty} \|Cf_n\| = 0$ ekanligini ko'rsatish kifoya, chunki

$$\lim_{n\to\infty} \|(B - \lambda I)f_n\| = 0$$

shart 4.16-misolda ko'rsatilgan. Dastlab, $(Cf_n)(t)$ ni hisoblaymiz.

$$(Cf_n)(t) = \int_0^1 t\, s\, f_n(s)ds = \sqrt{n(n+1)}\,t \int_{\lambda+\frac{1}{n+1}}^{\lambda+\frac{1}{n}} s\, ds =$$

$$= \frac{\sqrt{n(n+1)}}{2n(n+1)} \left(2\lambda + \frac{2n+1}{n(n+1)}\right) t.$$

Agar $g(t) = t$ desak, $\|g\| = \dfrac{1}{\sqrt{3}}$ bo'lib,

$$\lim_{n\to\infty} \|Cf_n\| = \lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{2n(n+1)} \left(2\lambda + \frac{2n+1}{n(n+1)}\right) \frac{1}{\sqrt{3}} = 0$$

bo'ladi. 4.4-ta'rifga ko'ra $[0, 1) \subset \sigma_{ess}(A)$ bo'ladi. Agar $\lambda = 1$ bo'lsa, nolga kuchzis yaqinlashuvchi ketma-ketlik sifatida $\{g_n\}$ ni olish mumkin:

$$g_n(x) = \sqrt{n(n+1)}\,\chi_{B_n}(x), \quad B_n = \left[\frac{n-1}{n}, \frac{n}{n+1}\right), \quad (g_n, g_m) = \delta_{nm}.$$

Bu hol uchun ham $\lim_{n\to\infty} \|(A - I)g_n\| = 0$ shart bajariladi. Bu yerdan, $\lambda = 1$ soni A operatorning muhim spektriga qarashli ekanligi kelib chiqadi.

Demak,

$$\sigma_{ess}(A) = [0,\ 1], \quad \sigma_{pp}(A) = \{\lambda_0\}, \quad \sigma(A) = [0,\ 1] \cup \{\lambda_0\}. \qquad \square$$

4.26-4.30-misollarda keltirilgan operatorlarning o'z-o'ziga qo'shma ekanligini ko'rsating, xos qiymatlari mavjudligini tekshiring, muhim spektri va rezolventasini toping.

4.26. $(Ax)(t) = \cos t\, x(t) + \int\limits_{-\pi}^{\pi} \sin t \sin s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

4.27. $(Ax)(t) = \cos 2t\, x(t) - \int\limits_{-\pi}^{\pi} \sin t \sin s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

4.28. $(Ax)(t) = \cos 4t\, x(t) + \int\limits_{-\pi}^{\pi} \cos t \cos s\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

4.29. $(Ax)(t) = x(t) - \frac{1}{\pi} \int\limits_{-\pi}^{\pi} (1 + \sin t \sin s)\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

4.30. $(Ax)(t) = x(t) - \frac{1}{\pi} \int\limits_{-\pi}^{\pi} (1 + \cos(t - s))\, x(s)\, ds, \quad x \in L_2[-\pi,\ \pi].$

4.31. Banax fazosi ℓ_1 da berilgan

$$A : \ell_1 \to \ell_1, \quad Ax = (x_1 + x_2, x_1 + 2x_2, 4x_3, x_4, x_5, x_6, \ldots)$$

operatorning spektri va rezolventasini toping.

Yechish. a) A operatorning xos qiymatlarini topish uchun $\lambda \in \mathbb{C}$ songa mos $Ax = \lambda x$ yoki

$$(x_1 + x_2, x_1 + 2x_2, 4x_3, x_4, x_5, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots) \qquad (4.15)$$

tenglamani yechamiz. (4.15) tenglama quyidagi tenglamalar sistemaga teng kuchli:

$$x_1 + x_2 = \lambda x_1, \ \ x_1 + 2x_2 = \lambda x_2, \ \ 4x_3 = \lambda x_3, \ \ x_n = \lambda x_n, \ \ n \geq 4. \quad (4.16)$$

Agar $\lambda \notin \{1, 4\}$ bo'lsa, (4.16) tenglik bajarilishi uchun $x_3 = x_4 = x_5 = \ldots = 0$ bo'lishi kerak. U holda (4.16) tenglamalar sistemasi

$$\begin{cases} (1-\lambda)x_1 + x_2 = 0 \\ x_1 + (2-\lambda)x_2 = 0 \end{cases}$$

sistemaga teng kuchli. Bu sistema nolmas yechimga ega bo'lishi uchun uning determinanti $(1-\lambda)(2-\lambda) - 1 = 0$ yoki

$$\lambda^2 - 3\lambda + 1 = 0 \tag{4.17}$$

tenglik bajarilishi kerak. (4.17) tenglama $\lambda_1 = 2^{-1}(3-\sqrt{5})$, $\lambda_2 = 2^{-1}(3+\sqrt{5})$ ildizlarga ega. Bu yerdan kelib chiqadiki, (4.15) tenglama $\lambda_1 = 2^{-1}(3-\sqrt{5})$ songa mos nolmas

$$x^{(1)} = \left(1, \frac{1-\sqrt{5}}{2}, 0, 0, \ldots\right) \tag{4.18}$$

yechimga va $\lambda_2 = 2^{-1}(3+\sqrt{5})$ soniga mos nolmas

$$x^{(2)} = \left(1, \frac{1+\sqrt{5}}{2}, 0, 0, \ldots\right) \tag{4.19}$$

yechimga ega. Shunday qilib, $\lambda_1 = 2^{-1}(3-\sqrt{5})$, $\lambda_2 = 2^{-1}(3+\sqrt{5})$ sonlari A operatorning xos qiymatlari bo'ladi. (4.18) va (4.19) ko'rinishdagi $x^{(1)}$ va $x^{(2)}$ elementlar A operatorning λ_1 va λ_2 xos qiymatlariga mos xos vektorlari bo'ladi. Agar $\lambda_3 = 1$ bo'lsa $x^{(3)} = e_n$, $n \geq 4$ nolmas elementlar $Ax = 1 \cdot x$ tenglikni qanoatlantiradi, ya'ni 1 soni A operatorning cheksiz karrali xos qiymati va $x^{(3)} = e_n$, $n \geq 4$ ko'rinishdagi elementlar unga mos xos vektorlar bo'ladi. Xuddi shunday ko'rsatish mumkinki, $\lambda_4 = 4$ soni ham A operatorning xos qiymati bo'ladi va $x^{(4)} = e_3$, esa unga mos xos vektor bo'ladi. Shunday qilib, A operator $2^{-1}(3-\sqrt{5})$, $2^{-1}(3+\sqrt{5})$, 1, 4 xos qiymatlarga ega va uning boshqa xos qiymatlari yo'q. $2^{-1}(3-\sqrt{5})$, $2^{-1}(3+\sqrt{5})$, 4 xos qiymatlar operatorning oddiy xos qiymatlari bo'ladi.

b) Endi $\lambda \notin \{2^{-1}(3-\sqrt{5}),\ 2^{-1}(3+\sqrt{5}),\ 1,\ 4\}$ holni qaraymiz. 4.2-ta'rifga ko'ra, A operatorning λ nuqtadagi rezolventasi $A-\lambda I$ operatorning teskarisi sifatida aniqlanadi. $(A-\lambda I)x=y$, ya'ni

$$((1-\lambda)x_1+x_2, x_1+(2-\lambda)x_2, (4-\lambda)x_3, (1-\lambda)x_4,\ldots)=$$

$$=(y_1, y_2, y_3\ldots) \tag{4.20}$$

tenglikdan x ni topamiz. Buning uchun

$$\begin{cases} (1-\lambda)x_1+x_2=y_1, \\ x_1+(2-\lambda)x_2=y_2, \\ (4-\lambda)x_3=y_3, \\ (1-\lambda)x_n=y_n, \quad n\geq 4. \end{cases}$$

tenglamalar sistemasini yechib,

$$\begin{cases} x_1=\dfrac{(2-\lambda)y_1-y_2}{\lambda^2-3\lambda+1}, \\ x_2=\dfrac{-y_1+(1-\lambda)y_2}{\lambda^2-3\lambda+1}, \\ x_3=(4-\lambda)^{-1}y_3, \\ x_n=(1-\lambda)^{-1}y_n, \quad n\geq 4 \end{cases}$$

munosabatlarni olamiz, ya'ni (4.20) tenglama yagona

$$x=\left(\frac{(2-\lambda)y_1-y_2}{\lambda^2-3\lambda+1}, \frac{-y_1+(1-\lambda)y_2}{\lambda^2-3\lambda+1}, \frac{y_3}{4-\lambda}, \frac{y_4}{1-\lambda}, \frac{y_5}{1-\lambda}, \ldots\right)$$

yechimga ega. Shunday qilib, A operatorning rezolventasi quyidagi formula bilan aniqlanadi:

$$R_\lambda(A)x=\left(\frac{(2-\lambda)x_1-x_2}{\lambda^2-3\lambda+1}, \frac{-x_1+(1-\lambda)x_2}{\lambda^2-3\lambda+1}, \frac{x_3}{4-\lambda}, \frac{x_4}{1-\lambda}, \frac{x_5}{1-\lambda}, \ldots\right).$$

Ko'rinib turibdiki, agar $\lambda \notin \{2^{-1}(3-\sqrt{5}),\ 2^{-1}(3+\sqrt{5}),\ 1,\ 4\}$ bo'lsa, $D(R_\lambda(A))=\ell_1$. 4.2-ta'rifga ko'ra, barcha

$$\lambda \notin \left\{\frac{3-\sqrt{5}}{2}; \frac{3+\sqrt{5}}{2}, 1, 4\right\}$$

lar. A operator uchun regulyar qiymat bo'ladi. Bundan

$$\sigma(A) = \left\{\frac{3-\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 1,\ 4\right\}$$

tenglik kelib chiqadi. □

4.32-4.46-misollarda keltirilgan operatorlarning xos qiymatlari va xos vektorlarini toping.

4.32. $A : \mathbb{C}^4 \to \mathbb{C}^4,\ Ax = (x_1,\ x_2 + x_3,\ x_2 - x_3,\ x_4)$.

4.33. $A : C[0,\ 2] \to C[0,\ 2],\ (Ax)(t) = x(0)t^2 + x(t)t + x(2)$.

4.34. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \int\limits_0^1 (t^2 s + ts^2) x(s)\, ds$.

4.35. $A : C[0,\ 2\pi] \to C[0,\ 2\pi],\ (Ax)(t) = \frac{1}{2\pi}\int\limits_0^{\pi} \cos(t+s) x(s)\, ds$.

4.36. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^1 (1 + ts) x(s)\, ds$.

4.37. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = 2\,x(-1)\,t + 3\,x(1)\,t^2$.

4.38. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = \int\limits_{-1}^1 (t + s + st) x(s)\, ds$.

4.39. $A : \ell_2 \to \ell_2,\ Ax = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_4, \ldots, x_n, \ldots)$

4.40. $A : \ell_2 \to \ell_2,\ Ax = (3x_1, 4x_2, -2x_3, 5x_4, x_5, x_6, \ldots)$.

4.41. $A : \ell_1 \to \ell_1,\ Ax = (\frac{1}{2}x_1, \frac{2}{3}x_2, \ldots, \frac{n}{n+1}x_n, \ldots)$.

4.42. $A : \ell_1 \to \ell_1,\ Ax = (\frac{1}{4}x_1, \frac{1}{2}x_2, \ldots, \frac{2n-1}{4n}x_{2n-1}, \frac{1}{2n}x_{2n}, \ldots)$

4.43. $A : \ell_1 \to \ell_1,\ Ax = (x_1 + x_2, x_1 - x_2, x_2 - x_3, x_4, 0, 0, \ldots)$.

4.44. $A : \ell_3 \to \ell_3,\ Ax = (x_1, 2x_2, x_3, 3x_4, x_5, x_6, \ldots)$.

4.45. $A : m \to m,\ Ax = (x_1, 2x_2 + x_3, x_2 + 3x_3, x_4, 0, 0, \ldots)$.

4.46. $A: m \to m, \quad Ax = (6\,x_1, 5\,x_2, 4\,x_3, 3\,x_4, 2\,x_5, \, x_6, x_7, \ldots)$.

4.47-4.64-misollarda keltirilgan operatorlarning spektri va rezolventasini toping.

4.47. $A: C[1,\ 3] \to C[1,\ 3], \quad (Ax)(t) = x(2)\,t + x(3)\,t^2$.

4.48. $A: C[0,\ 2] \to C[0,\ 2], \quad (Ax)(t) = x(1) + x(t)t + x(2)t$.

4.49. $A: C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t^2\,x(t) + x(0)$.

4.50. $A: C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t\,x(t) + x(0)t^2 + x(1)t^3$.

4.51. $A: C[-2,\ 2] \to C[-2,\ 2], \quad (Ax)(t) = |t|\,x(t)$.

4.52. $A: L_2[0,\ 1] \to L_2[0,\ 1], \quad (Ax)(t) = (t+2)\,x(t)$.

4.53. $A: L_2(-\infty, \infty) \to L_2(-\infty, \infty), \quad (Ax)(t) = \operatorname{arctg} t \cdot x(t)$.

4.54. $A: C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t^2\,x(t) + t\int\limits_0^1 s\,x(s)\,ds$.

4.55. $A: L_2[0, \infty) \to L_2[0, \infty), \quad (Ax)(t) = e^{-t}x(t) + \int\limits_0^\infty 2^{-t-s}x(s)ds$.

4.56. $A: L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (Ax)(t) = t\,x(t) + \int\limits_{-1}^1 t\,s\,x(s)\,ds$.

4.57. $A: \ell_2 \to \ell_2, \quad Ax = (-x_1, x_2, -x_3, \ldots, -x_{2n-1}, x_{2n}, \ldots)$.

4.58. $A: \ell_2 \to \ell_2, \quad Ax = \left(\frac{1}{3}\,x_1, \frac{1}{4}\,x_2, \ldots, \frac{1}{n+2}\,x_n, \ldots\right)$.

4.59. $A: \ell_2 \to \ell_2, \quad Ax = (0,\ x_2, \frac{x_3}{2}, \ldots, \frac{x_n}{n-1}, \ldots)$.

4.60. $A: \ell_1 \to \ell_1, \quad Ax = (x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots)$.

4.61. $A: \ell_1 \to \ell_1, \quad Ax = (0, x_1, x_2, \ldots, x_n, \ldots)$.

4.62. $A: \ell_1 \to \ell_1, \quad Ax = (x_2, x_3, \ldots, x_n, \ldots)$.

lar, A operator uchun regulyar qiymat bo'ladi. Bundan

$$\sigma(A)=\left\{\frac{3-\sqrt{5}}{2},\ \frac{3+\sqrt{5}}{2},\ 1,\ 4\right\}$$

tenglik kelib chiqadi. □

4.32-4.46-misollarda keltirilgan operatorlarning xos qiymatlari va xos vektorlarini toping.

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4.36. $A:C[-1,\ 1]\to C[-1,\ 1]$, $(Ax)(t)=\int\limits_{-1}^1(1+ts)x(s)\,ds$.

4.37. $A:C[-1,\ 1]\to C[-1,\ 1]$, $(Ax)(t)=2\,x(-1)\,t+3\,x(1)\,t^2$.

4.38. $A:L_2[-1,\ 1]\to L_2[-1,\ 1]$, $(Ax)(t)=\int\limits_{-1}^1(t+s+st)x(s)\,ds$.

4.39. $A:\ell_2\to\ell_2$, $Ax=(x_1+x_2,x_2+x_3,x_3+x_1,x_4,\ldots,x_n,\ldots)$.

4.40. $A:\ell_2\to\ell_2$, $Ax=(3x_1,4x_2,-2x_3,5x_4,x_5,x_6,\ldots)$.

4.41. $A:\ell_1\to\ell_1$, $Ax=(\frac{1}{2}\,x_1,\frac{2}{3}\,x_2,\ldots,\frac{n}{n+1}\,x_n,\ldots)$.

4.42. $A:\ell_1\to\ell_1$, $Ax=(\frac{1}{4}\,x_1,\frac{1}{2}\,x_2,\ldots,\frac{2n-1}{4n}\,x_{2n-1},\frac{1}{2n}\,x_{2n},\ldots)$.

4.43. $A:\ell_1\to\ell_1$, $Ax=(x_1+x_2,x_1-x_2,x_2-x_3,x_4,\ 0,\ 0,\ldots)$.

4.44. $A:\ell_3\to\ell_3$, $Ax=(x_1,2x_2,x_3,3x_4,x_5,x_6,\ldots)$.

4.45. $A:m\to m$, $Ax=(x_1,2x_2+x_3,x_2+3x_3,x_4,\ 0,\ 0,\ldots)$.

4.46. $A : m \to m, \quad Ax = (6\,x_1, 5\,x_2, 4\,x_3, 3\,x_4, 2\,x_5, x_6, x_7, \ldots)$.

4.47-4.64-misollarda keltirilgan operatorlarning spektri va rezolventasini toping.

4.47. $A : C[1,\ 3] \to C[1,\ 3], \quad (Ax)(t) = x(2)\,t + x(3)\,t^2.$

4.48. $A : C[0,\ 2] \to C[0,\ 2], \quad (Ax)(t) = x(1) + x(t)t + x(2)t.$

4.49. $A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t^2\,x(t) + x(0).$

4.50. $A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t\,x(t) + x(0)t^2 + x(1)t^3.$

4.51. $A : C[-2,\ 2] \to C[-2,\ 2], \quad (Ax)(t) = |t|\,x(t).$

4.52. $A : L_2[0,\ 1] \to L_2[0,\ 1], \quad (Ax)(t) = (t+2)\,x(t).$

4.53. $A : L_2(-\infty, \infty) \to L_2(-\infty, \infty), \quad (Ax)(t) = \mathrm{arctg}\,t \cdot x(t).$

4.54. $A : C[0,\ 1] \to C[0,\ 1], \quad (Ax)(t) = t^2\,x(t) + t\int\limits_0^1 s\,x(s)\,ds.$

4.55. $A : L_2[0, \infty) \to L_2[0, \infty), \quad (Ax)(t) = e^{-t}x(t) + \int\limits_0^\infty 2^{-t-s}x(s)ds.$

4.56. $A : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (Ax)(t) = t\,x(t) + \int\limits_{-1}^1 t\,s\,x(s)\,ds.$

4.57. $A : \ell_2 \to \ell_2, \quad Ax = (-x_1, x_2, -x_3, \ldots, -x_{2n-1}, x_{2n}, \ldots).$

4.58. $A : \ell_2 \to \ell_2, \quad Ax = \left(\frac{1}{3}\,x_1, \frac{1}{4}\,x_2, \ldots, \frac{1}{n+2}\,x_n, \ldots\right).$

4.59. $A : \ell_2 \to \ell_2, \quad Ax = (0,\ x_2, \frac{x_3}{2}, \ldots, \frac{x_n}{n-1}, \ldots).$

4.60. $A : \ell_1 \to \ell_1, \quad Ax = (x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots).$

4.61. $A : \ell_1 \to \ell_1, \quad Ax = (0, x_1, x_2, \ldots, x_n, \ldots).$

4.62. $A : \ell_1 \to \ell_1, \quad Ax = (x_2, x_3, \ldots, x_n, \ldots).$

4.63. $A : m \to m, \quad Ax = (x_1 + x_2, x_2 + x_1, x_3, x_4, x_5, \dots)$.

4.64. $A : m \to m, \quad Ax = (2\,x_1, \frac{3}{2}\,x_2, \frac{4}{3}\,x_3, \dots, \frac{n+1}{n}\,x_n, \dots)$.

4.65. O'z-o'ziga qo'shma $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operatorning xos qiymatlari soni (karraligi bilan qo'shib hisoblanganda) n ga tengligini isbotlang.

4.66. Shunday $A : \mathbb{C}^3 \to \mathbb{C}^3$ chiziqli operatorga misol keltiringki, uning yagona oddiy xos qiymati bo'lsin.

4.67. Shunday $A : \mathbb{C}^3 \to \mathbb{C}^3$ chiziqli operatorga misol keltiringki, uning bitta ikki karrali xos qiymati bo'lsin.

4.68. Faraz qilaylik $A : X \to X$ chiziqli operator va A^{-1} mavjud bo'lsin. A va A^{-1} operatorlar bir xil xos vektorlarga ega. Isbotlang.

4.69. Faraz qilaylik $A \in L(X)$ va A^2 ning xos vektori mavjud bo'lsin, u holda A ham xos vektorga ega. Isbotlang.

4.70. A va $R_\lambda(A)$ lar o'rin almashinuvchi operatorlardir. Isbotlang.

4.71. Faraz qilaylik $\lambda, \mu \in \rho(A)$ bo'lsin, u holda Hilbert ayniyati

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu)R_\lambda(A)R_\mu(A) = (\lambda - \mu)R_\mu(A)R_\lambda(A)$$

ni isbotlang.

4.72. Faraz qilaylik $A, B \in L(X), \quad \lambda \in \rho(A) \cap \rho(B)$ bo'lsin.

$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)(B - A)R_\lambda(B)$$

tenglikni isbotlang.

4.73. Faraz qilaylik $A \in L(X)$ bo'lsin. Biror $\lambda \in \rho(A)$ uchun $R_\lambda(A)$ to'la uzluksiz (kompakt) bo'lishi mumkinmi?

4.74. $C[0,\ 2\pi]$ fazoni o'zini-o'ziga akslantiruvchi $(Ax)(t) = e^{it}\,x(t)$ operatorni qaraymiz. $\sigma(A) = \{\lambda \in C : |\lambda| = 1\}$ tenglikni isbotlang.

4.75. $\ell_2(\mathbb{Z})$ fazoni o'zini-o'ziga akslantiruvchi $(Ax)(n) = e^{in}\,x(n)$ operatorni unitar ekanligini ko'rsating, uning spektri $\sigma(A)$ ni toping.

4.76. $\ell_2(\mathbb{Z})$ fazoda shunday unitar operatorga misol keltiringki, uning spektri $\sigma(A)$ ikki nuqtali to'plam bo'lsin.

4.77. Istalgan unitar operatorning spektrini saqlovchi minimal to'plamni toping.

4.78. Shunday o'z-o'ziga qo'shma operatorga misol keltiringki uning spektri $\sigma(A) = [m,\ M]$ bo'lsin. $m = \inf\limits_{\|x\|=1} (Ax, x)$, $M = \sup\limits_{\|x\|=1} (Ax, x)$.

4.79. Shunday o'z-o'ziga qo'shma operatorga misol keltiringki uning spektri $\sigma(A) = \{m,\ M\}$ bo'lsin.

4.80. $C[0,\ 1]$ fazoni o'zini-o'ziga akslantiruvchi $(Ax)(t) = x(0) + t\,x(1)$ operatorni qaraymiz. $\sigma(A)$ va $R_\lambda(A)$ larni toping.

4.81. L to'plam H Hilbert fazosining qism fazosi, P esa L ga ortogonal proyeksiyalash operatori bo'lsin. P operatorning spektrini toping, $R_\lambda(P)$ ni P orqali ifodalang.

4.82. H Hilbert fazosi, $\{e_n\}$, $n \in \mathbb{N}$ undagi ixtiyoriy ortonormal bazis bo'lsin. $A : H \to H$ operatorni quyidagicha aniqlaymiz:
$Ae_1 = 0, \quad Ae_{k+1} = e_k, \quad k \in \mathbb{N}$. Quyidagilarni isbotlang.

a) A chiziqli chegaralangan operator;

b) $A^* e_k = e_{k+1}$ tenglik o'rinli;

c) $\sigma(A) = \{\lambda \in C : \ |\lambda| \le 1\}$ tenglik o'rinli;

d) $\sigma(A) = \{\lambda \in C : \ |\lambda| < 1\}$ to'plamning ixtiyoriy nuqtasi A operatorning xos qiymati bo'ladi;

e) $\sigma(A^*) = \{\lambda \in C: \ |\lambda| \leq 1\}$ tenglik o'rinli;

f) A^* operator xos qiymatlarga ega emas, ya'ni $\sigma_{pp}(A^*) = \emptyset$.

4.83. $C[0, 1]$ fazoda differensiallash operatori $(Ax)(t) = \dfrac{dx(t)}{dt}$ ni qaraymiz. Quyidagilarni isbotlang.

a) agar $D(A) = \{x \in C^{(1)}[0, 1]: \ x(0) = 0\}$ bo'lsa, $\sigma(A) = \emptyset$.

b) agar $D(A) = C^{(1)}[0, 1]$ bo'lsa, u holda $\sigma(A) = \mathbb{C}$ tenglik o'rinli, hamda istalgan kompleks son A operatorning xos qiymati bo'ladi.

c) agar $D(A) = \{x \in C^{(1)}[0, 1] : x(0) = x(1)\}$ bo'lsa, u holda $\sigma(A)$ faqat $2\pi i n$, $n \in \mathbb{Z}$ ko'rinishdagi xos qiymatlardan iborat.

4.84. Faraz qilaylik $A, B \in L(X)$ bo'lsin. $\sigma(A{\cdot}B)$ va $\sigma(B{\cdot}A)$ to'plamlarning noldan farqli elementlari bir xil ekanligini isbotlang.

4.85. Faraz qilaylik, $A \in L(X)$ bo'lsin. $\lambda \in \mathbb{C}$ soni A operatorning spektriga qarashli bo'lishi uchun, shunday $x_n \in D(A)$, $n \in \mathbb{N}$, $\|x_n\| = 1$ ketma-ketlik mavjud bo'lib, $\|Ax_n - \lambda x_n\| \to 0$ munosabatning bajarilishi zarur va yetarli. Isbotlang.

4.86. Faraz qilaylik $A \in L(X)$, $\lambda \in \sigma(A)$ bo'lsin. Istalgan $n \in \mathbb{N}$ uchun $\lambda^n \in \sigma(A^n)$ ekanligini isbotlang.

4.87. Faraz qilaylik, $A \in L(X)$ uchun uzluksiz teskari operator mavjud bo'lsin. $\lambda \in \sigma(A^{-1})$ bo'lishi uchun $\lambda^{-1} \in \sigma(A)$ bo'lishi zarur va yetarli. Isbotlang.

4.88. Unitar ekvivalent $\hat{A}$ va $A = U\hat{A}U^{-1}$ operatorlar uchun quyidagilarni isbotlang.

a) $\sigma(A) = \sigma(\hat{A})$, b) $\sigma_{pp}(A) = \sigma_{pp}(\hat{A})$, c) $\sigma_{ess}(A) = \sigma_{ess}(\hat{A})$.

4.89. Har bir $s \in \mathbb{Z}$ uchun $\hat{U}_s : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(\hat{U}_s f)(n) = f(n+s)$ deymiz. $F : \ell_2(\mathbb{Z}) \to L_2[-\pi, \pi]$ esa 3.58-misolda aniqlangan Furye almashtirishi. $U_s = F\hat{U}_s F^{-1}$ uchun $(U_s f)(p) = e^{-isp} f(p)$, $f \in L_2[-\pi, \pi]$ tenglikni isbotlang.

4.90. $\hat{\Delta} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(\hat{\Delta}\hat{f})(n) = \hat{f}(n+1) + \hat{f}(n-1) - 2\hat{f}(n)$ operator *ayirmali Laplas operatori* deyiladi. $\Delta = F\hat{\Delta}F^{-1}$ operatorning $f \in L_2[-\pi, \pi]$ elementga ta'sirini toping.

Yechish: Ayirmali Laplas operatori $\hat{\Delta}$ uchun $\hat{\Delta} = \hat{U}_1 + \hat{U}_{-1} - 2\hat{U}_0$ tenglik o'rinli. Demak, $\Delta = F\hat{\Delta}F^{-1} = U_1 + U_{-1} - 2U_0$. 4.89-misol natijasiga ko'ra Δf, $f \in L_2[-\pi, \pi]$ uchun

$$(\Delta f)(p) = (U_1 + U_{-1} - 2U_0)f(p) = (e^{ip} + e^{-ip} - 2)f(p) = 2(\cos p - 1)f(p).$$

tenglikni olamiz. □

4.91. $(\hat{\Delta}\hat{f})(n) = \sum_{|s|=1} (\hat{f}(n+s) - \hat{f}(n))$, $\hat{f} \in \ell_2(\mathbb{Z}^\nu)$ operatorga unitar ekvivalent bo'lgan $\Delta = F\hat{\Delta}F^{-1}$ operatorning $f \in L_2([-\pi, \pi]^\nu)$ elementga ta'sirini toping.

4.92. $(\hat{V}\hat{f})(n) = \hat{v}(n)\hat{f}(n)$, $\hat{f} \in \ell_2(\mathbb{Z})$ operatorga unitar ekvivalent bo'lgan $V = F\hat{V}F^{-1}$ operatorning $f \in L_2[-\pi, \pi]$ elementga ta'sirini toping.

4.93. O'z-o'ziga qo'shma $A : H \to H$ operator uchun H_1 qism fazo invariant bo'lsin. U holda $H_2 = H \ominus H_1$ qism fazoning A uchun invariant ekanligini hamda $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$, $A_i = A|_{H_i}$, $i = 1, 2$, tenglikni isbotlang.

4.94. Agar H Hilbert fazosi uchun $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ yoyilma o'rinli bo'lib, H_k, $k \in \{1, 2, \ldots, n\}$ qism fazolar o'z-o'ziga qo'shma A operator uchun invariant bo'lsa, $\sigma(A) = \bigcup_{k=1}^{n} \sigma(A_k)$ tenglikni isbotlang.

4.95. $(Af)(p) = (2-2\cos p)f(p) - \frac{\mu}{2\pi}\int_{-\pi}^{\pi}\cos(p-s)f(s)\,ds, \quad f \in L_2[-\pi,\ \pi]$ operator uchun juft funksiyalardan iborat qism fazo $L_2^+[-\pi,\ \pi]$ va toq funksiyalardan iborat qism fazo $L_2^-[-\pi,\ \pi]$ ning invariant ekanligini ko'rsating. $A^+ = A|_{L_2^+[-\pi,\ \pi]}$ operatorning $f^+ \in L_2^+[-\pi,\ \pi]$ elementga, $A^- = A|_{L_2^-[-\pi,\ \pi]}$ operatorning $f^- \in L_2^-[-\pi,\ \pi]$ elementga ta'sirini toping.

4.96. $(Vf)(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty} e^{-n}\cos n(p-s)f(s)\,ds, \quad f \in L_2[-\pi,\ \pi]$ operator uchun 4.95-misolda aniqlangan $L_2^+[-\pi,\ \pi]$ va $L_2^-[-\pi,\ \pi]$ qism fazolarning invariant ekanligini ko'rsating. Quyidagilarni toping:

a) $V^+ = V|_{L_2^+[-\pi,\ \pi]}$, $V^- = V|_{L_2^-[-\pi,\ \pi]}$ operatorlarning xos qiymatlari va xos funksiyalarini toping.

b) V operatorning barcha xos qiymatlari va xos funksiyalarini toping. Oddiy va karrali xos qiymatlarini ajrating.

4.97. Aniqlanish sohasi $L_2[0,\ 1]$ Hilbert fazosi bo'lgan o'z-o'ziga qo'shma $(Af)(t) = tf(t)$ operatorning spekrtal proyektorlarini toping.

Yechish. Spektral proyektorlar sifatida $(E_\lambda f)(t) = \chi_{A_\lambda}(t)\, f(t)$ larni olamiz, bu yerda $A_\lambda = [0,\ \lambda]\cap[0,\ 1]$. Osongina tekshirish mumkinki, E_λ uchun spektral proyektorlarning 1-3 shartlari bajariladi. $A = \int_{-\infty}^{\infty} \lambda d\, E_\lambda$ tenglikni tekshiramiz. Ixtiyoriy $f,\ g \in L_2[0,\ 1]$ elementlarni olaylik. U holda

$$(Af, g) = \int_0^1 tf(t)\overline{g(t)}\,dt = \int_0^1 \lambda\, f(\lambda)\,\overline{g(\lambda)}d\lambda =$$

$$= \int_0^1 \lambda\, d\int_0^{\lambda} f(t)\,\overline{g(t)}dt = \int_0^1 \lambda d\left(\int_0^1 \chi_{[0,\lambda]}(t)f(t)\,\overline{g(t)}dt\right) =$$

$$= \int_{-\infty}^{\infty} \lambda\, d\,(E_\lambda f(t), g(t)). \qquad \square$$

4.98. $(Af)(t) = \cos t\, f(t),\ f \in L_2[-\pi,\ \pi]$ operatorga mos spektral proyektorlarni toping.

4.99. Agar φ funksiya $[-\pi,\ \pi]$ da o'lchovli va chegaralangan bo'lsa, u holda $(Af)(t) = \varphi(t)\, f(t),\ f \in L_2[-\pi,\ \pi]$ operatorga mos spektral proyektorlarni $(E_\lambda f)(t) = \chi_{\varphi^{-1}(-\infty,\lambda]}(t)\, f(t)$ shaklda tanlash mumkinligini isbotlang.

I bobni takrorlash uchun test savollari

1. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Af)(x) = \int\limits_{-1}^{1} xyf(y)dy$ operator yadrosini toping.

A) $KerA = \{f : f(x) = const\}$ B) $KerA = \{f : f(x) = \alpha + \beta x\}$

C) $KerA = \{f : \int\limits_{-1}^{1} yf(y)dy = 0\}$ D) $KerA = \{0\}$

2. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Af)(x) = \int\limits_{-1}^{1} (1 + xy)f(y)dy$ operatorning qiymatlar sohasini toping.

A) $ImA = \{f : f(x) = const\}$ B) $ImA = \{f : f(x) = \alpha + \beta x\}$

C) $ImA = \{f : \int\limits_{-1}^{1} yf(y)dy = 0\}$ D) $ImA = \{0\}$

3. $A : C[a,\ b] \to C[a,\ b],\ (Af)(x) = f'(x)$ differensial operator yadrosini toping.

A) $KerA = \{f : f(x) = const\}$ B) $KerA = \{f : f(x) = \alpha + \beta x\}$

C) $KerA = \{f : \int\limits_{a}^{b} f(x)dx = 0\}$ D) $KerA = \{0\}$

4. $A : C[a,\ b] \to C[a,\ b],\ (Af)(x) = f'(x)$ differensial operatorning aniqlanish sohasini toping.

A) $D(A) = C[a,\ b]$ B) $D(A) = \{f : f(x) = const\}$

C) $D(A) = C^{(1)}[a,\ b]$ D) $D(A) = \{f : \int_a^b f(x)dx = 0\}$

5. $A : C[0,\ 1] \to C[0,\ 1],\ (Af)(x) = (x + 1)f(x)$ operatorning kvadratini toping.

A) $(A^2f)(x) = (x+1)^2 f^2(x)$ B) $(A^2f)(x) = (x+1)^2 f(x)$
C) $(A^2f)(x) = (x^2+1) f(x)$ D) $(A^2f)(x) = (x+1)^2 f(x^2)$

6. Qachon $\lambda \in \mathbb{C}$ soni A operator uchun regulyar nuqta deyiladi?
A) Shunday C operator mavjud bo'lib, $A = \lambda C$ bo'lsa.
B) Agar $(A - \lambda I)^{-1}$ operator mavjud va chegaralangan bo'lsa.
C) Agar $Ax = \lambda x$ tenglama nolmas yechimga ega bo'lsa.
D) Agar $Ax = \lambda x$ tenglama yagona $x = 0$ yechimga ega bo'lsa.

7. Qachon $\lambda \in \mathbb{C}$ soni A operatorning xos qiymati deyiladi?
A) Shunday C operator mavjud bo'lib, $A = \lambda C$ bo'lsa.
B) Agar $(A - \lambda I)^{-1}$ operator mavjud va chegaralangan bo'lsa.
C) Agar $Ax = \lambda x$ tenglama noldan farqli yechimga ega bo'lsa.
D) Agar $Ax = \lambda x$ tenglama yagona $x = 0$ yechimga ega bo'lsa.

8. $A : X \to X$ operator spektri ta'rifini keltiring.
A) Barcha xos qiymatlar to'plami operatorning spektri deyiladi.
B) Regulyar bo'lmagan barcha $\lambda \in \mathbb{C}$ lar to'plami operatorning spektri deyiladi.
C) Barcha regulyar nuqtalar to'plami A operatorning spektri deyiladi.
D) Barcha $|\lambda| > \|A\|$ lar to'plami A operatorning spektri deyiladi.

9. $\{A_n\} \subset L(X,Y)$ operatorlar ketma-ketligining $A \in L(X,Y)$ operatorga tekis yaqinlashish ta'rifini toping.
A) agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo'lsa.
B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa.
C) ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$ bo'lsa.
D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = (Ax, y)$ bo'lsa.

10. $\{A_n\} \subset L(X,Y)$ operatorlar ketma-ketligining $A \in L(X,Y)$ operatorga kuchli yaqinlashish ta'rifini toping.

A) agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo'lsa.

B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa.

C) ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$ bo'lsa.

D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = (Ax, y)$ bo'lsa.

11. $\{A_n\} \subset L(X,Y)$ operatorlar ketma-ketligining $A \in L(X,Y)$ operatorga kuchsiz yaqinlashish ta'rifini toping.

A) agar $\lim\limits_{n\to\infty} \|A_n - A\| = 0$ bo'lsa.

B) ixtiyoriy $x \in X$ uchun $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$ bo'lsa.

C) ixtiyoriy $f \in Y^*$ va barcha $x \in X$ lar uchun $\lim\limits_{n\to\infty} f(A_n x) = f(Ax)$ bo'lsa.

D) ixtiyoriy $x, y \in H = X = Y$ uchun $\lim\limits_{n\to\infty} (A_n x, y) = |(Ax, y)|$ bo'lsa.

12. Nol operatorga kuchsiz ma'noda yaqinlashuvchi, lekin kuchli ma'noda yaqinlashmaydigan operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2, \quad A_n x = (\underbrace{0, 0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2, \quad Q_n x = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (A_n f)(x) = \sin^n x f(x)$

D) $P_n : \ell_2 \to \ell_2; \quad P_n x = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

13. Nol operatorga kuchli ma'noda yaqinlashuvchi, lekin tekis yaqinlashmaydigan operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2, \quad A_n x = (\underbrace{0, 0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2, \quad Q_n x = (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1,\ 1] \to L_2[-1,\ 1], \quad (A_n f)(x) = \sin^n x f(x)$

D) To'g'ri javob keltirilmagan.

14. Nol operatorga tekis yaqinlashuvchi operatorlar ketma-ketligini ko'rsating.

A) $A_n : \ell_2 \to \ell_2, \quad A_n x = (\underbrace{0, \ldots, 0}_{n}, x_1, x_2, x_3, \ldots)$

B) $Q_n : \ell_2 \to \ell_2, \quad Q_n x = (0, \ldots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

C) $A_n : L_2[-1, 1] \to L_2[-1, 1], \quad (A_n f)(x) = \sin^n x f(x)$

D) $P_n : \ell_2 \to \ell_2, \quad P_n x = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

15. Noto'g'ri tasdiqni toping.

A) Agar A chiziqli operator bo'lsa, A^{-1} ham chiziqli operator bo'ladi.

B) Agar $A \in L(X, Y)$ bo'lsa, u holda $A^{-1} \in L(Y, X)$ bo'ladi.

C) Agar A chiziqli operator bo'lsa, u holda A^* ham chiziqlidir.

D) Agar $A \in L(X, Y)$ bo'lsa, u holda $A^* \in L(Y^*, X^*)$ bo'ladi.

16. $A : X \to Y$ chiziqli operator teskarilanuvchan bo'lishi uchun quyidagi shartlardan qaysi birining bajarilishi zarur va yetarli.

A) $\|A\| \leq q < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim Ker A = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \geq m \|x\|$ bo'lishi

17. $A : X \to Y$ operatorga chegaralangan teskari operator mavjud bo'lishining zarur va yetarli shartini keltiring.

A) $\|A\| \leq q < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim Ker A = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \geq m \|x\|$ bo'lishi

18. $I - A : X \to X$ operatorga chegaralangan teskari operator mavjud bo'lishining yetarli shartini keltiring.

A) $\|A\| < 1$ B) $Ax = 0 \Longleftrightarrow x = 0$ C) $\dim Ker A = 1$

D) biror $m > 0$ va barcha $x \in D(A)$ larda $\|Ax\| \geq m \|x\|$ bo'lishi

19. $A - A' : X \to X$ operatorga chegaralangan teskari operator mavjud bo'lishining yetarli shartini keltiring.

A) $\|A\| < 1$ B) $\|A'\| < \|A^{-1}\|^{-1}$

C) $Ker A = \{0\}$ D) $\|A'\| < \|A\|^{-1}$

20. $A : \mathbb{R}^3 \to \mathbb{R}^3$, $Ax = (x_1, 2x_2, 3x_3)$ operatorga teskari operatorni topimg.

A) $A^{-1}x = (3x_3, 2x_2, x_1)$ B) $A^{-1}x = (x_1, 2^{-1}x_2, 3^{-1}x_3)$

C) $A^{-1}x = (x_1, 2^{-2}x_2, 3^{-2}x_3)$ D) $A^{-1}x = (x_1, 2x_2^{-1}, 3x_3^{-1})$

21. $A : X \to Y$ chiziqli operator teskarilanuvchan bo'lishi uchun quyidagi shartlardan qaysi birining bajarilishi zarur va yetarli.

A) $\|A\| < 1$ B) $Ker A = \{0\}$ C) $\dim Ker A = 1$ D) $\|A\| \geq 1$

22. A operator chiziqli bo'lishini ta'minlaydigan shartlarni ajrating:

1) $A(x + y) = Ax + Ay$, 2) $A(\alpha x) = \alpha Ax$, 3) $A(\alpha x) = \overline{\alpha} Ax$.

A) 1, 2   B) 2, 3   C) 1, 2, 3   D) 1, 3

23. Chiziqli bo'lmagan $A : C[a, b] \to C[a, b]$ operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

24. $C[a, b]$ ni $C[a, b]$ ga akslantiruvchi birlik operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

25. $C[a, b]$ ni $C[a, b]$ ga akslantiruvchi nol operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

26. $C[a, b]$ ni $C[a, b]$ ga akslantiruvchi chegaralanmagan operatorni toping.

A) $(Af)(x) = f'(x)$ B) $(Af)(x) = f(x)$

C) $(Af)(x) = 0$ D) $(Af)(x) = f(x) + 1$

27. Quyidagilar ichidan A chiziqli chegaralangan operator normasini hisoblash formulalarini ajrating:

1) $\|A\| = \sup\limits_{\|x\|=1} \|Ax\|$, 2) $\|A\| = \sup\limits_{x\neq 0} \dfrac{\|Ax\|}{\|x\|}$, 3) $\|A\| = \inf\limits_{\|x\|=1} \|Ax\|$.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

28. Quyidagilar ichidan to'g'ri tasdiqlarni ajrating:

1) Operatorlarni qo'shish kommutativ.

2) Operatorlarni ko'paytirish kommutativ.

3) Operatorlarni ko'paytirish assotsiativ.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

29. Quyidagilar ichidan to'g'rilarini ajrating:

1) $\|A + B\| \leq \|A\| + \|B\|$,

2) $\|A \cdot B\| \leq \|A\| \cdot \|B\|$,

3) $\|A \cdot B\| = \|A\| \cdot \|B\|$.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

30. $A : X \to Y$ - chiziqli operator. Teng kuchli tasdiqlarni ajrating:

1) A operator biror x_0 nuqtada uzluksiz.

2) A operator uzluksiz.

3) A operator chegaralangan.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

31. $C[-1, 1]$ fazoda normasi 1 bo'lgan operatorlarni ko'rsating.

1) $(Af)(x) = xf(x)$. 2) $(Bf)(x) = f(x)$. 3) $(Cf)(x) = 0$.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

32. $\mathbb{R}^n$ fazoga qo'shma fazoni ko'rsating.

A) $\mathbb{R}^n$ B) $\mathbb{R}^n_\infty$ C) $\mathbb{R}^n_q$ D) $\mathbb{R}^n_p$

33. $\mathbb{R}^n_p$, $p > 1$ fazoga qo'shma fazoni ko'rsating.

A) $\mathbb{R}^n$ B) $\mathbb{R}^n_\infty$ C) $\mathbb{R}^n_q,\ p^{-1}+q^{-1}=1$ D) $\mathbb{R}^n_p$

34. $\mathbb{R}^n_1$ va $\mathbb{R}^n_\infty$ fazolarga qo'shma fazolarni ko'rsating.

A) $\mathbb{R}^n$ va $\mathbb{R}^n_1$ B) $\mathbb{R}^n_\infty$ va $\mathbb{R}^n_1$

C) $\mathbb{R}^n_1$ va $\mathbb{R}^n_\infty$ D) $\mathbb{R}^n_1$ va $\mathbb{R}^n$

35. $C[a, b]$ fazoga qo'shma fazoni ko'rsating.

A) $C[a, b]$ B) $V_0[a, b]$ C) $L_q[a, b]$ D) $L_2[a, b]$

36. $L_p[a, b]$, $p > 1$ fazoga qo'shma fazoni ko'rsating.

A) $C[a, b]$ B) $V_0[a, b]$ C) $L_q[a, b],\ p^{-1}+q^{-1}=1$ D) $L_2[a, b]$

37. $L_2[a, b]$ fazoga qo'shma fazoni ko'rsating.

A) $C[a, b]$ B) $V_0[a, b]$ C) $L_q[a, b],\ p^{-1}+q^{-1}=1$ D) $L_2[a, b]$

38. ℓ_2 va ℓ_1 fazolarga qo'shma fazolarni ko'rsating.

A) ℓ_2 va m B) ℓ_1 va ℓ_2 C) ℓ_1 va m D) ℓ_2 va c

39. c va c_0 fazolarga qo'shma fazolarni ko'rsating.

A) ℓ_2 va ℓ_1 B) ℓ_1 va ℓ_1 C) m va m D) ℓ_1 va c

40. ℓ_p, $p > 1$ fazoga qo'shma fazoni ko'rsating.

A) ℓ_p B) ℓ_∞ C) $\ell_q,\ p^{-1}+q^{-1}=1$ D) ℓ_1

41. $T : \ell_1 \to \ell_1,\ Tx = (0, x_1, x_2, x_3, \ldots, x_{n-1}, \ldots)$ ga qo'shma operatorni toping.

A) $T^* : m \to m,\ T^*x = (x_2, x_3, x_4, \ldots, x_{n+1}, \ldots)$

B) $T^* : \ell_2 \to \ell_2,\ T^*x = (x_2, x_3, x_4, \ldots, x_{n+1}, \ldots)$

C) $T^* : \ell_p \to \ell_q,\ T^*y = (y_2, y_3, y_4, \ldots, y_{n+1}, \ldots)$

D) $T^* : \ell_2 \to m,\ T^*y = (y_2, y_3, y_4, \ldots, y_{n+1}, \ldots)$

42. $T : \ell_2 \to \ell_2,\ Tx = (0, x_1, x_2, x_3, \ldots, x_{n-1}, \ldots)$ ga qo'shma operatorni toping.

A) $T^* : m \to m,\ T^*x = (x_2, x_3, x_4, \dots, x_{n+1}, \dots)$

B) $T^* : \ell_1 \to \ell_1,\ T^*x = (x_2, x_3, x_4, \dots, x_{n+1}, \dots)$

C) $T^* : \ell_p \to \ell_q,\ T^*y = (y_2, y_3, y_4, \dots, y_{n+1}, \dots)$

D) $T^* : \ell_2 \to \ell_2,\ T^*y = (y_2, y_3, y_4, \dots, y_{n+1}, \dots)$

43. $T : \ell_2 \to \ell_2,\ Tx = (a_1x_1, a_2x_2, a_3x_3, \dots, a_nx_n, \dots)$ operatorga qo'shma operatorni toping.

A) $T^* : m \to m,\ T^*x = (a_1x_1, a_2x_2, a_3x_3, \dots, a_nx_n, \dots)$

B) $T^* : \ell_2 \to \ell_2,\ T^*x = (\overline{a_1}x_1, \overline{a_2}x_2, \overline{a_3}x_3, \dots, \overline{a_n}x_n, \dots)$

C) $T^* : \ell_2 \to \ell_2,\ T^*x = (x_2, x_3, x_4, \dots, x_{n+1}, \dots)$

D) $T^* : \ell_2 \to \ell_2,\ T^*x = (\frac{1}{a_1}x_1, \frac{1}{a_2}x_2, \frac{1}{a_3}x_3, \dots, \frac{1}{a_n}x_n, \dots)$

44. $T : L_2[a,\ b] \to L_2[a,\ b],\ (Tf)(x) = u(x)f(x)$ operatorga Hilbert ma'nosidagi qo'shma operatorni toping.

A) $(T^*f)(x) = u(x)f(x)$ B) $(T^*f)(x) = \overline{u(x)}f(x)$

C) $(T^*f)(x) = \overline{u(x)f(x)}$ D) $(T^*f)(x) = \dfrac{f(x)}{u(x)}$

45. $T : L_2[a,\ b] \to L_2[a,\ b],\ (Tf)(x) = \int_a^b K(x,y)f(y)dy$ operatorga Hilbert ma'nosidagi qo'shma operatorni toping.

A) $(T^*f)(x) = \int_a^b \overline{K(y,x)}f(y)dy$ B) $(T^*f)(x) = \int_a^b K(y,x)f(y)dy$

C) $(T^*f)(x) = \int_a^b \overline{K(x,y)}f(y)dy$ D) $(T^*f)(x) = \int_a^b K^*(y,x)f(y)dy$

46. $A : \mathbb{C}^n \to \mathbb{C}^n$ chiziqli operator spektri haqidagi tasdiqlarning qaysi biri to'g'ri.

A) $\sigma(A)$ faqat chekli sondagi chekli karrali xos qiymatlardan iborat.

B) A ning spektri biror kesmani to'la to'ldiradi.

C) A ning spektri $(-\infty; \infty)$ to'plamning qismi.

D) $\sigma(A)$ doim nolni saqlaydi.

47. $A : \ell_2 \to \ell_2,\ \ Ax = (a_1x_1, a_2x_2, \dots, a_nx_n, \dots)$ operatorning spektrini toping.

A) $\sigma(A) = \{a_1, a_2, a_3, \dots, a_n, \dots\}$

B) $\sigma(A) = \overline{\{a_1, a_2, a_3, \dots, a_n, \dots\}}$

C) $\sigma(A) = \{\overline{a_1}, \overline{a_2}, \overline{a_3}, \dots, \overline{a_n}, \dots\}$

D) $\sigma(A) = \{1/a_1,\ 1/a_2,\ 1/a_3, \dots,\ 1/a_n, \dots\}$

48. $A : \ell_2 \to \ell_2,\ \ Ax = (a_1x_1, a_2x_2, \dots, a_nx_n, \dots)$ operatorning barcha xos qiymatlarini toping.

A) $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ B) $\overline{\{a_1, a_2, a_3, \dots, a_n, \dots\}}$

C) $\{\overline{a_1}, \overline{a_2}, \overline{a_3}, \dots, \overline{a_n}, \dots\}$ D) $\{1/a_1,\ 1/a_2,\ 1/a_3, \dots,\ 1/a_n, \dots\}$

49. $A : L_2[a, b] \to L_2[a, b]$. $(Af)(x) = xf(x)$ operatorning spektri haqida to'liq ma'lumotni toping.

A) $\sigma(A) = [a,\ b]$, $\sigma_{pp}(A) = \emptyset$, $\sigma_{qol}(A) = \emptyset$, $\sigma_{ess}(A) = [a,\ b]$

B) $\sigma(A) = [a,\ b]$, $\sigma_{pp}(A) = \emptyset$, $\sigma_{qol}(A) = [a,\ b]$, $\sigma_{ess}(A) = \emptyset$

C) $\sigma(A) = [a,\ b]$, $\sigma_{pp}(A) = [a,\ b]$, $\sigma_{qol}(A) = \emptyset$, $\sigma_{ess}(A) = \emptyset$

D) $\sigma(A) = [a,\ b]$, $\sigma_{pp}(A) = [a,\ b]$, $\sigma_{qol}(A) = \emptyset$, $\sigma_{ess}(A) = [a,\ b]$

50. $A : L_2[0,\ 1] \to L_2[0,\ 1]$. $(Af)(x) = xf(x)$ operatorning $\lambda \in C\backslash[0,\ 1]$ nuqtadagi rezolventasini toping.

A) $R_\lambda(A)f(x) = (x - \lambda)f(x)$ B) $R_\lambda(A)f(x) = (x - \lambda)^{-1}f(x)$

C) $R_\lambda(A)f(x) = (x - \overline{\lambda})^{-1}f(x)$ D) $R_\lambda(A)f(x) = |x - \lambda|^{-1}f(x)$

Bu bob ikki paragrafdan iborat. 5 – § kompakt operatorlarga, 6 – § integral tenglamalarga oid masalalarni o'z ichiga oladi

5-§. Kompakt operatorlar

Metrik va normalangan fazolarda kompakt, nisbiy kompakt to'plam ta'riflari II qism 7 – § da berilgan. Bu ta'riflarni esga olamiz. Chunki kompakt operatorlar shu tushunchalar asosida ta'riflanadi. Bizga $X-$ Banax fazosi va $M \subset X$ to'plam berilgan bo'lsin. Agar M to'plamdan olingan ixtiyoriy $\{x_n\}$ ketma-ketlikdan M da yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, M ga *kompakt to'plam* deyiladi. Agar N to'plamning yopig'i $[N]$ kompakt to'plam bo'lsa, u holda N *nisbiy kompakt to'plam* deyiladi. *To'plam nisbiy kompakt bo'lishi uchun uning to'la chegaralangan bo'lishi zarur va yetarli* (II qism 7.1-teorema). *Chekli o'lchamli fazolarda to'plam kompakt bo'lishi uchun uning chegaralangan va yopiq bo'lishi zarur va yetarlidir* (II qism 7.3-teorema). Asosiy funksional fazolardan biri $C[a, \ b]$ fazodir. Bu fazodagi to'plamning nisbiy kompaktlik kriteriysi Arsela teoremasi yordamida bayon qilingan (II qism 7.2-teorema). Agar $A \in L(X, Y)$ va dim $ImA < \infty$ bo'lsa, u holda A ga *chekli o'lchamli operator* (4.6-ta'rif) deyiladi. Agar dim $ImA = n$ bo'lsa, u holda A ga n *o'lchamli operator* deyiladi. Agar A operator X dagi har qanday chegaralangan to'plamni Y dagi nisbiy kompakt to'plamga akslantirsa, u holda A *kompakt operator yoki to'la uzluksiz operator* (4.7-ta'rif) deyiladi. Bu ta'rifga teng kuchli bo'lgan quyidagi ta'riflarni keltiramiz.

5.1-ta'rif. *Agar $A : X \to Y$ chiziqli operator X fazodagi birlik sharni Y fazodagi nisbiy kompakt to'plamga akslantirsa, u holda A kompakt operator deyiladi.*

5.2-ta'rif. *$A \in L(X, Y)$ (X, $Y-$ Banax fazolari) operator va ixtiyoriy $\{x_n\} \subset X$ chegaralangan ketma-ketlik berilgan bo'lsin. Agar $\{Ax_n\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa, u holda A ga kompakt operator deyiladi.*

5.3-ta'rif. *Agar H Hilbert fazosida aniqlangan A operator har qanday kuchsiz yaqinlashuvchi ketma-ketlikni kuchli yaqinlashuvchi ketma-ketlikka akslantirsa, u holda A kompakt operator deyiladi.*

Agar X Banax fazosini Y Banax fazosiga akslantiruvchi barcha kompakt operatorlar to'plamini $K(X, Y)$ orqali belgilasak, u holda $K(X, Y)$ Banax fazosi bo'ladi.

5.1-teorema. *Chekli o'lchamli operator kompaktdir.*

5.2-teorema. *Kompakt operatorga qo'shma operator kompaktdir.*

5.3-teorema. *Agar $\{A_n\}$ kompakt operatorlar ketma-ketligi A operatorga tekis yaqinlashsa, u holda A ham kompakt operator bo'ladi.*

5.4-teorema *(Hilbert-Shmidt). H Hilbert fazosida kompakt, o'z-o'ziga qo'shma, chiziqli A operator berilgan bo'sin. U holda H fazoda shunday $\{\phi_n\}$ to'la ortonormal sistema mavjudki, $A\phi_n = \lambda_n \phi_n$ va $\lim\limits_{n\to\infty} \lambda_n = 0$ tengliklar o'rinli.*

5.5-teorema. *Cheksiz o'lchamli fazodagi $A : X \to Y$ kompakt operatorning chegaralangan teskarisi mavjud emas.*

5.6-teorema. *A kompakt operatorning spektriga qarashli noldan farqli λ soni A uchun chekli karrali xos qiymat bo'ladi.*

Kompakt (to'la uzluksiz) operatorlar sinfi bir qator ajoyib xossalarga ega. Bu paragrafda shu xossalarga doir misollar qaraladi.

Endi operatorlarning kompakt yoki kompakt emasligini tekshirishga doir bir nechta misollar qaraymiz.

5.1. $A : \ell_p \to \ell_p (p \geq 1)$ operator $Ax = (a_1x_1, a_2x_2, \ldots, a_nx_n, \ldots,)$ tenglik bilan aniqlangan. Bu operator kompakt bo'lishi uchun $\lim\limits_{n\to\infty} a_n = 0$ munosabatning bajarilishi zarur va yetarli. Isbotlang.

Isbot. *Yetarliligi.* $\lim\limits_{n\to\infty} a_n = 0$ bo'lsin. Quyidagi A_n, $n \in \mathbb{N}$ operatorlar ketma-ketligini qaraymiz:

$$A_n x = (a_1 x_1, a_2 x_2, \ldots, a_n x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell_p.$$

Istalgan n natural son uchun A_n chekli o'lchamli operatordir (chunki $\dim Im A_n \le n$). 5.1-teoremaga ko'ra A_n kompakt operator bo'ladi. Bu $\{A_n\}$ ketma-ketlikning berilgan A operatorga tekis yaqinlashishini ko'rsatamiz. Buning uchun $n \to \infty$ da $\|A_n - A\| \to 0$ bo'lishini ko'rsatish kifoya. Har bir $x \in \ell_p$ element uchun

$$\|A_n x - Ax\| = \left(\sum_{k=n+1}^{\infty} |a_k|^p \, |x_k|^p \right)^{\frac{1}{p}} \le$$

$$\le \sup_{n+1 \le k < \infty} |a_k| \left(\sum_{k=n+1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \le \|x\| \cdot \sup_{n+1 \le k < \infty} |a_k|.$$

Bu yerdan $\|A_n - A\| \le \sup\limits_{n+1 \le k < \infty} |a_k|$ tengsizlik kelib chiqadi. $\lim\limits_{n\to\infty} a_n = 0$ shartdan

$$\lim_{n\to\infty} \|A_n - A\| \le \lim_{n\to\infty} \sup_{n+1 \le k < \infty} |a_k| = 0$$

ekanligi kelib chiqadi. U holda $\lim\limits_{n\to\infty} \|A_n - A\| = 0$. 5.3-teoremaga ko'ra, limitik operator A kompakt bo'ladi.

Zaruriyligi. Teskarisini faraz qilaylik, ya'ni A kompakt operator bo'lsin, ammo $\lim\limits_{n\to\infty} a_n = 0$ shart bajarilmasin. U holda shunday $a > 0$ soni va n_k $(n_k \to \infty)$ natural sonlar ketma-ketligi mavjud bo'lib, $|a_{n_k}| \ge a$ tengsizliklar bajariladi. $M = \left\{x^{(k)} \in \ell_p : x^{(k)} = e_{n_k}, \ k \in \mathbb{N}\right\}$ to'plamni qaraymiz. Bu to'plam chegaralangan to'plamdir, chunki ixtiyoriy $x^{(k)} \in M$ uchun $\|x^{(k)}\| = 1$ tenglik o'rinli. Bu to'plamning A akslantirishdagi tasviri $A(M)$ ning nisbiy kompakt emasligini ko'rsatamiz. Haqiqatan ham,

$$y^{(k)} = Ax^{(k)} = (0, 0.., 0, 0, a_{n_k}, 0, \ldots) = a_{n_k} e_{n_k}$$

tenglikdan, hamda $|a_{n_k}| \geq a$ dan $i \neq j$ da,

$$\left\|y^{(i)} - y^{(j)}\right\| = \left\|a_{n_i}e_{n_i} - a_{n_j}e_{n_j}\right\| \geq \sqrt[p]{2}\,a \tag{5.1}$$

tengsizlik kelib chiqadi. (5.1) tengsizlik $A(M)$ to'plamdan olingan $\{y^{(k)}\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratib olish mumkin emasligini ko'rsatadi, ya'ni $A(M)$ nisbiy kompakt to'plam emas. A operator chegaralangan M to'plamni nisbiy kompakt bo'lmagan $A(M)$ to'plamga akslantirgani uchun A operator kompakt emas. Bu ziddiyat $\lim\limits_{n\to\infty} a_n = 0$ munosabatni isbotlaydi. □

5.2. $A : C[a,\ b] \to C[a,\ b]$, $(Ax)(s) = \int\limits_a^b K(s,t)\,x(t)dt$ operator berilgan. Bu yerda $K(s,t)$, $T = [a,\ b] \times [a,\ b]$ kvadratda uzluksiz bo'lgan biror funksiya. Shu operatorning kompakt ekanligini ko'rsating.

Yechish. $K(s,\,t)$ funksiya T yopiq to'plamda uzluksiz bo'lganligi sababli Veyershtrass teoremasiga ko'ra ixtiyoriy $\varepsilon > 0$ soni uchun shunday n natural son va

$$P_n(s,t) = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{kl}\, s^k t^l$$

ko'phad mavjud bo'lib,

$$\|K - P_n\| \leq \max_{a \leq s,t \leq b} |\,K(s,t) - P_n(s,t)\,| < \varepsilon$$

tengsizlik bajariladi. Aytaylik, k, l – ixtiyoriy natural sonlar bo'lsin. $C[a,\ b]$ ni bir o'lchamli $c\,t^k$ ko'rinishdagi funksiyalar fazosiga akslantiruvchi

$$(A_{k,l}x)(s) = s^k \int\limits_a^b t^l\, x(t)\, dt$$

operatorni qaraymiz. $A_{k,l}$ chegaralangan va bir o'lchamli operator bo'lganligi uchun 5.1-teoremaga ko'ra u kompakt operator bo'ladi. Ixtiyoriy

n uchun

$$(A_n x)(s) = \int_a^b P_n(s,t)\, x(t)\, dt = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{kl}(A_{k,l}x)(s)$$

operator $A_{k,l}$ ko'rinishdagi kompakt operatorlarning chekli chiziqli kombinatsiyasidan iborat bo'lganligi uchun kompakt operatordir. Bundan tashqari,

$$\|A_n x - Ax\| = \max_{a\le s\le b}\left|\int_a^b P_n(s,t)\,x(t)\,dt - \int_a^b K(s,t)\,x(t)\,dt\right| \le$$

$$\max_{a\le s\le b}\int_a^b |P_n(s,t) - K(s,t)|\cdot|x(t)|dt \le (b-a)\cdot\|P_n - K\|\cdot\|x\|.$$

Bu yerdan

$$\|A_n - A\| \le (b-a)\cdot\|P_n - K\| \le \varepsilon\,(b-a) \qquad (5.2)$$

tengsizlik kelib chiqadi. (5.2) dagi $\varepsilon > 0$ ixtiyoriy bo'lganligi sababli shunday $P_n(s,t)$ ko'phadlar ketma-ketligini tanlash mumkinki,

$$\lim_{n\to\infty}\|A_n - A\| = 0.$$

Shunday qilib, $\{A_n\}$ kompakt operatorlar ketma-ketligi A operatorga tekis yaqinlashar ekan. U holda 5.3-teoremaga ko'ra, A ning kompakt operator ekanligini olamiz. □

Izoh. 5.2-misolda keltirilgan A operatorning kompaktligini $K(s,t)$ ning $[a,\ b]\times[a,\ b]$ kvadratda tekis uzluksizligidan va Arsela teoremasidan ham keltirib chiqarish mumkin.

5.3-5.22-misollarda keltirilgan operatorlarning kompaktligini ko'rsating.

5.3. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \int_0^1 (e^{s+t} + st)\,x(t)\,dt.$

5.4. $A : C[0,\ \pi] \to C[0,\ \pi],\ (Ax)(s) = \int_0^\pi \cos(s+t)\,x(t)\,dt.$

5.5. $A : C[0,\ 1] \to C[0,\ 1],\ \ (Ax)(t) = x(0)\,t + x(1)\,t^2.$

5.6. $A : C[0,\ 2\pi] \to C[0,\ 2\pi],\ \ (Ax)(s) = \int\limits_0^{2\pi} \sin(s + t)\,x(t)\,dt.$

5.7. $A : C[0,\ \pi] \to C[0,\ \pi],\ \ (Ax)(s) = \int\limits_0^{\pi} \cos(s - t)\,x(t)\,dt.$

5.8. $A : C[0,\ 3] \to C[0,\ 3],\ \ (Ax)(t) = x(0) + x(1)\,t + x(2)\,t^2 + x(3)\,t^3.$

5.9. $A : C[0,\ 1] \to C[0,\ 1],\ \ (Ax)(s) = \int\limits_0^1 \frac{1}{1 + s\,t}\ x(t)\,dt.$

5.10. $A : C[-1,\ 1] \to C[-1,\ 1],\ \ (Ax)(s) = \int\limits_{-1}^1 \frac{1}{9 - s^2t^2}\ x(t)\,dt.$

5.11. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ \ (Ax)(s) = \int\limits_0^1 (s^2t + s\,t^2)\,x(t)\,dt.$

5.12. $A : L_2[0,\ \pi] \to L_2[0,\ \pi],\ \ (Ax)(s) = \int\limits_0^{\pi} (s\,\cos t + t\,\cos\,s)\,x(t)\,dt.$

5.13. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ \ (Ax)(s) = \int\limits_0^1 \frac{1}{2 - s\,t}\ x(t)\,dt.$

5.14. $A : \ell_1 \to \ell_1\,,\ \ Ax = (0, \frac{x_2}{2}, 0, \frac{x_4}{4}, \ldots, 0, \frac{x_{2n}}{2n}, \ldots).$

5.15. $A : \ell_1 \to \ell_1\,,\ \ Ax = (x_1, 2x_2, 3x_3, \ldots, 9x_9, 10x_{10}, 0, 0, \ldots).$

5.16. $A : \ell_2 \to \ell_2\,,\ \ Ax = (x_1, \frac{x_2}{\ln 2}, \frac{x_3}{\ln 3}, \ldots, \frac{x_n}{\ln n}, \ldots).$

5.17. $A : \ell_2 \to \ell_2,\ \ \ Ax = (5x_1, 4x_2, 3x_3, 2x_4, x_5, \frac{1}{2}x_6, \ldots, \frac{1}{n-4}x_n, \ldots).$

5.18. $A : \ell_2 \to \ell_2,\ \ Ax = (100x_1, 99x_2, \ldots, 2x_{99}, x_{100}, \frac{1}{101}x_{101}, \ldots, \frac{1}{n}x_n, \ldots).$

5.19. $A : \ell_3 \to \ell_3\,,\ \ Ax = \left(\ln 2 \cdot x_1,\ \ln(1 + \frac{1}{2}) \cdot x_2, \ldots, \ln(1 + \frac{1}{n}) \cdot x_n, \ldots\right).$

5.20. $A : \ell_4 \to \ell_4,\ \ Ax = (0, 0, 0, 0, x_1, \frac{1}{2}\,x_2, \frac{1}{3}\,x_3, \frac{1}{4}\,x_4, \ldots, \frac{1}{n}\,x_n, \ldots).$

5.21. $A : \ell_5 \to \ell_5\,,\ \ Ax = \left(\mathrm{arctg}1 \cdot x_1,\ \mathrm{arctg}\frac{1}{2} \cdot x_2, \ldots,\ \mathrm{arctg}\frac{1}{n} \cdot x_n, \ldots\right).$

5.22. $A: m \to m$, $Ax = (x_1, x_1 + x_2, x_2 + x_3, x_3 + x_4, x_5, x_6, 0, 0, 0, \dots)$.

5.12-misolning yechimi. Ixtiyoriy $x \in L_2[0, \pi]$ uchun

$$(Ax)(s) = s\int_0^{\pi} \cos t\, x(t)\, dt + \cos s\int_0^{\pi} t\, x(t)\, dt = \alpha\, s + \beta \cos s \qquad (5.3)$$

tenglik bajariladi. Bu yerda

$$\alpha = \int_0^{\pi} \cos t\, x(t)\, dt, \quad \beta = \int_0^{\pi} t\, x(t)\, dt.$$

(5.3) tenglikdan $\dim ImA \le 2$ ni olamiz. Shunday x_1, x_2 ni topish mumkinki, $\alpha(x_1) = \beta(x_2) = 0$, $\alpha(x_2) = \beta(x_1) = 1$ bo'ladi, ya'ni A ikki o'lchamli operator. 5.1-teoremaga ko'ra u kompakt operator bo'ladi. □

5.23. $A : C[0, 1] \to C[0, 1]$, $(Ax)(t) = (t + 1)x(t)$ operatorni kompaktlikka tekshiring.

Yechish. A ni kompakt operator deb faraz qilaylik. A ga teskari operator mavjud va chegaralangan:

$$(A^{-1}y)(t) = \frac{y(t)}{t+1}, \qquad D(A^{-1}) = C[0, 1].$$

5.5-teoremaga ko'ra, cheksiz o'lchamli fazoda kompakt operatorning chegaralangan teskarisi mavjud emas. Demak, A kompakt operator emas. □

5.24. $C[0, 1]$ Banax fazosida

$$(Ax)(t) = \begin{cases} \frac{1}{t}\int_0^t x(s)ds, & t \neq 0, \\ (Ax)(0) = x(0) \end{cases}$$

formula bilan aniqlangan A operatorning chegaralangan, ammo kompakt emasligini ko'rsating.

Yechish. a) A operator uchun $D(A) = C[0,\ 1]$ va $ImA \subset C[0,\ 1]$ ekanligini, ya'ni ixtiyoriy $x \in C[0,\ 1]$ uchun $Ax \in C[0,\ 1]$ ekanligini ko'rsatamiz. Ixtiyoriy $x \in C[0,\ 1]$ uchun $y(t) = (Ax)(t)$ funksiyaning $t = 0$ nuqtada o'ngdan uzluksizligi x funksiyaning $t = 0$ nuqtada uzluksizligidan va

$$|y(t) - y(0)| = \left|\frac{1}{t}\int_0^t x(s)\,ds - \frac{1}{t}\int_0^t x(0)\,ds\right| \le \frac{1}{t}\int_0^t |x(s) - x(0)|\,ds$$

tengsizlikdan kelib chiqadi. Endi $t_0 \neq 0$ bo'lsin deb faraz qilaylik. U holda ixtiyoriy $0 < t \le 1$ uchun quyidagiga ega bo'lamiz:

$$y(t) - y(t_0) = \frac{1}{t}\int_0^t x(s)\,ds - \frac{1}{t_0}\int_0^{t_0} x(s)\,ds = \frac{1}{t}\int_{t_0}^t x(s)\,ds + \left(\frac{1}{t} - \frac{1}{t_0}\right)\int_0^{t_0} x(s)\,ds$$

yoki

$$|\,y(t) - y(t_0)\,| \le \frac{\|x\|}{t}\ |\,t - t_0\,| + \left|\frac{1}{t} - \frac{1}{t_0}\right| \|x\|\,. \qquad (5.4)$$

(5.4) dan $\lim\limits_{t \to t_0} y(t) = y(t_0)$ tenglik kelib chiqadi. Shunday qilib, y funksiya $[0,\ 1]$ da uzluksiz ekan, ya'ni $y = Ax \in C[0,\ 1]$.

b) Endi A ning chegaralangan operator ekanligini ko'rsatamiz.

$$\|Ax\| = \max_{0 \le t \le 1}\left|\frac{1}{t}\int_0^t x(s)\,ds\right| \le \|x\| \max_{0 \le t \le 1}\frac{1}{t}\int_0^t ds = \|x\|\,.$$

Demak, $\|A\| \le 1$, ya'ni A chegaralangan operator ekan.

c) A ning kompakt operator emasligini ko'rsatamiz. Uzluksiz funksiyalar ketma-ketligi $x_n(t),\ \ n = 0, 1, 2, \ldots$ ni quyidagicha tanlaymiz:

$$x_n(t) = \begin{cases} 0, & agar\ \ t \notin (2^{-n-1},\ 2^{-n}), \\ 1 - 2^{n+2}\,|t - 3 \cdot 2^{-n-2}|, & agar\ \ t \in (2^{-n-1},\ 2^{-n})\,. \end{cases}$$

$\{x_n\}$ chegaralangan ketma-ketlikdir, chunki ixtiyoriy n uchun

$$\|x_n\| = \max_{0 \le t \le 1} |x_n(t)| = 1.$$

$y_n(t) = (Ax_n)(t), \quad n = 1, 2, \ldots,$ funksiyalarni topamiz:

$$y_n(t) = \begin{cases} 0, & t \in [0, \ 2^{-n-1}], \\ \frac{1}{t}\int\limits_0^t (1 - 2^{n+2}\,|s - 3 \cdot 2^{-n-2}|)\, ds, & t \in (2^{-n-1}, 2^{-n}), \\ \frac{1}{2^{n+2}\,t}, & t \in [2^{-n}, 1]. \end{cases}$$

U holda $y_n(2^{-n}) = 2^n \cdot 2^{-n-2} = 2^{-2}$, $y_{n-m}(2^{-n}) = 0$, $n = 2, 3, 4, \ldots$, $m = 1, 2, 3, \ldots$, $n \neq m$ chunki $y_{n-m}(t) = 0$, $t \in [0, \ 2^{-n+m-1}]$. Bu tengliklardan $n \neq m$ bo'lganda

$$\|y_n - y_m\| \geq \frac{1}{4} \tag{5.5}$$

kelib chiqadi. (5.5) tengsizlikdan ko'rinadiki, $\{y_n = Ax_n\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratib olish mumkin emas. Bundan A kompakt operator emas degan xulosaga kelamiz. □

5.25. $A : C[0, 1] \to C[0, 1]$, $(Ax)(t) = x(t^4)$ operatorning kompakt emasligini ko'rsating.

Yechish. Dastlab A teskarilanuvchan operator ekanligini ko'rsatamiz. $Ax = 0$ yoki $x(t^4) = 0$ tenglama $t^4 = s$ almashtirishdan keyin $x(s) = 0$ tenglamaga keladi. Shuning uchun, $Ax = 0$ tenglama faqat $x = 0$ yechimga ega, shunday ekan, A teskarilanuvchan operator. Ixtiyoriy $y \in C[0, 1]$ uchun $Ax = y$ tenglama yoki $x(t^4) = y(t)$ tenglamani yechamiz. Agar $s = t^4$ almashtirishni olsak, $0 \leq t \leq 1$ bo'lganda $0 \leq s \leq 1$ bo'ladi va $x(t^4) = y(t)$ tenglama $x(s) = y(\sqrt[4]{s})$ ko'rinishni oladi, ya'ni $Ax = y$ tenglama yechimi $x(t) = y(\sqrt[4]{t})$ ko'rinishga ega. Bu yerdan A^{-1} operator $C[0, 1]$ fazoning hamma yerida aniqlanganligi va $(A^{-1}y)(t) = y(\sqrt[4]{t})$ formula vositasida ta'sir qilishi kelib chiqadi. Endi ixtiyoriy $y \in C[0, 1]$ uchun

$$\|A^{-1}y\| = \max_{0 \leq t \leq 1} \left| y(\sqrt[4]{t}) \right| = \max_{0 \leq \sqrt[4]{t} \leq 1} \left| y(\sqrt[4]{t}) \right| = \max_{0 \leq s \leq 1} |y(s)| = \|y\|$$

munosobatlar o'rinli bo'lgani uchun A^{-1} chegaralangan bo'ladi. 5.5-teoremaga ko'ra, cheksiz o'lchamli fazoda kompakt operatorning chegaralangan teskarisi mavjud emas. Demak, A kompakt operator emas. □

Masalani $\{x_n(t) = t^n\}$ chegaralangan ketma-ketlikning tasviri nisbiy kompakt emasligini ko'rsatish orqali ham yechish mumkin.

5.26-5.45-misollarda keltirilgan operatorlarning kompakt emasligini ko'rsating.

5.26. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (t+1)\,x(t).$

5.27. $A : C[-1,1] \to C[-1,1],\ (Ax)(t) = (t^2+1)\,x(t).$

5.28. $A : C[0,1] \to C[0,\ 1],\ (Ax)(t) = \sqrt{t+1}x(t).$

5.29. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = (1+2t)\,x(t).$

5.30. $A : C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = x(t^2).$

5.31. $A : C[-1,\ 1] \to C[-1,\ 1],\ (Ax)(t) = x(t^3).$

5.32. $A : C[-1,1] \to C[-1,1],\ (Ax)(t) = \frac{1}{2}[x(t) + x(-t)].$

5.33. $A : L_2[0,\ 1] \to L_2[0,\ 1],\ (Ax)(t) = (\sin t + \cos t)x(t).$

5.34. $A : L_2[-1,\ 1] \to L_2[-1,\ 1],\ (Ax)(t) = (t^2+2t+3)\,x(t).$

5.35. $A : L_2[0,\ \infty) \to L_2[0,\ \infty),\ (Ax)(t) = \frac{t+3}{t+4}\,x(t).$

5.36. $A : \ell_1 \to \ell_1,\ Ax = (0, x_2, 0, x_4, \ldots, 0, x_{2n}, \ldots).$

5.37. $A : \ell_1 \to \ell_1,\ Ax = \left(\sin\frac{\pi}{4}\cdot x_1, \sin\frac{2\pi}{4}\cdot x_2, \ldots, \sin\frac{n\pi}{4}\cdot x_n, \ldots\right).$

5.38. $A : \ell_1 \to \ell_1,\ Ax = (x_1, (1+\frac{1}{4})^2 x_2, (1+\frac{1}{9})^3 x_3, \ldots, (1+\frac{1}{n^2})^n x_n, \ldots).$

5.39. $A : \ell_2 \to \ell_2,\ Ax = (2x_1, 0, 2x_3, 0, \ldots, 2x_{2n-1}, 0, \ldots).$

5.40. $A : \ell_2 \to \ell_2,\ Ax = \left(\frac{1}{5}x_1, \frac{2}{9}x_2, \frac{3}{13}x_3, \ldots, \frac{n}{4n+1}x_n, \ldots\right).$

5.41. $A: \ell_2 \to \ell_2,\ Ax = (x_1, \frac{1}{2}x_2, \ldots, \frac{1}{10}x_{10}, x_{11}, x_{12}, \ldots).$

5.42. $A: \ell_3 \to \ell_3,\ Ax = (x_1, \frac{1}{2}x_2, x_3, \frac{1}{4}x_4, \ldots, x_{2n+1}, \frac{1}{2n}x_{2n}, \ldots).$

5.43. $A: \ell_4 \to \ell_4,\ Ax = \left(2x_1, (1+\frac{1}{2})^2 x_2, \ldots, (1+\frac{1}{n})^n x_n, \ldots\right).$

5.44. $A: \ell_5 \to \ell_5,\ Ax = (x_1, 0, 0, 0, x_5, 0, 0, 0, x_9, \ldots, x_{4n+1}, 0, 0, 0, \ldots).$

5.45. $A: \ell_5 \to \ell_5,\ Ax = (0, 0, 0, 0, 0, x_1, x_2, x_3, \ldots, x_n, \ldots).$

5.32-misolning yechimi. Ko'rsatamizki $\lambda = 1$ soni A operator uchun cheksiz karrali xos qiymat bo'ladi. Bu esa 5.6-teorema bilan birgalikda A operatorning kompakt emasligini isbotlaydi. Endi $\lambda = 1$ soni A operatorning cheksiz karrali xos qiymati ekanligini ko'rsatamiz. $C[-1,\ 1]$ fazoni juft funksiyalardan iborat $C^+[-1,\ 1]$ va toq funksiyalardan tashkil topgan $C^-[-1,\ 1]$ qism fazolarning yig'indisiga yoyish mumkin, ya'ni

$$C[-1,\ 1] = C^+[-1,\ 1] \oplus C^-[-1,\ 1].$$

Bu fazolar A operator uchun invariant qism fazolar bo'ladi va quyidagi tengliklar o'rinli:

$$Ax^+ = x^+,\ x \in C^+[-1,\ 1], \quad Ax^- = 0,\ x \in C^-[-1,\ 1].$$

Bu yerdan kelib chiqadiki $Ker(A - I) = C^+[-1,\ 1]$ va

$$\dim Ker(A - I) = \dim C^+[-1,\ 1] = \infty.$$

Ya'ni $\lambda = 1$ soni A operatorning cheksiz karrali xos qiymati ekan. □

5.46. $\varphi \in C[0,\ 1]$ nolmas funksiyaga qanday shartlar qo'yilganda $A: C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \varphi(t)\,x(t)$ operator kompakt bo'ladi.

5.47. $A: C[0,\ 1] \to C[0,\ 1],\ (Ax)(t) = \int\limits_0^1 K(t,s)\,x(s)\,ds + \sum\limits_{k=1}^{n} \varphi_k(t)\,x(t_k)$ operator berilgan, bu yerda $K(s,t),\ T = \{(s,t):\ 0 \le s,t \le 1\}$

birlik kvadratda uzluksiz bo'lgan biror funksiya. $\varphi_k \in C[0,1]$, $t_k \in [0,\ 1]$, $k = 1, 2, \ldots, n$. Bu operatorning kompaktligini isbotlang.

5.48. $(Ax)(t) = x'(t)$ differensial operator

a) $C^{(1)}[0,\ 1]$ ni $C[0,\ 1]$ ga;

b) $C^{(2)}[0,\ 1]$ ni $C^{(1)}[0,\ 1]$ ga;

c) $C^{(2)}[0,\ 1]$ ni $C[0,\ 1]$ ga, akslantiruvchi operator sifatida kompakt bo'ladimi?

5.49. $A : L_2[a,\ b] \to L_2[a,\ b]$, $(Ax)(t) = \int\limits_a^t x(\tau)\, d\tau$ operatorning kompakt ekanligini isbotlang.

5.50. Quyidagi operatorlardan qaysilari kompakt operator bo'ladi?

a) $A : \ell_2 \to \ell_2$, $Ax = (0, x_1, x_2, \ldots)$;

b) $A : \ell_2 \to \ell_2$, $Ax = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$;

c) $A : \ell_2 \to \ell_2$, $Ax = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$.

5.51. Quyidagi ichiga joylashtirish operatorlarining kompakt bo'lishini isbotlang:

a) $I : C^{(1)}[a,\ b] \to C[a,\ b]$, $Ix = x$,

b) $I : H^{(1)}[a,\ b] \to C[a,\ b]$, $Ix = x$.

Bu yerda $H^{(1)}[a,\ b]$ – $[a,\ b]$ kesmada uzluksiz differensiallanuvchi funksiyalar fazosi bo'lib, unda skalyar ko'paytma

$$(x, y) = \int\limits_a^b [x(t)\,\overline{y(t)} + x'(t)\,\overline{y'(t)}\,]\, dt$$

tenglik bilan aniqlanadi.

5.52. $A : H^{(1)}[a, b] \to L_2[a, b]$, $(Ax)(t) = x'(t)$ operatorning kompakt emasligini isbotlang.

5.53. $A : L_2[-1,1] \to L_2[-1,1]$, $(Ax)(t) = \int\limits_{-1}^{1} t^2 s\, x(s)\, ds$ operatorning kompakt ekanligini isbotlang va spektrini toping.

5.54. $A : L_2[0,1] \to L_2[0,1]$, $(Ax)(t) = \int\limits_{0}^{1} t\, s\, (1 - ts)\, x(s)\, ds$ operatorning kompakt ekanligini isbotlang va spektrini toping.

5.55. O'z-o'ziga qo'shma operatorning har xil xos qiymatlariga mos keluvchi xos vektorlari o'zaro ortogonal ekanligini isbotlang.

5.56. Cheksiz o'lchamli H Hilbert fazosida berilgan o'z-o'ziga qo'shma kompakt A operator cheklita xos qiymatlarga ega bo'lsin. U holda $\lambda = 0$ son A operatorning xos qiymati bo'lishini isbotlang.

5.57. H separabel Hilbert fazosida A kompakt operator berilgan bo'lsin.

a) A^*A ning o'z-o'ziga qo'shma kompakt operator bo'lishini isbotlang

b) $A^*Ax = \sum\limits_n \mu_n (x, h_n) h_n$ tasvirda barcha n larda $\mu_n > 0$ ekanligini isbotlang.

c) $\lambda_n = \sqrt{\mu_n}$, $e_n = \frac{1}{\lambda_n} Ah_n$ bo'lsin. $\{e_n\}$ lar ortonormal sistema tashkil qilishini va ixtiyoriy $x \in H$ uchun

$$Ax = \sum_n \lambda_n (x, h_n) e_n$$

tasvir o'rinli ekanligini isbotlang. λ_n sonlar A operatorning *singulyar sonlari* deyiladi.

5.58. $V : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, $(Vf)(n) = v(n) f(n)$ operatorning singulyar sonlarini toping. Bu yerda barcha $n \in \mathbb{Z}$ lar uchun $v(n) > 0$ deb faraz qilinadi.

6-§. Integral tenglamalar

Funksional fazoda tenglama berilgan bo'lib, noma'lum element funksiyadan iborat bo'lsa, bunday tenglama *funksional tenglama* deyiladi. Agar funksional tenglamada noma'lum funksiya integral ostida bo'lsa, u holda tenglama *integral tenglama* deyiladi.

$$\int_a^b K(s,t)\phi(t)dt + f(s) = 0, \tag{6.1}$$

$$\phi(s) = \int_a^b K(s,t)\phi(t)dt + f(s). \tag{6.2}$$

Bu yerda $\phi-$ noma'lum funksiya. $K(s,t)$ va $f(s)$ berilgan funksiyalar. (6.1) va (6.2) tenglamalar mos ravishda *birinchi va ikkinchi tur Fredholm tenglamalari* deyiladi. Xususan, $K(s,t)$ funksiya $t > s$ qiymatlar uchun $K(s,t) = 0$ shartni qanoatlantirsa, u holda (6.1) va (6.2) tenglamalar mos ravishda *birinchi va ikkinchi tur Volterra tenglamalari* deyiladi. (6.2) tenglamaning *yadrosi* deb nomlanuvchi $K(s,t)$ funksiyadan quyidagilar talab qilinadi, u – o'lchovli va

$$\int_a^b \int_a^b |K(s,t)|^2 dsdt < \infty \tag{6.3}$$

shartni qanoatlantiradi. Agar $K(s,t) = \sum\limits_{k=1}^{n} a_k(s)\, b_k(t)$ ko'rinishda bo'lsa (6.2) tenglama *ajralgan yadroli* integral tenglama deyiladi. $L_2[a, b]$ Hilbert fazosida aniqlangan

$$(T\phi)(s) = \int_a^b K(s,t)\phi(t)dt \tag{6.4}$$

operator K *yadroli Fredholm operatori* deyiladi.

6.1-ta'rif. *Agar biror* $\lambda \in \mathbb{C}$ *uchun*

$$\phi(s) = \lambda \int_a^b K(s,t)\phi(t)dt \Longleftrightarrow \phi(s) = \lambda(T\phi)(s)$$

tenglama noldan farqli yechimga ega bo'lsa, λ integral tenglamaning xarakteristik soni deyiladi. Tenglamaning nolmas yechimi esa λ xarakteristik songa mos xos funksiya deyiladi.

Agar $\lambda \neq 0$ integral tenglamaning xarakteristik soni bo'lsa, u holda $\frac{1}{\lambda}$ soni T operatorning xos qiymati bo'ladi.

6.1-teorema. *Agar $K(s,t)$ yadro (6.3) shartni qanoatlantirsa, u holda $L_2[a, b]$ fazoda (6.4) tenglik bilan aniqlanuvchi T operator chiziqli, kompakt va uning normasi uchun quyidagi tengsizlik o'rinli*

$$\| T \| \leq \sqrt{\int_a^b \int_a^b |K(s,t)|^2 ds dt}. \tag{6.5}$$

$L_2[a, b]$ fazoda (6.4) tenglik bilan aniqlanuvchi T operator o'z-o'ziga qo'shma (3.19-misolga qarang) bo'lishi uchun, deyarli barcha $s, t \in [a, b]$ larda $K(s,t) = \overline{K(t,s)}$ tenglikning bajarilishi zarur va yetarli.

6.2-teorema. *Agar 1 soni $T = T^*$ operator uchun xos qiymat bo'lmasa, u holda (6.2) tenglama ixtiyoriy f uchun yagona yechimga ega. Agar 1 soni T operator uchun xos qiymat bo'lsa, u holda (6.2) tenglama yechimga ega bo'lishi uchun f funksiya 1 soniga mos keluvchi barcha xos funksiyalarga ortogonal bo'lishi zarur va yetarli.*

X Banax fazosida biror T kompakt (to'la uzluksiz) operatorni olib,

$$x - Tx = y \tag{6.6}$$

ko'rinishdagi tenglamani qaraymiz. (6.6) tenglama bilan bir qatorda bir jinsli bo'lgan

$$x - Tx = 0 \tag{6.7}$$

tenglamani va ularga qo'shma bo'lgan

$$f - T^* f = g \tag{6.8}$$

$$f - T^* f = 0 \tag{6.9}$$

tenglamalarni qaraymiz.

Quyida keltiriladigan Fredholm teoremalari shu to'rt tenglamaning yechimlari orasidagi bog'lanishlarni ifodalaydi.

6.3-teorema *(6.6) tenglama berilgan $y \in X$ da yechimga ega bo'lishi uchun (6.9) bir jinsli tenglamaning yechimi bo'lgan har bir $f \in X^*$ da $f(y) = 0$ shartning bajarilishi zarur va yetarli.*

6.4-teorema *(Fredholm alternativasi). Yo (6.6) tenglama ixtiyoriy $y \in X$ da yagona yechimga ega, yo (6.7) bir jinsli tenglama noldan farqli yechimga ega.*

6.5-teorema. *Bir jinsli (6.7) va (6.9) tenglamalarning chiziqli erkli yechimlari soni chekli va o'zaro teng. Boshqacha qilib aytganda,*

$$\dim Ker(I - T) = \dim Ker(I - T^*) < \infty.$$

Integral tenglamalarga oid topshiriqlarni bajarish uchun Fredholm integral tenglamasi, uning turlari va Fredholm teoremalari haqida qo'shimcha ma'lumotlarni [11] ning $19 - 20 - \S$ laridan qarab olish mumkin.

Integral tenglamalarni yechishga doir bir nechta misollar qaraymiz.

6.1. Ushbu

$$x(s) - \int_{-1}^{1} (s\,t + s^2)\, x(t)\, dt = 1$$

ajralgan yadroli integral tenglamani yeching.

Yechish. Berilgan integral tenglamani quyidagicha yozib olamiz:

$$x(s) - s\int_{-1}^{1} t\, x(t)\, dt - s^2 \int_{-1}^{1} x(t)\, dt = 1. \tag{6.10}$$

Agar

$$\alpha_1 = \int_{-1}^{1} t\, x(t)\, dt, \qquad \alpha_2 = \int_{-1}^{1} x(t)\, dt \tag{6.11}$$

belgilashlarni kiritsak, (6.10) dan $x(s)$ uchun

$$x(s) = 1 + \alpha_1 s + \alpha_2 s^2 \tag{6.12}$$

ifodani hosil qilamiz. Agar (6.12) dagi α_1 va α_2 o'zgarmaslar aniqlansa, (6.12) tenglik bilan aniqlangan x funksiya berilgan integral tenglamaning yechimi bo'ladi. α_1 va α_2 o'zgarmaslarni aniqlash uchun (6.12) ni (6.11) ga qo'yib, quyidagi chiziqli tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} \alpha_1 = \int\limits_{-1}^{1} t\,(1 + \alpha_1 t + \alpha_2 t^2)\,dt = \frac{2}{3}\alpha_1 \\ \alpha_2 = \int\limits_{-1}^{1} (1 + \alpha_1 t + \alpha_2 t^2)\,dt = 2 + \frac{2}{3}\alpha_2. \end{cases} \tag{6.13}$$

Biz bu yerda

$$\int\limits_{-1}^{1} dt = 2, \quad \int\limits_{-1}^{1} t\,dt = 0, \quad \int\limits_{-1}^{1} t^2 dt = \frac{2}{3}, \quad \int\limits_{-1}^{1} t^3 dt = 0$$

tengliklardan foydalandik. (6.13) sistemani quyidagicha yozish mumkin:

$$\begin{cases} \frac{1}{3}\alpha_1 = 0, \\ \frac{1}{3}\alpha_2 = 2. \end{cases}$$

Bu yerdan $\alpha_1 = 0$, $\alpha_2 = 6$ ni olamiz. Demak, berilgan tenglama yechimi $x(s) = 1 + 6s^2$ funksiyadan iborat bo'ladi.

6.2. Agar

a) $a = -\frac{\pi}{4}$, $b = \frac{\pi}{4}$, $K(t,s) = \operatorname{tg} s$, $f(t) = 1$;

b) $a = 0$, $b = \frac{\pi}{2}$, $K(t,s) = \sin t \cos s$, $f(t) = \sin t$;

c) $a = 0$, $b = \pi$, $K(t,s) = \sin t \cos s$, $f(t) = \sin t$;

d) $a = 0$, $b = 1$, $K(t,s) = t + s - 2ts$, $f(t) = t + t^2$;

e) $a = -1$, $b = 1$, $K(t,s) = ts - t^2 s^2$, $f(t) = t^2 + t^4$;

f) $a = 0$, $b = 2\pi$, $K(t,s) = |\pi - s| \sin t$, $f(t) = t$;

g) $a = 0$, $b = \pi$, $K(t,s) = \sin s + s \cos t$, $f(t) = 1 - \dfrac{2t}{\pi}$;

h) $a = 0$, $b = \pi$, $K(t,s) = \sin(t - 2s)$, $f(t) = \cos 2t$;

bo'lsa, $C[a,\ b]$ fazoda

$$x(t) - \int_a^b K(t,s)x(s)ds = f(t)$$

tenglama yechimini toping.

6.3. $C[a,\ b]$ fazoda bir jinsli

$$x(s) - \lambda \int_{-\pi}^{\pi} \sin(3s + t)x(t)dt = 0$$

ajralgan yadroli integral tenglamani yeching.

Yechish. Agar $\sin(3s + t) = \sin 3s \cdot \cos t + \cos 3s \cdot \sin t$ ayniyatni hisobga olsak, berilgan integral tenglamani quyidagicha yozish mumkin:

$$x(s) = \lambda \sin 3s \int_{-\pi}^{\pi} \cos t\ x(t)\,dt + \lambda \cos 3s \int_{-\pi}^{\pi} \sin t\ x(t)\,dt. \qquad (6.14)$$

Bu yerda

$$\alpha_1 = \int_{-\pi}^{\pi} \cos t\, x(t)dt, \qquad \alpha_2 = \int_{-\pi}^{\pi} \sin t\, x(t)dt \qquad (6.15)$$

belgilashlarni kiritsak, (6.14) dan $x(s)$ uchun

$$x(s) = \lambda\alpha_1 \sin 3s + \lambda\alpha_2 \cos 3s \qquad (6.16)$$

ifodani olamiz. Endi α_1 va α_2 o'zgarmaslarni topish uchun (6.16) ni (6.15) tengliklarga qo'yib,

$$\begin{cases} \alpha_1 = \lambda\alpha_1 \int\limits_{-\pi}^{\pi} \sin 3t \cdot \cos t\,dt + \lambda\alpha_2 \int\limits_{-\pi}^{\pi} \cos 3t \cdot \cos t\,dt \\ \alpha_2 = \lambda\alpha_1 \int\limits_{-\pi}^{\pi} \sin 3t \cdot \sin t\,dt + \lambda\alpha_2 \int\limits_{-\pi}^{\pi} \sin t \cdot \cos 3t\,dt \end{cases} \qquad (6.17)$$

algebraik tenglamalar sistemasini hosil qilamiz. Agar (6.17) da

$$\int_{-\pi}^{\pi} \sin 3t \cdot \cos t\, dt = 0, \quad \int_{-\pi}^{\pi} \cos 3t \cdot \cos t\, dt = 0, \quad \int_{-\pi}^{\pi} \sin 3t \cdot \sin t\, dt = 0,$$

$$\int_{-\pi}^{\pi} \sin t \cdot \cos 3t\, dt = 0, \quad \int_{-\pi}^{\pi} \sin t \cdot \cos t\, dt = 0$$

ekanligini e'tiborga olsak, $\alpha_1 = 0$, $\alpha_2 = 0$ larni hosil qilamiz. Demak tekshirilayotgan integral tenglama λ parametrning barcha qiymatlar uchun yagona $x(s) = \lambda\alpha_1 \sin 3s + \lambda\alpha_2 \cos 3s = 0$ nol yechimga ega.

6.4. Agar

a) $a = 0$, $b = 2\pi$, $K(t,s) = \sin(t+s)$;

b) $a = 0$, $b = \pi$, $K(t,s) = \cos(t+s)$;

c) $a = 0$, $b = 1$, $K(t,s) = 2ts - 4t^2$;

d) $a = -1$, $b = 1$, $K(t,s) = ts + t^2s^2$

bo'lsa, $C[a, b]$ fazoda

$$x(t) - \lambda \int_a^b K(t,s)x(s)ds = 0$$

tenglamaning noldan farqli yechimlarini toping.

6.5. Agar

a) $a = -1$, $b = 1$, $K(t.s) = ts$, $f(t) = \alpha t^2 + \beta t + \gamma$;

b) $a = 0$, $b = \pi$, $K(t,s) = \cos(t+s)$, $f(t) = \alpha \sin t + \beta$;

c) $a = -1$, $b = 1$, $K(t,s) = t^2 - 2ts$, $f(t) = \alpha t^2 - \beta t$;

d) $a = -1$, $b = 1$, $K(t,s) = 3t + ts - 5t^2s^2$, $f(t) = \alpha t$

bo'lsa, bu tenglama ozod hadiga kiruvchi α, β, γ parametrlarning barcha qiymatlarida $C[a,b]$ fazoda

$$x(t) - \lambda \int_a^b K(t,s)x(s)ds = f(t)$$

tenglamaning yechimini toping.

6.6. Ixtiyoriy $\lambda \in \mathbb{C}$ va ixtiyoriy $f \in L_2[0,\ 2\pi]$ uchun

$$x(t) - \lambda \int_0^{2\pi} \sin(t-2s)\,x(s)\,ds = f(t)$$

tenglama yechimga ega ekanligini isbotlang va yechimni toping.

6.7. Agar

a) $a = 0,\quad b = 1,\quad K(t,s) = \begin{cases} t, & agar\ \ 0 \le t \le s \le 1, \\ s, & agar\ \ 0 \le s < t \le 1; \end{cases}$

b) $a = 0,\ b = \pi/2,\ K(t,s) = \begin{cases} \sin t\,\cos s, & agar\ \ 0 \le t \le s \le \dfrac{\pi}{2}, \\ \sin s\,\cos t, & agar\ \ 0 \le s < t \le \dfrac{\pi}{2}; \end{cases}$

c) $a = 0,\ b = \pi,\ K(t,s) = \begin{cases} \sin t\,\cos s, & agar\ \ 0 \le t \le s \le \pi, \\ \sin s\,\cos t, & agar\ \ 0 \le s < t \le \pi; \end{cases}$

d) $a = 0,\ b = 1,\ K(t,s) = \begin{cases} s(t+1), & agar\ \ 0 \le t \le s \le 1, \\ t(s+1), & agar\ \ 0 \le s < t \le 1; \end{cases}$

e) $a = 0,\ b = 1,\quad K(t,s) = e^{-|t-s|}$;

f) $a = 0,\ b = 1,\ K(t,s) = \begin{cases} (t+1)(s-2), & agar\ \ 0 \le t \le s \le 1, \\ (s+1)(t-2), & agar\ \ 0 \le s < t \le 1 \end{cases}$

bo'lsa, $L_2[a,b]$ kompleks Hilbert fazosida

$$x(t) - \lambda \int_a^b K(t,s)x(s)ds = 0$$

integral tenglamaning λ_n xarakteristik sonlari ($\frac{1}{\lambda_n}$ lar T operatorning xos qiymatlari) va φ_n xos funksiyalarini toping.

6.7-misol c) qismining yechimi. Maqsadimiz

$$x(t) - \lambda \int_0^{\pi} K(t,s)x(s)ds = 0 \Longleftrightarrow x(t) = \lambda(Tx)(t)$$

tenglamaning xarakteristik sonlari λ_n va ularga mos φ_n xos funksiyalarini topishdan iborat. Integral operatorning yadrosi $K(t,s)$ haqiqiy qiymatli va simmetrik $K(t,s) = K(s,t)$ bo'lganligi uchun, $K(t,s)$ yordamida (6.4) tenglik bilan aniqlangan T operator o'z-o'ziga qo'shma kompakt operator bo'ladi. Bu yerdan va 4.4-teoremaning b) bandiga ko'ra, λ_n lar haqiqiy bo'ladi. $K(t,s)$ ning berilishidan foydalanib, integral tenglamani quyidagicha yozib olamiz:

$$x(t) = \lambda \cos t \int_0^t \sin s\, x(s)ds + \lambda \sin t \int_t^{\pi} \cos s\, x(s)ds. \qquad (6.18)$$

Agar $x \in L_2[0,\ \pi]$ bo'lsa, $\int_0^t \sin s\, x(s)ds$ va $\int_t^{\pi} \cos s\, x(s)ds$ lar absolyut uzluksiz funksiyalar bo'ladi. Agar $x \in AC[0,\ \pi]$ funksiya (6.18) tenglamaning yechimi bo'lsa, u differensiallanuvchi bo'ladi. Xuddi shunday ko'rsatish mumkinki $x \in C^{(2)}[0,\ \pi]$ bo'ladi va (6.18) tenglikda $t = 0$ deb, $x(0) = 0$ ni olamiz. (6.18) tenglikni t bo'yicha differensiallab va o'xshash hadlarni ixchamlab

$$x'(t) = \lambda \cos t \int_t^{\pi} \cos s\, x(s)ds - \lambda \sin t \int_0^t \sin s\, x(s)ds \qquad (6.19)$$

ni olamiz. (6.19) tenglikda $t = \pi$ deb, $x'(\pi) = 0$ ni olamiz. (6.19) tenglikdan t bo'yicha hosila olib,

$$x''(t) = -\lambda\left(\sin t\int_t^{\pi}\cos s\, x(s)ds + \cos t\int_0^t \sin s\, x(s)ds\right) -$$

$$-\lambda(\cos^2 t + \sin^2 t)x(t) \Longleftrightarrow x''(t) + (\lambda+1)x(t) = 0 \qquad (6.20)$$

ni hosil qilamiz. Shunday qilib,

$$\begin{cases} x''(t) + (\lambda+1)x(t) = 0 \\ x(0) = x'(\pi) = 0 \end{cases} \qquad (6.21)$$

chegaraviy masalaga keldik. Agar $\lambda + 1 < 0$ bo'lsa, (6.20) differensial tenglamaning umumiy yechimi $x(t) = C_1 \operatorname{sh}\sqrt{\lambda+1}\,t + C_2 \operatorname{ch}\sqrt{\lambda+1}\,t$ bo'ladi. Chegaraviy shartlardan foydalanib, $x(0) = C_2 = 0$ va $x'(\pi) = C_1 \operatorname{ch}\sqrt{\lambda+1}\,\pi = 0$ ni, ya'ni $x(t) = 0$ ni olamiz. Bu yerdan (6.18) integral tenglama -1 dan kichik xarakteristik sonlarga ega emas degan xulosaga kelamiz. Xuddi shunday $\lambda + 1 = 0$ bo'lsa, (6.20) differensial tenglamaning umumiy yechimi $x(t) = C_1\, t + C_2$ dan chegaraviy shartlarni qanoatlantiruvchi yechimni ajratsak, $x(t) = 0$ ni olamiz. Endi $\lambda + 1 > 0$ bo'lsin. Agar $\lambda + 1 = w^2\,(w \in \mathbb{R})$ desak, (6.20) differensial tenglamaning umumiy yechimi $x(t) = C_1 \cos w\,t + C_2 \sin w\,t$ bo'ladi. Chegaraviy shartlardan foydalanib, $x(0) = C_1 = 0$ va $x'(\pi) = C_2\, w \cos w\pi = 0$ ni olamiz. Bu yerdan

$$w_k\,\pi = \frac{\pi}{2} + k\pi \Longleftrightarrow w_k = \frac{1}{2} + k, \quad k \in \mathbb{Z}_+$$

ni olamiz. Shunday qilib, $x_k(t) = C\sin\left(\frac{1}{2} + k\right)t$ funksiya (6.21) chegaraviy masalaning nolmas yechimi bo'ladi. Demak, (6.18) integral tenglamaning xarakteristik sonlari $\lambda_k = \left(\frac{1}{2} + k\right)^2 - 1$, $k \in \mathbb{Z}_+$ lar, ularga mos xos funksiyalar $x_k(t) = C\sin\left(\frac{1}{2}+k\right)t$ lar bo'ladi. Shunday qilib, $\frac{1}{\lambda_k}$, $k \in \mathbb{Z}_+$ lar T operatorning xos qiymatlari $\varphi_k(t) = \sqrt{\frac{2}{\pi}}\sin\left(\frac{1}{2} + k\right)t$ lar normasi bir bo'lgan xos funksiyalar bo'ladi. Ko'rsatish mumkinki, $\{\varphi_k\}$, $k \in \mathbb{Z}_+$ sistema $L_2[0, \pi]$ fazoda to'la ortonormal sistema bo'ladi. □

6.8. Agar

a) $f(t) = t, \quad K(t,s) = \begin{cases} t(s-1), & agar \ \ 0 \le t \le s \le 1, \\ s(t-1), & agar \ \ 0 \le s < t \le 1; \end{cases}$

b) $f(t) = \cos \pi t, \qquad K(t,s) = \begin{cases} (t+1)s, & agar \ \ 0 \le t \le s \le 1, \\ (s+1)t, & agar \ \ 0 \le s < t \le 1 \end{cases}$

bo'lsa, $L_2\,[a,b]$ kompleks Hilbert fazosida

$$x(t) - \lambda \int_0^1 K(t,s)x(s)ds = f(t) \tag{6.22}$$

integral tenglamaning yechimini toping.

6.8-misol a) qismining yechimi. Dastlab biz, bir jinsli

$$x(t) = \lambda \int_0^t s(t-1)\,x(s)ds + \lambda \int_t^1 t(s-1)\,x(s)ds \tag{6.23}$$

tenglamaning xarakteristik sonlari λ_n, ya'ni T operatorning xos qiymatlari $\dfrac{1}{\lambda_n}$ larni topamiz. Integral operatorning yadrosi $K(t,s)$ haqiqiy qiymatli va simmetrik $K(t,s) = K(s,t)$ bo'lganligi uchun, $K(t,s)$ yordamida (6.4) tenglik bilan aniqlangan T operator o'z-o'ziga qo'shma kompakt operator bo'ladi. Bu yerdan, λ_n larning haqiqiy ekanligi kelib chiqadi. Xuddi 6.7-misoldagi kabi, (6.23) integral tenglamadan

$$\begin{cases} x''(t) - \lambda\, x(t) = 0 \\ x(0) = x(1) = 0 \end{cases}$$

differensial tenglamaga kelamiz. Bu chegaraviy masalaning xarakteristik sonlari $\lambda_k = -k^2\,\pi^2$, $k \in \mathbb{N}$ lar, ularga mos xos funksiyalar $\varphi_k(t) = \sqrt{2}\sin k\,\pi\,t$ lardir. Demak, T operatorning xos qiymatlari $\dfrac{1}{\lambda_k} = -\dfrac{1}{k^2\,\pi^2}$,

$k \in \mathbb{N}$ sonlar, ularga mos normasi bir bo'lgan xos funksiyalar $\varphi_k(t) = \sqrt{2}\sin k\pi t$, $k \in \mathbb{N}$ lar bo'ladi. Bu sistema $L_2[0,\ 1]$ fazoda to'la ortonormal sistema bo'ladi. Endi (6.22) tenglamani yechishga Hilbert-Shidt teoremasini qo'llaymiz. $f(t) = t$ funksiyaning $\{\varphi_k\}$, $k \in \mathbb{N}$ ortonormal sistemadagi Furye koeffitsiyentlarini topamiz:

$$c_k = (f, \varphi_k) = \int_0^1 t\,\varphi_k(t)dt = \sqrt{2}\int_0^1 t\,\sin k\,\pi t\,dt = \frac{(-1)^{k-1}\sqrt{2}}{k\,\pi}.$$

Shunday qilib,

$$f(t) = t = \sum_{k=1}^{\infty} c_k\,\varphi_k(t), \quad x(t) = \sum_{k=1}^{\infty} x_k\,\varphi_k(t)$$

larni (6.22) ga qo'yib, quyidagi tenglamani olamiz:

$$\sum_{k=1}^{\infty} x_k\,\varphi_k - \lambda\sum_{k=1}^{\infty}\frac{x_k}{\lambda_k}\varphi_k = \sum_{k=1}^{\infty} c_k\,\varphi_k.$$

$\{\varphi_k\}_{k=1}^{\infty}$ ning ortonormal sistema ekanligidan foydalansak,

$$x_k\left(1 - \frac{\lambda}{\lambda_k}\right) = c_k, \quad k \in \mathbb{N} \tag{6.24}$$

tengliklarga kelamiz. Agar $\lambda \in \mathbb{C} \setminus \{\lambda_1,\ \lambda_2, \ldots, \lambda_n, \ldots\}$ bo'lsa, (6.24) sistema yagona $x_k = \dfrac{\lambda_k\,c_k}{\lambda_k - \lambda}$, $k \in \mathbb{N}$ yechimga ega. Bu yerdan (6.22) tenglama yechimi $x(t)$ uchun

$$x(t) = \sum_{k=1}^{\infty} x_k\,\varphi_k(t) = \sqrt{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,k\,\pi}{k^2\pi^2 + \lambda}\ \sin\ k\,\pi\,t$$

ifodani olamiz. Agar $\lambda \in \{\lambda_1,\ \lambda_2, \ldots, \lambda_n, \ldots\}$ bo'lsa, u holda (6.24) sistema yechimga ega emas, bu esa o'z navbatida (6.22) tenglamaning yechimga ega emasligini bildiradi. □

6.9-6.28-misollarda berilgan integral tenglama $\lambda \in \mathbb{C}$ parametrning qanday qiymatlarida yechimga ega, qanday qiymatlarida yechim mavjud emas, qanday qiymatlarida yechim cheksiz ko'p. Yechim mavjud bo'lgan hollarda yechimni toping.

6.9. $x(s) - \lambda \int\limits_0^1 s\,(1+t)x(t)dt = s^2.$

6.10. $x(s) - \lambda \int\limits_0^1 (s + s^2 t)x(t)dt = s^2 + 1.$

6.11. $x(s) - \lambda \int\limits_0^1 s\,x(t)dt = \sin 2\pi\, s.$

6.12. $x(s) - \lambda \int\limits_0^1 (t + s\,t)x(t)dt = s^2 - 1.$

6.13. $x(s) - \lambda \int\limits_0^1 (1 + 2s)x(t)dt = 1 - \frac{3}{2}s.$

6.14. $x(s) - \lambda \int\limits_{-1}^1 (t + s + s^2 t)x(t)dt = s^2 + 2s.$

6.15. $x(s) - \lambda \int\limits_0^1 s\,\sin 2\pi t\, x(t)dt = s.$

6.16. $x(s) - \lambda \int\limits_0^1 (t + s\,t + s^2 t)x(t)dt = 2s^2 + s.$

6.17. $x(s) - \lambda \int\limits_{-\pi/4}^{\pi/4} \operatorname{tg} t \cdot x(t)dt = \cos\, s.$

6.18. $x(s) - \lambda \int\limits_{-\pi}^{\pi} \cos(s + t)x(t)dt = \sin\, s.$

6.19. $x(s) - \lambda \int\limits_{-\pi}^{\pi} \sin s \cdot \cos t\, x(t)dt = \cos\, s.$

6.20. $x(s) - \int\limits_{-1}^1 (s\,t + s^2 t^2)\, x(t)dt = 1 + s^2.$

6.21. $x(s) - \lambda \int\limits_{-1}^1 (s + s^2 t)x(t)dt = \sin \pi\, s.$

6.22. $x(s) - \lambda \int\limits_0^\pi \sin(s+t)\, x(t)dt = \cos\, s.$

6.23. $x(s) - \lambda \int\limits_{-1}^1 (s+t)\, x(t)dt = \frac{1}{2} + \frac{3}{2}s.$

6.24. $x(s) - \lambda \int\limits_{-1}^1 (1+st+t^2)\, x(t)dt = 1 + s.$

6.25. $x(s) - \lambda \int\limits_0^1 \arccos t \cdot x(t)dt = \frac{1}{\sqrt{1-s^2}}.$

6.26. $x(s) - \lambda \int\limits_0^1 c^{s+t}\, x(t)dt = e^{2s}.$

6.27. $x(s) - \lambda \int\limits_{-1}^1 (1+t+st^2)\, x(t)dt = s^2.$

6.28. $x(s) - \lambda \int\limits_0^1 (1+t+s+st)\, x(t)dt = 2s + s^2.$

6.18-misolning yechimi. $\cos(s+t) = \cos s\, \cos t - \sin s \sin t$ ayniyatdan foydalansak, berilgan integral tenglamani quyidagicha yozish mumkin:

$$x(s) = \lambda \cos s \int_{-\pi}^{\pi} \cos t\, x(t)dt - \lambda \sin s \int_{-\pi}^{\pi} \sin t\, x(t)dt + \sin\, s. \qquad (6.25)$$

(6.15) belgilashdan foydalanib, (6.25) ni quyidagicha yozamiz:

$$x(s) = \lambda\alpha_1 \cos s - \lambda\alpha_2 \sin s + \sin s. \qquad (6.26)$$

$x(s)$ uchun hosil qilingan (6.26) ni (6.15) tengliklarga qo'yib,

$$\begin{cases} \alpha_1 = \lambda\, \alpha_1\, \pi \\ \alpha_2 = -\lambda\, \alpha_2\, \pi + \pi \end{cases} \qquad (6.27)$$

sistemani olamiz. Bu sistema $\lambda \neq \pm\frac{1}{\pi}$ da yagona $\alpha_1 = 0$ va $\alpha_2 = \frac{\pi}{1+\lambda\pi}$ yechimga ega. α_1 va α_2 larning bu qiymatlarini (6.26) ga qo'yib, berilgan tenglama yechimi (yagona) uchun

$$x(s) = \frac{-\lambda\pi}{1+\lambda\pi}\sin s + \sin s = \frac{\sin s}{1+\lambda\pi}$$

ifodani olamiz. Agar $\lambda = -\frac{1}{\pi}$ bo'lsa, (6.27) sistema yechimga ega emas, bundan berilgan tenglama ham yechimga ega emas degan xulosa chiqadi. Agar $\lambda = \frac{1}{\pi}$ bo'lsa, (6.27) sistema cheksiz ko'p yechimga ega bo'lib, bunda α_1 – ixtiyoriy son, $\alpha_2 = \frac{\pi}{2}$ dir. Ularning bu qiymatlarini (6.26) ga qo'yib,

$$x(s) = C\cos s - \frac{1}{\pi}\cdot\frac{\pi}{2}\sin s + \sin s = C\cos s + \frac{1}{2}\sin s$$

yechimni olamiz. Bu yerda C ixtiyoriy o'zgarmas son. □

6.29. Ushbu

$$x(s) = 1 + s + \int_0^s (s-t)\,x(t)\,dt \tag{6.28}$$

Volterra tipidagi integral tenglamani ketma-ket yaqinlashishlar usuli yordamida yeching.

Yechish. Boshlang'ich yaqinlashish sifatida $x_0(s) \equiv 1$ funksiyani olib, keyingi yaqinlashishlarni

$$x_n(s) = 1 + s + \int_0^s (s-t)\,x_{n-1}(t)\,dt, \quad n = 1,\ 2, \ldots$$

iteratsion formula yordamida topamiz:

$$x_1(s) = 1 + s + \int_0^s (s-t)\,dt = 1 + s + \frac{s^2}{2!},$$

$$x_2(s) = 1 + s + \int_0^s (s-t)\left(1 + t + \frac{t^2}{2!}\right)dt = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!},$$

$$x_3(s) = 1 + s + \int_0^s (s-t)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) dt =$$

$$= 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \frac{s^6}{6!}.$$

Bu jarayonni n marta takrorlash natijasida quyidagiga ega bo'lamiz:

$$x_n(s) = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^{2n-1}}{(2n-1)!} + \frac{s^{2n}}{(2n)!}.$$

Bu yerdan ko'rinib turibdiki, $x_n(s)$ funksiya $\sum\limits_{k=0}^{\infty} \frac{s^k}{k!} = e^s$ qatorning $2n$ – xususiy yig'indisidan iborat. Shuning uchun

$$x(s) = \lim_{n\to\infty} x_n(s) = e^s.$$

Demak, (6.28) integral tenglama yechimi $x(s) = e^s$ funksiyadan iborat ekan.

6.30-6.48-misollarda Volterra yoki Fredholm integral tenglamasi berilgan. Ularni ketma-ket yaqinlashish usuli yordamida yeching. Nolinchi yaqinlashish berilgan. Iteratsiyaning ikkinchi hadi $x_2(s)$ ni toping.

6.30. $x(s) = s + \int\limits_0^s x(t)dt, \qquad x_0(s) = s.$

6.31. $x(s) = 1 + \int\limits_0^s (s-t)x(t)dt, \qquad x_0(s) = 1.$

6.32. $x(s) = 2s^2 + 2 - \int\limits_0^s s\,x(t)dt, \qquad x_0(s) = 2.$

6.33. $x(s) = s + 1 + \int\limits_0^s x(t)dt, \qquad x_0(s) = 2s.$

6.34. $x(s) = s + \int\limits_0^s (s-t)x(t)dt, \qquad x_0(s) = 0.$

6.35. $x(s) = 1 + \int\limits_0^s (t-s)x(t)dt, \qquad x_0(s) = 0.$

6.36. $x(s) = \frac{5}{6}s - \frac{1}{9} + \frac{1}{3}\int\limits_0^1 (s+t)x(t)dt, \qquad x_0(s) = 0.$

6.37. $x(s) = 1 + \int\limits_0^s x(t)dt, \qquad x_0(s) = 0.$

6.38. $x(s) = e^s - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int\limits_0^1 x(t)dt, \qquad x_0(s) = 0.$

6.39. $x(s) = \frac{1}{2}\int\limits_0^\pi t\,s\,x(t)dt + \frac{5}{6}s, \qquad x_0(s) = s.$

6.40. $x(s) = \frac{1}{3}\int\limits_0^1 (s+t)x(t)dt + \frac{5}{6}s - \frac{1}{9}, \qquad x_0(s) = 2s.$

6.41. $x(s) = \frac{1}{3}\int\limits_0^1 x(t)dt + s, \qquad x_0(s) = 3s.$

6.42. $x(s) = \frac{1}{2}\int\limits_0^1 tx(t)dt + s + 1, \qquad x_0(s) = 1.$

6.43. $x(s) = \frac{1}{\pi}\int\limits_0^\pi \cos^2 t \cdot x(t)dt + 1, \qquad x_0(s) = 1.$

6.44. $x(s) = \pi\int\limits_0^1 (1-s)\sin 2\pi t\, x(t)dt + \frac{1}{2}(1-s), \qquad x_0(s) = 2.$

6.45. $x(s) = \frac{1}{2\pi}\int\limits_0^\pi \sin s \cdot t\, x(t)dt + 2\sin s, \qquad x_0(s) = 1.$

6.46. $x(s) = \frac{1}{2}\int\limits_0^1 x(t)dt + \sin\pi s, \qquad x_0(s) = 3.$

6.47. $x(s) = s + \int\limits_0^s x(t)dt, \qquad x_0(s) = s^2.$

6.48. $x(s) = s + 1 + s^2\int\limits_0^1 x(t)dt, \qquad x_0(s) = 2s.$

II bobni takrorlash uchun test savollari

1. ℓ_2 ni ℓ_2 ga akslantiruvchi A, B, C operatorlar berilgan:

$Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$,

$Bx = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$,

$Cx = (x_{1,}, x_{2,}, x_3, 0, \ldots, 0, \ldots)$,

Kompakt operatorlar keltirilgan javobni toping.

A) AB va BC B) B va C C) A va B D) AC va BC

2. $L_2[-1, 1]$ ni $L_2[-1, 1]$ ga akslantiruvchi A, B, I operatorlar berilgan:

$(Af)(x) = x\,f(x)$, $\quad (Bf)(x) = \int\limits_{-1}^{1}(1+xy)f(y)dy$, $\quad (If)(x) = f(x)$.

Kompakt operatorlar keltirilgan javobni toping.

A) AB va B B) B va I C) A va B D) A va I

3. $L_2[-1, 1]$ ni $L_2[-1, 1]$ ga akslantiruvchi

$$(Af)(x) = 3\int_{-1}^{1} x\,y\,f(y)\,dy$$

kompakt operatorning xos qiymatlarini toping.

A) 0, 2 B) 2 C) 0, 1, 2 D) 2, 3

4. $L_2[-1, 1]$ fazoni o'zini-o'ziga akslantiruvchi

$$(Af)(x) = 3\int_{-1}^{1} x\,y\,f(y)\,dy$$

kompakt operatorning xos funksiyalari ko'rsatilgan javobni toping.

A) $f_0(x) = 1,\ f_3(x) = x$ B) $f_0(x) = 1 + x,\ f_3(x) = x^2$

C) $f_0(x) = 3 + x,\ f_3(x) = 5x^2$ D) $f_0(x) = 4 + x,\ f_3(x) = x^4$

5. Chekli o'lchamli $A : \ell_2 \to \ell_2$ operatorni toping.

A) $Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$

B) $Ax = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$

C) $Ax = (x_{1,}, x_{2,}, x_3, 0, \ldots, 0, \ldots)$

D) $Ax = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

6. Kompakt $A : \ell_2 \to \ell_2$ operatorni toping.

A) $Ax = (0, 0, \ldots, 0, x_{1,}, x_{2,}, x_3, \ldots)$

B) $Ax = (0, 0, \ldots, 0, x_{n+1,}, x_{n+2,}, x_{n+3}, \ldots)$

C) $Ax = (a_1x_1, a_2x_2, a_3x_3, \ldots, a_nx_n, \ldots), \quad \lim\limits_{n\to\infty} a_n = 0$

D) $Ax = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$

7. ℓ_2 fazoda berilgan $Ax = (a_1x_1, a_2x_2, a_3x_3, \ldots, a_nx_n, \ldots)$ operatorning kompaktlik kriteriysini toping.

A) $\sup\limits_{n\geq 1} |a_n| < \infty$

B) shunday $n_0 \in \mathbb{N}$ mavjud bo'lib, barcha $n > n_0$ larda $a_n = 0$ bo'lishi

C) $\lim\limits_{n\to\infty} a_n = 0$

D) $\{a_n\}$ ning nolga yaqinlashuvchi qismiy ketma-ketlikni saqlashi

8. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Chekli o'lchamli $A \in L(X, Y)$ operator kompakt bo'ladi.

2) Chiziqli $A : \mathbb{C}^n \to \mathbb{C}^n$ operator kompakt bo'ladi.

3) Birlik $I : X \to X, \quad \dim X < \infty$ operator kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

9. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorlarning yig'indisi kompakt bo'ladi.

2) Kompakt operatorning songa ko'paytmasi kompakt bo'ladi.

3) Kompakt operatorning chegaralangan operatorga ko'paytmasi kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

10. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorga qo'shma operator kompakt bo'ladi.

2) Kompakt operatorga teskari operator kompakt bo'ladi.

3) Birlik $I : X \to X$, $\dim X = \infty$ operator kompakt bo'ladi.

A) 1, 2 B) 2, 3 C) 1 D) 1, 3

11. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorning xos qiymatlari oddiy bo'ladi.

2) O'z-o'ziga qo'shma operatorning xos qiymatlari haqiqiy bo'ladi.

3) O'z-o'ziga qo'shma kompakt operatorning har xil xos qiymatlariga mos xos vektorlari ortogonal bo'ladi.

A) 1, 2 B) 2, 3 C) 1, 2. 3 D) 1, 3

12. Quyidagi tasdiqlar ichidan to'g'rilarini ajrating.

1) Kompakt operatorning noldan farqli xos qiymatlari chekli karralidir.

2) $A : \ell_2 \to \ell_2$ kompakt operatorning spektri nolni saqlaydi.

3) Birlik $I : C[a, b] \to C[a, b]$ operator kompakt emas.

A) 1, 2 B) 2, 3 C) 1 D) 1, 2, 3

13. $C[-1, 1]$ fazoda chekli o'lchamli operatorlarni ko'rsating.

$$(Af)(x) = x\,f(x), \quad (Bf)(x) = \int_{-1}^{1} (x-y) f(y)\,dy, \quad (Cf)(x) = f(0)x^2$$

A) A, B B) A, C C) B, C D) A, B, C

14. ℓ_2 fazoda berilgan $Ax = (a_1x_1, a_2x_2, a_3x_3, \ldots, a_nx_n, \ldots)$ operator uchun quyidagilardan qaysilari invariant qism fazo bo'ladi.

1) $L_1 = \{x \in \ell_2 : x_1 = x_2 = 0\}$

2) $L_2 = \{x \in \ell_2 : x_3 = x_4 = x_5 = 0\}$

3) $L_3 = \{x \in \ell_2 : x_n = 0, \quad n \geq 6\}$

A) 1, 2 B) 2, 3 C) 1, 2, 3 D) 1, 3

15. $L_2[-\pi, \pi]$ fazoda kompakt operatorni ko'rsating.

A) $(Af)(x) = xf(x)$ B) $(Af)(x) = \int\limits_{-\pi}^{\pi} \cos(x-y)\, f(y)\, dy$

C) $(Af)(x) = f(x)$ D) $(Af)(x) = (x^2+1)\, f(x)$

16. $\{A_n\} \subset K(X)$ kompakt operatorlar ketma-ketligi, $A \in L(X)$ bo'lsin. Quyidagi tasdiqlardan qay, biri to'g'ri.

A) Agar $A_n \overset{u}{\Longrightarrow} A$ (tekis) bo'lsa, u holda A kompakt bo'ladi.

B) Agar $A_n \xrightarrow{s} A$ (kuchli) bo'lsa, u holda A kompakt bo'ladi.

C) Agar $A_n \xrightarrow{w} A$ (kuchsiz) bo'lsa, u holda A kompakt bo'ladi.

D) Agar $A_n \to A$ (nuqtali) bo'lsa, u holda A kompakt bo'ladi.

17. Chekli o'lchamli operator ta'rifini toping.

A) Agar $A \in L(X, Y)$ bo'lib, $\dim ImA < \infty$ bo'lsa

B) Agar A har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

C) Agar A har qanday nisbiy kompakt to'plamni kompakt to'plamga akslantirsa

D) Agar $A : \mathbb{C}^n \to \mathbb{C}^n$ bo'lib, $\dim Im\, A < n$ bo'lsa

18. Kompakt operator ta'rifini toping.

A) Agar $A \in L(X, Y)$ bo'lib, $\dim ImA < \infty$ bo'lsa

B) Agar $A \in L(X, Y)$ har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

C) Agar $A \in L(X, Y)$ har qanday nisbiy kompakt to'plamni kompakt to'plamga akslantirsa

D) Agar $A : \mathbb{C}^n \to \mathbb{C}^n$ bo'lib, $\dim Im\, A < n$ bo'lsa

19. Quyidagilar ichidan kompakt operator ta'riflarini ajrating.

1) Agar $A \in L(X, Y)$ har qanday chegaralangan to'plamni nisbiy kompakt to'plamga akslantirsa

2) Agar $A \in L(X, Y)$ operator X dagi birlik sharni nisbiy kompakt

to'plamga akslantirsa

3) Agar $A \in L(H)$ operator H dagi ixtiyoriy kuchsiz yaqinlashuvchi ketma-ketlikni kuchli yaqinlashuvchi ketma-ketlikka akslantirsa.

A) 1, 2, 3 B) 2, 3 C) 1, 3 D) 1, 2

20. u ga nisbatan Fredholmning I tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

21. u ga nisbatan Fredholmning II tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

22. u ga nisbatan Volterraning I tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

23. u ga nisbatan Volterraning II tur integral tenglamasini toping.

A) $u(x) = \int_a^b K(x,y)u(y)dy + f(x)$ B) $f(x) = \int_a^b K(x,y)u(y)dy$

C) $u(x) = \int_a^x K(x,y)u(y)dy + f(x)$ D) $f(x) = \int_a^x K(x,y)u(y)dy$

24. $T^* : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$ kompakt operator bo'lib, 1 uning oddiy xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiya esa $\cos 2x$ bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'ladigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) $1 - \cos 2x$ D) $\cos^2 x$

25. $T^* : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$ kompakt operator bo'lib, 1 uning ikki karrali xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiyalar esa $\cos x$ va $\sin x$ lar bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'ladigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) $\cos x + \sin x$ D) $\cos x - \sin x$

26. $T^* : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$ kompakt operator bo'lib, 1 uning oddiy xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiya esa $\cos 2x$ bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'lmaydigan f ni toping:

A) $\cos x$ B) $\cos 2x$ C) 1 D) $\sin x$

27. $T^* : L_2[-\pi, \pi] \to L_2[-\pi, \pi]$ kompakt operator bo'lib, 1 uning ikki karrali xos qiymati bo'lsin. 1 xos qiymatga mos keluvchi xos funksiyalar esa $\cos x$ va $\sin x$ lar bo'lsin. $u = Tu + f$ tenglama yechimga ega bo'lmaydigan f ni toping:

A) $\cos x + \sin x$ B) $\cos 2x$ C) 1 D) $\sin 2x$

28. $L_2[-\pi, \pi]$ fazoda $u(x) = 1 + \int_{-\pi}^{\pi} \cos x \sin y\, u(y)dy$ integral tenglama yechimini toping.

A) $1 + \cos x$ B) $1 + \sin x$ C) 1 D) $1 + \pi \cos x$

29. $L_2[-\pi, \pi]$ fazoda $u(x) = \sin x + \int_{-\pi}^{\pi} \cos x \sin y\, u(y)dy$ integral tenglama yechimini toping.

A) $\sin x + \cos x$ B) $1 + \sin x$ C) 1 D) $\sin x + \pi \cos x$

30. $L_2[-\pi, \pi]$ fazoda $u(x) = \cos x + \int_{-\pi}^{\pi} \cos x \sin y\, u(y)dy$ integral tenglama yechimini toping.

A) $\sin x + \cos x$ B) $1 + \sin x$ C) $\cos x$ D) $\sin x + \pi \cos x$

31. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos y\, u(y)dy$ chiziqli operator yadrosining o'lchamini toping.

A) $\dim Ker A = 0$ B) $\dim Ker A = 1$

C) $\dim Ker A = 2$ D) $\dim Ker A = \infty$

32. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos(x-y)u(y)dy$ chiziqli operator yadrosining o'lchamini toping.

A) $\dim Ker A = 0$ B) $\dim Ker A = 1$

C) $\dim Ker A = 2$ D) $\dim Ker A = \infty$

33. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = u(x) - \frac{1}{\pi}\int\limits_{-\pi}^{\pi}[\frac{1}{2} + \cos(x-y)]u(y)dy$ chiziqli operator yadrosining o'lchamini toping.

A) $\dim Ker A = 1$ B) $\dim Ker A = 2$

C) $\dim Ker A = 3$ D) $\dim Ker A = \infty$

34. $L_2[-\pi, \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}[\frac{1}{2} + \cos(x-y)]\,u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog'lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

35. $L_2[-\pi, \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos(x-y)\,u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog'lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

36. $L_2[-\pi, \pi]$ fazoda $u(x) = f(x) + \frac{1}{\pi}\int\limits_{-\pi}^{\pi}\cos x\,\cos y\,u(y)\,dy$ chiziqli integral tenglamaga mos bir jinsli tenglamaning chiziqli bog'lanmagan yechimlari sonini toping.

A) 1 B) 2 C) 3 D) ∞

37. $L_2[-\pi, \pi]$ fazoda $(Au)(x) = \int\limits_{-\pi}^{\pi}\cos(x-y)\,u(y)\,dy$, $(Bu)(x) = \int\limits_{-\pi}^{\pi}(\alpha\cos x\cos y - \beta\sin x\sin y)\,u(y)\,dy$ integral operatorlar berilgan. $A^* = B$ tenglik o'rinli bo'ladigan $\alpha \in \mathbb{R}$ va $\beta \in \mathbb{R}$ parametrlarning qiymatlarini toping.

A) $\alpha = 1, \beta = 1$ B) $\alpha = 1, \beta = -1$
C) $\alpha = -1, \beta = 1$ D) bunday qiymatlar yo'q

38. T – $L_2[a, b]$ fazodagi kompakt operator, 1 soni uning xos qiymati bo'lsin. $u = f + Tu$ (1) tenglama uchun quyidagi tasdiqlardan qaysilari to'g'ri?

1) (1) tenglama ba'zi $f \in L_2[a, b]$ larda yechimga ega emas.

2) (1) tenglama yechimga ega bo'lishi uchun f funksiya $u = T^*u$ tenglamaning barcha yechimlariga ortogonal bo'lishi zarur va yetarli.

3) (1) tenglama yechimga ega bo'lishi uchun $\dim KerT = \dim KerT^*$ bo'lishi zarur va yetarli.

A) 1, 2, 3 B) 2, 3 C) 3 D) 1, 2

39. $L_2[-\pi, \pi]$ fazoda ajralgan yadroli integral tenglamalarni ko'rsating.

1) $u(x) = \int_{-\pi}^{\pi} \cos(x - y)\, u(y)\, dy$

2) $u(x) = \int_{-\pi}^{\pi} (\alpha \cos x \cos y - \beta \sin x \sin y)\, u(y)\, dy$

3) $u(x) = \int_{-\pi}^{\pi} \ln(1 + |x - y|)\, u(y)\, dy.$

A) 1, 2, 3 B) 2, 3 C) 3 D) 1, 2

40. $L_2[-1, 1]$ fazoda ajralgan yadroli integral tenglamalarni ko'rsating.

1) $u(x) = \int_{-1}^{1} (x - y)^3 u(y)\, dy$

2) $u(x) = \int_{-1}^{1} (1 + xy)^2 u(y)\, dy$

3) $u(x) = \int_{-1}^{1} ln(1 + |x - y|)\, u(y)\, dy.$

A) 1, 2, 3 B) 2, 3 C) 3 D) 1, 2

1-§. Chiziqli uzluksiz operatorlar

5. Bu operator chiziqli, lekin uzluksiz emas.

Agar $A : C^{(1)}(G) \to C(G)$ sifatida qaralsa, u uzluksiz bo'ladi.

6. Bu operator chiziqli ham uzluksiz.

7. Bu operator chiziqli ham uzluksiz.

8. Bu operator chiziqli emas.

9. $D(A) = C^{(1)}(G, \mathbb{R}^3)$. Bu operator chiziqli, lekin uzluksiz emas.

10. Bu operator chiziqli, lekin uzluksiz emas.

13. $\|A\| = 2e^2$. **14.** $\|A\| = \dfrac{4}{\sqrt{15}}$. **15.** $\|A\| = 2$. **16.** $\|A\| = \sqrt{3}$.

17. $\|A\| = \dfrac{1}{\sqrt{3}}$. **18.** $\|A\| = 2$. **19.** $\|A\| = \sqrt{3}$. **20.** $\|A\| = \sqrt[5]{2}$.

21. $\|A\| = 1$. **22.** $\|A\| = 1$. **23.** $\|A\| = 2$. **24.** $\|A\| = 1$.

25. $\|A\| = e$. **26.** $\|A\| = 2$. **27.** $\|A\| = \dfrac{1}{5}$. **28.** $\|A\| = 1$.

29. $\|A\| = 1$. **30.** $\|A\| = 1$. **31.** $\|A\| = 4$. **32.** $\|A\| = 3$.

35. $D(A) = C^{(1)}[0,1] \neq C[0,1]$. 2) uzluksiz emas, chegaralangan emas.

37-54- misollarda $D(A) \neq X$. 2) uzluksiz emas, chegaralangan emas.

55. $X = C[0,1]$, $(Af)(x) = xf(x)$, $(Bf)(x) = \int\limits_0^1 f(t)dt$.

57. Yo'q. Masalan, $X = Y = \mathbb{R}^2$, $Ax = (0, x_2)$, $Bx = (0, 2x_2)$.

58. Umuman olganda yo'q. $A : C^{(1)}[0,1] \to C[0,1]$, $Ax(t) = x(t)$ misoldan foydalaning.

61. $\dim L(\mathbb{R}^n, \mathbb{R}^m) = n \cdot m$.

62. Yo'q. **63.** Ha. **65.** Ha. **66.** Ha.

70. A_n va B_n operatorlar ketma-ketligi A operatorga tekis yaqinlashadi. Demak ular A ga kuchli va kuchsiz ma'noda ham yaqinlashadi.

75. Istalgan noldan farqli ortogonal P va Q operatorlarni olish mumkin. Masalan: $P, Q : \ell_2 \to \ell_2$, $Px = (x_1, x_2, 0, 0, \ldots)$, $Qx = (0, 0, x_3, x_4, \ldots)$.

76. $A \in L(\ell_2)$, $Ax = (a_1x_1, a_2x_2, \ldots)$, $a = (a_1, a_2, \ldots, a_n, \ldots) \in m$,

$B : \ell_2 \to \ell_2$, $Bx = (x_1, 2x_2, \ldots, n\,x_n, \ldots)$. Agar $\{ na_n \}$ chegaralangan bo'lsa, u holda $A \cdot B$ va $B \cdot A$ operatorlar ham chegaralangan bo'ladi, agar $\{ na_n \}$ chegaralanmagan bo'lsa, u holda $A \cdot B$ va $B \cdot A$ operatorlar ham chegaralanmagan bo'ladi. Masalan, $a_n = \frac{1}{n}$ va $a_n = \frac{n-1}{n}$ hollarni qarang.

77. $\|P\| = 1$.

78. a) $R(A) = L_1^+[-1, 1]$ - juft funksiyalar to'plami, $R(B) = L_1^-[-1, 1]$ - toq funksiyalardan iborat to'plami. Ikkalasi ham qism fazo tashkil qiladi. b) $\|A\| = \|B\| = 1$. c) $A^2 = A$, $B^2 = B$, A va B lar ortogonal proeksiyalash operatorlari bo'ladi. d) $A \cdot B = 0$, $B \cdot A = 0$.

2-§. Teskari operatorlar

4. Operator 2.3-teorema shartlarini qanoatlantiradi. A teskarilanuvchan operator.

5. Operator 2.4-teorema shartlarini qanoatlantirmaydi. A^{-1} operator mavjud ammo, chegaralanmagan.

6. Operator 2.4-teorema shartlarini qanoatlantiradi. A^{-1} operator mavjud va chegaralangan.

8. $A^{-1}y = \frac{1}{2}(y_1 + y_2 - y_3, -y_1 + y_2 + y_3, y_1 - y_2 + y_3)$.

9. $A^{-1}y = (y_1, y_1 - y_2, y_1 - y_2 - y_3)$.

10. $A^{-1}y = (y_4, y_1, y_2, y_3)$.

11. $A^{-1}y = \left(\frac{1}{3}(2y_1 + y_2 - 2y_3), \frac{1}{3}(y_1 - y_2 - y_3), y_3, y_4, y_5\right)$.

12. $A^{-1}y = \left(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_2 - y_1), y_3 - \frac{1}{2}(y_2 - y_1), y_4, y_5, y_6, y_7\right)$.

13. $A^{-1}y = \left(y_1, \frac{1}{2}(y_1 + y_2 - y_3), \frac{1}{2}(-y_1 + y_2 + y_3), y_4, y_5 \ldots\right)$.

14. $A^{-1}y = (y_1, 2y_2, \ldots, ny_n, \ldots)$.

15. $A^{-1}y = (y_1, y_2, y_3 - y_2, y_4 - y_3 + y_2, y_5 - y_4, y_6, y_7, \ldots)$.

16. $A^{-1}y = (y_1, \sqrt{2}\,y_2, \ldots, \sqrt{n}\,y_n, \ldots)$.

17. $A^{-1}y = (y_1, 2y_2, \frac{3}{2}y_3, \ldots, \frac{n}{n-1}y_n, \ldots)$.

18. $(A^{-1}y)_n = 2\int\limits_0^1 y(t)\sin 2\pi n\, t\, dt, \quad A^{-1} : C[0,1] \to \ell_1$

19. $(A^{-1}y)_n = n^2 \int\limits_{-1}^{1} y(t)\cos \pi\, n\, t\, dt, \quad A^{-1} : C[-1,1] \to m.$

20. $(A^{-1}y)\,(s) = \int\limits_0^s y(t)\, dt.$

21. $A^{-1}y(t) = \frac{1}{t}y'(t), \quad y(0) = 0.$

22. $A^{-1}y(t) = \frac{y(t)}{t+2} - \frac{1}{2t+4}\frac{1}{1-\ln\frac{3}{2}}\int\limits_0^1 \frac{s\,y(s)ds}{s+2}.$

23. $A^{-1}y(t) = \frac{y(t)}{t+1} - \frac{t}{t+1}\frac{2}{1+2\ln 2}\int\limits_0^1 \frac{sy(s)ds}{s+1}.$

24. $A^{-1}y(t) = \frac{y(t)}{1+\sin t}.$

25. $A^{-1}y(t) = \frac{y(t)}{t+1} - \frac{3-t}{6(t+1)}y(0) - \frac{t}{3(t+1)}y(1).$

26. $A^{-1}y(t) = \frac{y(t)}{t^2} - \frac{y(1)}{2t^2}.$

27. $A^{-1}y(t) = y(t) - \frac{2\sin t}{2+\pi}\int\limits_0^\pi y(s)\sin s\, ds.$

29-48 misollar uchun berilgan operatorning chiziqli ekanligini va $Ax = 0$ tenglamaning noldan farqli yechimi mavjudligini ko'rsating.

51. Mavjud.

55. Mavjud, lekin chegaralanmagan.

56. b) $(A^{-1}x)(t) = x(t) - \int\limits_0^t e^{s-t}x(s)ds\,.$

57. $(A^{-1}y)\,(t) = y(t) - \frac{2}{2-\pi}\int\limits_0^\pi \cos(t-s)y(s)\, ds.$

58. b). $A^{-1}x(t) = \int\limits_0^t x(s)\sin\,(t-s)ds.$

3-§. Qo'shma operatorlar

2. $T^*y = (\lambda_1 y_1,\ \lambda_2 y_2,\ \lambda_3 y_3,\ \lambda_4 y_3)\,.$

3. $T^*y = (y_2, 2y_3, (2+i)y_4, 0).$

4. $T^*y = (0, \lambda_2 y_1, \ \lambda_3 y_2, \ldots, \lambda_n y_{n-1}, \ldots)$.

5. $T^*y = (\mu_1 y_2, \ \mu_2 y_3, \ldots, \mu_n y_{n+1}, \ldots)$.

6. $T^*x = (x_1, x_2, \ldots, x_n, 0, 0 \ldots)$.

7. $T^*y = (y_n, 0, 0, \ldots)$.

8. $T^*x = (e^{-i}x_1, e^{-2i}x_2, \ldots, e^{-ni}x_n, \ldots)$.

9. $T^*x = (0, \frac{1}{2}x_2, \frac{2}{3}x_3, \ldots, \frac{n}{n-1}x_n, \ldots)$.

10. $T^*x = (x_1, 2x_2, \ldots, 50x_{50}, 0, 0, \ldots)$.

11. $T^*x = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots)$.

13. $T^*x = (2\,x_1, -i\,x_2, (1-i)x_3, 0, 0, \ldots)$

14. $T^*x = (\bar{\lambda}_1 x_1, \bar{\lambda}_2 x_2, \ldots, \bar{\lambda}_n x_n, \ldots)$.

15. $T^*x = (x_1, x_2, x_3 + x_1, x_4 + x_2, \ldots, x_n + x_{n-2}, \ldots)$.

16. $T^*x = (x_1, x_2 + 2x_1, x_3 + 2x_2 + x_1, \ldots, x_n + 2x_{n-1} + x_{n-2}, \ldots)$.

17. $T^* = T$.

18. $T^* = T$.

22. $(T^*y)(t) = \int_0^1 [ts - i\cos(t+s)]x(s)ds$.

23. $(T^*x)(t) = \int_0^1 (s^2 + s + t)\, x(s)ds$.

24. $(T^*y)(t) = \int_t^1 t\, x(s)ds$.

25. $(T^*y)(t) = (\cos t - i\sin t)\, x(t) + \int_{-1}^1 (ts + it^2 s^2)\, x(s)ds$.

26. $(T^*y)(t) = (t - it^2)\, x(t) + \int_0^1 (s - it)\, x(s)ds$.

27. $(T^*y)(t) = y(t-h)$.

28. $(T^*y)(t) = u(t-h)y(t-h)$.

29. $T^*x = (\overline{\mu_1}x_1, \ldots, \overline{\mu_n}x_n, \ldots)$.

31. $(T^*x)(t) = \int_a^b \overline{K(s,t)}x(s)\, ds$.

32. $(T^*x)(t) = \left(\overline{u(t)} - i\overline{v(t)}\right) x(t)$.

33. $\beta \in \mathbb{R}, \ \alpha = ia, \ a \in \mathbb{R}$.

34. u va v lar haqiqiy funksiyalar bo'lgan holda $\alpha \in \mathbb{R}$, $\beta = ib$, $b \in \mathbb{R}$.

35. $\alpha \in \mathbb{R}$, $\beta = ib$, $b \in \mathbb{R}$.

36. $\overline{\alpha} = \beta$.

37. $\alpha\beta = 1$.

75. 3.64-misoldagi $U : \ell_2 \to \ell_2$ operator.

4-§. Chiziqli operatorning spektri

2. $\lambda_1 = 0$, $x^{(1)} = (1, -1, 0)$, $\lambda_2 = 2$, $x^{(2)} = (1, 1, 0)$, $\lambda_3 = 3$, $x^{(3)} = (0, 0, 1)$.

3. $\lambda_1 = 1$, $x^{(1)} = (1, 0, 0)$, $\lambda_2 = 1 + i$, $x^{(2)} = (0, 1, i)$, $\lambda_3 = 1 - i$, $x^{(3)} = (0, 1, -i)$,

4. $\lambda_1 = 2$, $x^{(1)} = (1,\ 1,\ 1)$, $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}\right)$, $\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}\right)$.

5. $\lambda_1 = 1$, $x^{(1)} = (1, 0, 0, 0)$, $\lambda_2 = 2$, $x^{(2)} = (0, 1, 0, 0)$, $\lambda_3 = 3$, $x^{(3)} = (0, 0, 1, 0)$, $\lambda_4 = 4$, $x^{(4)} = (0, 0, 0, 1)$.

6. $\lambda_1 = 2$, $x^{(1)} = (1, 1, 1, 0)$, $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}, 0\right)$, $\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}, 0\right)$, $\lambda_4 = 5$, $x^{(4)} = (0, 0, 0, 1)$.

7. $\lambda_1 = 0$, $x^{(1)} = (1, -1, 1, -1)$, $\lambda_2 = 2$, $x^{(2)} = (1, 1, 1, 1)$, $\lambda_3 = 1 - i$, $x^{(3)} = (1, -i, -1, i)$, $\lambda_4 = 1 + i$, $x^{(4)} = (1, i, -1, -i)$.

9. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar $\int_{-1}^{1} s\, x_0(s)\, ds = 0$ shartni qanoatlantiradi. $\lambda_1 = \frac{2}{3}$ oddiy xos qiymat, unga mos xos funksiya $x_1(t) = t$.

10. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar, $\int_{-1}^{1} x_0(s)\, ds = \int_{-1}^{1} s\, x_0(s)\, ds = 0$ shartni qanoatlantiradi. $\lambda_1 = \frac{2}{3}$ va $\lambda_2 = 2$ lar oddiy xos qiymatlar bo'lib, ularga mos xos funksiyalar $x_1(t) = t$ va $x_2(t) = 1$ dir.

11. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \cos nt,\ n \in \mathbb{N},\ x_0(t) = \sin nt,\ n \geq 2.$ $\lambda_1 = 2\pi$ va $\lambda_2 = \pi$ lar oddiy xos qiymatlar bo'lib, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = \sin t$ dir.

12. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2,\ x_0(t) = 1.$ $\lambda_1 = 1$ ikki karrali xos qiymat bo'lib, unga mos xos funksiyalar: $x_1(t) = \alpha\cos t + \beta\sin t$.

13. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2,\ x_0(t) = 1.$ $\lambda_1 = 1$ va $\lambda_2 = -1$ lar oddiy xos qiymatlar bo'lib, ularga mos xos funksiyalar $x_1(t) = \sin t + \cos t$ va $x_2(t) = \sin t - \cos t$.

14. $\lambda_0 = 0$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = \alpha\cos nt + \beta\sin nt,\ n \geq 2.$ $\lambda_1 = 1$ ikki karrali xos qiymat, unga mos xos funksiyalar: $x_1(t) = \alpha\cos t + \beta\sin t.$ $\lambda_2 = 2$ oddiy xos qiymat bo'lib, unga mos xos funksiya $x_2(t) = 1$ dir.

15. $\lambda_0 = 1$ cheksiz karrali xos qiymat, unga mos $x_0(t)$ xos funksiyalar: $x_0(t) = 1,\ x_0(t) = \sin nt,\ n \in \mathbb{N},\ x_0(t) = \cos nt,\ n \geq 2.$ $\lambda_1 = 2$ oddiy xos qiymat bo'lib, unga mos xos funksiya $x_1(t) = \cos t$ dir.

18. $\sigma(A) = \sigma_{qol}(A) = [1,\ 2]$, $\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{1 + t^2 - \lambda}$.

19. $\sigma(A) = \sigma_{qol}(A) = [-1,\ 1]$, $\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{t - \lambda} + \dfrac{t\,x(0)}{\lambda(t - \lambda)}$.

20. $\sigma(A) = \sigma_{qol}(A) = [-1,\ 1]$, $\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{\sin t - \lambda} - \dfrac{x(0)\cos t}{(1 - \lambda)(\sin t - \lambda)}$

21. $\sigma(A) = \sigma_{ess}(A) = [1,\ \infty)$, $\quad R_\lambda(A)x(t) = \dfrac{x(t)}{1 + t^2 - \lambda}$.

22. $\sigma(A) = \sigma_{ess}(A) = [0,\ 2]$, $\quad R_\lambda(A)x(t) = \dfrac{x(t)}{1 - \cos t - \lambda}$.

23. $\sigma(A) = \sigma_{ess}(A) = [-4,\ 6]$, $\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{1 - 2\cos t + 3\cos 2t - \lambda}$.

24. $\sigma(A) = \sigma_{ess}(A) = [0,\ \infty)$, $\quad R_\lambda(A)\,x(t) = \dfrac{x(t)}{t^2 - \lambda}$,

26. Operatorning 1 dan katta yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda) =$

$1+\int\limits_{-\pi}^{\pi}\frac{\sin^2 s}{\cos s-\lambda}\,ds$ funksiyaning noli. $\sigma_{ess}(A)=[-1,\,1]$,

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos t-\lambda}-\frac{\sin t}{\cos t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\sin s\,x(s)}{\cos s-\lambda}\,ds.$$

27. Operatorning -1 dan kichik yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda)=1-\int\limits_{-\pi}^{\pi}\frac{\sin^2 s}{\cos 2s-\lambda}\,ds$ funksiyaning noli, $\sigma_{ess}(A)=[-1,\,1]$,

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos 2t-\lambda}+\frac{\sin t}{\cos 2t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\sin s\,x(s)}{\cos 2s-\lambda}\,ds.$$

28. Operatorning 1 dan katta yagona oddiy xos qiymati bo'lib, u $\Delta(\lambda)=1+\int\limits_{-\pi}^{\pi}\frac{\cos^2 s}{\cos 4s-\lambda}\,ds$ funksiyaning noli, $\sigma_{ess}(A)=[-1,\,1]$,

$$R_\lambda(A)\,x(t)=\frac{x(t)}{\cos 4t-\lambda}-\frac{\cos t}{\cos 4t-\lambda}\,\frac{1}{\Delta(\lambda)}\int\limits_{-\pi}^{\pi}\frac{\cos s\,x(s)}{\cos 4s-\lambda}\,ds.$$

29. $\lambda_1=-1$ va $\lambda_2=0$ sonlari operatorning oddiy xos qiymatlari, $\lambda_3=1$ cheksiz karrali xos qiymat, shuning uchun $\sigma_{ess}(A)=\{1\}$.

$$R_\lambda(A)\,x(t)=\frac{x(t)}{1-\lambda}-\frac{1}{\pi(1-\lambda^2)}\int\limits_{-\pi}^{\pi}x(s)\,ds-\frac{\sin t}{\pi\,\lambda(1-\lambda)}\int\limits_{-\pi}^{\pi}\sin\,s\,x(s)\,ds.$$

30. $\lambda_1=-1$ oddiy, $\lambda_2=0$ esa ikki karrali xos qiymatdir, $\lambda_3=1$ cheksiz karrali xos qiymat, shuning uchun $\sigma_{ess}(A)=\{1\}$.

$$R_\lambda(A)\,x(t)=\frac{x(t)}{1-\lambda}-\frac{1}{\pi(1-\lambda^2)}\int\limits_{-\pi}^{\pi}x(s)\,ds-\frac{1}{\pi\,\lambda(1-\lambda)}\int\limits_{-\pi}^{\pi}\cos(t-s)x(s)\,ds.$$

32. $\lambda_1=\lambda_2=1$, $x^{(1)}=(1,0,0,0)$, $x^{(2)}=(0,0,0,1)$; $\lambda_3=\sqrt{2}$, $x^{(3)}=(0,\,1,\sqrt{2}-1,\,0)$; $\lambda_4=-\sqrt{2}$, $x^{(4)}=(0,\,1,-\sqrt{2}-1,\,0)$.

33. $\lambda_1=-1$, $x_1(t)=1-t$, $\lambda_2=4$, $x_2(t)=\frac{t^2+4}{4-t}$.

34. $\lambda_1=\frac{15-4\sqrt{15}}{60}$, $x_1(t)=60t^2-12\sqrt{15}t$; $\lambda_2=\frac{15+4\sqrt{15}}{60}$; $x_2(t)=60t^2+12\sqrt{15}t$, $\lambda=0$ cheksiz karrali xos qiymat, $\int\limits_0^1 s\,x(s)\,ds=\int\limits_0^1 s^2\,x(s)\,ds=0$ shartni qanoatlantiruvchi barcha funksiyalar $\lambda=0$ xos qiymatga mos keluvchi xos vektorlar bo'ladi.

35. $\lambda_1=1$ va $\lambda_2=-1$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t)=\cos t$ va $x_2(t)=\sin t$. $\lambda_3=0$ cheksiz karrali xos qiymat, unga mos keluvchi xos funksiyalar $x_3(t) = 1$, $x_3(t) = \sin nt$, $y_3(t) = \cos nt$, $n \geq 2$.

36. $\lambda_1 = 2$ va $\lambda_2 = \frac{2}{3}$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = t$. $\lambda = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $\int_0^1 s\,x(s)\,ds = \int_{-1}^1 s\,x(s)\,ds = 0$ shartni qanoatlantiruvchi funksiyalardir.

37. $\lambda_1 = -3$ va $\lambda_2 = 4$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 2t - t^2$ va $x_2(t) = \frac{1}{2}t + \frac{3}{2}t^2$. $\lambda_3 = 0$ cheksiz karrali xos qiymat, unga mos xos funksiyalar $x_3(t) = (t^2 - 1)\,y(t)$, $y \in C[-1,\ 1]$.

38. $\lambda_1 = 2$ va $\lambda_2 = \frac{2}{3}$ lar oddiy xos qiymatlar, ularga mos xos funksiyalar $x_1(t) = 1$ va $x_2(t) = t$. $\lambda_3 = 0$ cheksiz karrali xos qiymat, unga mos xos funksiyalar $x_1(t) = 1$, $x_2(t) = t$ larga ortogonal bo‘lgan funksiyalardir.

39. $\lambda_1 = 2$, $x^{(1)} = (1,\ 1,\ 1, 0, \ldots)$, $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$,
$x^{(2)} = \left(1 + i\sqrt{3},\ -2,\ 1 - i\sqrt{3}. 0, \ldots\right)$, $\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$,
$x^{(3)} = \left(1 - i\sqrt{3},\ -2,\ 1 + i\sqrt{3}, 0, \ldots\right)$. $\lambda_4 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(4)} = e_n$, $n \geq 4$.

40. $\lambda_1 = 3$, $x^{(1)} = e_1$, $\lambda_2 = 4$. $x^{(2)} = e_2$, $\lambda_3 = -2$, $x^{(3)} = e_3$, $\lambda_4 = 5$, $x^{(4)} = e_4$. $\lambda_5 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(n)} = e_n$, $n \geq 5$.

41. Istalgan $n \in \mathbb{N}$ uchun $\lambda_n = \frac{n}{n+1}$ oddiy xos qiymat bo‘lib, unga mos xos vektor e_n, $n \in \mathbb{N}$.

42. Istalgan $n \in \mathbb{N}$ uchun $\lambda_{2n-1} = \frac{2n-1}{4n}$ va $\lambda_{2n} = \frac{1}{2n}$ lar oddiy xos qiymatlar bo‘lib, ularga mos xos vektorlar $x^{(2n-1)} = e_{2n-1}$ va $x^{(2n)} = e_{2n}$, $n \in \mathbb{N}$.

43. $\lambda_1 = -1$, $x^{(1)} = (0, 0, 1, 0, 0, \ldots)$; $\lambda_2 = -\sqrt{2}$. $x^{(2)} = (1,\ -\sqrt{2} - 1,\ 3 + 2\sqrt{2}, 0, 0, \ldots)$; $\lambda_3 = \sqrt{2}$, $x^{(3)} = (1,\ \sqrt{2} - 1,\ 3 - 2\sqrt{2}, 0, 0, \ldots)$;

$\lambda_4 = 1$, $x^{(4)} = e_4$; $\lambda_5 = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(5)} = e_n$, , $n \geq 5$.

44. $\lambda_1 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(1)} = e_n$, $n = 1, 3, n \geq 5$, $\lambda_2 = 2$ va $\lambda_3 = 3$ lar oddiy xos qiymatlar bo'lib, ularga mos xos vektorlar $x^{(2)} = e_2$ va $x^{(3)} = e_4$.

45. $\lambda_1 = 1$ ikki karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(1)} = \alpha e_1 + \beta e_4$, $\lambda_2 = \frac{5}{2} - \frac{\sqrt{5}}{2}$, $x^{(2)} = (0, 2, 1 - \sqrt{5}, 0, 0, \ldots)$; $\lambda_3 = \frac{5}{2} + \frac{\sqrt{5}}{2}$, $x^{(3)} = (0, 2, 1 + \sqrt{5}, 0, 0, \ldots)$; $\lambda_4 = 0$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(4)} = e_n$, $n \geq 5$.

46. $\lambda_1 = 6$, $\lambda_2 = 5$, $\lambda_3 = 4$, $\lambda_4 = 3$, $\lambda_5 = 2$ lar oddiy xos qiymatlar bo'lib, ularga mos xos vektorlar $x^{(1)} = e_1$, $x^{(2)} = e_2$, $x^{(3)} = e_3$, $x^{(4)} = e_4$, $x^{(5)} = e_5$. $\lambda_6 = 1$ cheksiz karrali xos qiymat, unga mos keluvchi xos vektorlar $x^{(6)} = e_n$, $n \geq 6$.

47. $\sigma(A) = \left\{0, \frac{11 - \sqrt{97}}{2}, \frac{11 + \sqrt{97}}{2}\right\}$; $R_\lambda(A)\, x(t) =$

$$= \frac{1}{\lambda}\left[-x(t) + \frac{(9 - \lambda)\, x(2) - 4x(3)}{\lambda^2 - 11\lambda + 6}\, t - \frac{3x(2) + (\lambda - 2)x(3)}{\lambda^2 - 11\lambda + 6}\, t^2\right].$$

48. $\sigma(A) = [0, 2] \bigcup \{3 + \sqrt{2}\}$; $R_\lambda(A)\, x(t) =$

$$= \frac{1}{t - \lambda}\left[x(t) + \frac{x(1) + (\lambda - 2)x(2)}{\lambda^2 - 6\lambda + 7}\, t + \frac{(\lambda - 4)x(1) + x(2)}{\lambda^2 - 6\lambda + 7}\right].$$

49. $\sigma(A) = [0, 1]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{t^2 - \lambda} - \frac{x(0)}{(1 - \lambda)(t^2 - \lambda)}$.

50. $\sigma(A) = [0, 1] \bigcup \{2\}$,

$$R_\lambda(A)\, x(t) = \frac{x(t)}{t - \lambda} + \frac{x(0)}{\lambda(t - \lambda)}\, t^2 - \frac{\lambda\, x(1) + x(0)}{\lambda(2 - \lambda)(t - \lambda)}\, t^3.$$

51. $\sigma(A) = [0, 2]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{|t| - \lambda}$

52. $\sigma(A) = [2, 3]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{t + 2 - \lambda}$.

53. $\sigma(A) = [\frac{-\pi}{2}, \frac{\pi}{2}]$, $R_\lambda(A)\, x(t) = \frac{x(t)}{\mathrm{arctg}t - \lambda}$.

54. $\sigma(A) = [0, 1] \bigcup \{\lambda : \ \Delta(\lambda) = 0\}$, $\Delta(\lambda) = 2 + \dfrac{1}{2}\ln\left|\dfrac{1-\lambda}{1+\lambda}\right|$,

$$R_\lambda(A)\,x(t) = \frac{x(t)}{t^2-\lambda} - \frac{t}{\Delta(\lambda)(t^2-\lambda)}\int_0^1 \frac{s\,x(s)}{s^2-\lambda}\,ds.$$

55. $\sigma(A) = [0, 1] \bigcup \{\lambda : \ \Delta(\lambda) = 0\}$, $\Delta(\lambda) = 1 + \displaystyle\int_0^\infty \frac{2^{-2s}}{e^{-s}-\lambda}\,ds$.

$$R_\lambda(A)\,x(t) = \frac{x(t)}{e^{-t}-\lambda} - \frac{2^{-t}}{\Delta(\lambda)(e^{-t}-\lambda)}\int_0^\infty \frac{2^{-s}\,x(s)}{e^{-s}-\lambda}\,ds.$$

56. $\sigma(A) = [-1, 1] \bigcup \{\lambda : \ \Delta(\lambda) = 0\}$, $\Delta(\lambda) = 1 + 2\lambda + \lambda^2\ln\left|\dfrac{1-\lambda}{1+\lambda}\right|$, $R_\lambda(A)\,x(t) = \dfrac{x(t)}{t-\lambda} - \dfrac{t}{\Delta(\lambda)(t-\lambda)}\displaystyle\int_{-1}^1 \frac{s\,x(s)}{s-\lambda}\,ds$.

57. $\sigma(A) = \{-1, 1\}$.

$$R_\lambda(A)x = \left(-\frac{x_1}{1+\lambda}, \frac{x_2}{1-\lambda}, \ldots, -\frac{x_{2n-1}}{1+\lambda}, \frac{x_{2n}}{1-\lambda}, \ldots\right).$$

58. $\sigma(A) = \left\{0, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n}, \ldots\right\}$.

$$R_\lambda(A)\,x = \left(\frac{3\,x_1}{1-3\lambda}, \frac{4\,x_2}{1-4\lambda}, \ldots, \frac{(n+2)\,x_n}{1-(n+2)\lambda}, \ldots\right).$$

59. $\sigma(A) = \left\{0, 1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n}, \ldots\right\}$.

$$R_\lambda(A)x = \left(-\frac{1}{\lambda}x_1, \frac{1}{1-\lambda}x_2, \frac{2}{1-2\lambda}x_3, \ldots, \frac{n}{1-n\lambda}x_{n+1}, \ldots\right).$$

60. $\sigma(A) = \left\{0, 1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n}, \ldots\right\}$.

$$R_\lambda(A)x = \left(\frac{1}{1-\lambda}x_1, \frac{2}{1-2\lambda}x_2, \frac{3}{1-3\lambda}x_3, \ldots, \frac{n}{1-n\lambda}x_n, \ldots\right).$$

61. $\sigma(A) = \{\lambda \in \mathbb{C} : \ |\lambda| \le 1\}$. $R_\lambda(A)\,x =$

$$= \left(-\frac{x_1}{\lambda}, -\frac{x_1}{\lambda^2} - \frac{x_2}{\lambda}, \ldots, -\frac{x_1}{\lambda^n} - \frac{x_2}{\lambda^{n-1}} - \ldots - \frac{x_n}{\lambda}, \ldots\right).$$

62. $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. $(R_\lambda(A)\,x)_n = -\displaystyle\sum_{k=0}^\infty \frac{x_{n+k}}{\lambda^{k-1}}$, $n \in \mathbb{N}$.

63. $\sigma(A) = \{0, 1, 2\}$.

$$R_\lambda(A)\,x = \left(\frac{x_2-(1-\lambda)x_1}{2\lambda-\lambda^2}, \frac{x_1-(1-\lambda)x_2}{2\lambda-\lambda^2}, \frac{x_3}{1-\lambda}, \ldots, \frac{x_n}{1-\lambda}, \ldots\right).$$

64. $\sigma(A) = \left\{1, 2, \dfrac{3}{2}, \dfrac{1}{3}, \ldots, \dfrac{n+1}{n}, \ldots\right\}$.

$$R_\lambda(A)\,x = \left(\frac{1}{2-\lambda}x_1,\ \frac{2}{3-2\lambda}x_2,\ \ldots,\ \frac{n}{n+1-n\lambda}x_n,\ \ldots\right).$$

66. $A:\mathbb{C}^3 \to \mathbb{C}^3,\ \ Ax = (x_1 + x_2,\ x_2 + x_3,\ x_3)$.

67. $A:\mathbb{C}^3 \to \mathbb{C}^3,\ \ Ax = (x_1 + x_2,\ x_2,\ x_3)$.

73. Agar $\dim X < \infty$ bo'lsa mumkin. $\dim X = \infty$ bo'lsa, mumkin emas.

75. $\sigma(A) = \overline{\{e^{i},\ e^{2i},\ldots,e^{ni},\ldots\}}$.

76. $(Af)(n) = (-1)^n f(n),\ \ f \in \ell_2(\mathbb{Z})$.

77. Istalgan U unitar operatorning spektri $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.

78. $(Af)(x) = xf(x),\ \ f \in L_2[a,\ b]$.

79. Birlik operatordan farqli istalgan P proyeksiyalash operatori uchun $\sigma(P) = \{0,\ 1\} = \{m,\ M\}$ tenglik o'rinli.

80. $\sigma(A) = \{0,\ 1\},\ \ R_\lambda(A)\,x(t) = -\dfrac{x(t)}{\lambda} + \dfrac{(1-\lambda-t)x(0)}{\lambda(1-\lambda)^2} + \dfrac{x(1)\,t}{\lambda(1-\lambda)}$.

81. $\sigma(A) = \{0,\ 1\},\ \ R_\lambda(P) = \dfrac{1}{\lambda(1-\lambda)}\,P - \dfrac{1}{\lambda}\,I$.

91. $(\Delta f)(p) = 2\sum\limits_{j=1}^{\nu}(\cos p_j - 1)f(p),\ \ f \in L_2([-\pi,\ \pi]^\nu)$.

92. $(Vf)(p) = \dfrac{1}{\sqrt{2\pi}}\int\limits_{-\pi}^{\pi} v(p-q)f(q)\,dq,\ \ v = F\hat{v}$.

95. $(A^+ f^+)(p) = (2 - 2\cos p)f^+(p) - \dfrac{\mu}{2\pi}\int\limits_{-\pi}^{\pi}\cos p\cos s f^+(s)\,ds$.

$(A^- f^-)(p) = (2 - 2\cos p)f^-(p) - \dfrac{\mu}{2\pi}\int\limits_{-\pi}^{\pi}\sin p\sin s f^-(s)\,ds$.

96. a) V^+ operatorning xos qiymatlari $\lambda_0 = 1$ va $\lambda_n = e^{-n},\ \ n \in \mathbb{N}$ bo'lib, unga mos xos funksiya $f_0^+(p) = 1,\ \ f_n^+(p) = \cos np,\ \ \ V^-$ operatorning xos qiymatlari $\lambda_n = e^{-n},\ \ n \in \mathbb{N}$ bo'lib, unga mos xos funksiya $f_n^-(p) = \sin np$. b) V operator uchun $\lambda_0 = 1$ oddiy xos qiymat, $\lambda_n = e^{-n},\ \ n \in \mathbb{N}$ larning har biri ikki karrali xos qiymat bo'ladi.

98. Agar $\lambda < -1$ bo'lsa, $E_\lambda = 0$ ga, agar $\lambda > 1$ bo'lsa, $E_\lambda = I$ ga, agar $\lambda \in [-1,\ 1]$ bo'lsa, $(E_\lambda f)(t) = \chi_{A(\lambda)}(t)f(t)$, bu yerda $A(\lambda) = \{t \in [-\pi,\ \pi] : \cos t \le \lambda\}$.

3 - 7 - misollarda $\dim ImA = 2$ ekanligini ko'rsating va 5.1-teoremadan foydalaning.

8. $\dim ImA = 4$ ekanligini ko'rsating va 5.1-teoremadan foydalaning. 5.1-misoldan foydalaning.

9, 10-misollarda Artsela teoremasidan foydalaning.

11, 12-misollarda $\dim ImA = 2$ ekanligini ko'rsating va 5.1-teoremadan foydalaning.

13- misolda 6.1-teoremadan foydalaning.

14, 15-misollarda 5.1-misoldan foydalaning.

16 - 21- misollarda 5.1-misoldan foydalaning.

22-misolda $\dim ImA = 6$ ekanligini ko'rsating va 5.1-teoremadan foydalaning.

26 - 29 - misollarda 5.23-misoldan va 5.5-teoremadan foydalaning.

30, 31-misollarda 5.25-misoldan foydalaning.

33-35. 5.23-misoldan va 5.5-teoremadan foydalaning.

36-44 -misollarda 5.1-misoldan foydalaning.

45. $\{Ae_n\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin emasligini ko'rsating.

46. Hech qanday shartda ham bu operator kompakt bo'lmaydi.

48. a) yo'q. b) yo'q. c) ha.

50. b) va c).

53. $\sigma(A) = \{0\}$.

54. $\sigma(A) = \{0, \frac{2}{3}, -\frac{2}{5}\}$.

58. Singulyar sonlar $v(n)$, $n \in \mathbb{N}$.

6-§. Integral tenglamalar

2. a) $x(t) = 1$, b) $x(t) = 2\sin t$, c) $x(t) = \sin t$,
d) $x(t) = t^2 - t + \frac{1}{6}$, e) $x(t) = t^4 + \frac{25}{49}t^2$, f) $x(t) = t = \pi^3 \sin t$,

g) $x(t) = 1 - \frac{2t}{\pi} - \frac{\pi^2 \cos t}{18}$, h) $x(t) = \cos 2t + \frac{3\pi}{10} \sin t$,

4. a) $\lambda = \frac{1}{\pi}$ da yechim $x(t) = \alpha \sin t + \alpha \cos t$ ga, $\lambda = -\frac{1}{\pi}$ da yechim $x(t) = \alpha \sin t - \alpha \cos t$ bo'ladi.

b) $\lambda = \frac{2}{\pi}$ da yechim $x(t) = \alpha \cos t$ ga, $\lambda = -\frac{2}{\pi}$ da yechim $x(t) = \alpha \sin t$ ga teng bo'ladi.

c) $\lambda = -3$ da yechim $x(t) = \alpha t - 2\alpha t^2$ ga teng bo'ladi.

d) $\lambda = \frac{3}{2}$ da yechim $x(t) = \alpha t$ ga, $\lambda = \frac{5}{2}$ da yechim $x(t) = \alpha t^2$ ga teng bo'ladi.

5. a) Agar $\lambda \neq \frac{3}{2}$ bo'lsa, yechim yagona va u $x(t) = \alpha t^2 + \frac{3\beta}{3 - 2\lambda} t + \gamma$ ko'rinishda bo'ladi. Agar $\lambda = \frac{3}{2}$ va $\beta \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = \frac{3}{2}$ va $\beta = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \alpha t^2 + C t + \gamma$ ko'rinishda bo'ladi.

b) Agar $\lambda \neq \pm\frac{2}{\pi}$ bo'lsa, yechim yagona va u $x(t) = \beta + \frac{2\alpha - 4\beta\lambda}{2 + \lambda\pi} \sin t$ ko'rinishda bo'ladi. Agar $\lambda = -\frac{2}{\pi}$ va $\frac{\alpha\pi}{2} + 2\beta \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = -\frac{2}{\pi}$ va $\frac{\alpha\pi}{2} + 2\beta = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \beta + C \sin t$ ko'rinishda bo'ladi. Agar $\lambda = \frac{2}{\pi}$ bo'lsa, yechim cheksiz ko'p va u $x(t) = \beta + \frac{\alpha\pi - 4\beta}{2\pi} \sin t + C \cos t$ ko'rinishda bo'ladi.

c) Agar $\lambda \neq -\frac{3}{4}$, $\lambda \neq \frac{3}{2}$ bo'lsa, yechim yagona va u $x(t) = \frac{3\alpha}{3 - 2\lambda} t^2 - \frac{3\beta}{3 + 4\lambda} t$ ko'rinishda bo'ladi. Agar $\lambda = -\frac{3}{4}$ va $\beta \neq 0$ yoki $\lambda = \frac{3}{2}$ va $\alpha \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = -\frac{3}{4}$ va $\beta = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = C t + \frac{2\alpha}{3} t^2$ ko'rinishda bo'ladi. Agar $\lambda = \frac{3}{2}$ va $\alpha = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = -\frac{19\beta}{27} t + C t^2$ ko'rinishda bo'ladi.

d) Agar $\lambda \neq -\frac{1}{2}$, $\lambda \neq \frac{3}{2}$ bo'lsa, yechim yagona va u $x(t) = \frac{3\alpha t}{3-2\lambda}$ ko'rinishda bo'ladi. Agar $\lambda = \frac{3}{2}$ va $\alpha \neq 0$ bo'lsa, yechim mavjud emas. Agar $\lambda = \frac{3}{2}$ va $\alpha = 0$ bo'lsa, yechim cheksiz ko'p va u $x(t) = C\,t$ ko'rinishda bo'ladi. Agar $\lambda = -\frac{1}{2}$ bo'lsa, yechim cheksiz ko'p va u $x(t) = -\frac{3}{4}(\alpha - C)\,t + C\,t^2$ ko'rinishda bo'ladi.

6. $x(t) = \lambda \int\limits_0^{2\pi} \sin(t-2s)\, f(s)\, ds + f(t)$.

7. a) $\lambda_k = \pi^2(k+0,5)^2$, $k \in \mathbb{Z}$, $x_k(t) = C \sin \pi(k+0,5)\,t$.

b) $\lambda_k = 4k^2 - 1$, $k \in \mathbb{N}$, $x_k(t) = C \sin 2\,k\,t$.

c) $\lambda_k = (k+0,5)^2 - 1$, $k \in \mathbb{Z}_+$, $x_k(t) = C \sin(k+0,5)\,t$.

d) $\lambda_k = -w_k^2$, bu yerda w_k, $\operatorname{tg} w = \frac{2w}{1-w^2}$ tenglamaning ildizlari, $x_k(t) = w_k \cos w_k\, t + \sin w_k\, t$.

e) $\lambda_k = w_k^2 + 1$, bu yerda w_k, $\operatorname{tg} w = -\frac{2w}{1+w^2}$ tenglamaning ildizlari, $x_k(t) = w_k \cos w_k\, t + \sin w_k\, t$.

f) $\lambda_k = k^2\pi^2$, $x_k(t) = k\,\pi \cos k\,\pi\,t + \sin k\,\pi\,t$.

8. b) agar $\lambda \neq \pi^2$ va $\lambda \neq -4\,\pi^2\,n^2$, $n \geq 2$ bo'lsa, yechim $x_k(t) = \frac{\pi^3(\pi \cos \pi\,t + \sin \pi\,t)}{(\pi^2 - \lambda)(\pi^2+1)} - \sum\limits_{n=2}^{\infty} \frac{16\sqrt{2}\,\pi^2\,n^3(2\pi\,n \cos 2\pi\,n\,t + \sin 2\pi\,n\,t)}{(4n^2-1)(4\pi^2\,n^2+\lambda)\sqrt{4\pi^2\,n^2+1}}$ bo'ladi. Agar $\lambda \in \{\pi^2, -4\,\pi^2\,n^2,\ n \geq 2\}$ bo'lsa, yechim mavjud emas.

9. Parametr λ ning $\frac{6}{5}$ dan farqli barcha qiymatlarida tenglama yagona yechimga ega va u $x(s) = s^2 + \frac{7\lambda s}{2(6-5\lambda)}$ ko'rinishga ega.

10. Parametr $\lambda \in \mathbb{C}$ ning $\pm\frac{3}{2}$ dan farqli barcha qiymatlarida tenglama yagona yechimga ega va u $x(s) = s^2 + 1 + \frac{24\lambda s}{9-4\lambda^2} + \frac{16\lambda^2 s^2}{9-4\lambda^2}$ ko'rinishga ega.

11. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yechimga ega. $\lambda \neq 2$ da yechim yagona $x(s) = \sin 2\pi s$, $\lambda = 2$ da yechim cheksiz ko'p bo'lib, uning ko'rinishi $x(s) = \sin 2\pi s + as$. $a \in \mathbb{C}$ - ixtiyoriy son.

12. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yechimga ega. $\lambda \neq 3/2$ bo'lsa, $x(s) = s^2 - 1$ tenglamaning yagona yechimi bo'ladi. Agar $\lambda = 3/2$ bo'lsa, tenglama yechimi cheksiz ko'p bo'lib, ular $x(s) = s^2 - 1 + \alpha(1+s)$, $\alpha \in \mathbb{C}$ ko'rinishga ega.

13. Parametr $\lambda \in \mathbb{C}$ ning barcha $\lambda \neq 1/2$ qiymatlarida tenglama $x(s) = 1 - \frac{3}{2}s + \frac{\lambda(1+2s)}{4(1-2\lambda)}$ ko'rinishdagi yagona yechimga ega. Agar $\lambda = 1/2$ bo'lsa, tenglama yechimga ega emas.

14. Agar $\lambda = \pm 3/4$ bo'lsa, tenglama yechimga ega emas. Agar $\lambda \neq \pm 3/4$ bo'lsa, tenglama yagona $x(s) = s^2 + 2s + \lambda\alpha s + \lambda\beta(1+s^2)$, bu yerda $\alpha = \frac{2}{3} + \frac{8\lambda}{3} + \frac{12}{9-16\lambda^2} + \frac{4\lambda}{9-16\lambda^2}$, $\beta = \frac{12}{9-16\lambda^2} + \frac{4\lambda}{9-16\lambda^2}$.

15. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = s$ yechimga ega.

16. Parametr $\lambda \in \mathbb{C}$ ning barcha $\lambda \neq 3/2$ qiymatlarida tenglama $x(s) = 2s^2 + s + \frac{2\lambda}{3-2\lambda}(1+s+s^2)$ ko'rinishdagi yagona yechimga ega. Agar $\lambda = 3/2$ bo'lsa, tenglama yechimga ega emas.

17. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = \cos s$ yechimga ega.

18. Agar $\lambda \neq \pm\frac{1}{\pi}$ bo'lsa, tenglama yagona $x(s) = \frac{1}{1+\lambda\pi}\sin s$ yechimga ega. Agar $\lambda = -\frac{1}{\pi}$ bo'lsa, tenglama yechimga ega emas. Agar $\lambda = \frac{1}{\pi}$ bo'lsa, tenglama cheksiz ko'p $x(s) = \frac{1}{2}\sin s + \alpha\cos s$, $\alpha \in \mathbb{C}$ yechimlarga ega.

19. Parametr $\lambda \in \mathbb{C}$ ning barcha qiymatlarida tenglama yagona $x(s) = \cos s + \frac{\lambda\pi}{2}$ yechimga ega.

20. Agar $\lambda \notin \left\{\frac{3}{2}, \frac{5}{3}\right\}$ bo'lsa, tenglama yagona $x(s) = 1 + s^2 + \frac{16\lambda s^2}{3(5-3\lambda)}$ yechimga ega. Agar $\lambda = \frac{5}{3}$ bo'lsa, tenglama yechimga ega emas. Agar

$\lambda = \frac{3}{2}$ bo'lsa, tenglama cheksiz ko'p $x(s) = 1 + 17 \cdot s^2 + \alpha \cdot s$, $\forall \alpha \in \mathbb{C}$ yechimlarga ega.

21. Agar $\lambda \neq \pm\frac{3}{2}$ bo'lsa, tenglama yagona $x(s) = \sin \pi s$ yechimga ega. Agar $\lambda = \pm\frac{3}{2}$ bo'lsa, tenglama cheksiz ko'p $x(s) = \sin \pi s + \alpha s + \beta s^2$, $\forall \alpha, \beta \in \mathbb{C}$ yechimlarga ega.

22. Agar $\lambda \neq \pm\frac{2}{\pi}$ bo'lsa, tenglama yagona $x(s) = \cos s + \frac{2\pi\lambda}{4 - \pi^2\lambda^2} \sin s + \frac{\pi^2\lambda^2}{4 - \pi^2\lambda^2} \cos s$ yechimga ega. Agar $\lambda = \pm\frac{2}{\pi}$ bo'lsa, tenglama yechimga ega emas.

23. Agar $\lambda \neq \pm\frac{\sqrt{3}}{2}$ bo'lsa, tenglama yagona $x(s) = \frac{1}{2} + \frac{\lambda(3 + 2\lambda)}{3 - 4\lambda^2} + \frac{2s(3 + \lambda - 2\lambda^2)}{2(3 - 4\lambda^2)}$ yechimga ega. Agar $\lambda = \pm\frac{\sqrt{3}}{2}$ bo'lsa, tenglama yechimga ega emas.

24. Agar $\lambda \neq \frac{3}{2}$, $\lambda \neq \frac{3}{8}$ bo'lsa, tenglama yagona $x(s) = \frac{3}{3 - 8\lambda} + \frac{3s}{3 - 2\lambda}$ yechimga ega. Agar $\lambda = \frac{3}{2}$, $\lambda = \frac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

25. Agar $\lambda \neq 1$ bo'lsa, tenglama yagona $x(s) = \frac{1}{\sqrt{1 - s^2}} + \frac{\pi\lambda}{2(1 - \lambda)}$ yechimga ega. Agar $\lambda = \frac{3}{2}$, $\lambda = \frac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

26. Agar $\lambda \neq \frac{2}{e^2 - 1}$ bo'lsa, tenglama yagona $x(s) = e^{2s} + \frac{2\lambda(e^2 - 1)e^s}{3(2 - \lambda e^2 + \lambda)}$ yechimga ega. Agar $\lambda = \frac{2}{e^2 - 1}$ bo'lsa, tenglama yechimga ega emas.

27. Agar $\lambda \neq \frac{-9 \pm 3\sqrt{13}}{4}$ bo'lsa, tenglama yagona $x(s) = \frac{6\lambda}{9 - 18\lambda - 4\lambda^2} + \frac{9 - 14\lambda - 4\lambda^2}{9 - 18\lambda - 4\lambda^2} s$ yechimga ega. Agar $\lambda = \frac{-9 \pm 3\sqrt{13}}{4}$ bo'lsa, tenglama yechimga ega emas.

28. Agar $\lambda \neq \frac{3}{8}$ bo'lsa, tenglama yagona $x(s) = s + s^2 + \frac{6\lambda(1 + s)}{3 - 8\lambda}$ yechimga ega. Agar $\lambda = \frac{3}{8}$ bo'lsa, tenglama yechimga ega emas.

30. $x_2(s) = s + \dfrac{s^2}{2} + \dfrac{s^3}{6}$; $x(s) = e^s - 1$.

31. $x_2(s) = 1 + \dfrac{s^2}{2} + \dfrac{s^4}{24}$; $x(s) = \operatorname{ch} s$.

32. $x_2(s) = 2$; $x(s) = 2$.

33. $x_2(s) = 1 + 2s + \dfrac{s^2}{2} + \dfrac{s^3}{3}$; $x(s) = 2e^s - 1$.

34. $x_2(s) = s + \dfrac{s^3}{6}$; $x(s) = \operatorname{sh} s$.

35. $x_2(s) = 1 - \dfrac{s^2}{2}$; $x(s) = \cos s$.

36. $x_2(s) = \dfrac{101}{108}s - \dfrac{1}{27}$; $x(s) = \dfrac{461}{474}s - \dfrac{65}{948}$.

37. $x_2(s) = 1 + s$; $x(s) = e^s$.

38. $x_2(s) = e^s + \dfrac{1-e}{4}$; $x(s) = e^s$.

39. $x_2(s) = x(s) = s$.

40. $x_2(s) = s - \dfrac{1}{162}$; $x_2(s) = \dfrac{101}{108}s - \dfrac{1}{27}$; $x(s) = \dfrac{461}{474}s - \dfrac{65}{948}$.

41. $x_2(s) = s + \dfrac{1}{3}$; $x(s) = s + \dfrac{1}{4}$.

42. $x_2(s) = s + 1\dfrac{23}{48}$; $x(s) = s + 1\dfrac{5}{9}$.

43. $x_2(s) = 1\dfrac{3}{4}$; $x(s) = 2$.

44. $x_2(s) = \dfrac{3}{4}(1 - s)$; $x(s) = 1 - s$.

45. $x_2(s) = \dfrac{\pi + 24}{8}\sin s$; $x(s) = 4\sin s$.

46. $x_2(s) = \dfrac{1}{\pi} + \dfrac{3}{4} + \sin \pi s$; $x(s) = \dfrac{2}{\pi} + \sin \pi s$.

47. $x_2(s) = s + \dfrac{s^2}{2} + \dfrac{s^4}{12}$; $x(s) = e^s - 1$.

48. $x_2(s) = 1 + s + \dfrac{11}{6}s^2$; $x(s) = 1 + s + \dfrac{9}{4}s^2$.

Test javoblari

I bobda keltirilgan test javoblari

1-C 2-B 3-A 4-C 5-B 6-B 7-C 8-B 9-A 10-B 11-C 12-A 13-B 14-C 15-B 16-B 17-D 18-A 19-B 20-B 21-B 22-A 23-D 24-B 25-C 26-A 27-A 28-D 29-A 30-C 31-A 32-A 33-C 34-B 35-B 36-C 37-D 38-A 39-B 40-C. 41-A 42-D 43-B 44-B 45-A 46-A 47-B 48-A 49-A 50-B.

II bobda keltirilgan test javoblari

1-D 2-A 3-A 4-A 5-C 6-C 7-C 8-C 9-C 10-C 11-B 12-D 13-C 14-C 15-B 16-A 17-A 18-B 19-A 20-B 21-A 22-D 23-C 24-A 25-B 26-B 27-A 28-C 29-D 30-C 31-B 32-C 33-C 34-C 35-B 36-A 37-B 38-D 39-D 40-D.

Foydalanilgan adabiyotlar

1. Колмогоров А.Н., Фомин С.В. Элементы теории функций и функционального анализа. Москва: Наука. 1989.
2. Sarimsoqov T.A. Funksional analiz kursi. Toshkent: O'qituvchi. 1986.
3. Люстерник Л.А., Соболев В.И. Элементы функционального анализа. Москва: Наука. 1965.
4. Треногин В.А. Функциональный анализ. Москва: Наука. 1980.
5. В.А. Треногин, Б.М. Писаревский, Т.С. Соболева. Задачи и упражнения по функциональному анализу. Москва: Наука. 1984.
6. Sh.A. Ayupov, M.A. Berdiqulov, R.M. Turg'unboyev. Funksional analiz. Toshkent. 2008.
7. Ayupov Sh. A., Ibragimov M.M., Kudaybergenov K.K. Funksional analizdan misol va masalalar. O'quv qo'llanma. Nukus. Bilim. 2009.
8. А.Б. Антоневич, П.Н. Князов, Я.В. Радыно. Задачи и упражнения по функциональному анализу. Минск. Высшая школа. 1978.
9. Городецкий В.В., Нагнибида Н.И., Настасиев П.П. Методы решения задач по функциональному анализу. Киев 1990.
10. Ю.М. Березанский, Г.Ф. Ус., З.Г. Шефтель. Функциональный анализ. Киев: Выщая школа. 1990.
11. J.I. Abdullayev, R.N. G'anixo'jayev, M.H. Shermatov, O.I. Egamberdiyev. Funksional analiz. O'quv qo'llanma. Toshkent-Samarqand. 2009.

J.I. Abdullayev, R.N. G'anixo'jayev, I.A. Ikromov

FUNKSIONAL ANALIZDAN MASALALAR TO'PLAMI

uslubiy qo'llanma
III qism
CHIZIQLI OPERATORLAR

Toshkent – «TURON-IQBOL» – 2013
100182. Toshkent sh., H. Boyqaro ko'chasi, 51-uy
Tel.: 244-25-58. Faks.: 244-20-19

Muharrir *X. Alimov*
Badiiy muharrir *E. Muratov*
Texnik muharrir *I. Bozorov*
Musahhih: *S. Abdunabiyeva*

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Bosishga 05.06.2013 da ruxsat etildi. Bichimi $60x84^1/_{16}$.
«Bukvar» garniturasi. Ofset usulida bosildi.
Nashr t. 9,36. Shartli b.t. 8,37. Adadi 300 nusxa.
20-sonli buyurtma.

«TURON MATBAA» MCHJ bosmaxonasida chop etildi.
Toshkent, Olmazor tumani, Talabalar ko'chasi, 2-uy.