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ИНТЕГРАЛЫ И РЯДЫ



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ИНТЕГРАЛЫ И РЯДЫ ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ



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ПРЕДИСЛОВИЕ

Книга содержит неопределенные и определенные (в том числе кратные) интегралы, конечные суммы, ряды и произведения с элементарными функциями. Она написана с целью удовлетворения запросов широкого круга специалистов в различных областях знаний и включает результаты, изложенные в аналогичных изданиях, а также в научной и периодической литературе, опубликованной в последнее время. Ряд результатов публикуется впервые. Отметим, что, интерпретируя некоторые интегралы и ряды как обобщенные функции, интегральные преобразования обобщенных функций или рассматривая их регуляризацию, можно придать им смысл в более широкой области изменения параметров.

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Найти вполне оптимальную систему упорядочения материала книги вряд ли представляется возможным, но мы надеемся, что с помощью оглавления читатель сумеет отыскать нужные формулы.

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Для компактности изложения в ряде формул используется сокращенная запись. Например, формула

$$\int_0^1 x^{\alpha-1} \begin{cases} \arcsin x \\ \arccos x \end{cases} dx = \frac{\pi}{2\alpha} \begin{pmatrix} (1) \\ (0) \end{pmatrix} \mp \frac{1}{\sqrt{\pi}} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \quad [\operatorname{Re} \alpha > -\begin{pmatrix} (1) \\ (0) \end{pmatrix}]$$

представляет собой сокращенную запись двух формул

$$\int_0^1 x^{\alpha-1} \arcsin x dx = \frac{\pi}{2\alpha} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \right) \quad [\operatorname{Re} \alpha > -1]$$

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1.1. ВВЕДЕНИЕ

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В ряде формул, где при интегрировании возникает логарифм абсолютной величины, знак абсолютной величины для простоты опущен.

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5. $\int \sin x \, dx = -\cos x.$
6. $\int \cos x \, dx = \sin x.$

7. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$ 8. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$
 9. $\int \frac{\sin x}{\cos^2 x} dx = \sec x.$ 10. $\int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x.$
 11. $\int \operatorname{tg} x dx = -\ln |\cos x|.$ 12. $\int \operatorname{ctg} x dx = \ln |\sin x|.$
 13. $\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right|.$ 14. $\int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$
 15. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a}$ [a ≠ 0].
 16. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \operatorname{Arth} \frac{x}{a} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$ [a ≠ 0].
 17. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arcsin} \frac{x}{a}$ [a ≠ 0].
 18. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{Arsh} \frac{x}{a} = \ln (x + \sqrt{x^2 + a^2})$ [a ≠ 0].
 19. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{Arch} \frac{x}{a} = \ln |x + \sqrt{x^2 - a^2}|$ [a ≠ 0].
 20. $\int \operatorname{sh} x dx = \operatorname{ch} x.$ 21. $\int \operatorname{ch} x dx = \operatorname{sh} x.$
 22. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$ 23. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$
 24. $\int \operatorname{th} x dx = \ln \operatorname{ch} x.$ 25. $\int \operatorname{cth} x dx = \ln |\operatorname{sh} x|.$
 26. $\int \frac{dx}{\operatorname{sh} x} = \ln \left| \operatorname{th} \frac{x}{2} \right|.$ 27. $\int \frac{dx}{\operatorname{ch} x} = 2 \operatorname{arctg} e^x.$

1.1.3. Общие формулы.

28. $\int (af(x) + bg(x)) dx = a \int f(x) dx + b \int g(x) dx.$
 29. $\int \frac{d}{dx} f(x) dx = f(x).$
 30. $\int \frac{df(x)}{dx} g(x) dx = f(x) g(x) - \int f(x) \frac{dg(x)}{dx} dx$ [интегрирование по частям].
 31. $\int f(x) dx = \int f(g(y)) g'(y) dy$ [интегрирование подстановкой; $x = g(y)$].

1.2. СТЕПЕННАЯ И АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

1.2.1. Введение.

В этом пункте кратко изложены некоторые сведения общего характера, которые могут быть использованы при вычислении неопределенных интегралов.

1. Если $P_1(x)$ и $Q_1(x)$ — произвольные многочлены, то их отношение может быть представлено в виде

$$\frac{P_1(x)}{Q_1(x)} = P_2(x) + \frac{P(x)}{Q(x)},$$

где $P_2(x)$, $P(x)$, $Q(x)$ — многочлены, причем степень $P(x)$ меньше степени $Q(x)$. Пусть a_i , $i=1, \dots, m$, — корни многочлена $Q(x)$, n_i — соответствующие кратности корней. Тогда

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n_1} \frac{A_k^{(1)}}{(x-a_1)^k} + \sum_{k=1}^{n_2} \frac{A_k^{(2)}}{(x-a_2)^k} + \dots + \sum_{k=1}^{n_m} \frac{A_k^{(m)}}{(x-a_m)^k},$$

где

$$A_k^{(i)} = \frac{1}{(n_i - k)!} \frac{d^{n_i - k}}{dx^{n_i - k}} \left[\frac{P(x)}{Q(x)} (x - a_i)^{n_i} \right] \Big|_{x=a_i}.$$

Если $n_i = 1$ для всех i , то

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^m \frac{A^{(i)}}{x - a_i},$$

где $A^{(i)} = P(a_i)/Q'(a_i)$.

При объединении пар слагаемых, соответствующих комплексно сопряженным корням, получаются дроби вида $\frac{Ax+B}{x^2+bx+c}$, где трехчлен x^2+bx+c имеет

действительные коэффициенты. Поэтому вычисление интеграла $\int \frac{P(x)}{Q(x)} dx$ сводится к интегрированию рациональных дробей: $\int \frac{dx}{(x-a)^n}$, $\int \frac{Ax+B}{(x^2+bx+c)^n} dx$.

2. Интегралы вида $\int x^p (ax^r + b)^q dx$ (p, q, r — рациональные числа) выражаются в виде конечной комбинации элементарных функций только в следующих случаях:

- а) q — целое число;
- б) $(p+1)/r$ — целое число; тогда

$$\int x^p (ax^r + b)^q dx = \frac{1}{r} \int t^{(p+1)/r-1} (at+b)^q dt \quad [t = x^r];$$

- в) $\frac{p+1}{r} + q$ — целое число; тогда

$$\int x^p (ax^r + b)^q dx = \frac{1}{r} \int \left(\frac{at+b}{t} \right)^q t^{(p+1)/r+q-1} dt \quad [t = x^r].$$

3. Интегралы вида $\int R \left(x, \left(\frac{ax+b}{cx+d} \right)^p, \left(\frac{ax+b}{cx+d} \right)^q, \dots \right) dx$, где $p, q, \dots$ — рациональные числа, приводятся к интегралам от рациональных функций подстановкой

$$\frac{ax+b}{cx+d} = t^m,$$

где m — общий знаменатель дробей $p, q, \dots$.

4. Интегралы вида

$$I = \int \frac{R_1(x) + R_2(x) \sqrt{P(x)}}{R_3(x) + R_4(x) \sqrt{P(x)}} dx,$$

где R_1, R_2, R_3, R_4 — рациональные функции, а $P(x)$ — многочлен 3-й или 4-й степени с действительными коэффициентами и простыми корнями, приводятся к виду

$$I = \int R_5(x) dx + \int \frac{R(x)}{\sqrt{P(x)}} dx,$$

$$R_3(x) = \frac{R_1(x) R_2(x) - R_2(x) R_4(x) P(x)}{R_1^2(x) - R_4^2(x) P(x)},$$

$$R(x) = \frac{(R_2(x) R_3(x) - R_1(x) R_4(x)) P(x)}{R_1^2(x) - R_4^2(x) P(x)}.$$

Эллиптический интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ при помощи замен переменных которые указаны ниже, может быть приведен к форме, содержащей рациональные функции от эллиптических функций Якоби. Интегралы от эллиптических функций Якоби см. в [24].

1) Пусть $P(x) = a_0(x^2 \pm a^2)(x^2 \pm b^2)$. Тогда интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации трех интегралов: $\int \frac{dx}{\sqrt{P(x)}}$,

$$\int \frac{x^2 dx}{\sqrt{P(x)}}, \quad \int \frac{dx}{(x^2 - \rho^2) \sqrt{P(x)}}$$

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\sqrt{(a^2 - x^2)(x^2 - b^2)}, \quad \sqrt{(a^2 - x^2)(b^2 - x^2)}, \quad \sqrt{(a^2 + x^2)(b^2 - x^2)},$$

$$\sqrt{(a^2 + x^2)(x^2 - b^2)}, \quad \sqrt{(x^2 + a^2)(x^2 + b^2)}, \quad \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)},$$

где a^2, b^2 — действительные числа, ρ^2 и $\bar{\rho}^2$ — комплексно сопряженные числа. В первых пяти случаях подстановка

$$x^2 = \frac{A_1 + A_2 \operatorname{sn}^2 u}{B_1 + B_2 \operatorname{sn}^2 u} \quad (0 \leq u \leq K),$$

где $\operatorname{sn} u$ — эллиптический синус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные, приводит интегралы к виду $\int R_0(\operatorname{sn} u) du$, где R_0 — рациональная функция.

В шестом случае следует использовать замену

$$x^2 = \frac{A_1 + A_2 \operatorname{cn} u}{B_1 + B_2 \operatorname{cn} u} \quad (0 \leq u \leq 2K),$$

где $\operatorname{cn} u$ — эллиптический косинус Якоби и A_1, A_2, B_1, B_2 — некоторые постоянные.

2) Пусть $P(x) = a_0(x + a_1)(x + a_2)(x + a_3)$, где все a_i различны. Интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации трех интегралов:

$$\int \frac{dx}{\sqrt{P(x)}}, \quad \int \frac{x dx}{\sqrt{P(x)}}, \quad \int \frac{dx}{(x - \rho) \sqrt{P(x)}}.$$

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих пяти способов:

$$\sqrt{(x - a)(x - b)(x - c)}, \quad \sqrt{(a - x)(x - b)(x - c)}, \quad \sqrt{(a - x)(b - x)(x - c)},$$

$$\sqrt{(x - a)[(x - b_1)^2 + b_2^2]}, \quad \sqrt{(a - x)[(x - b_1)^2 + b_2^2]},$$

где a, b, c, b_1, b_2 действительны и $a > b > c$. В первых трех случаях к интегралам от эллиптических функций Якоби приводит подстановка

$$x = \frac{A_1 + A_2 \operatorname{sn}^2 u}{B_1 + B_2 \operatorname{sn}^2 u} \quad (0 \leq u \leq K), \quad (*)$$

где A_1, A_2, B_1, B_2 — некоторые постоянные. Если два корня многочлена $P(x)$ комплексны, то следует сделать замену

$$x = \frac{A_1 + A_2 \operatorname{cn} u}{B_1 + B_2 \operatorname{cn} u} \quad (0 \leq u \leq 2K). \quad (**)$$

3) Пусть $P(x) = a_0(x+a_1)(x+a_2)(x+a_3)(x+a_4)$, где все a_i различны. Интеграл $\int \frac{R(x)}{\sqrt{P(x)}} dx$ может быть представлен в виде линейной комбинации четырех интегралов: $\int \frac{dx}{\sqrt{P(x)}}, \int \frac{x dx}{\sqrt{P(x)}}, \int \frac{x^2 dx}{\sqrt{P(x)}}, \int \frac{dx}{(x-p)\sqrt{P(x)}}$.

Выражение $\sqrt{P(x)}$ можно записать (после вынесения множителя $\sqrt{|a_0|}$) одним из следующих шести способов:

$$\begin{aligned} & \sqrt{(x-a)(x-b)(x-c)(x-d)}, \quad \sqrt{(a-x)(x-b)(x-c)(x-d)}, \\ & \sqrt{(a-x)(b-x)(x-c)(x-d)}, \quad \sqrt{(x-a)(x-b)[(x-b_1)^2+b_2^2]}, \\ & \sqrt{(x-a)(b-x)[(x-b_1)^2+b_2^2]}, \quad \sqrt{[(x-b_1)^2+b_2^2][(x-b_2)^2+b_3^2]}, \end{aligned}$$

где $a, b, c, d, b_1, b_2, b_3, b_4$ действительны и $a > b > c > d$.

В первых трех случаях следует сделать замену (*), а в остальных — замену (**).

1.2.2. Интегралы вида $\int x^p(ax^r+b)^q dx$.

1. $\int x^p(ax^r+b)^q dx = \frac{x^{p+1}(ax^r+b)^q}{qr+p+1} + \frac{qrb}{qr+p+1} \int x^p(ax^r+b)^{q-1} dx.$
2. $= -\frac{x^{p+1}(ax^r+b)^{q+1}}{(q+1)rb} + \frac{qr+r+p+1}{(q+1)rb} \int x^p(ax^r+b)^{q+1} dx.$
3. $= \frac{x^{p+1}(ax^r+b)^q}{p+1} - \frac{qra}{p+1} \int x^{p+r}(ax^r+b)^{q-1} dx.$
4. $= \frac{x^{p-r+1}(ax^r+b)^{q+1}}{(q+1)ra} - \frac{p-r+1}{(q+1)ra} \int x^{p-r}(ax^r+b)^{q+1} dx.$
5. $= \frac{x^{p-r+1}(ax^r+b)^{q+1}}{(qr+p+1)a} - \frac{(p-r+1)b}{(qr+p+1)a} \int x^{p-r}(ax^r+b)^q dx.$
6. $= \frac{x^{p+1}(ax^r+b)^{q+1}}{(p+1)b} - \frac{(qr+r+p+1)a}{(p+1)b} \int x^{p+r}(ax^r+b)^q dx.$
7. $\int x^p(ax^r+b)^n dx = \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^{p+kr+1}}{p+kr+1}.$
8. $\int \frac{x^p dx}{(ax^r+b)^q} = \frac{x^{p+1}}{(q-1)rb(ax^r+b)^{q-1}} + \frac{(q-1)r-p-1}{(q-1)rb} \int \frac{x^p dx}{(ax^r+b)^{q-1}}.$
9. $= \frac{-x^{p-r+1}}{(q-1)ra(ax^r+b)^{q-1}} + \frac{p-r+1}{(q-1)ra} \int \frac{x^{p-r} dx}{(ax^r+b)^{q-1}}.$
10. $\int \frac{x^{r-1} dx}{(ax^r+b)^q} = \frac{-1}{(q-1)ra(ax^r+b)^{q-1}}.$
11. $\int \frac{x^{r-1} dx}{ax^r+b} = \frac{1}{ra} \ln |ax^r+b|.$

$$12. \int \frac{dx}{x^p (ax^r + b)^q} = \frac{1}{(q-1)rbx^{p-1}(ax^r + b)^{q-1}} + \frac{p+(q-1)r-1}{(q-1)rb} \int \frac{dx}{x^p (ax^r + b)^{q-1}}.$$

$$13. \int \frac{dx}{x^p (ax^r + b)} = -\frac{1}{(p-1)bx^{p-1}} - \frac{a}{b} \int \frac{dx}{x^{p-r}(ax^r + b)}.$$

$$14. \int \frac{dx}{x(ax^r + b)} = \frac{1}{rb} \ln \left| \frac{x^r}{ax^r + b} \right|.$$

1.2.3. Интегралы вида $\int \frac{x^m dx}{x^n \pm a^n}$.

$$1. \int \frac{dx}{x^{2n} + a^{2n}} = -\frac{1}{2na^{2n-1}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n} \pi + a^2 \right) + \frac{1}{na^{2n-1}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n} \pi}{a \sin \frac{2k+1}{2n} \pi}.$$

$$2. \int \frac{dx}{x^{2n+1} + a^{2n+1}} = \frac{1}{(2n+1)a^{2n}} \ln |x+a| - \frac{1}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) + \frac{2}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

$$3. \int \frac{dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n-1}} \ln \left| \frac{x-a}{x+a} \right| + \frac{1}{2na^{2n-1}} \sum_{k=1}^{n-1} \cos \frac{k\pi}{n} \ln \left(x^2 - 2ax \cos \frac{k\pi}{n} + a^2 \right) - \frac{1}{na^{2n-1}} \sum_{k=1}^{n-1} \sin \frac{k\pi}{n} \operatorname{arctg} \frac{x - a \cos \frac{k\pi}{n}}{a \sin \frac{k\pi}{n}}.$$

$$4. \int \frac{dx}{x^{2n+1} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n}} \ln |x-a| - \frac{1}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \cos \frac{2k+1}{2n+1} \pi \ln \left(x^2 + 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) - \frac{2}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \sin \frac{2k+1}{2n+1} \pi \operatorname{arctg} \frac{x + a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi}.$$

$$5. \int \frac{x^m}{x^{2n+1} + a^{2n+1}} dx = \frac{1}{na^{2n+1}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n} \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n} \pi}{a \sin \frac{2k+1}{2n} \pi} -$$

$$- \frac{1}{2na^{2n+1}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n} \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n} \pi + a^2 \right) \quad [1 \leq m \leq 2(n-1)].$$

$$6. \int \frac{x^m dx}{x^{2n+1} + a^{2n+1}} = \frac{(-1)^m}{(2n+1)a^{2n+1}} \ln |x+a| -$$

$$- \frac{1}{(2n+1)a^{2n+1}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n+1} \pi \ln \left(x^2 - 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) +$$

$$+ \frac{2}{(2n+1)a^{2n+1}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n+1} \pi \operatorname{arctg} \frac{x - a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi} \quad [1 \leq m \leq 2n-1].$$

$$7. \int \frac{x^m dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n+1}} \{ \ln |x-a| + (-1)^{m+1} \ln |x+a| \} +$$

$$+ \frac{1}{2na^{2n+1}} \sum_{k=1}^{n-1} \cos \frac{(m+1)k\pi}{n} \ln \left(x^2 - 2ax \cos \frac{k\pi}{n} + a^2 \right) -$$

$$- \frac{1}{na^{2n+1}} \sum_{k=1}^{n-1} \sin \frac{(m+1)k\pi}{n} \operatorname{arctg} \frac{x - a \cos \frac{k\pi}{n}}{a \sin \frac{k\pi}{n}} \quad [1 \leq m \leq 2(n-1)].$$

$$8. \int \frac{x^m dx}{x^{2n} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n+1}} \ln |x-a| +$$

$$+ \frac{(-1)^{m+1}}{(2n+1)a^{2n+1}} \sum_{k=0}^{n-1} \cos \frac{(m+1)(2k+1)\pi}{2n+1} \ln \left(x^2 + 2ax \cos \frac{2k+1}{2n+1} \pi + a^2 \right) +$$

$$+ \frac{2(-1)^{m+1}}{(2n+1)a^{2n+1}} \sum_{k=0}^{n-1} \sin \frac{(m+1)(2k+1)\pi}{2n+1} \operatorname{arctg} \frac{x + a \cos \frac{2k+1}{2n+1} \pi}{a \sin \frac{2k+1}{2n+1} \pi} \quad [1 \leq m \leq 2n-1].$$

1.2.1. Интегралы вида $\int \frac{x^p dx}{(x+a)^q}$.

$$1. \int \frac{x^p dx}{(x+a)^q} = -\frac{x^p}{(q-1)(x+a)^{q-1}} + \frac{p}{q-1} \int \frac{x^{p-1} dx}{(x+a)^{q-1}}.$$

$$2. \int_0^x \frac{x^{\lambda-1} dx}{x+a} = \frac{1}{a} x^\lambda \Phi \left(-\frac{x}{a}, 1, \lambda \right) \quad [\operatorname{Re} \lambda > 0; x < a].$$

$$3. \int_0^x \frac{x^{\lambda-1} dx}{(x+a)^{\nu}} = \frac{1}{\lambda} a^{-\nu} x^{\lambda} {}_2F_1(\nu, \lambda; 1+\lambda; -\frac{x}{a}) \quad [\operatorname{Re} \lambda > 0].$$

$$4. \int_x^{\infty} \frac{x^{\lambda-1} dx}{(x+a)^{\nu}} = \frac{x^{\lambda-\nu}}{\nu-\lambda} {}_2F_1(\nu, \nu-\lambda; 1+\nu-\lambda; -\frac{a}{x}) \quad [\operatorname{Re} \lambda < \operatorname{Re} \nu].$$

$$5. \int \frac{x^m dx}{(x+a)^q} = \sum_{k=0}^m \binom{m}{k} \frac{(-a)^k (x+a)^{m-q-k+1}}{m-q-k+1};$$

если $k=m-q+1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1} (-a)^{m-q+1} \ln |x+a|$.

$$6. \int \frac{x^m dx}{(x+a)^q} = - \sum_{k=0}^m \binom{m}{k} \frac{(-a)^{m-k}}{(q-k-1)(x+a)^{q-k-1}};$$

если $k=q-1$, то вместо соответствующего члена в сумме следует взять $\binom{m}{q-1} (-a)^{m-q+1} \ln |x+a|$.

$$7. \int \frac{dx}{(x+a)^n} = - \frac{1}{(n-1)(x+a)^{n-1}}. \quad 8. \int \frac{dx}{x+a} = \ln |x+a|.$$

$$9. \int \frac{x dx}{(x+a)^n} = - \frac{1}{(n-2)(x+a)^{n-2}} + \frac{1}{(n-1)(x+a)^{n-1}}.$$

$$10. \int \frac{x^2 dx}{(x+a)^n} = - \frac{1}{(n-3)(x+a)^{n-3}} + \frac{2a}{(n-2)(x+a)^{n-2}} - \frac{a^2}{(n-1)(x+a)^{n-1}}.$$

$$11. \int \frac{x^3 dx}{(x+a)^n} = - \frac{1}{(n-4)(x+a)^{n-4}} + \frac{3a}{(n-3)(x+a)^{n-3}} - \frac{3a^2}{(n-2)(x+a)^{n-2}} + \frac{a^3}{(n-1)(x+a)^{n-1}}.$$

$$12. \int \frac{x^m dx}{x+a} = \sum_{k=0}^{m-1} \frac{(-1)^k a^k x^{m-k}}{m-k} + (-a)^m \ln |x+a|.$$

$$13. \int \frac{x dx}{x+a} = x-a \ln |x+a|. \quad 14. \int \frac{x^2 dx}{x+a} = \frac{x^2}{2} - ax + a^2 \ln |x+a|.$$

$$15. \int \frac{x^3 dx}{x+a} = \frac{x^3}{3} - \frac{ax^2}{2} + a^2 x - a^3 \ln |x+a|.$$

$$16. \int \frac{x^m dx}{(x+a)^2} = \sum_{k=1}^{m-1} (-1)^k \frac{ka^{k-1} x^{m-k}}{m-k} + \frac{(-1)^{m-1} a^m}{x+a} + (-1)^{m-1} m a^{m-1} \ln |x+a|.$$

$$17. \int \frac{x dx}{(x+a)^2} = \frac{a}{x+a} + \ln |x+a|.$$

$$18. \int \frac{x^2 dx}{(x+a)^2} = x - \frac{a^2}{x+a} - 2a \ln |x+a|.$$

$$19. \int \frac{x^2 dx}{(x+a)^2} = \frac{x^2}{2} - 2ax + \frac{a^2}{x+a} + 3a^2 \ln |x+a|.$$

$$20. \int \frac{x dx}{(x+a)^2} = -\frac{1}{x+a} + \frac{a}{2(x+a)^2}.$$

$$21. \int \frac{x^2 dx}{(x+a)^2} = \frac{2a}{x+a} - \frac{a^2}{2(x+a)^2} + \ln |x+a|.$$

$$22. \int \frac{x^3 dx}{(x+a)^2} = x - \frac{3a^2}{x+a} + \frac{a^3}{2(x+a)^2} - 3a \ln |x+a|.$$

1.2.5. Интегралы вида $\int \frac{dx}{x^p (x+a)^q}.$

$$1. \int \frac{dx}{x^p (x+a)^q} = \frac{-1}{(p-1) a x^{p-1} (x+a)^{q-1}} - \frac{p+q-2}{(p-1) a} \int \frac{dx}{x^{p-1} (x+a)^q}.$$

$$2. = \frac{1}{(q-1) a x^{p-1} (x+a)^{q-1}} + \frac{p+q-2}{(q-1) a} \int \frac{dx}{x^p (x+a)^{q-1}}.$$

$$3. \int \frac{dx}{x^m (x+a)^n} = \frac{1}{a^{m+n-1}} \sum_{k=0}^{m+n-2} (-1)^{k+1} \binom{m+n-2}{k} \frac{(x+a)^{m-k-1}}{(m-k-1) x^{m-k-1}};$$

если $k=m-1$, то вместо соответствующего члена в сумме следует взять $(-1)^m \binom{m+n-2}{m-1} \ln \left| \frac{x+a}{x} \right|$.

$$4. \int \frac{dx}{x (x+a)^n} = \sum_{k=1}^{n-1} \frac{1}{k a^{n-k} (x+a)^k} - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right|.$$

$$5. = \frac{1}{a^n} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n-1}{k} \left(\frac{x}{x+a} \right)^k - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right|.$$

$$6. \int \frac{dx}{x^2 (x+a)^n} = -\frac{1}{a x (x+a)^{n-1}} - \frac{n}{a} \left[\sum_{k=1}^{n-1} \frac{1}{k a^{n-k} (x+a)^k} - \frac{1}{a^n} \ln \left| \frac{x+a}{x} \right| \right].$$

$$7. \int \frac{dx}{x^3 (x+a)^n} = \frac{(n+1)x-a}{2a^2 x^2 (x+a)^{n-1}} + \frac{n(n+1)}{2a^2} \int \frac{dx}{x (x+a)^n}.$$

$$8. \int \frac{dx}{x^m (x+a)} = \sum_{k=1}^{m-1} \frac{(-1)^k}{(m-k) a^k x^{m-k}} + \frac{(-1)^m}{a^m} \ln \left| \frac{x+a}{x} \right|.$$

$$9. \int \frac{dx}{x (x+a)} = \frac{1}{a} \ln \left| \frac{x}{x+a} \right|.$$

$$10. \int \frac{dx}{x^2 (x+a)} = -\frac{1}{a x} + \frac{1}{a^2} \ln \left| \frac{x+a}{x} \right|.$$

$$11. \int \frac{dx}{x^3 (x+a)} = \frac{1}{a^2 x} - \frac{1}{2a x^2} - \frac{1}{a^2} \ln \left| \frac{x+a}{x} \right|.$$

$$12. \int \frac{dx}{x (x+a)^2} = \frac{1}{a} \left(\frac{1}{x+a} - \frac{1}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$13. \int \frac{dx}{x^2(x+a)^2} = -\frac{1}{a^2} \left(\frac{1}{x+a} + \frac{1}{x} - \frac{2}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$14. \int \frac{dx}{x^3(x+a)^2} = \frac{1}{a^2} \left(\frac{1}{x+a} + \frac{2}{x} - \frac{a}{2x^2} - \frac{3}{a} \ln \left| \frac{x+a}{x} \right| \right).$$

$$15. \int \frac{dx}{x(x+a)^2} = \frac{1}{a^2(x+a)} + \frac{1}{2a(x+a)} - \frac{1}{a^2} \ln \left| \frac{x+a}{x} \right|.$$

$$16. \int \frac{dx}{x^2(x+a)^2} = -\frac{1}{2a^2(x+a)^2} - \frac{2}{a^2(x+a)} - \frac{1}{a^2x} + \frac{3}{a^2} \ln \left| \frac{x+a}{x} \right|.$$

$$17. \int \frac{dx}{x^3(x+a)^2} = \frac{1}{a^2} \left[\frac{3}{x+a} + \frac{a}{2(x+a)^2} + \frac{3}{x} - \frac{a}{2x^2} - \frac{6}{a} \ln \left| \frac{x+a}{x} \right| \right].$$

1.2.6. Интегралы вида $\int (x+c)^p \left(\frac{x+a}{x+b} \right)^q dx$.

$$1. \int x^p \left(\frac{x+a}{x+b} \right)^q dx = \frac{(x+a)(x+b)x^{p-1}}{p+1} \left(\frac{x+a}{x+b} \right)^q - \\ - \frac{q(b-a)+p(b+a)}{p+1} \int x^{p-1} \left(\frac{x+a}{x+b} \right)^q dx - \frac{(p-1)ab}{p+1} \int x^{p-2} \left(\frac{x+a}{x+b} \right)^q dx.$$

$$2. \int x \left(\frac{x+a}{x+b} \right)^q dx = \\ = \frac{(x+a)^q}{2(x+b)^{q-2}} + \frac{b(x+a)^q}{(q-1)(x+b)^{q-1}} + \frac{q[(q-1)(a-b)-2b]}{2(q-1)} \int \left(\frac{x+a}{x+b} \right)^{q-1} dx.$$

$$3. = \frac{(x+a)^{q+1}}{2(x+b)^{q-1}} - \frac{q(b-a)+a+b}{2} \int \left(\frac{x+a}{x+b} \right)^q dx.$$

$$4. \int \frac{(x+a)^p}{(x+b)^q} dx = -\frac{(x+a)^p}{(q-1)(x+b)^{q-1}} + \frac{p}{q-1} \int \frac{(x+a)^{p-1}}{(x+b)^{q-1}} dx.$$

$$5. \int \frac{(x+a)^p}{(x+b)^{p+2}} dx = \frac{1}{(p+1)(b-a)} \left(\frac{x+a}{x+b} \right)^{p+1}.$$

$$6. \int \frac{(x+a)^p}{(x+b)^{p+2}} dx = \frac{(x+a)^{p+1} [x+(p+1)(b-a)+b]}{(p+1)(p+2)(b-a)^2(x+b)^{p+2}}.$$

$$7. \int \frac{1}{(x+c)^p} \left(\frac{x+a}{x+b} \right)^q dx = \frac{-1}{(p-1)(a-c)(b-c)} \left\{ \frac{1}{(x+c)^{p-1}} \cdot \frac{(x+a)^{q+1}}{(x+b)^{q-1}} + \right. \\ \left. + [(p-2)(a+b-2c)-q(b-a)] \int \frac{1}{(x+c)^{p-1}} \left(\frac{x+a}{x+b} \right)^q dx + \right. \\ \left. + (p-3) \int \frac{1}{(x+c)^{p-2}} \left(\frac{x+a}{x+b} \right)^q dx \right\}.$$

$$8. \int \frac{1}{x+c} \left(\frac{x+a}{x+b} \right)^n dx = \sum_{k=1}^{n-1} \frac{1}{n-k} \left[\left(\frac{a-c}{b-c} \right)^k - 1 \right] \frac{(x+a)^{n-k}}{(x+b)^{n-k}} - \\ - \left[\left(\frac{a-c}{b-c} \right)^n - 1 \right] \ln |x+b| + \left(\frac{a-c}{b-c} \right)^n \ln |x+c|.$$

$$9. \int \left(\frac{x+a}{x+b} \right)^n dx = \\ = - \sum_{k=0}^{n-2} \frac{n}{(n-k)(n-k-1)} \frac{(x+a)^{n-k}}{(x+b)^{n-k-1}} + n(x+a) - n(b-a) \ln |x+b|.$$

$$10. = - \sum_{k=1}^{n-1} \binom{n}{k+1} \frac{(a-b)^{k+1}}{k(x+b)^k} + x + n(a-b) \ln |x+b|.$$

$$11. \int \frac{x+a}{x+b} dx = x + (a-b) \ln |x+b|.$$

$$12. \int x \frac{x+a}{x+b} dx = \frac{x^2}{2} + (a-b) (x - b \ln |x+b|).$$

$$13. \int x^2 \frac{x+a}{x+b} dx = \frac{x^3}{3} + (a-b) \left(\frac{x^2}{2} - bx + b^2 \ln |x+b| \right).$$

$$14. \int \frac{x+a}{x(x+b)} dx = \frac{a}{b} \ln |x| - \frac{a-b}{b} \ln |x+b|.$$

$$15. \int \left(\frac{x+a}{x+b} \right)^2 dx = x - \frac{(a-b)^2}{x+b} + 2(a-b) \ln |x+b|.$$

$$16. \int x \left(\frac{x+a}{x+b} \right)^2 dx = \frac{x^2}{2} + 2(a-b)x + (a-b) \left[\frac{b(a-b)}{x+b} + (a-3b) \ln |x+b| \right].$$

$$17. \int x^2 \left(\frac{x+a}{x+b} \right)^2 dx = \frac{x^3}{3} + (a-b)x^2 + \frac{b(a-b)^2 x}{x+b} + (a-b) [(a-3b)(x+b) - 2b(a-2b) \ln |x+b|].$$

$$18. \int \frac{1}{x} \left(\frac{x+a}{x+b} \right)^2 dx = \frac{(a-b)^2}{b(x+b)} + \frac{a^2}{b^2} \ln |x| - \left(\frac{a^2}{b^2} - 1 \right) \ln |x+b|.$$

$$19. \int \frac{(x+a)^2}{x+b} dx = \frac{x^2}{2} + (2a-b)x + (a-b)^2 \ln |x+b|.$$

$$20. \int x \frac{(x+a)^2}{x+b} dx = \frac{x^3}{3} + (2a-b) \frac{x^2}{2} + (a-b)^2 (x - b \ln |x+b|).$$

$$21. \int x^2 \frac{(x+a)^2}{x+b} dx = \frac{x^4}{4} + (2a-b) \frac{x^3}{3} + (a-b)^2 \left(\frac{x^2}{2} - bx + b^2 \ln |x+b| \right).$$

$$22. \int \frac{(x+a)^2}{x(x+b)} dx = x + \frac{1}{b} (a^2 \ln |x| - (a-b)^2 \ln |x+b|).$$

$$23. \int \frac{x+a}{(x+b)^2} dx = -\frac{a-b}{x+b} + \ln |x+b|.$$

$$24. \int x \frac{x+a}{(x+b)^2} dx = x + \frac{b(a-b)}{x+b} + (a-2b) \ln |x+b|.$$

$$25. \int x^2 \frac{x+a}{(x+b)^2} dx = \frac{x^2}{2} + (a-2b)x - \frac{b^2(a-b)}{x+b} - b(2a-3b) \ln |x+b|.$$

$$26. \int \frac{x+a}{x(x+b)^2} dx = \frac{a-b}{b(x+b)} - \frac{a}{b^2} \ln \left| \frac{x+b}{x} \right|.$$

1.2.7. Интегралы вида $\int \frac{x^p dx}{(x+a)^q (x+b)^r}.$

$$1. \int \frac{x^p dx}{(x+a)^q (x+b)^r} = \frac{1}{p-q-r+1} \left\{ \frac{x^{p-1}}{(x+a)^q (x+b)^{r-1}} - [(p-r)a + (p-q)b] \int \frac{x^{p-1} dx}{(x+a)^q (x+b)^r} - (p-1)ab \int \frac{x^{p-2} dx}{(x+a)^q (x+b)^r} \right\}.$$

- $$\begin{aligned}
 2. \int \frac{x^k dx}{(x+a)^m (x+b)^n} = & - \sum_{p=1}^{m-1} \frac{(-1)^{m+k+p+1}}{p (x+a)^p} \left[\sum_{q=0}^{m-p-1} \binom{k}{q} \binom{m+n-p-q-2}{n-1} \frac{a^{k-q}}{(b-a)^{m+n-p-q-1}} \right] - \\
 & - \sum_{p=1}^{n-1} \frac{(-1)^m}{p (x+b)^p} \left[\sum_{q=0}^{n-p-1} \binom{k}{q} \binom{m+n-p-q-2}{m-1} \frac{(-b)^{k-q}}{(b-a)^{m+n-p-q-1}} \right] + \\
 & + (-1)^{m+k+1} \sum_{p=0}^{m-1} \binom{k}{p} \binom{m+n-p-2}{n-1} \frac{a^{k-p}}{(b-a)^{m+n-p-1}} \ln |x+a| + \\
 & + (-1)^m \sum_{p=0}^{n-1} \binom{k}{p} \binom{m+n-p-2}{m-1} \frac{(-b)^{k-p}}{(b-a)^{m+n-p-1}} \ln |x+b|. \\
 3. \int \frac{dx}{(x+a)^q (x+b)^r} = & - \frac{1}{(r-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} - \frac{(q+r-2)}{(r-1)(a-b)} \int \frac{dx}{(x+a)^q (x+b)^{r-1}}. \\
 4. = & \frac{1}{(q-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} + \\
 & + \frac{q+r-2}{(q-1)(a-b)} \int \frac{dx}{(x+a)^{q-1}(x+b)^r}. \\
 5. \int \frac{dx}{(x+a)^m (x+b)^n} = & = \frac{(-1)^{m+n-2}}{(b-a)^{m+n-k}} \sum_{\substack{k=0 \\ k \neq n-1}}^{m+n-2} \frac{(-1)^k}{n-k-1} \binom{m+n-2}{k} \left(\frac{x+a}{x+b} \right)^{n-k-1} + \\
 & + \binom{m+n-2}{n-1} \frac{(-1)^{m-1}}{(b-a)^{m+n-1}} \ln \left| \frac{x+a}{x+b} \right|. \\
 6. = & \sum_{k=1}^{m-1} \frac{(-1)^{m+k}}{k} \binom{m+n-k-2}{n-1} \frac{1}{(b-a)^{m+n-k-1}(x+a)^k} - \\
 & - (-1)^m \sum_{k=1}^{n-1} \frac{1}{k} \binom{m+n-k-2}{m-1} \frac{1}{(b-a)^{m+n-k-1}(x+b)^k} + \\
 & + \frac{(-1)^{m+1}}{(b-a)^{m+n-1}} \binom{m+n-1}{n-1} \ln \left| \frac{x+a}{x+b} \right|. \\
 7. \int \frac{dx}{(x+a)(x+b)} = & - \frac{1}{a-b} \ln \left| \frac{x+a}{x+b} \right|. \\
 8. \int \frac{x dx}{(x+a)(x+b)} = & \frac{1}{a-b} (a \ln |x+a| - b \ln |x+b|). \\
 9. \int \frac{x^2 dx}{(x+a)(x+b)} = & x + \frac{1}{a-b} (b^2 \ln |x+b| - a^2 \ln |x+a|).
 \end{aligned}$$

- $$10. \int \frac{dx}{x(x+a)(x+b)} = \frac{1}{ab} \ln |x| + \frac{1}{a(a-b)} \ln |x+a| - \frac{1}{b(a-b)} \ln |x+b|.$$
- $$11. \int \frac{dx}{x^2(x+a)(x+b)} = -\frac{1}{abx} - \frac{a+b}{a^2b^2} \ln |x| -$$
- $$-\frac{1}{a^2(a-b)} \ln |x+a| + \frac{1}{b^2(a-b)} \ln |x+b|.$$
- $$12. \int \frac{dx}{(x+a)(x+b)^2} = \frac{-1}{(a-b)(x+b)} + \frac{1}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$
- $$13. \int \frac{xdx}{(x+a)(x+b)^2} = \frac{b}{(a-b)(x+b)} - \frac{a}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$
- $$14. \int \frac{x^2 dx}{(x+a)(x+b)^2} = \frac{-b^2}{(a-b)(x+b)} + \frac{a^2}{(a-b)^2} \ln |x+a| + \frac{b^2-2ab}{(a-b)^2} \ln |x+b|.$$
- $$15. \int \frac{dx}{x(x+a)(x+b)^2} = \frac{1}{b(a-b)(x+b)} + \frac{1}{ab^2} \ln |x| -$$
- $$-\frac{1}{a(a-b)^2} \ln |x+a| + \frac{2b-a}{b^2(a-b)} \ln |x+b|.$$
- $$16. \int \frac{dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2} \left(\frac{1}{x+a} + \frac{1}{x+b} \right) + \frac{2}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$
- $$17. \int \frac{xdx}{(x+a)^2(x+b)^2} = \frac{1}{(a-b)^2} \left(\frac{a}{x+a} + \frac{b}{x+b} \right) - \frac{a+b}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$
- $$18. \int \frac{x^2 dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2} \left(\frac{a^2}{x+a} + \frac{b^2}{x+b} \right) + \frac{2ab}{(a-b)^2} \ln \left| \frac{x+a}{x+b} \right|.$$

1.2.8. Интегралы вида $\int R(x, ax^2+bx+c) dx$.

- $$1. \int (ax^2+bx+c)^n dx = \frac{1}{2(2n+1)a} [(2ax+b)(ax^2+bx+c)^n -$$
- $$-n(b^2-4ac) \int (ax^2+bx+c)^{n-1} dx].$$
- $$2. = (-1)^n \frac{(n!)^2}{(2n+1)!} \frac{2ax+b}{2a} \sum_{k=0}^n (-1)^k \binom{2k}{k} \left(\frac{b^2-4ac}{a} \right)^{n-k} (ax^2+bx+c)^k.$$
- $$3. \int \frac{(ax^2+bx+c)^n dx}{x^m} = -\frac{(ax^2+bx+c)^{n+1}}{(m-1)cx^{m-1}} +$$
- $$+\frac{b(n-m+2)}{c(m-1)} \int \frac{(ax^2+bx+c)^n dx}{x^{m-1}} + \frac{a(2n-m+3)}{c(m-1)} \int \frac{(ax^2+bx+c)^n}{x^{m-2}} dx.$$
- $$4. \int \frac{x^m dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} +$$
- $$+\frac{(m-1)c}{(2n-m-1)a} \int \frac{x^{m-2} dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a} \int \frac{x^{m-1} dx}{(ax^2+bx+c)^n}.$$
- $$5. \int \frac{x^{2n-1} dx}{(ax^2+bx+c)^n} = \frac{1}{a} \int \frac{x^{2n-3} dx}{(ax^2+bx+c)^{n-1}} -$$
- $$-\frac{c}{a} \int \frac{x^{2n-3} dx}{(ax^2+bx+c)^n} - \frac{b}{a} \int \frac{x^{2n-2} dx}{(ax^2+bx+c)^n}.$$

$$6. \int \frac{x dx}{(ax^2 + bx + c)^n} = -\frac{1}{2(n-1)a(ax^2 + bx + c)^{n-1}} - \frac{b}{2a} \int \frac{dx}{(ax^2 + bx + c)^n}.$$

$$7. = \frac{bx + 2c}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} + \frac{(2n-3)b}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$8. \int \frac{x^3 dx}{(ax^2 + bx + c)^n} = -\frac{(b^2 - 2ac)(2ax + b) - b(b^2 - 4ac)}{2(n-1)a^2(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} - \frac{2(2n-3)(b^2 - 2ac) - 2(n-1)(b^2 - 4ac)}{2(n-1)a(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$9. \int \frac{dx}{x^m(ax^2 + bx + c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2 + bx + c)^{n-1}} - \frac{(m+n-2)b}{(m-1)c} \int \frac{dx}{x^{m-1}(ax^2 + bx + c)^n} - \frac{(m+2n-3)a}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^2 + bx + c)^n}.$$

$$10. \int \frac{dx}{x(ax^2 + bx + c)^n} = \frac{1}{2(n-1)c(ax^2 + bx + c)^{n-1}} - \frac{b}{2c} \int \frac{dx}{(ax^2 + bx + c)^n} + \frac{1}{c} \int \frac{dx}{x(ax^2 + bx + c)^{n-1}}.$$

$$11. \int \frac{dx}{(ax^2 + bx + c)^n} = \frac{-2ax - b}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} - \frac{2(2n-3)a}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}.$$

$$12. = \frac{2ax + b}{2n-1} \times \\ \times \sum_{k=0}^{n-2} \frac{2^k(2n-1)(2n-3)\dots(2n-2k-1)a^k}{(n-1)(n-2)\dots(n-k-1)(4ac-b^2)^{k+1}(ax^2 + bx + c)^{n-k-1}} + \\ + \frac{(2n-3)!!(2a)^{n-1}}{(n-1)!(4ac-b^2)^{n-1}} \int \frac{dx}{ax^2 + bx + c}.$$

$$13. \int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| \quad [b^2 - 4ac > 0].$$

$$14. = \frac{2}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2ax + b}{\sqrt{4ac - b^2}} \quad [b^2 - 4ac < 0].$$

$$15. = -\frac{2}{2ax + b} \quad [b^2 = 4ac].$$

$$16. = \frac{1}{a(p-q)} \ln \left| \frac{x-p}{x-q} \right|,$$

где p, q — действительные корни многочлена $ax^2 + bx + c$.

$$17. \int \frac{dx}{(ax^2 + bx + c)^2} = \frac{2ax + b}{(4ac - b^2)(ax^2 + bx + c)} + \frac{2a}{4ac - b^2} \int \frac{dx}{ax^2 + bx + c}.$$

$$18. \int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{2(4ac-b^2)(ax^2+bx+c)^2} + \frac{3a(2ax+b)}{(4ac-b^2)^2(ax^2+bx+c)} + \frac{6a^3}{(4ac-b^2)^2} \int \frac{dx}{ax^2+bx+c}.$$

$$19. \int \frac{x dx}{ax^2+bx+c} = \frac{1}{2a} \ln |ax^2+bx+c| - \frac{b}{a\sqrt{|b^2-4ac|}} \begin{cases} \frac{1}{2} \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| & [b^2 > 4ac], \\ \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}} & [4ac > b^2] \end{cases}$$

$$20. \int \frac{x^2 dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2} \ln |ax^2+bx+c| + \frac{b^2-2ac}{2a^2} \int \frac{dx}{ax^2+bx+c}.$$

$$21. \int \frac{x^3 dx}{ax^2+bx+c} = \frac{ax^2-2bx}{2a^2} + \frac{b^2-ac}{2a^2} \ln |ax^2+bx+c| - \frac{b(b^2-3ac)}{2a^3} \int \frac{dx}{ax^2+bx+c}.$$

$$22. \int \frac{x dx}{(ax^2+bx+c)^2} = \frac{bx+2c}{(b^2-4ac)(ax^2+bx+c)} + \frac{b}{b^2-4ac} \int \frac{dx}{ax^2+bx+c}.$$

$$23. \int \frac{x^2 dx}{(ax^2+bx+c)^2} = \frac{(2ac-b^2)x-bc}{a(b^2-4ac)(ax^2+bx+c)} - \frac{2c}{b^2-4ac} \int \frac{dx}{ax^2+bx+c}.$$

$$24. \int \frac{x^3 dx}{(ax^2+bx+c)^2} = \frac{c(2ac-b^2)+b(3ac-b^2)x}{a^2(4ac-b^2)} + \frac{1}{2a^2} \ln |ax^2+bx+c| + \frac{b(6ac-b^2)}{2a^2(b^2-4ac)} \int \frac{dx}{ax^2+bx+c}.$$

$$25. \int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c} \ln \frac{x^2}{|ax^2+bx+c|} - \frac{b}{c\sqrt{|b^2-4ac|}} \begin{cases} \frac{1}{2} \ln \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} & [b^2 > 4ac], \\ \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}} & [b^2 < 4ac]. \end{cases}$$

$$26. \int \frac{dx}{x^2(ax^2+bx+c)} = -\frac{1}{cx} - \frac{b}{2c^2} \ln \frac{x^2}{|ax^2+bx+c|} + \frac{b^2-2ac}{2c^2} \int \frac{dx}{ax^2+bx+c}.$$

$$27. \int \frac{dx}{x^4(ax^2+bx+c)} = \frac{2bx-c}{2c^4x^3} + \frac{b^2-ac}{2c^3} \ln \frac{x^2}{|ax^2+bx+c|} + \frac{b(3ac-b^2)}{2c^3} \int \frac{dx}{ax^2+bx+c}.$$

$$28. \int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} \left[1 + \frac{b(b+2ax)}{b^2-4ac} \right] + \frac{1}{2c^2} \ln \frac{x^2}{|ax^2+bx+c|} - \frac{b}{2c^2} \left(1 - \frac{2ac}{b^2-4ac} \right) \int \frac{dx}{ax^2+bx+c}.$$

$$29. \int \frac{dx}{x^3 (ax^3 + bx + c)^2} = \frac{(b^2 - 3ac)(b + 2ax)}{c^2 (4ac - b^2)(ax^3 + bx + c)} - \frac{c + bx}{c^2 x (ax^3 + bx + c)} - \\ - \frac{b}{c^3} \ln \left| \frac{x^2}{ax^3 + bx + c} \right| + \frac{1}{b^2 - 4ac} \left(\frac{b^4}{c^3} - \frac{6b^2 a}{c^2} + \frac{6a^2}{c} \right) \int \frac{dx}{ax^3 + bx + c}.$$

$$30. \int \frac{dx}{x^3 (ax^3 + bx + c)^2} = \\ = \frac{3bx - c}{2c^2 x^2 (ax^3 + bx + c)} + \frac{3b^2 - 2ac}{c^2} \int \frac{dx}{x (ax^3 + bx + c)^2} + \frac{9ab}{2c^2} \int \frac{dx}{(ax^3 + bx + c)^2}.$$

1.2.9. Интегралы вида $\int R(x+d, ax^3+bx+c) dx$.

$$1. \int (x+d)^m (ax^3+bx+c)^n dx = \frac{1}{(m+2n+1)a} \left[(x+d)^{m+1} (ax^3+bx+c)^{n+1} + \right. \\ \left. + (m+n)(2ad-b) \int (x+d)^{m-1} (ax^3+bx+c)^n dx - \right. \\ \left. - (m-1)(ad^2-bd+c) \int (x+d)^{m-2} (ax^3+bx+c)^n dx \right].$$

$$2. \int \frac{(ax^3+bx+c)^n}{(x+d)^m} dx = - \frac{(ax^3+bx+c)^n}{(m-2n-1)(x+d)^{m-1}} - \\ - \frac{2n(ad^2-bd+c)}{m-2n-1} \int \frac{(ax^3+bx+c)^{n-1}}{(x+d)^m} dx - \frac{n(b-2ad)}{m-2n-1} \int \frac{(ax^3+bx+c)^{n-1}}{(x+d)^{m-1}} dx.$$

$$3. = - \frac{1}{(m-1)(ad^2-bd+c)} \left[\frac{(ax^3+bx+c)^{n+1}}{(x+d)^{m-1}} + \right. \\ \left. + (m-n-2)(b-2ad) \int \frac{(ax^3+bx+c)^n}{(x+d)^{m-1}} dx + (m-2n-3)a \int \frac{(ax^3+bx+c)^n}{(x+d)^{m-2}} dx \right].$$

$$4. = - \frac{(ax^3+bx+c)^n}{(m-1)(x+d)^{m-1}} + \\ + \frac{n(b-2ad)}{m-1} \int \frac{(ax^3+bx+c)^{n-1}}{(x+d)^{m-1}} dx + 2na \int \frac{(ax^3+bx+c)^{n-1}}{(x+d)^{m-2}} dx.$$

$$5. \int \frac{(ax^3+bx+c)^n}{x+d} dx = (ad^2-bd+c)^n \ln|x+d| + \\ + \int (ax+b-ad) \left[\sum_{k=0}^{n-1} (ad^2-bd+c)^{n-k-1} (ax^3+bx+c)^k \right] dx.$$

$$6. \int \frac{(x+d)^m dx}{(ax^3+bx+c)^n} = \frac{(2ad-b)(2ax+b) + (4ac-b^2)}{2(n-1)(4ac-b^2)(ad^2-bd+c)} \frac{(x+d)^{m+1}}{(ax^3+bx+c)^{n-1}} + \\ + \frac{2(m-2n+4)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)} \frac{(x+d)^{m+1}}{(ax^3+bx+c)^{n-2}} + \\ + \left[\frac{4(2n-3)a}{(n-1)(4ac-b^2)} - \frac{2(m-n+2)}{(n-1)(ad^2-bd+c)} \right] \int \frac{(x+d)^m dx}{(ax^3+bx+c)^{n-1}} - \\ - \frac{2(m-2n+4)(m-2n+5)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)} \int \frac{(x+d)^m dx}{(ax^3+bx+c)^{n-2}}.$$

$$7. = \frac{1}{(m-2n+1)a} \left[\frac{(x+d)^{m-1}}{(ax^3+bx+c)^{n-1}} + \right. \\ \left. + (m-n)(2ad-b) \int \frac{(x+d)^{m-2} dx}{(ax^3+bx+c)^n} - (m-1)(ad^2-bd+c) \int \frac{(x+d)^{m-2} dx}{(ax^3+bx+c)^n} \right].$$

$$8. = \frac{1}{(n-1)(4ac-b^2)} \left[\frac{(2ax+b)(x+d)^m}{(ax^2+bx+c)^{n-1}} - \right. \\ \left. - 2(m-2n+3)a \int \frac{(x+d)^m dx}{(ax^2+bx+c)^{n-1}} - (b-2ad)m \int \frac{(x+d)^{m-1} dx}{(ax^2+bx+c)^{n-1}} \right].$$

$$9. \int \frac{dx}{(x+d)^m (ax^2+bx+c)^n} = \frac{-1}{(m-1)(ad^2-bd+c)} \times \\ \times \left[\frac{1}{(x+d)^{m-1} (ax^2+bx+c)^n} + 2(m+n-2)(b-2ad) \times \right. \\ \left. \times \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} + (m+2n-3)a \int \frac{dx}{(x+d)^{m-2} (ax^2+bx+c)^n} \right].$$

$$10. = \frac{1}{2(ad^2-bd+c)} \left[\frac{1}{(n-1)(x+d)^{m-1} (ax^2+bx+c)^{n-1}} + 2(ad-b) \times \right. \\ \left. \times \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} + \frac{m+2n-3}{n-1} \int \frac{dx}{(x+d)^m (ax^2+bx+c)^{n-1}} \right].$$

$$11. = \frac{1}{2(m+n-1)(ad-b)} \left[\frac{1}{(x+d)^m (ax^2+bx+c)^{n-1}} + \right. \\ \left. + (m+2n-2)a \int \frac{dx}{(x+d)^{m-1} (ax^2+bx+c)^n} \right] \quad [ad^2-bd+c=0].$$

$$12. \int \frac{dx}{(x+d)(ax^2+bx+c)^n} = \frac{1}{ad^2-bd+c} \left[\frac{1}{2(n-1)(ax^2+bx+c)^{n-1}} + \right. \\ \left. + (ad-b) \int \frac{dx}{(ax^2+bx+c)^n} + \int \frac{dx}{(x+d)(ax^2+bx+c)^{n-1}} \right].$$

$$13. \int \frac{dx}{(x+d)(ax^2+bx+c)} = \frac{1}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \\ + \frac{2ad-b}{2(ad^2-bd+c)\sqrt{b^2-4ac}} \begin{cases} \operatorname{arctg} \frac{2ax+b}{\sqrt{b^2-4ac}} & [b^2-4ac < 0], \\ \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| & [b^2-4ac > 0]. \end{cases}$$

$$14. \int \frac{dx}{(x+d)^2 (ax^2+bx+c)} = \\ = -\frac{1}{ad^2-bd+c} \left[\frac{1}{x+d} + \frac{2ad-b}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \right. \\ \left. + \frac{2c(ad^2-bd+c)-(2ad-b^2)}{2(ad^2-bd+c)^2} \int \frac{dx}{ax^2+bx+c} \right].$$

1.2.10. Интегралы вида $\int \frac{x^m dx}{(x^2 \pm a^2)^n}$.

$$1. \int \frac{x^m dx}{(x^2 \pm a^2)^n} = \pm \frac{x^{m+1}}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \mp \frac{(m-2n+3)}{2(n-1)a^2} \int \frac{x^m dx}{(x^2 \pm a^2)^{n-1}}.$$

$$2. = \frac{x^{m-1}}{(m-2n+1)(x^2 \pm a^2)^{n-1}} \mp \frac{(m-1)a^2}{m-2n+1} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^n}.$$

$$3. = \frac{-x^{m-1}}{2(n-1)(x^2 \pm a^2)^{n-1}} + \frac{m-1}{2(n-1)} \int \frac{x^{m-2}}{(x^2 \pm a^2)^{n-1}}.$$

$$4. = \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n-1}} \mp a^2 \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^n}.$$

5.
$$\int \frac{x^{2m} dx}{(x^2 + a^2)^n} = \frac{x^{2m+2}}{2(n-1)a^2} \left[\frac{1}{(x^2 + a^2)^{n-1}} + \right. \\ \left. + \sum_{k=1}^{n-2} \frac{(2n-2m-3)(2n-2m-5) \dots (2n-2m-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \frac{1}{a^{2k} (x^2 + a^2)^{n-k-1}} \right] + \\ + (-1)^m a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5) \dots (3-2m)(1-2m)}{2^{n-1} (n-1)!} \times \\ \times \left[\sum_{k=1}^m \frac{(-1)^k}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + \operatorname{arctg} \frac{x}{a} \right].$$
6.
$$\int \frac{x^{2m} dx}{(x^2 - a^2)^n} = -\frac{x^{2m+2}}{2(n-1)a^2} \left[\frac{1}{(x^2 - a^2)^{n-1}} - \right. \\ \left. - \sum_{k=1}^{n-2} (-1)^k \frac{(2n-2m-3)(2n-2m-5) \dots (2n-2m-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \times \right. \\ \left. \times \frac{1}{a^{2k} (x^2 - a^2)^{n-k-1}} \right] + \\ + (-1)^{n+1} a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5) \dots (3-2m)(1-2m)}{2^{n-1} (n-1)!} \times \\ \times \left[\sum_{k=1}^m \frac{1}{2k-1} \left(\frac{x}{a} \right)^{2k-1} - \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right].$$
7.
$$\int \frac{x^{2m+1} dx}{(x^2 \pm a^2)^n} = - \sum_{k=0}^m \frac{(-1)^k}{2} \binom{m}{k} \frac{a^{2k}}{(n-m+k-1)(x^2 \pm a^2)^{n-m+k-1}}.$$
8.
$$= -\frac{1}{2(\pm a^2)^{n-m-1}} \sum_{k=0}^{n-m-2} (-1)^k \binom{n-m-2}{k} \frac{1}{m+k+1} \times \\ \times \left(\frac{x^2}{x^2 \pm a^2} \right)^{m+k+1} \quad [n \geq m+2].$$
9.
$$\int \frac{x^{2m} dx}{x^2 \pm a^2} = \sum_{k=0}^{m-1} \frac{(-1)^k}{2} \frac{a^{2k} x^{2m-2k-1}}{2m-2k-1} + \frac{(-1)^m}{2} a^{2m-1} \left\{ \operatorname{arctg} \frac{x}{a}, \right. \\ \left. \left. \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right\}.$$
10.
$$\int \frac{x^{2m+1} dx}{x^2 \pm a^2} = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(-1)^k}{m-k} \frac{a^{2k} x^{2m-2k}}{m-k} + \frac{1}{2} \frac{(-1)^m}{(\pm a^2)^m} \ln |x^2 \pm a^2|.$$
11.
$$\int \frac{x^{2m} dx}{(x^2 \pm a^2)^n} = \pm \frac{x^{2m+2}}{2a^2 (x^2 \pm a^2)} + \frac{1}{2} (1-2m) a^{2m-2} \times \\ \times \left[\pm \sum_{k=1}^m \frac{(-1)^{m+k}}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + (-1)^m \left\{ \operatorname{arctg} \frac{x}{a}, \right. \right. \\ \left. \left. \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right\} \right].$$

$$12. \int \frac{dx}{(x^2 \pm a^2)^n} = \pm \frac{x}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1}}.$$

$$13. \int \frac{dx}{(x^2 \pm a^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} (\pm 1)^k \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)a^{2k} (x^2 \pm a^2)^{n-k}} +$$

$$+ (\pm 1)^n \frac{(2n-3)!!}{2^{n-1}(n-1)!a^{2n-1}} \left\{ \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right| \right\}.$$

$$14. \int \frac{dx}{x^2 \pm a^2} = \frac{1}{a} \left\{ \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right| \right\}.$$

$$15. \int \frac{dx}{(x^2 \pm a^2)^2} = \frac{x}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^3} \left\{ \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right| \right\}.$$

$$16. \int \frac{dx}{(x^2 \pm a^2)^3} = \pm \frac{x}{4a^2(x^2 \pm a^2)^2} + \frac{3x}{8a^4(x^2 \pm a^2)} + \frac{3}{8a^5} \left\{ \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right| \right\}.$$

$$17. \int \frac{x dx}{x^2 \pm a^2} = \frac{1}{2} \ln |x^2 \pm a^2|.$$

$$18. \int \frac{x^2 dx}{x^2 \pm a^2} = x - a \left\{ \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right| \right\}.$$

$$19. \int \frac{x^3 dx}{x^2 \pm a^2} = \frac{x^3}{2} \mp \frac{a^2}{2} \ln |x^2 \pm a^2|.$$

$$20. \int \frac{x^4 dx}{x^2 \pm a^2} = \frac{x^3}{3} \mp a^2 x \pm \left\{ \frac{a^2 \operatorname{arctg} \frac{x}{a}}{2} \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

$$21. \int \frac{x dx}{(x^2 \pm a^2)^2} = -\frac{1}{2(x^2 \pm a^2)}.$$

$$22. \int \frac{x^2 dx}{(x^2 \pm a^2)^2} = -\frac{x}{2(x^2 \pm a^2)} + \left\{ \frac{1}{2a} \operatorname{arctg} \frac{x}{a}, \left| \frac{1}{4a} \ln \left| \frac{x-a}{x+a} \right| \right| \right\}.$$

$$23. \int \frac{x^3 dx}{(x^2 \pm a^2)^2} = \pm \frac{a^2}{2(x^2 \pm a^2)} + \frac{1}{2} \ln |x^2 \pm a^2|.$$

$$24. \int \frac{x^4 dx}{(x^2 \pm a^2)^3} = x \pm \frac{a^2 x}{2(x^2 \pm a^2)} - \left\{ \frac{3a}{2} \operatorname{arctg} \frac{x}{a} \right. \\ \left. - \frac{3x}{4} \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

1.2.11. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^n}$.

$$1. \int \frac{dx}{x^{2m} (x^2 \pm a^2)^n} = \mp \frac{1}{(2m-1) a^2 x^{2m-1} (x^2 \pm a^2)^{n-1}} \mp \\ \mp \frac{2m+2n-3}{(2m-1) a^2} \int \frac{dx}{x^{2m-2} (x^2 \pm a^2)^n}.$$

$$2. = \frac{\pm 1}{(2m+2n-1) (x^2 \pm a^2)^{n-1}} \times$$

$$\times \sum_{k=1}^m \frac{(\mp 1)^k (2m+2n-1) (2m-2n-3) \dots (2m+2n-2k+1)}{(2m-1) (2m-3) \dots (2m-2k+1) a^{2k}} \frac{1}{x^{2m-2k+1}} + \\ + (\mp 1)^m \frac{(2n-1) (2n+1) \dots (2n+2m-3)}{(2m-1)!! a^{2m}} \int \frac{dx}{(x^2 \pm a^2)^n} \quad [\text{см. 1.2.10}].$$

$$3. \int \frac{dx}{x^{2m+1} (x^2 \pm a^2)^n} = \mp \frac{1}{2ma^2 x^{2n} (x^2 \pm a^2)^{n-1}} \mp \\ \mp \frac{m+n-1}{ma^2} \int \frac{dx}{x^{2m-1} (x^2 \pm a^2)^n}.$$

$$4. = \frac{1}{2(\mp a^2)^{m+n}} \sum_{k=0}^{m+n-1} (-1)^k \binom{m+n-1}{k} \frac{1}{k-n+1} \left(\frac{x^2 \pm a^2}{x^2} \right)^{k-n+1}.$$

$$5. \int \frac{dx}{x (x^2 \pm a^2)^n} = \frac{\pm 1}{2(n-1) a^2 (x^2 \pm a^2)^{n-1}} \pm \frac{1}{a^2} \int \frac{dx}{x (x^2 \pm a^2)^{n-1}}.$$

$$6. = \frac{1}{2} \sum_{k=1}^{n-1} \frac{(\pm 1)^k}{(n-k) a^{2k} (x^2 \pm a^2)^{n-k}} + \frac{(\pm 1)^n}{2a^{2n}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$7. \int \frac{dx}{x^{2m} (x^2 \pm a^2)} = \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k}}{(2k+1) a^{2(m-k)} x^{2k+1}} + \frac{(\mp 1)^m}{a^{2m+1}} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. - \frac{1}{2} \ln \left| \frac{x-a}{x+a} \right| \right\}.$$

$$8. \int \frac{dx}{x^{2m+1} (x^2 \pm a^2)} = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(\mp 1)^{k+1}}{(m-k) a^{2(k+1)} x^{2(m-k)}} + \frac{(\mp 1)^{m+1}}{2a^{2(m+1)}} \ln \frac{x^2}{|x^2 \pm a^2|}.$$

$$9. \int \frac{dx}{x^{2m} (x^2 \pm a^2)^2} = \mp \sum_{k=0}^{m-1} \frac{(\mp 1)^{m-k} (m-k)}{(2k+1) a^{2m-2k+2} x^{2k+1}} - \\ - \frac{(\mp 1)^{m+1} x}{2a^{2m+2} (x^2 \pm a^2)} + \frac{(\mp 1)^m (2m+1)}{2a^{2m+3}} \left\{ \operatorname{arctg} \frac{x}{a} \right. \\ \left. - \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

$$10. \int \frac{dx}{x(x^2 \pm a^2)} = \pm \frac{1}{2a^2} \ln \left| \frac{x^2}{x^2 \pm a^2} \right|.$$

$$11. \int \frac{dx}{x^2(x^2 \pm a^2)} = \mp \frac{1}{a^2 x} \mp \frac{1}{a^2} \left\{ \frac{\operatorname{arctg} \frac{x}{a}}{\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right|} \right\}.$$

$$12. \int \frac{dx}{x^3(x^2 \pm a^2)} = \mp \frac{1}{2a^2 x^2} - \frac{1}{2a^4} \ln \left| \frac{x^2}{x^2 \pm a^2} \right|.$$

$$13. \int \frac{dx}{x^4(x^2 \pm a^2)} = \mp \frac{1}{3a^2 x^3} + \frac{1}{a^4 x} + \frac{1}{a^6} \left\{ \frac{\operatorname{arctg} \frac{x}{a}}{\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right|} \right\}.$$

$$14. \int \frac{dx}{x(x^2 \pm a^2)^2} = \frac{1}{2a^2(a^2 \pm x^2)} + \frac{1}{2a^4} \ln \left| \frac{x^2}{x^2 \pm a^2} \right|.$$

$$15. \int \frac{dx}{x^2(x^2 \pm a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2 \pm a^2)} - \frac{3}{2a^6} \left\{ \frac{\operatorname{arctg} \frac{x}{a}}{\frac{1}{2} \ln \left| \frac{x-a}{x+a} \right|} \right\}.$$

$$16. \int \frac{dx}{x^3(x^2 \pm a^2)^2} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^4(x^2 \pm a^2)} \mp \frac{1}{a^6} \ln \left| \frac{x^2}{x^2 \pm a^2} \right|.$$

$$17. \int \frac{dx}{x^4(x^2 \pm a^2)^2} = -\frac{1}{3a^4 x^3} \pm \frac{2}{a^6 x} \pm \frac{x}{2a^6(x^2 \pm a^2)} + \frac{5}{2a^8} \left\{ \frac{\operatorname{arctg} \frac{x}{a}}{\frac{1}{2} \ln \left| \frac{x+a}{x-a} \right|} \right\}.$$

$$18. \int \frac{dx}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2} \left[\ln |x+b| - \frac{1}{2} \ln (x^2+a^2) + \frac{b}{a} \operatorname{arctg} \frac{x}{a} \right].$$

$$19. \int \frac{dx}{(x+b)(x^2-a^2)} = \\ = -\frac{1}{a^2-b^2} \ln |x+b| + \frac{1}{2a(a+b)} \ln |x-a| + \frac{1}{2a(a-b)} \ln |x+a|.$$

1.2.12. Интегралы вида $\int \frac{x^{\pm m} dx}{(x^3+a^3)^n}$.

$$1. \int \frac{x^m dx}{(x^3+a^3)^n} = \frac{x^{m+1}}{3(n-1)a^3(x^3+a^3)^{n-1}} - \frac{m-3n+4}{3(n-1)a^3} \int \frac{x^m dx}{(x^3+a^3)^{n-1}}.$$

$$2. = \frac{x^{m-2}}{(m-3n+1)(x^3+a^3)^{n-1}} - \frac{(m-2)a^3}{m-3n+1} \int \frac{x^{m-3} dx}{(x^3+a^3)^n}.$$

$$3. \int \frac{x^m dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3 \int \frac{x^{m-3} dx}{x^3+a^3}.$$

$$4. \int \frac{dx}{(x^3+a^3)^n} = \frac{x}{3a^2(n-1)(x^3+a^3)^{n-1}} + \frac{3n-4}{3a^3(n-1)} \int \frac{dx}{(x^3+a^3)^{n-1}}.$$

$$5. \int \frac{x dx}{(x^3+a^3)^n} = \frac{x^2}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-5}{3a^3(n-1)} \int \frac{x dx}{(x^3+a^3)^{n-1}}.$$

6. $\int \frac{dx}{x^2 + a^2} = \frac{1}{6a^2} \ln \frac{(x+a)^2}{x^2 - ax + a^2} + \frac{1}{a^2 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
7. $\int \frac{x dx}{x^2 + a^2} = \frac{1}{6a} \ln \frac{x^2 - ax + a^2}{(x+a)^2} + \frac{1}{a \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
8. $\int \frac{x^2 dx}{x^2 + a^2} = \frac{1}{3} \ln |x^2 + a^2|.$
9. $\int \frac{x^2 dx}{x^3 + a^3} = x + \frac{a}{6} \ln \frac{x^2 - ax + a^2}{(x+a)^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
10. $\int \frac{dx}{x(x^2 + a^2)} = \frac{1}{3a^3} \ln \left| \frac{x^3}{x^2 + a^2} \right|.$
11. $\int \frac{dx}{x^2(x^2 + a^2)} = -\frac{1}{a^2 x} + \frac{1}{6a^4} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{1}{a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
12. $\int \frac{dx}{x^3(x^2 + a^2)} = -\frac{1}{2a^2 x^2} - \frac{1}{6a^4} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{1}{a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
13. $\int \frac{dx}{(x^3 + a^3)^2} = -\frac{x}{3a^2(x^2 + a^2)} + \frac{1}{9a^4} \ln \frac{(x+a)^2}{x^2 - ax + a^2} + \frac{2}{3a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
14. $\int \frac{x dx}{(x^3 + a^3)^2} = \frac{x^2}{3a^2(x^2 + a^2)} + \frac{1}{18a^4} \ln \frac{x^2 - ax + a^2}{(x+a)^2} + \frac{1}{3a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
15. $\int \frac{x^2 dx}{(x^3 + a^3)^2} = -\frac{1}{3(x^2 + a^2)}.$
16. $\int \frac{dx}{x(x^3 + a^3)^2} = \frac{1}{3a^2(x^2 + a^2)} + \frac{1}{3a^4} \ln \left| \frac{x^3}{x^2 + a^2} \right|.$
17. $\int \frac{dx}{x^2(x^3 + a^3)^2} = \frac{1}{3a^2 x(x^2 + a^2)} - \frac{4}{3a^4 x} +$
 $+\frac{2}{9a^2} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{4}{3a^2 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$
18. $\int \frac{dx}{x^3(x^2 + a^2)^2} = -\frac{1}{3a^2 x^2(x^2 + a^2)} - \frac{5}{6a^4 x^2} -$
 $-\frac{5}{18a^4} \ln \frac{(x+a)^2}{x^2 - ax + a^2} - \frac{5}{3a^4 \sqrt{3}} \operatorname{arctg} \frac{2x-a}{a \sqrt{3}}.$

1.2.13. Интегралы вида $\int \frac{x^{n+m} dx}{(x^4 \pm a^4)^n}, \int \frac{x^{n+m} dx}{(ax^4 + bx^2 + c)^n}.$

1. $\int \frac{x^m dx}{(x^4 \pm a^4)^n} = \pm \frac{x^{m+1}}{4(n-1)a^4(x^4 \pm a^4)^{n-1}} \pm \frac{4n-m-5}{4(n-1)a^4} \int \frac{x^m dx}{(x^4 \pm a^4)^{n-1}}.$
2. $\int \frac{x^m dx}{(x^4 \pm a^4)^n} = \frac{x^{m-3}}{(m-4n+1)(x^4 \pm a^4)^{n-1}} - \frac{(m-3)a^4}{(m-4n+1)} \int \frac{x^{m-4} dx}{(x^4 \pm a^4)^n}.$
3. $\int \frac{x^m dx}{x^4 \pm a^4} = \frac{x^{m-3}}{m-3} \pm a^4 \int \frac{x^{m-4} dx}{x^4 \pm a^4}.$

$$4. \int \frac{dx}{x^m (x^4 \pm a^4)^n} = \mp \frac{1}{(m-1) a^4 x^{m-1} (x^4 \pm a^4)^{n-1}} \mp \frac{4n+m-5}{(m-1) a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

$$5. = \pm \frac{1}{a^4} \int \frac{dx}{x^m (x^4 \pm a^4)^{n-1}} - \frac{1}{a^4} \int \frac{dx}{x^{m-4} (x^4 \pm a^4)^n}.$$

$$6. \int \frac{dx}{x^4 \pm a^4} = \left\{ \frac{1}{4a^3 \sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right\} \\ + \frac{1}{4a^3} \ln \left| \frac{x+a}{x-a} \right| + \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a}.$$

$$7. \int \frac{x dx}{x^4 \pm a^4} = \left\{ \frac{1}{2a^3} \operatorname{arctg} \frac{x^2}{a^2} \right. \\ \left. - \frac{1}{4a^3} \ln \left| \frac{x^2 - a^2}{x^2 + a^2} \right| \right\}.$$

$$8. \int \frac{x^3 dx}{x^4 \pm a^4} = \left\{ \frac{1}{4a\sqrt{2}} \ln \frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right\} \\ - \frac{1}{4a} \ln \left| \frac{x+a}{x-a} \right| + \frac{1}{2a} \operatorname{arctg} \frac{x}{a}.$$

$$9. \int \frac{x^3 dx}{x^4 \pm a^4} = \frac{1}{4} \ln |x^4 \pm a^4|.$$

$$10. \int \frac{dx}{x(x^4 \pm a^4)} = \pm \frac{1}{4a^4} \ln \frac{x^4}{|x^4 \pm a^4|}.$$

$$11. \int \frac{dx}{x^3 (x^4 \pm a^4)} = \mp \frac{1}{a^4 x} \mp \frac{1}{a^4} \int \frac{x^2 dx}{x^4 \pm a^4}.$$

$$12. \int \frac{dx}{x^5 (x^4 \pm a^4)} = \mp \frac{1}{2a^4 x^3} \mp \frac{1}{4a^4} \left\{ 2 \operatorname{arctg} \frac{x^2}{a^2} \right. \\ \left. - \ln \left| \frac{x^2 - a^2}{x^2 + a^2} \right| \right\}.$$

$$13. \int \frac{x^m dx}{(x^4 \pm a^4)^2} = \frac{x^{m+1}}{4a^4 (a^4 \pm x^4)} \pm \frac{3-m}{4a^4} \int \frac{x^m dx}{x^4 \pm a^4}.$$

$$14. \int \frac{dx}{(x^4 \pm a^4)^2} = \frac{x}{4a^4 (a^4 \pm x^4)} \pm \\ \pm \left\{ \frac{3}{16a^7 \sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{3}{8a^7 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right\} \\ \pm \left\{ \frac{3}{16a^7} \ln \left| \frac{x+a}{x-a} \right| + \frac{3}{8a^7} \operatorname{arctg} \frac{x}{a} \right\}.$$

$$15. \int \frac{x dx}{(x^4 \pm a^4)^2} = \frac{x^3}{4a^4 (a^4 \pm x^4)} + \left\{ \frac{1}{4a^4} \operatorname{arctg} \frac{x^2}{a^2} \right. \\ \left. - \frac{1}{8a^4} \ln \left| \frac{x^2 - a^2}{x^2 + a^2} \right| \right\}.$$

- $$16. \int \frac{x^2 dx}{(x^4 \pm a^4)^2} = \frac{x^3}{4a^4(a^4 \pm x^4)} \pm \left\{ \frac{1}{16a^4 \sqrt{2}} \ln \frac{x^2 - ax\sqrt{2} + a^2}{x^2 + ax\sqrt{2} + a^2} + \frac{1}{8a^4 \sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2} \right. \\ \left. \pm \left[\frac{1}{16a^4} \ln \left| \frac{x-a}{x+a} \right| + \frac{1}{8a^4} \operatorname{arctg} \frac{x}{a} \right] \right\}.$$
- $$17. \int \frac{x^2 dx}{(x^4 \pm a^4)^2} = -\frac{1}{4(x^4 \pm a^4)}.$$
- $$18. \int \frac{dx}{x(x^4 \pm a^4)^2} = \frac{1}{4a^4(a^4 \pm x^4)} + \frac{1}{4a^8} \ln \frac{x^4}{|x^4 \pm a^4|}.$$
- $$19. \int \frac{dx}{x^2(x^4 \pm a^4)^2} = -\frac{1}{a^6 x} - \frac{x^3}{4a^6(x^4 \pm a^4)} - \frac{5}{4a^8} \int \frac{x^2 dx}{x^4 \pm a^4}.$$
- $$20. \int \frac{x^m dx}{(ax^4 + bx^2 + c)^n} = \frac{x^{m-3}}{(m-4n+1)a(ax^4 + bx^2 + c)^{n-1}} - \\ - \frac{(m-3)c}{(m-4n+1)a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^n} - \frac{(m-2n-1)b}{(m-4n+1)a} \int \frac{x^{m-2} dx}{(ax^4 + bx^2 + c)^n}.$$
- $$21. = \frac{1}{a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^{n-1}} - \frac{c}{a} \int \frac{x^{m-4} dx}{(ax^4 + bx^2 + c)^n} - \\ - \frac{b}{a} \int \frac{x^{m-2} dx}{(ax^4 + bx^2 + c)^n}.$$
- $$22. \int \frac{dx}{(ax^4 + bx^2 + c)^n} = \frac{abx^3 + (b^2 - 2ac)x}{2(n-1)c(b^2 - 4ac)(ax^4 + bx^2 + c)^{n-1}} + \\ + \frac{(4n-7)ab}{2(n-1)c(b^2 - 4ac)} \int \frac{x^2 dx}{(ax^4 + bx^2 + c)^{n-1}} + \\ + \frac{2(n-1)(b^2 - 4ac) + 2ac - b^2}{2(n-1)c(b^2 - 4ac)} \int \frac{dx}{(ax^4 + bx^2 + c)^{n-1}}.$$
- $$23. \int \frac{dx}{x^m(ax^4 + bx^2 + c)^n} = \frac{-1}{(m-1)cx^{m-1}(ax^4 + bx^2 + c)^{n-1}} - \\ - b \frac{m+2n-3}{(m-1)c} \int \frac{dx}{x^{m-2}(ax^4 + bx^2 + c)^n} - a \frac{m+4n-5}{(m-1)c} \int \frac{dx}{x^{m-4}(ax^4 + bx^2 + c)^n}.$$
- $$24. \int \frac{dx}{ax^4 + bx^2 + c} = \frac{2a}{\sqrt{b^2 - 4ac}} \left(\int \frac{dx}{2ax^2 + b - \sqrt{b^2 - 4ac}} - \right. \\ \left. - \int \frac{dx}{2ax^2 + b + \sqrt{b^2 - 4ac}} \right) \quad [b^2 > 4ac].$$
- $$25. = \frac{1}{3a \sin \alpha} \left[\sin \frac{\alpha}{2} \ln \frac{x^2 + 2x\sqrt{a/c} \cos(\alpha/2) + \sqrt{a/c}}{x^2 - 2x\sqrt{a/c} \cos(\alpha/2) + \sqrt{a/c}} + \right. \\ \left. + 2 \cos \frac{\alpha}{2} \operatorname{arctg} \frac{x^2 - \sqrt{a/c}}{2x\sqrt{a/c} \sin(\alpha/2)} \right] \quad \left[b^2 < 4ac, \cos \alpha = -\frac{b}{2\sqrt{ac}} \right].$$
- $$26. \int \frac{dx}{(ax^4 + bx^2 + c)^2} = \frac{abx^3 + (b^2 - 2ac)x}{2c(b^2 - 4ac)(ax^4 + bx^2 + c)} + \\ + \frac{b^2 - 6ac}{2a(b^2 - 4ac)} \int \frac{dx}{ax^4 + bx^2 + c} + \frac{ab}{2c(b^2 - 4ac)} \int \frac{x^2 dx}{ax^4 + bx^2 + c}.$$

$$27. \int \frac{x dx}{ax^4 + bx^2 + c} = \frac{1}{2\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax^2 + b - \sqrt{b^2 - 4ac}}{2ax^2 + b + \sqrt{b^2 - 4ac}} \right| \quad [b^2 > 4ac].$$

$$28. = \frac{1}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2ax^2 + b}{\sqrt{4ac - b^2}} \quad [b^2 < 4ac].$$

$$29. \int \frac{x^2 dx}{ax^4 + bx^2 + c} = \frac{b + \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b + \sqrt{b^2 - 4ac}} - \\ - \frac{b - \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b - \sqrt{b^2 - 4ac}} \quad [b^2 > 4ac].$$

$$30. = \frac{1}{4a(\sqrt{c/4a} - b/4a)^{1/2}} \left(\int \frac{x dx}{x^2 - 2(\sqrt{c/4a} - b/4a)^{1/2}x + 2\sqrt{c/4a}} - \right. \\ \left. - \int \frac{x dx}{x^2 + 2(\sqrt{c/4a} - b/4a)^{1/2}x + 2\sqrt{c/4a}} \right) \quad [b^2 < 4ac].$$

1.2.14. Интегралы вида $\int R(x, ax^{2k} + bx^k + c) dx$.

$$1. \int x^m (ax^{2k} + bx^k + c)^n dx = \frac{x^{m+1} (ax^{2k} + bx^k + c)^{n+1}}{(m+1)c} - \\ - \frac{(m+nk+k+1)b}{(m+1)c} \int x^{m+k} (ax^{2k} + bx^k + c)^n dx - \\ - \frac{(m+2nk+2k+1)a}{(m+1)c} \int x^{m+2k} (ax^{2k} + bx^k + c)^n dx.$$

$$2. = \frac{x^{m+1} (ax^{2k} + bx^k + c)^n}{m+1} - \frac{bnk}{m+1} \int x^{m+k} (ax^{2k} + bx^k + c)^{n-1} dx - \\ - \frac{2ank}{m+1} \int x^{m+2k} (ax^{2k} + bx^k + c)^{n-1} dx.$$

$$3. = \frac{x^{m-2k+1} (ax^{2k} + bx^k + c)^{n+1}}{(m+2nk+1)a} - \frac{(m-2k+1)c}{(m+2nk+1)a} \times \\ \times \int x^{m-2k} (ax^{2k} + bx^k + c)^n dx - \frac{(m+nk-k+1)b}{(m+2nk+1)a} \int x^{m-k} (ax^{2k} + bx^k + c)^n dx.$$

$$4. = \frac{x^{m+1} (ax^{2k} + bx^k + c)^n}{m+2nk+1} + \frac{2nkc}{m+2nk+1} \times \\ \times \int x^m (ax^{2k} + bx^k + c)^{n-1} dx + \frac{nbk}{m+2nk+1} \int x^{m+k} (ax^{2k} + bx^k + c)^{n-1} dx.$$

$$5. \int \frac{x^m dx}{(ax^{2k} + bx^k + c)^n} = \frac{x^{m-2k+1}}{(m-2nk+1)a (ax^{2k} + bx^k + c)^{n-1}} - \\ - \frac{(m-2k+1)c}{(m-2nk+1)a} \int \frac{x^{m-2k} dx}{(ax^{2k} + bx^k + c)^n} - \frac{(m-nk-k+1)b}{(m-2nk+1)a} \int \frac{x^{m-k} dx}{(ax^{2k} + bx^k + c)^n}.$$

$$6. \int \frac{dx}{x^m (ax^{2k} + bx^k + c)^n} = \frac{-1}{(m-1)c x^{m-1} (ax^{2k} + bx^k + c)^{n-1}} - \\ - \frac{(m+nk-k-1)b}{(m-1)c} \int \frac{dx}{x^{m-k} (ax^{2k} + bx^k + c)^n} - \\ - \frac{m+2nk-2k-1}{(m-1)c} \int \frac{dx}{x^{m-2k} (ax^{2k} + bx^k + c)^n}.$$

1.2.15. Интегралы вида $\int R(x^{1/2}, ax+b) dx$.

$$1. \int x^{m+1/2} (ax+b)^n dx = 2x^{m+3/2} \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^k}{2k+2m+3}.$$

$$2. \int \frac{(ax+b)^n}{x^{m+1/2}} dx = \frac{2}{x^{m-1/2}} \sum_{k=0}^n \binom{n}{k} \frac{a^k b^{n-k} x^k}{2k-2m+1}.$$

$$3. \int \frac{x^{m+1/2} dx}{(ax \pm b)^n} = -\frac{x^{m+1/2}}{(n-1)a(ax \pm b)^{n-1}} + \frac{2m+1}{2(n-1)a} \int \frac{x^{m-1/2} dx}{(ax \pm b)^{n-1}}.$$

$$4. = \frac{2}{a^n} \int \frac{t^{2m+2} dt}{(t^2 \pm b/a)^n} \quad [t = \sqrt{x}, \text{ см. 1.2.10}].$$

$$5. \int \frac{x^{m+1/2} dx}{ax \pm b} = 2 \sum_{k=0}^m \frac{(-1)^k b^k x^{m-k+1/2}}{(2m-2k+1)a^{k+1}} + \left(\mp \frac{b}{a}\right)^{m-1} \int \frac{dx}{x^{1/2}(ax \pm b)}.$$

$$6. \int \frac{dx}{x^{m+1/2}(ax \pm b)^n} = \mp \frac{2}{(2m-1)b x^{m-1/2}(ax \pm b)^{n-1}} \mp \frac{(2m+2n-3)a}{(2m-1)b} \int \frac{dx}{x^{m-1/2}(ax \pm b)^n} = \frac{2}{a^n} \int \frac{dr}{t^{2m}(t^2 \pm b/a)^n}.$$

$$7. \int \frac{x^{1/2} dx}{ax \pm b} = \frac{2x^{1/2}}{a} \mp \left(\frac{b}{a^2}\right)^{1/2} \left\{ 2 \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right| \right\}.$$

$$8. \int \frac{x^{3/2} dx}{ax \pm b} = \frac{2ax^{3/2} \mp 6bx^{1/2}}{3a^2} \pm \left(\frac{b}{a^3}\right)^{1/2} \left\{ 2 \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \right| \right\}.$$

$$9. \int \frac{x^{1/2} dx}{(ax \pm b)^2} = -\frac{x^{1/2}}{a(ax \pm b)} + \frac{1}{(a^2b)^{1/2}} \left\{ \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \frac{1}{2} \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right| \right\}.$$

$$10. \int \frac{x^{3/2} dx}{(ax \pm b)^2} = \frac{2ax^{3/2} \pm 3bx^{1/2}}{a^2(ax \pm b)} - 3 \left(\frac{b}{a^3}\right)^{1/2} \left\{ \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \frac{1}{2} \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \right| \right\}.$$

$$11. \int \frac{dx}{x^{1/2}(ax \pm b)} = \pm \frac{1}{(ab)^{1/2}} \left\{ 2 \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \right| \right\}.$$

$$12. \int \frac{dx}{x^{3/2}(ax \pm b)} = \mp \frac{2}{bx^{1/2}} + \left(\frac{a}{b^2}\right)^{1/2} \left\{ 2 \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right| \right\}.$$

$$13. \int \frac{dx}{x^{1/2}(ax \pm b)^2} = \pm \frac{x^{1/2}}{b(ax \pm b)} + \frac{1}{(ab^2)^{1/2}} \left\{ \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \frac{1}{2} \ln \left| \frac{b^{1/2} + (ax)^{1/2}}{b^{1/2} - (ax)^{1/2}} \right| \right| \right\}.$$

$$14. \int \frac{dx}{x^{3/2}(ax \pm b)^2} = \frac{-3ax \mp 2b}{b^2 x^{1/2}(ax \pm b)} - 3 \left(\frac{a}{b^2} \right)^{1/2} \left\{ \operatorname{arctg} \left(\frac{ax}{b} \right)^{1/2} \right. \\ \left. \left| \frac{1}{2} \ln \left| \frac{b^{1/2} - (ax)^{1/2}}{b^{1/2} + (ax)^{1/2}} \right| \right| \right\}.$$

1.2.16. Интегралы вида $\int R(x^{1/2}, x^2 \pm a^2) dx$.

Условие: $a > 0$.

$$1. \int \frac{x^{m+1/2}}{(x^2 \pm a^2)^n} dx = \pm \frac{x^{m+3/2}}{2(n-1)a^2(x^2 \pm a^2)^{n-1}} \pm \frac{4n-2m-7}{4(n-1)a^2} \int \frac{x^{m+1/2}}{(x^2 \pm a^2)^{n-1}} dx,$$

$$2. \int \frac{x^{m+1/2}}{x^2 \pm a^2} dx = \frac{2}{2m-1} x^{m-1/2} \mp a^2 \int \frac{x^{m-3/2}}{x^2 \pm a^2} dx,$$

$$3. \int \frac{x^{2m+1/2}}{x^2 \pm a^2} dx = 2 \sum_{k=1}^m \frac{(\mp a^2)^{k-1}}{4m-4k+3} x^{2m-2k+3/2} + (\mp a^2)^m \int \frac{x^{1/2}}{x^2 \pm a^2} dx.$$

$$4. \int \frac{x^{2m+3/2}}{x^2 \pm a^2} dx = 2 \sum_{k=0}^{m-1} \frac{(\mp a^2)^k}{4m-4k+1} x^{2m-2k+1/2} + (\mp a^2)^m \int \frac{x^{3/2}}{x^2 \pm a^2} dx.$$

$$5. \int \frac{x^{1/2}}{x^2 + a^2} dx = \frac{1}{2\sqrt{2a}} \ln \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} - \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

$$6. \int \frac{x^{1/2}}{x^2 - a^2} dx = \frac{1}{2\sqrt{a}} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| - \frac{1}{\sqrt{a}} \operatorname{arctg} \sqrt{\frac{a}{x}}.$$

$$7. \int \frac{x^{3/2}}{x^2 + a^2} dx = 2\sqrt{x} + \frac{\sqrt{2a}}{4} \ln \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} + \frac{\sqrt{2a}}{2} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

$$8. \int \frac{x^{3/2}}{x^2 - a^2} dx = 2\sqrt{x} + \frac{\sqrt{a}}{2} \ln \left| \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| + \sqrt{a} \operatorname{arctg} \sqrt{\frac{a}{x}}.$$

1.2.17. Интегралы вида $\int x^{\pm m} (ax+b)^{n+1/2} dx$.

$$1. \int x^m (ax+b)^{n+1/2} dx = \frac{2(ax+b)^{n+3/2}}{a^{m+1}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m+2n-2k+3} \binom{m}{k} (ax+b)^{m-k}.$$

$$2. \int (ax+b)^{n+1/2} dx = \frac{2}{(2n+3)a} (ax+b)^{n+3/2}.$$

3. $\int x^m (ax+b)^{1/2} dx = \frac{2}{(2m+3)a} x^m (ax+b)^{3/2} -$
 $-\frac{2mb}{(2m+3)a} \int x^{m-1} (ax+b)^{1/2} dx = \frac{2(ax+b)^{3/2}}{a^{m+1}} \sum_{k=0}^m \binom{m}{k} \frac{(-b)^{m-k}}{2k+3} (ax+b)^k.$
4. $\int \frac{(ax+b)^{n+1/2}}{x^m} dx = -\frac{(ax+b)^{n+3/2}}{(m-1)bx^{m-1}} +$
 $+ \frac{(2n-2m+5)a}{2(m-1)b} \int \frac{(ax+b)^{n+1/2}}{x^{m-1}} dx.$
5. $= \frac{2(ax+b)^{n+1/2}}{(2n-2m+3)x^{m-1}} + \frac{(2n+1)b}{2n-2m+3} \int \frac{(ax+b)^{n-1/2}}{x^m} dx.$
6. $\int \frac{(ax+b)^{n+1/2}}{x} dx = \frac{2(ax+b)^{n+1/2}}{2n+1} + b \int \frac{(ax+b)^{n-1/2}}{x} dx.$
7. $\int \frac{(ax+b)^{1/2}}{x^m} dx = -\frac{(ax+b)^{3/2}}{(m-1)bx^{m-1}} + \frac{(5-2m)a}{2(m-1)b} \int \frac{(ax+b)^{1/2}}{x^{m-1}} dx.$
8. $\int (ax+b)^{1/2} dx = \frac{2}{3a} (ax+b)^{3/2}.$
9. $\int x(ax+b)^{1/2} dx = \frac{2}{a^2} \left[\frac{ax+b}{5} - \frac{b}{3} \right] (ax+b)^{3/2}.$
10. $\int x^2(ax+b)^{1/2} dx = \frac{2}{a^3} \left[\frac{(ax+b)^2}{7} - \frac{2b(ax+b)}{5} + \frac{b^2}{3} \right] (ax+b)^{3/2}.$
11. $\int x^3(ax+b)^{1/2} dx =$
 $= \frac{2}{a^4} \left[\frac{(ax+b)^3}{9} - \frac{3b(ax+b)^2}{7} + \frac{3b^2(ax+b)}{5} - \frac{b^3}{3} \right] (ax+b)^{3/2}.$
12. $\int (ax+b)^{3/2} dx = \frac{2}{5a} (ax+b)^{5/2}.$
13. $\int x(ax+b)^{3/2} dx = \frac{2}{a^2} \left[\frac{(ax+b)}{7} - \frac{b}{5} \right] (ax+b)^{5/2}.$
14. $\int x^2(ax+b)^{3/2} dx = \frac{2}{a^3} \left[\frac{(ax+b)^3}{9} - \frac{2b(ax+b)}{7} + \frac{b^2}{5} \right] (ax+b)^{5/2}.$
15. $\int x^3(ax+b)^{3/2} dx = \frac{2}{a^4} \left[\frac{(ax+b)^3}{11} - \frac{3b(ax+b)^2}{9} + \right.$
 $\left. + \frac{3b^2(ax+b)}{7} - \frac{b^3}{5} \right] (ax+b)^{5/2}.$
16. $\int \frac{(ax+b)^{1/2}}{x} = 2(ax+b)^{1/2} + \begin{cases} b^{1/2} \ln \left| \frac{(ax+b)^{1/2} - b^{1/2}}{(ax+b)^{1/2} + b^{1/2}} \right| & [b > 0], \\ -2(-b)^{1/2} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$
17. $\int \frac{(ax+b)^{1/2}}{x^2} dx = -\frac{(ax+b)^{1/2}}{x} + \frac{a}{2} \int \frac{dx}{x(ax+b)^{1/2}}.$

$$18. \int \frac{(ax+b)^{1/2}}{x^3} dx = -\frac{(2b+ax)}{4bx^2} (ax+b)^{1/2} - \frac{a^2}{8b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$19. \int \frac{(ax+b)^{3/2}}{x} dx = \frac{2}{3} (ax+b)^{3/2} + 2b (ax+b)^{1/2} + b^2 \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$20. \int \frac{(ax+b)^{3/2}}{x^2} dx = -\frac{(ax+b)^{5/2}}{bx} + \frac{3a}{2b} \int \frac{(ax+b)^{3/2}}{x} dx.$$

$$21. \int \frac{(ax+b)^{3/2}}{x^3} dx = -\left(\frac{1}{2bx^2} + \frac{a}{4b^2x}\right) (ax+b)^{5/2} + \frac{3a^2}{8b^2} \int \frac{(ax+b)^{3/2}}{x} dx.$$

1.2.18. Интегралы вида $\int \frac{x^m dx}{(ax+b)^{n+1/2}}.$

$$1. \int \frac{x^m dx}{(ax+b)^{n+1/2}} = \frac{2}{a^{m+1} (ax+b)^{n-1/2}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m-2n-2k+1} \binom{m}{k} (ax+b)^{m-k}.$$

$$2. \int \frac{dx}{(ax+b)^{n+1/2}} = -\frac{2}{(2n-1)a(ax+b)^{n-1/2}}.$$

$$3. \int \frac{x^m dx}{(ax+b)^{1/2}} = \frac{2}{(2m+1)a} x^m (ax+b)^{1/2} - \frac{2mb}{(2m+1)a} \int \frac{x^{m-1} dx}{(ax+b)^{1/2}}.$$

$$4. = \frac{2(ax+b)^{1/2}}{a^{m+1}} \sum_{k=0}^m \frac{(-1)^k b^k}{2m-2k+1} \binom{m}{k} (ax+b)^{m-k}.$$

$$5. \int \frac{dx}{(ax+b)^{1/2}} = \frac{2}{a} (ax+b)^{1/2}.$$

$$6. \int \frac{x dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2} (ax+b)^{1/2}.$$

$$7. \int \frac{x^2 dx}{(ax+b)^{1/2}} = \frac{2}{a^3} \left[\frac{(ax+b)^3}{5} - \frac{2b(ax+b)}{3} + b^2 \right] (ax+b)^{1/2}.$$

$$8. \int \frac{x^3 dx}{(ax+b)^{1/2}} = \frac{2}{a^4} \left[\frac{(ax+b)^3}{7} - \frac{3b(ax+b)^2}{5} + b^2(ax+b) - b^3 \right] (ax+b)^{1/2}.$$

$$9. \int \frac{dx}{(ax+b)^{3/2}} = -\frac{2}{a(ax+b)^{1/2}}.$$

$$10. \int \frac{x dx}{(ax+b)^{3/2}} = \frac{2(ax+2b)}{a^2(ax+b)^{1/2}}.$$

$$11. \int \frac{x^2 dx}{(ax+b)^{3/2}} = \frac{2}{a^3} \left[\frac{(ax+b)^3}{3} - 2b(ax+b) - b^2 \right] \frac{1}{(ax+b)^{1/2}}.$$

$$12. \int \frac{x^3 dx}{(ax+b)^{3/2}} = \frac{2}{a^4} \left[\frac{(ax+b)^4}{5} - b(ax+b)^2 + 3b^2(ax+b) + b^3 \right] \frac{1}{(ax+b)^{1/2}}.$$

1.2.19. Интегралы вида $\int \frac{dx}{x^m (ax+b)^{n+1/2}}.$

1. $\int \frac{dx}{x^m (ax+b)^{n+1/2}} =$
 $= -\frac{1}{(m-1)x^{m-1}(ax+b)^{n+1/2}} - \frac{(2n+1)a}{2(m-1)} \int \frac{dx}{x^{m-1}(ax+b)^{n+3/2}}.$
2. $= -\frac{-2}{(2n-1)ax^m(ax+b)^{n-1/2}} - \frac{2m}{(2n-1)a} \int \frac{dx}{x^{m+1}(ax+b)^{n-1/2}}.$
3. $= 2a^{m-1} \int \frac{dt}{t^{2n}(t^2-b)^m} \quad [t=\sqrt{ax+b}, \text{ см. 1.2.10}].$
4. $\int \frac{dx}{x(ax+b)^{n+1/2}} = \frac{2}{(2n-1)b(ax+b)^{n-1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{n-1/2}}.$
5. $\int \frac{dx}{x^m(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{2(m-1)b} \int \frac{dx}{x^{m-1}(ax+b)^{1/2}}.$
6. $\int \frac{dx}{x(ax+b)^{1/2}} = \begin{cases} \frac{1}{b^{1/2}} \ln \left| \frac{(ax+b)^{1/2}-b^{1/2}}{(ax+b)^{1/2}+b^{1/2}} \right| & [b > 0], \\ \frac{2}{(-b)^{1/2}} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$
7. $\int \frac{dx}{x^3(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{bx} - \frac{a}{2b} \int \frac{dx}{x(ax+b)^{1/2}}.$
8. $\int \frac{dx}{x^3(ax+b)^{1/2}} = \frac{3ax-2b}{4b^2x^3} (ax+b)^{1/2} + \frac{3a^2}{8b^3} \int \frac{dx}{x(ax+b)^{1/2}}.$
9. $\int \frac{dx}{x(ax+b)^{3/2}} = \frac{2}{b(ax+b)^{1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{1/2}}.$
10. $\int \frac{dx}{x^3(ax+b)^{3/2}} = \frac{-3ax-b}{b^2x(ax+b)^{1/2}} - \frac{3a}{2b^3} \int \frac{dx}{x(ax+b)^{1/2}}.$
11. $\int \frac{dx}{x^3(ax+b)^{3/2}} = \left(-\frac{1}{2bx^3} + \frac{5a}{4b^2x} + \frac{15a^2}{4b^3} \right) \frac{1}{(ax+b)^{1/2}} +$
 $+ \frac{15a^2}{8b^3} \int \frac{dx}{x(ax+b)^{1/2}}.$
12. $\int \frac{(ax+b)^{1/2}+c}{(ax+b)^{1/2}+d} dx = x + 2 \frac{c-d}{a} [(ax+b)^{1/2} - d \ln |(ax+b)^{1/2}+d|].$

1.2.20. Интегралы вида $\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx.$

1. $\int (ax+b)^{m+1/2} (cx+d)^{n+1/2} dx = \frac{(ax+b)^{m+1/2} (cx+d)^{n+3/2}}{(m+n+2)c} -$
 $- \frac{(2m+1)(ad-bc)}{2(m+n+2)c} \int (ax+b)^{m-1/2} (cx+d)^{n+1/2} dx.$

$$\begin{aligned}
 2. \quad &= (cx+d)^{n+1/2} \sum_{k=0}^m \frac{(-1)^k (2m+1)(2m-1)\dots(2m-2k+3)(ad-bc)^k}{(m+n+2)(m+n+1)\dots(m+n-k+2)c^{k+1/2}} \times \\
 &\quad \times (ax+b)^{m-k+1/2} + \frac{m!!}{(-2c)^{m+1}} \sqrt{ax+b} \times \\
 &\quad \times \sum_{k=0}^n \frac{(2n+1)(2n-1)\dots(2n-2k+3)(ad-bc)^{m+k+1}}{2^k (m+n+2)(m+n+1)\dots(n-k+1)a^{k+1/2}} \times \\
 &\quad \times (cx+d)^{n-k+1/2} + \frac{m!! n!! (ad-bc)^{m+n+2}}{(m+n+2)! (-2c)^{m+1} (2a)^{n+1/2}} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n+1/2}} dx &= \frac{1}{(m-n+1)c} \cdot \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} - \\
 &\quad - \frac{(2m+1)(ad-bc)}{2(m-n+1)c} \int \frac{(ax+b)^{m-1/2}}{(cx+d)^{n+1/2}} dx.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad &= \frac{2}{(2n-1)(ad-bc)} \cdot \frac{(ax+b)^{m+3/2}}{(cx+d)^{n-1/2}} + \\
 &\quad + \frac{2(n-m-2)a}{(2n-1)(ad-bc)} \int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} dx.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad &= (ax+b)^{m+3/2} \sum_{k=0}^{n-m-2} \frac{(n-m-2)(n-m-3)\dots(n-m-k-1)2^{k+1}a^k}{(2n-1)(2n-3)\dots(2n-2k-1)(ad-bc)^{k+1/2}} \times \\
 &\quad \times \frac{1}{(cx+d)^{n-k-1/2}} \quad [n \geq m+2].
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int \frac{dx}{(ax+b)^{m+1/2}(cx+d)^{n+1/2}} &= \frac{-2}{(2m-1)(ad-bc)(ax+b)^{m-1/2}(cx+d)^{n-1/2}} - \\
 &\quad - \frac{2(m+n-1)}{(2m-1)(ad-bc)} \int \frac{dx}{(ax+b)^{m-1/2}(cx+d)^{n+1/2}}.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad &= \frac{1}{(cx+d)^{n-1/2}} \sum_{k=0}^{m-1} \frac{(m+n-1)(m+n-2)\dots(m+n-k)(-2)^{k+1}c^k}{(2m-1)(2m-3)\dots(2m-2k-1)(ad-bc)^{k+1/2}} \times \\
 &\quad \times \frac{1}{(ax+b)^{m-k-1/2}} + \\
 &\quad + \frac{(-2c)^m}{(m-1)!!} \sqrt{ax+b} \sum_{k=0}^{n-1} \frac{(m+n-1)(m+n-2)\dots(n-k)2^{k+1}a^k}{(2n-1)(2n-3)\dots(2n-2k-1)(ad-bc)^{n-k+1/2}} \times \\
 &\quad \times \frac{1}{(cx+d)^{n-k-1/2}}.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int \frac{dx}{\sqrt{(ax+b)(cx+d)}} &= \frac{2}{\sqrt{ac}} \ln(\sqrt{c(ax+b)} + \sqrt{a(cx+d)}) \quad \cdot \\
 &\quad (ac > 0; x > \max(-b/a, -d/c)).
 \end{aligned}$$

1.2.19. Интегралы вида $\int \frac{dx}{x^m (ax+b)^{n+1/2}}.$

$$\begin{aligned} \int \frac{dx}{x^m (ax+b)^{n+1/2}} &= \\ &= -\frac{1}{(m-1)x^{m-1}(ax+b)^{n+1/2}} - \frac{(2n+1)a}{2(m-1)} \int \frac{dx}{x^{m-1}(ax+b)^{n+3/2}}. \\ &= \frac{-2}{(2n-1)ax^m(ax+b)^{n-1/2}} - \frac{2m}{(2n-1)a} \int \frac{dx}{x^{m+1}(ax+b)^{n-1/2}}. \\ &= 2a^{m-1} \int \frac{dt}{t^{2n}(t^2-b)^m} \quad [t = \sqrt{ax+b}, \text{ см. 1.2.10}]. \end{aligned}$$

$$\int \frac{dx}{x(ax+b)^{n+1/2}} = \frac{2}{(2n-1)b(ax+b)^{n-1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{n-1/2}}.$$

$$\int \frac{dx}{x^m(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{2(m-1)b} \int \frac{dx}{x^{m-1}(ax+b)^{1/2}}.$$

$$\int \frac{dx}{x(ax+b)^{1/2}} = \begin{cases} \frac{1}{b^{1/2}} \ln \left| \frac{(ax+b)^{1/2} - b^{1/2}}{(ax+b)^{1/2} + b^{1/2}} \right| & [b > 0], \\ \frac{2}{(-b)^{1/2}} \operatorname{arctg} \frac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$$

$$\int \frac{dx}{x^3(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{bx} - \frac{a}{2b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$\int \frac{dx}{x^3(ax+b)^{3/2}} = \frac{3ax-2b}{4b^2x^2} (ax+b)^{1/2} + \frac{3a^2}{8b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$\int \frac{dx}{x(ax+b)^{3/2}} = \frac{2}{b(ax+b)^{1/2}} + \frac{1}{b} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$\int \frac{dx}{x^3(ax+b)^{3/2}} = \frac{-3ax-b}{b^2x(ax+b)^{1/2}} - \frac{3a}{2b^2} \int \frac{dx}{x(ax+b)^{1/2}}.$$

$$\begin{aligned} \int \frac{dx}{x^3(ax+b)^{5/2}} &= \left(-\frac{1}{2bx^2} + \frac{5a}{4b^2x} + \frac{15a^2}{4b^3} \right) \frac{1}{(ax+b)^{1/2}} + \\ &+ \frac{15a^2}{8b^3} \int \frac{dx}{x(ax+b)^{1/2}}. \end{aligned}$$

$$\frac{(ax+b)^{1/2}+c}{(ax+b)^{1/2}+d} dx = x + 2 \frac{c-d}{a} [(ax+b)^{1/2} - d \ln |(ax+b)^{1/2} + d|].$$

2.20. Интегралы вида $\int R(x, \sqrt{ax+b}, \sqrt{cx+d}) dx.$

$$\begin{aligned} \int (ax+b)^{m+1/2} (cx+d)^{n+1/2} dx &= \frac{(ax+b)^{m+1/2} (cx+d)^{n+3/2}}{(m+n+2)c} - \\ &- \frac{(2m+1)(ad-bc)}{2(m+n+2)c} \int (ax+b)^{m-1/2} (cx+d)^{n+1/2} dx. \end{aligned}$$

$$\begin{aligned}
 2. \quad &= (cx+d)^{m+1/2} \sum_{k=0}^m \frac{(-1)^k (2m+1)(2m-1) \dots (2m-2k+3) (ad-bc)^k}{(m+n+2)(m+n+1) \dots (m+n-k+2) c^{k+1/2}} \times \\
 &\quad \times (ax+b)^{m-k+1/2} + \frac{m!!}{(-2c)^{m+1/2}} \sqrt{ax+b} \times \\
 &\quad \times \sum_{k=0}^n \frac{(2n+1)(2n-1) \dots (2n-2k+3) (ad-bc)^{m+k+1}}{2^k (m+n+2)(m+n+1) \dots (n-k+1) a^{k+1}} \times \\
 &\quad \times (cx+d)^{n-k+1/2} + \frac{m!! n!! (ad-bc)^{m+n+2}}{(m+n+2)! (-2c)^{m+1/2} (2a)^{n+1/2}} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n+1/2}} dx &= \frac{1}{(m-n+1)c} \cdot \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} - \\
 &\quad - \frac{(2m+1)(ad-bc)}{2(m-n+1)c} \int \frac{(ax+b)^{m-1/2}}{(cx+d)^{n+1/2}} dx.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad &= \frac{2}{(2n-1)(ad-bc)} \cdot \frac{(ax+b)^{m+3/2}}{(cx+d)^{n-1/2}} + \\
 &\quad + \frac{2(n-m-2)a}{(2n-1)(ad-bc)} \int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} dx.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad &= (ax+b)^{m+3/2} \sum_{k=0}^{n-m-2} \frac{(n-m-2)(n-m-3) \dots (n-m-k-1) 2^{k+1} a^k}{(2n-1)(2n-3) \dots (2n-2k-1) (ad-bc)^{k+1/2}} \times \\
 &\quad \times \frac{1}{(cx+d)^{n-k-1/2}} \quad [n \geq m+2].
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \int \frac{dx}{(ax+b)^{m+1/2} (cx+d)^{n+1/2}} &= \frac{-2}{(2m-1)(ad-bc)(ax+b)^{m-1/2} (cx+d)^{n-1/2}} - \\
 &\quad - \frac{2(m+n-1)}{(2m-1)(ad-bc)} \int \frac{dx}{(ax+b)^{m-1/2} (cx+d)^{n+1/2}}.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad &= \frac{1}{(cx+d)^{n-1/2}} \sum_{k=0}^{m-1} \frac{(m+n-1)(m+n-2) \dots (m+n-k) (-2)^{k+1} c^k}{(2m-1)(2m-3) \dots (2m-2k-1) (ad-bc)^{k+1/2}} \times \\
 &\quad \times \frac{1}{(ax+b)^{m-k-1/2}} + \\
 &\quad + \frac{(-2c)^m}{(m-1)!!} \sqrt{ax+b} \sum_{k=0}^{n-1} \frac{(m+n-1)(m+n-2) \dots (n-k) 2^{k+1} a^k}{(2n-1)(2n-3) \dots (2n-2k-1) (ad-bc)^{n+k+1/2}} \times \\
 &\quad \times \frac{1}{(cx+d)^{n-k-1/2}}.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int \frac{dx}{\sqrt{(ax+b)(cx+d)}} &= \frac{2}{\sqrt{ac}} \ln(\sqrt{c(ax+b)} + \sqrt{a(cx+d)}) \quad \cdot \\
 &\quad [ac > 0; \quad x > \max(-b/a, -d/c)].
 \end{aligned}$$

$$= -\frac{2}{\sqrt{ac}} \ln (\sqrt{-c(ax+b)} + \sqrt{-a(cx+d)})$$

$$[ac > 0; x < \max(-b/a, -d/c)].$$

$$= -\frac{1}{\sqrt{-ac}} \arcsin \frac{c(ax+b) + a(cx+d)}{|ad-bc|}$$

$$[ac < 0].$$

$$\int \frac{dx}{(x-p)\sqrt{(ax+b)(cx+d)}} =$$

$$\frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{(cp+d)(ax+b)} - \sqrt{(ap+b)(cx+d)}]^2}{|x-p|}$$

$$[(ap+b)(cp+d) > 0; (cp+d)(ax+b) > 0; (ap+b)(cx+d) > 0].$$

$$= \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{-(cp+d)(ax+b)} - \sqrt{-(ap+b)(cx+d)}]^2}{|x-p|}$$

$$[(ap+b)(cp+d) > 0; (cp+d)(ax+b) < 0; (ap+b)(cx+d) < 0].$$

$$= \frac{1}{\sqrt{-(ap+b)(cp+d)}} \arcsin \frac{(cp+d)(ax+b) + (ap+b)(cx+d)}{|ad-bc||x-p|}$$

$$[(ap+b)(cp+d) < 0].$$

$$\int \frac{x^n}{A\sqrt{ax+b} + B\sqrt{cx+d}} dx =$$

$$= \frac{A}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{ax+b}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{ax+b} dx \right) -$$

$$- \frac{B}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{cx+d}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{cx+d} dx \right)$$

$$\left[p = \frac{-bA^2 + dB}{aA^2 - cB^2} \right].$$

$$= \frac{A}{bA^2 - dB^2} \int x^n \sqrt{ax+b} dx - \frac{B}{bA^2 - dB^2} \int x^n \sqrt{cx+d} dx$$

$$[aA^2 - cB^2 = 0].$$

$$\frac{\sqrt{ax+b}}{x-p} dx = 2\sqrt{ax+b} \mp$$

$$\mp \sqrt{|ap+b|} \begin{cases} \ln \left| \frac{\sqrt{ax+b} + \sqrt{ap+b}}{\sqrt{ax+b} - \sqrt{ap+b}} \right| \\ 2 \operatorname{arctg} \sqrt{\frac{|ap+b|}{ax+b}} \end{cases}$$

$$[ap+b > 0],$$

$$[ap+b < 0].$$

$$\frac{\alpha \sqrt{ax+b} + \beta \sqrt{cx+d}}{\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}} dx = \frac{\alpha\gamma a + \beta\delta c}{\gamma^2 a + \delta^2 c} x - \frac{\alpha\delta - \beta\gamma}{\gamma^2 a + \delta^2 c} \sqrt{(ax+b)(cx+d)} +$$

$$+ \frac{2\gamma\delta(\alpha\delta - \beta\gamma)(ad-bc)}{(\gamma^2 a - \delta^2 c)^2} \ln |\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}| -$$

$$- \frac{(\gamma^2 a + \delta^2 c)(\alpha\delta - \beta\gamma)(ad-bc)}{2(\gamma^2 a - \delta^2 c)^2} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

1.2.21. Интегралы вида $\int R(x, \sqrt[n]{ax+b}) dx$.

$$1. \int R(x, \sqrt[n]{ax+b}) dx = \frac{n}{a} \int R\left(\frac{t^n-b}{a}, t\right) t^{n-1} dt \quad [t = \sqrt[n]{ax+b}].$$

$$2. \int x^p (ax+b)^{1/n} dx = \frac{n}{(np+n+1)a} x^p (ax+b)^{1+1/n} - \frac{bnp}{(np+n+1)a} \int x^{p-1} (ax+b)^{1/n} dx.$$

$$3. \int x^m (ax+b)^{1/n} dx = \frac{1}{a^{m+1}} (ax+b)^{1+1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1+1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{1/n} dx = \frac{n}{(n+1)a} (ax+b)^{1+1/n}.$$

$$5. \int \frac{(ax+b)^{1/n}}{x^m} dx = -\frac{(ax+b)^{1+1/n}}{(m-1)bx^{m-1}} - \frac{(mn-2n-1)a}{(m-1)nb} \int \frac{(ax+b)^{1/n}}{x^{m-1}} dx.$$

$$6. \int \frac{(ax+b)^{1/n}}{x} dx = n(ax+b)^{1/n} + nb^{1/n} \int \frac{dt}{t^{n-1}} \quad [t = (ax/b + 1)^{1/n}].$$

$$7. \int \frac{x^m dx}{(ax+b)^{1/n}} = \frac{n}{(mn+n-1)a} x^m (ax+b)^{1-1/n} - \frac{mn b}{(mn+n-1)a} \int \frac{x^{m-1} dx}{(ax+b)^{1/n}}$$

$$8. = \frac{1}{a^{m+1}} (ax+b)^{1-1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1-1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$9. \int \frac{dx}{(ax+b)^{1/n}} = \frac{n}{(n-1)a} (ax+b)^{1-1/n}.$$

1.2.22. Интегралы вида $\int R[x, (ax+b)^{p/n}] dx$.

$$1. \int x^m (ax+b)^{p/n} dx = \frac{n}{(mn+n+p)a} x^m (ax+b)^{1+p/n} - \frac{mn b}{(mn+n+p)a} \int x^{m-1} (ax+b)^{p/n} dx.$$

$$2. = \frac{n}{a} (ax+b)^{1+p/n} \times$$

$$\times \sum_{k=0}^m \frac{m(m-1)\dots(m-k+1)}{(mn+p+n)(mn+p)\dots(mn+n+p-kn)} \left(-\frac{bn}{a}\right)^k x^{m-k}.$$

$$3. = \frac{n}{a^{m+1}} (ax+b)^{1+p/n} \sum_{k=0}^m \frac{\binom{m}{k} (-b)^k}{(m-k+1)n+p} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{p/n} dx = \frac{n}{(n+p)a} (ax+b)^{1+p/n}.$$

$$5. \int \frac{(ax+b)^{p/n}}{(x-\alpha)^m} dx = -\frac{1}{(m-1)(a\alpha+b)} \frac{(ax+b)^{1+p/n}}{(x-\alpha)^{m-1}} - \frac{[(m-2)n-p]a}{(m-1)n(a\alpha+b)} \int \frac{(ax+b)^{p/n}}{(x-\alpha)^{m-1}} dx.$$

$$9. \quad = -\frac{2}{\sqrt{ac}} \ln (\sqrt{-c(ax+b)} + \sqrt{-a(cx+d)})$$

[$ac > 0$; $x < \max(-b/a, -d/c)$].

$$10. \quad = -\frac{1}{\sqrt{-ac}} \arcsin \frac{c(ax+b) + a(cx+d)}{|ad-bc|}$$

[$ac < 0$].

$$11. \quad \int \frac{dx}{(x-p)\sqrt{(ax+b)(cx+d)}} =$$

$$= \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{(cp+d)(ax+b)} - \sqrt{(ap+b)(cx+d)}]P}{|x-p|}$$

[$(ap+b)(cp+d) > 0$; $(cp+d)(ax+b) > 0$; $(ap+b)(cx+d) > 0$].

$$12. \quad = \frac{1}{\sqrt{(ap+b)(cp+d)}} \ln \frac{[\sqrt{-(cp+d)(ax+b)} - \sqrt{-(ap+b)(cx+d)}]P}{|x-p|}$$

[$(ap+b)(cp+d) > 0$; $(cp+d)(ax+b) < 0$; $(ap+b)(cx+d) < 0$].

$$13. \quad = \frac{1}{\sqrt{-(ap+b)(cp+d)}} \arcsin \frac{(cp+d)(ax+b) + (ap+b)(cx+d)}{|ad-bc||x-p|}$$

[$(ap+b)(cp+d) < 0$].

$$14. \quad \int \frac{x^n}{A\sqrt{ax+b}B\sqrt{cx+d}} dx =$$

$$= \frac{A}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{ax+b}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{ax+b} dx \right) -$$

$$- \frac{B}{aA^2 - cB^2} \left(p^n \int \frac{\sqrt{cx+d}}{x-p} dx + \sum_{k=1}^n p^{n-k} \int x^{k-1} \sqrt{cx+d} dx \right)$$

[$p = \frac{-bA^2 + dB^2}{aA^2 - cB^2}$].

$$15. \quad = \frac{A}{bA^2 - dB^2} \int x^n \sqrt{ax+b} dx - \frac{B}{bA^2 - dB^2} \int x^n \sqrt{cx+d} dx$$

[$aA^2 - cB^2 = 0$].

$$16. \quad \int \frac{\sqrt{ax+b}}{x-p} dx = 2\sqrt{ax+b} \mp$$

$$\mp \sqrt{|ap+b|} \begin{cases} \ln \left| \frac{\sqrt{ax+b} + \sqrt{ap+b}}{\sqrt{ax+b} - \sqrt{ap+b}} \right| & [ap+b > 0], \\ 2 \operatorname{arctg} \sqrt{\frac{|ap+b|}{ax+b}} & [ap+b < 0]. \end{cases}$$

$$17. \quad \int \frac{\alpha \sqrt{ax+b} + \beta \sqrt{cx+d}}{\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}} dx = \frac{\alpha\gamma a + \beta\delta c}{\gamma^2 a + \delta^2 c} x - \frac{\alpha\delta - \beta\gamma}{\gamma^2 a + \delta^2 c} \sqrt{(ax+b)(cx+d)} +$$

$$+ \frac{2\gamma\delta(\alpha\delta - \beta\gamma)(ad-bc)}{(\gamma^2 a - \delta^2 c)^2} \ln |\gamma \sqrt{ax+b} + \delta \sqrt{cx+d}| -$$

$$- \frac{(\gamma^2 a + \delta^2 c)(\alpha\delta - \beta\gamma)(ad-bc)}{2(\gamma^2 a - \delta^2 c)^2} \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

1.2.21. Интегралы вида $\int R(x, \sqrt[n]{ax+b}) dx$.

$$1. \int R(x, \sqrt[n]{ax+b}) dx = \frac{n}{a} \int R\left(\frac{t^n-b}{a}, t\right) t^{n-1} dt \quad \left[t = \sqrt[n]{ax+b}\right].$$

$$2. \int x^p (ax+b)^{1/n} dx = \frac{n}{(np+n+1)a} x^p (ax+b)^{1+1/n} - \frac{bnp}{(np+n+1)a} \int x^{p-1} (ax+b)^{1/n} dx.$$

$$3. \int x^m (ax+b)^{1/n} dx = \frac{1}{a^{m+1}} (ax+b)^{1+1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1+1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{1/n} dx = \frac{n}{(n+1)a} (ax+b)^{1+1/n}.$$

$$5. \int \frac{(ax+b)^{1/n}}{x^m} dx = -\frac{(ax+b)^{1+1/n}}{(m-1)bx^{m-1}} - \frac{(mn-2n-1)a}{(m-1)nb} \int \frac{(ax+b)^{1/n}}{x^{m-1}} dx.$$

$$6. \int \frac{(ax+b)^{1/n}}{x} dx = n(ax+b)^{1/n} + nb^{1/n} \int \frac{dt}{t^{n+1}} \quad [t = (ax/b + 1)^{1/n}].$$

$$7. \int \frac{x^m dx}{(ax+b)^{1/n}} = \frac{n}{(mn+n-1)a} x^m (ax+b)^{1-1/n} - \frac{mn b}{(mn+n-1)a} \int \frac{x^{m-1} dx}{(ax+b)^{1/n}}$$

$$8. = \frac{1}{a^{m+1}} (ax+b)^{1-1/n} \sum_{k=0}^m \frac{(-b)^k}{m-k+1-1/n} \binom{m}{k} (ax+b)^{m-k}.$$

$$9. \int \frac{dx}{(ax+b)^{1/n}} = \frac{n}{(n-1)a} (ax+b)^{1-1/n}.$$

1.2.22. Интегралы вида $\int R[x, (ax+b)^{p/n}] dx$.

$$1. \int x^m (ax+b)^{p/n} dx = \frac{n}{(mn+n+p)a} x^m (ax+b)^{1+p/n} - \frac{mn b}{(mn+n+p)a} \int x^{m-1} (ax+b)^{p/n} dx.$$

$$2. = \frac{n}{a} (ax+b)^{1+p/n} \times \sum_{k=0}^m \frac{m(m-1)\dots(m-k+1)}{(mn+p+n)(mn+p)\dots(mn+n+p-kn)} \left(-\frac{bn}{a}\right)^k x^{m-k}.$$

$$3. = \frac{n}{a^{m+1}} (ax+b)^{1+p/n} \sum_{k=0}^m \frac{\binom{m}{k} (-b)^k}{(m-k+1)n+p} (ax+b)^{m-k}.$$

$$4. \int (ax+b)^{p/n} dx = \frac{n}{(n+p)a} (ax+b)^{1+p/n}.$$

$$5. \int \frac{(ax+b)^{p/n}}{(x-\alpha)^m} dx = -\frac{1}{(m-1)(a\alpha+b)} \frac{(ax+b)^{1+p/n}}{(x-\alpha)^{m-1}} - \frac{[(m-2)n-p]a}{(m-1)n(a\alpha+b)} \int \frac{(ax+b)^{p/n}}{(x-\alpha)^{m-1}} dx.$$

$$6. \quad = (ax+b)^{1+p/n} \times \\ \times \sum_{k=1}^{n-1} (-1)^k \frac{(m-p-2n)(mn-p-3n) \dots (mn-p-nk) a^{k-1}}{(m-1)(m-2) \dots (m-k) n^{k-1} (a\alpha+b)} \frac{1}{(x-\alpha)^{m-k}} - \\ - \frac{p(p+1) \dots (p+m-1) a^{m-1}}{n^{m-1} (m-1)! (a\alpha+b)^{m-1}} \int \frac{(ax+b)^{p/n}}{x-\alpha} dx.$$

$$7. \quad = \frac{-na^{m-1}}{(m-1)n-p} (ax+b)^{1-m+p/n} \quad [m\alpha+b=0].$$

$$8. \quad \int \frac{(ax+b)^{p/n}}{x-\alpha} dx = n(a\alpha+b)^{p/n} \int \frac{y^{n+p-1}}{y^n-1} dy \\ \left[y = \left(\frac{ax+b}{a\alpha+b} \right)^{1/n}; \quad p = \pm 1, \pm 2, \dots; \text{ см. 1.2.3} \right].$$

1.2.23. Интегралы вида $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$.

$$1. \quad \int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx = \frac{n(ad-bc)}{ac} \int R\left[\frac{-adt^n+bc}{ac(t^n-1)}, \sqrt[n]{\frac{a}{c}t}\right] \frac{t^{n-1}}{(t^n-1)^2} dt \\ \left[t = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}} \right].$$

$$2. \quad \int \frac{(ax+b)^{p/n}}{cx+d} dx = \frac{ax+b}{a} \frac{(ax+b)^{p/n}}{cx+d} + \frac{p(ad-bc)}{ac} \left(\frac{a}{c}\right)^{p/n} \int \frac{t^{n+p-1}}{t^n-1} dt \\ \left[t = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}}; \text{ см. 1.2.3} \right].$$

$$3. \quad \int x^m \frac{(ax+b)^{p/n}}{cx+d} dx = \frac{(ax+b)(cx+d)x^{m-1}}{(m+1)ac} \frac{(ax+b)^{p/n}}{cx+d} - \\ - \frac{(ad-bc)p+mn(ad+bc)}{(m+1)nac} \int x^{m-1} \frac{(ax+b)^{p/n}}{cx+d} dx - \\ - \frac{(m-1)bd}{(m+1)ac} \int x^{m-2} \frac{(ax+b)^{p/n}}{cx+d} dx.$$

$$4. \quad \int x \frac{(ax+b)^{p/n}}{cx+d} dx = \frac{(ax+b)(cx+d)}{2ac} \frac{(ax+b)^{p/n}}{cx+d} - \\ - \frac{(ad-bc)p+(ad+bc)n}{2nac} \int \frac{(ax+b)^{p/n}}{cx+d} dx.$$

$$5. \quad \int \frac{1}{(x-\alpha)^{m+1}} \frac{(ax+b)^{p/n}}{cx+d} dx = A \frac{(ax+b)(cx+d)}{(x-\alpha)^m} \frac{(ax+b)^{p/n}}{cx+d} + \\ + B \int \frac{1}{(x-\alpha)^m} \frac{(ax+b)^{p/n}}{cx+d} dx + D \int \frac{1}{(x-\alpha)^{m-1}} \frac{(ax+b)^{p/n}}{cx+d} dx, \\ A = \frac{-1}{m(a\alpha+b)(c\alpha+d)}, \quad B = \frac{-n(m-1)(ad+bc+2a\alpha)+p(ad-bc)}{mn(a\alpha+b)(c\alpha+d)},$$

$$D = \frac{-(m-2)ac}{m(a\alpha+b)(c\alpha+d)}.$$

$$6. \quad \int \frac{1}{x-\alpha} \frac{(ax+b)^{p/n}}{cx+d} dx = n \frac{(a\alpha+b)^{p/n}}{c\alpha+d} \int \frac{t^{p-1}}{t^n-1} dt - n \left(\frac{a}{c}\right)^{p/n} \int \frac{s^{p-1}}{s^n-1} ds \\ \left[t = \sqrt[n]{\frac{(a\alpha+b)(cx+d)}{(a\alpha+b)(c\alpha+d)}}, \quad s = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}}; \text{ см. 1.2.3} \right].$$

1.2.24. Интегралы вида $\int R(x^{1/n}, x^2 \pm a^2) dx$.

1. $\int \frac{x^{p/n} dx}{(x^2 \pm a^2)^{1/2}} = \frac{\pm x^{1+p/n}}{2(m-1)a^2(x^2 \pm a^2)^{m-1}} \pm \frac{(2m-3)n-p}{2(m-1)a^2n} \int \frac{x^{p/n} dx}{(x^2 \pm a^2)^{m-1}}.$
2. $= x^{1+p/n} \sum_{k=1}^{m-1} \frac{(2mn-3n-p)(2mn-5n-p) \dots (2mn-2nk+n-p)}{(m-1)(m-2) \dots (m-k)n^{k-1}(\pm 2a^2)^k} \times$
 $\times \frac{1}{(x^2 \pm a^2)^{m-k}} + \frac{(n-p)(3n-p) \dots (2mn-3n-p)}{(m-1)!(\pm 2a^2n)^{m-1}} \int \frac{x^{p/n} dx}{x^2 \pm a^2}.$
3. $\int \frac{x^{p/n} dx}{x^2 \pm a^2} = \frac{n}{p-n} x^{p/n-1} \mp a^2 \int \frac{x^{p/n-2} dx}{x^2 \pm a^2}.$
4. $\int \frac{dx}{x^{p/n}(x^2 \pm a^2)} = \frac{\pm nx^{1-p/n}}{(n-p)a^2} \mp \frac{1}{a^2} \int \frac{dx}{x^{p/n-2}(x^2 \pm a^2)}.$
5. $\int \frac{x^{p/n}}{x^2 \pm a^2} dx = \frac{1}{a^{1-p/n}} \sum_{k=0}^{n-1} \left(\sin \frac{4k+1}{2n} p\pi \ln |x^{1/n} - a^{1/n}| + \right.$
 $\left. + \cos \frac{4k+1}{2n} p\pi \operatorname{arctg} \frac{x^{1/n} - a^{1/n} \cos \frac{4k+1}{2n} \pi}{a^{1/n} \sin \frac{4k+1}{2n} \pi} \right) \quad [p = 1, 2, \dots, n-1; a > 0].$
6. $\int \frac{x^{p/n}}{x^2 - a^2} dx = \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n - 1} dt - \frac{na^{p/n-1}}{2} \int \frac{t^{n+p-1}}{t^n + 1} dt$
 $[t = (x/a)^{1/n}; p = \pm 1, \pm 2, \dots; a > 0; \text{ см. 1.2.3}].$

1.2.25. Интегралы вида $\int_a^x R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $x > a > b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{x-a}{x-b}}, k = \sqrt{\frac{b-c}{a-c}}.$

1. $\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$
2. $\int_a^x \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b)\Pi(\varphi, 1, k) + b^2 F(\varphi, k)].$
3. $\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^2(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k).$
4. $\int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] +$
 $+ \frac{2}{a-c} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$

$$\begin{aligned}
 5. \int_a^x \frac{dx}{(x-p) \sqrt{(x-a)(x-b)(x-c)}} = \\
 = \frac{2}{(b-p)(a-p) \sqrt{a-c}} \left[(b-a) \Pi \left(\varphi, \frac{b-p}{a-b}, k \right) + (a-p) F(\varphi, k) \right] \quad [p \neq a].
 \end{aligned}$$

$$\begin{aligned}
 6. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \\
 = \frac{2}{(a-b)(b-c)^3 \sqrt{a-c}} [(2a-b-c) E(\varphi, k) - 2(a-b) F(\varphi, k)] + \\
 + \frac{2}{(a-c)(b-c)} \sqrt{\frac{x-a}{(x-b)(x-c)}}.
 \end{aligned}$$

$$7. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)}} dx = -2 \sqrt{a-c} E(\varphi, k) + 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$\begin{aligned}
 8. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)}} dx = \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k) - 2 \sqrt{a-c} E(\varphi, k) + \\
 + 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.
 \end{aligned}$$

$$9. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)}} dx = 2 \sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$10. \int_a^x \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2 \sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)].$$

$$\begin{aligned}
 11. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \\
 = \frac{2 \sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2(a-b)}{(b-c) \sqrt{a-c}} F(\varphi, k) - 2 \sqrt{\frac{x-a}{(x-b)(x-c)}}.
 \end{aligned}$$

$$12. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k) - 2 \frac{b-c}{a-c} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$13. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2 \sqrt{a-c}}{a-b} E(\varphi, k).$$

$$\begin{aligned}
 14. \int_a^x \sqrt{\frac{(x-a)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] + \\
 + \frac{2}{3} (x+2c-a-2b) \sqrt{\frac{(x-a)(x-c)}{x-b}}.
 \end{aligned}$$

$$15. \int_a^x \sqrt{\frac{(x-a)(x-c)}{x-b}} dx = \frac{2}{3} \frac{\sqrt{(a-c)^3}}{b-c} [(a+c-2b) E(\varphi, k) - (a-b) F(\varphi, k)] + \\ + \frac{2(a-c)}{3(b-c)} (x+b-a-c) \sqrt{\frac{(x-a)(x-c)}{x-b}}.$$

$$16. \int_a^x \sqrt{\frac{(x-b)(x-c)}{x-a}} dx = \frac{2}{3} \sqrt{a-c} [2(a-b) F(\varphi, k) + (b+c-2a) E(\varphi, k)] + \\ + \frac{2}{3} (x+2a-2b-c) \sqrt{\frac{(x-a)(x-b)}{x-c}}.$$

1.2.26. Интегралы вида $\int_x^\infty R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $x \geq a > b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-c}{x-c}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].$$

$$3. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \\ - \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$4. \int_x^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \\ = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b) F(\varphi, k) - (a+b-2c) E(\varphi, k)] + \\ + \frac{2x-a-b}{(a-b)^2 \sqrt{(x-a)(x-b)(x-c)}} \quad [x > a].$$

$$\begin{aligned}
 6. \int_x^{\infty} \frac{dx}{\sqrt{(x-a)^2(x-b)(x-c)^3}} = \\
 = \frac{2}{(a-b)(b-c)\sqrt{a-c}} [(a+c-2b)E(\varphi, k) - (a-b)F(\varphi, k)] + \\
 + \frac{2}{(a-b)(a-c)} \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 7. \int_x^{\infty} \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \\
 = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(2a-b-c)E(\varphi, k) - 2(a-b)F(\varphi, k)] - \\
 - \frac{2}{(a-b)(b-c)} \sqrt{\frac{x-a}{(x-b)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_x^{\infty} \frac{dx}{\sqrt{(x-a)^2(x-b)^3(x-c)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
 \times [(a-b)(2a-b-c)F(\varphi, k) - 2(a^2+b^2+c^2-ab-ac-bc)E(\varphi, k)] + \\
 + \frac{2\{x(a+b-2c)-a(a-c)-b(b-c)\}}{(a-b)^2(a-c)(b-c)\sqrt{(x-a)(x-b)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_x^{\infty} \frac{dx}{(x-p)\sqrt{(x-a)(x-b)(x-c)}} = \\
 = \frac{2}{(p-c)\sqrt{a-c}} \left[\Pi\left(\varphi, \frac{p-c}{a-c}, k\right) - F(\varphi, k) \right].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_x^{\infty} \sqrt{\frac{x-c}{(x-a)^3(x-b)}} dx = \\
 = \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) + \frac{2(a-c)}{a-b} \sqrt{\frac{x-b}{(x-a)(x-c)}} \quad [x > a].
 \end{aligned}$$

$$\begin{aligned}
 11. \int_x^{\infty} \sqrt{\frac{x-b}{(x-a)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{x-b}{(x-a)(x-c)}} \\
 [x > a].
 \end{aligned}$$

$$12. \int_x^{\infty} \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - 2 \frac{b-c}{a-b} \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$13. \int_x^{\infty} \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{x-a}{(x-b)(x-c)}}.$$

$$14. \int_x^{\infty} \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k).$$

$$15. \int_x^{\infty} \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\varphi, k)$$

1.2.27. Интегралы вида $\int_x^a R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $a > x \geq b > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-x}{a-b}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\varphi, k) + 2 \frac{a}{b} \sqrt{a-c} E(\varphi, k).$$

$$3. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k) - \\ - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$4. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\varphi, k) - \\ - \frac{2}{(b-c)(a-c)} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$5. \int_x^a \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-p)\sqrt{a-c}} \Pi\left(\varphi, \frac{a-b}{a-p}, k\right) \quad [p \neq a].$$

$$6. \int_x^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \\ = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} [(b-c)F(\varphi, k) - 2(2a-b-c)E(\varphi, k)] + \\ + \frac{2(a-b-c+x)}{(a-b)(b-c)(a-c)} \sqrt{\frac{a-x}{(x-b)(x-c)}} \quad [x > b].$$

$$7. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)].$$

$$8. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\varphi, k),$$

$$9. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k),$$

$$10. \int_x^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{c-b} E(\varphi, k) + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$11. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \\ - \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$12. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-c} \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$13. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{x-b}} \quad [x > b].$$

$$14. \int_x^a \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

$$15. \int_x^a \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(a+c-2b)E(\varphi, k) + (b-c)F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

$$16. \int_x^a \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} \sqrt{(a-x)(x-b)(x-c)}.$$

1.2.28. Интегралы вида $\int_b^x R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}) dx$.

Условие: $a \geq x > b > c$.

Сбозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-b)}{(a-b)(x-c)}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_b^x \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c) \Pi(\varphi, k^2, k) + cF(\varphi, k)].$$

$$3. \int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad \{x < a\}.$$

$$4. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\varphi, k).$$

$$5. \int_b^x \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \\ = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} [(b-c)F(\varphi, k) - (2b-a-c)E(\varphi, k)] + \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad \{x < a\}.$$

$$6. \int_b^x \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)}} = \\ = \frac{2}{(c-p)(b-p)\sqrt{a-c}} \left[(c-b) \Pi\left(\varphi, k^2 \frac{c-p}{b-p}, k\right) + (b-p)F(\varphi, k) \right] \quad \{p \neq b\}.$$

$$7. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$8. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - \\ - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$9. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c} E(\varphi, k) - 2\sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$10. \int_b^x \sqrt{\frac{x-b}{(a-x)^2(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\varphi, k) + 2 \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

$$11. \int_b^x \sqrt{\frac{x-c}{(a-x)^2(x-b)}} dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k) - 1 - \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) + 2 \frac{a-c}{a-b} \sqrt{\frac{x-b}{(a-x)(x-c)}} \quad [x < a].$$

$$12. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$13. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$14. \int_b^x \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] + \\ + \frac{2}{3} (x+c-a-b) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$15. \int_b^x \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(b-c)F(\varphi, k) + (a+c-2b)E(\varphi, k)] + \\ + \frac{2}{3} (2b-a-2c+x) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

$$16. \int_b^x \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3} (b+2c-2a-x) \sqrt{\frac{(a-x)(x-b)}{x-c}}.$$

1.2.29. Интегралы вида $\int_x^b R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}) dx$.

Условие: $a > b > x \geq c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}}$, $k = \sqrt{\frac{b-c}{a-c}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c)\Pi(\varphi, k^2, k) + aF(\varphi, k)].$$

$$3. \int_x^b \frac{dx}{\sqrt{(a-x)^2(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\varphi, k),$$

$$4. \int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\varphi, l) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x > c],$$

$$5. \int_x^b \frac{dx}{\sqrt{(a-x)^2(b-x)(x-c)^3}} = \\ = \frac{2}{(b-c)(a-b)\sqrt{a-c}} \{ (a-b)F(\varphi, k) + (2b-a-c)E(\varphi, k) \} + \\ + \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad [x > c],$$

$$6. \int_x^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)}} = \\ = \frac{2}{(p-a)(p-b)\sqrt{a-c}} \left[(b-a) \Pi\left(\varphi, l; \frac{p-a}{p-b}, k\right) + (p-b)F(\varphi, k) \right] \quad [p \neq b],$$

$$7. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - 2 \sqrt{\frac{(b-x)(x-c)}{a-x}},$$

$$8. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k) - \\ - 2 \sqrt{\frac{(b-x)(x-c)}{a-x}},$$

$$9. \int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + \\ + 2 \sqrt{\frac{(b-x)(x-c)}{a-x}},$$

$$10. \int_x^b \sqrt{\frac{x-c}{(a-x)^2(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k),$$

$$11. \int_x^b \sqrt{\frac{b-x}{(a-x)^2(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)],$$

12. $\int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx =$
 $= 2 \frac{a-b}{(b-c) \sqrt{a-c}} F(\varphi, k) - \frac{2 \sqrt{a-c}}{b-c} E(\varphi, k) + 2 \frac{a-c}{b-c} \sqrt{\frac{b-x}{(x-x)(x-c)}}$
13. $\int_c^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\varphi, k) + 2 \sqrt{\frac{b-x}{(a-x)(x-c)}} \quad (x > c).$
14. $\int_x^b \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] +$
 $+ \frac{2}{3} (2c-2a-b+x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$
15. $\int_x^b \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(c-b) F(\varphi, k) + (2b-a-c) E(\varphi, k)] +$
 $+ \frac{2}{3} (2a+c-2b-x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$
16. $\int_x^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-a) F(\varphi, k) + (2a-b-c) E(\varphi, k)] +$
 $+ \frac{2}{3} (2c-b-x) \sqrt{\frac{(b-x)(x-c)}{a-x}}.$

1.2.30. Интегралы вида $\int_c^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}) dx$.

Условие: $a > b > x > c$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{x-c}{b-c}}, \quad k = \sqrt{\frac{b-c}{a-c}}.$

1. $\int_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$
2. $\int_c^x \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\varphi, k) - 2 \sqrt{a-c} E(\varphi, k).$
3. $\int_c^x \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b) \sqrt{a-c}} E(\varphi, k) -$
 $- \frac{2}{(a-b)(a-c)} \sqrt{\frac{(b-x)(x-c)}{a-x}}.$

$$4. \int_c^x \frac{dx}{(a-x)(b-x)^2(x-c)} = \frac{2}{(b-c)\sqrt{a-c}} F(\varphi, k) -$$

$$- \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) + \frac{2}{(a-b)(b-c)} \int \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

$$5. \int_c^x \frac{dx}{(a-x)^2(b-x)^2(x-c)} =$$

$$= \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a-b)F(\varphi, k) - (a+b-2c)E(\varphi, k)] +$$

$$+ \frac{2[a^2+b^2-ac-bc-x(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \int \sqrt{\frac{x-c}{(a-x)(b-x)}} \quad [x < b].$$

$$6. \int_c^x \frac{dx}{(p-x)(a-x)(b-x)(x-c)} =$$

$$= \frac{2}{(p-c)\sqrt{a-c}} \Pi\left(\varphi, \frac{b-c}{p-c}, k\right) \quad [p > c].$$

$$7. \int_c^x \int \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k).$$

$$8. \int_c^x \int \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\varphi, k) - \frac{2(a-b)}{\sqrt{a-c}} F(\varphi, k).$$

$$9. \int_c^x \int \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_c^x \int \sqrt{\frac{x-c}{(a-x)^2(b-x)}} dx =$$

$$= \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2}{a-b} \int \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$11. \int_c^x \int \sqrt{\frac{b-x}{(a-x)^2(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] +$$

$$+ \frac{2}{a-c} \int \sqrt{\frac{(b-x)(x-c)}{a-x}}.$$

$$12. \int_c^x \int \sqrt{\frac{a-x}{(b-x)^2(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\varphi, k) - E(\varphi, k)] +$$

$$+ \frac{2}{b-c} \int \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad [x < b].$$

$$13. \int_x^c \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = -\frac{2\sqrt{a-c}}{a-b} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{b-x}} \quad (x < b).$$

$$14. \int_c^x \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c) E(\varphi, k) - (a-b) F(\varphi, k)] + \\ + \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

$$15. \int_x^c \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c) E(\varphi, k) + (a-b) F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

$$16. \int_c^x \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c) E(\varphi, k) - 2(a-b) F(\varphi, k)] - \\ - \frac{2}{3} \sqrt{(a-x)(b-x)(x-c)}.$$

1.2.31. Интегралы вида $\int_x^c R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}) dx$.

Условие: $a > b > c > x$.

Обозначения: $\varphi = \operatorname{arcsin} \sqrt{\frac{c-x}{b-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_x^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)}} = \\ = \frac{2}{\sqrt{a-c}} [cF(\varphi, k) + (a-c) E(\varphi, k)] - 2 \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$3. \int_x^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \\ = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$4. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)^2(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k).$$

$$5. \int_x^c \frac{dx}{\sqrt{(a-x)^2(b-x)^2(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} [(a+b-2c)E(\varphi, k) - 2(b-c)F(\varphi, k)] + \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$6. \int_x^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(p-b)(p-c)\sqrt{a-c}} \times \\ \times \Pi\left(\varphi, \frac{p-b}{p-c}, k\right) + \frac{2}{(p-b)\sqrt{a-c}} F(\varphi, k) \quad (p \neq c).$$

$$7. \int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)}} dx = 2\sqrt{a-c} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$8. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)}} dx = \frac{2(b-c)}{\sqrt{a-c}} F(\varphi, k) - 2\sqrt{a-c} E(\varphi, k) + \\ + 2 \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$9. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)}} dx = -2\sqrt{a-c} E(\varphi, k) + 2 \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$10. \int_x^c \sqrt{\frac{b-x}{(a-x)^2(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k) - \frac{2(a-b)}{a-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$11. \int_x^c \sqrt{\frac{c-x}{(a-x)^2(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \\ - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k) - 2 \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$12. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^2}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)].$$

$$13. \int_x^c \sqrt{\frac{a-x}{(b-x)^2(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k).$$

$$14. \int_x^c \sqrt{\frac{(a-x)(b-x)}{c-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-c)F(\varphi, k) + \\ + (2-a-b)E(\varphi, k)] + \frac{2}{3}(a+2b-2c-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$15. \int_x^c \sqrt{\frac{(a-x)(c-x)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3}(a+c-b-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

$$16. \int_x^c \sqrt{\frac{(b-x)(c-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\varphi, k) - (b-c)F(\varphi, k)] + \\ + \frac{2}{3}(2b-2a+c-x) \sqrt{\frac{(a-x)(c-x)}{b-x}}.$$

1.2.32. Интегралы вида $\int_{-\infty}^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}) dx$.

Условие: $a > b > c \geq x$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-c}{a-x}}$, $k = \sqrt{\frac{a-b}{a-c}}$.

$$1. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\varphi, k).$$

$$2. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^2(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$3. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)^2(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\varphi, k) - \\ - \frac{2}{(a-b)\sqrt{a-c}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$4. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \\ = \frac{2}{(c-b)\sqrt{c-c}} E(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad \{x < c\}.$$

$$5. \int_{-\infty}^x \frac{dx}{(\nu-x)\sqrt{(a-x)(b-x)(c-x)}} = \\ = \frac{2}{(a-p)\sqrt{a-c}} \left[\Pi\left(\varphi, \frac{a-p}{a-c}, k\right) - F(\varphi, k) \right].$$

$$6. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^2(b-x)^2(c-x)}} =$$

$$= \frac{2}{(b-c)(a-b)^2 \sqrt{a-c}} [(a+b-2c) E(\varphi, k) - 2(b-c) F(\varphi, k)] -$$

$$- \frac{2}{(a-b)(b-c)} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$7. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)(b-x)^2(c-x)^2}} =$$

$$= \frac{2}{(a-b)(b-c)^2 \sqrt{a-c}} [(b-c) F(\varphi, k) - (a+b-2c) E(\varphi, k)] +$$

$$+ \frac{2(b+c-2x)}{(b-c)^2 \sqrt{(a-x)(b-x)(c-x)}} \quad [x < c].$$

$$8. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^2(b-x)(c-x)^2}} =$$

$$= \frac{2}{(a-b)(b-c) \sqrt{(a-c)^3}} [(2b-a-c) E(\varphi, k) - (b-c) F(\varphi, k)] +$$

$$+ \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

$$9. \int_{-\infty}^x \frac{dx}{\sqrt{(a-x)^2(b-x)^2(c-x)^2}} = \frac{2}{(a-b)^2(b-c)^2 \sqrt{(a-c)^3}} \times$$

$$\times [(b-c)(a+b-2c) F(\varphi, k) - 2(c^2+a^2+b^2-ab-ac-bc) E(\varphi, k)] +$$

$$+ \frac{2[c(a-c)+b(a-b)-x(2a-c-b)]}{(a-b)(a-c)(b-c)^2 \sqrt{(a-x)(b-x)(c-x)}} \quad [x < c].$$

$$10. \int_{-\infty}^x \sqrt{\frac{b-x}{(a-x)^2(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\varphi, k).$$

$$11. \int_{-\infty}^x \sqrt{\frac{c-x}{(a-x)^2(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\varphi, k) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\varphi, k).$$

$$12. \int_{-\infty}^x \sqrt{\frac{a-x}{(b-x)(c-x)^2}} dx = \frac{2}{\sqrt{a-c}} F(\varphi, k) - \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) +$$

$$+ \frac{2(a-c)}{b-c} \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

$$13. \int_{-\infty}^x \sqrt{\frac{a-x}{(b-x)^2(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\varphi, k) - \frac{2(a-b)}{b-c} \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

$$14. \int_{-\infty}^x \sqrt{\frac{b-x}{(a-x)(c-x)^3}} dx =$$

$$= \frac{2}{\sqrt{a-c}} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{b-x}{(a-x)(c-x)}} \quad [x < c].$$

$$15. \int_{-\infty}^x \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx =$$

$$= \frac{2\sqrt{a-c}}{a-b} [F(\varphi, k) - E(\varphi, k)] + 2 \sqrt{\frac{c-x}{(a-x)(b-x)}}.$$

1.2.33. Интегралы вида $\int_a^x R(\sqrt{x-a}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условие: $x > a > b > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(x-a)}{(a-d)(x-b)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_a^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_a^x \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\varphi, \frac{a-d}{b-d}, k \right) + b F(\varphi, k) \right].$$

$$3. \int_a^x \frac{dx}{x \sqrt{(x-a)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{ab \sqrt{(a-c)(b-d)}} \left\{ (b-a) \Pi \left[\varphi, \frac{b(a-d)}{a(b-d)}, k \right] + a F(\varphi, k) \right\}.$$

$$4. \int_a^x \frac{dx}{(p-x) \sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (a-b) \Pi \left[\varphi, \frac{(a-d)(p-b)}{(b-d)(p-a)}, k \right] + (p-a) F(\varphi, k) \right\}.$$

$$5. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{a-d}{b-d}, k \right) - F(\varphi, k) \right].$$

$$6. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{a-d}{b-d}, k \right).$$

$$7. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\varphi, \frac{a-d}{b-d}, k \right) + (b-c) F(\varphi, k) \right].$$

$$8. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b) \Pi \left(\varphi, \frac{a-d}{b-d}, k \right) + (b-d) F(\varphi, k) \right].$$

$$9. \int_a^x \sqrt{\frac{x-a}{(x-b)^2(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)^2(x-d)}} dx = \frac{2 \sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \\ - \frac{2(a-b)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{c-d} \sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

$$11. \int_a^x \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \frac{2}{c-d} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

$$12. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}} + \\ + \frac{2(a-b)}{(a-d) \sqrt{(a-c)(b-d)}} F(\varphi, k) + \frac{2 \sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\varphi, k).$$

$$13. \int_a^x \sqrt{\frac{x-b}{(x-a)(x-c)^2(x-d)}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(b-c)}{(a-c)(c-d)} \sqrt{\frac{(x-a)(x-d)}{(x-b)(x-c)}}.$$

$$14. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)^2(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$15. \int_a^x \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d} \sqrt{\frac{(x-a)(x-c)}{(x-b)(x-d)}}.$$

$$16. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)^2(x-c)}} dx =$$

$$= \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) - \frac{2(c-d)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$17. \int_a^x \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^2}} dx =$$

$$= \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(x-d)(x-a)}{(x-b)(x-c)}}.$$

1.2.34. Интегралы вида $\int_a^x R(\sqrt{a-x}, \sqrt{x-b}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условия: $a > x \geq b > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(a-x)}{(a-b)(x-d)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(x-c)(b-d)}}$.

$$1. \int_a^x \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{b-a}{b-d}, k \right) + d F(\varphi, k) \right].$$

$$2. \int_a^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$3. \int_a^x \frac{dx}{x \sqrt{(a-x)(x-b)(x-c)(x-d)}} =$$

$$= \frac{2}{a d \sqrt{(a-c)(b-d)}} \left\{ (d-a) \Pi \left[\varphi, \frac{d(b-a)}{a(b-d)}, k \right] + a F(\varphi, k) \right\}.$$

$$4. \int_a^x \frac{dx}{(p-x) \sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (a-d) \Pi \left[\varphi, \frac{(b-a)(p-d)}{(b-d)(p-a)}, k \right] + (p-a) F(\varphi, k) \right\} \quad [p \neq a].$$

$$5. \int_a^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{b-a}{b-d}, k \right) - F(\varphi, k) \right].$$

$$6. \int_a^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi \left(\varphi, \frac{b-a}{b-d}, k \right) - (b-d) F(\varphi, k) \right].$$

$$7. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{b-a}{b-d}, k \right) + (d-c) F(\varphi, k) \right].$$

$$8. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{b-a}{b-d}, k \right).$$

$$9. \int_x^a \sqrt{\frac{a-x}{(x-b)^2(x-c)(x-d)}} dx = \frac{2}{c-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}} \quad [x > b].$$

$$10. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)^2(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \\ - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{b-c} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$11. \int_x^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^2}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$12. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)^2(x-d)}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$13. \int_x^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^2}} dx = \\ = 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\varphi, k) - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$14. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)^2(x-d)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}}.$$

$$15. \int_x^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^2}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$16. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2\sqrt{(a-b)(b-d)}}{(a-b)(b-c)} E(\varphi, k) + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{(x-b)(x-d)}} \quad [x > b].$$

$$17. \int_x^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \\ = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(c-d)}{(a-c)(b-c)} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

1.2.35. Интегралы вида $\int_b^x R \sqrt{a-x} \sqrt{x-b} \sqrt{x-c} \sqrt{x-d} dx$.

Условие: $a > x > b > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-b)}{(a-b)(x-c)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_b^x \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_b^x \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi \left(\varphi, \frac{a-b}{a-c}, k \right) + c F(\varphi, k) \right].$$

$$3. \int_b^x \frac{dx}{x \sqrt{(a-x)(x-b)(x-c)(x-d)}} = \\ = \frac{2}{bc \sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left[\varphi, \frac{c(a-b)}{b(a-c)}, k \right] + b F(\varphi, k) \right\}.$$

$$4. \int_b^x \frac{dx}{(x-p) \sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(b-p)(p-c) \sqrt{(a-c)(b-d)}} \times \\ \times \left\{ (b-c) \Pi \left[\varphi, \frac{(a-b)(p-c)}{(a-c)(p-b)}, k \right] + (p-b) F(\varphi, k) \right\} \quad (p \neq b).$$

$$5. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi \left(\varphi, \frac{a-b}{a-c}, k \right) + (a-c) F(\varphi, k) \right].$$

$$6. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{a-b}{a-c}, k \right) - F(\varphi, k) \right].$$

$$7. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2(b-c)}{(a-c)(b-d)} \Pi\left(\varphi, \frac{a-b}{a-c}, k\right).$$

$$8. \int_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \\ = \frac{2}{(a-c)(b-d)} \left[(b-c) \Pi\left(\varphi, \frac{a-b}{a-c}, k\right) + (c-d) F(\varphi, k) \right].$$

$$9. \int_a^x \sqrt{\frac{x-d}{(a-x)^2(x-b)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{a-b} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}} \quad [x < a].$$

$$10. \int_b^x \sqrt{\frac{a-x}{(x-b)(x-c)^2(x-d)}} dx = \\ = \frac{2}{(b-c)(c-d)} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-d)}{(c-d)(a-c)(b-d)} F(\varphi, k).$$

$$11. \int_a^x \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^2}} dx = \\ = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$12. \int_a^x \sqrt{\frac{x-b}{(a-x)^2(x-c)(x-d)}} dx = \\ = \frac{2}{d-a} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{a-d} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}}.$$

$$13. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)^2(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$14. \int_b^x \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^2}} dx = \frac{2}{(a-d)(c-d)(a-c)(b-d)} \times \\ \times [(a-c)(b-d)F(\varphi, k) - (a-d)(b-c)F(\varphi, k)] - \frac{2}{a-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}}.$$

$$15. \int_b^x \sqrt{\frac{x-c}{(a-x)^2(x-b)(x-d)}} dx = \frac{2(b-c)}{(a-b)(a-c)(b-d)} F(\varphi, k) - \\ - \frac{2}{(a-b)(a-d)} \sqrt{\frac{(a-c)(b-d)}{(a-b)(a-d)}} E(\varphi, k) + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(x-b)(x-d)}{(a-x)(x-c)}} \quad [x < a].$$

$$16. \int_b^x \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx =$$

$$= \frac{2}{a-d} \left[\sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{c-d}{b-d} \sqrt{\frac{(a-x)(x-b)}{(x-c)(x-d)}} \right].$$

$$17. \int_b^x \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

1.2.36. Интегралы вида $\int_x^b R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}, \sqrt{x-d}) dx$

Условие: $a > b > x > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(b-x)}{(b-c)(a-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_x^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + aF(\varphi, k) \right].$$

$$3. \int_x^b \frac{dx}{x \sqrt{(a-x)(b-x)(x-c)(x-d)}} =$$

$$= \frac{2}{ab \sqrt{(a-c)(b-d)}} \left\{ (a-b) \Pi\left[\varphi, \frac{a(b-c)}{b(a-c)}, k\right] + bF(\varphi, k) \right\}.$$

$$4. \int_x^b \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (b-a) \Pi\left[\varphi, \frac{(b-c)(p-a)}{(a-c)(p-b)}, k\right] + (p-b) F(\varphi, k) \right\} \quad (p \neq b).$$

$$5. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{b-c}{a-c}, k\right).$$

$$6. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\varphi, \frac{b-c}{a-c}, k\right) - F(\varphi, k) \right].$$

$$7. \int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi\left(\varphi, \frac{b-c}{a-c}, k\right) + (a-c) F(\varphi, k) \right].$$

- $$8. \int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx =$$
- $$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a) \Pi \left(\varphi, \frac{b-c}{a-c}, k \right) + (a-d) F(\varphi, k) \right].$$
- $$9. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)^2(x-d)}} dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\varphi, k) -$$
- $$- 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) + \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$
- $$10. \int_x^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^2}} dx =$$
- $$= \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-d)}{(b-d)(c-d)} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$
- $$11. \int_x^b \sqrt{\frac{b-x}{(a-x)^2(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$
- $$12. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)^2(x-d)}} dx =$$
- $$= \frac{2}{d-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{c-d} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad [x > c].$$
- $$13. \int_x^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^2}} dx =$$
- $$= \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} [(a-c)(b-d)E(\varphi, k) - (a-b)(c-d)F(\varphi, k)] -$$
- $$- \frac{2}{c-d} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$
- $$14. \int_x^b \sqrt{\frac{x-c}{(a-x)^2(b-x)(x-d)}} dx =$$
- $$= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$
- $$15. \int_x^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^2}} dx =$$
- $$= \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$16. \int_a^b \sqrt{\frac{x-d}{(a-x)^2(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

$$17. \int_x^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{(b-x)(x-d)}{(a-x)(x-c)}} \quad \{x > c\}.$$

1.2.37. Интегралы вида $\int_c^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{x-c}, \sqrt{x-d}) dx$.

Условия: $a > b > x > c > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-d)(x-c)}{(b-c)(x-d)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_c^x \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_c^x \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\varphi, \frac{b-c}{b-d}, k \right) + d F(\varphi, k) \right].$$

$$3. \int_c^x \frac{dx}{x \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \\ = \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left[\varphi, \frac{d(b-c)}{c(b-d)}, k \right] + c F(\varphi, k) \right\}.$$

$$4. \int_c^x \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d) \sqrt{(a-c)(b-d)}} \times \\ \times \left\{ (c-d) \Pi \left[\varphi, \frac{(b-c)(p-d)}{(b-d)(p-c)}, k \right] + (p-c) F(\varphi, k) \right\} \quad \{p \neq c\}.$$

$$5. \int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi \left(\varphi, \frac{b-c}{b-d}, k \right) + (a-d) F(\varphi, k) \right].$$

$$6. \int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi \left(\varphi, \frac{b-c}{b-d}, k \right) + (b-d) F(\varphi, k) \right].$$

$$7. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2(c-d)}{(a-c)(b-d)} \left[\Pi\left(\varphi, \frac{b-c}{b-d}, k\right) - F(\varphi, k) \right].$$

$$8. \int_c^x \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{(a-c)(b-d)} \Pi\left(\varphi, \frac{b-c}{b-d}, k\right).$$

$$9. \int_c^x \sqrt{\frac{a-x}{(b-x)^2(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{b-c} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}} \quad [x < b].$$

$$10. \int_c^x \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$11. \int_c^x \sqrt{\frac{b-x}{(a-x)^2(x-c)(x-d)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$12. \int_c^x \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2}{(a-d)(c-d)} \sqrt{\frac{a-c}{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d)E(\varphi, k) - (a-b)(c-d)F(\varphi, k)].$$

$$13. \int_c^x \sqrt{\frac{x-c}{(a-x)^2(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \\ - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$14. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)^2(x-d)}} dx = \frac{2}{b-a} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) + \\ + \frac{2}{a-b} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}}.$$

$$15. \int_c^x \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^2}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$16. \int_c^x \sqrt{\frac{x-d}{(a-x)^2(b-x)(x-c)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(a-d)}{(a-b)(a-c)} \sqrt{\frac{(b-x)(x-c)}{(a-x)(x-d)}}.$$

$$17. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-x)(x-c)}{(b-x)(x-d)}} \quad \{x < b\}.$$

1.2.38. Интегралы вида $\int R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}) dx$.

Условие: $a > b > c > x > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(b-d)(c-x)}{(c-d)(b-x)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_x^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_x^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \times \\ \times \left[(c-b) \Pi \left(\varphi, \frac{c-d}{b-d}, k \right) + b F(\varphi, k) \right].$$

$$3. \int_x^c \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \\ = \frac{2}{bc \sqrt{(a-c)(b-d)}} \left\{ (b-c) \Pi \left[\varphi, \frac{b(c-d)}{c(b-d)}, k \right] + c F(\varphi, k) \right\}.$$

$$4. \int_x^c \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c) \sqrt{(a-c)(b-d)}} \times \\ \times \left\{ (c-b) \Pi \left[\varphi, \frac{(c-d)(p-b)}{(b-d)(p-c)}, k \right] + (p-c) F(\varphi, k) \right\} \quad \{p \neq c\}.$$

$$5. \int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi \left(\varphi, \frac{c-d}{b-d}, k \right) + (a-b) F(\varphi, k) \right].$$

$$6. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{c-d}{b-d}, k \right).$$

$$7. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{c-d}{b-d}, k \right) - F(\varphi, k) \right].$$

$$8. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi \left(\varphi, \frac{c-d}{b-d}, k \right) + (b-d) F(\varphi, k) \right]$$

$$9. \int_x^c \sqrt{\frac{a-x}{(b-x)^2(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\varphi, k).$$

$$10. \int_x^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{2}{c-d} \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \quad [x > d].$$

$$11. \int_x^c \sqrt{\frac{b-x}{(a-x)^2(c-x)(x-d)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) - \frac{2(a-b)}{(a-c)(a-d)} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$12. \int_x^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}} dx = \\ = \frac{2}{(a-d)(c-d) \sqrt{(a-c)(b-d)}} [(b-c)(a-d) F(\varphi, k) - (a-c)(b-d) E(\varphi, k)] + \\ + \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \quad [x > d].$$

$$13. \int_x^c \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)}} dx = \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} F(\varphi, k) - \\ - \frac{2(b-c)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{a-d} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$14. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)^2(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)].$$

$$15. \int_x^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}} dx = \\ = \frac{2}{a-d} \left[\sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \sqrt{\frac{(a-x)(c-x)}{(b-x)(x-d)}} \right] \quad [x > d].$$

$$16. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$

$$= \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$17. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)^2(c-x)}} dx =$$

$$= \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\varphi, k) - \frac{2(a-d)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$

1.2.39. Интегралы вида $\int_d^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}) dx$.

Условия: $a > b > c > x > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-d)}{(c-d)(a-x)}}$, $k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_d^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_d^x \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) + aF(\varphi, k) \right].$$

$$3. \int_d^x \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{ad \sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi\left[\varphi, \frac{a(d-c)}{d(a-c)}, k\right] + dF(\varphi, k) \right\}.$$

$$4. \int_d^x \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (d-a) \Pi\left[\varphi, \frac{(d-c)(p-a)}{(a-c)(p-d)}, k\right] + (p-d) F(\varphi, k) \right\} \quad (p \neq d).$$

$$5. \int_d^x \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{d-c}{a-c}, k\right).$$

$$6. \int_d^x \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) - (a-b) F(\varphi, k) \right].$$

$$7. \int_a^x \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - (a-c) F(\varphi, k) \right].$$

$$8. \int_a^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - F(\varphi, k) \right].$$

$$9. \int_a^x \sqrt{\frac{a-x}{(b-x)^2(c-x)(x-d)}} dx = -\frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(x-d)(c-x)}{(a-x)(b-x)}}.$$

$$10. \int_a^x \sqrt{\frac{a-x}{(b-x)(c-x)^2(x-d)}} dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\varphi, k) - 2\sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\varphi, k) + 2\frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$$

$$11. \int_a^x \sqrt{\frac{b-x}{(a-x)^2(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

$$12. \int_a^x \sqrt{\frac{b-x}{(a-x)(c-x)^2(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{c-d} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$$

$$13. \int_a^x \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$14. \int_a^x \sqrt{\frac{c-x}{(a-x)(b-x)^2(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$15. \int_a^x \sqrt{\frac{x-d}{(a-x)^2(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$16. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$

$$= \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [E(\varphi, k) - E(\varphi, k)] + \frac{2}{a-c} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$17. \int_x^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$

$$= \frac{2}{(a-b)(b-c)} E(\varphi, k) - \frac{2(a-d)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$

1.2.39. Интегралы вида $\int_d^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{x-d}) dx$.

Условия: $a > b > c > x > d$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(x-d)}{(c-d)(a-x)}}$, $k = \sqrt{\frac{(c-b)(c-d)}{(a-c)(b-d)}}$.

$$1. \int_d^x \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_d^x \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) + a F(\varphi, k) \right].$$

$$3. \int_d^x \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} =$$

$$= \frac{2}{ad \sqrt{(a-c)(b-d)}} \left\{ (a-d) \Pi\left[\varphi, \frac{a(d-c)}{d(a-c)}, k\right] + d F(\varphi, k) \right\}.$$

$$4. \int_d^x \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (d-a) \Pi\left[\varphi, \frac{(d-c)(p-a)}{(a-c)(p-d)}, k\right] + (p-d) F(\varphi, k) \right\} \quad [p \neq d].$$

$$5. \int_d^x \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\varphi, \frac{d-c}{a-c}, k\right).$$

$$6. \int_d^x \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\varphi, \frac{d-c}{a-c}, k\right) - (a-b) F(\varphi, k) \right].$$

- $$7. \int_a^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx =$$
- $$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - (a-c) F(\varphi, k) \right].$$
- $$8. \int_a^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$
- $$= \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{d-c}{a-c}, k \right) - F(\varphi, k) \right].$$
- $$9. \int_a^x \sqrt{\frac{a-x}{(b-x)^2(c-x)(x-d)}} dx =$$
- $$= \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(x-d)(c-x)}{(a-x)(b-x)}}.$$
- $$10. \int_a^x \sqrt{\frac{a-x}{(b-x)(c-x)^2(x-d)}} dx = \frac{2(a-d)}{(c-d) \sqrt{(a-c)(b-d)}} F(\varphi, k) -$$
- $$- 2 \sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\varphi, k) + 2 \frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$$
- $$11. \int_a^x \sqrt{\frac{b-x}{(a-x)^2(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$
- $$12. \int_a^x \sqrt{\frac{b-x}{(a-x)(c-x)^2(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] +$$
- $$+ \frac{2}{c-d} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}} \quad [x < c].$$
- $$13. \int_a^x \sqrt{\frac{c-x}{(a-x)^2(b-x)(x-d)}} dx =$$
- $$= \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) - \frac{2(b-c)}{(a-b) \sqrt{(a-c)(b-d)}} F(\varphi, k).$$
- $$14. \int_a^x \sqrt{\frac{c-x}{(a-x)(b-x)^2(x-d)}} dx =$$
- $$= \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$
- $$15. \int_a^x \sqrt{\frac{x-d}{(a-x)^2(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)].$$

$$16. \int_a^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-c)(b-d)}{(a-b)(b-c)}} E(\varphi, k) -$$

$$- \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{(c-x)(x-d)}{(a-x)(b-x)}}.$$

$$17. \int_a^x \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx =$$

$$= \frac{-2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{b-c} \sqrt{\frac{(b-x)(x-d)}{(a-x)(c-x)}}.$$

1.2.40. Интегралы вида $\int_a^x R(\sqrt{a-x}, \sqrt{b-x}, \sqrt{c-x}, \sqrt{d+x}) dx$.

Условие: $a > b > c > d > x$.

Сбозначения: $\varphi = \arcsin \sqrt{\frac{(a-c)(d-x)}{(a-d)(c-x)}}$, $k = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}$.

$$1. \int_a^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$2. \int_a^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[c \Pi\left(\varphi, \frac{a-d}{a-c}, k\right) - (c-d) F(\varphi, k) \right].$$

$$3. \int_a^d \frac{cx}{x \sqrt{(a-x)(b-x)(c-x)(d-x)}} =$$

$$= \frac{2}{cd \sqrt{(a-c)(b-d)}} \left\{ (c-d) \Pi\left[\varphi, \frac{c(a-d)}{d(a-c)}, k\right] + d F(\varphi, k) \right\}.$$

$$4. \int_a^d \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{(p-c)(p-d) \sqrt{(a-c)(b-d)}} \times$$

$$\times \left\{ (d-c) \Pi\left[\varphi, \frac{(a-d)(p-c)}{(a-c)(p-d)}, k\right] + (p-d) F(\varphi, k) \right\}$$

(p ≠ d).

$$5. \int_a^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx =$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi\left(\varphi, \frac{a-d}{a-c}, k\right) + (a-c) F(\varphi, k) \right].$$

$$6. \int_a^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \\ = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\varphi, \frac{a-d}{a-c}, k \right) + (b-c) F(\varphi, k) \right].$$

$$7. \int_a^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\varphi, \frac{a-d}{a-c}, k \right).$$

$$8. \int_a^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \\ = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\varphi, \frac{a-d}{a-c}, k \right) - F(\varphi, k) \right].$$

$$9. \int_a^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \\ = \frac{2}{b-d} \sqrt{\frac{a-c}{b-d}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{b-d} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$10. \int_a^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \\ = \frac{2 \sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\varphi, k) - \frac{a-b}{b-c} \frac{2}{\sqrt{(a-c)(b-d)}} F(\varphi, k).$$

$$11. \int_a^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \\ = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\varphi, k) - E(\varphi, k)] + \frac{2}{a-d} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$12. \int_a^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\varphi, k).$$

$$13. \int_a^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{(a-d) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \\ - \frac{2 \sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\varphi, k) + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$14. \int_a^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\varphi, k) - \frac{2(b-c)}{(a-b)(b-d)} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$15. \int_x^d \sqrt{\frac{d-x}{(a-x)^2(b-x)(c-x)}} dx = \\ = \frac{2}{b-a} \sqrt{\frac{b-d}{a-c}} E(\varphi, k) + \frac{2}{a-b} \sqrt{\frac{(b-x)(d-x)}{(a-x)(c-x)}}.$$

$$16. \int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)^2(c-x)}} dx = \frac{2}{(a-b)(b-c)} E(\varphi, k) - \\ - \frac{2(c-d)}{(b-c) \sqrt{(a-c)(b-d)}} F(\varphi, k) - \frac{2}{a-b} \sqrt{\frac{(a-x)(d-x)}{(b-x)(c-x)}}.$$

$$17. \int_x^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^2}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} \{F(\varphi, k) - E(\varphi, k)\}.$$

1.2.41. Интегралы вида $\int x^m (x^2 \pm a^2)^{n+1/2} dx$.

$$1. \int x^{2n} (x^2 \pm a^2)^{n+1/2} dx = \\ = \frac{x^{2n+1} (x^2 \pm a^2)^{n+3/2}}{m+2n+2} + \frac{(m-1)a^2}{m+2n+2} \int x^{2n-2} (x^2 \pm a^2)^{n+1/2} dx.$$

$$2. \int x^{2m} (x^2 \pm a^2)^{n+1/2} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+2m+2} \times \\ \times \left[x^{2m-1} + \sum_{k=1}^{m-1} (-1)^k \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m+n)(m+n-1) \dots (m+n-k+1)} a^{2k} x^{2m-2k-1} \right] + \\ + (-1)^m \frac{a^{2m} (2m-1)!!}{2^m (n+m+1)(n+m) \dots (n+2)} \int (x^2 \pm a^2)^{n+1/2} dx.$$

$$3. \int x^{2m+1} (x^2 \pm a^2)^{n+1/2} dx = \sum_{k=0}^m \binom{m}{k} \frac{(\pm a^2)^{m-k} (x^2 \pm a^2)^{n+k+3/2}}{2n+2k+3}.$$

$$4. \int x (x^2 \pm a^2)^{n+1/2} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{2n+3}.$$

$$5. \int (x^2 \pm a^2)^{n+1/2} dx = \frac{x(x^2 \pm a^2)^{n+1/2}}{2(n+1)} \pm \frac{(2n+1)a^2}{2(n+1)} \int (x^2 \pm a^2)^{n-1/2} dx.$$

$$6. = \frac{x(x^2 \pm a^2)^{1/2}}{2n+2} \times \\ \times \left[(x^2 \pm a^2)^{n+1} + \sum_{k=1}^n (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2^k n(n-1) \dots (n-k+1)} a^{2k} (x^2 \pm a^2)^{n-k} \right] + \\ + \frac{(-1)^{n+1} (2n+1)!!}{2^{n+1} (n+1)!} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$7. \int x^{2m} (x^2 \pm a^2)^{1/2} dx = \frac{(x^2 \pm a^2)^{3/2}}{2(m+1)} \times \\ \times \left[x^{2m+1} + \sum_{k=1}^{m-1} (-1)^k \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k m (m-1) \dots (m-k+1)} a^{2k} x^{2m-2k+1} \right] + \\ + (-1)^m \frac{a^{2m} (2m-1)!!}{2^m (m+1)!} \int (x^2 \pm a^2)^{1/2} dx.$$

$$8. \int (x^2 \pm a^2)^{1/2} dx = \frac{1}{2} x (x^2 \pm a^2)^{1/2} \pm \frac{a^2}{2} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$9. \int x (x^2 \pm a^2)^{1/2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2}.$$

$$10. \int x^2 (x^2 \pm a^2)^{1/2} dx = \\ = \frac{1}{4} x (x^2 \pm a^2)^{3/2} \mp \frac{a^2}{8} x (x^2 \pm a^2)^{1/2} - \frac{a^4}{8} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$11. \int x^2 (x^2 \pm a^2)^{1/2} dx = \frac{1}{5} (x^2 \pm a^2)^{5/2} \mp \frac{a^2}{3} (x^2 \pm a^2)^{3/2}.$$

$$12. \int (x^2 \pm a^2)^{3/2} dx = \\ = \frac{1}{4} x (x^2 \pm a^2)^{3/2} \mp \frac{3}{8} a^2 x (x^2 \pm a^2)^{1/2} + \frac{3}{8} a^4 \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$13. \int x (x^2 \pm a^2)^{3/2} dx = \frac{1}{5} (x^2 \pm a^2)^{5/2}.$$

$$14. \int x^2 (x^2 \pm a^2)^{3/2} dx = \\ = \frac{1}{6} x (x^2 \pm a^2)^{5/2} \mp \frac{a^2}{24} x (x^2 \pm a^2)^{3/2} - \frac{a^4}{16} (x^2 \pm a^2)^{1/2} \mp \\ \mp \frac{a^6}{16} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$15. \int x^3 (x^2 \pm a^2)^{3/2} dx = \frac{1}{7} (x^2 \pm a^2)^{7/2} \mp \frac{a^2}{5} (x^2 \pm a^2)^{5/2}.$$

1.2.42. Интегралы вида $\int \frac{(x^2 \pm a^2)^{n+1/2}}{x^m} dx.$

$$1. \int \frac{(x^2 \pm a^2)^{n+1/2}}{x^m} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{(m-1) a^2 x^{m-1}} \pm \frac{2n-m+1}{(m-1) a^2} \int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{m-2}} dx.$$

$$2. = -\frac{(x^2 \pm a^2)^{n+1/2}}{(m-1) x^{m-1}} + \frac{2n-1}{m-1} \int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{m-2}} dx.$$

$$3. \int \frac{(x^2 \pm a^2)^{n+1/2}}{x^{2m}} dx = \frac{(x^2 \pm a^2)^{n+3/2}}{(2m-1) a^2} \times$$

$$\times \left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1} \frac{(-1)^k 2^k (m-n-2)(m-n-3) \dots (m-n-k-1)}{(2m-3)(2m-5) \dots (2m-2k-1) a^{2k}} \frac{1}{x^{2m-2k-1}} \right] + \\ + \frac{(-1)^m 2^n (m-n-2)(m-n-3) \dots (-n)(-n-1)}{a^{2m} (2m-1)!} \int (x^2 \pm a^2)^{n+1/2} dx.$$

$$4. \quad = -\frac{(x^2 + a^2)^{n+3/2}}{(x^2 + a^2)^{m-n-1} x^{2n+3}} \sum_{k=0}^{m-n-2} (-1)^k \binom{m-n-2}{k} \frac{1}{2n+2k+3} \left(\frac{x^2 + a^2}{x^2} \right)^k$$

$1/m \geq n+2$.

$$5. \quad \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2n}} dx =$$

$$= - \sum_{k=0}^{n-1} \frac{(2n+1)(x^2 + a^2)^{n-k+1/2}}{[(2n-2k)^2 - 1] x^{2n-2k+1}} + (2n+1) \int (x^2 + a^2)^{1/2} dx.$$

$$6. \quad \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2m+1}} dx =$$

$$= -\frac{(x^2 + a^2)^{n+3/2}}{2n-2m+1} \sum_{k=1}^m (-1)^k \frac{(2n-2m+1)(2n-2m+3) \dots (2n-2m+2k-1)}{2^k m(m-1) \dots (m-k+1) a^{2k} x^{2m-2k+1}} +$$

$$+ (-1)^m \frac{(2n-2m+3)(2n-2m+5) \dots (2n+1)}{2^m a^{2m} m!} \int \frac{(x^2 + a^2)^{n+1/2}}{x} dx.$$

$$7. \quad \int \frac{(x^2 + a^2)^{n+1/2}}{x} dx =$$

$$= \sum_{k=0}^n \frac{(-1)^k (x^2 + a^2)^{n-k+1/2}}{2n-2k+1} + (-1)^n a^{2n+1} \left\{ \frac{1}{2} \ln \left| \frac{\sqrt{x^2 + a^2} - a}{\sqrt{x^2 + a^2} + a} \right| \right.$$

$$\left. + \arcsin \left| \frac{a}{x} \right| \right\}.$$

$$8. \quad \int \frac{(x^2 + a^2)^{1/2}}{x} dx = (x^2 + a^2)^{1/2} - a \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. + \arccos \left| \frac{a}{x} \right| \right\}.$$

$$9. \quad \int \frac{(x^2 + a^2)^{1/2}}{x^2} dx = -\frac{(x^2 + a^2)^{1/2}}{x} + \ln |x + (x^2 + a^2)^{1/2}|.$$

$$10. \quad \int \frac{(x^2 + a^2)^{1/2}}{x^3} dx = -\frac{(x^2 + a^2)^{1/2}}{2x^2} + \frac{1}{2a} \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. + \arccos \left| \frac{a}{x} \right| \right\}.$$

$$11. \quad \int \frac{(x^2 + a^2)^{3/2}}{x} dx =$$

$$= \frac{1}{3} (x^2 + a^2)^{3/2} + a^2 (x^2 + a^2)^{1/2} + a^2 \left\{ \ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| \right.$$

$$\left. + \arccos \left| \frac{a}{x} \right| \right\}.$$

$$12. \quad \int \frac{(x^2 + a^2)^{3/2}}{x^3} dx =$$

$$= -\frac{(x^2 + a^2)^{3/2}}{x} + \frac{3}{2} x (x^2 + a^2)^{1/2} + \frac{3}{2} a^2 \ln |x + (x^2 + a^2)^{1/2}|.$$

$$13. \int \frac{(x^2 + a^2)^{3/2}}{x^4} dx =$$

$$= -\frac{(x^2 + a^2)^{3/2}}{2x^2} + \frac{3}{2} \frac{(x^2 + a^2)^{1/2}}{x} - \frac{3a}{2} \left[\ln \left| \frac{a + (x^2 + a^2)^{1/2}}{x} \right| + \arccos \left| \frac{a}{x} \right| \right]$$

$$14. \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2n+2}} dx =$$

$$= - \sum_{k=0}^n \frac{1}{2n-2k+1} \left(\frac{x^2 + a^2}{x^2} \right)^{n-k+1/2} + \ln |x + \sqrt{x^2 + a^2}|.$$

$$15. \int \frac{(x^2 + a^2)^{n+1/2}}{x^{2n+4}} dx = -\frac{(x^2 + a^2)^{n+3/2}}{(2n+3)a^2 x^{2n+1}}.$$

1.2.43. Интегралы вида $\int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}}$.

$$1. \int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{-x^{m-1}}{(2n-1)(x^2 \pm a^2)^{n-1/2}} + \frac{m-1}{2n-1} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$2. = \pm \frac{x^{m+1}}{(2n-1)a^2(x^2 \pm a^2)^{n-1/2}} + \frac{2n-m-2}{(2n-1)a^2} \int \frac{x^m dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$3. = \frac{x^{m-1}}{(m-2n)(x^2 \pm a^2)^{n+1/2}} + \frac{(m-1)a^2}{m-2n} \int \frac{x^{m-2} dx}{(x^2 \pm a^2)^{n+1/2}}.$$

$$4. \int \frac{x^{2m} dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{x^{2m+1} \sqrt{x^2 \pm a^2}}{(2n-1)a^2} \times$$

$$\times \left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (-1)^k \frac{2^k (n-m-1)(n-m-2) \dots (n-m-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k}}{(x^2 \pm a^2)^{n-k}} \right] +$$

$$+ \frac{(-1)^n 2^n (n-m-1)(n-m-2) \dots (-m+1)(-m)}{a^{2n} (2n-1)!!} \int \frac{x^{2m} dx}{\sqrt{x^2 \pm a^2}}.$$

$$5. = (-1)^n m a^{2m-2n} \sum_{k=0}^{n-m-1} \frac{(-1)^k}{2m+2k+1} \binom{n-m-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{m+k+1/2}$$

($n \geq m+1$).

$$6. \int \frac{x^{2m} dx}{\sqrt{x^2 \pm a^2}} = \frac{\sqrt{x^2 \pm a^2}}{2m} \times$$

$$\times \left[x^{2m+1} + \sum_{k=1}^{m-1} \frac{(-1)^k (2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} a^{2k} x^{2m-2k-1} \right] +$$

$$+ (-1)^m a^{2m} \frac{(2m-1)!!}{2^m m!} \ln |x + \sqrt{x^2 \pm a^2}|.$$

$$7. \int \frac{x^{2n-2} dx}{(x^2 \pm a^2)^{n+1/2}} = \pm \frac{x^{2n-1}}{(2n-1)a^2 (x^2 \pm a^2)^{n-1/2}}.$$

$$8. \int \frac{x^{2m+1} dx}{(x^2 \pm a^2)^{n+1/2}} = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \frac{a^{2m-2k}}{(2n-2k-1)(x^2 \pm a^2)^{n-k-1/2}}.$$

$$9. \int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{1}{(2n-1)a^2(x^2 \pm a^2)^{n-1/2}} + \frac{2n-2}{(2n-1)a^2} \int \frac{dx}{(x^2 \pm a^2)^{n-1/2}}.$$

$$10. \int \frac{dx}{(x^2 \pm a^2)^{n+1/2}} = \frac{x(x^2 \pm a^2)^{1/2}}{(2n-1)a^2} \times \\ \times \left[\frac{1}{(x^2 \pm a^2)^n} + \sum_{k=1}^{n-1} (-1)^k \frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \cdot \frac{1}{a^{2k}(x^2 \pm a^2)^{n-k}} \right].$$

$$11. = \frac{(-1)^n}{a^{2n}} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{x^2 \pm a^2} \right)^{k+1/2}.$$

$$12. \int \frac{x dx}{(x^2 \pm a^2)^{n+1/2}} = -\frac{1}{(2n-1)(x^2 \pm a^2)^{n-1/2}}.$$

$$13. \int \frac{dx}{(x^2 \pm a^2)^{1/2}} = \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$14. \int \frac{x dx}{(x^2 \pm a^2)^{1/2}} = (x^2 \pm a^2)^{1/2}.$$

$$15. \int \frac{x^2 dx}{(x^2 \pm a^2)^{1/2}} = \frac{x(x^2 \pm a^2)^{1/2}}{2} \mp \frac{a^2}{2} \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$16. \int \frac{x^2 dx}{(x^2 \pm a^2)^{3/2}} = \frac{1}{3} (x^2 \pm a^2)^{3/2} \mp a^2 (x^2 \pm a^2)^{1/2}.$$

$$17. \int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \pm \frac{x}{a^2(x^2 \pm a^2)^{1/2}}.$$

$$18. \int \frac{x dx}{(x^2 \pm a^2)^{3/2}} = -\frac{1}{(x^2 \pm a^2)^{1/2}}.$$

$$19. \int \frac{x^2 dx}{(x^2 \pm a^2)^{3/2}} = -\frac{x}{(x^2 \pm a^2)^{1/2}} + \ln |x + (x^2 \pm a^2)^{1/2}|.$$

$$20. \int \frac{x^3 dx}{(x^2 \pm a^2)^{3/2}} = \frac{x^2 \pm 2a^2}{(x^2 \pm a^2)^{1/2}}.$$

1.2.44. Интегралы вида $\int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}}$.

$$1. \int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}} = \\ = \pm \frac{1}{(2n-1)a^2 x^{m-1} (x^2 \pm a^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1)a^2} \int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}}.$$

$$2. = \frac{-1}{(2n-1)x^{m+1}(x^2 \pm a^2)^{n-1/2}} - \frac{m+1}{2n-1} \int \frac{dx}{x^{m+3}(x^2 \pm a^2)^{n-1/2}}.$$

$$3. \quad \frac{-1}{(m-1)x^{m-1}(x^2 \pm a^2)^{n+1/2}} - \frac{2n-1}{m-1} \int \frac{dx}{x^m (x^2 \pm a^2)^{n+1/2}}.$$

$$4. \quad \int \frac{dx}{x^{2m} (x^2 \pm a^2)^{n+1/2}} = \frac{(-1)^{m+n}}{a^{2m+2n}} \sum_{k=0}^{m+n-1} \frac{(-1)^k (m+n-1)!}{2k-2m+1} \binom{m+n-1}{k} \frac{x^{2k-m-1/2}}{(x^2 \pm a^2)^{n-k+1/2}}.$$

$$5. \quad \int \frac{dx}{x (x^2 \pm a^2)^{n+1/2}} = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-2k-1) a^{2k+1} (x^2 \pm a^2)^{n-k+1/2}} - \frac{(-1)^{n+1}}{a^{2n+1}} \left[\ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. - \arccos \left| \frac{a}{x} \right| \right].$$

$$6. \quad \int \frac{dx}{x (x^2 \pm a^2)^{1/2}} = \frac{1}{a} \left[\ln \left| \frac{x}{a + (x^2 \pm a^2)^{1/2}} \right| \right. \\ \left. - \arccos \left| \frac{a}{x} \right| \right].$$

$$7. \quad \int \frac{dx}{x^2 (x^2 \pm a^2)^{1/2}} = \mp \frac{(x^2 \pm a^2)^{1/2}}{a^2 x}.$$

$$8. \quad \int \frac{dx}{x^2 (x^2 \pm a^2)^{1/2}} = \mp \frac{(x^2 \pm a^2)^{1/2}}{2a^2 x^2} + \frac{1}{2a^2} \left[\ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. - \arccos \left| \frac{a}{x} \right| \right].$$

$$9. \quad \int \frac{dx}{x (x^2 \pm a^2)^{3/2}} = \pm \frac{1}{a^2 (x^2 \pm a^2)^{1/2}} - \frac{1}{a^2} \left[\ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. - \arccos \left| \frac{a}{x} \right| \right].$$

$$10. \quad \int \frac{dx}{x^2 (x^2 \pm a^2)^{3/2}} = -\frac{2x^2 \pm a^2}{a^4 x (x^2 \pm a^2)^{1/2}}.$$

$$11. \quad \int \frac{dx}{x^3 (x^2 \pm a^2)^{3/2}} = \\ = -\frac{1}{2a^2 x^2 (x^2 \pm a^2)^{1/2}} - \frac{3}{2a^4 (x^2 \pm a^2)^{1/2}} \pm \frac{3}{2a^4} \left[\ln \left| \frac{a + (x^2 \pm a^2)^{1/2}}{x} \right| \right. \\ \left. - \arccos \left| \frac{a}{x} \right| \right].$$

1.2.45. Интегралы вида $\int \frac{dx}{(x+b)^n \sqrt{x^2 \pm a^2}}, \int \frac{dx}{(x^2 \pm b^2) \sqrt{x^2 \pm a^2}}.$

$$1. \quad \int \frac{dx}{(x+b)^n \sqrt{x^2 \pm a^2}} = \frac{-1}{(n-1)(b^2 \pm a^2)} \left| \frac{1}{(x+b)^{n-1}} - \right. \\ \left. - (2n-3)b \int \frac{dx}{(x+b)^{n-1} \sqrt{x^2 \pm a^2}} + (n-2) \int \frac{dx}{(x+b)^{n-2} \sqrt{x^2 \pm a^2}} \right|.$$

$$2. \int \frac{dx}{(x+b) \sqrt{x^2+a^2}} = \frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{a^2-bx - \sqrt{(a^2+b^2)(x^2+a^2)}}{x+b} \right|.$$

$$3. \int \frac{dx}{(x+b) \sqrt{x^2-a^2}} = \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{a^2+bx + \sqrt{(b^2-a^2)(x^2-a^2)}}{x+b} \right| \quad [b^2 > a^2].$$

$$4. = \frac{1}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{\sqrt{(a^2-b^2)(x^2-a^2)}}{a^2+bx} \quad [a^2 > b^2].$$

$$5. = \frac{1}{\sqrt{a^2-b^2}} \operatorname{arccos} \frac{a^2+bx}{a(x+b)} \quad [a^2 > b^2].$$

$$6. \int \frac{dx}{(x+a) \sqrt{x^2-a^2}} = \frac{1}{a} \sqrt{\frac{x-a}{x+a}}.$$

$$7. \int \frac{dx}{(x-a) \sqrt{x^2-a^2}} = -\frac{1}{a} \sqrt{\frac{x+a}{x-a}}.$$

$$8. \int \frac{dx}{(x^2+b^2)^m (x^2+a^2)^{n+1/2}} =$$

$$= \frac{x}{2(m-1)(a^2-b^2)b^2(x^2+b^2)^{m-1}(x^2+a^2)^{n+1/2}} +$$

$$+ \frac{(2m-3)a^2 - (2n+4m-6)b^2}{2(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-1}(x^2+a^2)^{n+1/2}} +$$

$$+ \frac{m+n-2}{(m-1)(a^2-b^2)b^2} \int \frac{dx}{(x^2+b^2)^{m-2}(x^2+a^2)^{n+1/2}}.$$

$$9. \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2-b^2)a^2(x^2+a^2)^{n+1/2}} +$$

$$+ \frac{(4n-3)a^2 - (2n-2)b^2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n+1/2}} -$$

$$- \frac{2n-2}{(2n-1)(a^2-b^2)a^2} \int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n-3/2}}.$$

$$10. \int \frac{dx}{(x^2+b^2) \sqrt{x^2+a^2}} = \frac{1}{b \sqrt{b^2-a^2}} \ln \frac{x \sqrt{b^2-a^2} + b \sqrt{x^2+a^2}}{\sqrt{x^2+b^2}} \quad [b^2 > a^2].$$

$$11. = \frac{1}{b \sqrt{a^2-b^2}} \operatorname{arctg} \frac{x \sqrt{a^2-b^2}}{b \sqrt{x^2+a^2}} \quad [a^2 > b^2].$$

$$12. \int \frac{dx}{(x^2-b^2) \sqrt{x^2+a^2}} = \frac{1}{2b \sqrt{a^2+b^2}} \ln \left| \frac{x \sqrt{a^2+b^2} - b \sqrt{x^2+a^2}}{\sqrt{x^2-b^2} + b \sqrt{x^2+a^2}} \right|.$$

$$13. \int \frac{dx}{(x^2+b^2) \sqrt{x^2-a^2}} = \frac{1}{b \sqrt{a^2+b^2}} \ln \frac{x \sqrt{a^2+b^2} + b \sqrt{x^2-a^2}}{\sqrt{x^2+b^2}}.$$

$$14. \int \frac{dx}{(x^2-b^2) \sqrt{x^2-a^2}} = \frac{1}{b \sqrt{b^2-a^2}} \ln \frac{x \sqrt{b^2-a^2} - b \sqrt{x^2-a^2}}{\sqrt{x^2-b^2}} \quad [b^2 > a^2].$$

$$15. \quad = \frac{1}{b \sqrt{a^2 - b^2}} \arcsin \frac{b \sqrt{x^2 - a^2}}{a \sqrt{x^2 - b^2}} \quad [a^2 > b^2].$$

$$16. \quad = \frac{1}{b \sqrt{a^2 - b^2}} \operatorname{arctg} \frac{b \sqrt{x^2 - a^2}}{x \sqrt{a^2 - b^2}} \quad [a^2 > b^2].$$

12.46. Интегралы вида $\int x^m (a^2 - x^2)^{n+1/2} dx$.

$$1. \quad \int x^m (a^2 - x^2)^{n+1/2} dx = -\frac{x^{m+1} (a^2 - x^2)^{n+3/2}}{m+2n+2} + \\ + \frac{(m-1)a^2}{m+2n+2} \int x^{m-2} (a^2 - x^2)^{n+1/2} dx.$$

$$2. \quad = \frac{x^{m+1} (a^2 - x^2)^{n+1/2}}{m+2n+2} + \frac{(2n+1)a^2}{m+2n+2} \int x^m (a^2 - x^2)^{n-1/2} dx.$$

$$3. \quad \int x^{2m} (a^2 - x^2)^{n+1/2} dx = -\frac{(a^2 - x^2)^{n+3/2}}{2n+2m+2} \times \\ \times \left[x^{2m+1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m+n)(m+n-1)\dots(m+n-k+1)} a^{2k} x^{2m-2k+1} \right] + \\ + \frac{a^{2m}(2m-1)!!}{2^n(m+n+1)(m+n)\dots(n+2)} \int (a^2 - x^2)^{n+1/2} dx.$$

$$4. \quad \int x^{2n+1} (a^2 - x^2)^{n+1/2} dx = \sum_{k=0}^n \frac{(-1)^{k+1} a^{2n-2k}}{2n+2k+3} \binom{m}{k} (a^2 - x^2)^{n+1/2+3/2}.$$

$$5. \quad \int x (a^2 - x^2)^{n+1/2} dx = -\frac{1}{2n+3} (a^2 - x^2)^{n+3/2}.$$

$$6. \quad \int (a^2 - x^2)^{n+1/2} dx = \frac{x}{2n+2} (a^2 - x^2)^{n+1/2} + \frac{(2n+1)a^2}{2n+2} \int (a^2 - x^2)^{n-1/2} dx.$$

$$7. \quad \int (a^2 - x^2)^{n+1/2} dx = \frac{x(a^2 - x^2)^{1/2}}{2n+2} \times \\ \times \left[(a^2 - x^2)^n + \sum_{k=1}^n \frac{(2n+1)(2n-1)\dots(2n-2k+3)}{2^k n(n-1)\dots(n-k+1)} a^{2k} (a^2 - x^2)^{n-k} \right] + \\ + \frac{(2n+1)!! a^{2n+2}}{2^{n+1}(n+1)!} \arcsin \frac{x}{|a|}.$$

$$8. \quad \int (a^2 - x^2)^{1/2} dx = \frac{x}{2} (a^2 - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{|a|}.$$

$$9. \quad \int x (a^2 - x^2)^{1/2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2}.$$

$$10. \quad \int x^2 (a^2 - x^2)^{1/2} dx = \frac{x}{8} (2x^2 - a^2) (a^2 - x^2)^{1/2} + \frac{a^3}{8} \arcsin \frac{x}{|a|}.$$

$$11. \quad \int x^3 (a^2 - x^2)^{1/2} dx = \frac{1}{5} (a^2 - x^2)^{5/2} - \frac{a^2}{3} (a^2 - x^2)^{3/2}.$$

$$12. \int (a^2 - x^2)^{3/2} dx = \frac{x}{4} (a^2 - x^2)^{3/2} + \frac{3a^2 x}{8} (a^2 - x^2)^{1/2} + \frac{3a^4}{8} \arcsin \frac{x}{a}.$$

$$13. \int x (a^2 - x^2)^{3/2} dx = -\frac{1}{5} (a^2 - x^2)^{5/2}.$$

$$14. \int x^2 (a^2 - x^2)^{3/2} dx = -\frac{x}{48} (8x^4 - 14a^2 x^2 + 3a^4) (a^2 - x^2)^{1/2} + \frac{a^6}{16} \arcsin \frac{x}{a}.$$

$$15. \int x^4 (a^2 - x^2)^{3/2} dx = \frac{1}{7} (a^2 - x^2)^{7/2} - \frac{a^2}{5} (a^2 - x^2)^{5/2}.$$

1.2.47. Интегралы вида $\int \frac{(a^2 - x^2)^{n+1/2}}{x^m} dx$.

$$1. \int \frac{(a^2 - x^2)^{n+1/2}}{x^m} dx = -\frac{(a^2 - x^2)^{n+1/2}}{(m-1)a^2 x^{m-1}} - \frac{2n-m+4}{(m-1)a^2} \int \frac{(a^2 - x^2)^{n+1/2}}{x^{m-2}} dx,$$

$$2. = -\frac{(a^2 - x^2)^{n+1/2}}{(m-1)x^{m-1}} - \frac{2n+1}{m-1} \int \frac{(a^2 - x^2)^{n-1/2}}{x^{m-2}} dx,$$

$$3. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2m}} dx = -\frac{(a^2 - x^2)^{n+3/2}}{(2m-1)a^2} \times \\ \times \left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{n-1} \frac{2^k (m-n-2)(m-n-3) \dots (m-n-k-1)}{(2^k-3)(2^k-5) \dots (2^k-2^k-1)} \frac{a^{-2k}}{x^{2m-2k-1}} \right] + \\ + \frac{2^n (m-n-2)(m-n-3) \dots (-n)(-n-1)}{a^{2m} (2m-1)!!} \int (a^2 - x^2)^{n-1/2} dx.$$

$$4. \int \frac{(a^2 - x^2)^{n+1/2}}{x} dx = \\ = \sum_{k=0}^n \frac{a^{2k}}{2n-2k+1} (a^2 - x^2)^{n-k+1/2} + \frac{a^{2n+1}}{2} \ln \left| \frac{a - (a^2 - x^2)^{1/2}}{a + (a^2 - x^2)^{1/2}} \right|.$$

$$5. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2n+2}} dx = \sum_{k=0}^n \frac{(-1)^{k+1}}{2n-2k+1} \left(\frac{a^2 - x^2}{x^2} \right)^{n-k+1/2} + (-1)^{n+1} \arcsin \frac{x}{a}.$$

$$6. \int \frac{(a^2 - x^2)^{n+1/2}}{x^{2n+1}} dx = -\frac{(a^2 - x^2)^{n+3/2}}{(2n+3)a^2 x^{2n+3}}.$$

$$7. \int \frac{(a^2 - x^2)^{1/2}}{x} dx = (a^2 - x^2)^{1/2} - a \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$8. \int \frac{(a^2 - x^2)^{1/2}}{x^2} dx = -\frac{(a^2 - x^2)^{1/2}}{x} - \arcsin \frac{x}{a}.$$

$$9. \int \frac{(a^2 - x^2)^{1/2}}{x^3} dx = -\frac{1}{2x^2} (a^2 - x^2)^{1/2} + \frac{1}{2a} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$10. \int \frac{(a^2 - x^2)^{3/2}}{x} dx = \frac{1}{3} (a^2 - x^2)^{3/2} + a^2 (a^2 - x^2)^{1/2} - a^3 \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$11. \int \frac{(a^2 - x^2)^{3/2}}{x^2} dx = -\frac{1}{x} (a^2 - x^2)^{3/2} - \frac{3x}{2} (a^2 - x^2)^{1/2} - \frac{3a^2}{2} \arcsin \frac{x}{a}.$$

$$12. \int \frac{(a^2 - x^2)^{3/2}}{x^3} dx = -\frac{(a^2 - x^2)^{3/2}}{2x^2} - \frac{3}{2} (a^2 - x^2)^{1/2} + \frac{3a}{2} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

1.2.48. Интегралы вида $\int \frac{x^m dx}{(a^2 - x^2)^{n+1/2}}$.

$$1. \int \frac{x^m dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{m-1}}{(2n-1)(a^2 - x^2)^{n+1/2}} - \frac{m-1}{2n-1} \int \frac{x^{m-2} dx}{(a^2 - x^2)^{n+1/2}}.$$

$$2. = \frac{x^{m-1}}{(2n-1)a^2(a^2 - x^2)^{n+1/2}} + \frac{2n-m-2}{(2n-1)a^2} \int \frac{x^m dx}{(a^2 - x^2)^{n+1/2}}.$$

$$3. = -\frac{x^{m-1}}{(m-2n)(a^2 - x^2)^{n+1/2}} + \frac{(m-1)a^2}{m-2n} \int \frac{x^{m-2} dx}{(a^2 - x^2)^{n+1/2}}.$$

$$4. \int \frac{x^{2m+1} dx}{(a^2 - x^2)^{n+1/2}} = \sum_{k=0}^m \frac{(-1)^k a^{2m-2k}}{2n-2k-1} \binom{m}{k} \frac{1}{(a^2 - x^2)^{n-k+1/2}}.$$

$$5. \int \frac{x^{2m} dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{2m+1}}{(2n-1)a^2} \times \\ \times \left[\frac{1}{(a^2 - x^2)^n} + \sum_{k=1}^{n-1} \frac{2^k (n-m-1)(n-m-2) \dots (n-m-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k}}{(a^2 - x^2)^{n-k}} \right] + \\ + \frac{2^n (n-m-1)(n-m-2) \dots (-m+1)(-m)}{a^{2n} (2n-1)!!} \int \frac{x^{2m} dx}{a^2 - x^2}.$$

$$6. \int \frac{x^{2m} dx}{(a^2 - x^2)^{n+1/2}} = \frac{1}{a^{2n-2m}} \sum_{k=0}^{n-m-1} \frac{1}{2m+2k+1} \binom{n-m-1}{k} \left(\frac{x^2}{a^2 - x^2} \right)^{m-k+1/2} \\ (n \geq m+1).$$

$$7. \int \frac{x^{2m-2} dx}{(a^2 - x^2)^{n+1/2}} = \frac{x^{2n-1}}{(2n-1)a^2(a^2 - x^2)^{n+1/2}}.$$

$$8. \int \frac{x^{2m} dx}{(a^2 - x^2)^{1/2}} = -\frac{(a^2 - x^2)^{1/2}}{2n} \times \\ \times \left[x^{-m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} \frac{a^{2k} x^{2m-2k-1}}{a^2 - x^2} \right] + \\ + \frac{a^{2m} (2m-1)!!}{2^{m+1} n!} \arcsin \frac{x}{|a|}.$$

$$9. \int \frac{dx}{(a^2 - x^2)^{n+1/2}} = \frac{x(a^2 - x^2)^{1/2}}{2n-1} \times \\ \times \left[\frac{1}{(a^2 - x^2)^n a^2} + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \frac{a^{-2k-2}}{(a^2 - x^2)^{n-k}} \right] \\ (n \geq 1).$$

$$10. \quad = \frac{1}{a^{2n}} \sum_{k=0}^{n-1} \frac{1}{2k+1} \binom{n-1}{k} \left(\frac{x^2}{a^2 - x^2} \right)^{k+1/2}.$$

$$11. \quad \int \frac{x dx}{(a^2 - x^2)^{n+1/2}} = \frac{1}{(2n-1)(a^2 - x^2)^{n-1/2}}.$$

$$12. \quad \int \frac{dx}{(a^2 - x^2)^{1/2}} = \arcsin \frac{x}{a}, \quad 13. \quad \int \frac{x dx}{(a^2 - x^2)^{1/2}} = -(a^2 - x^2)^{1/2}.$$

$$14. \quad \int \frac{x^2 dx}{(a^2 - x^2)^{1/2}} = -\frac{x}{2} (a^2 - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x}{a}.$$

$$15. \quad \int \frac{x^3 dx}{(a^2 - x^2)^{1/2}} = \frac{1}{3} (a^2 - x^2)^{3/2} - a^2 (a^2 - x^2)^{1/2}.$$

$$16. \quad \int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 (a^2 - x^2)^{1/2}}, \quad 17. \quad \int \frac{x dx}{(a^2 - x^2)^{3/2}} = \frac{1}{(a^2 - x^2)^{1/2}}.$$

$$18. \quad \int \frac{x^2 dx}{(a^2 - x^2)^{3/2}} = \frac{x}{(a^2 - x^2)^{1/2}} - \arcsin \frac{x}{a}.$$

$$19. \quad \int \frac{x^3 dx}{(a^2 - x^2)^{3/2}} = \frac{2a^2 - x^2}{(a^2 - x^2)^{1/2}}.$$

1.2.49. Интегралы вида $\int \frac{dx}{x^m (a^2 - x^2)^{n+1/2}}$.

$$1. \quad \int \frac{dx}{x^m (a^2 - x^2)^{n+1/2}} = \frac{1}{(2n-1) a^2 x^{n-1} (a^2 - x^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1) a^2} \int \frac{dx}{x^m (a^2 - x^2)^{n-1/2}}.$$

$$2. \quad = \frac{-1}{(m-1) a^2 x^{m-1} (a^2 - x^2)^{n-1/2}} + \frac{m+2n-2}{(m-1) a^2} \int \frac{dx}{x^{m-2} (a^2 - x^2)^{n-1/2}}.$$

$$3. \quad = \frac{1}{(2n-1) x^{m+1} (a^2 - x^2)^{n-1/2}} + \frac{m+1}{2n-1} \int \frac{dx}{x^{m+3} (a^2 - x^2)^{n-1/2}}.$$

$$4. \quad \int \frac{dx}{x^{2m} (a^2 - x^2)^{n+1/2}} = \frac{-1}{a^{2n+2m}} \sum_{k=0}^{m+n-1} \frac{1}{2m-2k-1} \binom{m+n-1}{k} \left(\frac{a^2 - x^2}{x^2} \right)^{m-k-1/2}.$$

$$5. \quad \int \frac{dx}{x (a^2 - x^2)^{n+1/2}} = \sum_{k=0}^{n-1} \frac{1}{(2n-2k-1) a^{2k+2} (a^2 - x^2)^{n-k-1/2}} - \frac{1}{a^{2n+1}} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$6. \quad \int \frac{dx}{x (a^2 - x^2)^{1/2}} = -\frac{1}{a} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$7. \int \frac{dx}{x^2 (a^2 - x^2)^{1/2}} = -\frac{1}{a^2 x} (a^2 - x^2)^{1/2}.$$

$$8. \int \frac{dx}{x^3 (a^2 - x^2)^{1/2}} = -\frac{1}{2a^2 x^2} (a^2 - x^2)^{1/2} - \frac{1}{2a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$9. \int \frac{dx}{x (a^2 - x^2)^{3/2}} = \frac{1}{a^2 (a^2 - x^2)^{1/2}} - \frac{1}{a^3} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

$$10. \int \frac{dx}{x^2 (a^2 - x^2)^{5/2}} = \frac{2x^2 - a^2}{a^4 x (a^2 - x^2)^{3/2}}.$$

$$11. \int \frac{dx}{x^3 (a^2 - x^2)^{3/2}} = -\frac{1}{2a^2 x^2 (a^2 - x^2)^{1/2}} + \frac{3}{2a^4 (a^2 - x^2)^{1/2}} - \frac{3}{2a^5} \ln \left| \frac{a + (a^2 - x^2)^{1/2}}{x} \right|.$$

1.2.50. Интегралы вида $\int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}}$,

$$\int \frac{dx}{(x^2 \pm b^2) (a^2 - x^2)^{n+1/2}}.$$

$$1. \int \frac{dx}{(x+b)^n \sqrt{a^2 - x^2}} = \frac{1}{(n-1)(b^2 - a^2)} \times \\ \times \left[\frac{1}{(x+b)^{n-1}} + (2n-3)b \int \frac{dx}{(x+b)^{n-1} \sqrt{a^2 - x^2}} - (n-2) \int \frac{dx}{(x+b)^{n-2} \sqrt{a^2 - x^2}} \right] \quad [b \neq a].$$

$$2. = \frac{1}{(2n-1)a} \left[\frac{1}{(x-a)^n} - (n-1) \int \frac{dx}{(x-a)^{n-1} \sqrt{a^2 - x^2}} \right] \quad [b = a].$$

$$3. \int \frac{dx}{(x+b) \sqrt{a^2 - x^2}} = \frac{-1}{\sqrt{a^2 - b^2}} \ln \left| \frac{bx + a^2 + \sqrt{(a^2 - b^2)(a^2 - x^2)}}{x+b} \right| \quad [a^2 > b^2].$$

$$4. = \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arccg} \frac{\sqrt{(b^2 - a^2)(a^2 - x^2)}}{a^2 + bx} \quad [a^2 < b^2].$$

$$5. = \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arcsin} \frac{a^2 + bx}{a(x+b)} \quad [a^2 < b^2].$$

$$6. \int \frac{dx}{(x+a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a-x}{a+x}} \quad [a > 0].$$

$$7. \int \frac{dx}{(x-a) \sqrt{a^2 - x^2}} = -\frac{1}{a} \sqrt{\frac{a+x}{a-x}} \quad [a > 0].$$

$$8. \int \frac{dx}{(b^2 - x^2)^n (a^2 - x^2)^{n+1/2}} = \frac{x}{2(m-1)(a^2 - b^2)b^2(b^2 - x^2)^{m-1}(a^2 - x^2)^{n-1/2}} + \\ + \frac{(2m-1)a^2 - 2(m+2n-3)b^2}{2(m-1)(a^2 - b^2)b^2} \int \frac{dx}{(b^2 - x^2)^{m-1}(a^2 - x^2)^{n-1/2}} + \\ + \frac{m+n-2}{(m-1)(a^2 - b^2)b^2} \int \frac{dx}{(b^2 - x^2)^m (a^2 - x^2)^{n+1/2}}.$$

$$9. \int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n+1/2}} = \frac{-x}{(2n-1)(a^2 - b^2)a^2(a^2 - x^2)^{n+1/2}} + \\ + \frac{(1n-2)a^2 - 2(n-1)b^2}{(2n-1)(a^2 - b^2)a^2} \int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n-1/2}} - \\ - \frac{2(n-1)}{(2n-1)(a^2 - b^2)a^2} \int \frac{dx}{(b^2 - x^2)(a^2 - x^2)^{n+3/2}}.$$

$$10. \int \frac{dx}{(b^2 - x^2) \sqrt{a^2 - x^2}} = \frac{1}{b \sqrt{a^2 - b^2}} \ln \frac{x \sqrt{a^2 - b^2} + b \sqrt{a^2 - x^2}}{x^2 - b^2} \quad |a^2 > b^2|.$$

$$11. = -\frac{1}{b \sqrt{b^2 - a^2}} \operatorname{arctg} \frac{x \sqrt{b^2 - a^2}}{b \sqrt{a^2 - x^2}} \quad |a^2 < b^2|.$$

$$12. = \frac{1}{b \sqrt{b^2 - a^2}} \arcsin \frac{x \sqrt{b^2 - a^2}}{a \sqrt{b^2 - x^2}} \quad |a^2 < b^2|.$$

$$13. \int \frac{dx}{(b^2 + x^2) \sqrt{a^2 - x^2}} = \frac{1}{b \sqrt{a^2 + b^2}} \operatorname{arctg} \frac{x \sqrt{a^2 + b^2}}{b \sqrt{a^2 - x^2}}.$$

1.2.51. Интегралы вида $\int x^{m-1} (ax^2 + bx + c)^{n+1/2} dx$.

Обозначение: $X = ax^2 + bx + c$.

$$1. \int x^m X^{n+1/2} dx = \frac{x^{m-1} X^{n+3/2}}{(m+2n+2)a} - \\ - \frac{(2m+2n+1)b}{2(m+2n+2)a} \int x^{m-1} X^{n+1/2} dx - \frac{(m-1)c}{(m+2n+2)a} \int x^{m-2} X^{n+1/2} dx.$$

$$2. \int x X^{n+1/2} dx = \frac{X^{n+3/2}}{(2n+3)a} - \frac{b}{2a} \int X^{n+1/2} dx.$$

$$3. \int X^{n+1/2} dx = \frac{2ax+b}{2(n+1)a} X^{n+1/2} + \frac{(2n+1)(4ac-b^2)}{8(n+1)a} \int X^{n-1/2} dx.$$

$$4. = \frac{(2ax+b) X^{1/2}}{4(n+1)a} \times \\ \times \left[X^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)} \left(\frac{4ac-b^2}{a}\right)^{k+1} X^{n-k-1} \right] + \\ + \frac{(2n+1)!!}{8^{n+1}(n+1)!} \left(\frac{4ac-b^2}{a}\right)^{n+1} \int \frac{dx}{X^{1/2}}.$$

$$5. \int x^m X^{1/2} dx = \frac{x^{m-1} X^{3/2}}{(m+2)a} - \frac{(2m+1)b}{2(m+2)a} \int x^{m-1} X^{1/2} dx - \\ - \frac{(m-1)c}{(m+2)a} \int x^{m-2} X^{1/2} dx.$$

$$6. \int X^{1/2} dx = \frac{2ax+b}{4a} X^{1/2} + \frac{4ac-b^2}{8a^{3/2}} \ln \left| \frac{2ax+b}{2|a|} + X^{1/2} \right| \quad |a > 0|.$$

$$7. = \frac{2ax+b}{4a} X^{1/2} + \frac{b^2-4ac}{8a \sqrt{|a|}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \quad |a < 0; b^2 > 4ac|.$$

8. $\int x \lambda^{1/2} dx = \frac{1}{3a} \lambda^{3/2} - \frac{b}{2a} \int \lambda^{1/2} dx.$
9. $\int x^2 \lambda^{1/2} dx = \frac{6ax - 5b}{24a^2} \lambda^{3/2} + \frac{5b^2 - 4ac}{16a^2} \int \lambda^{1/2} dx.$
10. $\int x^3 \lambda^{1/2} dx = \frac{48a^2x^2 - 42abx + 35b^2 - 32ac}{240a^3} \lambda^{3/2} - \frac{7b^3 - 12abc}{32a^3} \int \lambda^{1/2} dx.$
11. $\int \frac{\lambda^{n+1/2}}{x^m} dx = -\frac{1}{(m-1)cx^{m-1}} \lambda^{n+1/2} +$
 $+ \frac{(2n-2m-5)b}{2(m-1)c} \int \frac{\lambda^{n-1/2}}{x^{m-1}} dx + \frac{(2n-m+4)a}{(m-1)c} \int \frac{\lambda^{n-1/2}}{x^{m-2}} dx.$
12. $\int \frac{\lambda^{n+1/2}}{x} dx = \frac{\lambda^{n+1/2}}{2n+1} + \frac{b}{2} \int \lambda^{n-1/2} dx + c \int \frac{\lambda^{n-1/2}}{x} dx.$
13. $\int \frac{(ax^2+bx)^{n+1/2}}{x^m} dx = \frac{2}{(2n-2m+3)bx^m} (ax^2+bx)^{n+1/2} +$
 $+ \frac{2(m-2n-3)a}{(2n-2m+3)b} \int \frac{(ax^2+bx)^{n+1/2}}{x^{m-1}} dx.$
14. $\int \frac{\lambda^{1/2}}{x} dx = \lambda^{1/2} + \frac{b}{2} \int \frac{dx}{\lambda^{1/2}} + c \int \frac{dx}{x\lambda^{1/2}}.$
15. $\int \frac{\lambda^{1/2}}{x^2} dx = -\frac{\lambda^{1/2}}{x} + a \int \frac{dx}{\lambda^{1/2}} + \frac{b}{2} \int \frac{dx}{x\lambda^{1/2}}.$
16. $\int \frac{(ax^2+bx)^{1/2}}{x^2} dx = -\frac{2(ax^2+bx)^{1/2}}{x} + a \int \frac{dx}{(ax^2+bx)^{1/2}}.$
17. $\int \frac{\lambda^{1/2}}{x^3} dx = -\left(\frac{1}{2x^2} + \frac{b}{4cx}\right) \lambda^{1/2} + \left(\frac{a}{2} - \frac{b^2}{8c}\right) \int \frac{dx}{x\lambda^{1/2}}.$
18. $\int \frac{(ax^2+bx)^{1/2}}{x^3} dx = -\frac{2}{3hc^2} (ax^2+bx)^{3/2}.$
19. $\int \frac{\lambda^{3/2}}{x} dx = \frac{\lambda^{3/2}}{3} + \frac{2b(ax+b^2+8ac)}{8a} \lambda^{1/2} +$
 $+ \frac{(12ac-b^2)b}{16a} \int \frac{dx}{\lambda^{1/2}} + c^2 \int \frac{dx}{x\lambda^{1/2}}.$
20. $\int \frac{\lambda^{3/2}}{x^2} dx = -\frac{\lambda^{5/2}}{cx} + \frac{ax+b}{c} \lambda^{3/2} + \frac{3(2ax-b)}{4} \lambda^{1/2} +$
 $+ \frac{3(4ac-b^2)}{8} \int \frac{dx}{\lambda^{1/2}} + \frac{3bc}{2} \int \frac{dx}{x\lambda^{1/2}}.$
21. $\int \frac{(ax^2+bx)^{3/2}}{x^2} dx = \frac{(ax^2+bx)^{3/2}}{2x} + \frac{3b}{4} (ax^2+bx)^{1/2} + \frac{b^3}{8} \int \frac{dx}{(ax^2+bx)^{1/2}}.$

$$22. \int \frac{X^{3/2}}{x^3} dx = -\left(\frac{1}{2cx^2} + \frac{b}{4c^2x}\right) X^{5/2} + \\ + \frac{abx + 2ac + b^2}{4c^3} X^{3/2} + \frac{3(abx + 2ac + b^2)}{4c} X^{1/2} + \\ + \frac{3ab}{2} \int \frac{dx}{X^{1/2}} + \frac{3(4ac + b^2)}{8} \int \frac{dx}{xX^{1/2}}.$$

$$23. \int \frac{(ax^2 + bx)^{3/2}}{x^3} dx = \left(a - \frac{2b}{x}\right) (ax^2 + bx)^{1/2} + \frac{3ab}{2} \int \frac{dx}{(ax^2 + bx)^{1/2}}.$$

1.2.52. Интегралы вида $\int \frac{x^m dx}{(ax^2 + bx + c)^{n+1/2}}$.

Обозначение: $X = ax^2 + bx + c$.

$$1. \int \frac{x^m dx}{X^{n+1/2}} = \frac{x^{m-1}}{(m-2n) a X^{n-1/2}} - \frac{(2m-2n-1)b}{2(m-2n)a} \int \frac{x^{m-1} dx}{X^{n+1/2}} - \\ - \frac{(m-1)c}{(m-2n)a} \int \frac{x^{m-2} dx}{X^{n+1/2}}.$$

$$2. = \frac{2x^m}{(2n-1)bX^{n-1/2}} + \frac{2(2n-m-1)}{(2n-1)b} \int \frac{x^{m-1} dx}{X^{n+1/2}} - \frac{2c}{b} \int \frac{x^{m-1} dx}{X^{n+1/2}}.$$

$$3. = \frac{1}{a} \int \frac{x^{m-2} dx}{X^{n-1/2}} - \frac{c}{a} \int \frac{x^{m-2} dx}{X^{n+1/2}} - \frac{b}{a} \int \frac{x^{m-1} dx}{X^{n+1/2}}.$$

$$4. \int \frac{x^m dx}{X^{n+1/2}} = -\frac{x^{m-1}}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{x^{m-1} dx}{X^{n+1/2}} + \frac{1}{a} \int \frac{x^{m-2} dx}{X^{n-1/2}}.$$

$$5. \int \frac{x dx}{X^{n+1/2}} = -\frac{1}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{dx}{X^{n+1/2}}.$$

$$6. \int \frac{dx}{X^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} + \frac{8(n-1)a}{(2n-1)(4ac-b^2)} \int \frac{dx}{X^{n-1/2}}.$$

$$7. = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} \times \\ \times \left[1 + \sum_{k=1}^{n-1} \frac{8^k (n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \cdot \frac{a^k X^k}{(4ac-b^2)^k} \right].$$

$$8. \int \frac{dx}{X^{1/2}} = \frac{1}{\sqrt{a}} \ln \left| \frac{2ax+b}{2\sqrt{a}} + X \right| \quad (a > 0)$$

$$9. = \frac{1}{\sqrt{a}} \operatorname{Arsh} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (a > 0; b^2 < 4ac)$$

$$10. = \frac{-1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \quad (a < 0; b^2 > 4ac)$$

$$11. = \frac{1}{\sqrt{a}} \ln(2ax+b) \quad (a > 0; b^2 = 4ac)$$

$$12. \int \frac{x dx}{\lambda^{1/2}} = \frac{X^{1/2}}{a} - \frac{b}{2a} \int \frac{dx}{X^{1/2}}.$$

$$13. \int \frac{x^2 dx}{\lambda^{1/2}} = \frac{2ax - 3b}{4a^2} X^{1/2} + \frac{3b^2 - 4ac}{8a^2} \int \frac{dx}{\lambda^{1/2}}.$$

$$14. \int \frac{x^3 dx}{\lambda^{1/2}} = \frac{8a^2 x^2 - 10abx + 15b^2 - 16ac}{24a^3} X^{1/2} - \frac{5b^3 - 12abc}{16a^3} \int \frac{dx}{X^{1/2}}.$$

$$15. \int \frac{dx}{\lambda^{3/2}} = \frac{2(2ax + b)}{(4ac - b^2) X^{1/2}}.$$

$$16. \int \frac{x dx}{\lambda^{3/2}} = \frac{2(bx + 2c)}{(b^2 - 4ac) X^{1/2}}.$$

$$17. \int \frac{x^2 dx}{\lambda^{3/2}} = \frac{(4ac - 2b^2)x - 2bc}{a(b^2 - 4ac) X^{1/2}} + \frac{1}{a} \int \frac{dx}{X^{1/2}}.$$

$$18. \int \frac{x^3 dx}{\lambda^{3/2}} = \frac{a(4ac - b^2)x^2 + b(10ac - 3b^2)x + c(8ac - 3b^2)}{a^2(4ac - b^2) X^{1/2}} - \frac{3b}{2a^2} \int \frac{dx}{X^{1/2}}.$$

$$19. \int \frac{dx}{x^m \lambda^{n+1/2}} = -\frac{1}{(m-1)cx^{m-1}X^{n+1/2}} - \frac{(2m+2n-3)b}{2(m-1)c} \int \frac{dx}{x^{m-1}X^{n+1/2}} - \frac{(2n+m-2)a}{(m-1)c} \int \frac{dx}{x^{m-2}X^{n+1/2}}.$$

$$20. \int \frac{dx}{x\lambda^{n+1/2}} = \frac{1}{(2n-1)cX^{n+1/2}} - \frac{b}{2c} \int \frac{dx}{X^{n+1/2}} + \frac{1}{c} \int \frac{dx}{x\lambda^{n-1/2}}.$$

$$21. \int \frac{dx}{x^m (ax^2 + bx)^{n+1/2}} = -\frac{2}{(2m+2n-1)bx^m(a^2 + bx)^{n+1/2}} - \frac{2(m+2n-1)a}{(2m+2n-1)b} \int \frac{dx}{x^{m-1}(ax^2 + bx)^{n+1/2}}.$$

$$22. \int \frac{dx}{x\lambda^{1/2}} = -\frac{1}{\sqrt{c}} \ln \left| \frac{2c + bx + 2\sqrt{c} X^{1/2}}{x} \right| \quad [c > 0].$$

$$23. = -\frac{1}{\sqrt{c}} \operatorname{Arth} \frac{2c + bx}{2\sqrt{c} X^{1/2}} \quad [c > 0].$$

$$24. = -\frac{1}{\sqrt{c}} \operatorname{Arsh} \frac{2c + bx}{x\sqrt{4ac - b^2}} \quad [c > 0, b^2 < 4ac].$$

$$25. = \frac{1}{\sqrt{c}} \ln \left| \frac{x}{2c + bx} \right| \quad [c > 0, b^2 = 4ac].$$

$$26. = \frac{1}{\sqrt{-c}} \operatorname{arctg} \frac{2c + bx}{2\sqrt{-c} X^{1/2}} \quad [c < 0].$$

$$27. = \frac{1}{\sqrt{-c}} \operatorname{arcsin} \frac{2c + bx}{x\sqrt{b^2 - 4ac}} \quad [c < 0, b^2 > 4ac].$$

$$28. = -\frac{2\sqrt{bx + ar^2}}{bx} \quad [c = 0, b \neq 0].$$

29. $\int \frac{dx}{x^2 \lambda^{1/2}} = -\frac{\lambda^{1/2}}{cx} - \frac{b}{2c} \int \frac{dx}{x \lambda^{1/2}}.$
 30. $\int \frac{dx}{x^2 (ax^2 + bx)^{1/2}} = ? \quad \frac{2}{bx^2} + \frac{2a}{bx^3} + \frac{2a}{bx^2} (ax^2 + bx)^{1/2}.$
 31. $\int \frac{dx}{x^3 \lambda^{1/2}} = -\frac{1}{2cx^2} + \frac{3b}{4cx} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c} \right) \int \frac{dx}{x \lambda^{1/2}}.$
 32. $\int \frac{dx}{x^2 (ax^2 + bx)^{3/2}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{4a}{3b^2 x} - \frac{8a^2}{3b^3 x} \right) (ax^2 + bx)^{1/2}.$
 33. $\int \frac{dx}{x \lambda^{3/2}} = \frac{2 (mbx - 2ac + b^2)}{c (b^2 - 4ac) \lambda^{1/2}} + \frac{1}{c} \int \frac{dx}{x \lambda^{1/2}}.$
 34. $\int \frac{dx}{x (ax^2 + bx)^{3/2}} = \frac{2}{3} \left(-\frac{1}{bx} + \frac{4a}{b^2} - \frac{8ax}{b^3} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$
 35. $\int \frac{dx}{x^2 \lambda^{3/2}} = \left[-\frac{1}{cx} + \frac{2ab}{c (4ac - b^2)} + \frac{a (3b^2 - 3ac) x}{c^2 (4a^2 - b^2)} \right] \frac{1}{\lambda^{1/2}} - \frac{7b}{2c^2} \int \frac{dx}{x \lambda^{1/2}}.$
 36. $\int \frac{dx}{x^2 (ax^2 + bx)^{3/2}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{2a}{b^2 x} - \frac{8a^2}{b^3} - \frac{16ax}{b^4} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$
 37. $\int \frac{dx}{x^3 \lambda^{3/2}} = \left[-\frac{1}{cx^2} + \frac{5b}{2c^2 x} + \frac{15b^2 - 62acb^2 + 24a^2 c^2}{2c^3 (b^2 - 4ac)} + \right. \\ \left. + \frac{ab (15b^2 - 52ac) x}{2c^2 (b^2 - 4ac)} \right] \frac{1}{2 \lambda^{1/2}} + \frac{15b^2 - 12ac}{8c^3} \int \frac{dx}{x \lambda^{1/2}}.$
 38. $\int \frac{dx}{x^2 (ax^2 + bx)^{3/2}} = \frac{2}{7} \left(-\frac{1}{bx^2} + \frac{8a}{5b^2 x} - \frac{16a^2}{5b^3 x} + \frac{64x}{5b^4} + \right. \\ \left. + \frac{128c^2 x}{5b^2} \right) \frac{1}{(ax^2 + bx)^{1/2}}.$
- 1.2.53. Интегралы вида $\int R(x+p, ax^2+bx+c) dx$.
Обозначение: $X = ax^2 + bx + c$.
1. $\int \frac{dx}{(x+p)^m X^{n+1/2}} = -\frac{1}{(2m+2n-1)(ap^2-bp+c)} \frac{1}{(x+p)^{m-1} X^{n+1/2}} - \\ - \frac{2}{(2m+2n-3)(b-2ap)} \int \frac{dx}{(x+p)^{m-1} X^{n+1/2}} - \\ - \frac{2(m-1)(ap^2-bp+c)}{(m+2a-2p)a} \int \frac{dx}{(x+p)^{m-2} X^{n+1/2}}.$
 2. $= -\frac{1}{(2m+2n-1)(b-2ap)(x+p)^m X^{n-1/2}} - \\ - \frac{2}{(2m+2n-1)(b-2ap)} \int \frac{dx}{(x+p)^{m-1} X^{n-1/2}} - \\ - \frac{2(m-1)(ap^2-bp+c)}{(m+2a-2p)a} \int \frac{dx}{(x+p)^{m-2} X^{n-1/2}}.$

3. $\int \frac{dx}{(x+p) \lambda^{n+1/2}} = \frac{1}{(2n-1)(ap^2-bp+c)} \frac{1}{X^{n+1/2}} + \\ + \frac{1}{ap^2-bp+c} \int \frac{dx}{(x+p) X^{n+1/2}} + \frac{2ap-b}{2(ap^2-bp+c)} \int \frac{dx}{X^{n+1/2}}.$
4. $= \frac{1}{(2n+1) \lambda^{n+1/2}} + \frac{b-2ap}{2} \int \frac{dx}{X^{n+3/2}} - \\ - \frac{1}{X^{n+3/2}}.$
5. $\int \frac{dx}{(x+p)^m} = \frac{(m-1)(ap^2-bp+c)}{(2n-2m+3)(b-2ap)} \frac{1}{X^{n+1/2}} dx + \frac{(2n-m+1)a}{(m-1)(ap^2-bp+c)} \int \frac{X^{n+1/2}}{(x+p)^{m-1}} dx + \\ + \frac{2X^{n+3/2}}{(2n-2m+3)(b-2ap)(x+p)^m} - \\ - \frac{2(2n-m+3)a}{(2n-2m+3)(b-2ap)} \int \frac{X^{n+1/2}}{(x+p)^{m-1}} dx.$
7. $\int \frac{X^{n+1/2}}{x+p} dx = \frac{X^{n+1/2}}{2n+1} + \frac{b-2ap}{2} \int \frac{X^{n-1/2}}{(x+p)^2} dx + (ap^2-bp+c) \int \frac{X^{n-1/2}}{x+p} dx.$
8. $\int \frac{X^{1/2} dx}{(x+p)^n} = -\frac{X^{3/2}}{(n-1)(x+p)^{n+1}} + \frac{1}{2(n-1)} \int \frac{(b+2ax) dx}{(x+p)^{n+1} X^{1/2}}.$
9. $\int \frac{X^{1/2} dx}{x+p} = \int \frac{ax+b-2ap}{X^{1/2}} dx + (ap^2-bp+c) \int \frac{dx}{(x+p) X^{1/2}} - \\ - \frac{X^{1/2}}{(x+p)^2}.$
10. $\int \frac{dx}{(x+p)^n X^{1/2}} = \frac{(n-1)(ap^2-bp+c)(x+p)^{n-1}}{(2n-3)(b-2ap)} \int \frac{dx}{(x+p)^{n-1} X^{1/2}} - \\ - \frac{2(n-1)(ap^2-bp+c)}{(n-2)a} \int \frac{dx}{(x+p)^{n-2} X^{1/2}}.$
11. $= \frac{2X^{1/2}}{(2n-1)(2ap-b)(x+p)^n} + \frac{2(n-1)a}{(2n-1)(2ap-b)} \int \frac{dx}{(x+p)^{n-1} X^{1/2}} - \\ - \frac{2(n-1)(ap^2-bp+c)}{(n-2)a} \int \frac{dx}{(x+p)^{n-2} X^{1/2}}.$
12. $\int \frac{dx}{(x+p)^n X^{1/2}} = -\int \frac{1}{(x+p)^{n-1} X^{1/2}} dt$
13. $\int \frac{dx}{(x+p) X^{1/2}} = -\frac{1}{(ap^2-bp+c)^{1/2}} \ln \left| \frac{(ap^2-bp+c)^{1/2} + X^{1/2}}{x+p} \right| + \\ + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}} \left| \frac{1}{(ap^2-bp+c)^{1/2}} - X^{1/2} \right|$
14. $= \frac{1}{2(ap^2-bp+c)^{1/2}} \ln \left| \frac{(ap^2-bp+c)^{1/2} + X^{1/2}}{x+p} \right| + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}} \left| \frac{1}{(ap^2-bp+c)^{1/2}} - X^{1/2} \right|$
15. $= \frac{(bp-ap^2-c) \arcsin \frac{(b-2ap)x-bp+2c}{(x+p)(b^2-4ac)^{1/2}}}{(b^2-4ac)^{1/2}} + \frac{(b-2ap)x-bp+2c}{(x+p)(b^2-4ac)^{1/2}}.$

$$16. = -\frac{2\lambda^{1/2}}{(b-2ap)(x+p)} \quad [a:\lambda^2-bp+c=0].$$

$$17. \int \frac{dx}{(x^2+px+q)\lambda^{1/2}} = \frac{1}{(p^2-4q)^{1/2}} \left[\int \frac{dx}{\left(x + \frac{p-1}{2} \frac{p^2-4q}{2}\right)\lambda^{1/2}} - \int \frac{dx}{\left(x + \frac{p+1}{2} \frac{p^2-4q}{2}\right)\lambda^{1/2}} \right] \quad [p^2 > 4q].$$

$$18. \int \frac{dx}{(x^2+t^2)\lambda^{1/2}} = \int \frac{(\alpha-\beta)(t+1)dt}{[(x^2+p^2)t^2 + (\beta^2+t^2)[(a\alpha^2+b\alpha+c)t^2 + \alpha\beta^2 + b\beta + c]\lambda^{1/2}} \\ \left[t = \frac{1-\beta}{\alpha-1}; \alpha \text{ и } \beta \text{ определяются на системе уравнений} \right. \\ \left. b(\alpha+1) + 2c - 2ap^2 = 0, \alpha\beta + p^2 = 0 \right]$$

$$19. \int \frac{(\alpha x + \beta) dx}{(p + \lambda)^n \lambda^{1/2}} = \frac{\alpha}{a} \int \frac{du}{(p+u^2)^n} + \frac{2\beta a - \alpha b}{2a} \int \frac{(1 - au^2)^{n-1} du}{(p+c-b^2/4a-au^2)^n} \\ \left[u = \lambda^{1/2}; \quad c = \frac{b+2\alpha\beta}{2\lambda^{1/2}} \right]$$

1.2.51. Интегралы вида $\int f(x, \sqrt{x^2-x+1}) dx$.

Условие: $x > 1$.

Обозначение: $\varphi = \arcsin \frac{1}{\sqrt{x^2-x+1}}$.

$$1. \int_1^{\infty} \frac{dx}{\sqrt{x(x-1)(x^2-x+1)}} = F\left(\varphi, \frac{\sqrt{3}}{2}\right) \quad (x \geq 1)$$

$$2. \int_x^{\infty} \frac{dx}{\sqrt{x^3(x-1)^2(x^2-x+1)}} = \frac{2(2x-1)}{\sqrt{x(x-1)(x^2-x+1)}} - 4E\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$3. \int_x^{\infty} \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3} \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) \right] \quad (x \geq 1)$$

$$4. \int_x^{\infty} \frac{dx}{(2x-1)^2 \sqrt{x(x-1)(x^2-x+1)}} = \\ = \frac{4}{3} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{2}{2x-1} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$5. \int_x^{\infty} \frac{dx}{(2x-1)^3 \sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \\ - \frac{1}{27} F\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{8(3x^2-5x+2)}{9(2x-1)^3} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$6. \int_{-\infty}^{\infty} \frac{(2x-1)^2 dx}{\sqrt{x^3(x-1)^2(x^2-x+1)}} = 4 \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) + \frac{2x-1}{2\sqrt{x(x-1)(x^2-x+1)}} \right].$$

$$7. \int_{-\infty}^{\infty} \frac{(2x-1)^2 dx}{\sqrt{x(x-1)(x^2-x+1)^2}} = 4E\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$8. \int_{-\infty}^{\infty} \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}} dx = \frac{4}{3} E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\varphi, \frac{\sqrt{3}}{2}\right).$$

$$9. \int_{-\infty}^{\infty} \frac{1}{(2x-1)^2} \sqrt{\frac{x(x-1)}{x^2-x+1}} dx = \frac{1}{3} \left[F\left(\varphi, \frac{\sqrt{3}}{2}\right) - E\left(\varphi, \frac{\sqrt{3}}{2}\right) \right] + \frac{1}{2(2x-1)} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$10. \int_{-\infty}^{\infty} \frac{1}{(2x-1)^2} \sqrt{\frac{x^2-x+1}{x(x-1)}} dx = E\left(\varphi, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2x-1)} \sqrt{\frac{x(x-1)}{x^2-x+1}}.$$

$$11. \int_0^{\frac{1}{2}} \frac{1}{[1+(1+\sqrt{3})x]^2} \sqrt{\frac{x^2-x+1}{x(x+1)}} dx = \frac{1}{3} E\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) \quad [x > 0].$$

$$12. \int_0^{\frac{1}{2}} \sqrt{\frac{x(x+1)}{(x^2-x+1)^2}} dx = \frac{1}{3} E\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) - \frac{2-\sqrt{3}}{3} F\left(\arccos \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) - \frac{2(2+\sqrt{3})}{3} \frac{1+(1-\sqrt{3})x}{1+(1+\sqrt{3})x} \sqrt{\frac{x(1+x)}{x^2-x+1}} \quad [x > 0].$$

1.2.55. Интегралы вида $\int_0^x R(x, \sqrt{x^2+a^2}, \sqrt{x^2+b^2}) dx$.

Условия: $x > 0, a > b > 0$.

Обозначения: $\varphi = \arctg \frac{x}{b}, k = \sqrt{\frac{a^2-b^2}{a}}$.

$$1. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a} F(\varphi, k).$$

$$2. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = x \sqrt{\frac{x^2+a^2}{x^2+b^2}} - aE(\varphi, k).$$

3. $\int_0^x \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p-b^2)} \left[\frac{b^2}{p} E\left(q, \frac{p+b^2}{p}, k\right) + F(q, k) \right] \quad (a \neq 0).$
4. $\int_0^x \frac{dx}{(x^2+a^2)^2(x^2+b^2)^2} = \frac{1}{a(a^2-b^2)} [F(q, k) - E(q, k)] + \frac{x}{a^2 \sqrt{(x^2+a^2)(x^2+b^2)}}.$
5. $\int_0^x \frac{dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{1}{a(a^2-b^2)} [F(q, k) - E(q, k)] + \frac{x}{a^2 \sqrt{(x^2+a^2)(x^2+b^2)}}.$
6. $\int_0^x \frac{p dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{1}{ab^2(a^2-b^2)} [a^2 E(q, k) - b^2 F(q, k)].$
7. $\int_0^x \frac{x^2 dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{1}{a(a^2-b^2)} [a^2 E(q, k) - b^2 F(q, k)] - \frac{x}{(x^2+a^2)(x^2+b^2)}.$
8. $\int_0^x \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{a}{a^2-b^2} [F(q, k) - E(q, k)].$
9. $\int_0^x \frac{\sqrt{x^2+a^2}}{x^2+b^2} dx = a [F(q, k) - E(q, k)] + x \sqrt{\frac{x^2+a^2}{x^2+b^2}}.$
10. $\int_0^x \frac{\sqrt{x^2+b^2}}{x^2+a^2} dx = \frac{b^2}{a} F(q, k) - a E(q, k) + x \sqrt{\frac{x^2+a^2}{x^2+b^2}}.$
11. $\int_0^x \frac{\sqrt{x^2+a^2}}{(x^2+b^2)^2} dx = \frac{a}{b^2} E(q, k).$
12. $\int_0^x \frac{\sqrt{x^2+b^2}}{(x^2+a^2)^2} dx = \frac{1}{a} E(q, k) - \frac{b^2-b^2}{a^2} \frac{x}{(x^2+a^2)(x^2+b^2)}.$
13. $\int_0^x \frac{\sqrt{(x^2+a^2)(x^2+b^2)}}{x^2+b^2} dx = \frac{a}{3} [2a^2 F(q, k) - (a^2+b^2) E(q, k)] + \frac{x}{3} (x^2+a^2+2b^2) \sqrt{\frac{x^2+a^2}{x^2+b^2}} + \frac{x}{3} (x^2+a^2+2b^2) [F(q, k) - E(q, k)] \quad (a > 0).$

14. $\int_0^x \frac{dx}{(x^2+a^2-1)(x^2+a^2)} = \frac{1}{2(a+1)} \left[F\left(\arccos \frac{(x+1)^2-(a+1)x}{(x+1)^2+(a+1)x}, \frac{(1-a+1)^2}{2(a+1)}\right) + F\left(\arccos \frac{(x-1)^2-(a+1)x}{(x-1)^2+(a+1)x}, \frac{(1-a-1)^2}{2(a+1)}\right) \right] \cdot \left[\begin{matrix} a < 0; & 0 \leq x \leq 1/a \\ a > 0; & 1/a \leq x < \infty \end{matrix} \right].$
 15. $\int_0^x \frac{dx}{(x^2+1)(x^2+a^2)} = \frac{1}{2(a+1)} \left[F\left(\arccos \frac{(x+1)^2-(a+1)x}{(x+1)^2+(a+1)x}, \frac{(1-a+1)^2}{2(a+1)}\right) + F\left(\arccos \frac{(x-1)^2-(a+1)x}{(x-1)^2+(a+1)x}, \frac{(1-a-1)^2}{2(a+1)}\right) \right] \cdot \left[\begin{matrix} a < 0; & 0 \leq x \leq 1/a \\ a > 0; & 1/a \leq x < \infty \end{matrix} \right].$
 16. $\int_0^x \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-c} F\left(2 \arctg \sqrt{\frac{x}{c}}, \sqrt{\frac{c-a}{2c}}\right) \quad [c > 0; c = \sqrt{a^2+b^2}].$
 17. $\int_0^x \frac{dx}{(x^2+a^2)(x^2+b^2+c^2)} = \frac{1}{b^2-c} F\left(2 \arctg \sqrt{\frac{cx}{p(x+a)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2+q^2}{pq}}\right) \quad [c > 0; p = ab+c^2; q^2 = (b+a)^2+c^2].$
 18. $\int_0^x \frac{dx}{(x^2+a^2)(x^2+b^2+c^2)} = \frac{1}{b^2-c} F\left(2 \arctg \sqrt{\frac{q(a-x)}{px}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2+a^2}{2q}}\right) \quad [a > 0; p = ab+c^2; q^2 = (b-a)^2+c^2; q^2 = ab+c^2].$
- 1.2.56. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2+a^2}, \sqrt{x^2+b^2}) dx$.
- Здесь $x \geq 0, c > 0$.
- Соединения: $q = \arctg \frac{x}{a}, k = \sqrt{\frac{a^2-b^2}{a}}.$
1. $\int_x^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a} F(q, k).$
 2. $\int_x^\infty \frac{dx}{x^2 \sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{b^2 x} \sqrt{\frac{x^2+b^2}{x^2+a^2}} - \frac{1}{c b^2} E(q, k) \quad (c > 0).$

$$3. \int_x^{\infty} \frac{dx}{(\rho - x^2) \sqrt{(x^2 + a^2)(x^2 + b^2)}} = -\frac{1}{a(u^2 + \rho)} \left[\Pi\left(\varphi, \frac{a^2 + \rho}{a^2}, k\right) - F(\varphi, k) \right].$$

$$4. \int_x^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)^2 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \\ = \frac{1}{ab^2(a^2 - b^2)} [a^2 E(\varphi, k) - b^2 F(\varphi, k)] - \frac{x}{b^2 \sqrt{(x^2 + a^2)(x^2 + b^2)}}.$$

$$6. \int_x^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)^3}} = \\ = \frac{1}{ab^2(a^2 - b^2)^2} [(a^2 + b^2) E(\varphi, k) - 2b^2 F(\varphi, k)] - \frac{x}{b^2(a^2 - b^2) \sqrt{(x^2 + a^2)(x^2 + b^2)}}.$$

$$7. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3 (x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} [a^2 E(\varphi, k) - b^2 F(\varphi, k)].$$

$$8. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}.$$

$$9. \int_x^{\infty} \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\varphi, k) - \frac{(a^2 - b^2)x}{b^2 \sqrt{(x^2 + a^2)(x^2 + b^2)}}.$$

$$10. \int_x^{\infty} \sqrt{\frac{x^2 + b^2}{a^2(x^2 + a^2)^3}} dx = \frac{1}{a} E(\varphi, k).$$

$$11. \int_x^{\infty} \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} \frac{dx}{x^2} = \frac{1}{a} F(\varphi, k) - \frac{a}{b^2} E(\varphi, k) + \frac{a^2}{b^2 x} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} \quad (x > 0).$$

$$12. \int_x^{\infty} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} \frac{dx}{x^2} = \frac{1}{a} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} \quad (x > 0).$$

1.2.57. Интегралы вида $\int_b^x R(x, \sqrt{x^2 + a^2}, \sqrt{x^2 - b^2}) dx$.

Условие: $x > b > 0$.

Обозначения: $\varphi = \arccos \frac{b}{x}$, $k = \frac{a}{\sqrt{a^2 + b^2}}$.

$$1. \int_b^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varphi, k).$$

$$2. \int_a^x \frac{x^2 dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{b^2}{\sqrt{a^2+b^2}} F(\varphi, k) - \sqrt{a^2+b^2} E(\varphi, k) + \frac{1}{x} \sqrt{(x^2+a^2)(x^2-b^2)}.$$

$$3. \int_b^x \frac{dx}{x^2 \sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2+b^2}} [(a^2+b^2) E(\varphi, k) - b^2 F(\varphi, k)].$$

$$4. \int_a^x \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{p(p-b^2) \sqrt{a^2+b^2}} \left[b^2 \Pi \left(\varphi, \frac{p}{p-b^2}, k \right) + (p-b^2) F(\varphi, k) \right] \quad [p \neq b^2].$$

$$5. \int_a^x \frac{dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{a^2 \sqrt{a^2+b^2}} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{(a^2+b^2)x} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$6. \int_b^x \frac{x^2 dx}{\sqrt{(x^2+a^2)^3(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}} E(\varphi, k) - \frac{a^2}{x(a^2+b^2)} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$7. \int_a^x \sqrt{\frac{x^2+a^2}{x^2-b^2}} dx = \sqrt{a^2+b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{(x^2+a^2)(x^2-b^2)}.$$

$$8. \int_b^x \frac{1}{x^2} \sqrt{\frac{x^2+a^2}{x^2-b^2}} dx = \frac{\sqrt{a^2+b^2}}{b^2} E(\varphi, k).$$

$$9. \int_a^x \sqrt{\frac{x^2-b^2}{x^2+a^2}} dx = \frac{1}{x} \sqrt{(x^2+a^2)(x^2-b^2)} + \sqrt{a^2+b^2} E(\varphi, k).$$

$$10. \int_b^x \frac{1}{x^2} \sqrt{\frac{x^2-b^2}{x^2+a^2}} dx = \frac{\sqrt{a^2+b^2}}{a^2} [F(\varphi, k) - E(\varphi, k)].$$

$$11. \int_a^x \sqrt{\frac{x^2-b^2}{(x^2+a^2)^3}} dx = \frac{\sqrt{a^2+b^2}}{a^2} E(\varphi, k) - \frac{b^2}{a^2 \sqrt{a^2+b^2}} F(\varphi, k) - \frac{1}{x} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$12. \int_k^x \frac{1}{(x^2+a^2)(x^2-b^2)} dx = \frac{1}{3} \frac{1}{a^2+b^2} [(b^2-a^2) E(\varphi, k) - b^2 F(\varphi, k)] + \\ + \frac{x^2+a^2-b^2}{3x} \frac{1}{(x^2+a^2)(x^2-b^2)}.$$

1.2.58. Интегралы вида $\int_x^\infty R(x, \sqrt{x^2+a^2}, \sqrt{x^2-b^2}) dx$.

Условие: $x-b \geq 0$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a^2+b^2}{x^2+a^2}}$, $k = \frac{a}{\sqrt{a^2+b^2}}$.

$$1. \int_x^\infty \frac{dx}{(x^2+a^2)(x^2-b^2)} = \frac{1}{a^2+b^2} E(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{x^2 (x^2+a^2)(x^2-b^2)} = \\ = \frac{1}{a^2 b^2} \frac{1}{a^2+b^2} [(a^2+b^2) E(\varphi, k) - b^2 F(\varphi, k)] - \frac{1}{b^2 x} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$3. \int_x^\infty \frac{dx}{(x^2-p^2)(x^2+a^2)(x^2-b^2)} = \\ = \frac{1}{(a^2+p^2)(a^2+b^2)} \left[\Pi\left(\varphi, \frac{a^2+p^2}{a^2+b^2}, k\right) - F(\varphi, k) \right].$$

$$4. \int_x^\infty \frac{dx}{(x^2+a^2)^2 (x^2-b^2)} = \frac{1}{a^2} \frac{1}{a^2+b^2} [F(\varphi, k) - E(\varphi, k)].$$

$$5. \int_x^\infty \frac{x^2 dx}{(x^2+a^2)^2 (x^2-b^2)} = \frac{1}{a^2+b^2} E(\varphi, k).$$

$$6. \int_x^\infty \frac{dx}{(x^2+a^2)(x^2-b^2)^2} = \frac{x}{b^2 (x^2+a^2)(x^2-b^2)} - \frac{1}{b^2} \frac{1}{a^2+b^2} E(\varphi, k).$$

$$7. \int_x^\infty \frac{x dx}{(x^2+a^2)(x^2-b^2)^2} = \frac{1}{a^2+b^2} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{x}{(x^2+a^2)(x^2-b^2)} \quad [x > b]$$

$$8. \int_x^\infty \frac{dx}{(x^2+a^2)^2 (x^2-b^2)^2} = \frac{b^2-a^2}{a^2 b^2} \frac{1}{(a^2+b^2)^2} E(\varphi, k) - \\ - \frac{1}{a^2} \frac{1}{(a^2+b^2)^2} F(\varphi, k) + \frac{x}{b^2 (a^2+b^2) (x^2+a^2)(x^2-b^2)} \quad [x > b]$$

$$9. \int_x^\infty \frac{1}{\sqrt{\frac{x^2+a^2}{x^2-b^2}}} dx = \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k) - \frac{\sqrt{a^2+b^2}}{b^2} E(\varphi, k) + \\ + \frac{(a^2-b^2)x}{b^2 \sqrt{(x^2+a^2)(x^2-b^2)}} \quad [x > b].$$

$$10. \int_1^\infty \frac{1}{x^2} \sqrt{\frac{x^2+a^2}{x^2-b^2}} dx = \frac{\sqrt{a^2+b^2}}{b^2} E(\varphi, k) - \frac{a^2}{b^2 x} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$11. \int_x^\infty \frac{1}{x^2} \sqrt{\frac{x^2-b^2}{x^2+a^2}} dx = \frac{\sqrt{a^2+b^2}}{a^2} [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{\frac{x^2-b^2}{x^2+a^2}}.$$

$$12. \int_1^\infty \frac{1}{\sqrt{\frac{x^2-b^2}{(x^2+a^2)^3}}} dx = \frac{1}{a^2 \sqrt{a^2+b^2}} E(\varphi, k) - \frac{b^2}{a^2 \sqrt{a^2+b^2}} F(\varphi, k).$$

1.2.30. Интегралы вида $\int_0^x R(x, \sqrt{x^2+a^2}, \sqrt{b^2-x^2}) dx$.

Условие: $b \geq x > 0$.

Обозначения: $\varphi = \arcsin \frac{x}{b} \sqrt{\frac{a^2+b^2}{x^2+a^2}}$, $k = \frac{b}{\sqrt{a^2+b^2}}$.

$$1. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k).$$

$$2. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \sqrt{a^2+b^2} E(\varphi, k) - \frac{a^2}{\sqrt{a^2+b^2}} F(\varphi, k) - \\ - x \sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

$$3. \int_0^x \frac{1}{(p-x^2) \sqrt{(x^2+a^2)(b^2-x^2)}} dx = \\ = \frac{1}{p(p+a^2) \sqrt{a^2+b^2}} \left\{ a \operatorname{li} \left[\rho, \frac{b^2(p+a^2)}{p(a^2+b^2)}, k \right] + p F(\varphi, k) \right\} \quad [p \neq 0].$$

$$4. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3 (b^2-x^2)}} = \frac{1}{a^3 \sqrt{a^2+b^2}} E(\varphi, k).$$

$$5. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)^3 (b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}} [F(\varphi, k) - E(\varphi, k)].$$

$$6. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{b^2 \sqrt{a^2+b^2}} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{x}{b^2 \sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x < b]$$

$$7. \int_0^x \frac{x^2 dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{x}{\sqrt{(x^2+a^2)(b^2-x^2)}} - \frac{1}{\sqrt{a^2+b^2}} E(\varphi, k) \quad [x < b]$$

$$8. \int_0^x \frac{dx}{\sqrt{(x^2+a^2)^3(b^2-x^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2+b^2}} [a^2 F(\varphi, k) - (a^2-b^2) E(\varphi, k)] + \\ + \frac{x}{b^2 (a^2+b^2) \sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x < b]$$

$$9. \int_0^x \sqrt{\frac{x^2+a^2}{b^2-x^2}} dx = \sqrt{a^2+b^2} E(\varphi, k) - x \sqrt{\frac{b^2-x^2}{x^2+a^2}}$$

$$10. \int_0^x \sqrt{\frac{x^2+a^2}{(b^2-x^2)^3}} dx = \frac{a^2}{b^2 \sqrt{a^2+b^2}} F(\varphi, k) - \frac{\sqrt{a^2+b^2}}{b^3} E(\varphi, k) + \\ + \frac{(a^2+b^2)x}{b^2 \sqrt{(x^2+a^2)(b^2-x^2)}} \quad [x < b]$$

$$11. \int_0^x \sqrt{\frac{b^2-x^2}{x^2+a^2}} dx = \sqrt{a^2+b^2} [F(\varphi, k) - E(\varphi, k)] + x \sqrt{\frac{b^2-x^2}{x^2+a^2}}$$

$$12. \int_0^x \sqrt{\frac{b^2-x^2}{(x^2+a^2)^3}} dx = \frac{\sqrt{a^2+b^2}}{a^2} E(\varphi, k) - \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k)$$

$$13. \int_0^x \sqrt{(x^2+a^2)(b^2-x^2)} dx = \frac{1}{3} \sqrt{a^2+b^2} [a^2 F(\varphi, k) - (a^2-b^2) E(\varphi, k)] + \\ + \frac{x}{3} (x^2+2a^2-b^2) \sqrt{\frac{b^2-x^2}{x^2+a^2}}$$

1.2.60. Интегралы вида $\int_x^b R(x, \sqrt{x^2+a^2}, \sqrt{b^2-x^2}) dx$.

Условие: $b > x \geq 0$.

Обозначения: $\varphi = \arccos \frac{x}{b}$, $k = \frac{b}{\sqrt{a^2+b^2}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k).$$

$$2. \int_x^b \frac{x^2 dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \sqrt{a^2+b^2} E(\varphi, k) - \frac{a^2}{\sqrt{a^2+b^2}} F(\varphi, k).$$

$$3. \int_x^b \frac{dx}{x^2 \sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2+b^2}} [a^2 F(\varphi, k) - (a^2+b^2) E(\varphi, k)] + \\ + \frac{1}{a^2 b^2 x} \sqrt{(x^2+a^2)(b^2-x^2)} \quad [x > 0].$$

$$4. \int_x^b \frac{dx}{(p-x^2) \sqrt{(x^2+a^2)(b^2-x^2)}} = \\ = -\frac{1}{(p-b^2) \sqrt{a^2+b^2}} \Pi\left(\varphi, \frac{b^2}{b^2-p}, k\right) \quad [p \neq b^2].$$

$$5. \int_x^b \frac{dx}{\sqrt{(x^2+a^2)^3 (b^2-x^2)}} = \frac{1}{a^2 \sqrt{a^2+b^2}} E(\varphi, k) - \frac{x}{a^2 (a^2+b^2)} \sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

$$6. \int_x^b \frac{x^2 dx}{\sqrt{(x^2+a^2)^3 (b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{a^2+b^2} \sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

$$7. \int_x^b \sqrt{\frac{x^2+a^2}{b^2-x^2}} dx = \sqrt{a^2+b^2} E(\varphi, k).$$

$$8. \int_x^b \frac{1}{x^2} \sqrt{\frac{x^2+a^2}{b^2-x^2}} dx = \frac{\sqrt{a^2+b^2}}{b^2} [F(\varphi, k) - E(\varphi, k)] + \\ + \frac{1}{b^2 x} \sqrt{(b^2-x^2)(x^2+a^2)} \quad [x > 0].$$

$$9. \int_x^b \sqrt{\frac{b^2-x^2}{x^2+a^2}} dx = \sqrt{a^2+b^2} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_x^b \frac{1}{x^2} \sqrt{\frac{b^2-x^2}{x^2+a^2}} dx = \frac{\sqrt{(b^2-x^2)(x^2+a^2)}}{a^2 x} - \frac{\sqrt{a^2+b^2}}{a^2} E(\varphi, k) \quad [x > 0].$$

$$11. \int_x^b \sqrt{\frac{b^2-x^2}{(x^2+a^2)^3}} dx = \frac{\sqrt{a^2+b^2}}{a^2} E(\varphi, k) - \frac{1}{\sqrt{a^2+b^2}} F(\varphi, k) - \frac{x}{a^2} \sqrt{\frac{b^2-x^2}{x^2+a^2}}.$$

$$12. \int_x^b \frac{1}{\sqrt{(x^2+a^2)(b^2-x^2)}} dx = \frac{1}{3} \sqrt{a^2+b^2} [a^2 F(\varphi, k) + 2(b^2-a^2) E(\varphi, k)] + \\ + \frac{x}{3} \sqrt{(x^2+a^2)(b^2-x^2)}.$$

1.2.61. Интегралы вида $\int_a^x R(x, \sqrt{x^2-a^2}, \sqrt{x^2-b^2}) dx$.

Условие: $a > c > b > 0$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{x^2-a^2}{x^2-b^2}}$.

$$1. \int_a^x \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_a^x \frac{x^2 dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = a \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + x \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$3. \int_a^x \frac{dx}{x^2 \sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{a^2 x} \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$4. \int_a^x \frac{dx}{(p-x) \sqrt{(x^2-a^2)(x^2-b^2)}} =$$

$$= \frac{1}{a(p-a^2)(p-b^2)} \left[(a^2-b^2) \Pi\left(\varphi, \frac{p-b^2}{p-a^2}, \frac{b}{a}\right) + (p-a^2) F\left(\varphi, \frac{b}{a}\right) \right]$$

(1.2.62)

$$5. \int_a^x \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{b^3(a^2-b^2)} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{ab^3} F\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_a^x \frac{x^2 dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{a^2-b^2} E\left(\varphi, \frac{b}{a}\right).$$

$$7. \int_a^x \sqrt{\frac{x^2-a^2}{x^2-b^2}} dx = x \sqrt{\frac{x^2-a^2}{x^2-b^2}} - a E\left(\varphi, \frac{b}{a}\right).$$

$$8. \int_a^x \frac{1}{x^2} \sqrt{\frac{x^2-a^2}{x^2-b^2}} dx = \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{a^2-b^2}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{1}{x} \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$9. \int_a^x \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}} dx = \frac{a}{b^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$10. \int_a^x \sqrt{\frac{x^2-b^2}{x^2-a^2}} dx = \frac{a^2-b^2}{a} F\left(\varphi, \frac{b}{a}\right) - a E\left(\varphi, \frac{b}{a}\right) + x \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$11. \int_a^x \frac{1}{x^3} \sqrt{\frac{x^2-b^2}{x^2-a^2}} dx = \frac{1}{a} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{x} \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$12. \int_a^x \frac{1}{(x^2-a^2)(x^2-b^2)} dx = \frac{1}{3} \left[(a^2+b^2) E\left(\varphi, \frac{b}{a}\right) - \right. \\ \left. - (a^2-b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} (x^2-a^2-2b^2) \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$13. \int \frac{dx}{x(x^2-1)(x^2-a^2)} = -\frac{1}{2a(a+1)} \left[F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{x+1+a}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \pm \right. \\ \left. \pm F\left(\arcsin \frac{1}{|x-1-a|}, \frac{1}{\sqrt{2(a+1)}}\right) \right] \\ \left[a > 0: \begin{cases} 0 \leq x \leq \min(1, a) \\ \max(1, a) \leq x < \infty \end{cases} \right].$$

$$14. \int \frac{\sqrt{x} dx}{x(x^2-1)(x^2-a^2)} = \frac{1}{2(a+1)} \left[F\left(\arcsin \frac{\sqrt{(x-1)(x-a)}}{x+1+a}, \frac{|\sqrt{a}-1|}{\sqrt{2(a+1)}}\right) \pm \right. \\ \left. \pm F\left(\arcsin \frac{1}{|x-1-a|}, \frac{1}{\sqrt{2(a+1)}}\right) \right] \\ \left[a > 0: \begin{cases} 0 \leq x \leq \min(1, a) \\ \max(1, a) \leq x < \infty \end{cases} \right].$$

1.2.62. Интегралы вида $\int_a^{\infty} R(x, \sqrt{x^2-a^2}, \sqrt{x^2-b^2}) dx$.

Условие: $x > a > b > 0$.

Обозначение: $\varphi = \arcsin \frac{a}{x}$.

$$1. \int_a^{\infty} \frac{dx}{(x^2-a^2)(x^2-b^2)} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_a^{\infty} \frac{dx}{x^2 \sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$3. \int_a^{\infty} \frac{dx}{(x^2-p) \sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap} \left[H\left(\varphi, \frac{p}{a^2}, \frac{b}{a}\right) - F\left(\varphi, \frac{b}{a}\right) \right]$$

$(p \neq 0)$.

$$4. \int_a^{\infty} \frac{dx}{(x^2-a^2)^{3/2} \sqrt{(x^2-b^2)}} = \frac{1}{a(b^2-a^2)} \left[E\left(\varphi, \frac{b}{a}\right) - \frac{a}{x} \sqrt{\frac{x^2-b^2}{x^2-a^2}} \right]$$

$(a > b)$.

$$5. \int_a^{\infty} \frac{x^2 dx}{(x^2-a^2)^2 \sqrt{(x^2-b^2)}} = \frac{a}{a^2-b^2} \left[\frac{a}{x} \sqrt{\frac{x^2-b^2}{x^2-a^2}} - E\left(\varphi, \frac{b}{a}\right) \right] + \\ + \frac{1}{a} F\left(\varphi, \frac{b}{a}\right) \quad (a > b).$$

$$6. \int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} =$$

$$= \frac{1}{b^2(a^2-b^2)} \left[aE\left(\frac{b}{a}\right) + \frac{b^2}{x} \sqrt{\frac{x^2-a^2}{x^2-b^2}} - \frac{1}{a b^2} F\left(\frac{b}{a}\right) \right].$$

$$7. \int_x^a \frac{x^2 dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a^2-b^2} \left[aE\left(\frac{b}{a}\right) - \frac{b^2}{x} \sqrt{\frac{x^2-a^2}{x^2-b^2}} \right].$$

$$8. \int_x^a \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a b^2(a^2-b^2)} F\left(\frac{b}{a}\right) - \frac{1}{a b^2(a^2-b^2)} E\left(\frac{b}{a}\right) + \frac{1}{x(a^2-b^2)} \sqrt{\frac{x^2-a^2}{x^2-b^2}} \quad (x > a).$$

$$9. \int_x^a \frac{1}{x^2} \sqrt{\frac{x^2-a^2}{x^2-b^2}} dx = \frac{a}{b^2} E\left(\frac{b}{a}\right) - \frac{a^2-b^2}{a b^2} F\left(\frac{b}{a}\right).$$

$$10. \int_x^a \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}} dx = \frac{a}{b^2} \left[F\left(\frac{b}{a}\right) - E\left(\frac{b}{a}\right) \right] + \frac{1}{x} \sqrt{\frac{x^2-a^2}{x^2-b^2}}.$$

$$11. \int_x^a \frac{1}{x^2} \sqrt{\frac{x^2-b^2}{x^2-a^2}} dx = \frac{1}{a} E\left(\frac{b}{a}\right).$$

$$12. \int_x^a \sqrt{\frac{x^2-b^2}{(x^2-a^2)^3}} dx = \frac{1}{a} \left[F\left(\frac{b}{a}\right) - E\left(\frac{b}{a}\right) \right] + \frac{1}{x} \sqrt{\frac{x^2-b^2}{x^2-a^2}} \quad (x > a).$$

1.2.63. Интегралы вида $\int_x^a R(x) \sqrt{a^2-x^2} \sqrt{x^2-b^2} dx$.

Условия: $a > x > b > 0$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a^2-x^2}{a^2-b^2}}$, $k = \sqrt{\frac{a^2-b^2}{a^2}}$.

$$1. \int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a} F(\varphi, k),$$

$$2. \int_x^a \frac{x^2 dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = a E(\varphi, k),$$

$$3. \int_x^a \frac{dx}{x^2 \sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a b^2} E(\varphi, k) - \frac{1}{x b^2} \sqrt{\frac{a^2-x^2}{x^2-b^2}}.$$

$$4. \int_x^a \frac{dx}{(x^2-p)(x^2-x^2)(x^2-b^2)} = \frac{1}{a(a^2-p)} \Pi\left(\frac{a^2-b^2}{\eta}, \frac{a^2-b^2}{a^2-p}, k\right) \quad [p \neq a^2].$$

$$5. \int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} =$$

$$= \frac{1}{a b^2(a^2-b^2)} \left[b^2 F(\eta, k) - a^2 E(\eta, k) + a x \sqrt{\frac{a^2-x^2}{x^2-b^2}} \right] \quad [x > b].$$

$$6. \int_x^a \frac{x^2 dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{a^2-b^2} \left[a F(\eta, k) - a E(\eta, k) + x \sqrt{\frac{a^2-x^2}{x^2-b^2}} \right] \quad [x > b].$$

$$7. \int_x^a \sqrt{\frac{a^2-x^2}{x^2-b^2}} dx = a [F(\eta, k) - E(\eta, k)].$$

$$8. \int_x^a \frac{1}{x^2} \sqrt{\frac{a^2-x^2}{x^2-b^2}} dx = \frac{a}{b^2} E(\eta, k) - \frac{1}{a} F(\eta, k) - \frac{1}{b^2 x} \sqrt{\frac{a^2-x^2}{x^2-b^2}}.$$

$$9. \int_x^a \sqrt{\frac{a^2-x^2}{(x^2-b^2)^3}} dx = \frac{x}{b^2} \sqrt{\frac{a^2-x^2}{x^2-b^2}} - \frac{a}{b^2} E(\eta, k)$$

$$10. \int_x^a \sqrt{\frac{x^2-b^2}{a^2-x^2}} dx = a E(\eta, k) - \frac{b^2}{a} F(\eta, k),$$

$$11. \int_x^a \frac{1}{x^2} \sqrt{\frac{x^2-b^2}{a^2-x^2}} dx = \frac{1}{a} [F(\eta, k) - E(\eta, k)] + \frac{1}{a^2 x} \sqrt{\frac{a^2-x^2}{x^2-b^2}}.$$

$$12. \int_x^a \sqrt{\frac{a^2-x^2}{(x^2-b^2)^3}} dx = \frac{a}{3} [(a^2+b^2) E(\eta, k) - 2 b^2 F(\eta, k)] -$$

$$- \frac{x}{3} \sqrt{\frac{a^2-x^2}{x^2-b^2}}.$$

1.2.64. Интегралы вида $\int_0^x R(x) \sqrt{a^2-x^2} \sqrt{x^2-b^2} dx$.

Условия: $a > x > b > 0$.

Обозначения: $\eta = \arcsin \frac{a}{x} \sqrt{\frac{x^2-b^2}{a^2-b^2}}$, $k = \sqrt{\frac{a^2-b^2}{a^2}}$.

$$1. \int_0^x \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a} F(\eta, k).$$

$$2. \int_b^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = aE(\varphi, k) - \frac{1}{x} \sqrt{(a^2 - x^2)(x^2 - b^2)}.$$

$$3. \int_b^x \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\varphi, k).$$

$$4. \int_b^x \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \\ = \frac{1}{ap(p - b^2)} \left\{ b^2 H \left[\varphi, \frac{p(a^2 - b^2)}{a^2(p - b^2)}, k \right] + (p - b^2) F(\varphi, k) \right\} \quad [p > b^2].$$

$$5. \int_b^x \frac{dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \frac{1}{a(a^2 - b^2)} \left[F(\varphi, k) - E(\varphi, k) + \frac{a}{x} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} \right] \quad [x < a].$$

$$6. \int_b^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3 (x^2 - b^2)}} = \\ = \frac{1}{a(a^2 - b^2)} \left[b^2 F(\varphi, k) - a^2 E(\varphi, k) + \frac{a^3}{x} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} \right] \quad [x < a].$$

$$7. \int_b^x \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a [F(\varphi, k) - E(\varphi, k)] + \frac{1}{x} \sqrt{(a^2 - x^2)(x^2 - b^2)}.$$

$$8. \int_b^x \frac{1}{x^3} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = \frac{a}{b^3} E(\varphi, k) - \frac{1}{a} F(\varphi, k).$$

$$9. \int_b^x \frac{1}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = \frac{1}{a} [F(\varphi, k) - E(\varphi, k)].$$

$$10. \int_b^x \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = aE(\varphi, k) - \frac{b^2}{a} F(\varphi, k) - \frac{1}{x} \sqrt{(a^2 - x^2)(x^2 - b^2)}.$$

$$11. \int_b^x \sqrt{\frac{x^2 - b^2}{(a^2 - x^2)^3}} dx = \frac{1}{x} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} - \frac{1}{a} E(\varphi, k) \quad [x < a].$$

$$12. \int_b^x \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} [(a^2 + b^2) E(\varphi, k) - 2b^2 F(\varphi, k)] + \\ + \frac{x^2 - a^2 - b^2}{3x} \sqrt{(a^2 - x^2)(x^2 - b^2)}.$$

1.2.65. Интегралы вида $\int_0^x R(x, \sqrt{a^2 - x^2}, \sqrt{b^2 - x^2}) dx$.

Условие: $a > b > x > 0$.

Обозначение: $\varphi = \arcsin \frac{x}{b}$.

$$1. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$3. \int_0^x \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ap} \Pi\left(\varphi, \frac{b^2}{p}, \frac{b}{a}\right) \quad [p \neq b].$$

$$4. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)^2 (b^2 - x^2)}} = \frac{1}{a^2 (a^2 - b^2)} \left[aE\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$5. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)^2 (b^2 - x^2)}} = \frac{1}{a^2 - b^2} \left[aE\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right] - \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{ab^2} F\left(\varphi, \frac{b}{a}\right) - \frac{1}{b^2 (a^2 - b^2)} \left[aE\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} \right] \quad [x < b].$$

$$7. \int_0^x \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{a^2 - b^2} \left[x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} - aE\left(\varphi, \frac{b}{a}\right) \right] \quad [x < b].$$

$$8. \int_0^x \frac{dx}{\sqrt{(a^2 - x^2)^3 (b^2 - x^2)}} = \frac{1}{ab^2 (a^2 - b^2)} F\left(\varphi, \frac{b}{a}\right) - \frac{a^2 + b^2}{ab^2 (a^2 - b^2)^2} E\left(\varphi, \frac{b}{a}\right) + \frac{[a^4 + b^4 - (a^2 + b^2)x^2] x}{a^2 b^2 (a^2 - b^2)^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} \quad [x < b].$$

$$9. \int_0^x \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = aE\left(\varphi, \frac{b}{a}\right).$$

$$10. \int_0^x \sqrt{\frac{a^2 - x^2}{(b^2 - x^2)^3}} dx = \frac{a}{b^3} \left[F\left(\varphi, \frac{b}{a}\right) + E\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{b^3} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} \quad [x < b].$$

$$11. \int_0^x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = aE\left(\varphi, \frac{b}{a}\right) - \frac{a^2 - b^2}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$12. \int_0^x \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) + \frac{x}{a} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$13. \int_0^x \frac{1}{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \left[(a^2 + b^2) E\left(\varphi, \frac{b}{a}\right) - \right. \\ \left. - (a^2 - b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} \sqrt{(a^2 - x^2)(b^2 - x^2)}.$$

1.2.66. Интегралы вида $\int_x^b R(x, \sqrt{a^2 - x^2}, \sqrt{b^2 - x^2}) dx$.

Условие: $a > b > x \geq 0$.

Обозначение: $\varphi = \arcsin \frac{a}{b} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$.

$$1. \int_x^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$2. \int_x^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + x \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}.$$

$$3. \int_x^b \frac{dx}{x^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ab^2} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right] + \frac{1}{b^2 x} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}}$$

$[x > 0].$

$$4. \int_x^b \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \\ = \frac{1}{a(p - a^2)(p - b^2)} \left\{ (b^2 - a^2) \Pi\left[\varphi, \frac{b^2(p - a^2)}{a^2(p - b^2)}, \frac{b}{a}\right] + \right. \\ \left. + (p - b^2) F\left(\varphi, \frac{b}{a}\right) \right\} \quad [p \neq b^2].$$

$$5. \int_x^b \frac{dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{1}{a(a^2 - b^2)} E\left(\varphi, \frac{b}{a}\right).$$

$$6. \int_x^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{a}{a^2 - b^2} E\left(\varphi, \frac{b}{a}\right) - \frac{1}{a} F\left(\varphi, \frac{b}{a}\right).$$

$$7. \int_x^b \sqrt{\frac{b^2 - x^2}{b^2 - x^2}} dx = a \left[E\left(\varphi, \frac{b}{a}\right) - \frac{x}{a} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \right].$$

$$8. \int_x^a \frac{1}{x^2} \sqrt{\frac{a^2-x^2}{b^2-x^2}} dx = \frac{a^2-b^2}{ab^3} F\left(\varphi, \frac{b}{a}\right) - \frac{a}{b^2} E\left(\varphi, \frac{b}{a}\right) + \frac{c^2}{b^2 x} \sqrt{\frac{b^2-x^2}{a^2-x^2}} \quad [x > 0].$$

$$9. \int_x^b \sqrt{\frac{b^2-x^2}{c^2-x^2}} dx = a E\left(\varphi, \frac{b}{a}\right) - \frac{a^2-b^2}{a} F\left(\varphi, \frac{b}{a}\right) - x \sqrt{\frac{b^2-x^2}{a^2-x^2}}.$$

$$10. \int_x^b \sqrt{\frac{b^2-x^2}{(a^2-x^2)^3}} dx = \frac{1}{a} \left[F\left(\varphi, \frac{b}{a}\right) - E\left(\varphi, \frac{b}{a}\right) \right].$$

$$11. \int_x^b \frac{1}{x^2} \sqrt{\frac{b^2-x^2}{a^2-x^2}} dx = \frac{1}{x} \sqrt{\frac{b^2-x^2}{a^2-x^2}} - \frac{1}{a} E\left(\varphi, \frac{b}{a}\right) \quad [x > 0].$$

$$12. \int_x^b \sqrt{(a^2-x^2)(b^2-x^2)} dx = \frac{a}{3} \left[(a^2+b^2) E\left(\varphi, \frac{b}{a}\right) - (a^2-b^2) F\left(\varphi, \frac{b}{a}\right) \right] + \frac{x}{3} (x^2-2a^2-b^2) \sqrt{\frac{b^2-x^2}{a^2-x^2}}.$$

1.2.67. Интегралы вида $\int R(x, \sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}) dx$.

Обозначения: $\varphi = \arccos \frac{x^2-\rho\bar{\rho}}{x^2+\rho\rho}$, $k = \frac{1}{2} \sqrt{-\frac{(\rho-\bar{\rho})^2}{\rho\bar{\rho}}}$, ρ и $\bar{\rho}$ — комплексно сопряженные числа.

$$1. \int_x^\infty \frac{dx}{\sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}} = \frac{1}{\rho\bar{\rho}} F(\varphi, k).$$

$$2. \int_x^\infty \frac{x^2 dx}{(x^2-\rho\bar{\rho})^2 \sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}} = -\frac{2x \sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}}{(\rho+\bar{\rho})^2 (x^2-\rho^2\bar{\rho}^2)} - \frac{1}{(\rho+\bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E(\varphi, k).$$

$$3. \int_x^\infty \frac{x^2 dx}{(x^2+\rho\bar{\rho})^2 \sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}} = -\frac{1}{(\rho-\bar{\rho})^2 \sqrt{\rho\bar{\rho}}} [F(\varphi, k) - E(\varphi, k)].$$

$$4. \int_x^\infty \frac{x^2 dx}{\sqrt{(x^2+\rho^2)^3 (x^2+\bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho^2-\bar{\rho}^2)^2} E(\varphi, k) + \frac{1}{(\rho-\bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\varphi, k) - \frac{2x(x^2-\rho\bar{\rho})}{(\rho+\bar{\rho})^2 (x^2+\rho\bar{\rho}) \sqrt{(x^2+\rho^2)(x^2+\bar{\rho}^2)}}.$$

$$5. \int_{-\infty}^{\infty} \frac{(x^2 - \rho\bar{\rho}) dx}{(x^2 + \rho^2)^2 (x^2 + \bar{\rho}^2)} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} [F(\varphi, k) - E(\varphi, k)] + \frac{2x(x^2 - \rho\bar{\rho})}{(x^2 + \rho\bar{\rho}) \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}}.$$

$$6. \int_{-\infty}^{\infty} \frac{(x^2 - \rho\bar{\rho}) dx}{(x^2 + \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} E(\varphi, k) + \frac{(\rho + \bar{\rho})^2}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E_1(\varphi, k).$$

$$7. \int_{-\infty}^{\infty} \frac{(x^2 + \rho\bar{\rho})^2 dx}{[(x^2 + \rho\bar{\rho})^2 - 4\rho\bar{\rho}\rho\bar{\rho}] \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{1}{\sqrt{\rho\bar{\rho}}} \Pi(\varphi, \rho^2, k).$$

$$8. \int_{-\infty}^{\infty} \frac{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)} dx}{(x^2 + \rho\bar{\rho})^2} = \frac{1}{\sqrt{\rho\bar{\rho}}} F(\varphi, k).$$

1.2.68. Интегралы вида $\int R(x, \sqrt{x^2 + 1}) dx$.

Условие: $x > -1$.

Обозначения: $\varphi = \frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}$, $k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

$$1. \int \frac{x^n}{x^2+1} dx = \frac{2}{2n-1} x^{n-2} \sqrt{x^2+1} - \frac{2n-4}{2n-1} \int \frac{x^{n-2}}{x^2+1} dx.$$

$$2. \int \frac{dx}{x^2+1} = -\frac{1}{\sqrt{3}} F(\varphi, k).$$

$$3. \int \frac{x}{x^2+1} dx = \frac{2\sqrt{x^2+1}}{x+1+\sqrt{3}} - \frac{\sqrt{3}-1}{\sqrt{3}} F(\varphi, k) + 2\sqrt{3} E(\varphi, k).$$

$$4. \int \frac{x^2}{x^2+1} dx = \frac{2}{3} \sqrt{x^2+1}.$$

$$5. \int \frac{dx}{(x-\rho)\sqrt{x^2+1}} = \frac{1}{\sqrt{3}(\rho+1+\sqrt{3})} F(\varphi, k) - \frac{1}{2\sqrt{3}(\rho+1)} \left[\frac{\rho+1-\sqrt{3}}{\rho+1+\sqrt{3}} \Pi(\varphi, -\rho, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x) \sqrt{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(\rho+1+\sqrt{3})^2}{4\sqrt{3}(\rho+1)}; \rho \neq -1, -1-\sqrt{3}; \text{см. 1.5.33. 12-14} \right].$$

$$6. \int \frac{dx}{(x+1+\sqrt{3})\sqrt{x^2+1}} = -\frac{\sqrt{3}}{6} F(\varphi, k) + \frac{\sqrt{3}}{6k} \arcsin(k \sin \varphi).$$

$$7. \int \frac{dx}{(x-1)\sqrt{x^2+1} \sqrt{3(x+1)(x+1+\sqrt{3})}} = -\frac{\sqrt{3}}{3} [F(\varphi, k) - 2E(\varphi, k)].$$

*) См. также 1.2.73.

$$8. \int \frac{dx}{x \sqrt{x^3-1}} = \frac{1}{2} \frac{1}{\sqrt{3}} F(\varphi, k) + \frac{2-\sqrt{3}}{2\sqrt{3}} \Pi\left(\varphi, \frac{3-2\sqrt{3}}{6}, k\right) - \frac{1}{4} \ln \frac{x^2-2+2\sqrt{x^3+1}}{x^2-2-2\sqrt{x^3+1}}.$$

1.2.99. Интегралы вида $\int R(x, \sqrt{x^3-1}) dx$ *).

Условие: $x > 1$.

Обозначения: $\varphi = \arccos \frac{1}{\sqrt{3}} \frac{3+1-x}{1+3-1+x}$, $\psi = \arccos \frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$,

$$k = \sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

$$1. \int \frac{dx}{1-\sqrt{x^3-1}} = \frac{1}{\sqrt{3}} F(\varphi, k), \quad 2. \int \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt{3}} F(\psi, k).$$

$$3. \int \frac{x dx}{\sqrt{x^3-1}} = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) F(\varphi, k) - 2 \frac{1}{\sqrt{3}} E(\varphi, k) + \frac{2}{\sqrt{3-1+x}}.$$

$$4. \int \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt{3}} [F(\psi, k) - 2E(\psi, k)] + \frac{2}{\sqrt{3(x-1+\sqrt{3})\sqrt{x-1}}} \quad (x > 1).$$

$$5. \int \frac{(x-1) dx}{(1+\sqrt{3-x})^2 \sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt{3}} E(\varphi, k).$$

$$6. \int \frac{(x-1) dx}{(1+\sqrt{3-x})^2 \sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{x^3-1}}{x^2-2x-2} - \frac{2-\sqrt{3}}{\sqrt{3}} E(\psi, k).$$

$$7. \int \frac{(x-1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt{3}} [F(\varphi, k) - E(\varphi, k)].$$

$$8. \int \frac{(x-1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{2-\sqrt{3}}{\sqrt{3}} [F(\psi, k) - E(\psi, k)].$$

$$9. \int \frac{(x^2+x+1) dx}{(x-1+\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt{3}} E(\varphi, k).$$

$$10. \int \frac{(x^2+x+1) dx}{(x-1-\sqrt{3})^2 \sqrt{x^3-1}} = \frac{1}{\sqrt{3}} E(\psi, k).$$

*). См. также 1.2.73.

$$11. \int_1^x \frac{(x-1) dx}{(x^2+x+1) \sqrt{x^3-1}} = \frac{4}{\sqrt[3]{27}} E(\varphi, k) - \\ - \frac{2+\sqrt[3]{3}}{\sqrt[3]{27}} F(\varphi, k) - \frac{2-\sqrt[3]{3}}{\sqrt[3]{3}} \frac{2(x-1)(\sqrt[3]{3}+1-x)}{(\sqrt[3]{3}-1+x) \sqrt{x^3-1}}.$$

$$12. \int_1^x \frac{(x-1+\sqrt[3]{3})^2 dx}{[(x-1+\sqrt[3]{3})^2-4\sqrt[3]{3}\rho^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[3]{3}} \Pi(\varphi, \rho^2, k).$$

$$13. \int_x^\infty \frac{(x-1+\sqrt[3]{3})^2 dx}{[(x-1+\sqrt[3]{3})^2-4\sqrt[3]{3}\rho^2(x-1)] \sqrt{x^3-1}} = \frac{1}{\sqrt[3]{3}} \Pi(\varphi, \rho^2, k).$$

1.2.70. Интегралы вида $\int R(x, \sqrt[3]{1-x^3}) dx$.

Условие: $x \leq 1$.

Обозначения: $\varphi = \arccos \frac{x-1+\sqrt[3]{3}}{x-1-\sqrt[3]{3}}$, $\psi = \arccos \frac{\sqrt[3]{3}-1+x}{\sqrt[3]{3}+1-x}$.

$$k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt[3]{3}}}{2}.$$

$$1. \int \frac{x^m dx}{\sqrt[3]{1-x^3}} = \frac{-2x^{m+2} \sqrt[3]{1-x^3}}{2m-1} + \frac{2(m-2)}{2m-1} \int \frac{x^{m-2} dx}{\sqrt[3]{1-x^3}}.$$

$$2. \int_{-\infty}^x \frac{dx}{\sqrt[3]{1-x^3}} = \frac{1}{\sqrt[3]{3}} F(\varphi, k). \quad 3. \int_x^1 \frac{dx}{\sqrt[3]{1-x^3}} = \frac{1}{\sqrt[3]{3}} F(\psi, k).$$

$$4. \int \frac{x dx}{\sqrt[3]{1-x^3}} = \left(\frac{1}{\sqrt[3]{3}} - \sqrt[3]{3} \right) F(\psi, k) + 2\sqrt[3]{3} E(\psi, k) - \frac{2\sqrt[3]{1-x^3}}{\sqrt[3]{3}+1-x}.$$

$$5. \int \frac{x^2 dx}{\sqrt[3]{1-x^3}} = -\frac{2}{3} \sqrt[3]{1-x^3}.$$

$$6. \int \frac{dx}{(x-\rho) \sqrt[3]{1-x^3}} = \frac{-1}{\sqrt[3]{3}(\rho-1-\sqrt[3]{3})} F(\varphi, k) + \\ + \frac{1}{2\sqrt[3]{3}(\rho-1)} \left[\frac{\rho-1+\sqrt[3]{3}}{\rho-1-\sqrt[3]{3}} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x) \sqrt[3]{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(\rho-1-\sqrt[3]{3})^2}{4\sqrt[3]{3}(\rho-1)}; \rho \neq 1, 1+\sqrt[3]{3}; \text{ см. 1.5.33, 12-14} \right].$$

$$7. \int \frac{dx}{(x-1-\sqrt[3]{3}) \sqrt[3]{1-x^3}} = \frac{-\sqrt[3]{3}}{6} F(\varphi, k) + \frac{\sqrt[3]{3}}{6k} \arcsin(k \sin \varphi).$$

*) См. также 1.2.74.

- $$8. \int \frac{dx}{(x-1)\sqrt{1-x^3}} = \frac{2\sqrt{x^2+x+1}}{\sqrt{3}(x-1-\sqrt{3})\sqrt{1-x}} - \frac{1}{\sqrt{3}} \frac{1}{27} [F(\varphi, k) - 2E(\varphi, k)] \quad (x < 1).$$
- $$9. \int \frac{dx}{x\sqrt{1-x^3}} = \frac{\sqrt{3-1}}{2\sqrt[3]{3}} F(\varphi, k) + \frac{2-\sqrt{3}}{2\sqrt[3]{3}} \Pi\left(\varphi, \frac{3+2\sqrt{3}}{6}, k\right) + \frac{1}{4} \ln \frac{x^2+2-2\sqrt{1-x^3}}{x^3+2+2\sqrt{1-x^3}}.$$
- $$10. \int_{-\infty}^x \frac{(x-1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[3]{3}} \frac{1}{27} \left[E(\varphi, k) - \frac{2\sqrt[3]{3}\sqrt{1-x^3}}{x^3-2x-2} \right].$$
- $$11. \int_{-\infty}^x \frac{(x-1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[3]{3}} \frac{1}{27} [F(\varphi, k) - E(\varphi, k)].$$
- $$12. \int_x^1 \frac{(x-1)dx}{(x-1-\sqrt{3})^2\sqrt{1-x^3}} = \frac{\sqrt{3}-2}{\sqrt[3]{3}} \frac{1}{27} [F(\varphi, k) - E(\varphi, k)].$$
- $$13. \int_{-\infty}^x \frac{(x^2+x+1)dx}{(x-1-\sqrt{3})^3\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} E(\varphi, k).$$
- $$14. \int_x^1 \frac{(x^2+x+1)dx}{(x-1-\sqrt{3})^3\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} E(\varphi, k).$$
- $$15. \int_x^1 \frac{(x-1-\sqrt{3})^2 dx}{[(x-1-\sqrt{3})^2+4\sqrt[3]{3}\rho^2(x-1)]\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} \Pi(\varphi, \rho^2, k).$$
- $$16. \int_{-\infty}^x \frac{(x-1-\sqrt{3})^2 dx}{[(x-1-\sqrt{3})^2+4\sqrt[3]{3}\rho^2(x-1)]\sqrt{1-x^3}} = \frac{1}{\sqrt[3]{3}} \Pi(\varphi, \rho^2, k).$$
- $$17. \int \sqrt{1-x^3} dx = \frac{1}{5} \left[\frac{1}{\sqrt[3]{3}} \frac{1}{27} F(\varphi, k) - 2x\sqrt{1-x^3} \right].$$
- $$18. \int_0^x \sqrt{\frac{x}{1-x^3}} \frac{dx}{1-x} = \frac{1}{\sqrt[3]{3}} \frac{1}{27} \left[F\left(\arccos \frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) - 2E\left(\arccos \frac{1-(1+\sqrt{3})x}{1+(\sqrt{3}-1)x}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) \right] + \frac{2}{\sqrt[3]{3}} \frac{\sqrt{x(1+x+x^2)}}{\sqrt{1-x}(1+\sqrt{3}x-x^2)} \quad (x < 1).$$

1.2.71. Интегралы вида $\int R(x, \sqrt{x^2+1}) dx$.

Обозначения: $\varphi = \arccos \frac{1-x}{1+x}$, $\psi = \operatorname{arctg} \left[(1+\sqrt{2}) \frac{1-x}{1+x} \right]$,

$$k = \frac{1}{2} \sqrt{2(1-\sqrt{2})}.$$

$$1. \int \frac{x^n dx}{1+x^4+1} = \frac{x^{n-3}}{n-1} \sqrt{x^4+1} - \frac{n-3}{n-1} \int \frac{x^{n-4} dx}{\sqrt{x^4+1}} dx.$$

$$2. \int_0^x \frac{dx}{1+x^4+1} = \frac{1}{2} F \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) \quad (x \leq 1).$$

$$3. \int_x^\infty \frac{dx}{1+x^4+1} = \frac{1}{2} F \left(\varphi, \frac{\sqrt{2}}{2} \right) \quad (x \geq 1).$$

$$4. \int_x^1 \frac{dx}{1+x^4+1} = (2-\sqrt{2}) F(\psi, k) \quad (0 \leq x < 1).$$

$$5. \int \frac{x dx}{1+x^4+1} = \frac{1}{2} \ln(x^2 + \sqrt{x^4+1}).$$

$$6. \int_0^x \frac{x^2 dx}{1+x^4+1} = \frac{x \sqrt{x^4+1}}{x^2+1} + \frac{1}{2} F \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) - \\ - E \left(\arccos \frac{1-x^2}{1+x^2}, \frac{\sqrt{2}}{2} \right) \quad (0 \leq x \leq 1).$$

$$7. \int_1^x \frac{x^2 dx}{1+x^4+1} = \frac{x \sqrt{x^4+1}}{x^2+1} - \frac{1}{2} F \left(\arccos \frac{x^2-1}{x^2+1}, \frac{\sqrt{2}}{2} \right) + \\ + E \left(\arccos \frac{x^2-1}{x^2+1}, \frac{\sqrt{2}}{2} \right) \quad (x \geq 1).$$

$$8. \int \frac{x^3 dx}{1+x^4+1} = \frac{1}{2} \sqrt{x^4+1}. \quad 9. \int \frac{dx}{x \sqrt{x^4+1}} = \frac{1}{2} \ln \frac{x^3}{1+x^4+1}.$$

$$10. \int_x^\infty \frac{dx}{x^2 \sqrt{x^4+1}} = \frac{1}{2} \left[F \left(\varphi, \frac{\sqrt{2}}{2} \right) - 2E \left(\varphi, \frac{\sqrt{2}}{2} \right) \right] + \frac{\sqrt{x^4+1}}{x(x^2+1)} \quad (x \geq 1).$$

$$11. \int \frac{(1-x)^2 dx}{(x^2-x\sqrt{2}+1) \sqrt{x^4+1}} = \frac{1}{2} [F(\psi, k) - E(\psi, k)] \quad (0 \leq x < 1).$$

$$12. \int \frac{(1+x)^2 dx}{(x^2+x\sqrt{2}+1) \sqrt{x^4+1}} = \frac{3\sqrt{2}+4}{2} E(\psi, k) - \frac{3\sqrt{2}-4}{2} F(\psi, k) \quad (0 \leq x < 1).$$

$$13. \int_x^1 \frac{(x^2 + x\sqrt{2} + 1) dx}{(x^2 - x\sqrt{2} + 1)\sqrt{x^4 + 1}} = (2 + \sqrt{2}) E(\varphi, k) \quad [0 \leq x < 1].$$

$$14. \int_x^\infty \frac{x^2 dx}{(x^4 + 1)\sqrt{x^4 + 1}} = \frac{1}{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{1}{4} F\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{x(x^2 - 1)}{2(x^2 + 1)\sqrt{x^4 + 1}} \quad [x \geq 1].$$

$$15. \int_x^\infty \frac{x^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = \frac{1}{4} \left[F\left(\varphi, \frac{\sqrt{2}}{2}\right) - E\left(\varphi, \frac{\sqrt{2}}{2}\right) \right] \quad [x \geq 1].$$

$$16. \int_x^\infty \frac{x^2 dx}{(x^2 - 1)^2 \sqrt{x^4 + 1}} = \frac{x\sqrt{x^4 + 1}}{2(x^4 - 1)} - \frac{1}{4} E\left(\varphi, \frac{\sqrt{2}}{2}\right) \quad [x > 1].$$

$$17. \int_x^\infty \frac{(x^2 + 1)^2 dx}{[(x^2 + 1)^2 - 4x^2] \sqrt{x^4 + 1}} = \frac{1}{2} \Pi\left(\varphi, p^2, \frac{\sqrt{2}}{2}\right) \quad [x \geq 1].$$

$$18. \int_x^\infty \frac{(x^2 - 1)^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = E\left(\varphi, \frac{\sqrt{2}}{2}\right) - \frac{1}{2} F\left(\varphi, \frac{\sqrt{2}}{2}\right) \quad [x \geq 1].$$

$$19. \int_x^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 + 1)^2} = \frac{1}{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right) \quad [x \geq 1].$$

$$20. \int_x^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 - 1)^2} = \frac{1}{2} \left[F\left(\varphi, \frac{\sqrt{2}}{2}\right) - E\left(\varphi, \frac{\sqrt{2}}{2}\right) \right] + \frac{x\sqrt{x^4 + 1}}{x^4 - 1} \quad [x > 1].$$

1.2.72. Интегралы вида $\int R(x, \sqrt{\pm x^4 \mp 1}) dx$.

Обозначения: $\varphi = \arccos \frac{1}{x}$, $\psi = \arccos x$.

$$1. \int_1^x \frac{dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F\left(\varphi, \frac{\sqrt{2}}{2}\right) \quad [x > 1].$$

$$2. \int_1^x \frac{x^2 dx}{\sqrt{x^4 - 1}} = \frac{1}{\sqrt{2}} F\left(\varphi, \frac{\sqrt{2}}{2}\right) - \sqrt{2} E\left(\varphi, \frac{\sqrt{2}}{2}\right) + \frac{1}{x} \sqrt{x^4 - 1} \quad [x > 1].$$

$$3. \int_1^x \frac{x^4 dx}{\sqrt{x^4 - 1}} = \frac{1}{3} F\left(\varphi, \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{3} x \sqrt{x^4 - 1} \quad [x > 1].$$

$$4. \int_x^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{\sqrt{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) \quad [x < 1].$$

$$5. \int_x^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \sqrt[4]{2} E\left(\psi, \frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt[4]{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) \quad [x < 1],$$

$$6. \int_x^1 \frac{x^3 dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt[4]{2}} F\left(\psi, \frac{\sqrt{2}}{2}\right) + \frac{x}{3} \sqrt{1-x^4} \quad [x < 1],$$

1.2.73. Интегралы вида $\int f(x, \sqrt{x+x^3}) dx$.

Условие: $x \geq 0$ или $x \leq -1$.

Обозначения: $\varphi = \arccos \frac{(1-\sqrt[4]{3})x+1}{(1+\sqrt[4]{3})x+1}$, $k = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

$$1. \int \frac{dx}{\sqrt{x^4+x}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

$$2. \int \frac{x dx}{\sqrt{x^4+x}} = \frac{1}{2} \ln \frac{2x^2+1+2\sqrt{x^4+x}}{2x-1} + \frac{1-\sqrt[4]{3}}{2\sqrt[4]{3}} F(\varphi, k) - \frac{2-\sqrt[4]{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3+2\sqrt[4]{3}}{6}, k\right).$$

$$3. \int \frac{x^2 dx}{\sqrt{x^4+x}} = \frac{(1+\sqrt[4]{3})\sqrt{x^4+x}}{x+1+\sqrt[4]{3}x} + \frac{\sqrt[4]{3}-1}{2\sqrt[4]{3}} F(\varphi, k) - \sqrt[4]{3} E(\varphi, k).$$

$$4. \int \frac{dx}{(x-\rho)\sqrt{x^4+x}} = \frac{-1-\sqrt[4]{3}}{\sqrt[4]{3}(1+\rho+\sqrt[4]{3}\rho)} F(\varphi, k) - \frac{1}{2\sqrt[4]{3}(1+\rho)\rho} \left[\frac{1+\rho-\sqrt[4]{3}\rho}{1+\rho+\sqrt[4]{3}\rho} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x)\sqrt{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(1+\rho+\sqrt[4]{3}\rho)^2}{-4\sqrt[4]{3}\rho(1+\rho)}; \rho \neq 0, -1, \frac{1-\sqrt[4]{3}}{2}; \text{ см. 1.5.33, 12-14} \right].$$

$$5. \int \frac{dx}{(x+1)\sqrt{x^4+x}} = \frac{-2\sqrt{x^4+x}}{\sqrt[4]{3}(x+1+\sqrt[4]{3}x)(x+1)} + \frac{2\sqrt[4]{3}}{3} \left[\frac{\sqrt[4]{3}-1}{2} F(\varphi, k) + E(\varphi, k) \right].$$

$$6. \int \frac{dx}{(2x+\sqrt[4]{3}-1)\sqrt{x^4+x}} = \frac{\sqrt[4]{3}}{6} \left[F(\varphi, k) + \frac{2+\sqrt[4]{3}}{k} \arcsin(k \sin \varphi) \right].$$

$$7. \int \frac{dx}{x\sqrt{x^4+x}} = -\frac{2\sqrt{x^4+x}}{(x+1+\sqrt[4]{3}x)x} + \frac{\sqrt[4]{3}-1}{\sqrt[4]{3}} F(\varphi, k) - 2\sqrt[4]{3} E(\varphi, k).$$

1.2.74. Интегралы вида $\int R(x, \sqrt{x-x^3}) dx$.

Условие: $0 \leq x \leq 1$.

Обозначения: $\varphi = \arccos \frac{1-(\sqrt[4]{3}+1)x}{1+(\sqrt[4]{3}-1)x}$, $k = \frac{\sqrt{2-\sqrt[4]{3}}}{2}$.

$$1. \int \frac{dx}{\sqrt{x-x^3}} = \frac{1}{\sqrt[4]{3}} F(\varphi, k).$$

$$2. \int \frac{x dx}{\sqrt{x-x^4}} = -\operatorname{arctg} \frac{x-x^2}{\sqrt{x-x^4}} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) + \frac{2+\sqrt{3}}{2\sqrt[4]{3}} \Pi\left(\varphi, \frac{3-2\sqrt{3}}{6}, k\right).$$

$$3. \int \frac{x^2 dx}{\sqrt{x-x^4}} = \frac{(1-\sqrt{3})\sqrt{x-x^4}}{\sqrt{3}x-x+1} - \frac{1+\sqrt{3}}{2\sqrt[4]{3}} F(\varphi, k) + \sqrt[4]{3} E(\varphi, k).$$

$$4. \int \frac{dx}{(x-\rho)\sqrt{x-x^4}} = \frac{-\sqrt[4]{3}+1}{\sqrt[4]{3}(1-\rho+\sqrt{3}\rho)} F(\varphi, k) - \\ - \frac{1}{2\sqrt[4]{3}\rho(1-\rho)} \left[\frac{1-\rho-\sqrt{3}\rho}{1-\rho+\sqrt{3}\rho} \Pi(\varphi, -\rho_1, k) + \int \frac{\cos x dx}{(1+\rho_1 \sin^2 x)\sqrt{1-k^2 \sin^2 x}} \right] \\ \left[\rho_1 = \frac{(1-\rho+\sqrt{3}\rho)^2}{-4\sqrt{3}\rho(1-\rho)}; \rho \neq 0, 1, -\frac{1+\sqrt{3}}{2}; \text{см. 1.5.33. 12-14} \right].$$

$$5. \int \frac{dx}{(x-1)\sqrt{x-x^4}} = \frac{2\sqrt{x-x^4}}{\sqrt{3}(x-1)(\sqrt{3}x-x+1)} - \\ - \frac{2\sqrt[4]{3}}{3} \left[\frac{\sqrt[4]{3}+1}{2} F(\varphi, k) - E(\varphi, k) \right].$$

$$6. \int \frac{dx}{(2x+1+\sqrt{3})\sqrt{x-x^4}} = \frac{\sqrt[4]{3}}{6} \left[F(\varphi, k) + \frac{2-\sqrt{3}}{k} \arcsin(k \sin \varphi) \right].$$

$$7. \int \frac{dx}{x\sqrt{x-x^4}} = \frac{-2\sqrt{x-x^4}}{x(\sqrt{3}x-x+1)} + \frac{1+\sqrt{3}}{\sqrt[4]{3}} F(\varphi, k) - 2\sqrt[4]{3} E(\varphi, k).$$

1.2.75. Интегралы вида $\int R(x, \sqrt{x^4+2b^2x^2+a^4}) dx$.

Условия: $a > b > 0, x \geq 0$.

Обозначения: $\varphi = \arccos \frac{x^2-a^2}{x^2+a^2}, k = \frac{\sqrt{a^2-b^2}}{a\sqrt{2}}$.

$$1. \int_x^\infty \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a} F(\varphi, k).$$

$$2. \int_x^\infty \frac{dx}{x^3 \sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a^3} [F(\varphi, k) - 2E(\varphi, k)] + \frac{\sqrt{x^4+2b^2x^2+a^4}}{a^2x(x^2+a^2)}$$

$[x > 0].$

$$3. \int_x^\infty \frac{x^2 dx}{(x^2+a^2)^2 \sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{4a(a^2-b^2)} [F(\varphi, k) - E(\varphi, k)].$$

$$4. \int_x^\infty \frac{x^2 dx}{(x^2-a^2)^2 \sqrt{x^4+2b^2x^2+a^4}} = \frac{x\sqrt{x^4+2b^2x^2+a^4}}{2(a^2+b^2)(x^2-a^4)} - \frac{1}{4a(a^2+b^2)} E(\varphi, k).$$

$[x > a].$

$$5. \int_x^\infty \frac{(x^2-a^2)^2 dx}{(x^2+a^2)^2 \sqrt{x^4+2b^2x^2+a^4}} = \frac{a}{a^3-b^3} E(\varphi, k) - \frac{a^2+b^2}{2a(a^2-b^2)} F(\varphi, k).$$

$$6. \int_x^{\infty} \frac{(x^2 + a^2)^2 dx}{[(x^2 + a^2)^2 - 4a^2 p^2 x^2] \sqrt{x^4 + 2b^2 x^2 + a^4}} = \frac{1}{2a} \Pi(\varphi, p^2, k).$$

$$7. \int_x^{\infty} \frac{x^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \frac{a}{2(a^4 - b^4)} E(\varphi, k) - \frac{1}{4a(a^2 - b^2)} F(\varphi, k) - \\ - \frac{x(x^2 - a^2)}{2(a^2 + b^2)(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$8. \int_x^{\infty} \frac{(x^2 + a^2)^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\varphi, k) - \frac{(a^2 - b^2)x(x^2 - a^2)}{(a^2 + b^2)(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$9. \int_x^{\infty} \frac{(x^2 - a^2)^2 dx}{\sqrt{(x^4 + 2b^2 x^2 + a^4)^3}} = \\ = \frac{a}{a^2 - b^2} [F(\varphi, k) - E(\varphi, k)] + \frac{x(x^2 - a^2)}{(x^2 + a^2) \sqrt{x^4 + 2b^2 x^2 + a^4}}.$$

$$10. \int_x^{\infty} \frac{\sqrt{x^4 + 2b^2 x^2 + a^4}}{(x^2 + a^2)^2} dx = \frac{1}{2a} E(\varphi, k).$$

$$11. \int_x^{\infty} \frac{\sqrt{x^4 + 2b^2 x^2 + a^4}}{(x^2 - a^2)^2} dx = \frac{1}{2a} [F(\varphi, k) - E(\varphi, k)] + \frac{x}{x^2 - a^2} \sqrt{x^4 + 2b^2 x^2 + a^4}.$$

$$12. \int_x^a \frac{dx}{\sqrt{x^4 + 2b^2 x^2 + a^4}} = \\ = \frac{\sqrt{2}}{a \sqrt{2 + \sqrt{a^2 + b^2}}} F \left[\arctg \left(\frac{a \sqrt{2 + \sqrt{a^2 + b^2}}}{\sqrt{a^2 + b^2}} \frac{a - x}{a + x} \right), \frac{2 \sqrt{a \sqrt{2(a^2 - b^2)}}}{a \sqrt{2 + \sqrt{a^2 + b^2}}} \right] \\ [0 \leq x < a].$$

1.2.76. Интегралы вида $\int R(x, \sqrt{x^2 + 1}) dx$.

Обозначения: $\varphi = \arccos \frac{(1 - \sqrt{3})x^2 + 1}{(1 + \sqrt{3})x^2 + 1}$, $\psi = \arccos \frac{x^2 + 1 - \sqrt{3}}{x^2 + 1 + \sqrt{3}}$,

$$k = \sin \frac{5\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

$$1. \int \frac{dx}{\sqrt{x^2 + 1}} = \frac{1}{2\sqrt{3}} F(\varphi, k).$$

$$2. \int \frac{x}{\sqrt{x^2 + 1}} dx = \frac{-1}{2\sqrt{3}} F(\psi, k).$$

$$3. \int \frac{x^2}{\sqrt{x^2 + 1}} dx = \frac{1}{3} \ln(x^2 + \sqrt{x^2 + 1}).$$

- $$4. \int \frac{x^3}{\sqrt{x^6+1}} dx = \frac{\sqrt{x^6+1}}{x^2+1+\sqrt[3]{3}} - \sqrt[3]{3} \left[\frac{3-\sqrt{3}}{6} F(\psi, k) - E(\psi, k) \right],$$
- $$5. \int \frac{x^4}{\sqrt{x^6+1}} dx = \frac{(1+\sqrt{3})x\sqrt{x^6+1}}{2(1+\sqrt{3})x^2+2} + \frac{\sqrt{3}-1}{4\sqrt[3]{3}} F(\varphi, k) - \frac{\sqrt[3]{3}}{2} E(\varphi, k),$$
- $$6. \int \frac{x^5}{\sqrt{x^6+1}} dx = \frac{1}{3} \sqrt{x^6+1},$$
- $$7. \int \frac{dx}{(x-\rho)\sqrt{x^6+1}} = \frac{1}{2\sqrt[3]{3}} \left\{ \frac{1}{\rho^2+1+\sqrt{3}} F(\psi, k) - \right. \\ \left. - \frac{\rho^2+1-\sqrt{3}}{2(\rho^2+1)(\rho^2+1+\sqrt{3})} \Pi(\psi, -\rho_1, k) - \frac{1}{2(\rho^2+1)} I(\psi, \rho_1, k) \right\} + \\ + \frac{\rho}{2\sqrt[3]{3}} \left\{ \frac{-1-\sqrt{3}}{\rho^2+1+\sqrt{3}\rho^2} F(\varphi, k) - \frac{\rho^2+1-\sqrt[3]{3}\rho^2}{2\rho^2(\rho^2+1)(\rho^2+1+\sqrt{3}\rho^2)} \Pi(\varphi, -\rho_1, k) - \right. \\ \left. - \frac{1}{2\rho^2(\rho^2+1)} I(\varphi, \rho_1, k) \right\} \\ \left[I(\varphi, \rho, k) = \int_0^\varphi \frac{\cos \varphi \, d\varphi}{(1+\rho \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}; \quad x > 0; \quad \rho \neq 0; \right. \\ \left. \rho_1 = \frac{-(\rho^2+1+\sqrt{3})^2}{4\sqrt{3}(\rho^2+1)}; \quad \rho_2 = \frac{-(\rho^2+1+\sqrt[3]{3}\rho^2)^2}{4\sqrt[3]{3}\rho^2(\rho^2+1)}; \text{ см. 1.2.50} \right].$$
- $$8. \int \frac{dx}{x\sqrt{x^6+1}} = \frac{\sqrt{3}-1}{4\sqrt[3]{3}} F(\psi, k) + \frac{2-\sqrt{3}}{4\sqrt[3]{3}} \Pi\left(\psi, -\frac{3+2\sqrt{3}}{6}, k\right) - \\ - \frac{1}{8} \ln \frac{x^4+2+2\sqrt{x^6+1}}{x^4+2-2\sqrt{x^6+1}}.$$

1.2.77. Интегралы вида $\int R(x, \sqrt{1-x^6}) dx$.

Условие: $0 \leq x \leq 1$.

Обозначения: $\varphi = \arccos \frac{(1+\sqrt{3})x^2-1}{(-1+\sqrt{3})x^2+1}$, $k_1 = \sin \frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$;

$\psi = \arccos \frac{x^2-1+\sqrt{3}}{x^2-1-\sqrt{3}}$, $k_2 = \sin \frac{5\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

- $$1. \int \frac{dx}{\sqrt{1-x^6}} = \frac{-1}{2\sqrt[3]{3}} F(\varphi, k_1), \quad 2. \int \frac{x}{\sqrt{1-x^6}} dx = \frac{1}{2\sqrt[3]{3}} F(\psi, k_2),$$
- $$3. \int \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3} \arcsin(x^3),$$
- $$4. \int \frac{x^3}{\sqrt{1-x^6}} dx = \frac{-\sqrt{1-x^6}}{x^2-1-\sqrt{3}} - \frac{\sqrt{3}-1}{2\sqrt[3]{3}} F(\psi, k_2) + \sqrt[3]{3} E(\psi, k_2),$$
- $$5. \int \frac{x^4}{\sqrt{1-x^6}} dx = \frac{(1-\sqrt{3})x\sqrt{1-x^6}}{2(\sqrt{3}x^2-x^2+1)} + \frac{1+\sqrt{3}}{4\sqrt[3]{3}} F(\varphi, k_1) - \frac{\sqrt[3]{3}}{2} E(\varphi, k_1),$$
- $$6. \int \frac{x^5}{\sqrt{1-x^6}} dx = \frac{-1}{3} \sqrt{1-x^6}.$$

$$7. \int \frac{dx}{(x-\rho)\sqrt{1-x^2}} = \frac{1}{2} \int \frac{dt}{(t-\rho^2)\sqrt{1-t^2}} + \frac{\rho}{2} \int \frac{dt}{(t-\rho^2)\sqrt{t-t^2}} \\ [t=x^2; \text{ см. 1.2.70 и 1.2.74}].$$

$$8. \int \frac{dx}{x\sqrt{1-x^2}} = \frac{\sqrt{3}-1}{4\sqrt[4]{3}} F(\psi, k_2) + \frac{2-\sqrt{3}}{4\sqrt[4]{3}} \Pi\left(\psi, \frac{3+2\sqrt{3}}{6}, k_2\right) + \\ + \frac{1}{8} \ln \frac{x^4+2-2\sqrt{1-x^2}}{x^4+2+2\sqrt{1-x^2}}.$$

1.2.78. Интегралы вида $\int R(x, \sqrt[4]{x^2 \pm 1}) dx$.

$$1. \int R(x, \sqrt[4]{x^2+1}) dx = \\ = -2\sqrt[4]{27} \int R\left(2\sqrt[4]{27} \frac{\sin \varphi \sqrt{1-k^2 \sin^2 \varphi}}{(1-\cos \varphi)^2}, \frac{(\sqrt{3}-1)\cos \varphi + \sqrt{3}+1}{1-\cos \varphi}\right) \times \\ \times \frac{2+\cos \varphi - 2k^2 \sin^2 \varphi}{(1-\cos \varphi)^2 \sqrt{1-k^2 \sin^2 \varphi}} d\varphi \\ \left[\sin \varphi = \frac{2\sqrt[4]{3}(\sqrt[4]{x^2+1}-1)^{1/2}}{\sqrt[4]{x^2+1} + \sqrt{3}-1}; \cos \varphi = \frac{\sqrt[4]{x^2+1} - \sqrt{3}-1}{\sqrt[4]{x^2+1} + \sqrt{3}-1}; k = \frac{\sqrt{2-\sqrt{3}}}{2} \right].$$

$$2. \int R(x, \sqrt[4]{x^2-1}) dx = \\ = -2\sqrt[4]{27} \int R\left(2\sqrt[4]{27} \frac{\sin \varphi \sqrt{1-k^2 \sin^2 \varphi}}{(1-\cos \varphi)^2}, \frac{(1+\sqrt{3})\cos \varphi + \sqrt{3}-1}{1-\cos \varphi}\right) \times \\ \times \frac{2+\cos \varphi - 2k^2 \sin^2 \varphi}{(1-\cos \varphi)^2 \sqrt{1-k^2 \sin^2 \varphi}} d\varphi \\ \left[\sin \varphi = \frac{2\sqrt[4]{3}(\sqrt[4]{x^2-1}+1)^{1/2}}{\sqrt[4]{x^2-1} + 1 + \sqrt{3}}; \cos \varphi = \frac{\sqrt[4]{x^2-1} + 1 - \sqrt{3}}{\sqrt[4]{x^2-1} + 1 + \sqrt{3}}; k = \frac{\sqrt{2+\sqrt{3}}}{2} \right].$$

1.2.79. Интегралы вида $\int R(x, \sqrt[4]{x^2 \pm 1}) dx, \int R(x, \sqrt[4]{1-x^2}) dx$.

Обозначения: $\varphi = \arccos \frac{1}{\sqrt{x^2+1}}, \psi = \arccos \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}},$
 $\lambda = \arccos \sqrt[4]{1-x^2}.$

$$1. \int_0^x \frac{dx}{\sqrt[4]{x^2+1}} = V2 \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2x}{\sqrt[4]{x^2+1}} \quad [x > 0].$$

$$2. \int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^3}} = V2 F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > 0].$$

$$3. \int_0^x \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2V2 E\left(\varphi, \frac{1}{\sqrt{2}}\right) - V2 F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > 0].$$

$$4. \int_0^x \frac{x^2 dx}{\sqrt[4]{(x^2+1)^3}} = 2V2 \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2x}{\sqrt[4]{x^2+1}} \quad [x > 0].$$

$$5. \int_1^x \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\psi, \frac{1}{\sqrt{2}}\right) - 2E\left(\psi, \frac{1}{\sqrt{2}}\right) + \frac{2x\sqrt{x^2-1}}{1+\sqrt{x^2-1}} \quad [x > 1].$$

$$6. \int_1^x \frac{dx}{\sqrt{(x^2-1)^3}} = F\left(\psi, \frac{1}{\sqrt{2}}\right) \quad [x > 1].$$

$$7. \int_1^x \frac{dx}{x^2 \sqrt{x^2-1}} = E\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1}{2} F\left(\psi, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}}{x} \quad [x > 1].$$

$$8. \int_1^x \frac{dx}{(x^2+2\sqrt{x^2-1})\sqrt[4]{x^2-1}} = \frac{1}{2} \left[F\left(\psi, \frac{1}{\sqrt{2}}\right) - E\left(\psi, \frac{1}{\sqrt{2}}\right) \right] \quad [x > 1].$$

$$9. \int_1^x \frac{x^2 dx}{(x^2+2\sqrt{x^2-1})\sqrt{(x^2-1)^3}} = E\left(\psi, \frac{1}{\sqrt{2}}\right) \quad [x > 1].$$

$$10. \int_0^x \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] \quad [0 < x \leq 1].$$

$$11. \int_0^x \frac{x^2 dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] - \frac{2x}{5} \sqrt[4]{(1-x^2)^3} \quad [0 < x \leq 1].$$

$$12. \int_0^x \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} F\left(\lambda, \frac{1}{\sqrt{2}}\right) \quad [0 < x \leq 1].$$

$$13. \int_0^x \frac{x^2 dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\lambda, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} x \sqrt[4]{1-x^2} \quad [0 < x \leq 1].$$

$$14. \int_0^x \frac{dx}{(1+\sqrt{1-x^2})\sqrt[4]{1-x^2}} = \sqrt{2} \left[F\left(\lambda, \frac{1}{\sqrt{2}}\right) - E\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] + \frac{x \sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}} \quad [0 < x \leq 1].$$

$$15. \int_0^x \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} \left[2E\left(\lambda, \frac{1}{\sqrt{2}}\right) - F\left(\lambda, \frac{1}{\sqrt{2}}\right) \right] - \frac{2x \sqrt[4]{1-x^2}}{1+\sqrt{1-x^2}} \quad [0 < x \leq 1].$$

1.2.80. Интегралы вида $\int R(\sqrt[4]{\pm(x-a)}, \sqrt[4]{(x-b)}) dx$.

Обозначения: $\varphi = \arccos \frac{a-b-2\sqrt[4]{(x-a)(x-b)}}{a-b+2\sqrt[4]{(x-a)(x-b)}}$,

$$\psi = \arccos \sqrt[4]{\frac{4(a-x)(x-b)}{(a-b)^2}}.$$

$$1. \int_a^x \frac{dx}{\sqrt[4]{(x-a)(x-b)}} = \sqrt{\frac{a-b}{2}} F\left[\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\psi, \frac{1}{\sqrt{2}}\right)\right] + \\ + \frac{2(2x-a-b)\sqrt[4]{(x-a)(x-b)}}{a-b+2\sqrt[4]{(x-a)(x-b)}} \quad [x > a > b].$$

$$2. \int_a^x \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt[4]{a-b}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) \quad [x > a > b].$$

$$3. \int_b^x \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b} \left\{ 2\left[E\left(\frac{1}{\sqrt{2}}\right) + E\left(\psi, \frac{1}{\sqrt{2}}\right)\right] - \right. \\ \left. - K\left(\frac{1}{\sqrt{2}}\right) - F\left(\psi, \frac{1}{\sqrt{2}}\right) \right\} \quad [a > x > b].$$

$$4. \int_a^x \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt[4]{a-b}} \left[K\left(\frac{1}{\sqrt{2}}\right) + F\left(\psi, \frac{1}{\sqrt{2}}\right) \right] \quad [a > x > b].$$

1.2.81. Интегралы вида $\int R(\sqrt[4]{x^4+1}) dx$.

$$1. \int \frac{dx}{\sqrt[4]{x^4+1}} = \frac{1}{4} \ln \frac{\sqrt[4]{x^4+1} + x}{\sqrt[4]{x^4+1} - x} - \frac{1}{2} \operatorname{arctg} \frac{\sqrt[4]{x^4+1}}{x}.$$

$$2. \int x \sqrt[4]{x^4+1} dx = \frac{1}{3} x^3 \sqrt[4]{x^4+1} + \frac{1}{3\sqrt{2}} F\left(\arccos \frac{1}{\sqrt[4]{x^4+1}}, \frac{1}{\sqrt{2}}\right).$$

1.3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

1.3.1. Интегралы вида $\int f(e^{ax}) dx$.

$$1. \int f(e^{ax}) dx = \frac{1}{a} \int \frac{f(t)}{t} dt \quad [t = e^{ax}].$$

$$2. \int f(b^x) dx = \int f(e^{x \ln b}) dx \quad [b > 0; b \neq 1].$$

$$3. \int e^{ax} dx = \frac{1}{a} e^{ax}.$$

$$4. \int b^x dx = \frac{b^x}{\ln b} \quad [b > 0; b \neq 1].$$

$$5. \int \frac{dx}{e^{ax} + b} = \frac{1}{ab} [ax - \ln(e^{ax} + b)].$$

$$6. \int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}} \operatorname{arctg} \left(e^{ax} \sqrt{\frac{b}{c}} \right) \quad [bc > 0].$$

$$= \frac{1}{2a\sqrt{-bc}} \ln \frac{c + e^{ax} \sqrt{-bc}}{c - e^{ax} \sqrt{-bc}} \quad [bc < 0].$$

$$7. \int \frac{e^{ax} - 1}{e^{ax} + 1} dx = \frac{2}{a} \ln (e^{ax/2} + e^{-ax/2}).$$

$$8. \int \frac{dx}{\sqrt{b + ce^{ax}}} = \frac{1}{a\sqrt{b}} \ln \frac{\sqrt{b + ce^{ax}} - \sqrt{b}}{\sqrt{b + ce^{ax}} + \sqrt{b}} \quad [b > 0].$$

$$= \frac{2}{a\sqrt{-b}} \operatorname{arctg} \frac{\sqrt{b + ce^{ax}}}{\sqrt{-b}} \quad [b < 0].$$

1.3.2. Интегралы вида $\int f(x, e^{ax}) dx$.

$$1. \int x^\lambda e^{ax} dx = \frac{1}{a} x^\lambda e^{ax} - \frac{\lambda}{a} \int x^{\lambda-1} e^{ax} dx.$$

$$2. \int \frac{1}{x^\lambda} e^{ax} dx = -\frac{e^{ax}}{(\lambda-1)x^{\lambda-1}} + \frac{a}{\lambda-1} \int \frac{e^{ax}}{x^{\lambda-1}} dx.$$

$$3. \int_0^x x^{\lambda-1} e^{-ax} dx = \frac{1}{a^\lambda} \gamma(\lambda, ax) \quad [\operatorname{Re} \lambda > 0].$$

$$4. \int_x^\infty x^{\lambda-1} e^{-ax} dx = \frac{1}{a^\lambda} \Gamma(\lambda, ax) \quad [\operatorname{Re} \alpha > 0].$$

$$5. \int_x^\infty x^{\lambda-n-1} e^{-ax} dx = \frac{(-1)^n a^{n-1}}{(n-\lambda)(n-\lambda-1)\dots(1-\lambda)} \int_x^\infty x^{\lambda-1} e^{-ax} dx +$$

$$+ \frac{e^{-ax}}{x^{n-\lambda}} \sum_{k=0}^{n-1} \frac{(-1)^k (ax)^k}{(n-\lambda)(n-\lambda-1)\dots(n-\lambda-k)} \quad [\operatorname{Re} \alpha > 0].$$

$$6. \int x^n e^{ax} dx = e^{ax} \left[\frac{x^n}{a} + \sum_{k=1}^n (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k} \right].$$

$$7. \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right).$$

$$8. \int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$$

$$9. \int P(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m \frac{(-1)^k}{a^k} \frac{d^k}{dx^k} P(x),$$

где $P(x)$ — многочлен степени m .

10. $\int x^{n-1/2} e^{-ax} dx = (2n-1)!! \sqrt{\pi} 2^{-n} a^{-n-1/2} \operatorname{erf}(\sqrt{ax}) -$

$$- \frac{1}{a} x^{n-1/2} e^{-ax} \sum_{k=0}^{n-1} \frac{(2n-1)!!}{(2n-2k-1)!!} (2ax)^{-k}.$$
11. $\int \frac{e^{ax}}{x^n} dx = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \operatorname{Ei}(ax).$
12. $\int \frac{e^{ax}}{x} dx = \operatorname{Ei}(ax) \quad [a \neq 0].$
13. $= \ln|x| + \sum_{k=1}^{\infty} \frac{(ax)^k}{k!k}.$
14. $\int_x^{\infty} \frac{e^{-ax}}{x} dx = -\operatorname{Ei}(-ax) \quad [a > 0].$
15. $\int \frac{e^{ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erfi}(\sqrt{ax}) \quad [a > 0].$
16. $\int \frac{e^{-ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erf}(\sqrt{ax}) \quad [a > 0].$
17. $\int \frac{e^{-ax}}{x^{3/2}} dx = -\frac{2}{\sqrt{x}} e^{-ax} - 2\sqrt{\pi a} \operatorname{erf}(\sqrt{ax}) \quad [a > 0].$
18. $\int_x^{\infty} x^{-n-1/2} e^{-ax} dx = \frac{(-2)^n a^{n-1/2} \sqrt{\pi}}{(2n-1)!!} \operatorname{erfc}(\sqrt{ax}) +$

$$+ 2 \frac{e^{-ax}}{x^{n-1/2}} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!! (-2ax)^k}{(2n-1)!!} \quad [a > 0].$$
19. $\int_x^{\infty} \frac{e^{-ax}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{a}} \operatorname{erfc}(\sqrt{ax}) \quad [a > 0].$
20. $\int \frac{e^{ax}}{x^2} dx = -\frac{e^{ax}}{x} + a \operatorname{Ei}(ax).$
21. $\int \frac{e^{ax}}{x^3} dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2} \operatorname{Ei}(ax).$
22. $\int \frac{e^{ax} dx}{x+b} = e^{-ab} \operatorname{Ei}(ab+ax).$
23. $\int \frac{x^m e^{ax}}{(x+b)^n} dx = (-1)^m b^m e^{-ab} \sum_{k=0}^m \frac{(-1)^k}{b^k} \binom{m}{k} \int t^{k-n} e^{at} dt \quad [t = x+b].$

$$24. \int \frac{x e^{ax} dx}{x+b} = \frac{e^{ax}}{a} - b e^{-ab} \operatorname{Ei}(ax+ab).$$

$$25. \int \frac{x^2 e^{ax} dx}{x+b} = \frac{ax-ab-1}{a^2} e^{ax} + b^2 e^{-ab} \operatorname{Ei}(ax+ab).$$

$$26. \int \frac{e^{ax} dx}{(x+b)^2} = -\frac{e^{ax}}{x+b} + a e^{-ab} \operatorname{Ei}(ax+ab).$$

$$27. \int \frac{x e^{ax} dx}{(x+b)^2} = -\frac{b e^{ax}}{x+b} + (1-ab) e^{-ab} \operatorname{Ei}(ax+ab).$$

$$28. \int \frac{x^2 e^{ax} dx}{(x+b)^2} = \left(\frac{1}{a} - \frac{b^2}{x+b} \right) e^{ax} + (ab^2-2b) e^{-ab} \operatorname{Ei}(ax+ab).$$

$$29. \int \frac{e^{ix} dx}{x^2+a^2} = \frac{i}{2a} [e^a \operatorname{Ei}(-a+ix) - e^{-a} \operatorname{Ei}(a+ix)].$$

$$30. \int \frac{x e^{ix} dx}{x^2+a^2} = \frac{1}{2} [e^{-a} \operatorname{Ei}(a+ix) + e^a \operatorname{Ei}(-a+ix)].$$

$$31. \int \frac{e^x dx}{x^2+a^2} = \frac{1}{a} \operatorname{Im} [e^{ia} \operatorname{Ei}(x-ia)].$$

$$32. \int \frac{x e^x dx}{x^2+a^2} = \operatorname{Re} [e^{ia} \operatorname{Ei}(x-ia)].$$

$$33. \int_0^x \frac{e^x-1}{x} dx = \operatorname{Ei}(x) - C - \ln x$$

[x > 0].

$$34. \int_1^x \frac{(1-e^{-ix}) dx}{x} = C + \ln x - \operatorname{ci}(x) + i \operatorname{Si}(x).$$

1.3.3. Интегралы вида $\int f(x, e^{-a^2 x^2}) dx$.

$$1. \int_0^x e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erf}(ax).$$

$$2. \int_0^x e^{a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erfi}(ax).$$

$$3. \int_x^\infty e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a} \operatorname{erfc}(ax).$$

$$4. \int x^p e^{-ax^2} dx = -\frac{x^{p-1}}{2a} e^{-ax^2} + \frac{p-1}{2a} \int x^{p-2} e^{-ax^2} dx.$$

$$5. \int x^{2n} e^{-a^2 x^2} dx =$$

$$= -\frac{x^{2n-1} e^{-a^2 x^2} (2n-1)!!}{2a^2} \sum_{k=0}^{n-1} \frac{1}{2^k (2n-2k-1)!! (ax)^{2k}} + \\ + \frac{\sqrt{\pi} (2n-1)!!}{2^{n+1} a^{2n+1}} \operatorname{erf}(ax).$$

$$6. \int x^{2n+1} e^{-ax^2} dx = -\frac{x^{2n} e^{-ax^2}}{2a} \sum_{k=0}^n \frac{n!}{(n-k)! a^k x^{2k}}.$$

$$7. \int x e^{-a x^2} dx = -\frac{1}{2a} e^{-a x^2}.$$

$$8. \int x^2 e^{-a^2 x^2} dx = -\frac{1}{2a^2} x e^{-a^2 x^2} + \frac{\sqrt{\pi}}{4a^3} \operatorname{erf}(ax).$$

$$9. \int x^3 e^{-ax^2} dx = -\frac{ax^2 + 1}{2a^2} e^{-ax^2}.$$

$$10. \int \frac{e^{-ax^2}}{x^p} dx = -\frac{e^{-ax^2}}{(p-1)x^{p-1}} - \frac{2a}{p-1} \int \frac{e^{-ax^2}}{x^{p-2}} dx.$$

$$11. \int \frac{e^{-a^2 x^2}}{x^{2n}} dx = \frac{1}{2a^2 \Gamma(n+1/2)} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k \Gamma\left(n-k+\frac{1}{2}\right) a^{2k} x^{2k-2n-1} + \frac{(-1)^n \pi a^{2n-1}}{2\Gamma(n+1/2)} \operatorname{erf}(ax).$$

$$12. \int \frac{e^{-a^2 x^2}}{x^{2n+1}} dx = \frac{1}{2a^2 n!} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k (n-k)! a^{2k} x^{2k-2n-2} + \frac{(-a^2)^n}{2(n!)} \operatorname{Ei}(-a^2 x^2).$$

$$13. \int \frac{e^{-a^2 x^2}}{x} dx = \frac{1}{2} \operatorname{Ei}(-a^2 x^2).$$

$$14. \int \frac{e^{-a^2 x^2}}{x^2} dx = -\frac{1}{x} e^{-a^2 x^2} - \sqrt{\pi} a \operatorname{erf}(ax).$$

$$15. \int \frac{e^{-a^2 x^2}}{x^3} dx = -\frac{1}{2x^2} e^{-a^2 x^2} - \frac{a^2}{2} \operatorname{Ei}(-a^2 x^2).$$

$$16. \int f(x) e^{-(ax^2+bx+c)} dx = \frac{1}{\sqrt{a}} e^{(b^2-4ac)/(4a)} \int f\left(\frac{2\sqrt{a}t-b}{2a}\right) e^{-t^2} dt \quad \left[t = \sqrt{a}\left(x + \frac{b}{2a}\right), a > 0\right].$$

$$17. \int e^{-(ax^2+bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)} \operatorname{erf}\left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right).$$

$$18. \int (x-a)^n e^{-c(x-b)^2} dx = \sum_{k=0}^n \binom{n}{k} \frac{(b-a)^{n-k}}{c^{k+1}} \int t^k e^{-t^2} dt \quad [t = c(x-b), c \neq 0].$$

$$19. \int x^n e^{-a^2 x^2 + bx} dx = \frac{1}{a^{n+1}} e^{b^2/(4a^2)} \sum_{k=0}^n \binom{n}{k} \left(\frac{b}{2a}\right)^{n-k} \int t^k e^{-t^2} dt \quad [t = ax - \frac{b}{2a}].$$

$$20. \int_0^x e^{-a^2 t^2 - b^2/x^2} dx = \frac{\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erf} \left(ax + \frac{b}{x} \right) + e^{-2ab} \operatorname{erf} \left(ax - \frac{b}{x} \right) - \right. \\ \left. - e^{2ab} + e^{-2ab} \right] \quad [\operatorname{Re} b^2 > 0].$$

1.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1.4.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

$$\begin{aligned} 1. \int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx &= \int R \left(\frac{t^2-1}{2t}, \frac{t^2+1}{2t}, \frac{t^2-1}{t^2+1}, \frac{t^2+1}{t^2-1} \right) \frac{dt}{t} \\ &\quad [t = e^x]. \\ 2. \int R(\operatorname{sh} x, \operatorname{ch}^2 x) \operatorname{ch} x dx &= \int R(t, 1+t^2) dt \\ &\quad [t = \operatorname{sh} x]. \\ 3. \int R(\operatorname{sh}^2 x, \operatorname{ch} x) \operatorname{sh} x dx &= \int R(t^2-1, t) dt \\ &\quad [t = \operatorname{ch} x]. \\ 4. \int R(\operatorname{th} x, \operatorname{cth} x) dx &= \int R \left(t, \frac{1}{t} \right) \frac{dt}{1-t^2} \\ &\quad [t = \operatorname{th} x]. \end{aligned}$$

1.4.2. Интегралы вида $\int \operatorname{sh}^p x dx, \int \operatorname{ch}^p x dx$.

$$\begin{aligned} 1. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx &= \frac{1}{p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \mp \frac{p-1}{p} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p-2} dx. \\ 2. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2m} dx &= (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \\ &\quad + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \frac{(-1)^k}{2m-2k} \binom{2m}{k} \left\{ \frac{\operatorname{sh} (2m-2k)x}{\operatorname{ch} (2m-2k)x} \right\}. \\ 3. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2m+1} dx &= \sum_{k=0}^m \frac{(-1)^{m+k}}{2k+1} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k+1}. \\ 4. &= \frac{1}{2^{2m}} \sum_{k=0}^m \frac{(-1)^k}{2m-2k+1} \binom{2m+1}{k} \left\{ \frac{\operatorname{ch} (2m-2k+1)x}{\operatorname{sh} (2m-2k+1)x} \right\}. \\ 5. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx &= \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}. \\ 6. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx &= \frac{1}{4} \operatorname{sh} 2x \mp \frac{x}{2}. \\ 7. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx &= \mp \frac{3}{4} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{1}{12} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} = \mp \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^3. \\ 8. \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^4 dx &= \frac{3}{8} x \mp \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x = \\ &= \frac{3}{8} x \mp \frac{3}{8} \operatorname{sh} x \operatorname{ch} x + \frac{1}{4} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}. \end{aligned}$$

6. $\int x^{2n+1} e^{-ax^2} dx = -\frac{x^{2n} e^{-ax^2}}{2a} - \sum_{k=0}^n \frac{n!}{(n-k)! a^k x^{2k}}.$
7. $\int x e^{-ax^2} dx = -\frac{1}{2a} e^{-ax^2}.$
8. $\int x^2 e^{-a^2 x^2} dx = -\frac{1}{2a^3} x e^{-a^2 x^2} + \frac{\sqrt{\pi}}{4a^3} \operatorname{erf}(ax).$
9. $\int x^3 e^{-ax^2} dx = -\frac{ax^2 + 1}{2a^2} e^{-ax^2}.$
10. $\int \frac{e^{-ax^2}}{x^p} dx = -\frac{e^{-ax^2}}{(p-1)x^{p-1}} - \frac{2a}{p-1} \int \frac{e^{-ax^2}}{x^{p-2}} dx.$
11. $\int \frac{e^{-a^2 x^2}}{x^{2n}} dx = \frac{1}{2a^2 \Gamma(n+1/2)} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k \Gamma\left(n-k+\frac{1}{2}\right) a^{2k} x^{2k-2n-1} +$
 $+ \frac{(-1)^n \pi a^{2n-1}}{2\Gamma(n+1/2)} \operatorname{erf}(ax).$
12. $\int \frac{e^{-a^2 x^2}}{x^{2n+1}} dx = \frac{1}{2a^2 n!} e^{-a^2 x^2} \sum_{k=1}^n (-1)^k (n-k)! a^{2k} x^{2k-2n-2} + \frac{(-a^2)^n}{2(n!)} \operatorname{Ei}(-a^2 x^2).$
13. $\int \frac{e^{-a^2 x^2}}{x} dx = \frac{1}{2} \operatorname{Ei}(-a^2 x^2).$
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15. $\int \frac{e^{-a^2 x^2}}{x^3} dx = -\frac{1}{2x^2} e^{-a^2 x^2} - \frac{a^2}{2} \operatorname{Ei}(-a^2 x^2).$
16. $\int f(x) e^{-(ax^2+bx+c)} dx =$
 $= \frac{1}{\sqrt{a}} e^{(b^2-4ac)/(4a)} \int f\left(\frac{2\sqrt{a}t-b}{2a}\right) e^{-t^2} dt \quad \left[t = \sqrt{a}\left(x + \frac{b}{2a}\right), a > 0\right].$
17. $\int e^{-(ax^2+bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)} \operatorname{erf}\left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right).$
18. $\int (x-a)^n e^{-c^2(x-b)^2} dx =$
 $= -\sum_{k=0}^n \binom{n}{k} \frac{(b-a)^{n-k}}{c^{k+1}} \int t^k e^{-t^2} dt \quad [t = c(x-b), c \neq 0].$
19. $\int x^n e^{-a^2 x^2+bx} dx =$
 $= \frac{1}{a^{n+1}} e^{b^2/(4a^2)} \sum_{k=0}^n \binom{n}{k} \left(\frac{b}{2a}\right)^{n-k} \int t^k e^{-t^2} dt \quad \left[t = ax - \frac{b}{2a}\right].$

$$20. \int_0^x e^{-a^2 x^2 - b^2/x^2} dx = \frac{\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erf} \left(ax + \frac{b}{x} \right) + e^{-2ab} \operatorname{erf} \left(ax - \frac{b}{x} \right) - \right. \\ \left. - e^{2ab} + e^{-2ab} \right] \quad [\operatorname{Re} b > 0]$$

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1.4.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

1. $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx = \int R \left(\frac{t^2-1}{2t}, \frac{t^2+1}{2t}, \frac{t^2-1}{t^2+1}, \frac{t^2+1}{t^2-1} \right) \frac{dt}{t} \quad \begin{matrix} [t = e^x]. \\ [t = \operatorname{sh} x]. \end{matrix}$
2. $\int R(\operatorname{sh} x, \operatorname{ch}^2 x) \operatorname{ch} x dx = \int R(t, 1+t^2) dt$
3. $\int R(\operatorname{sh}^2 x, \operatorname{ch} x) \operatorname{sh} x dx = \int R(t^2-1, t) dt \quad [t = \operatorname{ch} x].$
4. $\int R(\operatorname{th} x, \operatorname{cth} x) dx = \int R \left(t, \frac{1}{t} \right) \frac{dt}{1-t^2} \quad [t = \operatorname{th} x].$

1.4.2. Интегралы вида $\int \operatorname{sh}^p x dx, \int \operatorname{ch}^p x dx$.

1. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^p dx = \frac{1}{p} \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^{p-1} \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right) \mp \frac{p-1}{p} \int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^{p-2} dx.$
2. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^{2m} dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} +$
 $+ \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \frac{(-1)^k}{2m-2k} \binom{2m}{k} \left(\frac{\operatorname{sh} (2m-2k)x}{\operatorname{ch} (2m-2k)x} \right).$
3. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^{2m+1} dx = \sum_{k=0}^m \frac{(-1)^{m+k}}{2k+1} \binom{m}{k} \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right)^{2k+1}.$
4. $= \frac{1}{2^{2m}} \sum_{k=0}^m \frac{(-1)^k}{2m-2k+1} \binom{2m+1}{k} \left(\frac{\operatorname{ch} (2m-2k+1)x}{\operatorname{sh} (2m-2k+1)x} \right).$
5. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right) dx = \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right).$
6. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^2 dx = \frac{1}{4} \operatorname{sh} 2x \mp \frac{x}{2}.$
7. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^3 dx = \mp \frac{3}{4} \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right) + \frac{1}{12} \left(\frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right) = \mp \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right) + \frac{1}{3} \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right)^3.$
8. $\int \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^4 dx = \frac{3}{8} x \mp \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x =$
 $= \frac{3}{8} x \mp \frac{3}{8} \operatorname{sh} x \operatorname{ch} x + \frac{1}{4} \left(\frac{\operatorname{sh} x}{\operatorname{ch} x} \right)^3 \left(\frac{\operatorname{ch} x}{\operatorname{sh} x} \right).$

$$22. \int \operatorname{ch} ax \operatorname{ch} bx \operatorname{ch} cx \, dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} + \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

1.4.4. Интегралы вида $\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x} dx$, $\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x} dx$.

$$1. \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^q x} dx = \frac{1}{p-q} \frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} - \frac{p-1}{p-q} \int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^q x} dx.$$

$$2. = \frac{1}{q-1} \frac{\operatorname{sh}^{p+1} x}{\operatorname{ch}^{q-1} x} - \frac{p-q+2}{q-1} \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{q-2} x} dx.$$

$$3. = -\frac{1}{q-1} \frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} + \frac{p-1}{q-1} \int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^{q-2} x} dx.$$

$$4. \int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x} dx = \frac{1}{q-p} \frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{q-p} \int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^p x} dx.$$

$$5. = -\frac{1}{p-1} \frac{\operatorname{ch}^{q+1} x}{\operatorname{sh}^{p-1} x} + \frac{q-p+2}{p-1} \int \frac{\operatorname{ch}^q x}{\operatorname{sh}^{p-2} x} dx.$$

$$6. = -\frac{1}{p-1} \frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{p-1} \int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^{p-2} x} dx.$$

$$7. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-2n} x}{\operatorname{ch}^p x \operatorname{sh}^{-2n} x} \right\} dx = \pm \frac{1}{2n-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} \left[\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+1} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+2k+1} \right] + \\ + (\pm 1)^n \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx.$$

$$8. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-2n-1} x}{\operatorname{ch}^p x \operatorname{sh}^{-2n-1} x} \right\} dx = \pm \frac{1}{2n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} \left[\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n} + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(\pm 1)^k (2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2n+2k} \right] + \\ + (\pm 1)^n \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch}^{-1} x}{\operatorname{ch}^p x \operatorname{sh}^{-1} x} \right\} dx.$$

$$9. \int \left\{ \frac{\operatorname{sh}^{2m} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-1} + (\mp 1)^m \left\{ \frac{\operatorname{arctg} \operatorname{sh} x}{\ln |\operatorname{th}(x/2)|} \right\}.$$

$$10. \int \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k} + \\ + (\mp 1)^m \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\} - \sum_{k=1}^m \frac{(\mp 1)^{m+k}}{2k} \binom{m}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k} + (\mp 1)^m \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

$$11. \int \frac{\{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n} x\}}{\{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n} x\}} dx = \sum_{k=0}^m \frac{(\mp 1)^{m+k}}{2k-2n+1} \binom{m}{k} \frac{\{\operatorname{ch} x\}^{2k-2n+1}}{\{\operatorname{sh} x\}}.$$

$$12. \int \frac{\{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-2n-1} x\}}{\{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-2n-1} x\}} dx = \sum_{\substack{k=0 \\ k \neq n}}^m \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \frac{\{\operatorname{ch} x\}^{2k-2n}}{\{\operatorname{sh} x\}} + \\ + (\mp 1)^{m+n} \binom{m}{n} \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \quad [n \leq m].$$

$$13. = \sum_{k=0}^m \frac{(\mp 1)^{m+k}}{2k-2n} \binom{m}{k} \frac{\{\operatorname{ch} x\}^{2k-2n}}{\{\operatorname{sh} x\}} \quad [n > m].$$

$$14. \int \frac{\{\operatorname{sh} x \operatorname{ch}^{-\rho} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-\rho} x\}} dx = -\frac{1}{\rho-1} \frac{\{\operatorname{ch} x\}^{-\rho+1}}{\{\operatorname{sh} x\}}.$$

$$15. \int \frac{\{\operatorname{sh}^{\rho} x \operatorname{ch}^{-\rho-2} x\}}{\{\operatorname{ch}^{\rho} x \operatorname{sh}^{-\rho-2} x\}} dx = \pm \frac{1}{\rho+1} \frac{\{\operatorname{th} x\}^{\rho+1}}{\{\operatorname{cth} x\}}.$$

$$16. \int \frac{\{\operatorname{sh} x \operatorname{ch}^{-1} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-1} x\}} dx = \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$17. \int \frac{\{\operatorname{sh}^2 x \operatorname{ch}^{-1} x\}}{\{\operatorname{ch}^2 x \operatorname{sh}^{-1} x\}} dx = \frac{\{\operatorname{sh} x\}}{\{\operatorname{ch} x\}} \mp \frac{\{\operatorname{arctg}(\operatorname{sh} x)\}}{\{\ln |\operatorname{th}(x/2)|\}}.$$

$$18. \int \frac{\{\operatorname{sh}^3 x \operatorname{ch}^{-1} x\}}{\{\operatorname{ch}^3 x \operatorname{sh}^{-1} x\}} dx = \frac{1}{2} \operatorname{ch}^2 x \mp \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$19. \int \frac{\{\operatorname{sh}^4 x \operatorname{ch}^{-1} x\}}{\{\operatorname{ch}^4 x \operatorname{sh}^{-1} x\}} dx = \frac{1}{3} \frac{\{\operatorname{sh} x\}^3}{\{\operatorname{ch} x\}} \mp \frac{\{\operatorname{sh} x\}}{\{\operatorname{ch} x\}} + \frac{\{\operatorname{arctg}(\operatorname{sh} x)\}}{\{\ln |\operatorname{th}(x/2)|\}}.$$

$$20. \int \frac{\{\operatorname{sh} x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-2} x\}} dx = -\frac{\{\operatorname{ch} x\}^{-1}}{\{\operatorname{sh} x\}}.$$

$$21. \int \frac{\{\operatorname{sh}^2 x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch}^2 x \operatorname{sh}^{-2} x\}} dx = x - \frac{\{\operatorname{th} x\}}{\{\operatorname{cth} x\}}.$$

$$22. \int \frac{\{\operatorname{sh}^3 x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch}^3 x \operatorname{sh}^{-2} x\}} dx = \frac{\{\operatorname{ch} x\}}{\{\operatorname{sh} x\}} \pm \frac{\{\operatorname{ch} x\}^{-1}}{\{\operatorname{sh} x\}}.$$

$$23. \int \frac{\{\operatorname{sh}^4 x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch}^4 x \operatorname{sh}^{-2} x\}} dx = \mp \frac{3}{2} x + \frac{1}{4} \operatorname{sh} 2x \pm \frac{\{\operatorname{th} x\}}{\{\operatorname{cth} x\}}.$$

$$24. \int \frac{\{\operatorname{sh} x \operatorname{ch}^{-3} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-3} x\}} dx = \pm \frac{1}{2} \frac{\{\operatorname{th} x\}^2}{\{\operatorname{cth} x\}}.$$

$$25. \int \frac{\{\operatorname{sh}^2 x \operatorname{ch}^{-3} x\}}{\{\operatorname{ch}^2 x \operatorname{sh}^{-3} x\}} dx = -\frac{1}{2} \frac{\{\operatorname{sh} x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-2} x\}} + \frac{1}{2} \frac{\{\operatorname{arctg}(\operatorname{sh} x)\}}{\{\ln |\operatorname{th}(x/2)|\}}.$$

$$26. \int \frac{\{\operatorname{sh}^3 x \operatorname{ch}^{-3} x\}}{\{\operatorname{ch}^3 x \operatorname{sh}^{-3} x\}} dx = -\frac{1}{2} \frac{\{\operatorname{th} x\}^2}{\{\operatorname{cth} x\}} + \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$27. \int \frac{\{\operatorname{sh}^4 x \operatorname{ch}^{-3} x\}}{\{\operatorname{ch}^4 x \operatorname{sh}^{-3} x\}} dx = \pm \frac{1}{2} \frac{\{\operatorname{sh} x \operatorname{ch}^{-2} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-2} x\}} + \frac{\{\operatorname{sh} x\}}{\{\operatorname{ch} x\}} \mp \frac{3}{2} \frac{\{\operatorname{arctg}(\operatorname{sh} x)\}}{\{\ln |\operatorname{th}(x/2)|\}}.$$

$$28. \int \frac{\{\operatorname{sh} x \operatorname{ch}^{-4} x\}}{\{\operatorname{ch} x \operatorname{sh}^{-4} x\}} dx = -\frac{1}{3} \frac{\{\operatorname{ch} x\}^{-3}}{\{\operatorname{sh} x\}}.$$

$$29. \int \left\{ \frac{\operatorname{sh}^2 x \operatorname{ch}^{-4} x}{\operatorname{ch}^2 x \operatorname{sh}^{-4} x} \right\} dx = \pm \frac{1}{3} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^3 +$$

$$30. \int \left\{ \frac{\operatorname{sh}^3 x \operatorname{ch}^{-4} x}{\operatorname{ch}^3 x \operatorname{sh}^{-4} x} \right\} dx = - \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} \pm \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3}.$$

$$31. \int \left\{ \frac{\operatorname{sh}^4 x \operatorname{ch}^{-4} x}{\operatorname{ch}^4 x \operatorname{sh}^{-4} x} \right\} dx = - \frac{1}{3} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^3 - \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} + x.$$

1.4.5. Интегралы вида $\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x}$.

$$1. \int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x} = - \frac{1}{(p-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} - \frac{p+q-2}{p-1} \int \frac{dx}{\operatorname{sh}^{p-2} x \operatorname{ch}^q x}.$$

$$2. = \frac{1}{(q-1) \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^{q-2} x}.$$

$$3. \int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2m-2k-1} \binom{m+n-1}{k} \operatorname{th}^{2k-2m+1} x.$$

$$4. \int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \operatorname{th}^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln |\operatorname{th} x|.$$

$$5. \int \left\{ \frac{\operatorname{sh}^{-2m} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m} x \operatorname{sh}^{-1} x} \right\} dx = \\ = \sum_{k=1}^m \frac{(-1)^k}{2m-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k-1} + (-1)^m \left\{ \operatorname{arctg}(\operatorname{sh} x) \right\} \\ + \ln |\operatorname{th}(x/2)|.$$

$$6. \int \left\{ \frac{\operatorname{sh}^{-2m-1} x \operatorname{ch}^{-1} x}{\operatorname{ch}^{2m-1} x \operatorname{sh}^{-1} x} \right\} dx = \sum_{k=1}^m \frac{(-1)^k}{2m-2k+2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2m+2k-2} + (-1)^m \ln |\operatorname{th} x|.$$

$$7. \int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln |\operatorname{th} x|.$$

$$8. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-2} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-2} x} \right\} dx = \pm \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} \pm \left\{ \operatorname{arctg}(\operatorname{sh} x) \right\}.$$

$$9. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-3} x} \right\} dx = - \frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 + \ln \left| \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} \right|.$$

$$10. \int \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-4} x} \right\} dx = \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} \pm \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3} + \left\{ \operatorname{arctg}(\operatorname{sh} x) \right\}.$$

$$11. \int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2 \operatorname{cth} 2x.$$

$$12. \int \left\{ \frac{\operatorname{sh}^{-2} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-2} x \operatorname{sh}^{-3} x} \right\} dx = - \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} - \frac{1}{2} \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} - \frac{3}{2} \left\{ \operatorname{arctg}(\operatorname{sh} x) \right\}.$$

$$13. \int \left\{ \frac{\operatorname{sh}^{-2} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-2} x \operatorname{sh}^{-4} x} \right\} dx = \pm \frac{1}{3} \left\{ \frac{\operatorname{sh}^{-1} x \operatorname{ch}^{-3} x}{\operatorname{ch}^{-1} x \operatorname{sh}^{-3} x} \right\} \mp \frac{8}{3} \operatorname{cth} 2x.$$

$$14. \int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^3 x} = \frac{1}{2} \operatorname{th}^2 x - \frac{1}{2} \operatorname{cth}^2 x - 2 \ln |\operatorname{th} x|.$$

$$15. \int \left\{ \frac{\operatorname{sh}^{-3} x \operatorname{ch}^{-4} x}{\operatorname{ch}^{-3} x \operatorname{sh}^{-4} x} \right\} dx = \\ = -\frac{1}{2} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} - \frac{1}{3} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-3} - \frac{1}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-2} x}{\operatorname{sh} x \operatorname{ch}^{-2} x} \right\} + \frac{5}{2} \left\{ \ln \left| \operatorname{th} (x/2) \right| \right\} \\ + \frac{5}{2} \left\{ \operatorname{arctg} (\operatorname{sh} x) \right\}.$$

$$16. \int \frac{dx}{\operatorname{sh}^6 x \operatorname{ch}^4 x} = 8 \operatorname{cth} 2x - \frac{8}{3} \operatorname{cth}^3 2x.$$

1.4.6. Интегралы вида $\int \operatorname{th}^p x dx$, $\int \operatorname{cth}^p x dx$.

$$1. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^p dx = -\frac{1}{p-1} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{p-1} + \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{p-2} dx.$$

$$2. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n} dx = -\sum_{k=1}^n \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n-2k+1} + x.$$

$$3. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n+1} dx = -\sum_{k=1}^n \frac{1}{2n-2k+2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n-2k+2} + \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$4. = -\frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-2k} + \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$5. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} dx = \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$6. \int \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 dx = x - \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}.$$

1.4.7. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \operatorname{th} x, \operatorname{cth} x) dx$.

$$1. \int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{(a+b \operatorname{ch} x+c \operatorname{sh} x)^n} dx = \\ = \frac{Bc-Cb+(Ac-Ca) \operatorname{ch} x+(Ab-Ba) \operatorname{sh} x}{(1-n)(a^2-b^2+c^2)(a+b \operatorname{ch} x+c \operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2-b^2+c^2)} \times \\ \times \int \frac{(n-1)(Aa-Bb+Cc)-(n-2)(Ab-Ba) \operatorname{ch} x-(n-2)(Ac-Ca) \operatorname{sh} x}{(a+b \operatorname{ch} x+c \operatorname{sh} x)^{n-1}} dx$$

$[a^2+c^2 \neq b^2].$

$$2. = \frac{Bc-Cb-Ca \operatorname{ch} x-Ba \operatorname{sh} x}{(n-1)a(a+b \operatorname{ch} x+c \operatorname{sh} x)^n} + \\ + \left[\frac{A}{a} + \frac{n(Bb-Cc)}{(n-1)a^2} \right] (c \operatorname{ch} x+b \operatorname{sh} x) \frac{(n-1)!}{(2n-1)!!} \times \\ \times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)! a^k} \cdot \frac{1}{(a+b \operatorname{ch} x+c \operatorname{sh} x)^{n-k}}$$

$[a^2+c^2=b^2].$

$$3. \int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{a+b \operatorname{ch} x+c \operatorname{sh} x} dx = \frac{Cb-Bc}{b^2-c^2} \ln(a+b \operatorname{ch} x+c \operatorname{sh} x) + \\ + \frac{Bb-Cc}{b^2-c^2} x + \left(A-a \frac{Bb-Cc}{b^2-c^2} \right) \int \frac{dx}{a+b \operatorname{ch} x+c \operatorname{sh} x}$$

$[b^2 \neq c^2].$

$$4. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x \pm b \operatorname{sh} x} dx = \frac{C \mp B}{2a} (\operatorname{ch} x \mp \operatorname{sh} x) + \\ + \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2} \right] x + \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2} \right] \ln (a + b \operatorname{ch} x \pm b \operatorname{sh} x) \\ [ab \neq 0].$$

$$5. \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \operatorname{arctg} \frac{(b-a) \operatorname{th} \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} \\ [b^2 > a^2 + c^2; a \neq b].$$

$$6. = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \operatorname{th} \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \operatorname{th} \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} \\ [b^2 < a^2 + c^2; a \neq b].$$

$$7. = \frac{1}{c} \ln \left(a + c \operatorname{th} \frac{x}{2} \right) \\ [a = b, c \neq 0].$$

$$8. = \frac{2}{(a-b) \operatorname{th} \frac{x}{2} + c} \\ [b^2 = a^2 + c^2].$$

$$9. \int \frac{dx}{(a \operatorname{ch} x + b \operatorname{sh} x)^n} = \frac{1}{(a^2 - b^2)^{n/2}} \int \frac{dx}{\operatorname{ch}^n \left(x + \operatorname{Arth} \frac{b}{a} \right)} \\ [a > |b|].$$

$$10. = \frac{1}{(b^2 - a^2)^{n/2}} \int \frac{dx}{\operatorname{sh}^n \left(x + \operatorname{Arth} \frac{a}{b} \right)} \\ [b > |a|].$$

$$11. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a \operatorname{ch} x + b \operatorname{sh} x} dx = \frac{A}{\sqrt{a^2 - b^2}} \operatorname{arctg} \left| \operatorname{sh} \left(x + \operatorname{Arth} \frac{b}{a} \right) \right| + \\ + \frac{1}{a^2 - b^2} \left[(Ca - Bb) \ln \operatorname{ch} \left(x + \operatorname{Arth} \frac{b}{a} \right) + (Ba - Cb)x \right] \\ [a > |b|].$$

$$12. = \frac{A}{\sqrt{b^2 - a^2}} \ln \left| \operatorname{th} \frac{x + \operatorname{Arth} \frac{a}{b}}{2} \right| + \\ + \frac{1}{b^2 - a^2} \left[(Cb - Ba)x + (Bb - Ca) \ln \operatorname{sh} \left(x + \operatorname{Arth} \frac{a}{b} \right) \right] \\ [b > |a|].$$

$$13. = \frac{1}{a} \left[\frac{B+C}{2} x - \frac{B-C}{4} e^{-2x} - Ae^{-x} \right] \\ [a = b].$$

$$14. = \frac{1}{a} \left[\frac{B-C}{2} x + \frac{B+C}{4} e^{2x} + Ae^x \right] \\ [a = -b].$$

$$15. \int \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a \operatorname{th} \frac{x}{2} - b + \sqrt{a^2 + b^2}}{a \operatorname{th} \frac{x}{2} - b - \sqrt{a^2 + b^2}}.$$

$$16. = \frac{2}{\sqrt{a^2+b^2}} \operatorname{Arth} \frac{a \operatorname{th} \frac{x}{2} - b}{\sqrt{a^2+b^2}}.$$

$$17. \int \frac{dx}{a+b \operatorname{ch} x} = \frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a \operatorname{ch} x}{a+b \operatorname{ch} x} \quad [b^2 > a^2; x < 0].$$

$$18. = -\frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a \operatorname{ch} x}{a+b \operatorname{ch} x} \quad [b^2 > a^2; x > 0].$$

$$19. = \frac{1}{\sqrt{a^2-b^2}} \ln \frac{a+b+\sqrt{a^2-b^2} \operatorname{th} \frac{x}{2}}{a+b-\sqrt{a^2-b^2} \operatorname{th} \frac{x}{2}} \quad [a^2 > b^2].$$

$$20. \int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{(1 \pm \operatorname{ch} x)^n} dx = \frac{B \operatorname{sh} x}{(1-n)(1 \pm \operatorname{ch} x)^n} \pm \frac{C}{1-n} \frac{1}{(1 \pm \operatorname{ch} x)^{n-1}} +$$

$$+ \left(\frac{n}{n-1} B \pm A \right) \frac{(n-1)!}{(2n-1)!!} \operatorname{sh} x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \frac{1}{(1 \pm \operatorname{ch} x)^{n-k}}.$$

$$21. \int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{1 \pm \operatorname{ch} x} dx = \pm Bx \pm C \ln |1 \pm \operatorname{ch} x| + (A \mp B) \frac{\operatorname{ch} x \mp 1}{\operatorname{sh} x}.$$

$$22. \int \frac{A+B \operatorname{ch} x+C \operatorname{sh} x}{(a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x)(a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x)} dx =$$

$$= A_0 \ln \frac{a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x}{a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x} + A_1 \int \frac{dx}{a_1+b_1 \operatorname{ch} x+c_1 \operatorname{sh} x} +$$

$$+ A_2 \int \frac{dx}{a_2+b_2 \operatorname{ch} x+c_2 \operatorname{sh} x},$$

где

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ b_1 & c_1 & A \\ B & C & C \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ C & A \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ C & B & C \\ c_2 & b_2 & C \end{vmatrix} \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}.$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \right].$$

$$\begin{aligned}
 23. \quad \int \frac{A \operatorname{ch}^2 x + 2B \operatorname{sh} x \operatorname{ch} x + C \operatorname{sh}^2 x}{a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x} dx = \\
 = \frac{1}{4b^2 - (a+c)^2} \{ [4Bb - (A+C)(a+c)]x + \\
 + [(A+C)b - B(a+c)] \ln(a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x) + \\
 + [2(A-C)b^2 + 2Bb(a-c) + (Ca - Ac)(a+c)] f(x) \},
 \end{aligned}$$

где

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{th} x + b - \sqrt{b^2 - ac}}{c \operatorname{th} x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac];$$

$$= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{th} x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac];$$

$$= -\frac{1}{c \operatorname{th} x + b} \quad [b^2 = ac].$$

$$\begin{aligned}
 24. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (a + b \operatorname{sh} x)} dx = \\
 = \frac{1}{a} \left[A \ln \left| \operatorname{th} \frac{x}{2} \right| + B \ln \left| \frac{\operatorname{sh} x}{a + b \operatorname{sh} x} \right| + (Ca - Ab) \int \frac{dx}{a + b \operatorname{sh} x} \right].
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (a + b \operatorname{ch} x)} dx = \\
 = \frac{1}{a^2 - b^2} \left[(Aa + Bb) \ln \left| \operatorname{th} \frac{x}{2} \right| + (Ab - Ba) \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{sh} x} \right| \right] + C \int \frac{dx}{a + b \operatorname{ch} x}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{sh} x (1 \pm \operatorname{ch} x)} dx = \\
 = \frac{B \pm A}{2} \ln \left| \left\{ \operatorname{th} (x/2) \right\} \right| \pm \frac{A \mp B}{4} \left\{ \operatorname{th} (x/2) \right\}^2 + C \frac{\operatorname{ch} x \mp 1}{\operatorname{sh} x}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{ch} x (a + b \operatorname{sh} x)} dx = \\
 = \frac{1}{a^2 + b^2} \left[(Aa + Cb) \operatorname{arctg} (\operatorname{sh} x) + (Ab - Ca) \ln \left| \frac{a + b \operatorname{sh} x}{\operatorname{ch} x} \right| \right] + B \int \frac{dx}{a + b \operatorname{sh} x}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{\operatorname{ch} x (a + b \operatorname{ch} x)} dx = \\
 = \frac{1}{a} \left[A \operatorname{arctg} (\operatorname{sh} x) - C \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{ch} x} \right| - (Ab - Ba) \int \frac{dx}{a + b \operatorname{ch} x} \right].
 \end{aligned}$$

$$29. \quad \int \frac{dx}{a + b \operatorname{sh}^2 x} = \frac{1}{\sqrt{a(b-a)}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} \frac{1}{a} \operatorname{th} x \right) \quad [b/a > 1].$$

$$\begin{aligned}
 30. \quad &= \frac{1}{\sqrt{a(a-b)}} \operatorname{Arth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \\
 &\quad [0 < b/a < 1] \text{ или } [b/a < 0; \operatorname{sh}^2 x < -a/b].
 \end{aligned}$$

$$31. \quad = \frac{1}{\sqrt{a(a-b)}} \operatorname{Arcth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \quad [b/a < 0; \operatorname{sh}^2 x > -a/b].$$

$$32. \quad = \frac{1}{a} \operatorname{th} x \quad [a = b].$$

$$33. = \frac{1}{a\sqrt{2}} \left\{ \operatorname{Arth}(\sqrt{2} \operatorname{th} x) \right\} \quad [a = -b; \operatorname{sh}^2 x \leq 1].$$

$$34. \int \frac{dx}{a+b \operatorname{ch}^2 x} = \frac{1}{\sqrt{-a(a+b)}} \operatorname{arctg} \left(\sqrt{-\left(1+\frac{b}{a}\right)} \operatorname{cth} x \right) \quad [b/a < -1].$$

$$35. = \frac{1}{\sqrt{a(a+b)}} \operatorname{Arth} \left(\sqrt{1+\frac{b}{a}} \operatorname{cth} x \right) \quad [-1 < b/a < 0; \operatorname{ch}^2 x > -a/b].$$

$$36. = \frac{1}{\sqrt{a(a+b)}} \operatorname{Arcth} \left(\sqrt{1+\frac{b}{a}} \operatorname{cth} x \right) \quad [b/a > 0] \text{ или } [-1 < b/a < 0; \operatorname{ch}^2 x < -a/b].$$

$$37. = \frac{1}{a\sqrt{2}} \operatorname{Arcth}(\sqrt{2} \operatorname{cth} x) \quad [a = b].$$

$$38. = \frac{1}{a} \operatorname{cth} x \quad [a = -b].$$

$$39. \int \left(a+b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right)^{-2} dx = \\ = \frac{1}{2a(b+a)} \left[\pm b \operatorname{sh} x \operatorname{ch} x \left(a+b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 \right)^{-1} + (b \mp 2a) \int \left(a+b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 \right)^{-1} dx \right].$$

$$40. \int \frac{dx}{a+b \operatorname{th}^2 x} = \frac{1}{a+b} \left[x + \sqrt{\frac{b}{a}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} \operatorname{th} x \right) \right] \quad [ab > 0; a+b \neq 0].$$

$$41. = \frac{1}{a+b} \left[x + \frac{b}{2\sqrt{-ab}} \ln \frac{\sqrt{-ab}-b \operatorname{th} x}{\sqrt{-ab}+b \operatorname{th} x} \right] \quad [ab < 0; a+b \neq 0].$$

$$42. = \frac{1}{2a} x + \frac{1}{4a} \operatorname{sh} 2x \quad [a+b=0].$$

1.4.8. Интегралы вида $\int R(\operatorname{sh}(ax+b), \operatorname{ch}(cx+d)) dx$.

$$1. \int \frac{\operatorname{sh}(ax+b) \operatorname{sh}(cx+d)}{\operatorname{ch}(ax+b) \operatorname{ch}(cx+d)} dx = \\ = \frac{1}{2(a+c)} \operatorname{sh}[(a+c)x+b+d] + \frac{1}{2(a-c)} \operatorname{sh}[(a-c)x+b-d].$$

$$2. \int \operatorname{sh}(ax+b) \operatorname{ch}(cx+d) dx = \\ = \frac{1}{2(a+c)} \operatorname{ch}[(a+c)x+b+d] + \frac{1}{2(a-c)} \operatorname{ch}[(a-c)x+b-d].$$

$$3. \int \frac{\operatorname{sh}(ax+b) \operatorname{sh}(ax+d)}{\operatorname{ch}(ax+b) \operatorname{ch}(ax+d)} dx = \mp \frac{x}{2} \operatorname{ch}(b-d) + \frac{1}{4a} \operatorname{sh}(2ax+b+d).$$

$$4. \int \operatorname{sh}(ax+b) \operatorname{ch}(ax+d) dx = \frac{x}{2} \operatorname{sh}(b-d) + \frac{1}{4a} \operatorname{ch}(2ax+b+d).$$

1.4.9. Интегралы вида $\int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx, \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx$.

$$1. \int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{p+a} \left[\operatorname{sh}^p x \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - p \int \operatorname{sh}^{p-1} x \left\{ \frac{\operatorname{ch}(a-1)x}{\operatorname{sh}(a-1)x} \right\} dx \right].$$

$$2. \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{p+a} \left[\operatorname{ch}^p x \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} + p \int \operatorname{ch}^{p-1} x \left\{ \frac{\operatorname{sh} (a-1)x}{\operatorname{ch} (a-1)x} \right\} dx \right].$$

$$3. \int \operatorname{sh} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx = \left\{ \frac{2n}{(-1)^{n+1}} \right\} \left[\frac{1}{p+2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+2} + \right. \\ \left. + \sum_{k=1}^{n-1} (\pm 1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots (4n^2-4k^2)}{(2k+1)!(2k+p+2)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+2} \right].$$

[$p \neq -2, -4, \dots, -2n$].

$$4. \int \operatorname{ch} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx = (\pm 1)^n \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^p dx + \\ + \sum_{k=1}^n (\pm 1)^k \frac{4n^2(4n^2-2^2) \dots [4n^2-(2k-2)^2]}{(2k)!} \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p} dx.$$

$$5. \int \operatorname{sh}^p x \left\{ \frac{\operatorname{sh} (p+2)x}{\operatorname{ch} (p+2)x} \right\} dx = \frac{1}{p+1} \operatorname{sh}^{p+1} x \left\{ \frac{\operatorname{sh} (p+1)x}{\operatorname{ch} (p+1)x} \right\}.$$

$$6. \int \operatorname{ch}^p x \left\{ \frac{\operatorname{sh} (p+2)x}{\operatorname{ch} (p+2)x} \right\} dx = \frac{1}{p+1} \operatorname{ch}^{p+1} x \left\{ \frac{\operatorname{sh} (p+1)x}{\operatorname{sh} (p+1)x} \right\}.$$

$$7. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{sh} (2n+1)x}{\operatorname{ch}^p x \operatorname{ch} (2n+1)x} \right\} dx = (\pm 1)^n (2n+1) \left[\int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} dx + \right. \\ \left. + \sum_{k=1}^n (\pm 1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+1} dx \right].$$

$$8. \int \left\{ \frac{\operatorname{sh}^p x \operatorname{ch} (2n+1)x}{\operatorname{ch}^p x \operatorname{sh} (2n+1)x} \right\} dx = \frac{(\pm 1)^n}{p+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{p+1} + \\ + \sum_{k=1}^n (\pm 1)^{n-k} \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times \\ \times \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k+p+1} \quad [p \neq -3, -5, \dots, -(2n+1)].$$

$$9. \int \left\{ \frac{\operatorname{sh} (2n+1)x \operatorname{sh}^{-1} x}{\operatorname{ch} (2n+1)x \operatorname{ch}^{-1} x} \right\} dx = 2 \sum_{k=0}^{n-1} (\pm 1)^k \frac{\operatorname{sh} (2n-2k)x}{2n-2k} + (\pm 1)^n x.$$

$$10. \int \left\{ \frac{\operatorname{sh} (2n+1)x \operatorname{ch}^{-1} x}{\operatorname{ch} (2n+1)x \operatorname{sh}^{-1} x} \right\} dx = 2 \sum_{k=0}^{n-1} (\pm 1)^k \frac{\operatorname{ch} (2n-2k)x}{2n-2k} + (\pm 1)^n \ln \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$11. \int \operatorname{sh} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = 2 \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \left\{ \frac{\operatorname{sh} (2n-2k-1)x}{\operatorname{ch} (2n-2k-1)x} \right\}.$$

- $$12. \int \operatorname{ch} 2nx \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx =$$
- $$= 2 \sum_{k=0}^{n-1} \frac{(\pm 1)^k}{2n-2k-1} \left\{ \frac{\operatorname{ch} (2n-2k-1)x}{\operatorname{sh} (2n-2k-1)x} \right\} + (\pm 1)^n \left\{ \frac{\ln |\operatorname{th} (x/2)|}{\operatorname{arcsin} \operatorname{th} x} \right\}.$$
- $$13. \int \operatorname{sh} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-n} dx = \frac{2}{2-n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2-n}.$$
- $$14. \int \operatorname{sh} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = 2 \ln \left\{ \left| \frac{\operatorname{sh} x}{\operatorname{ch} x} \right| \right\}.$$
- $$15. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-1} dx = 2 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \pm \left\{ \frac{\ln |\operatorname{th} (x/2)|}{\operatorname{arcsin} \operatorname{th} x} \right\}.$$
- $$16. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = - \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} + 2x.$$
- $$17. \int \operatorname{ch} 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} dx = - \frac{1}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-2} x}{\operatorname{sh} x \operatorname{ch}^{-2} x} \right\} + \frac{3}{2} \left\{ \frac{\ln |\operatorname{th} (x/2)|}{\operatorname{arcsin} \operatorname{th} x} \right\}.$$
- $$18. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-1} x}{\operatorname{ch} 3x \operatorname{ch}^{-1} x} \right\} dx = \operatorname{sh} 2x \pm x.$$
- $$19. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-2} x}{\operatorname{ch} 3x \operatorname{ch}^{-2} x} \right\} dx = 4 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \pm 3 \left\{ \frac{\ln |\operatorname{th} (x/2)|}{\operatorname{arcsin} \operatorname{th} x} \right\}.$$
- $$20. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{sh}^{-3} x}{\operatorname{ch} 3x \operatorname{ch}^{-3} x} \right\} dx = 4x - 3 \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\}.$$
- $$21. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-n} x}{\operatorname{ch} 3x \operatorname{sh}^{-n} x} \right\} dx = \frac{4}{3-n} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{3-n} + \frac{1}{1-n} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{1-n}.$$
- $$22. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-1} x}{\operatorname{ch} 3x \operatorname{sh}^{-1} x} \right\} dx = 2 \operatorname{sh}^2 x \mp \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$
- $$23. \int \left\{ \frac{\operatorname{sh} 3x \operatorname{ch}^{-3} x}{\operatorname{ch} 3x \operatorname{sh}^{-3} x} \right\} dx = - \frac{1}{2} \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^2 + 4 \ln \left\{ \frac{\operatorname{ch} x}{|\operatorname{sh} x|} \right\}.$$

1.4.10. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, \sqrt{\operatorname{sh} 2ax}) dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arccos \frac{1 - \operatorname{sh} 2ax}{1 + \operatorname{sh} 2ax}$.

- $$1. \int_0^x \sqrt{\operatorname{sh} 2ax} dx = \frac{1}{2a} \left[F \left(\varphi, \frac{1}{\sqrt{2}} \right) - 2E \left(\varphi, \frac{1}{\sqrt{2}} \right) \right] + \frac{1}{a} \frac{\sqrt{\operatorname{sh} 2ax (1 + \operatorname{sh}^2 2ax)}}{1 + \operatorname{sh} 2ax}.$$
- $$2. \int_0^x \frac{\sqrt{\operatorname{sh} 2ax}}{(1 + \operatorname{sh} 2ax)^2} dx = \frac{1}{4a} \left[F \left(\varphi, \frac{1}{\sqrt{2}} \right) - E \left(\varphi, \frac{1}{\sqrt{2}} \right) \right].$$

$$3. \int_0^x \frac{dx}{V \operatorname{sh} 2ax} = \frac{1}{2a} F\left(\varphi, \frac{1}{V2}\right).$$

$$4. \int_0^x \frac{\operatorname{ch}^2 2ax \, dx}{(1 + \operatorname{sh} 2ax)^2 V \operatorname{sh} 2ax} = \frac{1}{2a} E\left(\varphi, \frac{1}{V2}\right).$$

$$5. \int_0^x \frac{(1 - \operatorname{sh} 2ax)^2 \, dx}{(1 + \operatorname{sh} 2ax)^3 V \operatorname{sh} 2ax} = \frac{1}{2a} \left[2E\left(\varphi, \frac{1}{V2}\right) - F\left(\varphi, \frac{1}{V2}\right) \right].$$

$$6. \int_0^x \frac{(1 + \operatorname{sh} 2ax)^2 \, dx}{[(1 + \operatorname{sh} 2ax)^2 - 4\rho^2 \operatorname{sh} 2ax] V \operatorname{sh} 2ax} = \frac{1}{2a} \Pi\left(\varphi, \rho^2, \frac{1}{V2}\right).$$

1.4.11. Интегралы вида $\int R(\operatorname{sh} 2ax, \operatorname{ch} 2ax, V \operatorname{ch} 2ax) \, dx$.

Условие: $ax > 0$.

Обозначение: $\varphi = \arcsin V \frac{\operatorname{ch} 2ax - 1}{\operatorname{ch} 2ax}$.

$$1. \int_0^x V \operatorname{ch} 2ax \, dx = \frac{1}{a V 2} \left[F\left(\varphi, \frac{1}{V2}\right) - 2E\left(\varphi, \frac{1}{V2}\right) \right] + \frac{\operatorname{sh} 2ax}{a V \operatorname{ch} 2ax}.$$

$$2. \int_0^x \frac{V \operatorname{ch} 2ax \, dx}{\rho^2 + (1 - \rho^2) \operatorname{ch} 2ax} = \frac{1}{a V 2} \Pi\left(\varphi, \rho^2, \frac{1}{V2}\right).$$

$$3. \int_0^x \frac{dx}{V \operatorname{ch} 2ax} = \frac{1}{a V 2} F\left(\varphi, \frac{1}{V2}\right).$$

$$4. \int_0^x \frac{\operatorname{sh} 2ax \, dx}{V \operatorname{ch} 2ax} = \frac{1}{a} [V \operatorname{ch} 2ax - 1].$$

$$5. \int_0^x \frac{\operatorname{sh}^2 2ax \, dx}{V \operatorname{ch} 2ax} = -\frac{V 2}{3a} F\left(\varphi, \frac{1}{V2}\right) + \frac{1}{3a} \operatorname{sh} 2ax V \operatorname{ch} 2ax.$$

$$6. \int_0^x \frac{\operatorname{th}^2 2ax \, dx}{V \operatorname{ch} 2ax} = \frac{V 2}{3a} F\left(\varphi, \frac{1}{V2}\right) - \frac{\operatorname{th} 2ax}{3a V \operatorname{ch} 2ax}.$$

$$7. \int_0^x \frac{dx}{V \operatorname{ch}^3 2ax} = \frac{1}{a V 2} \left[2E\left(\varphi, \frac{1}{V2}\right) - F\left(\varphi, \frac{1}{V2}\right) \right].$$

1.4.12. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a+b \operatorname{sh} x}) dx$.

Условия: $a, b > 0, x > x_1$.

Обозначения: $\varphi = \arccos \frac{\sqrt{a^2+b^2}-a-b \operatorname{sh} x}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x}$,

$$k = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}}, \quad x_1 = -\operatorname{Arsh} \frac{a}{b}.$$

$$1. \int_{x_1}^x \sqrt{a+b \operatorname{sh} x} dx = \sqrt[4]{a^2+b^2} [F(\varphi, k) - 2E(\varphi, k)] + \frac{2b \operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x}.$$

$$2. \int_{x_1}^x \frac{\sqrt{a+b \operatorname{sh} x}}{\operatorname{ch}^2 x} dx = \sqrt[4]{a^2+b^2} E(\varphi, k) - \frac{\sqrt{a^2+b^2}-a}{2\sqrt[4]{a^2+b^2}} F(\varphi, k) - \\ - \frac{a+\sqrt{a^2+b^2}}{b} \frac{\sqrt{a^2+b^2}-a-b \operatorname{sh} x}{\sqrt{a^2+b^2}+a+b \operatorname{sh} x} \frac{\sqrt{a+b \operatorname{sh} x}}{\operatorname{ch} x}.$$

$$3. \int \frac{\operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{p+q \operatorname{sh} x} dx = 2\sqrt{\frac{aq-bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{aq-bp}} \\ [b \operatorname{sh} x > 0; (aq-bp)/q > 0].$$

$$4. = 2\sqrt{\frac{aq-bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{aq-bp}} \\ [b \operatorname{sh} x < 0; (aq-bp)/q > 0].$$

$$5. = 2\sqrt{\frac{bp-aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{sh} x)}{bp-aq}} \\ [(aq-bp)/q < 0].$$

$$6. \int_{x_1}^x \frac{\sqrt{a+b \operatorname{sh} x} dx}{[\sqrt{a^2+b^2}-a-b \operatorname{sh} x]^2} = \\ = -\frac{1}{\sqrt[4]{a^2+b^2}(\sqrt{a^2+b^2}-a)} E(\varphi, k) + \frac{b}{\sqrt{a^2+b^2}-a} \frac{\operatorname{ch} x \sqrt{a+b \operatorname{sh} x}}{a^2+b^2-(a+b \operatorname{sh} x)^2}.$$

$$7. \int_{x_1}^x \frac{dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{1}{\sqrt[4]{a^2+b^2}} F(\varphi, k).$$

$$8. \int_{x_1}^x \frac{\operatorname{ch} x dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{2}{b} \sqrt{a+b \operatorname{sh} x}.$$

$$9. \int_{x_1}^x \frac{\operatorname{ch}^2 x dx}{(\sqrt{a^2+b^2}+a+b \operatorname{sh} x)^2 \sqrt{a+b \operatorname{sh} x}} = \frac{1}{b^2 \sqrt[4]{a^2+b^2}} E(\varphi, k).$$

$$10. \int_{x_1}^x \frac{\operatorname{cth} x dx}{\sqrt{a+b \operatorname{sh} x}} = \frac{2}{\sqrt{a}} \operatorname{Arth} \sqrt{1+\frac{b}{a} \operatorname{sh} x}$$

$[b \operatorname{sh} x > 0; a > 0].$

$$11. \quad -\frac{2}{\sqrt{a}} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \quad \{b \operatorname{sh} x < 0; a > 0\}.$$

$$12. \quad = \frac{2}{\sqrt{-a}} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{sh} x\right)} \quad \{a < 0\}.$$

1.4.13. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{b \operatorname{ch} x - a}) dx$.

Условия: $b > a > 0, x > 0$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{b(\operatorname{ch} x - 1)}{b \operatorname{ch} x - a}}, k = \sqrt{\frac{a+b}{2b}}$.

$$1. \quad \int_0^x \sqrt{b \operatorname{ch} x - a} dx = (b-a) \sqrt{\frac{2}{b}} F(\varphi, k) - 2 \sqrt{2b} E(\varphi, k) + \frac{2b \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

$$2. \quad \int_0^x \frac{\sqrt{b \operatorname{ch} x - a}}{\operatorname{ch} x - 1} dx = \sqrt{a+b} [F(\varphi, k) - E(\varphi, k)].$$

$$3. \quad \int_0^x \frac{\sqrt{b \operatorname{ch} x - a} dx}{\rho^2 b - a + b(1 - \rho^2) \operatorname{ch} x} = \sqrt{\frac{2}{b}} \Pi(\varphi, \rho^2, k).$$

$$4. \quad \int_0^x \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} F(\varphi, k).$$

$$5. \quad \int_0^x \frac{\operatorname{ch} x dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} [F(\varphi, k) - 2E(\varphi, k)] + \frac{2 \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

$$6. \quad \int_0^x \frac{\operatorname{sh} x dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{b} [\sqrt{b \operatorname{ch} x - a} - \sqrt{b - a}].$$

$$7. \quad \int_0^x \frac{dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{1}{b^2 - a^2} \sqrt{\frac{2}{b}} [2bE(\varphi, k) - (b-a)F(\varphi, k)].$$

$$8. \quad \int_0^x \frac{(\operatorname{ch} x + 1) dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{2}{b-a} \sqrt{\frac{2}{b}} E(\varphi, k).$$

1.4.14. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{b \operatorname{ch} x - a}) dx$.

Условия: $a > b > 0, x > x_1$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{b \operatorname{ch} x - a}{b(\operatorname{ch} x - 1)}}, k = \sqrt{\frac{2b}{a+b}}, x_1 = \operatorname{Arch} \frac{a}{b}.$

$$1. \quad \int_{x_1}^x \sqrt{b \operatorname{ch} x - a} dx = -2 \sqrt{a+b} E(\varphi, k) + 2 \operatorname{cth} \frac{x}{2} \sqrt{b \operatorname{ch} x - a}.$$

$$2. \int_{x_1}^x \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{\sqrt{a+b}} F(\varphi, k),$$

$$3. \int_{x_1}^x \frac{\operatorname{cth}^2 \frac{x}{2} dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\varphi, k),$$

$$4. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x + 1) \sqrt{b \operatorname{ch} x - a}} = \frac{1}{\sqrt{a+b}} [F(\varphi, k) - E(\varphi, k)] + \frac{2\sqrt{b \operatorname{ch} x - a}}{(a+b) \operatorname{sh} x},$$

$$5. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x - 1) \sqrt{b \operatorname{ch} x - a}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{1}{\sqrt{a+b}} F(\varphi, k),$$

$$6. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{b \operatorname{ch} x - a}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+2b) F(\varphi, k) - (a+3b) E(\varphi, k)] + \\ + \frac{\sqrt{b \operatorname{ch} x - a}}{3(a+b) \operatorname{sh} x} \left(2 \frac{a+3b}{a+b} - \operatorname{th}^2 \frac{x}{2} \right).$$

$$7. \int_{x_1}^x \frac{dx}{(\operatorname{ch} x - 1)^2 \sqrt{b \operatorname{ch} x - a}} = \\ = \frac{1}{3(a-b)^2 \sqrt{a+b}} [(a-2b)(a-b) F(\varphi, k) + (3a-b)(a+b) E(\varphi, k)] + \\ + \frac{a+b}{6b(a-b) \operatorname{sh}^2(x/2)} \sqrt{b \operatorname{ch} x - a}.$$

1.4.15. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a-b \operatorname{ch} x}) dx$.

Условия: $a > b > 0$, $0 < x < x_1$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{a-b \operatorname{ch} x}{a-b}}$, $k = \sqrt{\frac{a-b}{a+b}}$, $x_1 = \operatorname{Arch} \frac{a}{b}$.

$$1. \int_x^{x_1} \sqrt{a-b \operatorname{ch} x} dx = 2\sqrt{a+b} [F(\varphi, k) - E(\varphi, k)],$$

$$2. \int_x^{x_1} \frac{dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k),$$

$$3. \int_x^{x_1} \frac{\operatorname{sh} x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2}{b} \sqrt{a-b \operatorname{ch} x}.$$

$$4. \int_x^{x_1} \frac{\operatorname{ch} x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\varphi, k) - \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$5. \int_x^{x_1} \frac{\operatorname{ch}^2 x \, dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2(b-2a)}{3b \sqrt{a+b}} F(\varphi, k) + \frac{4a \sqrt{a+b}}{3b^2} E(\varphi, k) + \frac{2}{3b} \operatorname{sh} x \sqrt{a-b \operatorname{ch} x}.$$

$$6. \int_x^{x_1} \frac{dx}{\operatorname{ch} x \sqrt{a-b \operatorname{ch} x}} = \frac{2b}{a \sqrt{a+b}} \Pi\left(\varphi, \frac{a-b}{a}, k\right).$$

$$7. \int_x^{x_1} \frac{dx}{(1+\operatorname{ch} x) \sqrt{a-b \operatorname{ch} x}} = \frac{1}{\sqrt{a+b}} E(\varphi, k) - \frac{1}{a+b} \operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}.$$

$$8. \int_x^{x_1} \frac{dx}{(1+\operatorname{ch} x)^2 \sqrt{a-b \operatorname{ch} x}} = \frac{1}{3 \sqrt{(a+b)^3}} [(a+3b) E(\varphi, k) - b F(\varphi, k)] - \frac{1}{3(a+b)^2} \frac{\operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}}{\operatorname{ch} x + 1} [2a + 4b + (a+3b) \operatorname{ch} x].$$

$$9. \int_x^{x_1} \frac{dx}{(a-b-arp^2+bp^2 \operatorname{ch} x) \sqrt{a-b \operatorname{ch} x}} = \frac{2}{(a-b) \sqrt{a+b}} \Pi(\varphi, p^2, k).$$

1.4.16. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a+b \operatorname{ch} x}) dx$.

Условия: $a > b > 0, x > 0$.

Обозначения: $\varphi = \arcsin\left(\operatorname{th} \frac{x}{2}\right), k = \sqrt{\frac{a-b}{a+b}}$.

$$1. \int_0^x \sqrt{a+b \operatorname{ch} x} \, dx = 2 \sqrt{a+b} [F(\varphi, k) - E(\varphi, k)] + 2 \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x}.$$

$$2. \int_0^x \frac{\sqrt{a+b \operatorname{ch} x}}{\operatorname{ch} x + 1} dx = \sqrt{a+b} E(\varphi, k).$$

$$3. \int \frac{\operatorname{sh} x \sqrt{a+b \operatorname{ch} x}}{p+q \operatorname{ch} x} dx = 2 \sqrt{\frac{aq-bp}{q}} \operatorname{Arcth} \sqrt{\frac{q(a+b \operatorname{ch} x)}{aq-bp}} \quad [(aq-bp)/q > 0].$$

$$4. = 2 \sqrt{\frac{bp-aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a+b \operatorname{ch} x)}{bp-aq}} \quad [(aq-bp)/q < 0].$$

$$5. \int_0^x \frac{dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k).$$

$$6. \int_0^x \frac{\operatorname{sh} x \, dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{b} (\sqrt{a+b \operatorname{ch} x} - \sqrt{a+b}).$$

7. $\int_0^x \frac{\operatorname{ch} x dx}{\sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\varphi, k) - \frac{2\sqrt{a+b}}{b} E(\varphi, k) + \frac{2}{b} \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x}.$
8. $\int_0^x \frac{\operatorname{ch} x - 1}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{2}{b} \left[\operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x} - \sqrt{a+b} E(\varphi, k) \right].$
9. $\int_0^x \frac{(\operatorname{ch} x - 1)^2}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{4\sqrt{a+b}}{3b^2} [(a+3b) E(\varphi, k) - bF(\varphi, k)] +$
 $+ \frac{4}{3b^2} \left[b \operatorname{ch}^2 \frac{x}{2} - (a+3b) \right] \operatorname{th} \frac{x}{2} \sqrt{a+b \operatorname{ch} x}.$
10. $\int_0^x \frac{\operatorname{th} x dx}{\sqrt{a+b \operatorname{ch} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x}.$
11. $\int_0^x \frac{\operatorname{th}^2(x/2)}{\sqrt{a+b \operatorname{ch} x}} dx = \frac{2\sqrt{a+b}}{a-b} [F(\varphi, k) - E(\varphi, k)].$
12. $\int_0^x \frac{dx}{(\operatorname{ch} x + 1) \sqrt{a+b \operatorname{ch} x}} = \frac{\sqrt{a+b}}{a-b} E(\varphi, k) - \frac{2b}{(a-b)\sqrt{a+b}} F(\varphi, k).$
13. $\int_0^x \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{a+b \operatorname{ch} x}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} [b(5b-a) F(\varphi, k) +$
 $+ (a-3b)(a+b) E(\varphi, k)] + \frac{1}{6(a-b)} \frac{\operatorname{sh}(x/2)}{\operatorname{ch}^3(x/2)} \sqrt{a+b \operatorname{ch} x}.$
14. $\int_0^x \frac{(1 + \operatorname{ch} x) dx}{|1 + p^2 + (1 - p^2) \operatorname{ch} x| \sqrt{a+b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} \Pi(\varphi, p^2, k).$

1.4.17. Интегралы вида

$$\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a^2 \operatorname{sh}^2 x \pm b^2}, \sqrt{b^2 - a^2 \operatorname{sh}^2 x}, \sqrt{a^2 \operatorname{ch}^2 x \pm b^2}, \sqrt{b^2 - a^2 \operatorname{ch}^2 x}) dx.$$

1. $\int \frac{1}{\sqrt{\operatorname{sh}^2 x + a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{sh}^2 x + a^2} \right).$
2. $\int \frac{1}{\sqrt{\operatorname{sh}^2 x - a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{sh}^2 x - a^2} \right) \quad [\operatorname{sh}^2 x > a^2].$
3. $\int \frac{1}{\sqrt{a^2 - \operatorname{sh}^2 x}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \begin{cases} \arcsin [(\operatorname{ch} x)/\sqrt{a^2 + 1}] \\ \arcsin [(\operatorname{sh} x)/a] \end{cases} \quad [\operatorname{sh}^2 x < a^2].$
4. $\int \frac{1}{\sqrt{\operatorname{ch}^2 x + a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{ch}^2 x + a^2} \right).$
5. $\int \frac{1}{\sqrt{\operatorname{ch}^2 x - a^2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \ln \left(\left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \sqrt{\operatorname{ch}^2 x - a^2} \right) \quad [\operatorname{ch}^2 x > a^2].$

$$6. \int \frac{1}{\sqrt{a^2 - \operatorname{ch}^2 x}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \begin{cases} \arcsin [(\operatorname{ch} x)/a] \\ \arcsin [(\operatorname{sh} x)/\sqrt{a^2 - 1}] \end{cases} \quad [\operatorname{ch}^2 x < a^2].$$

$$7. \int_0^x \frac{dx}{\sqrt{1 + k^2 \operatorname{sh}^2 x}} = F(\arcsin \operatorname{th} x, \sqrt{1 - k^2}) \quad [x > 0; k^2 < 1].$$

$$8. \int_0^x \frac{dx}{\sqrt{k^2 + (1 - k^2) \operatorname{ch}^2 x}} = F(\arcsin \operatorname{th} x, k) \quad [x > 0; k^2 < 1].$$

$$9. \int_0^x \frac{dx}{\sqrt{1 - k^2 \operatorname{ch}^2 x}} = F\left(\arcsin \frac{\operatorname{th} x}{\sqrt{1 - k^2}}, \sqrt{1 - k^2}\right) \quad [0 < x < \operatorname{ch}(1/\sqrt{1 - k^2}); k^2 < 1].$$

$$10. \int_x^\infty \frac{dx}{\sqrt{\operatorname{ch}^2 x - k^2}} = F(\arcsin \operatorname{sech} x, k) \quad [x > 0; k^2 < 1].$$

$$11. \int_x^\infty \frac{dx}{\sqrt{\operatorname{sh}^2 x + k^2}} = F(\arcsin \operatorname{sech} x, \sqrt{1 - k^2}) \quad [x > 0; k^2 < 1].$$

1.4.18. Интегралы вида $\int R(\operatorname{sh} x, \operatorname{ch} x, \sqrt{a \operatorname{sh} x + b \operatorname{ch} x}) dx$.

Условия: $b > a > 0$, $x > x_1$.

Обозначения: $\varphi = \arccos \frac{\sqrt{b^2 - a^2}}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}$, $x_1 = -\operatorname{Arsh} \frac{a}{\sqrt{b^2 - a^2}}$.

$$1. \int_{x_1}^x \sqrt{a \operatorname{sh} x + b \operatorname{ch} x} dx = \sqrt[4]{4(b^2 - a^2)} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 2E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{2(a \operatorname{ch} x + b \operatorname{sh} x)}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}.$$

$$2. \int_{x_1}^x \frac{dx}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}} = \sqrt[4]{\frac{4}{b^2 - a^2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$3. \int_{x_1}^x \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = \sqrt[4]{\frac{4}{(b^2 - a^2)^3}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$4. \int_{x_1}^x \frac{(\sqrt{b^2 - a^2} + a \operatorname{sh} x + b \operatorname{ch} x) dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = 2 \sqrt[4]{\frac{4}{b^2 - a^2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

1.4.19. Интегралы вида $\int \sqrt{\operatorname{th} x} dx$, $\int \sqrt{\operatorname{cth} x} dx$.

$$1. \int \sqrt{\operatorname{th} x} dx = \operatorname{Arth} \sqrt{\operatorname{th} x} - \operatorname{arctg} \sqrt{\operatorname{th} x}.$$

$$2. \int \sqrt{\operatorname{cth} x} dx = \operatorname{Arcth} \sqrt{\operatorname{cth} x} - \operatorname{arctg} \sqrt{\operatorname{cth} x}.$$

1.4.20. Интегралы вида $\int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx$.

$$1. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx = \frac{1}{q} x^p \left\{ \frac{\operatorname{sh}^{q-1} x \operatorname{ch} x}{\operatorname{ch}^{q-1} x \operatorname{sh} x} \right\} - \frac{p}{q^2} x^{p-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q + \\ + \frac{p(p-1)}{q^2} \int x^{p-2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx + \frac{q-1}{q} \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{q-2} dx.$$

$$2. \int x^m \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = (-1)^n \binom{2n}{n} \frac{x^{m+1}}{2^{2n} (m+1)} + \\ + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \int x^m \operatorname{ch} (2n-2k)x dx.$$

$$3. \int x^m \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+2} dx = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \int x^m \left\{ \frac{\operatorname{sh} (2n-2k+1)x}{\operatorname{ch} (2n-2k+1)x} \right\} dx.$$

$$4. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = \frac{1}{2n+1} \sum_{k=0}^{n-1} (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+1)}{2^{k+1} n (n-1) \dots (n-k)} \times \\ \times \left[\frac{-1}{2n-2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n-2k} + x \left\{ \frac{\operatorname{sh}^{2n-2k-1} x \operatorname{ch} x}{\operatorname{ch}^{2n-2k-1} x \operatorname{sh} x} \right\} \right] + (-1)^n \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2}.$$

$$5. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = \\ = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \frac{2^k (n+1)n \dots (n-k+1)}{(2n+1)(2n-1) \dots (2n-2k+1)} \left[\frac{-1}{2n-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n-2k+1} + \right. \\ \left. + x \left\{ \frac{\operatorname{sh}^{2n-2k} x \operatorname{ch} x}{\operatorname{ch}^{2n-2k} x \operatorname{sh} x} \right\} \right].$$

$$6. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - p \int x^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} dx.$$

$$7. \int x^{2n} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (2n)! \left[\sum_{k=0}^n \frac{x^{2k}}{(2k)!} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \sum_{k=0}^{n-1} \frac{x^{2k+1}}{(2k+1)!} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$8. \int x^{2n+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (2n+1)! \sum_{k=0}^n \left[\frac{x^{2k+1}}{(2k+1)!} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \frac{x^{2k}}{(2k)!} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$9. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$10. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = (x^2 + 2) \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} - 2x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$11. \int x^n \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = -\frac{x^{n+1}}{2(n+1)} + \\ + \frac{n!}{4} \left[\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2k}}{2^{2k} (n-2k)!} \operatorname{sh} 2x - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{x^{n-2k-1}}{2^{2k+1} (n-2k-1)!} \operatorname{ch} 2x \right].$$

$$12. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = \frac{x}{4} \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x + \frac{x^2}{4}.$$

$$13. \int_0^x x^{\alpha-1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \frac{x^{\alpha+\delta}}{\alpha+\delta} {}_2F_2 \left(\frac{\alpha+\delta}{2}; \frac{\alpha+\delta}{2}+1, \frac{1}{2}+\delta; \frac{x^2}{4} \right) \\ \left[\operatorname{Re} \alpha > -\delta; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$14. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^2 dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \operatorname{sh} 2x - \frac{x}{4} \operatorname{ch} 2x + \frac{x^3}{6}.$$

$$15. \int x^n \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \frac{n!}{4} \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \left[\frac{1}{3^{2k+1}} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} + \right. \\ \left. + 3 \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} \right] - \frac{n!}{4} \sum_{k=0}^{[(n-1)/2]} \frac{x^{n-2k-1}}{(n-2k-1)!} \left[\frac{1}{3^{2k+2}} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{ch} 3x} \right\} + 3 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right].$$

$$16. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = \pm \frac{3}{4} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} - \frac{1}{36} \left\{ \frac{\operatorname{sh} 3x}{\operatorname{ch} 3x} \right\} + \frac{3}{4} x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \frac{x}{12} \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\}.$$

$$17. \int x^2 \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^3 dx = + \left(\frac{3}{4} x^2 + \frac{3}{2} \right) \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \left(\frac{x^2}{12} + \frac{1}{54} \right) \left\{ \frac{\operatorname{ch} 3x}{\operatorname{sh} 3x} \right\} \pm \\ \pm \frac{3}{2} x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} - \frac{x}{18} \left\{ \frac{\operatorname{sh} 3x}{\operatorname{ch} 3x} \right\}.$$

1.4.21. Интегралы вида $\int \frac{1}{x^\rho} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx$.

$$1. \int \frac{1}{x^\rho} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx = - \frac{1}{(\rho-1)x^{\rho-1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q - \frac{q}{(\rho-1)(\rho-2)x^{\rho-2}} \left\{ \frac{\operatorname{sh}^{q-1} x \operatorname{ch} x}{\operatorname{ch}^{q-1} x \operatorname{sh} x} \right\} \pm \\ \pm \frac{q(q-1)}{(\rho-1)(\rho-2)} \int \frac{1}{x^{\rho-2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{q-2} dx + \frac{q^2}{(\rho-1)(\rho-2)} \int \frac{1}{x^{\rho-2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^q dx.$$

$$2. \int \frac{1}{x^{2n}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = - \frac{1}{x(2n-1)!} \left[\sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right] + \frac{1}{(2n-1)!} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}.$$

$$3. \int \frac{1}{x^{2n+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = - \frac{1}{x(2n)!} \left[\sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\} + \right. \\ \left. + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right] + \frac{1}{(2n)!} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$4. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \left(\frac{-1}{2} \right)^k \binom{2n}{k} \operatorname{chi} (2n-2k)x + \frac{(-1)^n}{2^{2n}} \binom{2n}{n} \ln |x|.$$

$$5. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^n \left(\frac{-1}{2} \right)^k \binom{2n+1}{k} \left\{ \operatorname{shi} (2n-2k+1)x \right\} \cdot \left\{ \operatorname{chi} (2n-2k+1)x \right\}.$$

$$6. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n} dx = -\frac{(-1)^n}{2^{2n} x} \binom{2n}{n} - \\ - \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \left(\frac{-1}{2} \right)^k \binom{2n}{k} \left[\frac{\operatorname{ch} (2n-2k)x}{x} - (2n-2k) \operatorname{shi} (2n-2k)x \right].$$

$$7. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2n+1} dx = -\frac{1}{2^{2n}} \sum_{k=0}^n \left(\frac{-1}{2} \right)^k \binom{2n+1}{k} \times \\ \times \left[\frac{1}{x} \left\{ \frac{\operatorname{sh} (2n-2k+1)x}{\operatorname{ch} (2n-2k+1)x} \right\} - (2n-2k+1) \left\{ \operatorname{chi} (2n-2k+1)x \right\} \right] \cdot \left\{ \operatorname{shi} (2n-2k+1)x \right\}.$$

$$8. \int \frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \left\{ \frac{\operatorname{shi} x}{\operatorname{chi} x} \right\}. \quad 9. \int_0^x \frac{\operatorname{sh} x}{x} dx = \operatorname{shi} x.$$

$$10. \int_0^x \frac{\operatorname{ch} x - 1}{x} dx = \operatorname{chi} x - \ln x - C.$$

$$11. \int \frac{1}{x^2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = -\frac{1}{x} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} + \left\{ \frac{\operatorname{chi} x}{\operatorname{shi} x} \right\}.$$

1.4.22. Интегралы вида $\int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^m dx$.

$$1. \int x^p \operatorname{th} x dx = \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(2k+p)(2k)!} x^{2k+p} \quad [|x| < \pi/2; p \geq -1].$$

$$2. \int x^p \operatorname{cth} x dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k+p)(2k)!} x^{2k+p} \quad [|x| < \pi; p \geq 1].$$

$$3. \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n} dx = \frac{1}{p+1} x^{p+1} + \sum_{k=0}^{n-1} \left(\frac{-1}{2} \right)^{n+k} \binom{n}{k} \int x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} dx.$$

$$4. \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\}^{2n+1} dx = \int x^p \left\{ \frac{\operatorname{th} x}{\operatorname{cth} x} \right\} dx + \\ + \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{2^{n-2k}} \binom{n}{k} \left[-x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} + p \int x^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-2n} dx \right].$$

1.4.23. Интегралы вида $\int x^r \operatorname{sh}^p x \operatorname{ch}^q x dx$.

$$1. \int x^r \operatorname{sh}^p x \operatorname{ch}^q x dx = \frac{1}{(p+q)^2} [(p+q) x^r \operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x - \\ - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r+1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x dx + \\ + r p \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x dx + (q-1)(p+q) \int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x dx].$$

$$2. = \frac{1}{(p+q)^2} [(p+q) x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x - \\ - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r-1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x dx - \\ - r q \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x dx - (p-1)(p+q) \int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x dx].$$

$$3. \int x^p \left\{ \frac{\operatorname{sh}^{2m} x \operatorname{ch}^{1-n} x}{\operatorname{ch}^{2m} x \operatorname{sh}^{-n} x} \right\} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int x^p \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{2k-n} dx.$$

$$4. \int x^p \left\{ \frac{\operatorname{sh}^{2m+1} x \operatorname{ch}^{-n} x}{\operatorname{ch}^{2m+1} x \operatorname{sh}^{-n} x} \right\} dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int x^p \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{2k-n} x}{\operatorname{ch} x \operatorname{sh}^{2k-n} x} \right\} dx.$$

$$5. \int x^p \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-n} x}{\operatorname{ch} x \operatorname{sh}^{-n} x} \right\} dx = -\frac{x^p}{n-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-n+1} + \frac{p}{n-1} \int x^{p-1} \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-n+1} dx.$$

$$6. \int x \left\{ \frac{\operatorname{sh} x \operatorname{ch}^{-2} x}{\operatorname{ch} x \operatorname{sh}^{-2} x} \right\} dx = -x \left\{ \frac{\operatorname{ch} x}{\operatorname{sh} x} \right\}^{-1} + \{ \operatorname{arctg} \operatorname{sh} x \}.$$

1.4.24. Интегралы вида $\int \frac{x^p}{\operatorname{sh}^q x} dx$, $\int \frac{x^p}{\operatorname{ch}^q x} dx$.

$$1. \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q} dx = \mp \frac{p x^{p-1}}{(q-1)(q-2)} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} \mp \frac{x^p}{q-1} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-q+1} x}{\operatorname{sh} x \operatorname{ch}^{-q+1} x} \right\} \pm \\ \pm \frac{p(p-1)}{(q-1)(q-2)} \int x^{p-2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx \mp \frac{q-2}{q-1} \int x^p \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx.$$

$$2. \int \frac{x^p}{\operatorname{sh} x} dx = \sum_{k=0}^{\infty} \frac{(2-2^{2k}) B_{2k}}{(2k)! (\rho+2k)} x^{p+2k} \quad [|x| < \pi; \rho > 0].$$

$$3. \int \frac{x^p}{\operatorname{ch} x} dx = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)! (\rho+2k+1)} x^{p+2k+1} \quad [|x| < \pi/2; \rho \geq 0].$$

$$4. \int \frac{x^p}{\operatorname{sh}^2 x} dx = -x^p \operatorname{cth} x + p \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(\rho+2k-1)(2k)!} x^{p+2k-1} \quad [|x| < \pi; \rho > 1].$$

$$5. \int \frac{x^p}{\operatorname{ch}^2 x} dx = x^p \operatorname{th} x - p \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k}-1) B_{2k}}{(\rho+2k-1)(2k)!} x^{p+2k-1} \quad [|x| < \pi/2; \rho > 1].$$

$$6. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2n} dx = \frac{1}{2n} \sum_{k=1}^{n-1} (-1)^k \frac{2n(2n-2) \dots (2n-2k+2)}{(2n-1)(2n-3) \dots (2n-2k+1)} \times \\ \times \left[x \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{2k-2n-1} x}{\operatorname{sh} x \operatorname{ch}^{2k-2n-1} x} \right\} + \frac{1}{2n-2k} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-2n} \right] + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx.$$

$$7. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2n-1} dx = \frac{1}{2n+1} \sum_{k=1}^n (-1)^k \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2n(2n-2) \dots (2n-2k+2)} \times \\ \times \left[x \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{2k-2n-2} x}{\operatorname{sh} x \operatorname{ch}^{2k-2n-2} x} \right\} + \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{2k-2n-1} \right] + \\ + (-1)^n \frac{(2n-1)!!}{(2n)!!} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx.$$

$$8. \int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q} dx = \pm \frac{p}{(q-1)(q-2)x^{p+1}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} \mp \\ \mp \frac{1}{(q-1)x^p} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-q+1} x}{\operatorname{sh} x \operatorname{ch}^{-q+1} x} \right\} - \frac{q-2}{q-1} \int \frac{1}{x^p} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx \pm \\ \pm \frac{p(p+1)}{(q-1)(q-2)} \int \frac{1}{x^{p+2}} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-q+2} dx.$$

$$9. \int \frac{dx}{x^n \operatorname{sh} x} = -[1 + (-1)^n] \frac{2^{n-2}-1}{n!} B_n \ln |x| + \sum_{\substack{k=0 \\ k \neq n/2}}^{\infty} \frac{2-2^{2k}}{(2k)!(2k-n)} B_{2k} x^{2k-n} \\ [|x| < \pi; n \geq 1].$$

$$10. \int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0 \\ k \neq (n-1)/2}}^{\infty} \frac{E_{2k}}{(2k)!(2k-n+1)} x^{2k-n+1} + \\ + \frac{1}{2} [1 - (-1)^n] \frac{E_{n-1}}{(n-1)!} \ln |x| \quad [|x| < \pi/2].$$

$$11. \int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1 - (-1)^n] \frac{2^n n}{(n+1)!} B_{n+1} \ln |x| - \\ - \frac{n}{x^{n+2}} \sum_{\substack{k=0 \\ k \neq (n+1)/2}}^{\infty} \frac{B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \quad [|x| < \pi].$$

$$12. \int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + [1 - (-1)^n] \frac{2^n (2^{n+1}-1)n}{(n+1)!} B_{n+1} \ln |x| + \\ + \frac{n}{x^{n+2}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(2^{2k}-1) B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \quad [|x| < \pi/2].$$

$$13. \int \frac{x dx}{\operatorname{sh} x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k)!(2k+1)} B_{2k} x^{2k+3} \quad [|x| < \pi].$$

$$14. \int \frac{x dx}{\operatorname{ch} x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k)! (2k+2)}$$

$$[|x| < \pi/2].$$

$$15. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx = \mp x \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} \pm \ln \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

$$16. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} dx = \mp \frac{x}{2} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-3} x}{\operatorname{sh} x \operatorname{ch}^{-3} x} \right\} \mp \frac{1}{2} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} \mp \frac{1}{2} \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-2} dx.$$

$$17. \int x \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-4} dx = \\ = \mp \frac{x}{3} \left\{ \frac{\operatorname{ch} x \operatorname{sh}^{-4} x}{\operatorname{sh} x \operatorname{ch}^{-4} x} \right\} \mp \frac{1}{6} \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}^{-3} + \frac{2}{3} x \left\{ \frac{\operatorname{cth} x}{\operatorname{th} x} \right\} - \frac{2}{3} \ln \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}.$$

1.4.25. Интегралы вида $\int R \left(x^p, \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\}, a + b \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} \right) dx$.

$$1. \int \frac{x^p \operatorname{ch} x dx}{(a + b \operatorname{sh} x)^q} = - \frac{x^p}{(q-1)b(a + b \operatorname{sh} x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \operatorname{sh} x)^{q-1}}.$$

$$2. \int \frac{x^p \operatorname{sh} x dx}{(a + b \operatorname{ch} x)^q} = - \frac{x^p}{(q-1)b(a + b \operatorname{ch} x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \operatorname{ch} x)^{q-1}}.$$

$$3. \int \frac{x dx}{1 \pm \operatorname{ch} x} = x \left\{ \frac{\operatorname{th}(x/2)}{\operatorname{cth}(x/2)} \right\} - 2 \ln \left\{ \frac{\operatorname{ch}(x/2)}{\operatorname{sh}(x/2)} \right\}.$$

$$4. \int \frac{x \operatorname{sh} x dx}{(1 \pm \operatorname{ch} x)^2} = - \frac{x}{\operatorname{ch} x \pm 1} \pm \left\{ \frac{\operatorname{th}(x/2)}{\operatorname{cth}(x/2)} \right\}.$$

1.4.26. Интегралы вида $\int (bx+c)^{\pm n} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx$.

$$1. \int (bx+c)^n \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{b^n}{a^{n+1}} \left(\operatorname{ch} \frac{ac}{b} \int t^n \left\{ \frac{\operatorname{sh} t}{\operatorname{ch} t} \right\} dt - \operatorname{sh} \frac{ac}{b} \int t^n \left\{ \frac{\operatorname{ch} t}{\operatorname{sh} t} \right\} dt \right) \quad \left[\frac{a}{b} t = bx+c \right].$$

$$2. \int (bx+c) \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{a} (bx+c) \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{b}{a^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$3. \int (bx+c)^2 \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{a} \left[(bx+c)^2 + \frac{2b^2}{a^2} \right] \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{2b(bx+c)}{a^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$4. \int (bx+c)^3 \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{bx+c}{a} \left[(bx+c)^2 + \frac{6b^2}{a^2} \right] \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} - \frac{3b}{a^2} \left[(bx+c)^2 + \frac{2b^2}{a^2} \right] \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}.$$

$$5. \int \frac{1}{(bx+c)^n} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \\ = \frac{a^{n-1}}{b^n} \left(\operatorname{ch} \frac{ac}{b} \int \frac{1}{t^n} \left\{ \frac{\operatorname{sh} t}{\operatorname{ch} t} \right\} dt - \operatorname{sh} \frac{ac}{b} \int \frac{1}{t^n} \left\{ \frac{\operatorname{ch} t}{\operatorname{sh} t} \right\} dt \right) \quad \left[\frac{a}{b} t = bx+c \right].$$

$$6. \int \frac{1}{bx+c} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{b} \operatorname{ch} \frac{ac}{b} \left\{ \frac{\operatorname{shi}(ax+ac/b)}{\operatorname{chi}(ax+ac/b)} \right\} - \frac{1}{b} \operatorname{sh} \frac{ac}{b} \left\{ \frac{\operatorname{chi}(ax+ac/b)}{\operatorname{shi}(ax+ac/b)} \right\} = \\ = \frac{1}{2b} e^{-ac/b} \operatorname{Ei}(ax+ac/b) \mp \frac{1}{2b} e^{ac/b} \operatorname{Ei}(-ax - \frac{ac}{b}).$$

$$7. \int \frac{1}{(bx+c)^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{b(bx+c)} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} + \frac{a}{b} \int \frac{1}{bx+c} \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} dx.$$

$$8. \int \frac{1}{(bx+c)^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{2b(bx+c)^2} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} - \\ - \frac{a}{2b^2(bx+c)} \left\{ \frac{\operatorname{ch} ax}{\operatorname{sh} ax} \right\} + \frac{a^2}{2b^2} \int \frac{1}{bx+c} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx.$$

1.4.27. Интегралы вида $\int R(x, e^{ax}, \operatorname{sh} bx, \operatorname{ch} bx) dx$.

$$1. \int e^{ax} \left\{ \frac{\operatorname{sh}(bx+c)}{\operatorname{ch}(bx+c)} \right\} dx = \frac{e^{ax}}{a^2-b^2} \left[a \left\{ \frac{\operatorname{sh}(bx+c)}{\operatorname{ch}(bx+c)} \right\} - b \left\{ \frac{\operatorname{ch}(bx+c)}{\operatorname{sh}(bx+c)} \right\} \right].$$

$$2. \int e^{ax} \left\{ \frac{\operatorname{sh}(ax+c)}{\operatorname{ch}(ax+c)} \right\} dx = \mp \frac{1}{2} x e^{\mp c} + \frac{1}{4a} e^{2ax+c}.$$

$$3. \int e^{-ax} \left\{ \frac{\operatorname{sh}(ax+c)}{\operatorname{ch}(ax+c)} \right\} dx = \frac{1}{2} x e^c \pm \frac{1}{4a} e^{-1(2ax+c)}.$$

$$4. \int x^p e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} \int x^p e^{(a+b)x} dx \mp \frac{1}{2} \int x^p e^{(a-b)x} dx \quad [\text{см. 1.3.2}].$$

$$5. \int_0^x x^{\alpha-1} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [(a-b)^{-\alpha} \gamma(\alpha, ax-bx) \mp (a+b)^{-\alpha} \gamma(\alpha, ax+bx)] \\ \left[\operatorname{Re} \alpha > -\delta, \ell = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$6. \int_x^\infty x^{\alpha-1} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [(a-b)^{-\alpha} \Gamma(\alpha, ax-bx) \mp (a+b)^{-\alpha} \Gamma(\alpha, ax+bx)] \\ [a > b > 0].$$

$$7. \int x e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{ax}}{a^2-b^2} \left[\left(ax - \frac{a^2+b^2}{a^2-b^2} \right) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} - \right. \\ \left. - \left(bx - \frac{2ab}{a^2-b^2} \right) \left\{ \frac{\operatorname{ch} bx}{\operatorname{sh} bx} \right\} \right].$$

$$8. \int x^2 e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{ax}}{a^3-b^3} \left[ax^3 - \frac{2(a^3+b^3)}{a^2-b^2} x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2} \right] \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} - \\ - \frac{e^{ax}}{a^3-b^3} \left[bx^3 - \frac{4ab}{a^2-b^2} x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2} \right] \left\{ \frac{\operatorname{ch} bx}{\operatorname{sh} bx} \right\}.$$

$$9. \int x^2 e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) \mp \frac{x^3}{4}.$$

$$10. \int x^2 e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{e^{2ax}}{4a} \left(x^3 + \frac{x}{a} + \frac{1}{2a^2} \right) \mp \frac{x^3}{6}.$$

$$11. \int \frac{1}{x} e^{ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} [\operatorname{Ei}(ax+bx) \mp \operatorname{Ei}(ax-bx)] \quad [a^2 \neq b^2].$$

$$12. \int_x^\infty \frac{1}{x} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = -\frac{1}{2} [\operatorname{Ei}(bx-ax) \mp \operatorname{Ei}(-ax-bx)] \quad [a > |b|].$$

$$13. \int_0^x \frac{1}{x} e^{-ax} \operatorname{sh} bx \, dx = \frac{1}{2} \ln \frac{a+b}{a-b} + \frac{1}{2} [\operatorname{Ei}(bx-ax) - \operatorname{Ei}(-ax-bx)] \quad [a > |b|].$$

$$14. \int \frac{1}{x} e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{2} [\operatorname{Ei}(2ax) \mp \ln x].$$

$$15. \int_0^x \frac{1}{x} e^{\pm x} \operatorname{sh} x \, dx = \pm \frac{1}{2} [\operatorname{Ei}(\pm 2x) - (C + \ln 2x)].$$

$$16. \int \frac{1}{x} e^{-ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{2} [\ln x \mp \operatorname{Ei}(-2ax)].$$

$$17. \int \frac{1}{x^2} e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = a \operatorname{Ei}(2ax) - \frac{1}{2x} (e^{2ax} \mp 1).$$

$$18. \int \frac{1}{x^3} e^{-ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{2x} \pm \left[\frac{1}{2x} e^{-2ax} + a \operatorname{Ei}(-2ax) \right].$$

$$19. \int \frac{x \, dx}{\operatorname{ch}^2 ax} = \frac{x}{a} \operatorname{th} ax - \frac{1}{a^2} \ln |\operatorname{ch} ax|.$$

$$20. \int_x^\infty \frac{1}{x} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}^{-1} dx = -2 \sum_{k=0}^{\infty} (\pm 1)^k \operatorname{Ei}[-(2k+1)ax].$$

$$21. \int_0^x \frac{x \, dx}{\cos t - \operatorname{ch} ax} = \frac{2}{a^2 \sin t} \left[L\left(\theta - \frac{t}{2}\right) - L\left(\theta + \frac{t}{2}\right) + 2L\left(\frac{t}{2}\right) \right] \\ \left[\theta = \operatorname{arctg}\left(\operatorname{th} \frac{ax}{2} \operatorname{ctg} \frac{t}{2}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

$$22. \int_0^x \frac{x \operatorname{ch}(ax/2)}{\cos t - \operatorname{ch} ax} \, dx = \frac{2}{a^2 \sin(t/2)} \left[L\left(\frac{\theta-t}{4}\right) - L\left(\frac{\theta+t}{4}\right) - \right. \\ \left. - L\left(\pi - \frac{\psi+t}{4}\right) - L\left(\frac{\psi-t}{4}\right) + 2L\left(\frac{t}{4}\right) + 2L\left(\frac{2\pi-t}{4}\right) \right] \\ \left[\theta = 4 \operatorname{arctg}\left(\operatorname{th} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right); \quad \psi = 4 \operatorname{arctg}\left(\operatorname{cth} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

1.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.5.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) \, dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

$$1. \int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) \, dx = \\ = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}, \frac{1-t^2}{2t}\right) \frac{2 \, dt}{1+t^2} \quad \left[t = \operatorname{tg} \frac{x}{2} \right].$$

Если

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

ТО

$$2. \int R(\sin x, \cos x) dx = - \int R(\sqrt{1-t^2}, t) \frac{dt}{\sqrt{1-t^2}} \quad \{t = \cos x\}.$$

Если

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

ТО

$$3. \int R(\sin x, \cos x) dx = \int R(t, \sqrt{1-t^2}) \frac{dt}{\sqrt{1-t^2}} \quad \{t = \sin x\}.$$

Если

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

ТО

$$4. \int R(\sin x, \cos x) dx = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right) \frac{dt}{1+t^2} \quad \{t = \tan x\}.$$

1.5.2. Интегралы вида $\int \sin^p x dx$.

$$1. \int \sin^p x dx = -\frac{1}{p} \sin^{p-1} x \cos x + \frac{p-1}{p} \int \sin^{p-2} x dx.$$

$$2. \int \sin^{2n} x dx =$$

$$= -\frac{\cos x}{2n} \left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sin^{2n-2k-1} x \right] + \frac{(2n-1)!!}{2^n n!} x.$$

$$3. = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}.$$

$$4. \int \sin^{2n+1} x dx =$$

$$= -\frac{\cos x}{2n+1} \left[\sin^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n (n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \sin^{2n-2k-2} x \right].$$

$$5. = (-1)^{n+1} \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n-2k+1)x}{2n-2k+1}.$$

$$6. = \sum_{k=0}^n \frac{(-1)^{k+1}}{2k+1} \binom{n}{k} \cos^{2k+1} x.$$

$$7. \int \sin x dx = -\cos x.$$

$$8. \int \sin^3 x dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$13. \int_0^x \frac{1}{x} e^{-ax} \operatorname{sh} bx \, dx = \frac{1}{2} \ln \frac{a+b}{a-b} + \frac{1}{2} [\operatorname{Ei}(bx-ax) - \operatorname{Ei}(-ax-bx)] \quad [a > |b|].$$

$$14. \int \frac{1}{x} e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{2} [\operatorname{Ei}(2ax) + \ln x].$$

$$15. \int_0^x \frac{1}{x} e^{\pm x} \operatorname{sh} x \, dx = \pm \frac{1}{2} [\operatorname{Ei}(\pm 2x) - (C + \ln 2x)].$$

$$16. \int \frac{1}{x} e^{-ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = \frac{1}{2} [\ln x + \operatorname{Ei}(-2ax)].$$

$$17. \int \frac{1}{x^2} e^{ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = a \operatorname{Ei}(2ax) - \frac{1}{2x} (e^{2ax} + 1).$$

$$18. \int \frac{1}{x^2} e^{-ax} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\} dx = -\frac{1}{2x} \pm \left[\frac{1}{2x} e^{-2ax} + a \operatorname{Ei}(-2ax) \right].$$

$$19. \int \frac{x \, dx}{\operatorname{ch}^2 ax} = \frac{x}{a} \operatorname{th} ax - \frac{1}{a^2} \ln |\operatorname{ch} ax|.$$

$$20. \int_x^\infty \frac{1}{x} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}^{-1} dx = -2 \sum_{k=0}^{\infty} (\pm 1)^k \operatorname{Ei}[-(2k+1)ax].$$

$$21. \int_0^x \frac{x \, dx}{\cos t - \operatorname{ch} ax} = \frac{2}{a^2 \sin t} \left[L\left(\theta - \frac{t}{2}\right) - L\left(\theta + \frac{t}{2}\right) + 2L\left(\frac{t}{2}\right) \right] \\ \left[\theta = \operatorname{arctg}\left(\operatorname{th} \frac{ax}{2} \operatorname{ctg} \frac{t}{2}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

$$22. \int_0^x \frac{x \operatorname{ch}(ax/2)}{\cos t - \operatorname{ch} ax} \, dx = \frac{2}{a^2 \sin(t/2)} \left[L\left(\frac{\theta-t}{4}\right) - L\left(\frac{\theta+t}{4}\right) - \right. \\ \left. - L\left(\pi - \frac{\psi+t}{4}\right) - L\left(\frac{\psi-t}{4}\right) + 2L\left(\frac{t}{4}\right) + 2L\left(\frac{2\pi-t}{4}\right) \right] \\ \left[\theta = 4 \operatorname{arctg}\left(\operatorname{th} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right); \quad \psi = 4 \operatorname{arctg}\left(\operatorname{cth} \frac{ax}{4} \operatorname{ctg} \frac{t}{4}\right); \quad t \neq 2\pi k; \quad k=0, \pm 1, \pm 2, \dots \right].$$

1.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.5.1. Введение.

Если R — рациональная функция своих аргументов, то интегралы вида $\int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) \, dx$ приводятся к интегралам от рациональных функций при помощи следующих формул:

$$1. \int R(\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x) \, dx = \\ = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}, \frac{1-t^2}{2t}\right) \frac{2 \, dt}{1+t^2} \quad \left[t = \operatorname{tg} \frac{x}{2}\right].$$

Если

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

то

$$2. \int R(\sin x, \cos x) dx = - \int R(\sqrt{1-t^2}, t) \frac{dt}{\sqrt{1-t^2}} \quad [t = \cos x].$$

Если

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

то

$$3. \int R(\sin x, \cos x) dx = \int R(t, \sqrt{1-t^2}) \frac{dt}{\sqrt{1-t^2}} \quad [t = \sin x].$$

Если

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

то

$$4. \int R(\sin x, \cos x) = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right) \frac{dt}{1+t^2} \quad [t = \operatorname{tg} x].$$

1.5.2. Интегралы вида $\int \sin^p x dx$.

$$1. \int \sin^p x dx = -\frac{1}{p} \sin^{p-1} x \cos x + \frac{p-1}{p} \int \sin^{p-2} x dx.$$

$$2. \int \sin^{2n} x dx =$$

$$= -\frac{\cos x}{2n} \left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sin^{2n-2k-1} x \right] + \frac{(2n-1)!!}{2^n n!} x.$$

$$3. = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}.$$

$$4. \int \sin^{2n+1} x dx =$$

$$= -\frac{\cos x}{2n+1} \left[\sin^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n (n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \sin^{2n-2k-2} x \right].$$

$$5. = (-1)^{n+1} \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n-2k+1)x}{2n-2k+1}.$$

$$6. = \sum_{k=0}^n \frac{(-1)^{k+1}}{2k+1} \binom{n}{k} \cos^{2k+1} x.$$

$$7. \int \sin x dx = -\cos x.$$

$$8. \int \sin^2 x dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$9. \int \sin^3 x \, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x.$$

$$10. \int \sin^4 x \, dx = \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \\ = -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3}{8} x.$$

$$11. \int \sin^5 x \, dx = -\frac{5}{8} \cos x + \frac{5}{48} \cos 3x - \frac{1}{80} \cos 5x = \\ = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{15} \cos^3 x - \frac{4}{5} \cos x.$$

$$12. \int \sin^6 x \, dx = \frac{5}{16} x - \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{192} \sin 6x = \\ = -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x.$$

$$13. \int \frac{dx}{\sin^p x} = -\frac{\cos x}{(p-1) \sin^{p-1} x} + \frac{p-2}{p-1} \int \frac{dx}{\sin^{p-2} x}.$$

$$14. \int \frac{dx}{\sin^{2n} x} = \\ = -\frac{\cos x}{2n-1} \left[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right].$$

$$15. \int \frac{dx}{\sin^{2n+1} x} = \\ = -\frac{\cos x}{2n} \left[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x \right] + \\ + \frac{(2n-1)!!}{2^n n!} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$16. \int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| = \ln \left| \frac{1 - \cos x}{\sin x} \right| = \ln \left| \frac{\sin x}{1 + \cos x} \right| = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right|.$$

$$17. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$18. \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$19. \int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3 \sin^3 x} - \frac{2}{3} \operatorname{ctg} x = -\frac{1}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

$$20. \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4 \sin^4 x} - \frac{3 \cos x}{8 \sin^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$21. \int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \operatorname{ctg}^3 x - \frac{4}{5} \operatorname{ctg} x = -\frac{1}{5} \operatorname{ctg}^5 x - \frac{2}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

1.5.3. Интегралы вида $\int \cos^p x \, dx$.

$$1. \int \cos^p x \, dx = \frac{1}{p} \sin x \cos^{p-1} x + \frac{p-1}{p} \int \cos^{p-2} x \, dx.$$

$$2. \int \cos^{2n} x \, dx =$$

$$= \frac{\sin x}{2n} \left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \cos^{2n-2k-1} x \right] + \frac{(2n-1)!!}{2^n n!} x$$

$$3. = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}.$$

$$4. \int \cos^{2n+1} x \, dx =$$

$$= \frac{\sin x}{2n+1} \left[\cos^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \cos^{2n-2k-2} x \right].$$

$$5. = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}.$$

$$6. = \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} \sin^{2k+1} x.$$

$$7. \int \cos x \, dx = \sin x.$$

$$8. \int \cos^3 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$9. \int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x.$$

$$10. \int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \frac{3}{8} x + \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x.$$

$$11. \int \cos^5 x \, dx = \frac{5}{8} \sin x + \frac{5}{48} \sin 3x + \frac{1}{80} \sin 5x = \frac{4}{5} \sin x - \frac{4}{15} \sin^3 x + \frac{1}{5} \cos^4 x \sin x.$$

$$12. \int \cos^6 x \, dx = \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x = \frac{5}{16} x + \frac{5}{16} \sin x \cos x + \frac{5}{24} \sin x \cos^3 x + \frac{1}{6} \sin x \cos^5 x.$$

$$13. \int \frac{dx}{\cos^p x} = \frac{\sin x}{(p-1) \cos^{p-1} x} + \frac{p-2}{p-1} \int \frac{dx}{\cos^{p-2} x}.$$

$$14. \int \frac{dx}{\cos^{2n} x} = \frac{\sin x}{2n-1} \left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2) \dots (n-k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right].$$

$$15. \int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n} \left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right] + \frac{(2n-1)!!}{2^n n!} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| = \ln \left| \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|.$$

$$17. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$18. \int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$19. \int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \operatorname{tg} x = \frac{1}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

$$20. \int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$21. \int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \operatorname{tg}^3 x + \frac{4}{5} \operatorname{tg} x = \frac{1}{5} \operatorname{tg}^5 x + \frac{2}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

1.5.4. Интегралы вида $\int \sin^p x \cos^q x dx$.

$$1. \int \sin^p x \cos^q x dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x dx.$$

$$2. = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx.$$

$$3. = \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x dx.$$

$$4. = \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x dx.$$

$$5. = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x dx.$$

$$6. = -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x dx.$$

$$7. = \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left(\sin^2 x - \frac{q-1}{p+q-2} \right) + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x dx.$$

$$\begin{aligned}
 8. \int \sin^p x \cos^{2n} x dx &= \\
 &= \frac{\sin^{p+1} x}{2n+p} \left[\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \right] + \\
 &+ \frac{(2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \sin^p x dx \quad [p \neq -2, -4, \dots, -2n].
 \end{aligned}$$

$$\begin{aligned}
 9. \int \sin^p x \cos^{2n+1} x dx &= \\
 &= \frac{\sin^{p+1} x}{2n+p+1} \left[\cos^{2n} x + \sum_{k=1}^n \frac{2^k n (n-1) \dots (n-k+1) \cos^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right] \\
 &\quad [p \neq -1, -3, \dots, -(2n+1)].
 \end{aligned}$$

$$\begin{aligned}
 10. \int \cos^p x \sin^{2n} x dx &= \\
 &= -\frac{\cos^{p+1} x}{2n+p} \left[\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3) \dots (2n-2k+1) \sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \right] + \\
 &+ \frac{(2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \cos^p x dx \quad [p \neq -2, -4, \dots, -2n].
 \end{aligned}$$

$$\begin{aligned}
 11. \int \cos^p x \sin^{2n+1} x dx &= \\
 &= -\frac{\cos^{p+1} x}{2n+p+1} \left[\sin^{2n} x + \sum_{k=1}^n \frac{2^k n (n-1) \dots (n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right] \\
 &\quad [p \neq -1, -3, \dots, -(2n+1)].
 \end{aligned}$$

$$12. \int \sin x \cos^p x dx = -\frac{1}{p+1} \cos^{p+1} x \quad [p \neq -1].$$

$$13. \int \sin^p x \cos x dx = \frac{1}{p+1} \sin^{p+1} x \quad [p \neq -1].$$

$$14. \int \sin^2 x \cos^2 x dx = -\frac{1}{8} \left(\frac{1}{4} \sin 4x - x \right).$$

$$\begin{aligned}
 15. \int \sin^2 x \cos^3 x dx &= -\frac{1}{16} \left(\frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right) = \\
 &= \frac{\sin^2 x}{5} \left(\cos^2 x + \frac{2}{3} \right) = \frac{\sin^2 x}{5} \left(\frac{5}{3} - \sin^2 x \right).
 \end{aligned}$$

$$16. \int \sin^2 x \cos^4 x dx = \frac{x}{16} + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x.$$

$$17. \int \sin^3 x \cos^2 x dx = \frac{1}{16} \left(\frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right) = \frac{1}{5} \cos^3 x - \frac{1}{3} \cos^5 x.$$

$$18. \int \sin^3 x \cos^3 x dx = \frac{1}{32} \left(\frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right).$$

$$19. \int \sin^3 x \cos^4 x dx = \frac{1}{7} \cos^3 x \left(-\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right).$$

$$20. \int \sin^4 x \cos^2 x dx = \frac{1}{16} x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x.$$

$$21. \int \sin^4 x \cos^3 x dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right).$$

$$22. \int \sin^4 x \cos^4 x dx = \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x.$$

$$23. \int \sin ax \sin bx \sin cx dx =$$

$$= -\frac{1}{4} \left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right].$$

$$24. \int \sin ax \cos bx \cos cx dx =$$

$$= -\frac{1}{4} \left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right].$$

$$25. \int \cos ax \sin bx \sin cx dx =$$

$$= \frac{1}{4} \left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right].$$

$$26. \int \cos ax \cos bx \cos cx dx =$$

$$= \frac{1}{4} \left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right].$$

1.5.5. Интегралы вида $\int \frac{\sin^p x}{\cos^q x} dx$.

$$1. \int \frac{\sin^p x}{\cos^q x} dx = -\frac{\sin^{p-1} x}{(p-q) \cos^{q-1} x} + \frac{p-1}{p-q} \int \frac{\sin^{p-2} x}{\cos^q x} dx,$$

$$2. = \frac{\sin^{p+1} x}{(q-1) \cos^{q-1} x} - \frac{p-q+2}{q-1} \int \frac{\sin^p x}{\cos^{q-2} x} dx,$$

$$3. = \frac{\sin^{p-1} x}{(q-1) \cos^{q-1} x} - \frac{p-1}{q-1} \int \frac{\sin^{p-2} x}{\cos^{q-2} x} dx,$$

$$4. \int \frac{\sin^p x}{\cos^{2n+1} x} dx = \frac{\sin^{p+1} x}{2n} \times$$

$$\times \left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \sec^{2n-2k} x \right] +$$

$$+ \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\sin^p x}{\cos x} dx.$$

$$5. \int \frac{\sin^p x}{\cos^{2n} x} dx = \frac{\sin^{p+1} x}{2n-1} \times$$

$$\times \left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \sec^{2n-2k-1} x \right] +$$

$$+ \frac{(2n-p-2)(2n-p-4) \dots (-p+2)(-p)}{(2n-1)!!} \int \sin^p x dx.$$

- $$6. \int \frac{\sin^{2m+1} x}{\cos^{2n+1} x} dx = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq n}}^m (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k} + (-1)^{m+1} \binom{m}{n} \ln |\cos x|$$
- [$n \leq m$].
- $$7. = \frac{1}{2} \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\cos^{2k-2n} x}{n-k}$$
- [$n > m$].
- $$8. \int \frac{\sin^{2m+1} x}{\cos^{2n} x} dx = \sum_{k=0}^m (-1)^{k+1} \binom{m}{k} \frac{\cos^{2k-2n+1} x}{2k-2n+1}$$
- $$9. \int \frac{\sin x}{\cos^q x} dx = \frac{1}{(q-1) \cos^{q-1} x}.$$
- $$10. \int \frac{\sin^p x}{\cos^{p+1} x} dx = \frac{1}{p+1} \operatorname{tg}^{p+1} x.$$
- $$11. \int \frac{\sin^{2n+1} x}{\cos x} dx = - \sum_{k=1}^n \frac{\sin^{2k} x}{2k} - \ln |\cos x|.$$
- $$12. = \sum_{k=1}^n \frac{(-1)^{k+1}}{2k} \binom{n}{k} \cos^{2k} x - \ln |\cos x|.$$
- $$13. \int \frac{\sin^{2n} x}{\cos x} dx = - \sum_{k=1}^n \frac{\sin^{2k-1} x}{2k-1} + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$
- $$14. \int \frac{\sin x}{\cos x} dx = - \ln |\cos x|.$$
- $$15. \int \frac{\sin^2 x}{\cos x} dx = - \sin x + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$
- $$16. \int \frac{\sin^3 x}{\cos x} dx = - \frac{\sin^2 x}{2} - \ln |\cos x| = \frac{1}{2} \cos^2 x - \ln |\cos x|.$$
- $$17. \int \frac{\sin^4 x}{\cos x} dx = - \frac{1}{3} \sin^3 x - \sin x + \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$
- $$18. \int \frac{\sin^p x}{\cos^2 x} dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-3} x dx.$$
- $$19. \int \frac{\sin x}{\cos^3 x} dx = \frac{1}{\cos x}.$$
- $$20. \int \frac{\sin^2 x}{\cos^2 x} dx = \operatorname{tg} x - x.$$
- $$21. \int \frac{\sin^3 x}{\cos^2 x} dx = \cos x + \frac{1}{\cos x}.$$
- $$22. \int \frac{\sin^4 x}{\cos^2 x} dx = \operatorname{tg} x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$23. \int \frac{\sin x}{\cos^3 x} dx = \frac{1}{2 \cos^2 x} = \frac{1}{2} \operatorname{tg}^2 x.$$

$$24. \int \frac{\sin^2 x}{\cos^3 x} dx = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$25. \int \frac{\sin^3 x}{\cos^3 x} dx = \frac{1}{2 \cos^2 x} + \ln |\cos x|.$$

$$26. \int \frac{\sin^4 x}{\cos^3 x} dx = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$27. \int \frac{\sin x}{\cos^4 x} dx = \frac{1}{3 \cos^3 x}. \quad 28. \int \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3} \operatorname{tg}^3 x.$$

$$29. \int \frac{\sin^3 x}{\cos^4 x} dx = -\frac{1}{\cos x} + \frac{1}{3 \cos^3 x}.$$

$$30. \int \frac{\sin^4 x}{\cos^4 x} dx = \frac{1}{3} \operatorname{tg}^3 x - \operatorname{tg} x + x.$$

1.5.6. Интегралы вида $\int \frac{\cos^q x}{\sin^p x} dx$.

$$1. \int \frac{\cos^q x}{\sin^p x} dx = \frac{\cos^{q-1} x}{(q-p) \sin^{p-1} x} + \frac{q-1}{q-p} \int \frac{\cos^{q-2} x}{\sin^p x} dx.$$

$$2. = -\frac{\cos^{q+1} x}{(p-1) \sin^{p-1} x} - \frac{q-p+2}{p-1} \int \frac{\cos^q x}{\sin^{p-2} x} dx.$$

$$3. = -\frac{\cos^{q-1} x}{(p-1) \sin^{p-1} x} - \frac{q-1}{p-1} \int \frac{\cos^{q-2} x}{\sin^{p-2} x} dx.$$

$$4. \int \frac{\cos^n x}{\sin^{2n+1} x} dx = -\frac{\cos^{n+1} x}{2n} [\operatorname{cosec}^{2n} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \operatorname{cosec}^{2n-2k} x] + \\ + \frac{(2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\cos^p x}{\sin x} dx.$$

$$5. \int \frac{\cos^p x}{\sin^{2n} x} dx = -\frac{\cos^{p+1} x}{2n-1} [\operatorname{cosec}^{2n-1} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x] + \\ + \frac{(2n-p-2)(2n-p-4) \dots (2-p)(-p)}{(2n-1)!} \int \cos^p x dx.$$

$$6. \int \frac{\cos^{2n+2} x}{\sin^{2m+1} x} dx = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq m}}^n (-1)^{k+1} \binom{n}{k} \frac{\sin^{2k-2m} x}{m-k} + (-1)^n \binom{n}{m} \ln |\sin x|$$

$[m \leq n]$

$$7. \quad = \frac{1}{2} \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \frac{\sin^{2k-2m} x}{m-k} \quad [m > n].$$

$$8. \quad \int \frac{\cos^{2n+1} x}{\sin^{2m} x} dx = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\sin^{2k-2m+1} x}{2k-2m+1}.$$

$$9. \quad \int \frac{\cos x}{\sin^p x} dx = -\frac{1}{(p-1) \sin^{p-1} x}.$$

$$10. \quad \int \frac{\cos^q x}{\sin^{q+2} x} dx = -\frac{1}{q+1} \operatorname{ctg}^{q+1} x.$$

$$11. \quad \int \frac{\cos^{2n+1} x}{\sin x} dx = \sum_{k=1}^n \frac{\cos^{2k} x}{2k} + \ln |\sin x|.$$

$$12. \quad = \frac{1}{2} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \sin^{2k} x + \ln |\sin x|.$$

$$13. \quad \int \frac{\cos^{2n} x}{\sin x} dx = \sum_{k=1}^n \frac{\cos^{2k-1} x}{2k-1} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$14. \quad \int \frac{\cos x}{\sin x} dx = \ln |\sin x|.$$

$$15. \quad \int \frac{\cos^2 x}{\sin x} dx = \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$16. \quad \int \frac{\cos^3 x}{\sin x} dx = \frac{\cos^2 x}{2} + \ln |\sin x|.$$

$$17. \quad \int \frac{\cos^4 x}{\sin x} dx = \frac{1}{3} \cos^3 x + \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$18. \quad \int \frac{\cos^p x}{\sin^2 x} dx = -\frac{\cos^{p-1} x}{\sin x} - (p-1) \int \cos^{p-2} x dx.$$

$$19. \quad \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}, \quad 20. \quad \int \frac{\cos^3 x}{\sin^2 x} dx = -\operatorname{ctg} x - x.$$

$$21. \quad \int \frac{\cos^3 x}{\sin^2 x} dx = -\sin x - \frac{1}{\sin x}.$$

$$22. \quad \int \frac{\cos^4 x}{\sin^2 x} dx = -\operatorname{ctg} x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$23. \quad \int \frac{\cos x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x}.$$

$$24. \quad \int \frac{\cos^2 x}{\sin^3 x} dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$25. \quad \int \frac{\cos^3 x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - \ln |\sin x|.$$

$$26. \int \frac{\cos^4 x}{\sin^3 x} dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$27. \int \frac{\cos x}{\sin^4 x} dx = -\frac{1}{3 \sin^3 x}. \quad 28. \int \frac{\cos^3 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^3 x.$$

$$29. \int \frac{\cos^3 x}{\sin^4 x} dx = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}.$$

$$30. \int \frac{\cos^4 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^3 x + \operatorname{ctg} x + x.$$

1.5.7. Интегралы вида $\int \frac{dx}{\sin^p x \cos^q x}$.

$$1. \int \frac{dx}{\sin^p x \cos^q x} = \frac{-1}{(p-1) \sin^{p-1} x \cos^{q-1} x} + \frac{p+q-2}{p-1} \int \frac{dx}{\sin^{p-2} x \cos^{q-1} x}.$$

$$2. = \frac{1}{(q-1) \sin^{p-1} x \cos^{q-2} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\sin^{p-1} x \cos^{q-2} x}$$

[см. также 1.5.4].

$$3. \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\operatorname{tg}^{2k-2m+1} x}{2k-2m+1}.$$

$$4. \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{\substack{h=0 \\ h \neq m}}^{m+n} \binom{m+n}{h} \frac{\operatorname{tg}^{2h-2m} x}{2h-2m} + \binom{m+n}{m} \ln |\operatorname{tg} x|.$$

$$5. \int \frac{dx}{\sin^{2m+1} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$6. \int \frac{dx}{\sin^{2m} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right|.$$

$$7. \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$8. \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$9. \int \frac{dx}{\sin x \cos x} = \ln |\operatorname{tg} x|. \quad 10. \int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$11. \int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2 \cos^2 x} + \ln |\operatorname{tg} x|.$$

$$12. \int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$13. \int \frac{dx}{\sin^2 x \cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| - \operatorname{cosec} x.$$

$$14. \int \frac{dx}{\sin^3 x \cos^3 x} = -2 \operatorname{ctg} 2x.$$

$$15. \int \frac{dx}{\sin^3 x \cos^3 x} = \left(\frac{1}{2 \cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{dx}{\sin^3 x \cos^3 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$17. \int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln |\operatorname{tg} x|.$$

$$18. \int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{1}{\cos x} \left(\frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$19. \int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2 \cos 2x}{\sin^2 2x} + 2 \ln |\operatorname{tg} x|.$$

$$20. \int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3 \cos^3 x} - \frac{\cos x}{2 \sin^2 x} + \frac{5}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$21. \int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3 \sin^3 x} + \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$22. \int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$23. \int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^2 x} + \frac{5}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$24. \int \frac{dx}{\sin^4 x \cos^4 x} = -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x.$$

1.5.8. Интегралы вида $\int \operatorname{tg}^p x dx$, $\int \operatorname{ctg}^p x dx$.

$$1. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^p dx = \pm \frac{1}{p-1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-1} - \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-2} dx.$$

$$2. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+1} + (-1)^n x.$$

$$3. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n+1} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+2} \mp (-1)^n \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$4. = \pm \sum_{k=1}^n \frac{(-1)^{n+k}}{2k} \binom{n}{k} \left\{ \frac{\cos x}{\sin x} \right\}^{-2k} \mp (-1)^n \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$5. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx = \mp \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}. \quad 6. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = \pm \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} - x.$$

$$7. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^3 dx = \pm \frac{1}{2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 \pm \ln \left\{ \left| \frac{\cos x}{\sin x} \right| \right\}.$$

$$26. \int \frac{\cos^4 x}{\sin^3 x} dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$27. \int \frac{\cos x}{\sin^4 x} dx = -\frac{1}{3 \sin^3 x}. \quad 28. \int \frac{\cos^3 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^2 x.$$

$$29. \int \frac{\cos^3 x}{\sin^4 x} dx = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}.$$

$$30. \int \frac{\cos^4 x}{\sin^4 x} dx = -\frac{1}{3} \operatorname{ctg}^3 x + \operatorname{ctg} x + x.$$

1.5.7. Интегралы вида $\int \frac{dx}{\sin^p x \cos^q x}$.

$$1. \int \frac{dx}{\sin^p x \cos^q x} = \frac{-1}{(p-1) \sin^{p-1} x \cos^{q-1} x} + \frac{p+q-2}{p-1} \int \frac{dx}{\sin^{p-2} x \cos^q x}.$$

$$2. = \frac{1}{(q-1) \sin^{p-1} x \cos^{q-1} x} + \frac{p+q-2}{q-1} \int \frac{dx}{\sin^p x \cos^{q-2} x}$$

[см. также 1.5.4].

$$3. \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\operatorname{tg}^{2k-2m+1} x}{2k-2n+1}.$$

$$4. \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq m}}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln |\operatorname{tg} x|.$$

$$5. \int \frac{dx}{\sin^{2m+1} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$6. \int \frac{dx}{\sin^{2m} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right|.$$

$$7. \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln |\operatorname{tg} x|.$$

$$8. \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$9. \int \frac{dx}{\sin x \cos x} = \ln |\operatorname{tg} x|. \quad 10. \int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$11. \int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2 \cos^2 x} + \ln |\operatorname{tg} x|.$$

$$12. \int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$13. \int \frac{dx}{\sin^2 x \cos x} = \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| - \operatorname{cosec} x.$$

$$14. \int \frac{dx}{\sin^2 x \cos^3 x} = -2 \operatorname{ctg} 2x.$$

$$15. \int \frac{dx}{\sin^2 x \cos^3 x} = \left(\frac{1}{2 \cos^2 x} - \frac{3}{2} \right) \frac{1}{\sin x} + \frac{3}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$16. \int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$17. \int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln |\operatorname{tg} x|.$$

$$18. \int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{1}{\cos x} \left(\frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$19. \int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2 \cos 2x}{\sin^2 2x} + 2 \ln |\operatorname{tg} x|.$$

$$20. \int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3 \cos^3 x} - \frac{\cos x}{2 \sin^2 x} + \frac{5}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$21. \int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3 \sin^3 x} + \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$22. \int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$23. \int \frac{dx}{\sin^5 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^4 x} + \frac{5}{2} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

$$24. \int \frac{dx}{\sin^4 x \cos^4 x} = -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x.$$

1.5.8. Интегралы вида $\int \operatorname{tg}^p x dx$, $\int \operatorname{ctg}^p x dx$.

$$1. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^p dx = \pm \frac{1}{p-1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-1} - \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-2} dx.$$

$$2. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+1} + (-1)^n x.$$

$$3. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n+1} dx = \mp \sum_{k=1}^n (-1)^k \frac{1}{2n-2k+2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n-2k+2} \mp (-1)^n \ln \left| \frac{\cos x}{\sin x} \right|.$$

$$4. = \pm \sum_{k=1}^n \frac{(-1)^{n+k}}{2k} \binom{n}{k} \left\{ \frac{\cos x}{\sin x} \right\}^{-2k} \mp (-1)^n \ln \left| \frac{\cos x}{\sin x} \right|.$$

$$5. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx = \mp \ln \left| \frac{\cos x}{\sin x} \right|. \quad 6. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = \pm \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} - x.$$

$$7. \int \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^3 dx = \pm \frac{1}{2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 \pm \ln \left| \frac{\cos x}{\sin x} \right|.$$

$$8. \int \sqrt{\lg x} dx = \frac{1}{\sqrt{2}} [\ln(\sin x + \cos x - \sqrt{\sin 2x}) + \arcsin(\sin x - \cos x)].$$

$$9. = \frac{1}{\sqrt{2}} \left[\frac{1}{2} \ln \frac{1 - \sqrt{2} \lg x + \lg x}{1 + \sqrt{2} \lg x + \lg x} + \operatorname{arctg} \frac{\sqrt{2} \lg x}{1 - \lg x} \right].$$

1.5.9. Интегралы вида $\int R(\sin x, \cos x, \lg x, \operatorname{ctg} x) dx$.

$$1. \int \frac{A + B \cos x + C \sin x}{(a + b \cos x + c \sin x)^n} dx =$$

$$= \frac{(Bc - Cb) + (Ac - Ca) \cos x - (Ab - Ba) \sin x}{(n-1)(a^2 - b^2 - c^2)(a + b \cos x + c \sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba) \cos x - (Ac - Ca) \sin x]}{(a + b \cos x + c \sin x)^{n-1}} dx$$

[$n \geq 2, a^2 \neq b^2 + c^2$].

$$2. = \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2} \right) \times$$

$$\times (-c \cos x + b \sin x) \frac{(n-1)!}{(2n-1)!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!}{(n-k-1)! a^k} \frac{1}{(a + b \cos x + c \sin x)^{n-k}}$$

[$n \geq 2, a^2 = b^2 + c^2$].

$$3. \int \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x} dx = \frac{Bc - Cb}{b^2 + c^2} \ln(a + b \cos x + c \sin x) +$$

$$+ \frac{Bb + Cc}{b^2 + c^2} x + \left(A - \frac{Bb + Cc}{b^2 + c^2} a \right) \int \frac{dx}{a + b \cos x + c \sin x}.$$

$$4. \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n} \quad [b = r \cos \alpha, c = r \sin \alpha].$$

$$5. \int \frac{dx}{a + b \cos x + c \sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \operatorname{arctg} \frac{(a-b) \operatorname{tg} \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}} \quad [a^2 > b^2 + c^2].$$

$$6. = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a-b) \operatorname{tg} \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a-b) \operatorname{tg} \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}} \quad [a^2 < b^2 + c^2].$$

$$7. = \frac{1}{c} \ln \left(a + c \operatorname{tg} \frac{x}{2} \right) \quad [a = b].$$

$$8. = \frac{-2}{c + (a-b) \operatorname{tg} \frac{x}{2}} \quad [a^2 = b^2 + c^2].$$

$$9. \int \frac{dx}{(a \cos x + b \sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{d\lambda}{\sin^n \left(x + \operatorname{arctg} \frac{a}{b} \right)} \quad [\text{см. 1.5.2.}].$$

$$10. \int \frac{A + B \cos x + C \sin x}{a \cos x + b \sin x} dx = \frac{Ba + Cb}{a^2 + b^2} x +$$

$$+ \frac{Bb - Ca}{a^2 + b^2} \ln \sin \left(x + \operatorname{arctg} \frac{a}{b} \right) + \frac{A}{\sqrt{a^2 + b^2}} \ln \operatorname{tg} \left[\frac{1}{2} \left(x + \operatorname{arctg} \frac{a}{b} \right) \right].$$

$$11. \int \frac{1}{a \cos x + b \sin x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{a^2 + b^2} \left[\left\{ \frac{b}{a} \right\} x + \left\{ \frac{a}{b} \right\} \ln (a \cos x + b \sin x) \right].$$

$$12. \int \frac{dx}{(a \cos x + b \sin x)^2} = -\frac{1}{a^2 + b^2} + \frac{a \sin x - b \cos x}{a \cos x + b \sin x}.$$

$$13. \int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{a \operatorname{tg} \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \quad [a^2 > b^2].$$

$$14. = \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \operatorname{tg} \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \operatorname{tg} \frac{x}{2} + b + \sqrt{b^2 - a^2}} \quad [a^2 < b^2].$$

$$15. \int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} \operatorname{tg} \frac{x}{2}}{a + b} \quad [a^2 > b^2].$$

$$16. = -\frac{1}{\sqrt{b^2 - a^2}} \ln \frac{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} + a + b}{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} - a - b} \quad [a^2 < b^2].$$

$$17. \int \frac{A + B \cos x + C \sin x}{(1 \pm \sin x)^n} dx = -\frac{1}{2^{n-1}} \left[2C \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \pm \right. \\ \left. \pm (A \mp C) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \right] + \frac{B}{n-1} \frac{1}{(1 \pm \sin x)^{n-1}} \quad [n \geq 2]$$

$$18. = \pm Cx + (A \mp C) \operatorname{tg} \left(\frac{\pi}{4} \mp \frac{x}{2} \right) \pm B \ln (1 \pm \sin x) \quad [n = 1].$$

$$19. \int \frac{A + B \cos x + C \sin x}{(1 \pm \cos x)^n} dx = \frac{1}{2^{n-1}} \left[2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{\pm(2k+1)} \frac{x}{2}}{2k+1} \pm \right. \\ \left. \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{\pm(2k+1)} \frac{x}{2}}{2k+1} \right] \pm \frac{C}{n-1} \frac{1}{(1 \pm \cos x)^{n-1}} \quad [n \geq 2].$$

$$20. = \pm Bx \pm (A \mp B) \operatorname{tg}^{\pm 1} \frac{x}{2} \mp C \ln (1 \pm \cos x) \quad [n = 1].$$

$$21. \int \frac{A + B \cos x + C \sin x}{\sin x (a + b \sin x)} dx = \\ = \frac{A}{a} \ln \operatorname{tg} \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} + \frac{Ca - Ab}{a} \int \frac{dx}{a + b \sin x}.$$

$$22. \int \frac{A + B \cos x + C \sin x}{\sin x (a + b \cos x)} dx = \\ = \frac{1}{a^2 - b^2} \left[(Aa - Bb) \ln \operatorname{tg} \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right] + C \int \frac{dx}{a + b \cos x}.$$

$$23. \int \frac{A+B \cos x+C \sin x}{\sin x(1 \pm \cos x)} dx = \frac{A \pm B}{2} \ln \operatorname{tg} \frac{x}{2} - \frac{B \mp A}{2} \frac{1}{1 \pm \cos x} \pm C \left(\operatorname{tg} \frac{x}{2} \right)^{\pm 1}.$$

$$24. \int \frac{A+B \cos x+C \sin x}{\cos x(a+b \sin x)} dx = \\ = \frac{1}{a^2-b^2} \left[(Aa-Cb) \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - (Ab-Ca) \ln \frac{a+b \sin x}{\cos x} \right] + B \int \frac{dx}{a+b \sin x}.$$

$$25. \int \frac{A+B \cos x+C \sin x}{\cos x(1 \pm \sin x)} dx = \\ = \frac{A \pm C}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp C}{2} \cdot \frac{1}{(1 \pm \sin x)} + B \operatorname{tg} \left(\frac{\pi}{4} \pm \frac{x}{2} \right).$$

$$26. \int \frac{A+B \cos x+C \sin x}{\cos x(a+b \cos x)} dx = \\ = \frac{A}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{C}{a} \ln \frac{a+b \cos x}{\cos x} + \left(B - \frac{Ab}{a} \right) \int \frac{dx}{a+b \cos x}.$$

$$27. \int \frac{dx}{a+b \left\{ \frac{\sin x}{\cos x} \right\}^2} = \frac{\pm \operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg} \left(\sqrt{\frac{a+b}{a}} \left\{ \operatorname{ctg} x \right\} \right) \quad \left[\frac{b}{a} > -1 \right].$$

$$28. \quad = \pm \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth} \left(\sqrt{-\frac{a+b}{a}} \left\{ \operatorname{tg} x \right\} \right) \\ \left[\frac{b}{a} < -1; \left\{ \frac{\sin x}{\cos x} \right\}^2 < -\frac{a}{b} \right].$$

$$29. \quad = \pm \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth} \left(\sqrt{-\frac{a+b}{a}} \left\{ \operatorname{ctg} x \right\} \right) \\ \left[\frac{b}{a} < -1; \left\{ \frac{\sin x}{\cos x} \right\}^2 > -\frac{a}{b} \right].$$

$$30. \int \frac{dx}{1 + \left\{ \frac{\sin x}{\cos x} \right\}^2} = \pm \frac{1}{\sqrt{2}} \operatorname{arctg} \left(\sqrt{2} \left\{ \operatorname{tg} x \right\} \right).$$

$$31. \int \frac{dx}{1 - \left\{ \frac{\sin x}{\cos x} \right\}^2} = \pm \left\{ \operatorname{tg} x \right\}.$$

$$32. \int \frac{dx}{\left(a+b \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a+b \left\{ \frac{\sin x}{\cos x} \right\}^2} \pm \frac{b \sin x \cos x}{a+b \left\{ \frac{\sin x}{\cos x} \right\}^2} \right].$$

$$33. \int \frac{dx}{\left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^2} = \frac{1}{4} \left[\frac{3}{\sqrt{2}} \operatorname{arctg} \left(\sqrt{2} \left\{ \operatorname{tg} x \right\} \right) \pm \frac{\sin x \cos x}{1 + \left\{ \frac{\sin x}{\cos x} \right\}^2} \right].$$

$$34. \int \frac{dx}{(a+b \sin^2 x)^3} = \frac{1}{8\rho a^3} \left[\left(3 + \frac{2}{\rho^2} + \frac{3}{\rho^4} \right) \operatorname{arctg} (\rho \operatorname{tg} x) + \right. \\ \left. + \left(3 + \frac{2}{\rho^2} - \frac{3}{\rho^4} \right) \frac{\rho \operatorname{tg} x}{1 + \rho^2 \operatorname{tg}^2 x} + \left(1 - \frac{2}{\rho^2} - \frac{1}{\rho^4} \operatorname{tg}^2 x \right) \frac{2\rho \operatorname{tg} x}{(1 + \rho^2 \operatorname{tg}^2 x)^2} \right] \\ \left[\rho^2 = 1 + \frac{b}{a} > 0 \right].$$

$$\begin{aligned}
 35. \quad &= \frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{tg} x) + \right. \\
 &\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{tg} x}{1 - q^2 \operatorname{tg}^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \operatorname{tg}^2 x \right) \frac{2q \operatorname{tg} x}{(1 - q^2 \operatorname{tg}^2 x)^2} \right] \\
 &\quad \left[q^2 = -1 - \frac{b}{a} > 0; \sin^2 x < -\frac{a}{b}; \text{ при } \sin^2 x > -\frac{a}{b} \right. \\
 &\quad \left. \text{следует } \operatorname{Arth}(q \operatorname{tg} x) \text{ заменить на } \operatorname{Arctg}(q \operatorname{tg} x) \right].
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int \frac{dx}{(a+b \cos^2 x)^2} &= -\frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{ctg} x) + \right. \\
 &\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{ctg} x}{1 + p^2 \operatorname{ctg}^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{ctg}^2 x \right) \frac{2p \operatorname{ctg} x}{(1 + p^2 \operatorname{ctg}^2 x)^2} \right] \\
 &\quad \left[p^2 = 1 + \frac{b}{a} > 0 \right].
 \end{aligned}$$

$$\begin{aligned}
 37. \quad &= -\frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{ctg} x) + \right. \\
 &\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{ctg} x}{1 - q^2 \operatorname{ctg}^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \operatorname{ctg}^2 x \right) \frac{2q \operatorname{ctg} x}{(1 - q^2 \operatorname{ctg}^2 x)^2} \right] \\
 &\quad \left[q^2 = -1 - \frac{b}{a} > 0; \cos^2 x < -\frac{a}{b}; \text{ при } \cos^2 x > -\frac{a}{b} \right. \\
 &\quad \left. \text{следует } \operatorname{Arth}(q \operatorname{ctg} x) \text{ заменить на } \operatorname{Arctg}(q \operatorname{ctg} x) \right].
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \int \frac{A + B \cos x + C \sin x}{(a_1 + b_1 \cos x + c_1 \sin x)(a_2 + b_2 \cos x + c_2 \sin x)} dx &= \\
 &= A_0 \ln \frac{a_1 + b_1 \cos x + c_1 \sin x}{a_2 + b_2 \cos x + c_2 \sin x} + A_1 \int \frac{dx}{a_1 + b_1 \cos x + c_1 \sin x} + \\
 &\quad + A_2 \int \frac{dx}{a_2 + b_2 \cos x + c_2 \sin x},
 \end{aligned}$$

где

$$\begin{aligned}
 &\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \\
 A_0 &= \frac{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}{\begin{vmatrix} B & C \\ b_1 & c_1 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} A & C \\ a_1 & c_1 \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix} \begin{vmatrix} B & A \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix}}, \\
 A_1 &= \frac{\begin{vmatrix} B & C \\ b_1 & c_1 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} A & C \\ a_1 & c_1 \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix} \begin{vmatrix} B & A \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \\
 A_2 &= \frac{\begin{vmatrix} C & B \\ c_2 & b_2 \\ a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} C & A \\ c_2 & a_2 \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix} \begin{vmatrix} A & B \\ a_2 & b_2 \\ b_1 & a_1 \\ c_1 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2} \\
 &\quad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2; \text{ см. 1.5.9.5} \right].
 \end{aligned}$$

$$39. \int \frac{dx}{a \cos^2 x + b \sin x \cos x + c \sin^2 x} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \frac{2c \operatorname{tg} x + b - \sqrt{b^2 - 4ac}}{2c \operatorname{tg} x + b + \sqrt{b^2 - 4ac}} \quad [b^2 > 4ac],$$

$$40. = \frac{2}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2c \operatorname{tg} x + b}{\sqrt{4ac - b^2}} \quad [b^2 < 4ac],$$

$$41. = -\frac{2}{2c \operatorname{tg} x + b} \quad [b^2 = 4ac],$$

$$42. \int \frac{A \cos^2 x + 2B \sin x \cos x + C \sin^2 x}{a \cos^2 x + 2b \sin x \cos x + c \sin^2 x} dx =$$

$$= \frac{1}{4b^2 + (a-c)^2} \{ [4Bb + (A-C)(a-c)]x + [(A-C)b - B(a-c)] \times$$

$$\times \ln(a \cos^2 x + 2b \sin x \cos x + c \sin^2 x) +$$

$$+ [2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)] f(x) \},$$

где

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{tg} x + b - \sqrt{b^2 - ac}}{c \operatorname{tg} x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac],$$

$$= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{tg} x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac],$$

$$= -\frac{1}{c \operatorname{tg} x + b} \quad [b^2 = ac].$$

$$43. \int \frac{dx}{(a \cos^2 x + b \sin x \cos x + c \sin^2 x)^2} =$$

$$= \frac{b \cos 2x + (c-a) \sin 2x}{(4ac - b^2)(a \cos^2 x + b \sin x \cos x + c \sin^2 x)} +$$

$$+ \frac{2(a+c)}{4ac - b^2} \int \frac{dx}{a \cos^2 x + b \sin x \cos x + c \sin^2 x} \quad [b^2 \neq 4ac].$$

$$44. = 16a^2 \int \frac{dx}{(2a \cos x + b \sin x)^4} \quad [b^2 = 4ac].$$

$$45. \int \frac{\operatorname{tg} x \, dx}{\operatorname{tg} x + a} = \frac{1}{a^2 + 1} [x - a \ln(a \cos x + \sin x)].$$

$$46. \int \frac{\operatorname{tg} x - a}{\operatorname{tg} x + a} dx = \frac{1 - a^2}{1 + a^2} x - \frac{2a}{1 + a^2} \ln \sin(x + \operatorname{arctg} a).$$

$$47. \int \frac{dx}{a^2 + \operatorname{tg}^2 x} = \frac{x}{a^2 - 1} - \frac{1}{(a^2 - 1)a} \operatorname{arctg} \frac{\operatorname{tg} x}{a}.$$

$$48. \int \frac{dx}{a^2 - \operatorname{tg}^2 x} = \frac{x}{a^2 + 1} - \frac{1}{2(a^2 + 1)a} \ln \left| \frac{\operatorname{tg} x - a}{\operatorname{tg} x + a} \right|.$$

$$49. \int \frac{dx}{1 + \operatorname{tg}^2 x} = \frac{x}{2} + \frac{1}{4} \sin 2x.$$

$$50. \int \frac{\operatorname{tg} x \, dx}{a^2 + \operatorname{tg}^2 x} = \frac{\ln(a^2 \cos^2 x + \sin^2 x)}{2(1 - a^2)}.$$

1.5.10. Интегралы вида $\int \frac{\sin(ax+b) \sin(cx+d)}{\cos(ax+b) \cos(cx+d)} dx$,

$$\int \sin(ax+b) \cos(cx+d) dx.$$

$$1. \int \frac{\sin(ax+b)}{\cos(ax+b)} dx = \mp \frac{1}{a} \frac{\cos(ax+b)}{\sin(ax+b)},$$

$$2. \int \frac{\sin(ax+b) \sin(cx+d)}{\cos(ax+b) \cos(cx+d)} dx = \\ = \frac{1}{2(a-c)} \sin[(a-c)x+b-d] \mp \frac{\sin[(a+c)x+b+d]}{2(a+c)},$$

$$3. \int \sin(ax+b) \cos(cx+d) dx = \\ = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)},$$

$$4. \int \frac{\sin(ax+b) \sin(ax+d)}{\cos(ax+b) \cos(ax+d)} dx = \frac{x}{2} \cos(b-d) \mp \frac{\sin(2ax+b+d)}{4a}.$$

$$5. \int \sin(ax+b) \cos(ax+d) dx = \frac{x}{2} \sin(b-d) - \frac{\cos(2ax+b+d)}{4a}.$$

1.5.11. Интегралы вида $\int \sin^p x \sin ax dx$.

$$1. \int \sin^p x \sin ax dx = \frac{-1 + \sin^p x \cos ax}{p+a} + \frac{p}{p+a} \int \sin^{p-1} x \cos(a-1)x dx,$$

$$2. = -\frac{\sin^p x \cos ax}{p+a} + \frac{p \sin^{p-1} x \sin(a-1)x}{(p+a)(p+a-2)} - \\ - \frac{p(p-1)}{(p+a)(p+a-2)} \int \sin^{p-2} x \sin(a-2)x dx.$$

$$3. \int \sin^p x \sin 2nx dx =$$

$$= 2n \left\{ \frac{\sin^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots [4n^2-(2k)^2]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x \right\} \\ [p \neq -2, -4, \dots, -2n]$$

$$4. \int \sin^p x \sin(2n+1)x dx = (2n+1) \left\{ \int \sin^{p+1} x dx + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \sin^{2k+p+1} x dx \right\}.$$

$$5. \int \sin^{p-1} x \sin(p+1)x dx = \frac{1}{p} \sin^p x \sin px.$$

$$6. \int \sin^{p-1} x \sin \left[(p+1) \left(\frac{\pi}{2} - x \right) \right] dx = \frac{1}{p} \sin^p x \cos p \left(\frac{\pi}{2} - x \right).$$

$$7. \int \frac{\sin ax}{\sin^p x} dx = 2 \int \frac{\cos(a-1)x}{\sin^{p-1} x} dx + \int \frac{\sin(a-2)x}{\sin^p x} dx.$$

8. $\int \frac{\sin 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin (2k-1)x}{2k-1}.$
9. $\int \frac{\sin (2n+1)x}{\sin x} dx = 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + x.$
10. $\int \frac{\sin 2x}{\sin^n x} dx = -\frac{2}{(n-2) \sin^{n-2} x}.$
11. $\int \frac{\sin 2x}{\sin^2 x} dx = 2 \ln |\sin x|.$ 12. $\int \frac{\sin 3x}{\sin x} dx = x + \sin 2x.$
13. $\int \frac{\sin 3x}{\sin^2 x} dx = 3 \ln \left| \operatorname{tg} \frac{x}{2} \right| + 4 \cos x.$
14. $\int \frac{\sin 3x}{\sin^3 x} dx = -3 \operatorname{ctg} x - 4x.$

1.5.12. Интегралы вида $\int \sin^p x \cos ax dx.$

1. $\int \sin^p x \cos ax dx = \frac{\sin^p x \sin ax}{p+a} - \frac{p}{p+a} \int \sin^{p-1} x \sin (a-1)x dx.$
2.
$$= \frac{\sin^p x \sin ax}{p+a} + \frac{p \sin^{p-1} x \cos (a-1)x}{(p+a)(p+a-2)} -$$

$$- \frac{p(p-1)}{(p+a)(p+a-2)} \int \sin^{p-2} x \cos (a-2)x dx,$$
3. $\int \sin^p x \cos 2nx dx = \int \sin^p x dx +$

$$+ \sum_{k=1}^n (-1)^k \frac{4n^2(4n^2-2^2) \dots [4n^2-(2k-2)^2]}{(2k)!} \int \sin^{2k+p} x dx,$$
4. $\int \sin^p x \cos (2n+1)x dx = \frac{\sin^{p+1} x}{p+1} +$

$$+ \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \sin^{2k+p+1} x$$

[$p \neq -1, -3, -5, \dots, -(2n+1)$].
5. $\int \sin^{p-1} x \cos (p+1)x dx = \frac{1}{p} \sin^p x \cos px.$
6. $\int \sin^{p-1} x \cos \left[(p+1) \left(\frac{\pi}{2} - x \right) \right] dx = -\frac{1}{p} \sin^p x \sin p \left(\frac{\pi}{2} - x \right).$
7. $\int \frac{\cos ax}{\sin^p x} dx = -2 \int \frac{\sin (a-1)x}{\sin^{p-1} x} dx + \int \frac{\cos (a-2)x}{\sin^p x} dx.$
8. $\int \frac{\cos 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos (2k-1)x}{2k-1} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$

$$9. \int \frac{\cos (2n+1)x}{\sin x} dx = \sum_{k=1}^n \frac{\cos 2kx}{k} + \ln |\sin x|.$$

$$10. \int \frac{\cos 2x}{\sin x} dx = 2 \cos x + \ln \left| \operatorname{tg} \frac{x}{2} \right|. \quad 11. \int \frac{\cos 2x}{\sin^2 x} dx = -\operatorname{ctg} x - 2x.$$

$$12. \int \frac{\cos 2x}{\sin^3 x} dx = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$13. \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}.$$

$$14. \int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln |\sin x|.$$

$$15. \int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln |\sin x|.$$

1.5.13. Интегралы вида $\int \cos^p x \sin ax dx$.

$$1. \int \cos^p x \sin ax dx = -\frac{\cos^p x \cos ax}{p+a} + \frac{p}{p+a} \int \cos^{p-2} x \sin (a-1)x dx.$$

$$2. \int \cos^p x \sin 2nx dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \right. \\ \left. + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2) \dots [4n^2-(2k)^2]}{(2k+1)! (2k+p+2)} \cos^{2k+p+2} x \right\} \\ [p \neq -2, -4, \dots, -2n].$$

$$3. \int \cos^p x \sin (2n+1)x dx = (-1)^{n+1} \left\{ \frac{\cos^{p+2} x}{p+1} + \right. \\ \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k)! (2k+p+1)} \times \right. \\ \left. \times \cos^{2k+p+2} x \right\} [p \neq -1, -3, -5, \dots, -(2n+1)].$$

$$4. \int \cos^{p-1} x \sin (p+1)x dx = -\frac{1}{p} \cos^p x \cos px.$$

$$5. \int \frac{\sin ax}{\cos^p x} dx = 2 \int \frac{\sin (a-1)x}{\cos^{p-1} x} dx - \int \frac{\sin (a-2)x}{\cos^p x} dx.$$

$$6. \int \frac{\sin (2n+1)x}{\cos x} dx = \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos 2kx}{k} + (-1)^{n+1} \ln |\cos x|.$$

$$7. \int \frac{\sin 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos (2k-1)x}{2k-1}.$$

$$8. \int \frac{\sin 2x}{\cos^n x} dx = \frac{2}{(n-2) \cos^{n-2} x}. \quad 9. \int \frac{\sin 2x}{\cos^3 x} dx = -2 \ln |\cos x|.$$

$$10. \int \frac{\sin 3x}{\cos^n x} dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}.$$

$$11. \int \frac{\sin 3x}{\cos x} dx = 2 \sin^2 x + \ln |\cos x|.$$

$$12. \int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2\cos^2 x} - 4 \ln |\cos x|.$$

1.5.14. Интегралы вида $\int \cos^p x \cos ax dx$.

$$1. \int \cos^p x \cos ax dx = \frac{\cos^p x \sin ax}{p+a} + \frac{p}{p+a} \int \cos^{p-1} x \cos (a-1)x dx.$$

$$2. \int \cos^n x \cos 2nx dx = (-1)^n \left\{ \int \cos^p x dx + \sum_{k=1}^n (-1)^k \frac{4n^2(2n^2-2^2) \dots [4n^2-(2k-2)^2]}{(2k)!} \int \cos^{2k+p} x dx \right\}.$$

$$3. \int \cos^p x \cos (2n+1)x dx = (-1)^n (2n+1) \left\{ \int \cos^{p+1} x dx + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2] \dots [(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \right. \\ \left. \times \int \cos^{2k+p+1} x dx \right\}.$$

$$4. \int \cos^{p-1} x \cos (p+1)x dx = \frac{1}{p} \cos^p x \sin px.$$

$$5. \int \frac{\cos ax}{\cos^p x} dx = 2 \int \frac{\cos (a-1)x}{\cos^{p-1} x} dx - \int \frac{\cos (a-2)x}{\cos^p x} dx.$$

$$6. \int \frac{\cos 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin (2k-1)x}{2k-1} + (-1)^n \ln \left| \lg \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$7. \int \frac{\cos (2n+1)x}{\cos x} dx = \sum_{k=1}^n (-1)^{n-k} \frac{\sin 2kx}{k} + (-1)^n x.$$

$$8. \int \frac{\cos 2x}{\cos x} dx = 2 \sin x - \ln \left| \lg \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$9. \int \frac{\cos 2x}{\cos^2 x} dx = 2x - \operatorname{tg} x.$$

$$10. \int \frac{\cos 2x}{\cos^3 x} dx = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2} \ln \left| \lg \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$11. \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x.$$

$$12. \int \frac{\cos 3x}{\cos^2 x} dx = 4 \sin x - 3 \ln \left| \lg \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$13. \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \operatorname{tg} x.$$

1.5.15. Интегралы вида $\int \left\{ \frac{\sin x}{\cos x} \right\}^m \left\{ \frac{\sin nx}{\cos nx} \right\}^{-1} dx$.

$$1. \int \frac{\sin^m x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \left[\frac{2k+1}{2(2n+1)} \pi \right] \ln \left| \frac{\sin \left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[\frac{k+n+1}{2(2n+1)} \pi - \frac{x}{2} \right]} \right|$$

[$m \leq 2n$].

$$2. \int \frac{\sin^{2m} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln |\cos x| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left| \cos^2 x - \sin^2 \frac{k\pi}{2n} \right| \right\}$$

[$m \leq n$].

$$3. \int \frac{\sin^{2m+1} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left| \operatorname{tg} \left(\frac{n+k}{4n} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{n-k}{4n} \pi - \frac{x}{2} \right) \right| \right\}$$

[$m < n$].

$$4. \int \frac{\sin^{2m} x dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left| \operatorname{tg} \left(\frac{2n+2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{2n-2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \right| \right\}$$

[$m \leq n$].

$$5. \int \frac{\sin^{2m+1} x dx}{\cos (2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln |\cos x| + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left| \cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right| \right\}$$

[$m \leq n$].

$$6. \int \frac{\sin^m x dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left(\frac{2k+1}{4n} \pi \right) \times \ln \left| \frac{\sin \left(\frac{2k-2n+1}{8n} \pi + \frac{x}{2} \right)}{\sin \left(\frac{2k+2n+1}{8n} \pi - \frac{x}{2} \right)} \right|$$

[$m < 2n$].

$$7. \int \frac{\cos^{2m+1} x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln |\sin x| + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left| \sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right| \right\}$$

[$m \leq n$].

$$8. \int \frac{\cos^{2m} x dx}{\sin (2n+1)x} = \frac{1}{2n+1} \left\{ \ln \left| \operatorname{tg} \frac{x}{2} \right| + \right. \\ \left. + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n+2} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n+2} \right) \right| \right\} \quad [m \leq n].$$

$$9. \int \frac{\cos^{2m+1} x}{\sin 2nx} dx = \\ = \frac{1}{2n} \left\{ \ln \left| \operatorname{tg} \frac{x}{2} \right| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n} \right) \right| \right\} \\ [m < n].$$

$$10. \int \frac{\cos^{2m} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln |\sin x| + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left| \sin^2 x - \sin^2 \frac{k\pi}{2n} \right| \right\} \\ [m \leq n].$$

$$11. \int \frac{\cos^m x}{\cos nx} dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n} \pi \ln \left| \frac{\sin \left(\frac{2k+1}{4n} \pi + \frac{x}{2} \right)}{\sin \left(\frac{2k+1}{4n} \pi - \frac{x}{2} \right)} \right| \\ [m \leq n].$$

1.5.16. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax}) dx$.

Обозначение: $\varphi = \arcsin \sqrt{\frac{2 \sin ax}{1 + \sin ax + \cos cx}}$.

$$1. \int \sin^{m+1/2} 2x \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \pm 2^{m+3/2} \int \frac{t^{2(m+2n+2)}}{(1+t^4)^{n+m+2}} dt \\ [t = \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{1/2}] \\ = \mp 2^{m+3/2} \int \frac{t^{2(m+1)} dt}{(1+t^4)^{n+m+2}} \quad [t = \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2}]$$

$$2. \int \sin^{m+1/2} 2x \left\{ \frac{\sec x}{\operatorname{cosec} x} \right\}^{2n+1} dx = \\ = \mp 2^{m+3/2} \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{1}{2m-4n+4k+1} \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{m-2n+2k} \\ [n \geq m+1].$$

$$3. \int \sin^{m+1/2} 2x \left\{ \frac{\sin x}{\cos x} \right\}^{2n} \left\{ \frac{\cos x}{\sin x} \right\} dx = \mp 2^{m+3/2} \int \frac{t^{2(m+2n)} dt}{(1+t^4)^{n+m+2}} \\ [t = \left\{ \frac{\operatorname{ctg} x}{\operatorname{tg} x} \right\}^{1/2}].$$

$$4. \int \sqrt{\sin 2x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \mp \frac{1}{2} \left\{ \frac{\cos x}{\sin x} \right\} \sqrt{\sin 2x} + \\ + \frac{1}{4} [\ln (\sin x + \cos x \pm \sqrt{\sin 2x}) + \arcsin (\sin x - \cos x)].$$

$$5. \int_0^x \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$6. \int \frac{1}{\sin^{n+1/2} 2x} \left\{ \frac{\sin x}{\cos x} \right\}^{2m+1} dx =$$

$$= \pm 2^{-n+1/2} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{3/2} \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} \frac{1}{4m-2n+4k+3} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2m-n+2k}$$

[$n \geq m+1$].

$$7. \int \frac{1}{\sqrt{\sin 2x}} \left\{ \frac{\sin x}{\cos x} \right\} dx =$$

$$= \frac{1}{2} [\ln(\sin x + \cos x \mp \sqrt{\sin 2x}) + \arcsin(\sin x - \cos x)].$$

$$8. \int \frac{1}{\sin^{n+1/2} 2x} \left\{ \frac{\sec x}{\operatorname{cosec} x} \right\}^{2m+1} dx =$$

$$= \pm 2^{1/2-n} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{1/2} \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{1}{4m+2n-4k+1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2m+n-2k}.$$

$$9. \int \frac{\sin x}{\sin^{n+1/2} 2x} \cos^{\pm 2m} x dx =$$

$$= 2^{-n+1/2} \operatorname{tg}^{3/2} x \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} \frac{\operatorname{tg}^{2k-n} x}{4k-2n+3} \quad [n \geq m+1].$$

$$10. \int \frac{\cos x}{\sin^{n+1/2} 2x} \sin^{\pm 2m} x dx =$$

$$= -2^{-n+1/2} \operatorname{ctg}^{3/2} x \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} \frac{\operatorname{ctg}^{2k-n} x}{4k-2n+3} \quad [n \geq m+1].$$

$$11. \int \frac{\operatorname{tg}^{\pm l} x dx}{\sin^{2m+1} x \sin^{n+1/2} 2x} =$$

$$= -2^{-n+1/2} \operatorname{ctg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{ctg}^{2m+n \mp l - 2k} x}{4m+2n \mp 2l - 4k + 1}.$$

$$12. \int \frac{\operatorname{tg}^{\pm l} x dx}{\cos^{2m+1} x \sin^{n+1/2} 2x} = 2^{-n+1/2} \operatorname{tg}^{1/2} x \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k \pm l - n} x}{4k \pm 2l - 2n + 1}.$$

$$13. \int_0^x \frac{\sin ax dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$14. \int_0^x \frac{\sin ax \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[\sqrt{\lg ax} - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right]$$

[$ax \neq \pi(2)$].

$$15. \int_0^x \frac{(1 + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$16. \int_0^x \frac{(1 + \cos ax) \, dx}{(1 - \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \sqrt{\lg ax} \right]$$

[$ax \neq \pi/2$].

$$17. \int_0^x \frac{(1 - \sin ax + \cos ax) \, dx}{(1 + \sin ax + \cos ax) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$18. \int_0^x \frac{(1 + \sin ax + \cos ax) \, dx}{[1 - \cos ax + (1 - 2r^2) \sin ax] \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

1.5.17. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx$.Условие: $0 < ax \leq \pi/4$.Обозначение: $\varphi = \arcsin(\sqrt{2} \sin ax)$.

$$1. \int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) \, dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{1 - 2\sin^2 t}) \, dt$$

[$t = ax$; см. 1.5.35].

$$2. \int_0^x \sqrt{\cos 2ax} \, dx = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$3. \int \cos^{n+1/2} 2x \frac{(\sin x)}{(\cos x)} \, dx = \mp \frac{\sqrt{\cos 2x}}{2(n+1)} \frac{(\cos x)}{(\sin x)} \times \\ \times \left[\cos^2 2x + \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{2^{k+1}n(n-1)\dots(n-k)} \cos^{n-k-1} 2x \right] + \\ + (-1)^n \frac{(2n+1)!!}{(2n+2)!! \sqrt{2}} \left\{ \ln \left(\cos x + \sqrt{\frac{\cos 2x}{2}} \right) \right\}.$$

$$4. \int \sqrt{\cos 2x} \frac{(\sin x)}{(\cos x)} \, dx = \mp \frac{1}{2} \sqrt{\cos 2x} \frac{(\cos x)}{(\sin x)} + \frac{1}{2\sqrt{2}} \left\{ \ln \left(\cos x + \sqrt{\frac{\cos 2x}{2}} \right) \right\}.$$

$$5. \int_0^x \frac{\sqrt{\cos 2ax}}{\cos^2 ax} \, dx = \frac{\sqrt{2}}{a} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{a} \lg ax \sqrt{\cos 2ax}.$$

$$6. \int_0^x \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$7. \int \frac{1}{\cos^{n-1/2} 2x} \frac{(\sin x)}{(\cos x)} dx = \frac{1}{(2n-1) \cos^{n-1/2} 2x} \frac{(\cos x)}{(\sin x)} \times$$

$$\times \left[1 + \sum_{k=0}^{n-2} (-1)^{k+1} \frac{2^{k+1} (n-1) (n-2) \dots (n-k-1)}{(2n-3) (2n-5) \dots (2n-2k-3)} \cos^{k+1} 2x \right].$$

$$8. = \frac{1}{\sqrt{\cos 2x}} \frac{(\cos x)}{(\sin x)} \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1} 2^k (n-1)}{2k+1} \binom{n-1}{k} \frac{1}{\cos^k 2x} \frac{(\cos x)^{2k}}{(\sin x)}.$$

$$9. \int \frac{1}{\sqrt{\cos 2x}} \frac{(\sin x)}{(\cos x)} dx = \frac{1}{\sqrt{2}} \left\{ \ln \left(\cos x - \sqrt{\frac{\cos 2x}{2}} \right) \right. \\ \left. - \arcsin \left(\sqrt{2} \sin x \right) \right\}.$$

$$10. \int_0^x \frac{\sin^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$11. \int_0^x \frac{\cos^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$12. \int_0^x \frac{\operatorname{tg}^2 ax}{\sqrt{\cos 2ax}} dx = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax}.$$

$$13. \int_0^x \frac{\operatorname{tg}^4 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{3a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}.$$

$$14. \int_0^x \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{tg} ax}{a} \sqrt{\cos 2ax}.$$

$$15. \int_0^x \frac{dx}{(1-2r^2 \sin^2 ax) \sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

$$16. \int_0^x \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} =$$

$$= \frac{2\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{(6 \cos^2 ax + 1) \sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax}.$$

$$17. \int_0^x \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}.$$

$$18. \int_0^x \frac{\sin^2 ax}{\sqrt{\cos^2 2ax}} dx = \frac{\sin 2ax}{2a \sqrt{\cos 2ax}} - \frac{1}{a \sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

1.5.18. Интегралы вида $\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx$.

Условие: $\pi/4 \leq ax < \pi/2$.

Обозначение: $\varphi = \arcsin(\sqrt{2} \cos ax)$.

$$1. \int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{2 \sin^2 t - 1}) dt \quad [t = ax; \text{ см. 1.5.38}].$$

$$2. \int_x^{\pi/(2a)} \sqrt{-\cos 2ax} dx = \frac{1}{a \sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right].$$

$$3. \int_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} F\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$4. \int_x^{\pi/(2a)} \frac{\sin^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right).$$

$$5. \int_x^{\pi/(2a)} \frac{\cos^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right]$$

$$6. \int_x^{\pi/(2a)} \frac{\operatorname{ctg}^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax}.$$

$$7. \int_x^{\pi/(2a)} \frac{\cos^4 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a \sqrt{2}} \left[\frac{5}{2} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - 3F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}.$$

$$8. \int_x^{\pi/(2a)} \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax}.$$

$$9. \int_x^{\pi/(2a)} \frac{dx}{(1 - 2r^2 \cos^2 ax) \sqrt{-\cos 2ax}} = \frac{1}{a \sqrt{2}} \Pi\left(\varphi, r^2, \frac{1}{\sqrt{2}}\right).$$

$$10. \int_x^{\pi/(2a)} \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} = \frac{-2}{3a \sqrt{2}} \left[F\left(\varphi, \frac{1}{\sqrt{2}}\right) - 6E\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{\cos ax}{3a \sin^3 ax} (6 \sin^2 ax + 1) \sqrt{-\cos 2ax}.$$

$$11. \int_x^{\pi/(2a)} \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} \left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right) \right] - \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}.$$

$$12. \int_x^{\pi/(2a)} \frac{\cos^2 ax \, dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} E\left(\varphi, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}}.$$

15.19. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a \pm b \sin x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{1 - \sin x}{2}}$,

$\psi = \arcsin \sqrt{\frac{b(1 - \sin x)}{a + b}}$, $k = \sqrt{\frac{2b}{a + b}}$.

$$1. \int_x^{\pi/2} \sqrt{a + b \sin x} \, dx = 2\sqrt{a + b} E(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right].$$

$$2. = (a - b) \sqrt{\frac{2}{b}} F\left(\psi, \frac{1}{k}\right) + 2\sqrt{2b} E\left(\psi, \frac{1}{k}\right) \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$3. \int_x^{\pi/2} \frac{dx}{\sqrt{a + b \sin x}} = \frac{2}{\sqrt{a + b}} F(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$4. = \sqrt{\frac{2}{b}} F\left(\psi, \frac{1}{k}\right) \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$5. \int_x^{\pi/2} \frac{\sin x \, dx}{\sqrt{a + b \sin x}} = \frac{2\sqrt{a + b}}{b} E(\varphi, k) - \frac{2a}{b\sqrt{a + b}} F(\varphi, k) \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$6. = \sqrt{\frac{2}{b}} \left\{ 2E\left(\psi, \frac{1}{k}\right) - F\left(\psi, \frac{1}{k}\right) \right\} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

$$7. \int_x^{\pi/2} \frac{\sin^2 x \, dx}{\sqrt{a + b \sin x}} = -\frac{4a\sqrt{a + b}}{3b^2} E(\varphi, k) + \frac{2(2a^2 + b^2)}{3b^2\sqrt{a + b}} F(\varphi, k) + \frac{2}{3b} \cos x \sqrt{a + b \sin x} \quad \left[a > b > 0; -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

$$8. = \sqrt{\frac{2}{b}} \left\{ \frac{2a + b}{3b} F\left(\psi, \frac{1}{k}\right) - \frac{4a}{3b} E\left(\psi, \frac{1}{k}\right) \right\} + \frac{2}{3b} \cos x \sqrt{a + b \sin x} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

- $$\begin{aligned}
 9. \int_x^{\pi/2} \frac{\operatorname{tg}^2 x \, dx}{\sqrt{a+b \sin x}} &= \frac{-1}{\sqrt{a+b}} F(\varphi, k) - \frac{a}{(a-b)\sqrt{a+b}} E(\varphi, k) + \\
 &+ \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < b < a; -\frac{\pi}{2} < x < \frac{\pi}{2}\right]. \\
 10. &= -\sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a+b)} F\left(\psi, \frac{1}{k}\right) + \frac{ab}{a^2-b^2} E\left(\psi, \frac{1}{k}\right) \right\} + \\
 &+ \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]. \\
 11. \int_x^{\pi/2} \frac{dx}{(2-p^2+p^2 \sin x) \sqrt{a+b \sin x}} &= \frac{1}{\sqrt{a+b}} \Pi(\varphi, p^2, k) \\
 &\quad \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right]. \\
 12. \int_x^{\pi/2} \frac{dx}{(a+b-p^2b+p^2b \sin x) \sqrt{a+b \sin x}} &= \frac{1}{a+b} \sqrt{\frac{2}{b}} \Pi\left(\psi, p^2, \frac{1}{k}\right) \\
 &\quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]. \\
 13. \int_x^{\pi/2} \frac{1-\sin x}{1+\sin x} \frac{dx}{\sqrt{a+b \sin x}} &= \frac{2}{a-b} \left\{ -\sqrt{a+b} E(\varphi, k) + \right. \\
 &\quad \left. + \operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right) \sqrt{a+b \sin x} \right\} \quad \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right]. \\
 14. \int_x^{\pi/2} \frac{dx}{\sqrt{(a+b \sin x)^3}} &= \frac{-2b \cos x}{(a^2-b^2) \sqrt{a+b \sin x}} + \frac{2}{(a-b) \sqrt{a+b}} E(\varphi, k) \\
 &\quad \left[0 < b < a; -\frac{\pi}{2} \leq x < \frac{\pi}{2}\right]. \\
 15. &= \sqrt{\frac{2}{b}} \left\{ \frac{1}{a+b} F\left(\psi, \frac{1}{k}\right) - \frac{2b}{b^2-a^2} E\left(\psi, \frac{1}{k}\right) \right\} - \frac{2b}{b^2-a^2} \frac{\cos x}{\sqrt{a+b \sin x}} \\
 &\quad \left[0 < |a| < b; -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]. \\
 16. \int (1 \pm \sin ax)^{n+1/2} dx &= \mp \frac{1}{a} \sqrt{1 \pm \sin x} \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k+1} \frac{(1 \mp \sin ax)^k}{2k+1}. \\
 17. \int \frac{dx}{(1 \pm \sin ax)^{n+1/2}} &= \\
 &= \frac{\cos ax}{na 2^n (1 \pm \sin ax)^{n+1/2}} \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} \times \right. \\
 &\quad \left. \times 2^{n-k-2} (1 \pm \sin ax)^{k+1} \right] \mp \frac{(2n-1)!!}{(2n)!!} \frac{1}{2^n \sqrt{2a}} \ln \frac{\sqrt{2+\sqrt{1 \mp \sin ax}}}{\sqrt{2-\sqrt{1 \mp \sin ax}}}.
 \end{aligned}$$

1.5.20. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a \pm b \cos x}) dx$,

Обозначения: $\varphi = \arcsin \sqrt{\frac{b(1 - \cos x)}{a+b}}$,

$$\psi = \arcsin \sqrt{\frac{(a+b)(1 - \cos x)}{2(a-b \cos x)}}, \quad k = \sqrt{\frac{2b}{a+b}}.$$

$$1. \int_0^x \sqrt{a+b \cos x} dx = 2 \sqrt{a+b} E\left(\frac{x}{2}, k\right) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$2. = \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\varphi, \frac{1}{k}\right) + 2bE\left(\varphi, \frac{1}{k}\right) \right\} \\ [b > |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right)].$$

$$3. \int_0^x \sqrt{a-b \cos x} dx = 2 \sqrt{a+b} E(\psi, k) - \frac{2b \sin x}{\sqrt{a-b \cos x}} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$4. \int_0^x \sqrt{\frac{a-b \cos x}{1+p \cos x}} dx = \frac{2(a-b)}{(1+p) \sqrt{a+b}} \Pi\left(\psi, \frac{2ap}{(a+b)(1+p)}, k\right) \\ [a > b > 0; 0 \leq x \leq \pi; p \neq -1].$$

$$5. \int_0^x \frac{dx}{\sqrt{a+b \cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2}, k\right) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$6. = \sqrt{\frac{2}{b}} F\left(\varphi, \frac{1}{k}\right) \quad [b > |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right)].$$

$$7. \int_0^x \frac{dx}{\sqrt{a-b \cos x}} = \frac{2}{\sqrt{a+b}} F(\psi, k) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$8. \int_0^x \frac{\cos x dx}{\sqrt{a+b \cos x}} = \frac{2}{b \sqrt{a+b}} \left\{ (a+b) E\left(\frac{x}{2}, k\right) - aF\left(\frac{x}{2}, k\right) \right\} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$9. = \sqrt{\frac{2}{b}} \left\{ 2E\left(\varphi, \frac{1}{k}\right) - F\left(\varphi, \frac{1}{k}\right) \right\} \\ [b > |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right)].$$

$$10. \int_0^x \frac{\cos x dx}{\sqrt{a-b \cos x}} = \frac{2}{b \sqrt{a+b}} \{ (b-a) \Pi(\psi, k^2, k) + aF(\psi, k) \} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$11. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, k\right) - 2a(a+b) E\left(\frac{x}{2}, k\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$12. = \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\varphi, \frac{1}{k}\right) - 4aE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$13. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F(\varphi, k) - 2a(a+b) E(\varphi, k) \right\} + \\ + \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} \quad [a > b > 0; 0 \leq x < \pi].$$

$$14. \int (1 \pm \cos ax)^{n+1/2} dx = \pm \sqrt{1 \mp \cos ax} \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k+1} \frac{(1 \mp \cos ax)^k}{2k+1}.$$

$$15. \int_0^x \frac{dx}{(2-p^2+p^2 \cos x) \sqrt{a+b \cos x}} = \frac{1}{\sqrt{a+b}} \Pi\left(\frac{x}{2}, p^2, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$16. \int_0^x \frac{dx}{(a+b-p^2b+p^2b \cos x) \sqrt{a+b \cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}} \Pi\left(\varphi, p^2, \frac{1}{k}\right) \\ \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$17. \int_0^x \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b \cos x}} = \frac{2}{a-b} \lg \frac{x}{2} \sqrt{a+b \cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$18. \int_0^x \frac{dx}{\sqrt{(a+b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E\left(\frac{x}{2}, k\right) - \frac{2b}{a^2-b^2} \frac{\sin x}{\sqrt{a+b \cos x}} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$19. = \frac{1}{a^2-b^2} \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\varphi, \frac{1}{k}\right) + 2bE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2b}{b^2-a^2} \frac{\sin x}{\sqrt{a+b \cos x}} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$20. \int_0^x \frac{dx}{\sqrt{(a-b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E(\varphi, k) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$21. \int \frac{dx}{(1 \pm \cos ax)^{n+1/2}} = \pm \frac{\sin ax}{na2^n (1 \pm \cos ax)^{n+1/2}} \times$$

$$\times \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} 2^{n-k-2} (1 \pm \cos ax)^{n+1} \right] \pm$$

$$\pm \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{a2^n \sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 \pm \cos ax}}{\sqrt{2} - \sqrt{1 \pm \cos ax}}.$$

1.5.21. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a+b \sin x+c \cos x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{\sqrt{b^2+c^2}-b \sin x-c \cos x}{2 \sqrt{b^2+c^2}}}$,

$$k = \sqrt{\frac{2 \sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}, \quad x_1 = \arcsin \frac{b}{\sqrt{b^2+c^2}}.$$

$$1. \int_x^{x_1} \sqrt{a+b \sin x+c \cos x} dx = 2 \sqrt{a+\sqrt{b^2+c^2}} E(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$2. = 2 \sqrt{2} \sqrt{b^2+c^2} E(\varphi, k) - \frac{\sqrt{2}(\sqrt{b^2+c^2}-a)}{\sqrt{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}; x_1 - \arccos \left(\frac{-a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$3. \int_x^{x_1} \frac{dx}{\sqrt{a+b \sin x+c \cos x}} = \frac{2}{\sqrt{a+\sqrt{b^2+c^2}}} F(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$4. = \frac{\sqrt{2}}{\sqrt{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}; x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$5. \int_x^{x_1} \frac{\sin x dx}{\sqrt{a+b \sin x+c \cos x}} =$$

$$= \frac{\sqrt{2} b}{\sqrt{(b^2+c^2)^3}} \{2E(\varphi, k) - F(\varphi, k)\} + \frac{2c}{b^2+c^2} \sqrt{a+b \sin x+c \cos x}$$

$$\left[0 < |a| < \sqrt{b^2+c^2}; x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$6. \int \frac{(b \cos x - c \sin x) dx}{\sqrt{a+b \sin x+c \cos x}} = 2 \sqrt{a+b \sin x+c \cos x}.$$

$$7. \int_x^{x_1} \frac{\sqrt{b^2+c^2}+b \sin x+c \cos x}{\sqrt{a+b \sin x+c \cos x}} dx = 2 \sqrt{a+\sqrt{b^2+c^2}} E(\varphi, k) -$$

$$- \frac{2(a-\sqrt{b^2+c^2})}{\sqrt{a+\sqrt{b^2+c^2}}} F(\varphi, k) \quad [0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$11. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F\left(\frac{x}{2}, k\right) - 2a(a+b) E\left(\frac{x}{2}, k\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$12. = \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b) F\left(\varphi, \frac{1}{k}\right) - 4aE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2}{3b} \sin x \sqrt{a+b \cos x} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$13. \int_0^x \frac{\cos^2 x \, dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2 \sqrt{a+b}} \left\{ (2a^2 + b^2) F(\psi, k) - 2a(a+b) E(\psi, k) \right\} + \\ + \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} \quad [a > b > 0; 0 \leq x < \pi].$$

$$14. \int (1 \pm \cos ax)^{n+1/2} dx = \pm \sqrt{1 \mp \cos ax} \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k+1} \frac{(1 \mp \cos ax)^k}{2k+1}.$$

$$15. \int_0^x \frac{dx}{(2-p^2+p^2 \cos x) \sqrt{a+b \cos x}} = \frac{1}{\sqrt{a+b}} \Pi\left(\frac{x}{2}, p^2, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$16. \int_0^x \frac{dx}{(a+b-p^2b+p^2b \cos x) \sqrt{a+b \cos x}} = \frac{\sqrt{2}}{(a+b) \sqrt{b}} \Pi\left(\varphi, p^2, \frac{1}{k}\right) \\ \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$17. \int_0^x \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b \cos x}} = \frac{2}{a-b} \lg \frac{x}{2} \sqrt{a+b \cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, k\right) \\ [a > b > 0; 0 \leq x < \pi].$$

$$18. \int_0^x \frac{dx}{\sqrt{(a+b \cos x)^3}} = \frac{2}{(a-b) \sqrt{a+b}} E\left(\frac{x}{2}, k\right) - \frac{2b}{a^2-b^2} \frac{\sin x}{\sqrt{a+b \cos x}} \\ [a > b > 0; 0 \leq x \leq \pi].$$

$$19. = \frac{1}{a^2-b^2} \sqrt{\frac{2}{b}} \left\{ (a-b) F\left(\varphi, \frac{1}{k}\right) + 2bE\left(\varphi, \frac{1}{k}\right) \right\} + \\ + \frac{2b}{b^2-a^2} \frac{\sin x}{\sqrt{a+b \cos x}} \quad \left[b \geq |a| > 0; 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

$$20. \int_0^x \frac{dx}{\sqrt{(a-b \cos x)^3}} = \frac{2}{(a-b) \sqrt{a+b}} E(\psi, k) \quad [a > b > 0; 0 \leq x \leq \pi].$$

$$21. \int \frac{dx}{(1 \pm \cos ax)^{n+1/2}} = \pm \frac{\sin ax}{na2^n (1 \pm \cos ax)^{n+1/2}} \times$$

$$\times \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} 2^{n-k-2} (1 \pm \cos ax)^{n+1} \right] \pm$$

$$\pm \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{a2^n \sqrt{2}} \ln \frac{\sqrt{2} + \sqrt{1 \mp \cos ax}}{\sqrt{2} - \sqrt{1 \mp \cos ax}}.$$

1.5.21. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a+b \sin x+c \cos x}) dx$.

Обозначения: $\varphi = \arcsin \sqrt{\frac{\sqrt{b^2+c^2}-b \sin x-c \cos x}{2 \sqrt{b^2+c^2}}}$,

$$k = \sqrt{\frac{2 \sqrt{b^2+c^2}}{a + \sqrt{b^2+c^2}}}, \quad x_1 = \arcsin \frac{b}{\sqrt{b^2+c^2}}.$$

$$1. \int_x^{x_1} \sqrt{a+b \sin x+c \cos x} dx = 2 \sqrt{a + \sqrt{b^2+c^2}} E(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$2. = 2 \sqrt{2} \sqrt{\sqrt{b^2+c^2}} E(\varphi, k) - \frac{\sqrt{2} (\sqrt{b^2+c^2} - a)}{\sqrt{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(\frac{-a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$3. \int_x^{x_1} \frac{dx}{\sqrt{a+b \sin x+c \cos x}} = \frac{2}{\sqrt{a + \sqrt{b^2+c^2}}} F(\varphi, k)$$

$$[0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$4. = \frac{\sqrt{2}}{\sqrt{b^2+c^2}} F(\varphi, k)$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$5. \int_x^{x_1} \frac{\sin x dx}{\sqrt{a+b \sin x+c \cos x}} =$$

$$= \frac{\sqrt{2} b}{\sqrt{(b^2+c^2)}} \{2E(\varphi, k) - F(\varphi, k)\} + \frac{2c}{b^2+c^2} \sqrt{a+b \sin x+c \cos x}$$

$$\left[0 < |a| < \sqrt{b^2+c^2}, x_1 - \arccos \left(-\frac{a}{\sqrt{b^2+c^2}} \right) \leq x < x_1 \right].$$

$$6. \int \frac{(b \cos x - c \sin x) dx}{\sqrt{a+b \sin x+c \cos x}} = 2 \sqrt{a+b \sin x+c \cos x}.$$

$$7. \int_x^{x_1} \frac{\sqrt{b^2+c^2} + b \sin x + c \cos x}{\sqrt{a+b \sin x+c \cos x}} dx = 2 \sqrt{a + \sqrt{b^2+c^2}} E(\varphi, k) -$$

$$- \frac{2(a - \sqrt{b^2+c^2})}{\sqrt{a + \sqrt{b^2+c^2}}} F(\varphi, k) \quad [0 < \sqrt{b^2+c^2} < a; x_1 - \pi \leq x < x_1].$$

$$8. \int_x^{x_1} \frac{\sqrt{b^2 + c^2} + b \sin x + c \cos x}{\sqrt{a + b \sin x + c \cos x}} dx = 2 \sqrt{2} \sqrt{b^2 + c^2} E(\varphi, k) \\ \left[0 < |a| < \sqrt{b^2 + c^2}; x_1 = \arccos \left(-\frac{a}{\sqrt{b^2 + c^2}} \right) \leq x < x_1 \right].$$

1.5.22. Интегралы вида $\int R(\sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \Delta^n dx = \frac{n-1}{n} (2 - k^2) \int \Delta^{n-2} dx - \\ - \frac{n-2}{n} (1 - k^2) \int \Delta^{n-4} dx + \frac{k^2}{n} \Delta^{n-2} \sin x \cos x.$$

$$2. \int \frac{dx}{\Delta^{n+1}} = \\ = -\frac{k^2 \sin x \cos x}{(n-1)(1-k^2)\Delta^{n-1}} + \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{(n-1)(1-k^2)} \int \frac{dx}{\Delta^{n-3}}.$$

$$3. \int \Delta dx = E(x, k).$$

$$4. \int \Delta^3 dx = \frac{2}{3} (2 - k^2) E(x, k) - \frac{1 - k^2}{3} F(x, k) + \frac{k^2}{3} \Delta \sin x \cos x.$$

$$5. \int \frac{dx}{\Delta} = F(x, k). \quad 6. \int \frac{dx}{\Delta^3} = \frac{1}{1 - k^2} E(x, k) - \frac{k^2}{1 - k^2} \frac{\sin x \cos x}{\Delta}.$$

1.5.23. Интегралы вида $\int R(\sin x, \sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{\sin x dx}{\Delta^p} = \frac{-\cos x}{(p-2)(1-k^2)\Delta^{p-1}} + \frac{p-3}{(p-2)(1-k^2)} \int \frac{\sin x dx}{\Delta^{p-1}}.$$

$$2. \int \frac{\sin x dx}{\Delta^{2n+1}} = -\cos x \sum_{l=0}^{n-1} \frac{2^l (n-1)(n-2) \dots (n-l-1)}{(2n-1)(2n-3) \dots (2n-2l-1)(1-k^2)^{l+1} \Delta^{2n-2l-1}}.$$

$$3. = -\frac{1}{(1-k^2)^n} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-k^2)^l \cos^{2l+1} x}{2l+1 \Delta^{2l+1}}.$$

$$4. \int \frac{\sin^p x dx}{\Delta^{p+3}} = \frac{-\sin^{p-2} x \cos x}{(p+1)k^2(1-k^2)\Delta^{p+1}} + \\ + \frac{2(p-1)k^2 - p + 2}{(p+1)k^2(1-k^2)} \int \frac{\sin^{p-2} x dx}{\Delta^{p+1}} + \frac{p-3}{(p+1)k^2(1-k^2)} \int \frac{\sin^{p-4} x dx}{\Delta^{p-1}}.$$

$$5. \int \Delta \sin x dx = -\frac{\Delta \cos x}{2} - \frac{1-k^2}{2k} \ln(k \cos x + \Delta).$$

$$6. \int \Delta \sin^2 x dx = -\frac{\Delta}{3} \sin x \cos x + \frac{1-k^2}{3k^2} F(x, k) + \frac{2k^2-1}{3k^2} E(x, k).$$

7. $\int \Delta \sin^2 x dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^2} \ln(k \cos x + \Delta).$
8. $\int \Delta \sin^4 x dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x -$
 $-\frac{2(2k^4 - k^2 - 1)}{15k^4} F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x, k).$
9. $\int \Delta^3 \sin x dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3(1 - k^2)^2}{8k} \ln(k \cos x + \Delta).$
10. $\int \Delta^3 \sin^2 x dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x +$
 $+\frac{(1 - k^2)(3 - 4k^2)}{15k^2} F(x, k) - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k).$
11. $\int \Delta^3 \sin^3 x dx = \frac{8k^4 \sin^3 x + 2k^2(5k^2 - 7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x -$
 $-\frac{5k^4 - 9k^2 + 3k^2 + 1}{16k^3} \ln(k \cos x + \Delta).$
12. $\int \frac{\sin^n x dx}{\Delta} = \frac{\sin^{n-2} x}{(n-1)k^2} \Delta \cos x +$
 $+\frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} dx.$
13. $\int \frac{\sin x dx}{\Delta} = \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} = -\frac{1}{k} \ln(k \cos x + \Delta) = \frac{1}{k} \ln(\Delta - k \cos x).$
14. $\int \frac{\sin^2 x dx}{\Delta} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k).$
15. $\int \frac{\sin^3 x dx}{\Delta} = \frac{\Delta \cos x}{2k^2} - \frac{1+k^2}{2k^3} \ln(k \cos x + \Delta).$
16. $\int \frac{\sin^4 x dx}{\Delta} = \frac{\Delta \sin x \cos x}{3k^2} + \frac{2+k^2}{3k^4} F(x, k) - \frac{2(1+k^2)}{3k^4} E(x, k).$
17. $\int \frac{\sin x dx}{\Delta^3} = -\frac{\cos x}{(1-k^2)\Delta}.$
18. $\int \frac{\sin^2 x dx}{\Delta^3} = \frac{1}{k^2(1-k^2)} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{1-k^2} \frac{\sin x \cos x}{\Delta}.$
19. $\int \frac{\sin^3 x dx}{\Delta^3} = -\frac{\cos x}{k^2(1-k^2)\Delta} + \frac{1}{k^2} \ln(k \cos x + \Delta).$
20. $\int \frac{\sin^4 x dx}{\Delta^3} = \frac{2-k^2}{k^2(1-k^2)} E(x, k) - \frac{2}{k^4} F(x, k) - \frac{\sin x \cos x}{k^2(1-k^2)\Delta}.$
21. $\int \frac{\Delta dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln(k \cos x + \Delta).$
22. $\int \frac{\Delta dx}{\sin^2 x} = (1-k^2) F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$
23. $\int \frac{\Delta dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{1-k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$

$$24. \int \frac{\Delta dx}{\sin^4 x} = \\ = \frac{1}{3} [-\Delta \operatorname{ctg}^3 x + (k^2 - 3) \Delta \operatorname{ctg} x + 2(1 - k^2) F(x, k) + (k^2 - 2) E(x, k)].$$

$$25. \int \frac{dx}{\Delta \sin^n x} = \\ = -\frac{\Delta \cos x}{(n-1) \sin^{n-1} x} + \frac{(n-2)(1+k^2)}{n-1} \int \frac{dx}{\Delta \sin^{n-2} x} - \frac{(n-3)k^2}{n-1} \int \frac{dx}{\Delta \sin^{n-4} x}.$$

$$26. \int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} = \ln \frac{\Delta + \cos x}{\sin x}.$$

$$27. \int \frac{dx}{\Delta \sin^3 x} = \int \frac{1 + \operatorname{ctg}^2 x}{\Delta} dx = F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$$

$$28. \int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1+k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$29. \int \frac{dx}{\Delta \sin^4 x} = \\ = \frac{1}{3} [-\Delta \operatorname{ctg}^3 x - \Delta(2k^2 + 3) \operatorname{ctg} x + (k^2 + 2) F(x, k) - 2(k^2 + 1) E(x, k)].$$

1.5.24. Интегралы вида $\int R(\cos x, \sqrt{1 - k^2 \sin^2 x}) dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{\cos x dx}{\Delta^p} = \frac{\sin x}{(p-2) \Delta^{p-2}} + \frac{p-3}{p-2} \int \frac{\cos x dx}{\Delta^{p-2}}.$$

$$2. \int \frac{\cos x dx}{\Delta^{2n+1}} = -\sin x \sum_{l=0}^{n-1} \frac{2^l (n-1)(n-2) \dots (n-l+1)}{(2n-1)(2n-3) \dots (2n-2l-1) \Delta^{2n-2l-1}}.$$

$$3. = \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{k^{2l}}{2l+1} \frac{\sin^{2l+1} x}{\Delta^{2l+1}}.$$

$$4. \int \Delta \cos x dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x).$$

$$5. \int \Delta \cos^2 x dx = \frac{\Delta}{3} \sin x \cos x - \frac{1-k^2}{3k^2} F(x, k) + \frac{k^2+1}{3k^2} E(x, k).$$

$$6. \int \Delta \cos^3 x dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^2} \arcsin(k \sin x).$$

$$7. \int \Delta \cos^4 x dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x + \\ + \frac{2(1-k^2)(1-3k^2)}{15k^4} F(x, k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x, k).$$

$$8. \int \Delta^3 \cos x dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin(k \sin x).$$

9. $\int \Delta^3 \cos^2 x \, dx = \frac{-3k^2 \sin^2 x + k^2 + 6}{15} \Delta \sin x \cos x -$
 $-\frac{(4-k^2)(k^2+3)}{15k^3} F(x, k) - \frac{2k^4-7k^2-3}{15k^3} E(x, k).$
10. $\int \Delta^3 \cos^3 x \, dx =$
 $= \frac{8k^4 \sin^4 x - 2k^4(6k^2+7) \sin^2 x + 30k^2+3}{48k^3} \Delta \sin x + \frac{6k^3-1}{16k^3} \arcsin(k \sin x).$
11. $\int \frac{\cos^n x \, dx}{\Delta} = \frac{\cos^{n-2} x}{(n-1)k^2} \Delta \sin x + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} \, dx +$
 $+ \frac{n-3}{n-1} \frac{1-k^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} \, dx.$
12. $\int \frac{\cos x \, dx}{\Delta} = \frac{1}{k} \arcsin(k \sin x) = \frac{1}{k} \operatorname{arctg} \frac{k \sin x}{\Delta}.$
13. $\int \frac{\cos^2 x \, dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{1-k^2}{k^2} F(x, k).$
14. $\int \frac{\cos^3 x \, dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{2k^2-1}{2k^3} \arcsin(k \sin x).$
15. $\int \frac{\cos x \, dx}{\Delta^3} = \frac{\sin x}{\Delta}.$
16. $\int \frac{\cos^2 x \, dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}.$
17. $\int \frac{\cos^3 x \, dx}{\Delta^3} = -\frac{(1-k^2) \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin(k \sin x).$
18. $\int \frac{\cos^4 x \, dx}{\Delta^3} = \frac{2-k^2}{k^4} E(x, k) - \frac{2(1-k^2)}{k^4} F(x, k) - \frac{(1-k^2) \sin x \cos x}{k^2 \Delta}.$
19. $\int \frac{\cos^4 x \, dx}{\Delta} = \frac{\Delta \sin x \cos x}{3k^2} + \frac{4k^2-2}{3k^4} E(x, k) + \frac{3k^4-5k^2+2}{3k^4} F(x, k).$
20. $\int \frac{\Delta \, dx}{\cos x} = \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} + k \arcsin(k \sin x).$
21. $\int \frac{\Delta \, dx}{\cos^3 x} = F(x, k) - E(x, k) + \Delta \operatorname{tg} x.$
22. $\int \frac{\Delta \, dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
23. $\int \frac{\Delta \, dx}{\cos^4 x} = \frac{1}{3} \left\{ \left[\operatorname{tg}^3 x - \frac{2k^2-3}{1-k^2} \operatorname{tg} x \right] \Delta + 2F(x, k) + \frac{k^2-2}{1-k^2} E(x, k) \right\}.$
24. $\int \frac{dx}{\Delta \cos x} = -\frac{1}{2 \sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$
25. $\int \frac{dx}{\Delta \cos^2 x} = F(x, k) - \frac{1}{1-k^2} E(x, k) + \frac{1}{1-k^2} \Delta \operatorname{tg} x.$
26. $\int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2(1-k^2) \cos^2 x} + \frac{2k^2-1}{4(1-k^2)^{3/2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}.$

$$27. \int \frac{dx}{\Delta \cos^4 x} = \\ = \frac{1}{3(1-k^2)} \left[\Delta \operatorname{tg}^3 x - \frac{5k^2-3}{1-k^2} \Delta \operatorname{tg} x - (3k^2-2) F(x, k) + \frac{2(2k^2-1)}{1-k^2} E(x, k) \right].$$

$$28. \int \frac{dx}{\Delta \cos^n x} = \\ = \frac{\Delta \sin x}{(n-1)(1-k^2) \cos^{n-1} x} - \frac{(n-2)(2k^2-1)}{(n-1)(1-k^2)} \int \frac{dx}{\Delta \cos^{n-2} x} + \\ + \frac{(n-3)k^2}{(n-1)(1-k^2)} \int \frac{dx}{\Delta \cos^{n-4} x}.$$

1.5.25. Интегралы вида $\int \sin^m x \cos^n x \sqrt{(1-k^2 \sin^2 x)^p} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \Delta^p \sin^m x \cos^n x dx = \frac{1}{(m+n+p)k^2} \{ \Delta^{p+2} \sin^{m-3} x \cos^{n+1} x + \\ + [m+n-2+(m+p-1)k^2] \int \Delta^p \sin^{m-2} x \cos^n x dx - \\ - (m-3) \int \Delta^p \sin^{m-4} x \cos^n x dx \}.$
2. $= \frac{1}{(m+n+p)k^2} \{ \Delta^{p+2} \sin^{m+1} x \cos^{n-3} x + \\ + [(n+p-1)k^2 - (m+n-2)(1-k^2)] \int \Delta^p \sin^m x \cos^{n-2} x dx + \\ + (n-3)(1-k^2) \int \Delta^p \sin^m x \cos^{n-4} x dx \} \quad [m+n+p \neq 0].$
3. $\int \Delta^p \sin x \cos^n x dx = -\frac{\Delta^{p+2} \cos^{n-1} x}{(n+p+1)k^2} - \frac{(n-1)(1-k^2)}{(n+p+1)k^2} \int \Delta^p \cos^{n-2} x \sin x dx.$
4. $\int \Delta^p \sin^m x \cos x dx = -\frac{\Delta^{p+2} \sin^{m-1} x}{(m+p+1)k^2} + \frac{m-1}{(m+p+1)k^2} \int \Delta^p \sin^{m-2} x \cos x dx.$
5. $\int \Delta^p \sin^3 x \cos^n x dx = \\ = \frac{(n+p+1)k^2 \cos^2 x + [(p+2)k^2 + n+1] \Delta^{p+2} \cos^{n-1} x - \\ - [(p+2)k^2 + n+1](n-1)(1-k^2)}{(n+p+1)(n+p+3)k^4} \int \Delta^p \cos^{n-2} x \sin x dx.$
6. $\int \Delta^p \sin^m x \cos^3 x dx = \\ = \frac{(m+p+1)k^2 \sin^2 x - [(p-m+1)k^2 + m+1] \Delta^{p+2} \sin^{m-1} x + \\ + [(p-m+1)k^2 + m+1](m-1)}{(m+p+1)(m+p+3)k^4} \int \Delta^p \sin^{m-2} x \cos x dx.$
7. $\int \Delta \sin x \cos x dx = -\frac{\Delta^3}{3k^2}.$
8. $\int \Delta \sin x \cos^2 x dx = -\frac{2k^2 \cos^2 x + 1 - k^2}{8k^2} \Delta \cos x + \frac{(1-k^2)^2}{8k^4} \ln(k \cos x + \Delta).$

- $$9. \int \Delta \sin x \cos^3 x dx = - \frac{3k^4 \sin^4 x - k^2 (5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta.$$
- $$10. \int \Delta \sin x \cos^4 x dx =$$
- $$= - \frac{8k^4 \sin^4 x + 2k^2 (7k^2 + 1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x -$$
- $$- \frac{(1-k^2)^2}{16k^3} \ln(k \cos x + \Delta).$$
- $$11. \int \Delta \sin^2 x \cos x dx = \frac{2k^2 \sin^2 x - 1}{8k^3} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x).$$
- $$12. \int \Delta \sin^2 x \cos^2 x dx = - \frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x -$$
- $$- \frac{(1-k^2)(2-k^2)}{15k^3} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k).$$
- $$13. \int \Delta \sin^2 x \cos^3 x dx =$$
- $$= - \frac{8k^4 \sin^4 x + 2k^2 (6k^2 + 1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x +$$
- $$+ \frac{2k^2 - 1}{16k^3} \arcsin(k \sin x).$$
- $$14. \int \Delta \sin^3 x \cos x dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$$
- $$15. \int \Delta \sin^3 x \cos^2 x dx =$$
- $$= \frac{8k^4 \sin^4 x - 2k^2 (k^2 + 1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x +$$
- $$+ \frac{(1-k^2)^2 (k^2 + 1)}{16k^3} \ln(k \cos x + \Delta).$$
- $$16. \int \Delta \sin^4 x \cos x dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^3} \arcsin(k \sin x).$$
- $$17. \int \Delta^3 \sin x \cos x dx = - \frac{\Delta^3}{5k^2}.$$
- $$18. \int \Delta^3 \sin x \cos^3 x dx =$$
- $$= - \frac{8k^4 \sin^4 x + 2k^2 (k^2 + 7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \Delta \cos x +$$
- $$+ \frac{(1-k^2)^2}{16k^3} \ln(k \cos x + \Delta).$$
- $$19. \int \Delta^3 \sin^2 x \cos x dx =$$
- $$= - \frac{8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + \frac{1}{16k^3} \arcsin(k \sin x).$$

15.26. Интегралы вида $\int \frac{\sin^m x}{\cos^n x} \sqrt{1-k^2 \sin^2 x} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

1. $\int \frac{\sin x}{\cos x} \Delta dx = \int \Delta \operatorname{tg} x dx = -\Delta + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
2. $\int \frac{\sin x}{\cos^2 x} \Delta dx = \frac{\Delta}{\cos x} - k \ln(k \cos x + \Delta).$
3. $\int \frac{\sin x}{\cos^3 x} \Delta dx = \frac{\Delta}{2 \cos^2 x} + \frac{k^2}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
4. $\int \frac{\sin x}{\cos^4 x} \Delta dx = \frac{-(2k^2+1)k^2 \sin^2 x + 3k^4 - k^2 + 1}{3(1-k^2) \cos^3 x} \Delta.$
5. $\int \frac{\sin^2 x}{\cos x} \Delta dx =$
 $= -\frac{\Delta \sin x}{2} + \frac{2k^2-1}{2k} \arcsin(k \sin x) + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
6. $\int \frac{\sin^2 x}{\cos^3 x} \Delta dx = \int \Delta \operatorname{tg}^2 x dx = \Delta \operatorname{tg} x + F(x, k) - 2E(x, k).$
7. $\int \frac{\sin^2 x}{\cos^5 x} \Delta dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2-1}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - k \arcsin(k \sin x).$
8. $\int \frac{\sin^2 x}{\cos x} \Delta dx = -\frac{k^2 \sin x + 3k^2 - 1}{3k^2} \Delta + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$
9. $\int \frac{\sin^2 x}{\cos^3 x} \Delta dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2-1}{2k} \ln(k \cos x + \Delta).$
10. $\int \frac{\sin^4 x}{\cos x} \Delta dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \Delta \sin x +$
 $+ \frac{8k^4 - 4k^2 - 1}{8k^2} \arcsin(k \sin x) + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$
11. $\int \frac{\cos x}{\sin x} \Delta dx = \int \Delta \operatorname{ctg} x dx = \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$
12. $\int \frac{\cos^2 x}{\sin x} \Delta dx = \frac{\Delta \cos x}{2} + \frac{k^2+1}{2k} \ln(k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
13. $\int \frac{\cos^3 x}{\sin x} \Delta dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$
14. $\int \frac{\cos^4 x}{\sin x} \Delta dx = \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \Delta \cos x +$
 $+ \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^2} \ln(k \cos x + \Delta).$
15. $\int \frac{\cos x}{\sin^2 x} \Delta dx = -\frac{\Delta}{\sin x} - k \arcsin(k \sin x).$

$$16. \int \frac{\cos^2 x}{\sin^2 x} \Delta dx = \int \operatorname{ctg}^2 x \Delta dx = -\Delta \operatorname{ctg} x + (1-k^2) F(x, k) - 2E(x, k).$$

$$17. \int \frac{\cos^3 x}{\sin^2 x} \Delta dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2 + 1}{2k} \arcsin(k \sin x).$$

$$18. \int \frac{\cos x}{\sin^3 x} \Delta dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$$

$$19. \int \frac{\cos x}{\sin^4 x} \Delta dx = -\frac{\Delta^3}{3 \sin^3 x}.$$

$$20. \int \frac{\cos^3 x}{\sin^3 x} \Delta dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2 + 1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln(k \cos x + \Delta).$$

1.5.27. Интегралы вида $\int \frac{\sin^p x \cos^q x}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \int \frac{\sin^p x \cos^q x}{\Delta^r} dx = \\ = \frac{-k^2 \sin^{p+1} x \cos^{q+1} x}{(r-2)(1-k^2) \Delta^{r-2}} + \frac{p-q-(2-k^2)(p-r+3)}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-2}} dx + \\ + \frac{p+q-r+4}{(r-2)(1-k^2)} \int \frac{\sin^p x \cos^q x}{\Delta^{r-4}} dx.$$

$$2. = \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^r} dx - \frac{1}{k^2} \int \frac{\sin^{p-2} x \cos^q x}{\Delta^{r-2}} dx.$$

$$3. = \frac{k^2-1}{k^2} \int \frac{\sin^p x \cos^{q-2} x}{\Delta^r} dx + \frac{1}{k^2} \int \frac{\sin^p x \cos^{q-2} x}{\Delta^{r-2}} dx.$$

$$4. \int \frac{\sin x \cos x dx}{\Delta^r} = \frac{1}{(r-2)k^2 \Delta^{r-2}}.$$

$$5. \int \frac{\sin x \cos x dx}{\Delta} = -\frac{\Delta}{k^2}.$$

$$6. \int \frac{\sin x \cos^3 x dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{1-k^2}{2k^3} \ln(k \cos x + \Delta).$$

$$7. \int \frac{\sin x \cos^3 x dx}{\Delta} = -\frac{1}{3k^4} (k^2 \cos^3 x - 2 + 2k^2) \Delta.$$

$$8. \int \frac{\sin x \cos^5 x dx}{\Delta} = \frac{3-5k^2+2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^3} \ln(k \cos x + \Delta).$$

$$9. \int \frac{\sin^2 x \cos x dx}{\Delta} = -\frac{\Delta \sin x}{2k^3} + \frac{\arcsin(k \sin x)}{2k^3}.$$

$$10. \int \frac{\sin^2 x \cos^3 x dx}{\Delta} = -\frac{\Delta \sin x \cos x}{3k^3} + \frac{2-k^2}{3k^4} E(x, k) + \frac{2k^2-2}{3k^4} F(x, k).$$

$$11. \int \frac{\sin^2 x \cos^3 x dx}{\Delta} = -\frac{2k^2 \cos^3 x + 2k^2 - 3}{8k^4} \Delta \sin x + \frac{4k^2-3}{8k^3} \arcsin(k \sin x).$$

- $$12. \int \frac{\sin^2 x \cos^4 x dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \Delta \sin x \cos x +$$
- $$+ \frac{9k^4 - 17k^2 + 8}{15k^6} F(x, k) - \frac{3k^4 - 13k^2 + 8}{15k^6} E(x, k).$$
- $$13. \int \frac{\sin^3 x \cos x dx}{\Delta} = -\frac{1}{3k^4} (2 + k^2 \sin^2 x) \Delta.$$
- $$14. \int \frac{\sin^3 x \cos^2 x dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \Delta \cos x - \frac{k^4 + 2k^2 - 3}{8k^6} \ln(k \cos x + \Delta).$$
- $$15. \int \frac{\sin^3 x \cos^3 x dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4 - 4k^2) \sin^2 x - 10k^2 + 8}{15k^6} \Delta.$$
- $$16. \int \frac{\sin^3 x \cos^4 x dx}{\Delta} =$$
- $$= \frac{8k^4 \sin^4 x - 2k^2 (7k^2 - 5) \sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \Delta \cos x -$$
- $$- \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln(k \cos x + \Delta).$$
- $$17. \int \frac{\sin^4 x \cos x dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \Delta \sin x + \frac{3}{8k^5} \arcsin(k \sin x).$$
- $$18. \int \frac{\sin^4 x \cos^2 x dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \Delta \sin x \cos x +$$
- $$+ \frac{k^4 + 7k^2 - 8}{15k^6} F(x, k) - \frac{2k^4 + 3k^2 - 8}{15k^6} E(x, k).$$
- $$19. \int \frac{\sin^4 x \cos^3 x dx}{\Delta} =$$
- $$= \frac{8k^4 \sin^4 x - 2k^2 (6k^2 - 5) \sin^2 x - 18k^2 + 15}{48k^6} \Delta \sin x + \frac{6k^2 - 5}{16k^7} \arcsin(k \sin x).$$
- $$20. \int \frac{\sin^m x \cos^n x dx}{\Delta^2} = \frac{\sin^{m-1} x \cos^{n-1} x}{k^2 \Delta} -$$
- $$- \frac{m-1}{k^2} \int \frac{\sin^{m-2} x \cos^n x dx}{\Delta} + \frac{n-1}{k^2} \int \frac{\sin^m x \cos^{n-2} x dx}{\Delta} dx.$$
- $$21. \int \frac{\sin x \cos x dx}{\Delta^2} = \frac{1}{k^2 \Delta}.$$
- $$22. \int \frac{\sin x \cos^2 x dx}{\Delta^2} = \frac{\cos x}{k^2 \Delta} - \frac{1}{k^2} \ln(k \cos x + \Delta).$$
- $$23. \int \frac{\sin x \cos^3 x dx}{\Delta^2} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}.$$
- $$24. \int \frac{\sin x \cos^4 x dx}{\Delta^2} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3(1 - k^2)}{2k^6} \ln(k \cos x + \Delta).$$
- $$25. \int \frac{\sin^2 x \cos x dx}{\Delta^2} = \frac{\sin x}{k^2 \Delta} - \frac{1}{k^5} \arcsin(k \sin x).$$
- $$26. \int \frac{\sin^2 x \cos^2 x dx}{\Delta^2} = \frac{2 - k^2}{k^4} F(x, k) - \frac{2}{k^4} E(x, k) + \frac{\sin x \cos x}{k^2 \Delta}.$$
- $$27. \int \frac{\sin^2 x \cos^3 x dx}{\Delta^2} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^6} \arcsin(k \sin x).$$

$$28. \int \frac{\sin^3 x \cos x dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \Delta}.$$

$$29. \int \frac{\sin^3 x \cos^3 x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta).$$

$$30. \int \frac{\sin^4 x \cos x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \operatorname{arcsin}(k \sin x).$$

1.5.28. Интегралы вида $\int \frac{\sin^p x}{\cos^q x \sqrt{1 - k^2 \sin^2 x}} \frac{dx}{\Delta^r},$

$$\int \frac{\cos^p x}{\sin^q x \sqrt{1 - k^2 \sin^2 x}} \frac{dx}{\Delta^r}.$$

Условие: $0 < k^2 < 1.$

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}.$

$$1. \int \frac{\sin^p x dx}{\cos^q x \Delta} = \frac{\sin^{p+1} x}{(q-1)(1-k^2) \cos^{q-1} x \Delta^{r-2}} - \frac{p-q+2-(p-2q-r+5)k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x dx}{\cos^{q-2} x \Delta^r} - \frac{(p-q-r+4)k^2}{(q-1)(1-k^2)} \int \frac{\sin^p x dx}{\cos^{q-4} x \Delta^r}.$$

$$2. \int \frac{\cos^q x dx}{\sin^p x \Delta^r} = \frac{-\cos^{q+1} x}{(p-1) \sin^{p-1} x \Delta^{r-2}} + \frac{p-q-2+(p+r-3)k^2}{p-1} \times \\ \times \int \frac{\cos^q x dx}{\sin^{p-2} x \Delta^r} + \frac{(p-q+r-4)k^2}{p-1} \int \frac{\cos^q x dx}{\sin^{p-4} x \Delta^r}.$$

$$3. \int \frac{\sin x dx}{\cos^{2m} x \Delta^{2n+1}} = \\ = (-1)^{m+n} \frac{k^{2(m+n-1)}}{(1-k^2)^{m+n}} \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{(-1)^l}{(2n-2l-1)2^l} \left(\frac{\cos x}{\Delta} \right)^{2n-2l-1}.$$

$$4. \int \frac{\cos x dx}{\sin^{2n} x \Delta^{2n+1}} = \sum_{l=0}^{m+n-1} \binom{m+n-1}{l} \frac{k^{2(m+n-l-1)}}{2n-2l-1} \left(\frac{\sin x}{\Delta} \right)^{2n-2l-1}.$$

$$5. \int \frac{\sin x dx}{\cos x \Delta} = \int \operatorname{tg} x \frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$6. \int \frac{\sin x dx}{\cos^2 x \Delta} = \frac{\Delta}{(1-k^2) \cos x}.$$

$$7. \int \frac{\sin x dx}{\cos^3 x \Delta} = \int \operatorname{tg} x (1 + \operatorname{tg}^2 x) \frac{dx}{\Delta} = \\ = \frac{\Delta}{2(1-k^2) \cos^2 x} - \frac{k^2}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$8. \int \frac{\sin x dx}{\cos^4 x \Delta} = -\frac{2k^2 \cos^2 x - 1 + k^2}{3(1-k^2)^2 \cos^3 x} \Delta.$$

$$9. \int \frac{\sin^2 x dx}{\cos x \Delta} = \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - \frac{1}{k} \operatorname{arcsin}(k \sin x).$$

$$10. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{tg}^2 x}{\Delta} dx = \frac{\Delta}{(1-k^2)} \operatorname{tg} x - \frac{1}{(1-k^2)} E(x, k).$$

$$11. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2(1-k^2) \cos^2 x} - \frac{1}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}.$$

$$12. \int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$$

$$13. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{(1-k^2) \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$14. \int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \\ = \frac{\Delta \sin x}{2k^2} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} - \frac{2k^2 + 1}{2k^3} \arcsin(k \sin x).$$

$$15. \int \frac{\cos x}{\sin x} \frac{dx}{\Delta} = \int \operatorname{ctg} x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$$

$$16. \int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$17. \int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta}.$$

$$18. \int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2 - 1}{2k^3} \ln(k \cos x + \Delta).$$

$$19. \int \frac{\cos x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x}.$$

$$20. \int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{ctg}^2 x}{\Delta} dx = -\Delta \operatorname{ctg} x - E(x, k).$$

$$21. \int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x} - \frac{1}{k} \arcsin(k \sin x).$$

$$22. \int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$$

$$23. \int \frac{\cos^2 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{1-k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$24. \int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta.$$

$$25. \int \operatorname{tg}^n x \frac{dx}{\Delta} = \frac{\operatorname{tg}^{n-2} x}{(n-1)(1-k^2)} \frac{\Delta}{\cos^2 x} - \\ - \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-4} x}{\Delta} dx.$$

$$26. \int \operatorname{ctg}^n x \frac{dx}{\Delta} = -\frac{\operatorname{ctg}^{n-2} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1} (2-k^2) \times \\ \times \int \frac{\operatorname{ctg}^{n-2} x}{\Delta} dx - \frac{n-3}{n-1} (1-k^2) \int \frac{\operatorname{ctg}^{n-4} x}{\Delta} dx.$$

1.5.29. Интегралы вида $\int \frac{\sqrt{(1-k^2 \sin^2 x)^r}}{\sin^m x \cos^n x} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

1. $\int \frac{\Delta dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}$.
2. $\int \frac{\Delta dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$.
3. $\int \frac{\Delta dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta} + \frac{2-k^2}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}$.
4. $\int \frac{\Delta dx}{\sin x \cos^4 x} = \frac{(2k^2-3) \sin^2 x - 3k^2 + 4}{3(1-k^2) \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$.
5. $\int \frac{\Delta dx}{\sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1+k^2}{2 \sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}$.
6. $\int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \left(\frac{1}{1-k^2} \operatorname{tg} x - \operatorname{ctg} x \right) \Delta + 2F(x, k) - \frac{2-k^2}{1-k^2} E(x, k)$.
7. $\int \frac{\Delta dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2-3}{4 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x}$.
8. $\int \frac{\Delta dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}} + \frac{k^2-2}{4} \ln \frac{1+\Delta}{1-\Delta}$.
9. $\int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2-3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$.
10. $\int \frac{\Delta dx}{\sin^4 x \cos x} = -\frac{(3-k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta - \sqrt{1-k^2} \sin x}{\Delta + \sqrt{1-k^2} \sin x}$.

1.5.30. Интегралы вида $\int \frac{dx}{\sin^m x \cos^n x \sqrt{(1-k^2 \sin^2 x)^r}}$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

1. $\int \frac{dx}{\Delta \sin x \cos x} =$
 $= \int (\operatorname{tg} x + \operatorname{ctg} x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{1}{2 \sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}$.
2. $\int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{(1-k^2) \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$.
3. $\int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\operatorname{ctg} x + 2 \operatorname{tg} x + \operatorname{tg}^3 x) \frac{dx}{\Delta} =$
 $= \frac{\Delta}{2(1-k^2) \cos^2 x} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta} + \frac{2-3k^2}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}$.

4. $\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2 - 3) \sin^2 x - 6k^2 + 4}{3(1 - k^2)^2 \cos^3 x} \Delta - \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$
5. $\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta - \sqrt{1 - k^2} \sin x}{\Delta + \sqrt{1 - k^2} \sin x}.$
6. $\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \left(\frac{\lg x}{1 - k^2} - \operatorname{ctg} x \right) \Delta + \frac{k^2 - 2}{1 - k^2} E(x, k) + 2F(x, k).$
7. $\int \frac{dx}{\Delta \sin^2 x \cos^3 x} =$
 $= \frac{(3 - 2k^2) \sin^2 x - 2(1 - k^2)}{2(1 - k^2) \sin x \cos^2 x} \Delta - \frac{4k^2 - 3}{4(1 - k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1 - k^2} \sin x}{\Delta - \sqrt{1 - k^2} \sin x}.$
8. $\int \frac{dx}{\Delta \sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$
9. $\int \frac{dx}{\Delta \sin^3 x \cos^3 x} = \frac{(3 - k^2) \sin^2 x - 1 + k^2}{2(1 - k^2) \sin^2 x \cos x} \Delta + \frac{k^2 + 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$
10. $\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3 + 2k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta - \sqrt{1 - k^2} \sin x}{\Delta + \sqrt{1 - k^2} \sin x}.$

1.5.31. Интегралы вида

$$\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}, \sqrt{1 - q^2 \sin^2 x}) dx.$$

Условия: $0 < p^2 < q^2 < 1$, $0 < x \leq \pi/2$.

Обозначение: $\varphi = \arcsin \frac{\sqrt{1 - p^2} \sin x}{\sqrt{1 - p^2 \sin^2 x}}.$

1. $\int_0^x \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{1}{\sqrt{1 - p^2}} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$
2. $\int_0^x \frac{\lg x dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} =$
 $= \frac{\lg x \sqrt{1 - q^2 \sin^2 x}}{(1 - q^2) \sqrt{1 - p^2 \sin^2 x}} - \frac{1}{(1 - q^2) \sqrt{1 - p^2}} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$
3. $\int_0^x \frac{\sin^2 x dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)^3}} = \frac{\sqrt{1 - p^2}}{(1 - q^2)(q^2 - p^2)} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) -$
 $- \frac{1}{(q^2 - p^2) \sqrt{1 - p^2}} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) - \frac{\sin x \cos x}{(1 - q^2) \sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}}.$
4. $\int_0^x \frac{\cos^2 x dx}{\sqrt{(1 - p^2 \sin^2 x)^3(1 - q^2 \sin^2 x)}} =$
 $= \frac{\sqrt{1 - p^2}}{q^2 - p^2} E\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right) - \frac{1 - q^2}{(q^2 - p^2) \sqrt{1 - p^2}} F\left(\varphi, \sqrt{\frac{q^2 - p^2}{1 - p^2}}\right).$

$$5. \int_0^x \frac{1}{1-\rho^2 \sin^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-\rho^2 \sin^2 x}} dx = \frac{1}{\sqrt{1-\rho^2}} E\left(\varphi, \sqrt{\frac{q^2-\rho^2}{1-\rho^2}}\right).$$

$$6. \int_0^x \sqrt{\frac{1-\rho^2 \sin^2 x}{(1-q^2 \sin^2 x)^3}} dx = \\ = \frac{\sqrt{1-\rho^2}}{1-q^2} E\left(\varphi, \sqrt{\frac{q^2-\rho^2}{1-\rho^2}}\right) - \frac{q^2-\rho^2}{1-q^2} \frac{\sin x \cos x}{\sqrt{(1-\rho^2 \sin^2 x)(1-q^2 \sin^2 x)}}.$$

$$7. \int_0^x \frac{1}{1+(\rho^2 r^2 - \rho^2 - r^2) \sin^2 x} \sqrt{\frac{1-\rho^2 \sin^2 x}{1-q^2 \sin^2 x}} dx = \\ = \frac{1}{\sqrt{1-\rho^2}} \Pi\left(\varphi, r^2, \sqrt{\frac{q^2-\rho^2}{1-\rho^2}}\right).$$

1.5.32. Интегралы вида $\int \frac{(a + \sin x)^p}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \int \frac{(a + \sin x)^p}{\Delta} dx = \\ = \frac{1}{(p-1)k^2} \left[(a + \sin x)^{p-3} \cos x \Delta + 2(2p-3)ak^2 \int \frac{(a + \sin x)^{p-4}}{\Delta} dx + \right. \\ \left. + (p-2)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p-2}}{\Delta} dx - a(2p-5)(1+k^2-2a^2k^2) \times \right. \\ \left. \times \int \frac{(a + \sin x)^{p-3}}{\Delta} dx - (p-3)(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^{p-4}}{\Delta} dx \right] \\ \left[a^2 \neq 1, \frac{1}{k^2} \text{ при } p = -1, -2, \dots \right].$$

$$2. \int \frac{a + \sin x}{\Delta} dx = aF(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$3. \int \frac{(a + \sin x)^2}{\Delta} dx = \frac{1+k^2a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$4. \int \frac{dx}{(a + \sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^{n-1}} - \right. \\ - (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a + \sin x)^{n-1} \Delta} - \\ - (n-2)(6a^2k^2-k^2-1) \int \frac{dx}{(a + \sin x)^{n-2} \Delta} - \\ \left. - (10-4n)ak^2 \int \frac{dx}{(a + \sin x)^{n-3} \Delta} - (n-3)k^2 \int \frac{dx}{(a + \sin x)^{n-4} \Delta} \right].$$

$$5. \int \frac{dx}{(a + \sin x) \Delta} = \\ = \frac{1}{a} \Pi\left(x, \frac{1}{a^2}, k\right) + \frac{1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2} \Delta - \sqrt{1-k^2a^2} \cos x}{\sqrt{1-a^2} \Delta + \sqrt{1-k^2a^2} \cos x}.$$

$$6. \int \frac{dx}{(a + \sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\Delta \cos x}{a + \sin x} - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x) \Delta} - 2ak^2 \int \frac{(a + \sin x) dx}{\Delta} + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta} \right].$$

$$7. \int \frac{dx}{(a + \sin x)^3 \Delta} = \frac{1}{2(1-a^2)(1-a^2k^2)} \left[-\frac{\Delta \cos x}{(a + \sin x)^2} - 3a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)^2 \Delta} - (6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x) \Delta} + 2ak^2 F(x, k) \right].$$

$$8. \int \frac{dx}{(1 \pm \sin x)^n \Delta} = \frac{1}{(2n-1)(1-k^2)} \left[\mp \frac{\Delta \cos x}{(1 \pm \sin x)^n} + (n-1)(1-5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} + 2(2n-3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} - (n-2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right].$$

$$9. \int \frac{dx}{(1 \pm \sin x) \Delta} = \frac{\mp \Delta \cos x}{(1-k^2)(1 \pm \sin x)} + F(x, k) - \frac{1}{1-k^2} E(x, k).$$

$$10. \int \frac{dx}{(1 \pm \sin x)^2 \Delta} = \frac{1}{3(1-k^2)^2} \left[\mp \frac{(1-k^2) \Delta \cos x}{(1 \pm \sin x)^2} \mp \frac{(1-5k^2) \Delta \cos x}{1 \pm \sin x} + (1-3k^2)(1-k^2) F(x, k) - (1-5k^2) E(x, k) \right].$$

$$11. \int \frac{dx}{(1 \pm k \sin x)^n \Delta} = \frac{1}{(2n-1)(1-k^2)} \left[\pm \frac{k \Delta \cos x}{(1 \pm k \sin x)^n} + (n-1)(5-k^2) \int \frac{dx}{(1 \pm k \sin x)^{n-1} \Delta} - 2(2n-3) \int \frac{dx}{(1 \pm k \sin x)^{n-2} \Delta} + (n-2) \int \frac{dx}{(1 \pm k \sin x)^{n-3} \Delta} \right].$$

$$12. \int \frac{dx}{(1 \pm k \sin x) \Delta} = \pm \frac{k \Delta \cos x}{(1-k^2)(1 \pm k \sin x)} + \frac{1}{1-k^2} E(x, k).$$

$$13. \int \frac{dx}{(1 \pm k \sin x)^2 \Delta} = \frac{1}{3(1-k^2)^2} \left[\pm \frac{k(1-k^2) \Delta \cos x}{(1 \pm k \sin x)^2} \pm \frac{k(5-k^2) \Delta \cos x}{1 \pm k \sin x} - 2(1-k^2) F(x, k) + (5-k^2) E(x, k) \right].$$

1.5.33. Интегралы вида $\int \frac{(1+a \sin^2 x)^p}{\sqrt{1-k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1-k^2 \sin^2 x}$.

$$1. \int \frac{(1+a \sin^2 x)^p}{\Delta} dx = \frac{1}{(2p-1)k^2} \left[a^2(1+a \sin^2 x)^{p-2} \Delta \sin x \cos x + (2p-2)(3k^2+k^2a+a) \int \frac{(1+a \sin^2 x)^{p-1}}{\Delta} dx - (2p-3)(3k^2+2ak^2+2a+a^2) \int \frac{(1+a \sin^2 x)^{p-2}}{\Delta} dx + 2(p-2)a(a+1)(k^2+a) \int \frac{(1+a \sin^2 x)^{p-3}}{\Delta} dx \right].$$

$$2. \int \frac{1+a \sin^2 x}{\Delta} dx = \left(1 + \frac{a}{k^2}\right) F(x, k) - \frac{a}{k^2} E(x, k).$$

$$3. \int \frac{(1+a \sin^2 x)^2}{\Delta} dx = \frac{a^2}{3k^2} \Delta \sin x \cos x + \\ + \left[1 + \frac{2a}{k^2} + \frac{(2+k^2)a^2}{3k^4}\right] F(x, k) - \left[\frac{2a}{k^2} + \frac{2(k^2+1)a^2}{3k^4}\right] E(x, k).$$

$$4. \int \frac{dx}{(a + \sin^2 x)^n \Delta} = \frac{1}{2(a-b)(b+a)(a+k^2)} \left[\frac{\Delta \sin x \cos x}{(1+a \sin^2 x)^{n-1} \Delta} + \right. \\ + (2n-3)(a^2+2a+2ak^2+3k^2) \int \frac{dx}{(1+a \sin^2 x)^{n-1} \Delta} - \\ - (2n-4)(k^2a+a+3k^2) \int \frac{dx}{(1+a \sin^2 x)^{n-2} \Delta} + \\ \left. + (2n-5)k^2 \int \frac{dx}{(1+a \sin^2 x)^{n-3} \Delta} \right].$$

$$5. \int \frac{dx}{(1+a \sin^2 x) \Delta} = \Pi(x, -a, k).$$

$$6. \int \frac{dx}{(1+a \sin^2 x)^2 \Delta} = \\ = \frac{1}{2(1+a)(k^2+a^2)} \left[\frac{a^2 \Delta \sin x \cos x}{1+a \sin^2 x} + (a+k^2) F(x, k) + a E(x, k) + \right. \\ \left. + (a^2+2a+2ak^2+3k^2) \Pi(x, -a, k) \right].$$

$$7. \int_0^x \frac{\sin x dx}{(1+a \sin^2 x) \Delta} = \frac{1}{k(1+a)(k^2+a)} \ln \frac{\sqrt{1+a} \Delta - \sqrt{k^2+a} \cos x}{(\sqrt{1+a} - \sqrt{k^2+a}) \sqrt{1+a \sin^2 x}} \\ [a > -k^2].$$

$$8. = \frac{1}{2\sqrt{-(1+a)(k^2+a)}} \times \\ \times \left[\arcsin \frac{(1+a)\Delta^2 + (k^2+a)\cos^2 x}{(1-k^2)(1+a \sin^2 x)} - \arcsin \frac{1+2a+k^2}{1-k^2} \right] \quad [-1 < a < -k^2].$$

$$9. = -\frac{1}{\sqrt{(1+a)(k^2+a)}} \ln \frac{\sqrt{-(k^2+a)} \cos x - \sqrt{-(1+a)} \Delta}{[\sqrt{-(k^2+a)} - \sqrt{-(1+a)}] \sqrt{1+a \sin^2 x}} \\ [a < -1; 0 \leq \sin^2 x < -1/a].$$

$$10. \int_0^x \frac{\sin x dx}{(1-\sin^2 x) \Delta} = \frac{\Delta}{(1-k^2) \cos x} - \frac{1}{1-k^2}.$$

$$11. \int_0^x \frac{\sin x dx}{(1-k^2 \sin^2 x) \Delta} = -\frac{\cos x}{(1-k^2) \Delta} + \frac{1}{1-k^2}.$$

$$12. \int_0^x \frac{\cos x dx}{(1+a \sin^2 x) \Delta} = \frac{1}{\sqrt{k^2+a}} \operatorname{arctg} \frac{\sqrt{k^2+a} \sin x}{\Delta} \quad [a > -k^2].$$

$$13. \quad = \frac{1}{2\sqrt{-(k^2+a)}} \ln \frac{\Delta + \sqrt{-(k^2+a)} \sin x}{\Delta - \sqrt{-(k^2+a)} \sin x} \quad [a < -k^2; 0 \leq \sin^2 x < -1/a]$$

$$14. \quad \int_0^x \frac{\cos x \, dx}{(1-k^2 \sin^2 x) \Delta} = \frac{\sin x}{\Delta}.$$

1.5.34. Интегралы вида $\int \frac{(a + \cos x)^p}{\sqrt{1 - k^2 \sin^2 x}} \, dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \quad \int \frac{(a + \cos x)^p}{\Delta} \, dx = \frac{1}{(p-1)k^2} \left[(a + \cos x)^{p-3} \Delta \sin x + \right. \\ \left. + 2(2p-3)ak^2 \int \frac{(a + \cos x)^{p-1}}{\Delta} \, dx - (p-2)(1-2k^2+6a^2k^2) \times \right. \\ \left. \times \int \frac{(a + \cos x)^{p-2}}{\Delta} \, dx + (2p-5)a(1-2k^2+a^2k^2) \int \frac{(a + \cos x)^{p-3}}{\Delta} \, dx + \right. \\ \left. + (p-3)(1-a^2)(1-k^2a^2) \int \frac{(a + \cos x)^{p-4}}{\Delta} \, dx \right] \quad [p \neq 1; a^2 \neq 1, -(1-k^2)/k^2].$$

$$2. \quad \int \frac{a + \cos x}{\Delta} \, dx = aF(x, k) + \frac{1}{k} \arcsin(k \sin x).$$

$$3. \quad \int \frac{(a + \cos x)^2}{\Delta} \, dx = \frac{a^2k^2 + k^2 - 1}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2a}{k} \arcsin(k \sin x).$$

$$4. \quad \int \frac{dx}{(a + \cos x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-k^2+a^2k^2)} \left[\frac{\Delta \sin x}{(a + \cos x)^{n-1} \Delta} - \right. \\ \left. - (2n-3)(1-2k^2+2a^2k^2)a \int \frac{dx}{(a + \cos x)^{n-1} \Delta} - \right. \\ \left. - (n-2)(2k^2-1-6a^2k^2) \int \frac{dx}{(a + \cos x)^{n-2} \Delta} - \right. \\ \left. - (4n-10)ak^2 \int \frac{dx}{(a + \cos x)^{n-3} \Delta} + (n-3)k^2 \int \frac{dx}{(a + \cos x)^{n-4} \Delta} \right].$$

$$5. \quad = \frac{1}{(2n-1)a(1-2k^2+2a^2k^2)} \left[\frac{\sin x \Delta}{(a + \cos x)^n} + \right. \\ \left. + (n-1)(1-2k^2+6a^2k^2) \int \frac{dx}{(a + \cos x)^{n-1} \Delta} - \right. \\ \left. - 2(2n-3)k^2a \int \frac{dx}{(a + \cos x)^{n-2} \Delta} + (n-2)k^2 \int \frac{dx}{(a + \cos x)^{n-3} \Delta} \right] \\ [a = \pm 1, \pm \sqrt{1-k^2/(k^2)}].$$

$$6. \quad \int \frac{dx}{(1 \pm \cos x) \Delta} = \frac{\pm 1 - \cos x}{\sin x} \Delta + F(x, k) - E(x, k).$$

$$7. \quad \int \frac{dx}{(1 \pm \cos x \, ki / \sqrt{1-k^2}) \Delta} = \\ = - \frac{k^2 \sin x \cos x (1 \mp \cos ki/x \sqrt{1-k^2})}{\Delta} \mp \frac{ki}{\sqrt{1-k^2}} \Delta \sin x + E(x, k).$$

$$8. \int \frac{dx}{(a + \cos x) \Delta} = \frac{a}{a^2 - 1} \Pi \left(x, -\frac{1}{a^2 - 1}, k \right) + \\ + \frac{1}{2 \sqrt{(1 - a^2)(1 + k^2 a^2 - k^2)}} \ln \frac{\sqrt{1 - a^2} \Delta + k \sqrt{1 + k^2 a^2 - k^2} \sin x}{\sqrt{1 - a^2} \Delta - k \sqrt{1 + k^2 a^2 - k^2} \sin x}.$$

$$9. \int \frac{dx}{(a + \cos x)^2 \Delta} = \frac{1}{(1 - a^2)(1 - k^2 + a^2 k^2)} \left[\frac{\sin x \Delta}{a + \cos x} - \right. \\ \left. - (1 - 2k^2 + 2a^2 k^2) a \int \frac{dx}{(a + \cos x) \Delta} + 2ak^2 \int \frac{a + \cos x}{\Delta} dx - \right. \\ \left. - k^2 \int \frac{(a + \cos x)^2}{\Delta} dx \right].$$

$$10. \int \frac{dx}{(a + \cos x)^3 \Delta} = \frac{1}{2(1 - a^2)(1 - k^2 + a^2 k^2)} \left[\frac{\Delta \sin x}{(a + \cos x)^2} - \right. \\ \left. - 3a(1 - 2k^2 + 2k^2 a^2) \int \frac{dx}{(a + \cos x)^2 \Delta} - \right. \\ \left. - (2k^2 - 1 - 6a^2 k^2) \int \frac{dx}{(a + \cos x) \Delta} - 2ak^2 F(x, k) \right].$$

1.5.35. Интегралы вида $\int \frac{(a + \operatorname{tg} x)^p}{\sqrt{1 - k^2 \sin^2 x}} dx$.

Условие: $0 < k^2 < 1$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{(a + \operatorname{tg} x)^p}{\Delta} dx = \\ = \frac{1}{(p - 1)(1 - k^2)} \left[\frac{(a + \operatorname{tg} x)^{p-3} \Delta}{\cos^2 x} + 2(2p - 3)a(1 - k^2) \int \frac{(a + \operatorname{tg} x)^{p-4}}{\Delta} dx - \right. \\ \left. - (p - 2)(2 + 6a^2 - k^2 - 6k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-2}}{\Delta} dx + \right. \\ \left. + (2p - 5)a(2 + 2a^2 - k^2 - 6k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-3}}{\Delta} dx - \right. \\ \left. - (p - 3)(1 + a^2)(1 + a^2 - k^2 a^2) \int \frac{(a + \operatorname{tg} x)^{p-4}}{\Delta} dx \right].$$

$$2. \int \frac{a + \operatorname{tg} x}{\Delta} dx = aF(x, k) + \frac{1}{2\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}}.$$

$$3. \int \frac{(a + \operatorname{tg} x)^2}{\Delta} dx = \\ = \frac{1}{1 - k^2} \Delta \operatorname{tg} x + a^2 F(x, k) - \frac{1}{1 - k^2} E(x, k) + \frac{a}{\sqrt{1 - k^2}} \ln \frac{\Delta + \sqrt{1 - k^2}}{\Delta - \sqrt{1 - k^2}}.$$

$$4. \int \frac{dx}{(a + \operatorname{tg} x)^n \Delta} = \frac{1}{(n - 1)(1 + a^2)(1 + a^2 - k^2 a^2)} \times \\ \times \left[-\frac{\Delta}{(a + \operatorname{tg} x)^{n-1} \cos^2 x} + (2n - 3)a(2 + 2a^2 - k^2 - 2k^2 a^2) \times \right. \\ \times \int \frac{dx}{(a + \operatorname{tg} x)^{n-1} \Delta} - (n - 2)(2 + 6a^2 - k^2 - 6k^2 a^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-2} \Delta} + \\ \left. + (4n - 10)a(1 - k^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-3} \Delta} - (n - 3)(1 - k^2) \int \frac{dx}{(a + \operatorname{tg} x)^{n-4} \Delta} \right] \\ [n \neq 1, a^2 \neq -1, -1/(1 - k^2)].$$

$$3. \int \frac{dx}{(1-r^2 \sin^2 x) \sqrt{1-p^2 \sin^2 x}} = \frac{1}{p} \Pi \left(\varphi, \frac{r^2}{p^2}, \frac{1}{p} \right)$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{1-p^2 \sin^2 x}) dx$$

при $p^2 > 1$ можно пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

$$1) F(x, k) \rightarrow \frac{1}{p} F \left(\varphi, \frac{1}{p} \right);$$

$$2) E(x, k) \rightarrow pE \left(\varphi, \frac{1}{p} \right) - \frac{p^2-1}{p} F \left(\varphi, \frac{1}{p} \right)$$

$$3) k \rightarrow p.$$

1.5.37. Интегралы вида $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$.

Обозначение: $\varphi = \arcsin \frac{\sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x}}$.

$$1. \int \sqrt{1+p^2 \sin^2 x} dx = \sqrt{1+p^2} E \left(\varphi, \frac{1}{\sqrt{1+p^2}} \right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}.$$

$$2. \int \frac{\sqrt{1+p^2 \sin^2 x} dx}{1+(p^2-r^2 p^2-r^2) \sin^2 x} = \frac{1}{\sqrt{1+p^2}} \Pi \left(\varphi, r^2, \frac{p}{\sqrt{1+p^2}} \right).$$

$$3. \int \frac{dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{\sqrt{1+p^2}} F \left(\varphi, \frac{p}{\sqrt{1+p^2}} \right).$$

$$4. \int \frac{\sin x dx}{\sqrt{1+p^2 \sin^2 x}} = -\frac{1}{p} \arcsin \left(\frac{p \cos x}{\sqrt{1+p^2}} \right).$$

$$5. \int \frac{\cos x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

$$6. \int \frac{dx}{\sin x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{\sqrt{1+p^2 \sin^2 x} - \cos x}{\sqrt{1+p^2 \sin^2 x} + \cos x}.$$

$$7. \int \frac{dx}{\cos x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2} \sin x}.$$

$$8. \int \frac{\operatorname{tg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2}}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2}}.$$

$$9. \int \frac{\operatorname{ctg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{1 - \sqrt{1+p^2 \sin^2 x}}{1 + \sqrt{1+p^2 \sin^2 x}}.$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$$

можно также пользоваться формулами пп. 1.5.22—1.5.30, произведя в них следующие замены [5]:

$$1) k^2 \rightarrow -p^2;$$

$$2) F(x, k) \rightarrow \frac{1}{\sqrt{1+p^2}} F \left(\varphi, \frac{p}{\sqrt{1+p^2}} \right);$$

$$3) E(x, k) \rightarrow \sqrt{1+p^2} E\left(\varphi, -\frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}};$$

$$4) \frac{1}{k} \ln(k \cos x + \Delta) \rightarrow \frac{1}{p} \operatorname{arcsin} \frac{p \cos x}{\sqrt{1+p^2}};$$

$$5) \frac{1}{k} \operatorname{arcsin}(k \sin x) \rightarrow \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

1.5.38. Интегралы вида $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx$.
Условие: $a^2 > 1$.

Обозначение: $\varphi = \operatorname{arcsin} \frac{a \cos x}{\sqrt{a^2 - 1}}$.

$$1. \int \sqrt{a^2 \sin^2 x - 1} dx = \frac{1}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right) - aE\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right).$$

$$2. \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right).$$

$$3. \int \frac{\sin x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\varphi}{a}.$$

$$4. \int \frac{\cos x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln(a \sin x + \sqrt{a^2 \sin^2 x - 1}).$$

$$5. \int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}}.$$

$$6. \int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2 \sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} \sin x - \sqrt{a^2 \sin^2 x - 1}}.$$

$$7. \int \frac{dx}{(1 - r^2 \sin^2 x) \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2 - 1)} \Pi\left(\varphi, \frac{r^2(a^2 - 1)}{a^2(r^2 - 1)}, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [r^2 > 1]$$

$$8. \int \frac{\lg x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2 \sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}}.$$

$$9. \int \frac{\operatorname{ctg} x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arcsin} \frac{1}{a \sin x}.$$

Для вычисления интегралов вида

$$\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx \quad (a^2 > 1)$$

можно также воспользоваться формулами пп. 1.5.22—1.5.30. Для этого надо произвести в них следующие замены [5]:

$$1) F(x, k) \rightarrow -\frac{1}{ia} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right);$$

$$2) E(x, k) \rightarrow \frac{i}{a} F\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right) - iaE\left(\varphi, \frac{\sqrt{a^2 - 1}}{a}\right);$$

$$3) \frac{1}{k} \ln(k \cos x + \Delta) \rightarrow -\frac{\varphi}{ia};$$

$$4) \frac{1}{k} \arcsin(k \sin x) \rightarrow \frac{1}{ia} \ln(a \sin x + \sqrt{a^2 \sin^2 x - 1});$$

$$5) \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} \rightarrow -\frac{1}{i} \operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}};$$

$$6) \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2} \sin x}{\Delta - \sqrt{1-k^2} \sin x} \rightarrow \frac{1}{2i\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} \sin x - \sqrt{a^2 \sin^2 x - 1}};$$

$$7) \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} \rightarrow -\frac{1}{i} \arcsin \frac{1}{a \sin x};$$

$$8) \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}} \rightarrow \frac{1}{2i\sqrt{a^2-1}} \ln \frac{\sqrt{a^2-1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2 \sin^2 x - 1}};$$

$$9) \text{ Оставшиеся } \Delta \rightarrow i\sqrt{a^2 \sin^2 x - 1};$$

$$10) k \rightarrow a;$$

$$11) \text{ умножить полученные равенства на } i.$$

1.5.39. Интегралы вида $\int f(\operatorname{tg} x) dx$, $\int f(\operatorname{ctg} x) dx$.

Обозначение: $t(x) = \begin{cases} \operatorname{tg} x \\ \operatorname{ctg} x \end{cases}$.

$$1. \int R(\sin^2 x, \cos^2 x, \operatorname{tg} x, \operatorname{ctg} x, \sqrt{a^2 \pm b^2 \operatorname{tg}^2 x}) dx = \int R\left(\frac{1}{1+\tau^2}, \frac{\tau^2}{1+\tau^2}, \tau, \frac{1}{\tau}, \sqrt{a^2 \pm b^2 \tau^2}\right) \frac{d\tau}{1+\tau^2} \quad [\tau = \operatorname{tg} x].$$

$$2. \int R(\sin^2 x, \cos^2 x, \operatorname{tg} x, \operatorname{ctg} x, \sqrt{a^2 \pm b^2 \operatorname{ctg}^2 x}) dx = \int R\left(\frac{\tau^2}{1+\tau^2}, \frac{1}{1+\tau^2}, \frac{1}{\tau}, \tau, \sqrt{a^2 \pm b^2 \tau^2}\right) \frac{d\tau}{1+\tau^2} \quad [\tau = \operatorname{ctg} x].$$

$$3. \int [a^2 + b^2 t^2(x)]^{n+1/2} t(x) dx = \pm \sqrt{a^2 + b^2 t^2(x)} \sum_{k=0}^n (a^2 - b^2)^k \frac{[a^2 + b^2 t^2(x)]^{n-k}}{2n - 2k + 1} + \begin{cases} \pm \frac{1}{2} (a^2 - b^2)^{n+1/2} \ln \frac{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2 t^2(x)}} & [a^2 > b^2], \\ (-1)^{n+1} (b^2 - a^2)^{n+1/2} \begin{cases} \operatorname{arctg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \\ \operatorname{arccotg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \end{cases} & [a^2 < b^2]. \end{cases}$$

$$\begin{aligned}
 4. \int [a^2 - b^2 t^2(x)]^{n+1/2} t(x) dx &= \\
 &= \pm \sqrt{a^2 - b^2 t^2(x)} \sum_{k=0}^n (a^2 + b^2)^k \frac{[a^2 - b^2 t^2(x)]^{n-k}}{2n-2k+1} \pm \\
 &\pm \frac{1}{2} (a^2 + b^2)^{n+1/2} \ln \frac{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2 t^2(x)}}.
 \end{aligned}$$

$$\begin{aligned}
 5. \int \frac{t(x) dx}{[a^2 + b^2 t^2(x)]^{n+1/2}} &= \\
 &= \pm \frac{1}{\sqrt{a^2 + b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1) (a^2 - b^2)^{n-k} [a^2 + b^2 t^2(x)]^k} + \\
 &+ \begin{cases} \pm \frac{1}{2 (a^2 - b^2)^{n+1/2}} \ln \frac{\sqrt{a^2 - b^2} - \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} + \sqrt{a^2 + b^2 t^2(x)}} & [a^2 > b^2], \\ \frac{(-1)^n}{(b^2 - a^2)^{n+1/2}} \begin{pmatrix} \operatorname{arctg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \\ \operatorname{arcctg} \sqrt{\frac{a^2 + b^2 t^2(x)}{b^2 - a^2}} \end{pmatrix} & [a^2 < b^2]. \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 6. \int \frac{t(x) dx}{[a^2 - b^2 t^2(x)]^{n+1/2}} &= \\
 &= \pm \frac{1}{\sqrt{a^2 - b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1) (a^2 + b^2)^{n-k} [a^2 - b^2 t^2(x)]^k} \pm \\
 &\pm \frac{1}{2 (a^2 + b^2)^{n+1/2}} \ln \frac{\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} + \sqrt{a^2 - b^2 t^2(x)}}.
 \end{aligned}$$

$$7. \int \frac{dx}{\sqrt{a^2 + b^2 t^2(x)}} = \frac{1}{2 \sqrt{b^2 - a^2}} \ln \frac{\sqrt{b^2 - a^2} t(x) \pm \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2} t(x) \mp \sqrt{a^2 + b^2 t^2(x)}} \quad [a^2 < b^2].$$

$$8. = \frac{1}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} t(x)}{\sqrt{a^2 + b^2 t^2(x)}} \quad [a^2 > b^2].$$

$$9. \int \frac{dx}{\sqrt{a^2 - b^2 t^2(x)}} = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} \operatorname{arctg} \frac{\sqrt{a^2 + b^2} t(x)}{\sqrt{a^2 - b^2 t^2(x)}} \\ \operatorname{arcctg} \frac{\sqrt{a^2 + b^2} t(x)}{\sqrt{a^2 - b^2 t^2(x)}} \end{pmatrix}.$$

$$10. \int \frac{dx}{\sqrt{b^2 t^2(x) - a^2}} = \frac{1}{2 \sqrt{a^2 + b^2}} \ln \frac{\sqrt{a^2 + b^2} t(x) \pm \sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2} t(x) \mp \sqrt{b^2 t^2(x) - a^2}}.$$

$$11. \int \frac{t(x) dx}{\sqrt{a^2 + b^2 t^2(x)}} = \frac{1}{\sqrt{b^2 - a^2}} \begin{pmatrix} \operatorname{arctg} \frac{\sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2}} \\ \operatorname{arcctg} \frac{\sqrt{a^2 + b^2 t^2(x)}}{\sqrt{b^2 - a^2}} \end{pmatrix} \quad [a^2 < b^2].$$

$$12. = \frac{1}{2 \sqrt{a^2 - b^2}} \ln \frac{\sqrt{a^2 - b^2} \mp \sqrt{a^2 + b^2 t^2(x)}}{\sqrt{a^2 - b^2} \pm \sqrt{a^2 + b^2 t^2(x)}} \quad (a^2 > b^2).$$

$$13. \int \frac{t(x) dx}{\sqrt{a^2 - b^2 t^2(x)}} = \frac{1}{2 \sqrt{a^2 + b^2}} \ln \frac{\sqrt{a^2 + b^2} \mp \sqrt{a^2 - b^2 t^2(x)}}{\sqrt{a^2 + b^2} \pm \sqrt{a^2 - b^2 t^2(x)}}.$$

$$14. \int \frac{t(x) dx}{\sqrt{b^2 t^2(x) - a^2}} = \frac{1}{\sqrt{a^2 + b^2}} \left[\operatorname{arctg} \frac{\sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2}} \right. \\ \left. - \operatorname{arccctg} \frac{\sqrt{b^2 t^2(x) - a^2}}{\sqrt{a^2 + b^2}} \right].$$

$$15. \int \frac{t(x) dx}{[a^2 \pm b^2 t^2(x)]^{n+1/2}} = \\ = \frac{1}{\sqrt{a^2 \pm b^2 t^2(x)}} \sum_{k=0}^{n-1} \frac{1}{(2k+1)(a^2 \mp b^2 t^2(x))^{n-k} (a^2 \pm b^2 t^2(x))^k} + \\ + \frac{1}{(a^2 - b^2 t^2(x))^{n+1/2}} \int \frac{t(x) dx}{\sqrt{a^2 \pm b^2 t^2(x)}}.$$

1.5.40. Интегралы вида $\int x^p \left\{ \frac{\sin x}{\cos x} \right\}^q dx$.

$$1. \int x^p \left\{ \frac{\sin x}{\cos x} \right\}^q dx = \frac{x^{p-1}}{q^2} \left\{ \frac{\sin x}{\cos x} \right\}^{q-1} \times \\ \times \left[p \left\{ \frac{\sin x}{\cos x} \right\} \mp q x \left\{ \frac{\cos x}{\sin x} \right\} \right] + \frac{q-1}{q} \int x^p \left\{ \frac{\sin x}{\cos x} \right\}^{q-2} dx - \frac{p(p-1)}{q^2} \int x^{p-2} \left\{ \frac{\sin x}{\cos x} \right\}^q dx.$$

$$2. \int x^m \left\{ \frac{\sin x}{\cos x} \right\}^{2n} dx = \binom{2n}{n} \frac{x^{m+1}}{2^{2n}(m+1)} + \\ + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n+k} \binom{2n}{k} \int x^m \cos(2n-2k)x dx.$$

$$3. \int x^m \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^{n+k} \binom{2n+1}{k} \int x^m \left\{ \frac{\sin(2n-2k+1)x}{\cos(2n-2k+1)x} \right\} dx.$$

$$4. \int x^n \left\{ \frac{\sin x}{\cos x} \right\} dx = \mp \sum_{k=0}^n k! \binom{n}{k} x^{n-k} \left\{ \frac{\cos(x+k\pi/2)}{\sin(x+k\pi/2)} \right\}.$$

$$5. \int x^{2n} \left\{ \frac{\sin x}{\cos x} \right\} dx = \\ = (2n)! \left[\mp \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \left\{ \frac{\cos x}{\sin x} \right\} + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \left\{ \frac{\sin x}{\cos x} \right\} \right].$$

$$6. \int x^{2n+1} \left\{ \frac{\sin x}{\cos x} \right\} dx = \\ = (2n+1)! \left[\mp \sum_{k=0}^n (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \left\{ \frac{\cos x}{\sin x} \right\} + \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \left\{ \frac{\sin x}{\cos x} \right\} \right].$$

7. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^{2n} dx =$

$$= \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2} + \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1) \dots (2n-2k+1)}{2^{k+1} n (n-1) \dots (n-k)} \times$$

$$\times \left[\frac{1}{2n-2k} \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k} + x \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k-1} \left\{ \frac{\cos x}{\sin x} \right\} \right].$$
8. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^{2n+1} dx = \frac{1}{n+1} \sum_{k=0}^n \frac{2^k (n+1) n \dots (n-k+1)}{(2n+1)(2n-1) \dots (2n-2k+1)} \times$

$$\times \left[\frac{1}{2n-2k+1} \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k+1} + x \left\{ \frac{\sin x}{\cos x} \right\}^{2n-2k} \left\{ \frac{\cos x}{\sin x} \right\} \right].$$
9. $\int x \left\{ \frac{\sin x}{\cos x} \right\} dx = \left\{ \frac{\sin x}{\cos x} \right\} + x \left\{ \frac{\cos x}{\sin x} \right\}.$
10. $\int x^2 \left\{ \frac{\sin x}{\cos x} \right\} dx = 2x \left\{ \frac{\sin x}{\cos x} \right\} + (x^2 - 2) \left\{ \frac{\cos x}{\sin x} \right\}.$
11. $\int x^3 \left\{ \frac{\sin x}{\cos x} \right\} dx = (3x^2 - 6) \left\{ \frac{\sin x}{\cos x} \right\} + (x^3 - 6x) \left\{ \frac{\cos x}{\sin x} \right\}.$
12. $\int x^n \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^{n+1}}{2(n+1)} +$

$$+ \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k} (n-2k)!} \sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1} (n-2k-1)!} \cos 2x \right\}.$$
13. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x.$
14. $\int x^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^3}{6} + \left(\frac{x^2}{4} - \frac{1}{8} \right) \sin 2x + \frac{x}{4} \cos 2x.$
15. $\int x^3 \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{x^4}{8} + \left(\frac{x^3}{4} - \frac{3x}{8} \right) \sin 2x + \left(\frac{3x^2}{8} - \frac{3}{16} \right) \cos 2x$
16. $\int x^n \left\{ \frac{\sin x}{\cos x} \right\}^2 dx =$

$$= \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{2n-2k}}{(n-2k)!} \left[\frac{1}{3^{2k+1}} \left\{ \frac{\cos 3x}{\sin 3x} \right\} + 3 \left\{ \frac{\cos x}{\sin x} \right\} \right] + \right.$$

$$\left. + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k x^{2n-2k-1}}{(n-2k-1)!} \left[\frac{1}{3^{2k+2}} \left\{ \frac{\sin 3x}{\cos 3x} \right\} + 3 \left\{ \frac{\sin x}{\cos x} \right\} \right] \right\}.$$
17. $\int x \left\{ \frac{\sin x}{\cos x} \right\}^3 dx = \frac{3}{4} \left\{ \frac{\sin x}{\cos x} \right\} + \frac{1}{36} \left\{ \frac{\sin 3x}{\cos 3x} \right\} + \frac{3}{4} x \left\{ \frac{\cos x}{\sin x} \right\} + \frac{x}{12} \left\{ \frac{\cos 3x}{\sin 3x} \right\}.$
18. $\int x^3 \left\{ \frac{\sin x}{\cos x} \right\} dx = + x^3 \left\{ \frac{\cos x}{\sin x} \right\} + 6 \int x^{3-1} \left\{ \frac{\cos x}{\sin x} \right\} dx.$

$$19. \int_0^x x^{\mu-1} \frac{\{\sin ax\}}{\{\cos ax\}} dx = \frac{1}{2} \frac{\{i\}}{\{1\}} [(ia)^{-\mu} \gamma(\mu, iax) - (-ia)^{-\mu} \gamma(\mu, -iax)]$$

$$\left[\operatorname{Re} \mu > \left\{ \frac{-1}{0} \right\}; x > 0 \right].$$

$$20. \int_x^\infty x^{\mu-1} \frac{\{\sin ax\}}{\{\cos ax\}} dx = \frac{1}{2a^{\mu}} \frac{\{i\}}{\{1\}} \left[\frac{1}{\pi} e^{i\mu\pi/2} \Gamma(\mu, -iax) + e^{-i\mu\pi/2} \Gamma(\mu, iax) \right]$$

$$[\operatorname{Re} \mu < 1; x > 0].$$

$$21. \int_x^\infty x^{\mu-1} \frac{\{\sin x\}}{\{\cos x\}} dx = \frac{\{S(x, \mu)\}}{\{C(x, \mu)\}}.$$

1.5.41. Интегралы вида $\int \frac{1}{x^p} \frac{\{\sin x\}}{\{\cos x\}}^q dx$.

$$1. \int \frac{1}{x^p} \frac{\{\sin x\}}{\{\cos x\}}^q dx =$$

$$= -\frac{1}{(p-1)x^{p-1}} \frac{\{\sin x\}}{\{\cos x\}}^q + \frac{q}{(p-1)(p-2)x^{p-2}} \frac{\{\sin x\}}{\{\cos x\}}^{q-1} \{\cos x\} +$$

$$+ \frac{q(q-1)}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \frac{\{\sin x\}}{\{\cos x\}}^{q-2} dx - \frac{q^2}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \frac{\{\sin x\}}{\{\cos x\}}^q dx.$$

$$2. \int \frac{1}{x^p} \frac{\{\sin x\}}{\{\cos x\}} dx = -\frac{1}{(p-1)x^{p-1}} \frac{\{\sin x\}}{\{\cos x\}} + \frac{1}{p-1} \int \frac{1}{x^{p-1}} \frac{\{\cos x\}}{\{\sin x\}} dx.$$

$$3. = -\frac{1}{(p-1)x^{p-1}} \frac{\{\sin x\}}{\{\cos x\}} + \frac{1}{(p-1)(p-2)x^{p-2}} \frac{\{\cos x\}}{\{\sin x\}} -$$

$$- \frac{1}{(p-1)(p-2)} \int \frac{1}{x^{p-2}} \frac{\{\sin x\}}{\{\cos x\}} dx.$$

$$4. \int \frac{1}{x^{2n}} \frac{\{\sin x\}}{\{\cos x\}} dx = \frac{(-1)^{n+1}}{x(2n)!} \left[\frac{1}{x} \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \frac{\{\cos x\}}{\{\sin x\}} + \right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \frac{\{\sin x\}}{\{\cos x\}} \right] + \frac{(-1)^{n+1}}{(2n-1)!} \frac{\{ci(x)\}}{\{\sin(x)\}}.$$

$$5. \int \frac{1}{x^{2n+1}} \frac{\{\sin x\}}{\{\cos x\}} dx = \frac{(-1)^{n+1}}{x(2n)!} \left[\frac{1}{x} \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \frac{\{\cos x\}}{\{\sin x\}} + \right.$$

$$\left. + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \frac{\{\sin x\}}{\{\cos x\}} \right] + \frac{(-1)^n}{(2n)!} \frac{\{si(x)\}}{\{\cos(x)\}}.$$

$$6. \int \frac{1}{x} \frac{\{\sin x\}}{\{\cos x\}}^{2m} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{m+k} \binom{2m}{k} ci[(2m-2k)x].$$

$$7. \int \frac{1}{x} \frac{\{\sin x\}}{\{\cos x\}}^{2m+1} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^{m+k} \binom{2m+1}{k} \{ci(2m-2k+1)x\}.$$

$$8. \int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\}^{2m} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} \pm \\ \pm \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{m+k+1} \binom{2m}{k} \left[\frac{\cos (2m-2k)x}{x} + (2m-2k) \operatorname{si} [(2m-2k)x] \right].$$

$$9. \int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\}^{2m+1} dx = \pm \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^{m+k+1} \binom{2m+1}{k} \times \\ \times \left[\frac{1}{x} \left\{ \frac{\sin (2m-2k+1)x}{\cos (2m-2k+1)x} \right\} \mp (2m-2k+1) \left\{ \frac{\operatorname{ci} (2m-2k+1)x}{\operatorname{si} (2m-2k+1)x} \right\} \right].$$

$$10. \int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} dx = \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}. \quad 11. \int_x^\infty \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}.$$

$$12. \int_0^x \frac{\sin x}{x} dx = \operatorname{Si}(x). \quad 13. \int_0^x \frac{1-\cos x}{x} dx = C + \ln x - \operatorname{ci}(x).$$

$$14. \int \frac{1}{x^2} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\} \pm \left\{ \frac{\operatorname{ci}(x)}{\operatorname{si}(x)} \right\}.$$

$$15. \int \frac{1}{x^3} \left\{ \frac{\sin x}{\cos x} \right\} dx = - \frac{1}{2x^2} \left\{ \frac{\sin x}{\cos x} \right\} \mp \frac{1}{2x} \left\{ \frac{\cos x}{\sin x} \right\} - \frac{1}{2} \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\}.$$

$$16. \int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{1}{2} \ln |x| \mp \frac{1}{2} \operatorname{ci}(2x).$$

$$17. \int \frac{1}{x} \left\{ \frac{\sin x}{\cos x} \right\}^3 dx = \frac{3}{4} \left\{ \frac{\operatorname{si}(x)}{\operatorname{ci}(x)} \right\} \mp \frac{1}{4} \left\{ \frac{\operatorname{si}(3x)}{\operatorname{ci}(3x)} \right\}.$$

1.5.42. Интегралы вида $\int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^q dx$.

$$1. \int x^p \operatorname{tg} x dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k} (2^{2k}-1)}{(p+2k)(2k)!} B_{2k} x^{p+2k} \quad [p > -1; |x| < \pi/2].$$

$$2. \int x^p \operatorname{ctg} x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p > 1; |x| < \pi].$$

$$3. \int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n} dx = (-1)^n \frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \int x^p \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} dx$$

[см. 1.5.44, 1.5.45].

$$4. \int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{2n+1} dx = \pm \sum_{k=0}^{n-1} \frac{(-1)^k \binom{n}{k}}{2n-2k} x^p \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} + \\ + (-1)^n \int x^p \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx \mp p \sum_{k=0}^{n-1} \frac{(-1)^k \binom{n}{k}}{2n-2k} \int x^{p-1} \left\{ \frac{\cos x}{\sin x} \right\}^{2k-2n} dx$$

[см. 1.5.44, 1.5.45].

$$5. \int x \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = \pm x \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} + \ln \left| \frac{\cos x}{\sin x} \right| - \frac{x^2}{2}.$$

1.5.43. Интегралы вида $\int x^r \sin^p x \cos^q x dx$.

$$1. \int x^r \sin^p x \cos^q x dx = \frac{1}{(p+q)^2} \{ (p+q) x^r \sin^{p+1} x \cos^{q-1} x + \\ + r x^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x dx - \\ - r p \int x^{r-1} \sin^{p-1} x \cos^{q-1} x dx + (q-1)(p+q) \int x^r \sin^p x \cos^{q-2} x dx \}.$$

$$2. = \frac{1}{(p+q)^2} \{ -(p+q) x^r \sin^{p-1} x \cos^{q+1} x + \\ + r x^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x dx + \\ + r q \int x^{r-1} \sin^{p-1} x \cos^{q-1} x dx + (p-1)(p+q) \int x^r \sin^{p-2} x \cos^q x dx \}.$$

$$3. \int x \sin^p x \cos^q x dx = \frac{1}{p+q} \left[x \sin^{p+1} x \cos^{q-1} x - \right. \\ \left. - \int \sin^{p+1} x \cos^{q-1} x dx + (q-1) \int x \sin^p x \cos^{q-2} x dx \right].$$

$$4. = \frac{1}{p+q} \left[-x \sin^{p-1} x \cos^{q+1} x + \int \sin^{p-1} x \cos^{q+1} x dx + \right. \\ \left. + (p-1) \int x \sin^{p-2} x \cos^q x dx \right].$$

$$5. \int x^p \frac{\sin^{2m} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\cos^{n-2k} x} \quad [\text{см. 1.5.45}].$$

$$6. \int x^p \frac{\sin^{2m+1} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x} dx \quad [\text{см. 1.5.46}].$$

$$7. \int x^p \frac{\sin x}{\cos^n x} dx = \frac{x^p}{(n-1) \cos^{n-1} x} - \frac{p}{n-1} \int \frac{x^{p-1}}{\cos^{n-1} x} dx \quad [\text{см. 1.5.45}].$$

$$8. \int x^p \frac{\cos^{2m} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p}{\sin^{n-2k} x} dx \quad [\text{см. 1.5.44}].$$

$$9. \int x^p \frac{\cos^{2m+1} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x} dx \quad [\text{см. 1.5.46}].$$

$$10. \int x^p \frac{\cos x}{\sin^n x} dx = -\frac{x^p}{(n-1) \sin^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}}{\sin^{n-1} x} dx \quad [\text{см. 1.5.44}].$$

$$11. \int \frac{x \cos x}{\sin^2 x} dx = -\frac{x}{\sin x} + \ln \left| \operatorname{tg} \frac{x}{2} \right|.$$

$$12. \int \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x} - \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right|.$$

1.5.44. Интегралы вида $\int \frac{x^p dx}{\sin^q x}$.

$$1. \int \frac{x^p dx}{\sin^q x} = -\frac{x^{p-1} [p \sin x + (q-2) x \cos x]}{(q-1)(q-2) \sin^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{x^p dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\sin^{q-2} x}.$$

$$2. \int \frac{x^p dx}{\sin x} = \frac{x^p}{p} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(p+2k)(2k)!} B_{2k} x^{p-2k} \quad [|x| < \pi; p \geq 0].$$

$$3. \int \frac{x^p dx}{\sin^3 x} = -x^p \operatorname{ctg} x + \frac{p}{p-1} x^{p-1} + p \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{p-2k-1}}{(p+2k-1)(2k)!} B_{2k} \\ [|x| < \pi; p > 1].$$

$$4. \int \frac{x dx}{\sin^q x} = -\frac{x \cos x}{(q-1) \sin^{q-1} x} - \frac{1}{(q-1)(q-2) \sin^{q-2} x} + \frac{q-2}{q-1} \int \frac{x dx}{\sin^{q-2} x}.$$

$$5. \int \frac{x dx}{\sin^{2n} x} = \\ = -\frac{1}{2n} \sum_{k=1}^{n-1} \frac{2n(2n-2) \dots (2n-2k+2) [\sin x + (2n-2k)x \cos x]}{(2n-1)(2n-3) \dots (2n-2k+3)(2n-2k+1)(2n-2k) \sin^{2n-2k+1} x} + \\ + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (\ln |\sin x| - x \operatorname{ctg} x).$$

$$6. \int \frac{x dx}{\sin^{2n+1} x} = \\ = -\frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1) \dots (2n-2k+1) [\sin x + (2n-2k-1)x \cos x]}{2n(2n-2) \dots (2n-2k+2)(2n-2k)(2n-2k-1) \sin^{2n-2k} x} + \\ + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\sin x}.$$

$$7. \int \frac{dx}{x^p \sin^q x} = \frac{p}{(q-1)(q-2)x^{p-1} \sin^{q-2} x} - \frac{\cos x}{(q-1)x^p \sin^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{dx}{x^p \sin^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)} \int \frac{dx}{x^{p+2} \sin^{q-2} x}.$$

$$8. \int \frac{dx}{x^n \sin x} = -\frac{1}{n x^n} - [1 + (-1)^n] (-1)^{n/2} \frac{2^{n-1}-1}{n!} B_n \ln x - \\ - \sum_{\substack{k=1 \\ k \neq n/2}}^{\infty} (-1)^k \frac{2(2^{2k-1}-1)}{(2k)!(2k-n)} B_{2k} x^{2k-n} \quad [n > 1; |x| < \pi].$$

$$9. \int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} - \\ - \{1 - (-1)^n\} (-1)^{(n+1)/2} \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)! (2k-n-1)} B_{2k} \quad (|x| < \pi).$$

$$10. \int \frac{x dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1}-1)}{(2k+1)!} B_{2k} x^{2k+1}.$$

$$11. \int \frac{x^3 dx}{\sin x} = \frac{x^3}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}-1}{(2k)! (k+1)} B_{2k} x^{2k+3}.$$

$$12. \int \frac{x dx}{\sin^2 x} = -x \operatorname{ctg} x + \ln |\sin x|.$$

$$13. \int \frac{x dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x dx}{\sin x}.$$

$$14. \int \frac{x dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \operatorname{ctg} x + \frac{2}{3} \ln |\sin x|.$$

1.5.45. Интегралы вида $\int \frac{x^p dx}{\cos^q x}$.

$$1. \int \frac{x^p dx}{\cos^q x} = -\frac{x^{p-1} [p \cos x - (q-2) x \sin x]}{(q-1)(q-2) \cos^{q-1} x} + \\ + \frac{q-2}{q-1} \int \frac{x^{p-2} dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cos^q x}.$$

$$2. \int \frac{x^p dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{p+2k+1}}{(p+2k+1)(2k)!} \quad (|x| < \pi/2; p \geq 0).$$

$$3. \int \frac{x^p dx}{\cos^2 x} = x^p \operatorname{tg} x + p \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} (2^{2k}-1) x^{p+2k-1}}{(p+2k-1)(2k)!} B_{2k} \\ (p > 1; |x| < \pi/2)$$

$$4. \int \frac{x dx}{\cos^{2n} x} = \\ = \frac{2n+1}{(2n+2) 2^n} \sum_{k=0}^{n-1} \frac{(2n+2) 2^n \dots (2n-2k+2) [(2n-2k) x \sin x - \cos x]}{(2n+1)(2n-1) \dots (2n-2k+3)(2n-2k+1)(2n-2k) \cos^{2n-2k-1} x} + \\ + \frac{2^{n-1} (n-1)!}{(2n-1)!} (x \operatorname{tg} x + \ln |\cos x|).$$

$$5. \int \frac{x dx}{\cos^{2n+1} x} = -\frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1) \{(2n-2k+1)x \sin x - \cos x\}}{2n(2n-2)\dots(2n-2k+2)(2n-2k)(2n-2k-1) \cos^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\cos x}.$$

$$6. \int \frac{dx}{x^p \cos^q x} = \frac{p}{(q-1)(q-2)x^{p-1} \cos^{q-2} x} + \frac{\sin x}{(q-1)x^p \cos^{q-1} x} + \frac{q-2}{q-1} \int \frac{dx}{x^p \cos^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)} \int \frac{dx}{x^{p+2} \cos^q x}.$$

$$7. \int \frac{dx}{x^n \cos x} = \frac{1}{2} [1 - (-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq (n-1)/2}}^{\infty} \frac{|E_{2k}| x^{2k-n-1}}{(2k-n+1)(2k)!} \quad [|\pi'| < \pi/2].$$

$$8. \int \frac{dx}{x^n \cos^2 x} = \frac{\lg x}{x^n} - [1 - (-1)^n] (-1)^{(n+1)/2} \frac{2^n n}{(n+1)!} (2^{n+1} - 1) B_{n+1} \ln x - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq (n+1)/2}}^{\infty} \frac{(-1)^k (2^{2k} - 1) (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \quad [|\pi'| < \pi/2].$$

$$9. \int \frac{x dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}. \quad 10. \int \frac{x dx}{\cos^2 x} = x \lg x + \ln |\cos x|.$$

$$11. \int \frac{x dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x dx}{\cos x}.$$

$$12. \int \frac{x dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \lg x + \frac{2}{3} \ln |\cos x|.$$

1.5.46. Интегралы вида $\int R(x^p, \sin x, \cos x, a + b \sin x + c \cos x) dx$.

$$1. \int \frac{x^p \cos x dx}{(a + b \sin x)^q} = -\frac{x^p}{(q-1)b(a + b \sin x)^{q-1}} + \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \sin x)^{q-1}}.$$

$$2. \int \frac{x^p \sin x dx}{(a + b \cos x)^q} = \frac{x^p}{(q-1)b(a + b \cos x)^{q-1}} - \frac{p}{(q-1)b} \int \frac{x^{p-1} dx}{(a + b \cos x)^{q-1}}.$$

$$3. \int \frac{x dx}{1 \pm \sin x} = \mp x \left\{ \lg [(\pi - 2x)/4] \right\} + 2 \ln \left| \frac{\cos [(\pi - 2x)/4]}{\sin [(\pi - 2x)/4]} \right|.$$

$$4. \int \frac{x dx}{1 \pm \cos x} = \pm x \left\{ \lg (x/2) \right\} + 2 \ln \left| \frac{\cos (x/2)}{\sin (x/2)} \right|.$$

$$5. \int \frac{x \cos x dx}{(1 \pm \sin x)^2} = \mp \frac{x}{1 \pm \sin x} + \left\{ \lg [(2x - \pi)/4] \right\}.$$

$$6. \int \frac{x \sin x dx}{(1 \pm \cos x)^2} = \pm \frac{x}{1 \pm \cos x} - \left\{ \lg (x/2) \right\}.$$

$$7. \int \frac{x dx}{(\cos x + a \sin x)^2} = \frac{1}{a^2 + 1} \ln (\cos x + a \sin x) + x \frac{\operatorname{tg} x - a}{(a^2 + 1)(1 + a \operatorname{tg} x)}.$$

$$8. \int \frac{x^2 dx}{[(ax - b) \sin x + (a + bx) \cos x]^2} = \frac{x \sin x + \cos x}{b [(ax - b) \sin x + (a + bx) \cos x]}.$$

$$9. \int \frac{\cos^2 x dx}{[a \cos x + (ax + b) \sin x]^2} = \frac{\operatorname{tg} x}{a [a + (ax + b) \operatorname{tg} x]}.$$

1.5.47. Интегралы вида $\int \frac{x \sin^m x \cos^n x}{\sqrt{1 - k^2 \sin^2 x}} dx$

Условие: $0 < k^2 < 1$.

Сбозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$.

$$1. \int \frac{x \sin x \cos x}{\Delta} dx = -\frac{x \Delta}{k^2} + \frac{1}{k^2} E(x, k).$$

$$2. \int \frac{x \sin^3 x \cos x}{\Delta} dx = \frac{1 - k^2}{9k^4} F(x, k) + \frac{2k^2 + 5}{9k^4} E(x, k) - \\ - \frac{1}{9k^4} [3(3 - \Delta^2)x + k^2 \sin x \cos x] \Delta.$$

$$3. \int \frac{x \sin x \cos^3 x}{\Delta} dx = -\frac{1 - k^2}{9k^4} F(x, k) + \frac{7k^2 - 5}{9k^4} E(x, k) - \\ - \frac{1}{9k^4} [3(\Delta^2 - 3 + 3k^2)x - k^2 \sin x \cos x] \Delta.$$

$$4. \int \frac{x \sin x}{\Delta^3} dx = -\frac{x \cos x}{(1 - k^2) \Delta} + \frac{1}{k(1 - k^2)} \arcsin(k \sin x).$$

$$5. \int \frac{x \cos x}{\Delta^3} dx = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$6. \int \frac{x \sin x \cos x}{\Delta^3} dx = \frac{x}{k^2 \Delta} - \frac{1}{k^3} F(x, k).$$

$$7. \int \frac{x \sin^3 x \cos x}{\Delta^3} dx = x \frac{2 - k^2 \sin^2 x}{k^4 \Delta} - \frac{1}{k^3} [E(x, k) + F(x, k)].$$

$$8. \int \frac{x \sin x \cos^3 x}{\Delta^3} dx = x \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta} + \frac{1 - k^2}{k^4} F(x, k) + \frac{1}{k^3} E(x, k).$$

1.5.48. Интегралы вида $\int (x + b)^{\pm n} \left\{ \frac{\sin ax}{\cos ax} \right\} dx$.

$$1. \int (x + b)^n \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \\ = \frac{n!}{a} \left[\mp \sum_{k=0}^{[n/2]} (-1)^k \left(\frac{1}{a} \right)^{2k} \frac{(x + b)^{n-2k}}{(n-2k)!} \left\{ \cos ax \right\} + \right. \\ \left. + \sum_{k=0}^{[(n-1)/2]} (-1)^k \left(\frac{1}{a} \right)^{2k+1} \frac{(x + b)^{n-2k-1}}{(n-2k-1)!} \left\{ \sin ax \right\} \right].$$

$$2. \int (x + b) \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \mp \frac{1}{a} (x + b) \left\{ \frac{\cos ax}{\sin ax} \right\} + \frac{1}{a^2} \left\{ \frac{\sin ax}{\cos ax} \right\}.$$

3. $\int (x+b)^2 \frac{\{\sin ax\}}{\{\cos ax\}} dx = -\frac{1}{a} \left[(x+b)^2 - \frac{2b}{a^2} \right] \frac{\{\cos ax\}}{\{\sin ax\}} + \frac{2}{a^2} (x+b) \frac{\{\sin ax\}}{\{\cos ax\}}.$
4. $\int (x+b)^2 \frac{\{\sin ax\}}{\{\cos ax\}} dx = -\frac{x+b}{a} \left[(x+b)^2 - \frac{6}{a^2} \right] \frac{\{\cos ax\}}{\{\sin ax\}} +$
 $+ \frac{3}{a^2} \left[(x+b)^2 - \frac{2}{a^2} \right] \frac{\{\sin ax\}}{\{\cos ax\}}.$
5. $\int \frac{1}{(x+b)^n} \frac{\{\sin ax\}}{\{\cos ax\}} dx = a^{n-1} \left(\cos ab \int \frac{1}{t^n} \frac{\{\sin t\}}{\{\cos t\}} dt - \right.$
 $\left. - \sin ab \int \frac{1}{t^n} \frac{\{\cos t\}}{\{\sin t\}} dt \right) \quad [t = a(x+b)].$
6. $\int_x^\infty \frac{1}{(x+b)^n} \frac{\{\sin ax\}}{\{\cos ax\}} dx =$
 $= \sum_{k=1}^{n-1} \frac{(k-1)!}{(n-1)!} (x+b)^{-k} (-a)^{n-k-1} \frac{\{\cos [\pi(n-k)/2 - ax]\}}{\{\sin [\pi(n-k)/2 - ax]\}} -$
 $- \frac{(-a)^{n-1}}{(n-1)!} \left[\text{ci}(ab+ax) \frac{\{\cos(ab+\pi n/2)\}}{\{\sin(ab+\pi n/2)\}} \pm \text{si}(ab+ax) \frac{\{\sin(ab+\pi n/2)\}}{\{\cos(ab+\pi n/2)\}} \right].$
7. $\int \frac{1}{x+b} \frac{\{\sin ax\}}{\{\cos ax\}} dx = \cos ab \frac{\{\text{si}(ax+ab)\}}{\{\text{ci}(ax+ab)\}} - \sin ab \frac{\{\text{ci}(ax+ab)\}}{\{\text{si}(ax+ab)\}}.$
8. $\int \frac{1}{(x+b)^2} \frac{\{\sin ax\}}{\{\cos ax\}} dx = -\frac{1}{x+b} \frac{\{\sin ax\}}{\{\cos ax\}} \pm a \int \frac{1}{x+b} \frac{\{\cos ax\}}{\{\sin ax\}} dx.$
9. $\int \frac{1}{(x+b)^3} \frac{\{\sin ax\}}{\{\cos ax\}} dx = -\frac{1}{2(x+b)^2} \frac{\{\sin ax\}}{\{\cos ax\}} - \frac{a}{2(x+b)} \frac{\{\cos ax\}}{\{\sin ax\}} -$
 $- \frac{a^2}{2} \int \frac{1}{x+b} \frac{\{\sin ax\}}{\{\cos ax\}} dx.$
10. $\int_x^\infty \frac{1}{x(x+b)} \sin ax dx = \frac{1}{b} [\cos ab \text{ si}(ab+ax) - \sin ab \text{ ci}(ab+ax) - \text{si}(ax)].$

1.5.49. Интегралы вида $\int e^{ax} \sin^p x \cos^q x dx.$

1. $\int e^{ax} \sin^p x \cos^q x dx =$
 $= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^p x \cos^{q-1} x [a \cos x + (p+q) \sin x] - \right.$
 $\left. - pa \int e^{ax} \sin^{p-1} x \cos^{q-1} x dx + (q-1)(p+q) \int e^{ax} \sin^p x \cos^{q-2} x dx \right\}.$
2. $= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] + \right.$
 $\left. + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}.$
3. $= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - \right.$
 $\left. - p \cos^2 x] + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}.$

$$4. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + \right. \\ \left. + q \sin^2 x - p \cos^2 x) + q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx - \right. \\ \left. - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}.$$

$$5. = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - \right. \\ \left. - p \cos^2 x) + p(p-1) \int e^{ax} \sin^p x \cos^{q-2} x dx + \right. \\ \left. + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^q x dx \right\}.$$

$$6. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^p dx = \\ = \frac{a}{a^2 + p^2 b^2} e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^p - \frac{pb}{a^2 + p^2 b^2} e^{ax} \left\{ \frac{\cos bx}{\cos bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{p-1} + \\ + \frac{p(p-1)b^2}{a^2 + p^2 b^2} \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{p-2} dx.$$

$$7. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n} dx = \\ = \sum_{k=0}^{n-1} \frac{(2n)! b^{2k} e^{ax}}{(2n-2k)! [a^2 + (2n)^2 b^2] [a^2 + (2n-2)^2 b^2] \dots [a^2 + (2n-2k)^2 b^2]} \times \\ \times \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k} - (2n-2k)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k-1} \right] + \\ + \frac{(2n)! b^{2n} e^{ax}}{[a^2 + (2n)^2 b^2] [a^2 + (2n-2)^2 b^2] \dots [a^2 + 4b^2] a}.$$

$$8. = \left(\frac{2n}{n} \right) \frac{e^{ax}}{2^{2n} a} + \\ + \frac{e^{ax}}{2^{2n-1}} \sum_{k=1}^n (-1)^k \binom{2n}{n-k} \frac{1}{a^2 + 4b^2 k^2} (a \cos 2b k x + 2b k \sin 2b k x).$$

$$9. \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n+1} dx = \\ = \sum_{k=0}^n \frac{(2n+1)! b^{2k} e^{ax}}{(2n-2k+1)! [a^2 + (2n+1)^2 b^2] [a^2 + (2n-1)^2 b^2] \dots [a^2 + (2n-2k+1)^2 b^2]} \times \\ \times \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k+1} - (2n-2k+1)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{2n-2k} \right].$$

$$10. = \frac{e^{ax}}{2^{2n}} \sum_{k=0}^n \frac{(-1)^k}{a^2 + (2k+1)^2 b^2} \binom{2n+1}{n-k} \times \\ \times \left[a \left\{ \frac{\sin (2k+1) bx}{\cos (2k+1) bx} \right\} - (2k+1)b \left\{ \frac{\cos (2k+1) bx}{\sin (2k+1) bx} \right\} \right].$$

11. $\int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{e^{ax}}{a^2 + b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right].$
12. $\int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^2 dx = \frac{e^{ax}}{2a} \mp \frac{e^{ax}}{a^2 + 4b^2} \left(\frac{a}{2} \cos 2bx + b \sin 2bx \right).$
13. $\int e^{ax} \sin bx \cos cx dx =$
 $= \frac{e^{ax}}{2} \left[\frac{a \sin (b+c)x - (b+c) \cos (b+c)x}{a^2 + (b+c)^2} + \frac{a \sin (b-c)x - (b-c) \cos (b-c)x}{a^2 + (b-c)^2} \right].$
14. $\int e^{ax} \sin^2 bx \cos cx dx =$
 $= \frac{e^{ax}}{4} \left[2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos (2b+c)x + (2b+c) \sin (2b+c)x}{a^2 + (2b+c)^2} - \right.$
 $\left. - \frac{a \cos (2b-c)x + (2b-c) \sin (2b-c)x}{a^2 + (2b-c)^2} \right].$
15. $\int e^{ax} \sin bx \cos^2 cx dx =$
 $= \frac{e^{ax}}{4} \left[2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin (b+2c)x - (b+2c) \cos (b+2c)x}{a^2 + (b+2c)^2} + \right.$
 $\left. + \frac{a \sin (b-2c)x - (b-2c) \cos (b-2c)x}{a^2 + (b-2c)^2} \right].$
16. $\int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p} dx =$
 $= - \frac{e^{ax}}{(p-1)(p-2)b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+2} \pm (p-2)b \left\{ \frac{\cos bx}{\sin bx} \right\} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+1} \right] +$
 $+ \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\}^{-p+2} dx.$
17. $\int e^{-\lambda^2 x} \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \frac{\sqrt{\pi}}{2} e^{-a^2/4} \left\{ \operatorname{Im} \operatorname{erf} (x - ia/2) \right\}.$

1.5.50. Интегралы вида $\int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^p dx.$

1. $\int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^p dx =$
 $= \pm \frac{e^{ax}}{p-1} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-1} \mp \frac{a}{p-1} \int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-1} dx - \int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^{p-2} dx.$
2. $\int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx = \frac{e^{ax}}{a} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \mp \frac{1}{a} \int e^{ax} \left\{ \frac{\cos x}{\sin x} \right\}^{-2} dx.$
3. $\int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\}^2 dx = - \frac{e^{ax}}{a} \pm e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} \mp a \int e^{ax} \left\{ \frac{\operatorname{tg} x}{\operatorname{ctg} x} \right\} dx.$

1.5.51. Интегралы вида $\int R(x, e^{ax}, \sin bx, \cos bx) dx.$

Обозначения: $\sin \varphi = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \varphi = \frac{a}{\sqrt{a^2 + b^2}}.$

1. $\int x^p e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{x^p e^{ax}}{a^2 + b^2} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right] -$
 $- \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} \left[a \left\{ \frac{\sin bx}{\cos bx} \right\} \mp b \left\{ \frac{\cos bx}{\sin bx} \right\} \right] dx.$

$$2. \quad = \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \left\{ \frac{\sin(bx + \varphi)}{\cos(bx + \varphi)} \right\} - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \left\{ \frac{\sin(bx + \varphi)}{\cos(bx + \varphi)} \right\} dx.$$

$$3. \quad \int_0^x x^{\lambda-1} e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [(a+ib)^{-\lambda} \gamma(\lambda, ax+ibx) \mp \\ \mp (a-ib)^{-\lambda} \gamma(\lambda, ax-ibx)] \quad \left[\operatorname{Re} \lambda > \left\{ \begin{matrix} -1 \\ 0 \end{matrix} \right\} \right].$$

$$4. \quad \int_x^\infty x^{\lambda-1} e^{-ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [(a+ib)^{-\lambda} \Gamma(\lambda, ax+ibx) \mp \\ \mp (a-ib)^{-\lambda} \Gamma(\lambda, ax-ibx)].$$

$$5. \quad \int x^n e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \left\{ \frac{\sin(bx + \varphi k)}{\cos(bx + \varphi k)} \right\}.$$

$$6. \quad \int x e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \left\{ \frac{\sin bx}{\cos bx} \right\} + \left(bx - \frac{2ab}{a^2 + b^2} \right) \left\{ \sin bx \right\} \right].$$

$$7. \quad \int x^2 e^{ax} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{e^{ax}}{a^2 + b^2} \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \left\{ \frac{\sin bx}{\cos bx} \right\} \mp \\ \mp \frac{e^{ax}}{a^2 + b^2} \left[bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \left\{ \sin bx \right\}.$$

$$8. \quad \int x^n e^{ax} \left\{ \frac{\sin(bx+c)}{\cos(bx+c)} \right\} dx = \\ = \frac{e^{ax}}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} (n+1)n \dots (n-k+2)}{(a^2 + b^2)^{k/2}} x^{n-k+1} \left\{ \frac{\sin(bx+c+k\varphi)}{\cos(bx+c+k\varphi)} \right\}.$$

$$9. \quad \int x e^{ax} \left\{ \frac{\sin(bx+c)}{\cos(bx+c)} \right\} dx = \\ = \frac{1}{\sqrt{a^2 + b^2}} x e^{ax} \left\{ \frac{\sin(bx+c+\varphi)}{\cos(bx+c+\varphi)} \right\} - \frac{1}{a^2 + b^2} e^{ax} \left\{ \frac{\sin(bx+c+2\varphi)}{\cos(bx+c+2\varphi)} \right\}.$$

$$10. \quad \int_0^x \frac{1}{x} e^{ix} \sin x dx = \operatorname{Si}(2x) + i [C + \ln(2x) - \operatorname{ci}(2x)].$$

1.5.52. Интегралы вида $\int \left\{ \frac{\operatorname{sh}(ax+b)}{\operatorname{ch}(ax+b)} \right\}^m \left\{ \frac{\sin(cx+d)}{\cos(cx+d)} \right\}^n dx.$

$$1. \quad \int \operatorname{sh}(ax+b) \left\{ \frac{\sin(cx+d)}{\cos(cx+d)} \right\} dx = \frac{a}{a^2 + c^2} \operatorname{ch}(ax+b) \left\{ \frac{\sin(cx+d)}{\cos(cx+d)} \right\} \mp \\ \mp \frac{c}{a^2 + c^2} \operatorname{sh}(ax+b) \left\{ \frac{\cos(cx+d)}{\sin(cx+d)} \right\}.$$

$$2. \int \operatorname{ch}(ax+b) \left\{ \frac{\sin(cx+d)}{\cos(cx+d)} \right\} dx = \frac{a}{a^2+c^2} \operatorname{sh}(ax+b) \left\{ \frac{\sin(cx+d)}{\cos(cx+d)} \right\} + \\ + \frac{c}{a^2+c^2} \operatorname{ch}(ax+b) \left\{ \frac{\cos(cx+d)}{\sin(cx+d)} \right\}.$$

$$3. \int \operatorname{sh} x \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{2} \operatorname{ch} x \left\{ \frac{\sin x}{\cos x} \right\} + \frac{1}{2} \operatorname{sh} x \left\{ \frac{\cos x}{\sin x} \right\}.$$

$$4. \int \operatorname{ch} x \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{1}{2} \operatorname{sh} x \left\{ \frac{\sin x}{\cos x} \right\} + \frac{1}{2} \operatorname{ch} x \left\{ \frac{\cos x}{\sin x} \right\}.$$

$$5. \int \operatorname{sh}^{2m}(ax+b) \sin^{2n}(cx+d) dx = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \\ + \frac{(-1)^{m+n}}{2^{2m+2n-1}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} \binom{2n}{k} \sin[(2n-2k)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \}.$$

$$6. \int \operatorname{sh}^{2m}(ax+b) \sin^{2n+1}(cx+d) dx = \\ = \frac{(-1)^{m+1}}{2^{2m+2n}} \binom{2m}{m} \sum_{k=0}^n \frac{(-1)^k}{(2n-2k+1)c} \binom{2n+1}{k} \cos[(2n-2k+1)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \sin[(2n-2k+1)(cx+d)] - \\ - (2n-2k+1)c \operatorname{ch}[(2m-2j)(ax+b)] \cos[(2n-2k+1)(cx+d)] \}.$$

$$7. \int \operatorname{sh}^{2m+1}(ax+b) \sin^{2n}(cx+d) dx = \\ = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{(-1)^j \binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{ch}[(2m-2j+1)(ax+b)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j+1)a \operatorname{ch}[(2m-2j+1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{sh}[(2m-2j+1)(ax+b)] \sin[(2n-2k)(cx+d)] \}.$$

$$\begin{aligned}
 8. \int \operatorname{sh}^{2m+1}(ax+b) \sin^{2n+1}(cx+d) dx = \\
 = \frac{(-1)^n}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^j \binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1)(ax+b)] \sin [(2n-2k+1)(cx+d)] - \\
 - (2n-2k+1) c \operatorname{sh} [(2m-2j+1)(ax+b)] \cos [(2n-2k+1)(cx+d)] \},
 \end{aligned}$$

$$\begin{aligned}
 9. \int \operatorname{sh}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \\
 + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m}{j}}{(2m-2j)a} \operatorname{sh} [(2m-2j)(ax+b)] + \\
 + \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin [(2n-2k)(cx+d)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k) c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 10. \int \operatorname{sh}^{2m}(ax+b) \cos^{2n+1}(cx+d) dx = \\
 = \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{\binom{2n+1}{k}}{(2n-2k+1)c} \sin [(2n-2k+1)(cx+d)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^j \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j) a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k+1)(cx+d)] + \\
 + (2n-2k+1) c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 11. \int \operatorname{sh}^{2m+1}(ax+b) \cos^{2n}(cx+d) dx = \\
 = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{(-1)^j \binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{ch} [(2m-2j+1)(ax+b)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k) c \operatorname{sh} [(2m-2j+1)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$12. \int \operatorname{sh}^{2m+1} (ax+b) \cos^{2n+1} (cx+d) dx =$$

$$= \frac{1}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^j \binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j+1) a \operatorname{ch} [(2m-2j+1) (ax+b)] \cos [(2n-2k+1) (cx+d)] + \\ + (2n-2k+1) c \operatorname{sh} [(2m-2j+1) (ax+b)] \sin [(2n-2k+1) (cx+d)] \}.$$

$$13. \int \operatorname{ch}^{2m} (ax+b) \sin^{2n} (cx+d) dx =$$

$$= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \sin [(2n-2k) (cx+d)] + \\ + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh} [(2m-2j) (ax+b)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j) a \operatorname{sh} [(2m-2j) (ax+b)] \cos [(2n-2k) (cx+d)] + \\ + (2n-2k) c \operatorname{ch} [(2m-2j) (ax+b)] \sin [(2n-2k) (cx+d)] \}.$$

$$14. \int \operatorname{ch}^{2m+1} (ax+b) \sin^{2n} (cx+d) dx =$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{\binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{sh} [(2m-2j+1) (ax+b)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{ (2m-2j+1) a \operatorname{sh} [(2m-2j+1) (ax+b)] \cos [(2n-2k) (cx+d)] + \\ + (2n-2k) c \operatorname{ch} [(2m-2j+1) (ax+b)] \sin [(2n-2k) (cx+d)] \}.$$

$$15. \int \operatorname{ch}^{2m} (ax+b) \sin^{2n+1} (cx+d) dx =$$

$$= \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{(-1)^{k+1} \binom{2n+1}{k}}{(2n-2k+1)c} \cos [(2n-2k+1) (cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{(-1)^k \binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j) a \operatorname{sh} [(2m-2j) (ax+b)] \sin [(2n-2k+1) (cx+d)] - \\ - (2n-2k+1) c \operatorname{ch} [(2m-2j) (ax+b)] \cos [(2n-2k+1) (cx+d)] \}.$$

$$\begin{aligned}
 16. \int \operatorname{ch}^{2m+1}(ax+b) \sin^{2n+1}(cx+d) dx = \\
 = \frac{(-1)^n}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^k \binom{2n+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j+1)a \operatorname{sh} [(2m-2j+1)(ax+b)] \sin [(2n-2k+1)(cx+d)] - \\
 - (2n-2k+1)c \operatorname{ch} [(2m-2j+1)(ax+b)] \cos [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 17. \int \operatorname{ch}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \\
 = \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin [(2n-2k)(cx+d)] + \\
 + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh} [(2m-2j)(ax+b)] + \\
 + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j)a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k)c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 18. \int \operatorname{ch}^{2m+1}(ax+b) \cos^{2n}(cx+d) dx = \\
 = \frac{\binom{2n}{n}}{2^{2m+2n}} \sum_{j=0}^m \frac{\binom{2m+1}{j}}{(2m-2j+1)a} \operatorname{sh} [(2m-2j+1)(ax+b)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^m \sum_{k=0}^{n-1} \frac{\binom{2m+1}{j} \binom{2n}{k}}{(2m-2j+1)^2 a^2 + (2n-2k)^2 c^2} \times \\
 \times \{ (2m-2j+1)a \operatorname{sh} [(2m-2j+1)(ax+b)] \cos [(2n-2k)(cx+d)] + \\
 + (2n-2k)c \operatorname{ch} [(2m-2j+1)(ax+b)] \sin [(2n-2k)(cx+d)] \}.
 \end{aligned}$$

$$\begin{aligned}
 19. \int \operatorname{ch}^{2m}(ax+b) \cos^{2n+1}(cx+d) dx = \\
 = \frac{\binom{2m}{m}}{2^{2m+2n}} \sum_{k=0}^n \frac{\binom{2n+1}{k}}{(2n-2k+1)c} \sin [(2n-2k+1)(cx+d)] + \\
 + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^n \frac{\binom{2m}{j} \binom{2n+1}{k}}{(2m-2j)^2 a^2 + (2n-2k+1)^2 c^2} \times \\
 \times \{ (2m-2j)a \operatorname{sh} [(2m-2j)(ax+b)] \cos [(2n-2k+1)(cx+d)] + \\
 + (2n-2k+1)c \operatorname{ch} [(2m-2j)(ax+b)] \sin [(2n-2k+1)(cx+d)] \}.
 \end{aligned}$$

$$20. \int \operatorname{ch}^{2m+1}(ax+b) \cos^{2n+1}(cx+d) dx =$$

$$= \frac{1}{2^{2m+2n}} \sum_{j=0}^m \sum_{k=0}^n \frac{\binom{2m+1}{j} \binom{2n+1}{k}}{(2m-2j+1)^2 a^2 + (2n-2k+1)^2 c^2} \times \\ \times \{ (2m-2j+1) a \operatorname{sh} \{ (2m-2j+1)(ax+b) \} \cos \{ (2n-2k+1)(cx+d) \} + \\ + (2n-2k+1) c \operatorname{ch} \{ (2m-2j+1)(ax+b) \} \sin \{ (2n-2k+1)(cx+d) \} \}.$$

1.5.53. Интегралы вида $\int x^p \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx$.

$$1. \int x^p \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{x^{p+1}}{2} \left\{ \frac{\cos x^2}{\sin x^2} \right\} + \frac{p-1}{2} \int x^{p-2} \left\{ \frac{\cos x^2}{\sin x^2} \right\} dx.$$

$$2. \int x^n \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = (n-1)!! \left[\sum_{k=0}^{[n/4]} (-1)^k \left(\frac{x^{n-4k+2}}{2^{2k+1} (n-4k+3)!!} \left\{ \frac{\cos x^2}{\sin x^2} \right\} + \right. \right. \\ \left. \left. + \frac{x^{n-4k+1}}{2^{2k} (n-4k+1)!!} \right) + \frac{(-1)^{[n/4]}}{2^{2[n/4]} (n-4[n/4]-1)!!} \int x^{n-4[n/4]} \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx \right].$$

$$3. \int \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left\{ \frac{S(x^2)}{C(x^2)} \right\}. \quad 4. \int x \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{1}{2} \left\{ \frac{\cos x^2}{\sin x^2} \right\}.$$

$$5. \int x^2 \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{x}{2} \left\{ \frac{\cos x^2}{\sin x^2} \right\} + \frac{1}{2} \sqrt{\frac{\pi}{2}} \left\{ \frac{C(x^2)}{S(x^2)} \right\}.$$

$$6. \int x^3 \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{x^2}{2} \left\{ \frac{\cos x^2}{\sin x^2} \right\} + \frac{1}{2} \left\{ \frac{\sin x^2}{\cos x^2} \right\}.$$

$$7. \int \frac{1}{x} \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = \frac{1}{2} \left\{ \frac{S(x^2)}{C(x^2)} \right\}.$$

$$8. \int \frac{1}{x^2} \left\{ \frac{\sin x^2}{\cos x^2} \right\} dx = -\frac{1}{x} \left\{ \frac{\sin x^2}{\cos x^2} \right\} + \sqrt{\frac{\pi}{2}} \left\{ \frac{C(x^2)}{S(x^2)} \right\}.$$

$$9. \int \left\{ \frac{\sin (ux^2 + bx + c)}{\cos (ux^2 + bx + c)} \right\} dx = \sqrt{\frac{\pi}{2u}} \left[\cos \frac{4ac - b^2}{4a} \left\{ \frac{S \{ (2ax + b)^2 / (4a) \}}{C \{ (2ax + b)^2 / (4a) \}} \right\} + \right. \\ \left. + \sin \frac{4ac - b^2}{4a} \left\{ \frac{C \{ (2ax + b)^2 / (4a) \}}{S \{ (2ax + b)^2 / (4a) \}} \right\} \right] \quad [a > 0].$$

1.6. ЛОГАРИФИЧЕСКАЯ ФУНКЦИЯ

1.6.1. Интегралы вида $\int x^p \ln^q x dx$.

$$1. \int f(x) \ln g(x) dx = F(x) \ln g(x) - \int \frac{F(x) g'(x)}{g(x)} dx \quad [F(x) = \int f(x) dx].$$

$$2. \int \ln g(x) dx = x \ln g(x) - \int \frac{x g'(x)}{g(x)} dx.$$

$$3. \int f(x) \ln^n x dx = F(x) \ln^n x - n \int \frac{F(x)}{x} \ln^{n-1} x dx \quad [F(x) = \int f(x) dx].$$

$$4. \int x^p \ln^q x dx = \frac{x^{p+1} \ln^{q+1} x}{q+1} - \frac{p+1}{q+1} \int x^p \ln^{q+1} x dx.$$

5.
$$= \frac{x^{p+1} \ln^q x}{p+1} - \frac{q}{p+1} \int x^p \ln^{q-1} x dx, \quad [p, q \neq -1].$$
6.
$$\int x^p \ln^n x dx = \frac{x^{p+1}}{n+1} \sum_{k=0}^n (-1)^k (n+1) n (n-1) \dots (n-k+1) \frac{\ln^{n-k} x}{(p+1)^{k+1}}.$$
7.
$$= \frac{\ln^{n+1} x}{n+1} \quad [p = -1].$$
8.
$$\int \frac{x^p}{\ln x} dx = \text{li}(x^{p+1}) \quad [p \neq -1].$$
9.
$$= \ln |\ln x| + \sum_{k=1}^{\infty} \frac{(p+1)^k}{k! k} \ln^k x.$$
10.
$$\int \frac{\ln^q x}{x^p} dx = -\frac{\ln^q x}{(p-1)x^{p-1}} + \frac{q}{p-1} \int \frac{\ln^{q-1} x}{x^p} dx \quad [p, -q \neq -1].$$
11.
$$\int \frac{\ln^q x}{x} dx = \frac{\ln^{q+1} x}{q+1}.$$
12.
$$\int \frac{\ln^n x}{x^m} dx = \frac{-1}{(n+1)x^{m-1}} \sum_{k=0}^n \frac{(n+1)n \dots (n-k+1)}{(m-1)^{k+1}} \ln^{n-k} x \quad [m \neq 1].$$
13.
$$\int \ln^q x dx = x \ln^q x - q \int \ln^{q-1} x dx \quad [q \neq -1].$$
14.
$$\int \ln^n x dx = x \sum_{k=0}^n (-1)^k \binom{n}{k} k! \ln^{n-k} x.$$
15.
$$\int \ln x dx = x \ln x - x.$$
16.
$$\int \ln^2 x dx = x \ln^2 x - 2x \ln x + 2x.$$
17.
$$\int \ln^3 x dx = x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x.$$
18.
$$\int x^p \ln x dx = x^{p+1} \left[\frac{\ln x}{p+1} - \frac{1}{(p+1)^2} \right].$$
19.
$$\int x^p \ln^2 x dx = x^{p+1} \left[\frac{\ln^2 x}{p+1} - \frac{2 \ln x}{(p+1)^2} + \frac{2}{(p+1)^3} \right].$$
20.
$$\int x^p \ln^3 x dx = x^{p+1} \left[\frac{\ln^3 x}{p+1} - \frac{3 \ln^2 x}{(p+1)^2} + \frac{6 \ln x}{(p+1)^3} - \frac{6}{(p+1)^4} \right].$$
21.
$$\int \frac{\ln x}{x} dx = \frac{\ln^2 x}{2}, \quad 22. \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x}.$$
23.
$$\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2}, \quad 24. \int \frac{\ln^2 x}{x} dx = \frac{\ln^3 x}{3}.$$
25.
$$\int \frac{\ln^3 x}{x^2} dx = -\frac{\ln^3 x}{x} - \frac{2 \ln x}{x} - \frac{2}{x}.$$
26.
$$\int \frac{\ln^2 x}{x^3} dx = -\frac{\ln^2 x}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2}.$$

1.6.2. Интегралы вида $\int \frac{x^p dx}{\ln^q x}$.

1. $\int \frac{x^p dx}{\ln^q x} = \frac{-x^{p+1}}{(q-1) \ln^{q-1} x} + \frac{p+1}{q-1} \int \frac{x^p dx}{\ln^{q-1} x}.$
2. $\int \frac{x^m dx}{\ln^n x} = \frac{-x^{m+1}}{n} \sum_{k=1}^{n-1} \frac{(m+1)^{k-1}}{(n-1)(n-2)\dots(n-k) \ln^{n-k} x} + \frac{(m+1)^{n-1}}{(n-1)!} \operatorname{li}(x^{m+1}).$
3. $\int \frac{x^m dx}{\ln x} = \operatorname{li}(x^{m+1}).$
4. $\int \frac{dx}{x^p \ln^q x} = \frac{-1}{(q-1) x^{p-1} \ln^{q-1} x} - \frac{p-1}{q-1} \int \frac{dx}{x^p \ln^{q-1} x}.$
5. $\int \frac{dx}{x^m \ln^n x} = \frac{1}{n x^{m-1}} \sum_{k=1}^{n-1} \frac{(-1)^k (m-1)^{k-1}}{(n-1)(n-2)\dots(n-k) \ln^{n-k} x} +$
 $+ (-1)^{n-1} \frac{(m-1)^{n-1}}{(n-1)!} \operatorname{li}\left(\frac{1}{x^{m-1}}\right) \quad [n > 1].$
6. $\int \frac{dx}{x^p \ln x} = \ln |\ln x| + \sum_{k=1}^{\infty} (-1)^k \frac{(p-1)^k}{k! k} \ln^k x.$
7. $\int \frac{dx}{x \ln x} = \ln |\ln x|.$
8. $\int \frac{dx}{x \ln^q x} = -\frac{1}{(q-1) \ln^{q-1} x}.$

1.6.3. Интегралы вида $\int (x+a)^q \ln x dx$.

1. $\int (x+a)^m \ln x dx = \frac{1}{m+1} (x+a)^{m+1} \ln x - \frac{1}{m+1} \int \frac{(x+a)^{m+1}}{x} dx.$
2. $= \frac{1}{m+1} [(x+a)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^m \binom{m}{k} \frac{a^{m-k} x^{k+1}}{(k+1)^2}.$
3. $\int (x+a) \ln x dx = \frac{1}{2} (x^2 + 2ax) \ln x - ax - \frac{x^2}{4}.$
4. $\int (x+a)^2 \ln x dx = \frac{1}{3} (x^3 + 3ax^2 + 3a^2x) \ln x - \frac{x^3}{9} - \frac{ax^2}{2} - a^2x.$
5. $\int (x+a)^3 \ln x dx = \frac{1}{4} [(x+a)^4 - a^4] \ln x - \frac{x^4}{16} - \frac{ax^3}{3} - \frac{3a^2x^2}{4} - a^3x.$
6. $\int \frac{\ln^n x dx}{(x+a)^m} = \frac{x \ln^n x}{(m-1)a(x+a)^{m-1}} - \frac{n}{(m-1)a} \int \frac{\ln^{n-1} x dx}{(x+a)^{m-1}} +$
 $+ \frac{m-2}{(m-1)a} \int \frac{\ln^n x dx}{(x+a)^{m-1}}.$
7. $\int \frac{\ln x dx}{(x+a)^n} =$
 $= -\frac{\ln x}{(n-1)(x+a)^{n-1}} + \frac{1}{(n-1)a^{n-1}} \left[\ln \frac{x}{x+a} + \sum_{k=1}^{n-2} \frac{(-1)^k}{k} \binom{n-2}{k} \left(\frac{x}{x+a}\right)^k \right],$

$$8. \int \frac{\ln x dx}{x+a} = \ln x \cdot \ln \frac{x+a}{a} + \text{Li}_2\left(-\frac{x}{a}\right).$$

$$9. \int \frac{\ln x dx}{(x+a)^2} = -\frac{\ln x}{x+a} + \frac{1}{a} \ln \frac{x}{x+a}.$$

$$10. \int \frac{\ln x dx}{(x+a)^3} = -\frac{\ln x}{2(x+a)^2} + \frac{1}{2a(x+a)} + \frac{1}{2a^2} \ln \frac{x}{x+a}.$$

$$11. \int \frac{\ln x dx}{(x+a)^{1/2}} = 2 \left[(\ln x - 2)(x+a)^{1/2} + a^{1/2} \ln \frac{(x+a)^{1/2} + a^{1/2}}{(x+a)^{1/2} - a^{1/2}} \right] \quad [a > 0].$$

$$12. = 2 \left[(\ln x - 2)(x+a)^{1/2} + 2(-a)^{1/2} \arctg \left(\frac{x+a}{-a} \right)^{1/2} \right] \quad [a < 0].$$

1.6.4. Интегралы вида $\int \frac{x^m \ln x}{(a^2 \pm x^2)^{n/2}} dx$.

Условие: $a > 0$.

$$1. \int \frac{\ln x}{x^2+a^2} dx = \frac{1}{a} \ln x \cdot \arctg \frac{x}{a} - \frac{1}{a} \text{Im Li}_2\left(\frac{ix}{a}\right).$$

$$2. \int_0^x \frac{x \ln x}{x^2+a^2} dx = \frac{1}{4} \text{Li}_2\left(-\frac{x^2}{a^2}\right) + \frac{1}{2} \ln x \cdot \ln\left(1+\frac{x^2}{a^2}\right).$$

$$3. \int \frac{x^2 \ln x}{x^2+a^2} dx = -\frac{a}{2} \text{Cl}_2\left(2 \arctg \frac{x}{a}\right) - \frac{a}{2} \text{Cl}_2\left(\pi - 2 \arctg \frac{x}{a}\right) + x \ln a - a \ln a \cdot \arctg \frac{x}{a}.$$

$$4. \int \frac{\ln x}{x^2-a^2} dx = \frac{1}{2a} \left[\ln x \cdot \ln \left| \frac{x-a}{x+a} \right| + \text{Li}_2\left(\frac{x}{a}\right) - \text{Li}_2\left(-\frac{x}{a}\right) \right].$$

$$5. \int_0^x \frac{x \ln x}{x^2-a^2} dx = -\frac{1}{4} \text{Li}_2\left(1-\frac{x^2}{a^2}\right) + \frac{1}{2} \ln a \cdot \ln(a^2-x^2) - \ln^2 a + \frac{\pi^2}{24}.$$

$$6. \int_0^x \frac{\ln x}{\sqrt{a^2-x^2}} dx = \ln \frac{a}{2} \arcsin \frac{x}{a} - \frac{1}{2} \text{Cl}_2\left(2 \arcsin \frac{x}{a}\right).$$

$$7. \int_x^a \frac{\ln x}{\sqrt{a^2-x^2}} dx = \ln \frac{a}{2} \cdot \arccos \frac{x}{a} + \frac{1}{2} \text{Cl}_2\left(\pi - 2 \arccos \frac{x}{a}\right).$$

$$8. \int \frac{\ln x}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} \text{Li}_2\left[\pm \frac{1}{a^2}(x - \sqrt{x^2 \pm a^2})^2\right] + \frac{1}{2} \ln^2 \frac{x + \sqrt{x^2 \pm a^2}}{2a} + \ln a \cdot \ln \sqrt{x^2 \pm a^2}.$$

1.6.5. Интегралы вида $\int x^p \ln(ax+b) dx$.

$$1. \int f(x) \ln^n(ax+b) dx = \frac{1}{a} \int f\left(\frac{t-b}{a}\right) \ln^n t dt \quad [t=ax+b].$$

2. $\int x^{\alpha-1} \ln(ax+b) dx = \frac{x^\alpha}{\alpha} \ln(ax+b) + \frac{ax^{\alpha+1}}{\alpha(\alpha+1)b} {}_2F_1\left(1, \alpha; \alpha+1; -\frac{ax}{b}\right).$
3. $\int x^m \ln(ax+b) dx = \frac{1}{m+1} \left(x^{m+1} + (-1)^m \frac{b^{m+1}}{a^{m+1}} \right) \ln(ax+b) +$
 $+ \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} b^{k-1}}{(m-k+2) a^{k-1}}.$
4. $\int \ln(ax+b) dx = \frac{1}{a} (ax+b) \ln(ax+b) - x.$
5. $\int x \ln(ax+b) dx = \frac{1}{2} \left(x^2 - \frac{b^2}{a^2} \right) \ln(ax+b) - \frac{1}{2} \left(\frac{x^2}{2} - \frac{bx}{a} \right).$
6. $\int x^2 \ln(ax+b) dx = \frac{1}{3} \left(x^3 + \frac{b^3}{a^3} \right) \ln(ax+b) - \frac{1}{3} \left(\frac{x^3}{3} - \frac{bx^2}{2a} + \frac{b^2x}{a^2} \right).$
7. $\int \frac{\ln(ax+b)}{x^p} dx = -\frac{\ln(ax+b)}{(p-1)x^{p-1}} + \frac{a}{p-1} \int \frac{dx}{(ax+b)x^{p-1}}.$
8. $\int \frac{\ln(ax+b)}{x^m} dx =$
 $= \frac{-1}{m-1} \frac{\ln(ax+b)}{x^{m-1}} + \frac{1}{m-1} \left(\frac{-a}{b} \right)^{m-1} \left[\ln \frac{ax+b}{x} + \sum_{k=1}^{m-2} \frac{1}{k} \left(\frac{-b}{ax} \right)^k \right].$
9. $\int \frac{\ln(ax+b)}{x} dx = \ln|b| \cdot \ln|x| - \text{Li}_2\left(\frac{-ax}{b}\right)$ [$ab \neq 0$].
10. $= \ln|x| \cdot \ln(ax+b) - a \int \frac{\ln|x| dx}{ax+b}.$
11. $= \ln|b| \cdot \ln|x| + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left(\frac{ax}{b} \right)^k$ [$|ax| < |b|$].
12. $= \frac{\ln^2|ax|}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \left(\frac{b}{ax} \right)^k$ [$|ax| > |b|$].
13. $\int \frac{\ln(ax+b)}{x^3} dx = \frac{a}{b} \ln|x| - \left(\frac{1}{x} + \frac{a}{b} \right) \ln(ax+b).$
14. $\int \frac{\ln(ax+b)}{x+c} dx = \ln(b-ac) \ln\left(\frac{x+c}{c}\right) - \text{Li}_2\left[\frac{a(x+c)}{ac-b}\right] + \text{Li}_2\left(\frac{ac}{ac-b}\right).$
15. $\int \frac{\ln(x+a)}{(x+b)^2+c^2} dx = \frac{1}{2c} \arctg \frac{cx}{b^2+c^2+bx} \cdot \ln[(a-b)^2+c^2] -$
 $- \frac{1}{2c} \text{Cl}_2(2\theta+2\varphi) + \frac{1}{2c} \text{Cl}_2(2\theta_0+2\varphi) - \frac{1}{2c} \text{Cl}_2(\pi-2\theta) +$
 $+ \frac{1}{2c} \text{Cl}_2(\pi-2\theta_0)$ [$\text{ig } \theta = \frac{x+b}{c}, \text{ig } \theta_0 = \frac{b}{c}, \text{ig } \varphi = \frac{a-b}{c}$].

1.6.6. Интегралы вида $\int x^{l+m} \ln \frac{x+a}{x-a} dx$.

$$1. \int x^p \ln \frac{x+a}{x-a} dx = \frac{1}{p+1} x^{p+1} \ln \frac{x+a}{x-a} + \frac{2a}{p+1} \int \frac{x^{p+1}}{x^2-a^2} dx.$$

$$2. \int x^m \ln \frac{x+a}{x-a} dx = \frac{1}{m+1} \{x^{m+1} - (-a)^{m+1}\} \ln(x+a) - \\ - \frac{1}{m+1} \{x^{m+1} - a^{m+1}\} \ln(x-a) + \frac{a^{m+1}}{m+1} \sum_{k=1}^{m+1} \frac{1 - (-1)^{m-k+1}}{k} \left(\frac{x}{a}\right)^k.$$

$$3. \int \ln \frac{x+a}{x-a} dx = (x+a) \ln(x+a) - (x-a) \ln(x-a) = x \ln \frac{x+a}{x-a} + a \ln(x^2-a^2).$$

$$4. \int x \ln \frac{x+a}{x-a} dx = \frac{1}{2} (x^2-a^2) \ln \frac{x+a}{x-a} + ax.$$

$$5. \int x^3 \ln \frac{x+a}{x-a} dx = \frac{1}{3} x^3 \ln \frac{x+a}{x-a} + \frac{1}{3} a^3 \ln(x^2-a^2) + \frac{ax^2}{3}.$$

$$6. \int x^5 \ln \frac{x+a}{x-a} dx = \frac{x^6-a^6}{6} \ln \frac{x+a}{x-a} + \frac{ax^5}{6} + \frac{a^3x}{2}.$$

$$7. \int \frac{1}{x^m} \ln \frac{x+a}{x-a} dx = \frac{1}{(m-1)x^{m-1}} \ln \frac{x-a}{x+a} + \\ + \frac{1}{(m-1)a^{m-1}} \left[(-1)^{m-1} \ln \frac{x+a}{x} - \ln \frac{x-a}{x} \sum_{k=1}^{m-2} \frac{1 - (-1)^{m-k-1}}{k} \left(\frac{a}{x}\right)^k \right].$$

$$8. \int \frac{1}{x} \ln \frac{a+x}{a-x} dx = \text{Li}_2\left(\frac{x}{a}\right) - \text{Li}_2\left(-\frac{x}{a}\right) \quad [a \neq 0].$$

$$9. = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{x}{a}\right)^{2k+1}.$$

$$10. \int \frac{1}{x^2} \ln \frac{x+a}{x-a} dx = \frac{1}{x} \ln \frac{x-a}{x+a} - \frac{1}{a} \ln \frac{|x^2-a^2|}{x^2}.$$

1.6.7. Интегралы вида $\int x^{l+m} \ln(x^2 \pm a^2) dx$.

$$1. \int x^{2n} \ln |x^2 \pm a^2| dx = \frac{1}{2n+1} \left[x^{2n+1} \ln |x^2 \pm a^2| - \right. \\ \left. - 2 \sum_{k=0}^n \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right] \pm \left[\frac{(-1)^n 2a^{2n+1} \arctg \frac{x}{a}}{a^{2n+1} \ln \left| \frac{x+a}{x-a} \right|} \right].$$

$$2. \int x^{2n+1} \ln |x^2 \pm a^2| dx = \\ = \frac{1}{2n+1} \left[\frac{1}{2} x^{2n+2} - \frac{1}{2} (-1)^{n+1} a^{2n+2} \ln |x^2 \pm a^2| - \sum_{k=1}^{n+1} \frac{(-1)^{n-k+1}}{k} a^{2n-2k+2} x^{2k} \right].$$

$$3. \int \ln |x^2 \pm a^2| dx = x \ln |x^2 \pm a^2| - 2x + \left\{ \frac{2a \operatorname{arctg} \frac{x}{a}}{a \ln \left| \frac{x+a}{x-a} \right|} \right\}.$$

$$4. \int x \ln |x^2 \pm a^2| dx = \frac{1}{2} [(x^2 \pm a^2) \ln |x^2 \pm a^2| - x^2].$$

$$5. \int x^2 \ln |x^2 \pm a^2| dx = \frac{1}{3} x^3 \ln |x^2 \pm a^2| - \frac{2}{9} x^3 \pm \frac{2}{3} a^2 x \mp \frac{1}{3} \left\{ \frac{2a^3 \operatorname{arctg} \frac{x}{a}}{a^2 \ln \left| \frac{x+a}{x-a} \right|} \right\}.$$

$$6. \int x^3 \ln |x^2 \pm a^2| dx = \frac{1}{4} \left[(x^4 - a^4) \ln |x^2 \pm a^2| - \frac{x^4}{2} \pm a^2 x^2 \right].$$

$$7. \int \frac{\ln |x^2 \pm a^2|}{x^{2m}} dx = \frac{-\ln |x^2 \pm a^2|}{(2m-1)x^{2m-1}} + \\ + \frac{(-1)^{m+1}}{(2m-1)a^{2m-1}} \sum_{k=0}^{m-2} \frac{(-1)^k}{2k+1} \left(\frac{a}{x} \right)^{2k+1} - \frac{(-1)^m}{(2m-1)a^{2m-1}} \left\{ 2 \operatorname{arctg} \frac{x}{a} \right. \\ \left. \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

$$8. \int \frac{\ln |x^2 \pm a^2|}{x^{2m+1}} dx = \\ = \frac{-\ln |x^2 \pm a^2|}{2mx^{2m}} + \frac{(-1)^m}{2ma^{2m}} \left[\ln \frac{|x^2 \pm a^2|}{x^2} + \sum_{k=1}^{m-1} \frac{(-1)^k}{k} \left(\frac{a}{x} \right)^{2k} \right].$$

$$9. \int \frac{\ln |x^2 \pm a^2|}{x} dx = \ln^2 x + \frac{1}{2} \operatorname{Li}_2 \left(\mp \frac{a^2}{x^2} \right) \quad \left[\left\{ \frac{a \neq 0}{x^2 > a^2} \right\} \right].$$

$$10. \quad = \ln a^2 \cdot \ln x - \frac{1}{2} \operatorname{Li}_2 \left(\mp \frac{x^2}{a^2} \right) \quad \left[\left\{ \frac{a \neq 0}{x^2 < a^2} \right\} \right].$$

$$11. \int \frac{\ln |x^2 \pm a^2|}{x^2} dx = -\frac{1}{x} \ln |x^2 \pm a^2| \pm \frac{1}{a} \left\{ \frac{2 \operatorname{arctg} \frac{x}{a}}{\ln \left| \frac{x+a}{x-a} \right|} \right\}.$$

$$12. \int \frac{\ln |x^2 \pm a^2|}{x^3} dx = -\frac{1}{2a^2} \frac{a^2 \pm x^2}{x^2} \ln |x^2 \pm a^2| \pm \frac{1}{a^2} \ln x.$$

$$13. \int \frac{\ln(1+x^2)}{1-x} dx = \frac{1}{4} \operatorname{Li}_2(-x^2) + \frac{1}{2} \operatorname{Li}_2 \left(\frac{2x}{1+x^2} \right) - \operatorname{Li}_2(x) + \\ + \frac{1}{4} \ln^2(1+x^2) - \ln(1-x) \ln(1+x^2).$$

$$14. \int \frac{\ln(x^2 + a^2)}{x^2 + a^2} dx = \\ = \frac{2 \ln a}{a} \operatorname{arctg} \frac{x}{a} + \frac{1}{2a} \ln 4 (x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{1}{a} \operatorname{Im} \operatorname{Li}_2 \left(\frac{a+ix}{2a} \right).$$

$$15. \int \frac{(x+a) \ln [(x+a)^2 + b^2]}{(x+a)^2 + c^2} dx = \\ = \frac{1}{2} \operatorname{Li}_2 \left[\frac{c^2 - b^2}{(x+a)^2 + c^2} \right] + \frac{1}{4} \ln \left[\frac{(x+a)^2 + c^2}{c^2} \right] \ln [c^2 (x+a)^2 + c^4].$$

$$\begin{aligned}
 16. \int \frac{\ln [(x+a)^2 + b^2]}{(x+c)^2 + d^2} dx &= \frac{\theta}{d} \ln 2 [(a-c)^2 + b^2 + d^2] - \frac{1}{d} \text{Cl}_2(\pi - 2\theta) + \\
 &+ \frac{(\theta - \varphi)}{d} \ln \frac{1 - \sqrt{1 - e^2}}{2} - \frac{\eta}{d} \ln \left(\frac{1 - \sqrt{1 - e^2}}{e} \right) + \frac{1}{2d} \text{Cl}_2(2\eta) + \\
 &+ \frac{1}{2d} \text{Cl}_2(4\theta - 4\varphi - 2\eta) - \frac{1}{2d} \text{Cl}_2(4\theta - 4\varphi) \\
 \left[e &= \sqrt{1 - \frac{bd}{(a-c)^2 + b^2 + d^2}}; \lg 2\varphi = \frac{2d(a-c)}{b^2 + (a-c)^2 + d^2}; \right. \\
 \lg \theta &= \frac{x+c}{d}, \lg \eta = \frac{e \sin(2\theta - 2\varphi)}{1 + e \cos(2\theta - 2\varphi) - \sqrt{1 - e^2}} \left. \right].
 \end{aligned}$$

1.6.8. Интегралы вида $\int x^{\pm m} \ln(x + \sqrt{x^2 \pm a^2}) dx$.

1. $\int f(x) \ln(x + \sqrt{x^2 + a^2}) dx =$
 $= -2a \ln a \int f\left(\frac{2at}{t^2-1}\right) \frac{t^2+1}{(t^2-1)^2} dt - 2a \int f\left(\frac{2a(y-1)}{y(y-2)}\right) \frac{y^2-2y+2}{y^2(y-2)^2} \ln y dy +$
 $+ 2a \int f\left(\frac{2a(z+1)}{z(z+2)}\right) \frac{z^2+2z+2}{z^2(z+2)^2} \ln z dz$
 $\left[x = \frac{2at}{t^2-1}; t = y-1 = z+1 = \frac{a + \sqrt{x^2 + a^2}}{x}; a > 0 \right].$
2. $\int f(x) \ln(x + \sqrt{x^2 - a^2}) dx =$
 $= -4a \ln a \int f\left(\frac{a(t^2+1)}{t^2-1}\right) \frac{t}{(t^2-1)^2} dt - 4a \int f\left(\frac{a(y^2-2y+2)}{y(y-2)}\right) \frac{y-1}{y^2(y-2)^2} \ln y dy +$
 $+ 4a \int f\left(\frac{a(z^2+2z+2)}{z(z+2)}\right) \frac{z+1}{z^2(z+2)^2} \ln z dz$
 $\left[x = a \frac{t^2+1}{t^2-1}; t = y-1 = z+1 = \sqrt{\frac{x+a}{x-a}}; a > 0 \right].$
3. $\int x^p \ln(x + \sqrt{x^2 \pm a^2}) dx = \frac{x^{p+1}}{p+1} \ln(x + \sqrt{x^2 \pm a^2}) - \frac{1}{p+1} \int \frac{x^{p+1} dx}{\sqrt{x^2 \pm a^2}}.$
4. $\int \ln(x + \sqrt{x^2 \pm a^2}) dx = x \ln(x + \sqrt{x^2 \pm a^2}) - \sqrt{x^2 \pm a^2}.$
5. $\int x \ln(x + \sqrt{x^2 \pm a^2}) dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right) \ln(x + \sqrt{x^2 \pm a^2}) - \frac{x}{4} \sqrt{x^2 \pm a^2}.$
6. $\int x^2 \ln(x + \sqrt{x^2 \pm a^2}) dx =$
 $= \frac{x^3}{3} \ln(x + \sqrt{x^2 \pm a^2}) - \frac{(x^2 \pm a^2)^{3/2}}{9} \pm \frac{a^2}{3} \sqrt{x^2 \pm a^2}.$
7. $\int x^3 \ln(x + \sqrt{x^2 \pm a^2}) dx = \left(\frac{x^4}{4} - \frac{3a^2}{32}\right) \ln(x + \sqrt{x^2 \pm a^2}) -$
 $- \frac{x^3}{16} \sqrt{x^2 \pm a^2} \pm \frac{3}{32} a^2 x \sqrt{x^2 \pm a^2}.$
8. $\int \frac{\ln(x + \sqrt{x^2 + a^2})}{x^p} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{dx}{x^{p-1} \sqrt{x^2 \pm a^2}}.$

$$9. \int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^{2n}} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{(2n-1)x^{2n-1}} \mp \frac{\sqrt{x^2 \pm a^2}}{(2n-1)(2n-2)} \times \\ \times \left[\frac{1}{a^2 x^{2n-2}} \pm \frac{1}{a^{2n}} \sum_{k=1}^{n-2} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{2^k(n-2)(n-3)\dots(n-k-1)} \left(\frac{a}{x}\right)^{2n-2k} \right] + \\ + \frac{(-1)^n}{(2n-1)a^{2n-1}} \frac{(2n-3)!!}{(2n-2)!!} \left\{ \ln \left| \frac{a + \sqrt{x^2 + a^2}}{x} \right| \right. \\ \left. + \arctg \frac{\sqrt{x^2 - a^2}}{a} \right\}.$$

$$10. \int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^{2n+1}} dx = -\frac{1}{2nx^{2n}} \ln(x + \sqrt{x^2 \pm a^2}) \mp \frac{\sqrt{x^2 \pm a^2}}{(2n-1)2n} \times \\ \times \left[\frac{1}{a^2 x^{2n-1}} \pm \frac{1}{a^{2n+1}} \sum_{k=1}^{n-1} (-1)^k \frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \left(\frac{a}{x}\right)^{2n-2k-1} \right].$$

$$11. \int \frac{\ln(x + \sqrt{x^2 + a^2})}{x} dx = \ln|a| \cdot \ln|x| \pm \frac{1}{2} \ln^2 \left| \frac{2x}{a} \right| \pm \\ \pm \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k-2)!! (2k)^3} \left(\frac{a}{x}\right)^{2k} \quad [x/a > 1].$$

$$12. = \ln|a| \cdot \ln|x| + \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!!}{(2k)!! (2k+1)^3} \left(\frac{x}{a}\right)^{2k+1} \quad [|x| < |a|].$$

$$13. \int \frac{\ln(x + \sqrt{x^2 - a^2})}{x} dx = \ln|a| \cdot \ln|x| + \frac{1}{2} \ln^2 \left| \frac{2x}{a} \right| + \\ + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k-2)!! (2k)^3} \left(\frac{a}{x}\right)^{2k} \quad [|x| > |a|].$$

$$14. \int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^3} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{x} \mp \frac{1}{a} \left\{ \ln \left| \frac{a + \sqrt{x^2 + a^2}}{x} \right| \right. \\ \left. + \arctg \frac{\sqrt{x^2 - a^2}}{a} \right\}.$$

$$15. \int \frac{\ln(x + \sqrt{x^2 \pm a^2})}{x^3} dx = -\frac{\ln(x + \sqrt{x^2 \pm a^2})}{2x^2} \mp \frac{\sqrt{x^2 \pm a^2}}{2a^2 x}.$$

1.6.9. Интегралы вида $\int \frac{x^p}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx$,
 $\int \ln^p(x + \sqrt{x^2 \pm 1}) dx$.

$$1. \int \frac{x^p}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \frac{x^{p-1}}{p} \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) - \\ - \frac{x^p}{p^2} \mp \frac{p-1}{p} \int \frac{x^{p-2}}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx.$$

$$2. \int \frac{1}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \frac{1}{2} \ln^2(x + \sqrt{x^2 \pm 1}).$$

$$3. \int \frac{x}{\sqrt{x^2 \pm 1}} \ln(x + \sqrt{x^2 \pm 1}) dx = \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) - x.$$

$$4. \int \ln^p(x + \sqrt{x^2 \pm 1}) dx = x \ln^p(x + \sqrt{x^2 \pm 1}) - \\ - p \sqrt{x^2 \pm 1} \ln^{p-1}(x + \sqrt{x^2 \pm 1}) + p(p-1) \int \ln^{p-2}(x + \sqrt{x^2 \pm 1}) dx.$$

$$5. \int \ln^n(x + \sqrt{x^2 \pm 1}) dx = \\ = \frac{1}{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} (n+1) n \dots (n-2k+1) x \ln^{n-2k}(x + \sqrt{x^2 \pm 1}) - \\ - \sum_{k=0}^{\lfloor n/2 \rfloor} n(n-1) \dots (n-2k) \sqrt{x^2 \pm 1} \ln^{n-2k-1}(x + \sqrt{x^2 \pm 1}).$$

$$6. \int \ln^2(x + \sqrt{x^2 \pm 1}) dx = x \ln^2(x + \sqrt{x^2 \pm 1}) - 2 \sqrt{x^2 \pm 1} \ln(x + \sqrt{x^2 \pm 1}) + 2x.$$

1.6.10. Интегралы вида $\int f(x, \ln x, e^{ax}, \sin x, \cos x, \dots) dx$.

$$1. \int x^{\alpha-1} e^{-ax} \ln x dx = \\ = a^{-\alpha} \gamma(\alpha, ax) \ln x - \alpha^{-2} x^{\alpha} {}_2F_2(\alpha, \alpha; \alpha+1, \alpha+1; -ax) \quad [a, \operatorname{Re} \alpha > 0].$$

$$2. \int_0^x e^{-ax} \ln x dx = \frac{1}{a} [\operatorname{Ei}(-ax) - e^{-ax} \ln x - \ln a - C].$$

$$3. \int_x^{\infty} e^{-ax} \ln x dx = \frac{1}{a} [e^{-ax} \ln x - \operatorname{Ei}(-ax)] \quad [\operatorname{Re} \alpha > 0].$$

$$4. \int_x^{\infty} \frac{1}{x} e^{-ax} \ln x dx = \frac{1}{2} [(C + \ln ax)^2 + \zeta(2) - 2 \ln x \operatorname{Ei}(-ax)] + \sum_{k=0}^{\infty} \frac{(-ax)^{k+1}}{k!(k+1)^2}.$$

$$5. \int_0^x e^{-ax} \ln(x+b) dx = \\ = \frac{1}{a} [\ln b - e^{-ax} \ln(x+b) - e^{ab} \operatorname{Ei}(-ab) + e^{ab} \operatorname{Ei}(-ax-ab)].$$

$$6. \int_0^x e^{-ax} \ln(b-x) dx = \\ = \frac{1}{a} [\ln b - e^{-ax} \ln(b-x) + e^{-ab} \operatorname{Ei}(ab-ax) - e^{-ab} \operatorname{Ei}(ab)] \quad [b > a].$$

$$7. \int_0^x \ln(b^2-x^2) e^{-ax} dx = \frac{1}{a} \{2 \ln b - e^{-ax} \ln(b^2-x^2) - \\ - e^{-ab} [\operatorname{Ei}(ab) - \operatorname{Ei}(ab-ax)] - e^{ab} [\operatorname{Ei}(-ab) - \operatorname{Ei}(-ab-ax)]\}.$$

$$8. \int_0^x \ln x \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \pm \frac{1}{a} \left[\left\{ \text{ci}(ax) \right\} - \ln x \left\{ \frac{\cos ax}{\sin ax} \right\} \right] - \frac{1}{a} \left\{ \frac{\ln a + C}{\pi/2} \right\}.$$

$$9. \int_0^x \ln(b-x) \left\{ \frac{\sin ax}{\cos ax} \right\} dx = \\ = \mp \frac{1}{a} \ln(b-x) \left\{ \frac{\cos ax}{\sin ax} \right\} \pm \frac{1}{a} [\text{ci}(ab-ax) - \text{ci}(ab)] \left\{ \frac{\cos ab}{\sin ab} \right\} + \\ + \frac{1}{a} [\text{Si}(ab-ax) - \text{Si}(ab)] \left\{ \frac{\sin ab}{\cos ab} \right\} + \frac{1}{a} \left\{ \frac{\ln b}{0} \right\}.$$

$$10. \int_0^x \frac{1}{x} \ln(1+ax^n) dx = -\frac{1}{n} \text{Li}_2(-ax^n).$$

$$11. \int_0^x \left\{ \frac{\ln \text{sh } x}{\ln \text{ch } x} \right\} dx = \frac{1}{2} \text{Li}_2(\pm e^{-2x}) - x \ln 2 + \frac{x^2}{2} \mp \frac{\pi^2}{24} \left\{ \frac{2}{1} \right\}.$$

$$12. \int_0^x \ln(\sin x) dx = -\frac{i}{2} \text{Li}_2(e^{-2ix}) - x \ln 2 + \frac{i}{2} \left(\frac{\pi}{2} - x \right)^2.$$

$$13. = x \ln x - x + \sum_{k=1}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)! 4k} (2x)^{2k+1} \quad (|x| < \pi).$$

$$14. \int_0^x \ln(\cos x) dx = \frac{i}{2} \text{Li}_2(-e^{2ix}) - x \ln 2 - \frac{i}{2} x^2.$$

$$15. = \sum_{k=1}^{\infty} (-1)^k \frac{(2^{2k}-1) B_{2k}}{(2k+1)! 4k} (2x)^{2k+1} \quad (|x| < \pi/2).$$

$$16. \int_0^x \left\{ \frac{\ln \sin x}{\ln \cos x} \right\} dx = -x \ln 2 \mp \frac{1}{2} \left\{ \frac{\text{Cl}_2(2x)}{\text{Cl}_2(\pi-2x)} \right\}.$$

$$17. \int_a^x \ln(\cos a - \cos x) dx = \text{Cl}_2(2a) - \text{Cl}_2(x+a) - \text{Cl}_2(x-a) - (x-a) \ln 2.$$

$$18. \int_0^x \ln(\sin a + \sin x) dx = -i \text{Li}_2(e^{-i(x+a)}) - i \text{Li}_2(e^{-i(x-a+\pi)}) + \\ + i \text{Li}_2(e^{-ia}) + i \text{Li}_2(e^{i(a-\pi)}) - x \ln 2 + \frac{i}{2} x(x-\pi).$$

$$19. \int_0^x \ln(\cos a + \cos x) dx = \text{Cl}_2(\pi+a-x) + \text{Cl}_2(\pi-a-x) - x \ln 2.$$

$$20. \int_0^x \ln(\text{tg } x + \text{tg } a) dx = \\ = -x \ln \cos a - \frac{1}{2} \text{Cl}_2(2x+2a) + \frac{1}{2} \text{Cl}_2(2a) - \frac{1}{2} \text{Cl}_2(\pi-2x).$$

$$21. \int_0^x \ln(1 + \sin a \cos x) dx = x \ln \sin^2 \frac{a}{2} - \eta \ln \lg^2 \frac{a}{2} - \\ - \text{Cl}_2(2x) + \text{Cl}_2(2x - 2\eta) + \text{Cl}_2(2\eta) \quad [\lg \eta = \sin x / (\lg(a/2) + \cos x)].$$

$$22. \int_0^x \ln(1 - 2a \cos x + a^2) dx = \text{Cl}_2(2x + 2\omega) - \text{Cl}_2(2x) - \text{Cl}_2(2\omega) - 2\omega \ln a \\ [\lg \omega = a \sin x / (1 - a \cos x)].$$

$$23. \int_0^x \ln(a \cos x + b \sin x) dx = \\ = x \ln \frac{\sqrt{a^2 + b^2}}{2} - \frac{i}{2} [\text{Li}_2(-e^{2i(\varphi-x)}) - \text{Li}_2(-e^{2i\varphi}) - x(x-2\varphi)] \\ [\lg \varphi = b/a; |\varphi| \leq \pi/2; |\varphi-x| \leq \pi/2].$$

$$24. \int_0^x \ln(1 - a^2 \sin^2 x) dx = (\pi - 2\theta) \ln[(1 + \sqrt{1-a^2})/a] + 2x \ln(a/2) - \\ - \frac{\pi}{2} \ln 2 + L(\theta + x) - L(\theta - x) + L(\pi/2 - 2x) \\ [|x| < \pi/2; \text{ctg } \theta = \lg x \sqrt{1-a^2}; |a| < 1].$$

$$25. = x \ln a - L[\pi/2 + x - \arcsin(1/a)] + L[\pi/2 - x - \arcsin(1/a)] \\ [|x| < \pi/2; a |\sin x| < 1; a > 0].$$

$$26. \int_0^x \frac{\sin 2x \ln \lg x}{1 - a^2 \sin^2 2x} dx = \\ = -\frac{1}{4a \sqrt{1-a^2}} \left[\frac{\pi}{2} \ln 2 + L(\theta - \arccos a) - L(\theta + \arccos a) - L(\pi/2 - 2 \arccos a) \right] \\ [\lg \theta = a \cos 2x / \sqrt{1-a^2}; 0 < x < \pi/2; |a| < 1].$$

$$27. \int \left\{ \frac{\sin \ln x}{\cos \ln x} \right\} dx = \frac{x}{2} (\sin \ln x + \cos \ln x).$$

$$28. \int \frac{dx}{(\ln x + a)^n} = -nx \sum_{k=1}^{n-1} \frac{1}{n(n-1) \dots (n-k+1) (\ln x + a)^{n-k}} + \frac{e^{-a}}{(n-1)!} \text{li}(xe^a).$$

$$29. \int \frac{dx}{\ln x + a} = e^{-a} \text{li}(xe^a).$$

1.6.11. Интегралы вида $\int \frac{\ln(ax+b) \ln(cx+d)}{jx+g} dx.$

$$1. \int_0^x \frac{\ln^2(1-x)}{x} dx = \\ = \ln x \ln^2(1-x) + 2 \ln(1-x) \text{Li}_2(1-x) - 2 \text{Li}_3(1-x) + 2 \text{Li}_3(1).$$

$$2. \int_0^x \frac{\ln^2(1+x)}{x} dx = \ln x \ln^2(1+x) - \frac{2}{3} \ln^3(1+x) - \\ - 2 \ln(1+x) \operatorname{Li}_2\left(\frac{1}{1+x}\right) - 2 \operatorname{Li}_2\left(\frac{1}{1+x}\right) + 2 \operatorname{Li}_2(1).$$

$$3. \int_0^x \frac{\ln x \ln(1-x)}{x} dx = \operatorname{Li}_2(x) - \ln x \operatorname{Li}_2(x).$$

$$4. \int_1^x \frac{\ln x \ln(x-1)}{x} dx = \frac{1}{3} \ln^3 x + \ln x \operatorname{Li}_2(1/x) - 2 \operatorname{Li}_2(1/x) + 2 \operatorname{Li}_2(1).$$

$$5. \int_0^x \frac{\ln^2 x \ln(1-x)}{x} dx = -2 \operatorname{Li}_4(x) + 2 \ln x \operatorname{Li}_3 x - \ln^2 x \operatorname{Li}_2(x).$$

$$6. \int_0^x \frac{\ln^3 x}{1-x} dx = -6 \operatorname{Li}_4(x) + 6 \ln x \operatorname{Li}_3(x) - 3 \ln^2 x \operatorname{Li}_2(x) - \ln^3 x \ln(1-x).$$

$$7. \int_0^x \frac{\ln^2 x \ln(1-x)}{1-x} dx = 2 \left[\operatorname{Li}_4(1-x) - \operatorname{Li}_4(x) - \operatorname{Li}_4\left(\frac{-x}{1-x}\right) - \operatorname{Li}_4(1) \right] + \\ + 2 \left[\ln(1-x) \operatorname{Li}_3 x - \ln x \operatorname{Li}_3(1-x) + \operatorname{Li}_3(1) \ln\left(\frac{x}{1-x}\right) \right] + \\ + 2 \ln x \ln(1-x) \operatorname{Li}_2(1-x) + \\ + \frac{1}{12} \ln^2(1-x) [6 \ln^2 x + 4 \ln x \ln(1-x) - \ln^2(1-x) - 2\pi^2].$$

$$8. \int_0^x \frac{\ln(1-x) \ln(1-ax)}{x} dx = \operatorname{Li}_2\left(\frac{1-ax}{1-x}\right) - \operatorname{Li}_2\left(\frac{1-ax}{a(1-x)}\right) - \operatorname{Li}_2(1-x) - \\ - \operatorname{Li}_2(1-ax) + \operatorname{Li}_2(1/a) + \operatorname{Li}_2(1) + \ln(1-ax) [\operatorname{Li}_2(1/a) - \operatorname{Li}_2(x)] + \\ + \ln(1-x) [\operatorname{Li}_2(1-ax) - \operatorname{Li}_2(1/a) + \pi^2/6] + \frac{1}{2} \ln a \ln^2(1-x).$$

$$9. \int \frac{\ln(ax+b) \ln(cx+d)}{x+g/f} dx = \ln \frac{f}{df-cg} \operatorname{Li}_2(y) + \ln \frac{f}{bf-ag} \operatorname{Li}_2(hy) + \\ + \ln \frac{f}{df-cg} \ln \frac{f}{bf-ag} \ln y + \int \ln(1-y) \ln(1-hy) \frac{dy}{y} \\ (h = (c/a)(b/-aR)/(df-cg); y = a(fx+R)/(ag-bf); ag-bf \neq 0; cg-df \neq 0).$$

$$10. = \ln \frac{af}{c(ag-bf)} \operatorname{Li}_2(y) + \ln \frac{af}{c(ag-bf)} \ln \frac{f}{bf-ag} \ln y - \\ - \frac{1}{2} \ln \frac{f}{ag-bf} \ln^2 y + \int \ln y \ln(1-y) \frac{dy}{y} \quad [ag-bf \neq 0; cg-df=0].$$

$$\begin{aligned}
 11. \quad &= \ln \frac{cf}{a(cg-df)} \operatorname{Li}_2(z) + \ln \frac{af}{a(cg-df)} \ln \frac{l}{df-cg} \ln z - \\
 &\quad - \frac{1}{2} \ln \frac{l}{df-cg} \ln^2 z + \int \ln z \ln(1-z) \frac{dz}{z} \\
 &\quad \left[z = \frac{c(lx+g)}{ag-df}; \quad ag-bf=0; \quad cg-df \neq 0 \right]. \\
 12. \quad &= \frac{1}{2} \ln \frac{c}{a} \ln^2 v + \frac{1}{3} \ln^3 v \quad [v = a(x+g)/g; \quad ag-bf=0; \quad cg-df=0].
 \end{aligned}$$

1.7. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1.7.1. Введение.

Интегралы от обратных тригонометрических функций сводятся к интегралам от тригонометрических функций при помощи следующих формул:

$$\begin{aligned}
 1. \quad &\int f\left(x, \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt \quad \left[x = \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}\right]. \\
 2. \quad &\int f\left(x, \begin{Bmatrix} \operatorname{arctg} x \\ \operatorname{arccotg} x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-2} dt \quad \left[x = \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}\right]. \\
 3. \quad &\int f\left(x, \sqrt{x^2+a^2}, \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix}\right) dx = \\
 &= \pm a \int f\left(a \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}, a \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-1}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}^{-2} dt \quad \left[x = a \begin{Bmatrix} \operatorname{tg} t \\ \operatorname{ctg} t \end{Bmatrix}\right]. \\
 4. \quad &\int f(x) \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix} dx = F(x) \begin{Bmatrix} \operatorname{arctg}(x/a) \\ \operatorname{arccotg}(x/a) \end{Bmatrix} \mp a \int \frac{F(x)}{x^2+a^2} dx \\
 &\quad [F(x) = \int f(x) dx]. \\
 5. \quad &\int f\left(x, \sqrt{1-x^2}, \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix}\right) dx = \pm \int f\left(\begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}, \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix}, t\right) \begin{Bmatrix} \cos t \\ \sin t \end{Bmatrix} dt \\
 &\quad \left[x = \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix}\right]. \\
 6. \quad &\int f(x) \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} dx = F(x) \begin{Bmatrix} \arcsin x \\ \arccos x \end{Bmatrix} \mp \int \frac{F(x)}{\sqrt{1-x^2}} dx \quad [F(x) = \int f(x) dx].
 \end{aligned}$$

1.7.2. Интегралы вида $\int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n dx$.

$$\begin{aligned}
 1. \quad &\int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n dx = x \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^n \pm \\
 &\quad \pm n \sqrt{a^2-x^2} \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-1} - n(n-1) \int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2} dx. \\
 2. \quad &= x \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2k)! \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2k} \mp \\
 &\quad \mp \sqrt{a^2-x^2} \sum_{k=1}^{[(n+1)/2]} (-1)^k \binom{n}{2k-1} (2k-1)! \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix}^{n-2k+1}. \\
 3. \quad &\int \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix} dx = x \begin{Bmatrix} \arcsin(x/a) \\ \arccos(x/a) \end{Bmatrix} \pm \sqrt{a^2-x^2}.
 \end{aligned}$$

$$4. \int \frac{\{\arcsin(x/a)\}^2}{\{\arccos(x/a)\}} dx = x \frac{\{\arcsin(x/a)\}^2}{\{\arccos(x/a)\}} \pm 2 \sqrt{a^2 - x^2} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} - 2x.$$

$$5. \int \frac{\{\arcsin(x/a)\}^3}{\{\arccos(x/a)\}} dx = x \frac{\{\arcsin(x/a)\}^3}{\{\arccos(x/a)\}} \pm \\ \pm 3 \sqrt{a^2 - x^2} \frac{\{\arcsin(x/a)\}^2}{\{\arccos(x/a)\}} - 6x \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \mp 6 \sqrt{a^2 - x^2}.$$

1.7.3. Интегралы вида $\int x^{\pm n} \frac{\{\arcsin x\}}{\{\arccos x\}} dx$.

$$1. \int x^n \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \frac{x^{n+1}}{n+1} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}}.$$

$$2. \int x^{2n} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \frac{x^{2n+1}}{2n+1} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \\ \pm \frac{\sqrt{a^2 - x^2}}{(2n+1)^2} \left[x^{2n} + a^{2n} \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1) \dots (n-k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \left(\frac{x}{a} \right)^{2n-2k-2} \right].$$

$$3. \int x^{2n+1} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \frac{x^{2n+2}}{2n+2} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \\ \pm \frac{\sqrt{a^2 - x^2}}{(2n+2)^2} \left[x^{2n+1} + a^{2n+1} \sum_{k=1}^n \frac{(2n+1)(2n-1) \dots (2n-2k+3)}{2^k n(n-1) \dots (n-k+1)} \left(\frac{x}{a} \right)^{2n-2k+1} \right] - \\ - \frac{(2n+1)!!}{(2n+2) 2^{n+1} (n+1)!} a^{2n+2} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}}.$$

$$4. \int x \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \frac{x}{4} \sqrt{a^2 - x^2}.$$

$$5. \int x^2 \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \frac{x^3}{3} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2}.$$

$$6. \int x^3 \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \left(\frac{x^4}{4} - \frac{3a^4}{32} \right) \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \frac{1}{32} (2x^3 + 3xa^2) \sqrt{a^2 - x^2}.$$

$$7. \int \frac{1}{x^n} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = \\ = -\frac{1}{(n-1)x^{n-1}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \frac{1}{n-1} \int \frac{dx}{x^{n-1} \sqrt{a^2 - x^2}}.$$

$$8. \int \frac{1}{x^{2n}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = -\frac{1}{(2n-1)x^{2n-1}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \mp \\ \mp \frac{\sqrt{a^2 - x^2}}{(2n-1)(2n-2)} \left[\frac{1}{a^2 x^{2n-2}} + \frac{1}{a^{2n}} \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5) \dots (2n-2k-1)}{2^k (n-2)(n-3) \dots (n-k-1)} \times \right. \\ \left. \times \left(\frac{a}{x} \right)^{2n-2k-2} \right] \mp \frac{(2n-3)!!}{(2n-1) 2^{n-1} a^{2n-1} (n-1)!} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| \quad (n \geq 2).$$

$$9. \int \frac{1}{x^{2n+1}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = -\frac{1}{2nx^{2n}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} +$$

$$+ \frac{\sqrt{a^2-x^2}}{2n(2n-1)} \left[\frac{1}{a^2x^{2n-1}} + \frac{1}{a^{2n+1}} \sum_{k=1}^{n-1} \frac{2^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \left(\frac{a}{x}\right)^{2n-2k-1} \right].$$

$$10. = -\frac{1}{2nx^{2n}} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} +$$

$$+ \frac{1}{2na^{2n}} \sum_{k=0}^{n-1} \frac{1}{2n-2k-1} \binom{n-1}{k} \left(\frac{a^2-x^2}{x^2}\right)^{n-k-1/2}.$$

$$11. \int \frac{1}{x} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx =$$

$$= \ln x \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} \pm \ln \frac{a^2+a\sqrt{a^2-x^2}}{2} \operatorname{arctg} \frac{a+\sqrt{a^2-x^2}}{x} +$$

$$+ 2 \operatorname{Im} \operatorname{Li}_2 \left(i \frac{a+\sqrt{a^2-x^2}}{2x} \right) \pm 2 \operatorname{Im} \operatorname{Li}_2 \left(\frac{x+ia+i\sqrt{a^2-x^2}}{2x} \right).$$

$$12. = \pm \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!(2k+1)^2} \left(\frac{x}{a}\right)^{2k+1} + \frac{\pi}{2} \{0\}.$$

$$13. \int_0^x \frac{\arcsin x}{x} dx = \frac{1}{2} \operatorname{Cl}_2(2 \arcsin x) + \arcsin x \ln 2x.$$

$$14. \int \frac{1}{x^2} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = -\frac{1}{x} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} + \frac{1}{a} \ln \left| \frac{a+\sqrt{a^2-x^2}}{x} \right|.$$

$$15. \int \frac{1}{x^3} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} dx = -\frac{1}{2x^2} \frac{\{\arcsin(x/a)\}}{\{\arccos(x/a)\}} + \frac{1}{2a^2x} \sqrt{a^2-x^2}.$$

1.7.4. Интегралы вида $\int (1 \pm x)^{\pm n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.$

$$1. \int (1+x)^{n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \frac{2}{2n+3} (1+x)^{n+3/2} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm$$

$$\pm \frac{2^{n+3} \sqrt{1-x}}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(1-x)^k}{2^k(2k+1)}.$$

$$2. \int (1-x)^{n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = -\frac{2}{2n+3} (1-x)^{n+3/2} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm$$

$$\pm \frac{2^{n+3} \sqrt{1+x}}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(1+x)^k}{2^k(2k+1)}.$$

$$\begin{aligned}
 3. \int \frac{1}{(1+x)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \\
 = -\frac{2}{(2n-1)(1+x)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} + \frac{\sqrt{1-x}}{(2n-1)(2n-2)2^{n-2}(1+x)^{n-1}} \left[2^{n-2} + \right. \\
 \left. + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{(2n-4)(2n-6)\dots(2n-2k-2)} 2^{n-2-k}(1+x)^k \right] \pm \\
 \pm \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{1}{(2n-1)2^{n-2}\sqrt{2}} \ln \frac{\sqrt{2}-\sqrt{1-x}}{\sqrt{1+x}}.
 \end{aligned}$$

$$\begin{aligned}
 4. \int \frac{1}{(1-x)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \\
 = \frac{2}{(2n-1)(1-x)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} + \frac{\sqrt{1+x}}{(2n-1)(2n-2)2^{n-2}(1-x)^{n-1}} \left[2^{n-2} + \right. \\
 \left. + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5)\dots(2n-2k-1)}{(2n-4)(2n-6)\dots(2n-2k-2)} 2^{n-2-k}(1-x)^k \right] \pm \\
 \pm \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{1}{(2n-1)2^{n-2}\sqrt{2}} \ln \frac{\sqrt{2}-\sqrt{1+x}}{\sqrt{1-x}}.
 \end{aligned}$$

$$5. \int \frac{\arcsin x}{(ax+b)^2} dx = -\frac{\arcsin x}{a(ax+b)^2} - \frac{2}{a\sqrt{b^2-a^2}} \operatorname{arctg} \sqrt{\frac{(b-a)(1-x)}{(a+b)(1+x)}} \quad [a^2 < b^2].$$

$$6. = -\frac{\arcsin x}{a(ax+b)} - \frac{1}{a\sqrt{a^2-b^2}} \ln \frac{\sqrt{(a+b)(1+x)} + \sqrt{(a-b)(1-x)}}{\sqrt{(a+b)(1+x)} - \sqrt{(a-b)(1-x)}} \quad [a^2 > b^2].$$

$$7. \int \frac{x \arcsin x}{(1+cx^2)^2} dx = \frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{2c\sqrt{c+1}} \operatorname{arctg} \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \quad [c > -1].$$

$$8. = -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \quad [c < -1].$$

1.7.5. Интегралы вида $\int x^p (1-x^2)^{q+1/2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\}^r dx$.

Интегралы этого пункта выражаются через элементарные функции в следующих случаях:

- 1) $p, r=0, 1, 2, \dots; \quad q=-1, 0, 1, 2, \dots;$
- 2) $r=1; \quad p=0, 1, 2, \dots; \quad q=-1, -2, \dots;$
- 3) $p=-2, -4, \dots;$
- 4) $p=0; \quad q=-1.$

$$\begin{aligned}
 1. \int x^m (1-x^2)^{n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \\
 = -\frac{x^{m+1} (1-x^2)^{n+3/2}}{m+2n+2} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm \frac{1}{m+2n+2} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{m+2k}}{m+2k} + \\
 + \frac{m-1}{m+2n+2} \int x^{m-2} (1-x^2)^{n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.
 \end{aligned}$$

$$\begin{aligned}
 2. \int x (1-x^2)^{n+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \\
 = -\frac{(1-x^2)^{n+3/2}}{2n+3} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm \frac{1}{2n+3} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k+1}}{2k+1}.
 \end{aligned}$$

$$\begin{aligned}
 3. \int (1-x^2)^{p+1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \frac{x(1-x^2)^{p+1/2}}{2p+2} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm \\
 \pm \frac{(1-x^2)^{p+1}}{4(p+1)^2} + \frac{2p+1}{2p+2} \int (1-x^2)^{p-1/2} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.
 \end{aligned}$$

$$4. \int \frac{1}{\sqrt{1-x^2}} \frac{\{\arcsin x\}^p}{\{\arccos x\}} dx = \pm \frac{1}{p+1} \frac{\{\arcsin x\}^{p+1}}{\{\arccos x\}}.$$

$$5. \int \frac{1}{\sqrt{1-x^2}} \frac{\{\arcsin x\}^{-1}}{\{\arccos x\}} dx = \pm \ln \left\{ \frac{\{\arcsin x\}}{\{\arccos x\}} \right\}.$$

$$\begin{aligned}
 6. \int \frac{\sqrt{1-x^2}}{x} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \\
 = \sqrt{1-x^2} \frac{\{\arcsin x\}}{\{\arccos x\}} - x + \int \frac{1}{x \sqrt{1-x^2}} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.
 \end{aligned}$$

$$\begin{aligned}
 7. \int \frac{(1-x^2)^{n+1/2}}{x^m} \frac{\{\arcsin x\}}{\{\arccos x\}} dx = \\
 = -\frac{(1-x^2)^{n+3/2}}{(m-1)x^{m-1}} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm \frac{1}{m-1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k-m+1}}{2k-m+2} - \\
 - \frac{2n-m+4}{m-1} \int \frac{(1-x^2)^{n+1/2}}{x^{m-2}} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.
 \end{aligned}$$

$$\begin{aligned}
 8. &= -\frac{(1-x^2)^{n+1/2}}{(m-1)x^{m-1}} \frac{\{\arcsin x\}}{\{\arccos x\}} \pm \\
 &\pm \frac{1}{m-1} \sum_{k=0}^n (-1)^k \frac{x^{2k-m+1}}{2k-m+2} - \frac{2n+1}{m-1} \int \frac{(1-x^2)^{n-1/2}}{x^{m-2}} \frac{\{\arcsin x\}}{\{\arccos x\}} dx.
 \end{aligned}$$

Если $k=m/2-1$, то соответствующий член в сумме следует заменить на $(-1)^{m/2-1} \frac{1}{m-1} \binom{n+1}{m/2-1} \ln x$.

9.
$$\int \frac{(1-x^2)^{n+1/2}}{x} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{(1-x^2)^{n+1/2}}{2n+1} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k+1}}{2k+1} + \int \frac{(1-x^2)^{n-1/2}}{x} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
10.
$$\int \frac{x}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = -\sqrt{1-x^2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm x.$$
11.
$$\int \frac{1}{(1-x^2)^{p+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x}{(2p-1)(1-x^2)^{p-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{(2p-1)(2p-2)(1-x^2)^{p-1}} + \frac{2p-2}{2p-1} \int \frac{1}{(1-x^2)^{p-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
12.
$$\int \frac{1}{(1-x^2)^{3/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \ln \sqrt{1-x^2}.$$
13.
$$\int \frac{x^m}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = -\frac{1}{m} x^{m-1} \sqrt{1-x^2} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \frac{x^m}{m} +$$

$$+ \frac{m-1}{m} \int \frac{x^{m-2}}{\sqrt{1-x^2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
14.
$$\int \frac{x}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= \frac{1}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp \frac{1}{2n-1} \int \frac{dx}{(1-x^2)^n}.$$
15.
$$\int \frac{x^m}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx = \frac{x^{m-1}}{(2n-m)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp$$

$$\mp \frac{1}{2n-m} \int \frac{x^{m-1} dx}{(1-x^2)^n} - \frac{m-1}{2n-m} \int \frac{x^{m-2}}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
16.
$$\int \frac{x^m}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= \frac{x^{m-1}}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \mp \frac{1}{2n-1} \int \frac{x^{m-1} dx}{(1-x^2)^n} -$$

$$- \frac{m-1}{2n-1} \int \frac{x^{m-2}}{(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$
17.
$$\int \frac{1}{x^m (1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx =$$

$$= -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} \pm \frac{1}{m-1} \int \frac{dx}{x^{m-1}(1-x^2)^n} +$$

$$+ \frac{2n+m-2}{m-1} \int \frac{1}{x^{m-2}(1-x^2)^{n+1/2}} \left\{ \begin{matrix} \arcsin x \\ \arccos x \end{matrix} \right\} dx.$$

$$18. \int \frac{1}{x(1-x^2)^{n+1/2}} \left\{ \frac{\arcsin x}{\arccos x} \right\} dx =$$

$$= \frac{1}{(2n-1)(1-x^2)^{n-1/2}} \left\{ \frac{\arcsin x}{\arccos x} \right\} \mp \frac{1}{2n-1} \int \frac{dx}{(1-x^2)^n} +$$

$$+ \int \frac{1}{x(1-x^2)^{n-1/2}} \left\{ \frac{\arcsin x}{\arccos x} \right\} dx.$$

$$19. \int \left\{ \frac{\arcsin x}{\arccos x} \right\}^p dx = x \left\{ \frac{\arcsin x}{\arccos x} \right\}^p \mp$$

$$\pm p \sqrt{1-x^2} \left\{ \frac{\arcsin x}{\arccos x} \right\}^{p-1} - p(p-1) \int \left\{ \frac{\arcsin x}{\arccos x} \right\}^{p-2} dx.$$

$$20. \int \frac{dx}{\arcsin x} = \ln(\arcsin x) + \sum_{k=1}^{\infty} (-1)^k \frac{\arcsin^{2k+1} x}{2k(2k)!}.$$

$$21. \int \frac{dx}{\arccos x} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\arccos^{2k+1} x}{(2k+1)(2k+1)!}.$$

1.7.6. Интегралы вида $\int x^p \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} dx.$

$$1. \int x^p \operatorname{arcsec} \frac{x}{a} dx = \frac{x^{p+1}}{p+1} \operatorname{arcsec} \frac{x}{a} \mp \frac{a}{p+1} \int \frac{x^p dx}{\sqrt{x^2-a^2}}$$

$$\left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$2. \int x^p \operatorname{arccosec} \frac{x}{a} dx = \frac{x^{p+1}}{p+1} \operatorname{arccosec} \frac{x}{a} \pm \frac{a}{p+1} \int \frac{x^p dx}{\sqrt{x^2-a^2}}$$

$$\left[\left\{ -\frac{\pi/2}{0} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

$$3. \int x^{2n} \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} dx = \frac{x^{2n+1}}{2n+1} \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} \mp$$

$$\mp \frac{ax^{2n-1} \sqrt{x^2-a^2}}{2n(2n+1)} \left[1 + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3) \dots (2n-2k-1)}{(2n-2)(2n-4) \dots (2n-2k-2)} \left(\frac{a}{x} \right)^{2k+2} \right] \mp$$

$$\mp \frac{1}{2n+1} \frac{(2n-1)!!}{(2n)!!} \ln \left| x + \sqrt{x^2-a^2} \right| \left[0 < \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} < \frac{\pi}{2} \right].$$

$$4. \int x^{2n+1} \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} dx = \frac{x^{2n+2}}{2n+2} \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} \mp$$

$$\mp \frac{ax^{2n} \sqrt{x^2-a^2}}{(2n+1)(2n+2)} \left[1 + \sum_{k=0}^{n-1} \frac{2n(2n-2) \dots (2n-2k)}{(2n-1)(2n-3) \dots (2n-2k-1)} \left(\frac{a}{x} \right)^{2k+2} \right].$$

$$5. = \frac{x^{2n+2}}{2n+2} \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} \mp \frac{a^{2n+1} \sqrt{x^2-a^2}}{2n+2} \sum_{k=0}^n \binom{n}{k} \frac{1}{2k+1} \left(\frac{x^2-a^2}{a^2} \right)^k$$

$$\left[0 < \left\{ \frac{\operatorname{arcsec}(x/a)}{\operatorname{arccosec}(x/a)} \right\} < \frac{\pi}{2} \right].$$

$$6. \int \frac{1}{x^{2n}} \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} dx = -\frac{1}{(2n-1)x^{2n-1}} \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} \pm$$

$$\pm \frac{\sqrt{x^2-a^2}}{(2n-1)a^{2n-1}x} \sum_{k=0}^{n-1} \frac{(-1)^k (n-1)!}{2k+1} \left(\frac{x^2-a^2}{x^2} \right)^k \quad \left[0 < \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} < \frac{\pi}{2} \right].$$

$$7. \int \frac{1}{x^{2n+1}} \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} dx = -\frac{1}{2nx^{2n}} \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} \pm$$

$$\pm \frac{\sqrt{x^2-a^2}}{(2n)^3 ax^{2n}} \left[1 + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3)\dots(2n-2k-1)}{(2n-2)(2n-4)\dots(2n-2k-2)} \left(\frac{x}{a} \right)^{2k+2} \right] +$$

$$+ \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n} \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} \quad \left[0 < \left\{ \operatorname{arcsec} \left(\frac{x}{a} \right) \right\} < \frac{\pi}{2} \right].$$

$$8. \int \operatorname{arcsec} \frac{x}{a} dx = x \operatorname{arcsec} \frac{x}{a} \mp a \ln |x + \sqrt{x^2-a^2}|$$

$$\left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$9. \int \operatorname{arccosec} \frac{x}{a} dx = x \operatorname{arccosec} \frac{x}{a} \pm a \ln |x + \sqrt{x^2-a^2}|$$

$$\left[\left\{ -\frac{0}{\pi/2} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

$$10. \int x \operatorname{arcsec} \frac{x}{a} dx = \frac{x^2}{2} \operatorname{arcsec} \frac{x}{a} \mp \frac{a}{2} \sqrt{x^2-a^2}$$

$$\left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$11. \int x \operatorname{arccosec} \frac{x}{a} dx = \frac{x^2}{2} \operatorname{arccosec} \frac{x}{a} \pm \frac{a}{2} \sqrt{x^2-a^2}$$

$$\left[\left\{ -\frac{0}{\pi/2} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

$$12. \int x^2 \operatorname{arcsec} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arcsec} \frac{x}{a} \mp \frac{ax}{6} \sqrt{x^2-a^2} \mp$$

$$\mp \frac{a^3}{6} \ln |x + \sqrt{x^2-a^2}| \quad \left[\left\{ -\frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$13. \int x^2 \operatorname{arccosec} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arccosec} \frac{x}{a} \pm \frac{ax}{6} \sqrt{x^2-a^2} \pm$$

$$\pm \frac{a^3}{6} \ln |x + \sqrt{x^2-a^2}| \quad \left[\left\{ \frac{0}{-\pi/2} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

$$14. \int \frac{1}{x} \operatorname{arcsec} \frac{x}{a} dx = \frac{\pi}{2} \ln |x| + \frac{a}{x} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)^2} \left(\frac{a}{x} \right)^{2k+1}$$

$$[0 < \operatorname{arcsec} (x/a) < \pi/2].$$

$$15. \int \frac{1}{x} \operatorname{arccosec} \frac{x}{a} dx = -\frac{a}{x} - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!! (2k+1)^2} \left(\frac{a}{x} \right)^{2k+1}$$

$$[-\pi/2 < \operatorname{arccosec} (x/a) < \pi/2].$$

$$16. \int \frac{1}{x^2} \operatorname{arcsec} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arcsec} \frac{x}{a} \pm \frac{\sqrt{x^2 - a^2}}{ax} \quad \left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$17. \int \frac{1}{x^2} \operatorname{arccosec} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arccosec} \frac{x}{a} \mp \frac{\sqrt{x^2 - a^2}}{ax} \quad \left[\left\{ -\frac{0}{\pi/2} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

$$18. \int \frac{1}{x^3} \operatorname{arcsec} \frac{x}{a} dx = -\frac{1}{2x^2} \operatorname{arcsec} \frac{x}{a} \pm \frac{\sqrt{x^2 - a^2}}{4ax^2} \pm \frac{1}{4a^2} \operatorname{arccos} \left| \frac{a}{x} \right| \quad \left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} \frac{x}{a} < \left\{ \frac{\pi/2}{\pi} \right\} \right].$$

$$19. \int \frac{1}{x^3} \operatorname{arccosec} \frac{x}{a} dx = -\frac{1}{2x^2} \operatorname{arccosec} \frac{x}{a} \mp \frac{\sqrt{x^2 - a^2}}{4ax^2} \mp \frac{1}{4a^2} \operatorname{arccos} \left| \frac{a}{x} \right| \quad \left[\left\{ -\frac{0}{\pi/2} \right\} < \operatorname{arccosec} \frac{x}{a} < \left\{ \frac{\pi/2}{0} \right\} \right].$$

1.7.7. Интегралы вида $\int (x \pm 1)^{\pm n + 1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx$.

Условие: $0 < \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} < \frac{\pi}{2}$.

$$1. \int (x+1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \frac{2}{2n+3} (x+1)^{n+3/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \mp \\ \mp \frac{4}{2n+3} \operatorname{arctg} \sqrt{x-1} \mp \frac{4}{2n+3} \sum_{k=1}^{n+1} \binom{n+1}{k} \int (y^2+1)^{k-1} dy \quad [y = \sqrt{x-1}].$$

$$2. \int (x-1)^{n+1/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = \frac{2}{2n+3} (x-1)^{n+2/2} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{2(-1)^n \ln \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}}{2n+3} \mp \frac{4(-1)^{n+1}}{2n+3} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int (y^2-1)^k dy \\ [y = \sqrt{x+1}].$$

$$3. \int \frac{1}{(x+1)^{n+1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = -\frac{2}{(2n-1)(x+1)^{n-1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{4}{2n-1} \operatorname{arctg} \sqrt{x-1} \mp \frac{4}{2n-1} \sum_{k=0}^{n-1} \int \frac{dy}{(y^2+2)^{n-k}} \quad [y = \sqrt{x-1}].$$

$$4. \int \frac{1}{(x-1)^{n+1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} dx = -\frac{2}{(2n-1)(x-1)^{n-1/2}} \left\{ \begin{matrix} \operatorname{arcsec} x \\ \operatorname{arccosec} x \end{matrix} \right\} \pm \\ \pm \frac{2(-1)^n \ln \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}}{2n-1} \pm \frac{4(-1)^n}{2n-1} \sum_{k=0}^{n-1} (-1)^k \int \frac{dy}{(y^2-2)^{n-k}} \quad [y = \sqrt{x+1}].$$

1.7.8. Интегралы вида $\int x^m (x^2-1)^{\pm n+1/2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx$.

Условие: $0 < \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} < \frac{\pi}{2}$.

Интегралы $\int x^m (x^2-1)^{n+1/2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx$ при целых m, n, l с помощью подстановок $y = \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\}$ сводятся к интегралам вида $\int y' \left\{ \frac{\sin y}{\cos y} \right\}^{2n+2} \times \times \left\{ \frac{\cos y}{\sin y} \right\}^{-(2n+m+1)} dy$. Результат выражается в виде конечной комбинации элементарных функций в следующих случаях [18]:

- 1) $l \geq 1, \quad n \geq -1, \quad m \leq -2n-3$;
- 2) $l = 1, \quad n < -1, \quad m \leq -2n-3$;
- 3) $l = 1, \quad m - \text{нечетное}, \quad m \geq -2n-3$;
- 4) $m = -1, \quad n = -1$.

$$1. \int \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} (x^2-1)^{n+1/2} dx = \frac{(x^2-1)^{n+1/2} x \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\}}{2(n+1)} \mp \\ \mp \frac{(-1)^n}{2(n+1)} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k+1}}{2k+1} - \frac{2n+1}{2n+2} \int \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} (x^2-1)^{n-1/2} dx.$$

$$2. \int x^m (x^2-1)^{n+1/2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx = \\ = \frac{x^{m-1} (x^2-1)^{n+3/2}}{m+2n+2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} \mp \frac{(-1)^{n+1}}{m+2n+2} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{m-2k-1}}{m-2k-1} + \\ + \frac{m-1}{m+2n+2} \int x^{m-2} (x^2-1)^{n+1/2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx.$$

$$3. \int x (x^2-1)^{n+1/2} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx = \frac{(x^2-1)^{n+3/2}}{2n+3} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} \pm \\ \pm \frac{(-1)^n}{2n+3} \ln x + \frac{(-1)^{n+1}}{2(2n+3)} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k}}{k}.$$

$$4. \int \frac{(x^2-1)^{n+1/2}}{x^m} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx = \\ = \frac{(x^2-1)^{n+3/2}}{(m-1)x^{m-1}} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} \mp \frac{(-1)^{n+1}}{m-1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{x^{2k-m+1}}{2k-m+1} - \\ - \frac{2n-m+4}{m-1} \int \frac{(x^2-1)^{n+1/2}}{x^{m-2}} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx.$$

$$5. = - \frac{(x^2-1)^{n+1/2}}{(m-1)x^{m-1}} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} - \frac{(-1)^n}{m-1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{x^{2k-m+1}}{2k-m+1} + \\ + \frac{2n+1}{m-1} \int \frac{(x^2-1)^{n-1/2}}{x^{m-2}} \left\{ \frac{\operatorname{arccsec} x}{\operatorname{arccosec} x} \right\} dx.$$

- $$\begin{aligned}
 6. \int \frac{(x^2-1)^{n+1/2}}{x} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{(x^2-1)^{n+1/2}}{2n+1} \left\{ \operatorname{arcsec} x \right\} \mp \\
 &\mp \frac{(-1)^n}{2n+1} \ln x \mp \frac{(-1)^n}{2(2n+1)} \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{x^{2k}}{k} - \int \frac{(x^2-1)^{n-1/2}}{x} \left\{ \operatorname{arcsec} x \right\} dx, \\
 7. \int \frac{\sqrt{x^2-1}}{x} \left\{ \operatorname{arcsec} x \right\} dx &= \mp \frac{1}{2} \left\{ \operatorname{arcsec} x \right\}^2 + \sqrt{x^2-1} \left\{ \operatorname{arcsec} x \right\} \mp \ln x, \\
 8. \int \frac{x^m}{\sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{1}{m} x^{m-1} \sqrt{x^2-1} \left\{ \operatorname{arcsec} x \right\} \mp \\
 &\mp \frac{x^{m-1}}{m(m-1)} \mp \frac{m-1}{m} \int \frac{x^{m-2}}{\sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx, \\
 9. \int \frac{x}{(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{-1}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \pm \\
 &\pm \frac{(-1)^n}{2(2n-1)} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n-1}{k} \left(\frac{x^2}{x^2-1} \right)^k \mp \frac{(-1)^n}{2n-1} \ln \frac{\sqrt{x^2-1}}{x}, \\
 10. \int \frac{1}{x^m (x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{1}{(m-1)x^{m-1}(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \mp \\
 &\mp \frac{1}{m-1} \int \frac{dx}{x^m (x^2-1)^n} + \frac{2n+m-2}{m-1} \int \frac{1}{x^{m-2}(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx, \\
 11. \int \frac{1}{x^m \sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{\sqrt{x^2-1}}{(m-1)x^{m-1}} \left\{ \operatorname{arcsec} x \right\} \pm \\
 &\pm \frac{1}{(m-1)^2 x^{m-1}} \mp \frac{m-2}{m-1} \int \frac{1}{x^{m-2} \sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx, \\
 12. \int \frac{1}{x^2 \sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx &= \frac{\sqrt{x^2-1}}{x} \left\{ \operatorname{arcsec} x \right\} \pm \frac{1}{x}, \\
 13. \int \frac{x}{\sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx &= \sqrt{x^2-1} \left\{ \operatorname{arcsec} x \right\} \mp \ln x, \\
 14. \int \frac{1}{(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx &= \mp \frac{x}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \pm \\
 &\pm \frac{1}{2n-1} \int \frac{dx}{(x^2-1)^n} - \frac{2n-2}{2n-1} \int \frac{1}{(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} dx, \\
 15. \int \frac{1}{(x^2-1)^{3/2}} \left\{ \operatorname{arcsec} x \right\} dx &= -\frac{x}{\sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} \pm \frac{1}{2} \ln \frac{x-1}{x+1}.
 \end{aligned}$$

$$16. \int \frac{x^m}{(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx =$$

$$= -\frac{x^{m-1}}{(2n-m)(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \mp \frac{1}{2n-m} \int \frac{x^{m-2} dx}{(x^2-1)^n} -$$

$$- \frac{m-1}{2n-m} \int \frac{x^{m-2}}{(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx.$$

$$17. = -\frac{x^{m-1}}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \mp \frac{1}{2n-1} \int \frac{x^{m-2} dx}{(x^2-1)^n} +$$

$$+ \frac{m-1}{2n-1} \int \frac{x^{m-2}}{(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx.$$

$$18. \int \frac{1}{x(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx =$$

$$= -\frac{1}{(2n-1)(x^2-1)^{n-1/2}} \left\{ \operatorname{arcsec} x \right\} \mp \frac{1}{2n-1} \int \frac{dx}{x(x^2-1)^n} -$$

$$- \int \frac{1}{x(x^2-1)^{n+1/2}} \left\{ \operatorname{arcsec} x \right\} dx.$$

$$19. \int \frac{1}{x\sqrt{x^2-1}} \left\{ \operatorname{arcsec} x \right\} dx = \pm \frac{1}{2} \left\{ \operatorname{arcsec} x \right\}^2.$$

1.7.9. Интегралы вида $\int x^p \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx$.

$$1. \int x^p \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{x^{p+1}}{p+1} \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp \frac{a}{p+1} \int \frac{x^{p+1} dx}{x^2+a^2}.$$

$$2. \int x^{2n} \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{x^{2n+1}}{2n+1} \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp$$

$$\mp \frac{a}{2(2n+1)} \left[n! \sum_{k=0}^{n-1} \frac{(-a^2)^k (x^2+a^2)^{n-k}}{(n-k)! k! (n-k)} + (-a^2)^n \ln(x^2+a^2) \right].$$

$$3. \int x^{2n+1} \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{x^{2n+2}}{2n+2} \left\{ \operatorname{arctg} \frac{x}{a} \right\} +$$

$$+ \frac{a^{2n+2}}{2n+2} \left[\sum_{k=1}^{n+1} \frac{(-1)^{n+k}}{2k-1} \left(\frac{x}{a} \right)^{2k-1} + (-1)^n \operatorname{arctg} \frac{x}{a} \right].$$

$$4. \int \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = x \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp \frac{a}{2} \ln(x^2+a^2).$$

$$5. \int x \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{x^2+a^2}{2} \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp \frac{1}{2} ax.$$

$$6. \int x^2 \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{x^3}{3} \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp \frac{ax^2}{6} \pm \frac{a^3}{6} \ln(x^2+a^2).$$

$$7. \int x^3 \left\{ \operatorname{arctg} \frac{x}{a} \right\} dx = \frac{1}{4} (x^4-a^4) \left\{ \operatorname{arctg} \frac{x}{a} \right\} \mp \frac{ax^3}{12} \pm \frac{a^3x}{4}.$$

8. $\int \frac{1}{x^n} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \frac{-1}{(n-1)x^{n-1}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \pm \frac{a}{n-1} \int \frac{dx}{x^{n-1}(x^2+a^2)}.$
9. $\int \frac{1}{x^{2n}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = -\frac{1}{(2n-1)x^{2n-1}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \mp$
 $\mp \frac{1}{2(2n-1)a^{2n-1}} \left[\sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{k+1} \left(\frac{a}{x} \right)^{2k+2} + (-1)^{n-1} \ln \frac{x^2+a^2}{x^2} \right].$
10. $\int \frac{1}{x^{2n+1}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = -\frac{1}{2nx^{2n}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \pm$
 $\pm \frac{1}{2na^{2n}} \left[\sum_{k=1}^n \frac{(-1)^k}{2n-2k+1} \left(\frac{a}{x} \right)^{2n-2k+1} + (-1)^n \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \right].$
11. $\int \frac{1}{x} \operatorname{arctg} \frac{x}{a} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{x}{a} \right)^{2k+1} \quad [|x| < |a|].$
12. $= \pm \frac{\pi}{2} \ln |x| + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [\pm x/a > 1].$
13. $\int \frac{1}{x} \operatorname{arctg} \frac{x}{a} dx = \frac{\pi}{2} \ln |x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{x}{a} \right)^{2k+1} \quad [|x| < |a|].$
14. $= - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [x/a > 1].$
15. $= \pi \ln |x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x} \right)^{2k+1} \quad [x/a < -1].$
16. $\int \frac{1}{x} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \pm \operatorname{Im} \operatorname{Li}_2 \left(\frac{ix}{a} \right) + \frac{\pi}{2} \left\{ \ln |x| \right\}.$
17. $\int \frac{1}{x^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \frac{-1}{x} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \pm \frac{1}{2a} \ln \frac{x^2}{x^2+a^2}.$
18. $\int \frac{1}{x^3} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = -\frac{1}{2} \left(\frac{1}{x^2} + \frac{1}{a^2} \right) \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} \mp \frac{1}{2ax}.$
19. $\int \frac{1}{(x+b)^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$
 $= \pm \frac{a}{a^2+b^2} \left[\ln \frac{x+b}{\sqrt{x^2+a^2}} - \frac{a^2-bx}{a(x+b)} \operatorname{arctg} \frac{x}{a} \right] - \frac{\pi}{2(x+b)} \left\{ \frac{0}{1} \right\}.$
20. $\int \frac{\operatorname{arctg}(ax+b)}{cx+d} dx = \theta \ln \frac{\sin(\theta+\varphi)}{\cos \theta} + \frac{1}{2} \operatorname{Cl}_2(2\varphi+2\theta) + \frac{1}{2} \operatorname{Cl}_2(\pi-2\theta)$
 $[\operatorname{tg} \varphi = ad-bc; \operatorname{tg} \theta = ax+b].$

1.7.10. Интегралы вида $\int x^p (x^2 + a^2)^q \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\}^r dx$.

Интегралы этого пункта выражаются через элементарные функции в следующих случаях [18]:

- 1) $p, r = 1, 2, 3, \dots$ и $q < -\frac{p}{2} - 1$;
- 2) $r = 1, p$ — четное;
- 3) $r = 1, 2, p$ — нечетное, $q = -1, 0, 1, 2, \dots$;
- 4) $p = 0, q = -1$.

$$1. \int x^m (x^2 + a^2)^n \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx = \\ = \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} \sum_{k=0}^n \binom{n}{k} \frac{a^{2n-2k} x^{m+2k+1}}{m+2k+1} \mp \sum_{k=0}^n \frac{a^{2n-2k+1}}{m+2k+1} \binom{n}{k} \int \frac{x^{m+2k+1}}{1+x^2} dx.$$

$$2. \int x^{2m} (x^2 + a^2) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx = x^{2m+1} \left(\frac{x^2}{2m+3} + \frac{a^2}{2m+1} \right) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} \mp \\ \mp \frac{a x^{2m+2}}{(2m+2)(2m+3)} \mp \frac{(-1)^m}{(2m+1)(2m+3)} \sum_{k=0}^m \frac{(-1)^k a^{2m-2k+3}}{k} \binom{m}{k} (x^2 + a^2)^k \mp \\ \mp \frac{(-1)^m a^{2m+3}}{(2m+1)(2m+3)} \ln (x^2 + a^2).$$

$$3. \int x^{2m+1} (x^2 + a^2) \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx = \\ = \frac{1}{2} \left[\frac{a^2 x^{2m+2}}{m+1} + \frac{x^{2m+4}}{m+2} + \frac{a^{2m+4}}{(m+1)(m+2)} \right] \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} \mp \\ \mp \frac{a x^{2m+3}}{2(m+2)(2m+3)} \mp \frac{a^3}{2(m+1)(m+2)} \sum_{k=0}^m (-1)^k \frac{a^{2k} x^{2m-2k+1}}{2m-2k+1}.$$

$$4. \int \frac{(x^2 + a^2)^n}{x^m} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx = \\ = \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} \sum_{k=0}^n \binom{n}{k} \frac{a^{2n-2k} x^{2k-m+1}}{2k-m+1} \mp \sum_{k=0}^n \frac{a^{2n-2k+1}}{2k-m+1} \binom{n}{k} \int \frac{x^{2k-m+1}}{x^2 + a^2} dx.$$

Если $k = (m-1)/2, k = 0, 1, \dots, n$, то соответствующие слагаемые в суммах заменяются на $a^{2n-m+1} \binom{n}{k} \ln \frac{x}{a} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\}$ и $a^{2n-m+2} \binom{n}{k} \int \frac{\ln (x/a)}{x^2 + a^2} dx$.

$$5. \int \frac{x^m}{(x^2 + a^2)^n} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx = \\ = -\frac{x^{m-1}}{2(n-1)(x^2 + a^2)^{n-1}} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} \pm \frac{a}{2(n-1)} \int \frac{x^{m-1} dx}{(x^2 + a^2)^n} + \\ + \frac{m-1}{2(n-1)} \int \frac{x^{m-2}}{(x^2 + a^2)^{n-1}} \left\{ \begin{matrix} \operatorname{arctg} (x/a) \\ \operatorname{arccctg} (x/a) \end{matrix} \right\} dx.$$

6. $\int \frac{1}{(x^2+a^2)^n} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx =$
 $= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} \pm \frac{1}{4(n-1)^2 a(x^2+a^2)^{n-1}} +$
 $+ \frac{2n-3}{(2n-2)a^3} \int \frac{1}{(x^2+a^2)^{n-1}} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx.$
7. $\int \frac{1}{(x^2+a^2)^n} \operatorname{arctg} \frac{x}{a} dx =$
 $= \operatorname{arctg} \frac{x}{a} \left[\sum_{k=1}^{n-1} \frac{(2n-2k-2)!! (2n-3)!!}{(2n-2)!! (2n-2k-1)!!} \frac{x}{a^{2k} (x^2+a^2)^{n-k}} + \frac{1}{2a^{2n-1}} \frac{(2n-3)!!}{(2n-2)!!} \right] +$
 $+ \frac{a}{2} \sum_{k=1}^{n-1} \frac{(2n-3)!! (2n-2k-2)!!}{(2n-2)!! (2n-2k-1)!! (n-k)} \frac{1}{a^{2k} (x^2+a^2)^{n-k}}.$
8. $\int \frac{x}{(x^2+a^2)^n} \left\{ \frac{\operatorname{arctg} x}{\operatorname{arccctg} x} \right\} dx =$
 $= \frac{1}{2(n-1)} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} \left[\frac{(2n-3)!!}{(2n-2)!! a^{2n-2}} - \frac{1}{(x^2+a^2)^{n-1}} \right] \pm$
 $\pm \frac{x}{4(n-1)^2 a (x^2+a^2)^{n-1}} \left[1 + \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5) \dots (2n-2k-1)}{(2n-4)(2n-6) \dots (2n-2k-2)} \left(\frac{x^2+a^2}{a^2} \right)^k \right].$
9. $\int \frac{x^n}{x^2+a^2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx =$
 $= \int x^{n-2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx - a^2 \int \frac{x^{n-2}}{x^2+a^2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx.$
10. $\int \frac{x^{2m}}{x^2+a^2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx = \pm \frac{(-1)^m a^{2m-1}}{2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\}^2 +$
 $+ \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} \sum_{k=1}^m (-1)^{m-k} \frac{a^{2m-2k} x^{2k-1}}{2k-1} \mp$
 $\mp \sum_{k=1}^m \frac{(-1)^{m-k} a^{2m-2k+1}}{2k-1} \int \frac{x^{2k-1}}{x^2+a^2} dx.$
11. $\int \frac{x^{2m+1}}{1+x^2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx =$
 $= \frac{1}{2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} \sum_{k=1}^m (-1)^{m-k} \frac{a^{2m-2k} x^{2k}}{k} \mp$
 $\mp \frac{1}{2} \sum_{k=1}^m \frac{(-1)^{m-k} a^{2m-2k-1}}{k} \int \frac{x^{2k} dx}{a^2+x^2} + (-1)^m 2^m \int \frac{x}{a^2+x^2} \left\{ \frac{\operatorname{arctg} x}{\operatorname{arccctg} x} \right\} dx.$
12. $\int \frac{1}{x^2+a^2} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\} dx = \pm \frac{1}{2a} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccctg}(x/a)} \right\}^2.$

- $$13. \int \frac{x}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx = \frac{1}{2} \ln \frac{\sqrt{x^2+a^2}}{2a} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} +$$
- $$+ \frac{i}{4} \left[\operatorname{Li}_2 \left(\frac{a+ix}{2a} \right) - \operatorname{Li}_2 \left(\frac{a-ix}{2a} \right) \right] + \frac{\pi}{8} \left\{ \ln (x^2+a^2) \right\}.$$
- $$14. = \frac{1}{2} \ln (x^2+a^2) \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} + \frac{a}{2} \int \frac{\ln (x^2+a^2)}{x^2+a^2} dx.$$
- $$15. \int \frac{x^2}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$
- $$= x \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} + \frac{a}{2} \ln (x^2+a^2) + \frac{a}{2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2.$$
- $$16. \int \frac{x^3}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$
- $$= -\frac{ax}{2} + \frac{1}{2} (x^2+a^2) \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} - a^2 \int \frac{x}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx.$$
- $$17. \int \frac{1}{(x^2+a^2)^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$
- $$= \frac{x}{2a^2 (x^2+a^2)} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} + \frac{1}{4a^3} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2 + \frac{1}{4a (x^2+a^2)}.$$
- $$18. \int \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2 dx =$$
- $$= x \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^2 - a \ln \frac{\sqrt{x^2+a^2}}{2a} \operatorname{arctg} \frac{x}{a} - a \ln \operatorname{Li}_2 \left(\frac{a+ix}{2a} \right) + \frac{\pi a}{2} \left\{ \ln (x^2+a^2) \right\}.$$
- $$19. \int \frac{1}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^r dx = \pm \frac{1}{(r+1)a} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^{r+1}.$$
- $$20. \int \frac{1}{x^2+a^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}^{-1} dx = \pm \frac{1}{a} \ln \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\}.$$
- $$21. \int \frac{x}{\sqrt{a^2-x^2}} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} dx =$$
- $$= -\sqrt{a^2-x^2} \left\{ \operatorname{arctg} \left(\frac{x}{a} \right) \right\} + a \arcsin \frac{x}{a} + \sqrt{2a} \left\{ \operatorname{arctg} \left[\sqrt{2x} (a^2-x^2)^{-1/2} \right] \right\}.$$
- $$22. \int \frac{\operatorname{arctg} x}{(ax^2+b)^{3/2}} dx = \frac{x \operatorname{arctg} x}{b(ax^2+b)^{1/2}} - \frac{1}{b\sqrt{a-b}} \operatorname{arctg} \sqrt{\frac{ax^2+b}{a-b}} \quad [a > b].$$
- $$23. = \frac{x \operatorname{arctg} x}{b\sqrt{ax^2+b}} - \frac{1}{2b\sqrt{b-a}} \ln \frac{\sqrt{ax^2+b} - \sqrt{b-a}}{\sqrt{ax^2+b} + \sqrt{b-a}} \quad [b > a].$$

1.8. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

Неопределенные интегралы от обратных гиперболических функций вычисляются по формулам раздела 1.6 после следующих замен:

$$\operatorname{Arsh} x = \ln (x + \sqrt{x^2+1}),$$

$$\operatorname{Arch} x = \pm \ln (x + \sqrt{x^2-1})$$

[$x \geq 1$].

$$\operatorname{Arth} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad [|x| < 1],$$

$$\operatorname{Arch} x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad [|x| > 1],$$

$$\operatorname{Arsech} x = \pm \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) \quad [0 < x \leq 1],$$

$$\operatorname{Arcosech} x = \ln \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) \quad [x > 0].$$

Глава 2. ОПРЕДЕЛЕННЫЕ ИНТЕГРАЛЫ

2.1. ВВЕДЕНИЕ

2.1.1. Предварительные сведения.

В этой главе содержатся определенные интегралы от элементарных функций. Интегралы, у которых один или оба предела интегрирования являются переменными, в случаях, когда подынтегральная функция от них не зависит (в частности, интегралы от иррациональных функций, сводящиеся к эллиптическим интегралам), помещены в главе 1.

Указанные в формулах условия сходимости интегралов обеспечивают их существование либо в обычном смысле, либо как несобственных интегралов, либо в смысле главного значения [20].

В начале соответствующих разделов помещены интегралы общего вида, зависящие от нескольких параметров и позволяющие получать более общие результаты, чем содержащиеся в последующих пунктах. Эти интегралы выражаются через функции $V_{\frac{r}{2}}^{\pm}$, $V_{\frac{r}{2}}$, приведенные в приложении III; параметры, входящие в них, указаны в каждой формуле в фигурных скобках.

В качестве примера вычислим интеграл

$$\int_0^a \frac{x^{\alpha-1} dx}{V a^2 - x^2 (bx + c)} = I \quad [a, b, c > 0].$$

В силу формулы 2.2.2.5

$$I = \frac{a^{\alpha-2}}{b} \begin{cases} V_1^{+} + V_{\frac{1}{2}}^{+} \\ V_1^{-} \end{cases} \quad \begin{matrix} |y| \leq 1, \\ |y| \geq 1, \end{matrix}$$

где $y = c/(ab)$, $r = 2$, $\beta = 1/2$, $\rho = 0$. Пусть, например, $|y| \geq 1$ ($c \geq ab$); тогда $I = \frac{a^{\alpha-2}}{b} V_1^{-}$. Подставляя значения r , β и ρ в параметры E_1^{-} , u , a_3 , b_3 , c_3 , z функции V_1^{-} и учитывая, что $p = 2$, $q = 1$, получаем

$$\begin{aligned} V_1^{-} &= \sum_{h=0}^1 \frac{(-1)^h}{h! 2^h} \Gamma \left[\begin{matrix} 1/2, (\alpha+h)/2, 1+h \\ 1, (1+\alpha+h)/2 \end{matrix} \right] y^{-h-1} \times \\ &\quad \times {}_4F_3 \left(\begin{matrix} 1, \Delta(2, 1+h), \Delta(1, (\alpha+h)/2); (-y)^{-2} \\ \Delta(2, 1+h), \Delta(1, (1+\alpha+h)/2) \end{matrix} \right) = \\ &= \frac{\sqrt{\pi}}{2y} \Gamma \left[\frac{\alpha/2}{(1+\alpha)/2} \right] {}_2F_1 \left(1, \frac{\alpha}{2}; \frac{1+\alpha}{2}; y^2 \right) - \\ &\quad - \frac{\sqrt{\pi}}{2y^2} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] {}_2F_1 \left(1, \frac{\alpha+1}{2}; 1+\frac{\alpha}{2}; y^2 \right). \end{aligned}$$

Таким образом,

$$\int_a^b \frac{x^{\alpha-1} dx}{\sqrt{a^2-x^2}(bx+c)} = \frac{a^{\alpha-1}}{2c} V_{\pi} \Gamma \left[\frac{\alpha/2}{(1+\alpha)/2} \right] {}_2F_1 \left(1, \frac{\alpha}{2}; \frac{\alpha+1}{2}; \frac{a^2 b^2}{c^2} \right) - \\ - \frac{V_{\pi} a^{\alpha} b}{2c^3} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] {}_2F_1 \left(1, \frac{\alpha+1}{2}; \frac{\alpha}{2}+1; \frac{a^2 b^2}{c^2} \right) \\ \left[\operatorname{Re} \alpha > 0, a > 0, \left| \arg \frac{c}{ab} \right| < \pi \right].$$

Функции $V_1^+ + V_2^+$ и V_1^- в формуле 2.2.2.5 аналитически продолжают друг друга, поэтому полученная формула справедлива не только при $|y| = \left| \frac{c}{ab} \right| \geq 1$, но и при $\left| \arg \frac{c}{ab} \right| < \pi$. Подставив, в частности, в эту формулу $\alpha = 1$ и используя соотношения $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \sqrt{\pi}/2$, ${}_2F_1(a, b; b; z) = (1-z)^{-a}$, ${}_2F_1(1, 1; \frac{3}{2}; \sin^2 z) = \frac{z}{\sin z \cos z}$, получим

$$\int_a^b \frac{dx}{\sqrt{a^2-x^2}(bx+c)} = \frac{\pi}{2c} \left(1 - \frac{a^2 b^2}{c^2} \right)^{-1/2} - \frac{1}{c} \left(1 - \frac{a^2 b^2}{c^2} \right)^{-1/2} \arcsin \frac{ab}{c}$$

[см. 1.2.50.5].

2.1.2. Общие формулы.

Некоторые свойства определенного интеграла:

1. $\int_a^b f(x) dx = - \int_b^a f(x) dx.$
2. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$

Формула Ньютона—Лейбница:

3. $\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$

$[F'(x) = f(x)].$

Интегрирование подстановкой:

4. $\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) g'(t) dt$

$[a = g(\alpha), b = g(\beta), g(t), g'(t) \text{ непрерывны, } g'(t) \neq 0 \text{ и } a < x = g(t) < b \text{ при } \alpha < t < \beta].$

Интегрирование по частям:

5. $\int_a^b f(x) g'(x) dx = f(x) g(x) \Big|_a^b - \int_a^b g(x) f'(x) dx.$

Теоремы о среднем:

6. $\int_a^b f(x) g(x) dx = f(\xi) \int_a^b g(x) dx$ $[f \text{ непрерывна на } [a, b], g(x) > 0, a < \xi < b].$

$$7. \int_a^b f(x) g(x) dx = f(a) \int_a^{\xi} g(x) dx \quad [f \geq 0, \text{ монотонна и не возрастает; } a \leq \xi \leq b].$$

$$8. \quad = f(b) \int_{\xi}^b g(x) dx \quad [f \geq 0, \text{ монотонна и не убывает; } a \leq \xi \leq b].$$

$$9. \int_a^b f(x) g(x) dx = f(a) \int_a^{\xi} g(x) dx + f(b) \int_{\xi}^b g(x) dx \quad [f \text{ монотонна; } a \leq \xi \leq b].$$

$$10. \quad = A \int_a^{\xi} g(x) dx + B \int_{\xi}^b g(x) dx$$

[$A \geq f(a+0)$, $B \leq f(b-0)$, если f монотонно убывает; $A \leq f(a+0)$, $B \geq f(b-0)$, если f монотонно возрастает; $a \leq \xi \leq b$].

Дифференцирование интеграла по параметру:

$$11. \frac{d}{dp} \int_a^b f(x, p) dx = f(b, p) \frac{db}{dp} - f(a, p) \frac{da}{dp} + \int_a^b \frac{\partial}{\partial p} f(x, p) dx.$$

$$12. \frac{d}{db} \int_a^b f(x) dx = f(b), \quad 13. \frac{d}{da} \int_a^b f(x) dx = -f(a).$$

Некоторые формулы замены переменных:

$$14. \int_a^b f(x) dx = \frac{b-a}{h} \int_0^h f\left(a + \frac{b-a}{h} x\right) dx = \frac{b-a}{d-c} \int_c^d f\left(\frac{ad-bc + (b-a)x}{d-c}\right) dx.$$

$$15. \int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$$

$$16. \int_a^b f(x) dx = (b-a) \int_0^{\infty} f\left(\frac{a+bx}{1+x}\right) \frac{dx}{(1+x)^2}.$$

$$17. \int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx = 0 \quad [f(-x) = -f(x)].$$

$$18. \quad = 2 \int_0^a f(x) dx \quad [f(-x) = f(x)].$$

$$19. \int_0^a f(x) dx = \int_0^a f(a-x) dx = \frac{1}{2} \int_0^a [f(x) + f(a-x)] dx.$$

$$20. \int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx = 0 \quad [f(2a-x) = -f(x)].$$

$$21. \quad = 2 \int_0^a f(x) dx \quad [f(2a-x) = f(x)].$$

$$22. \quad \int_0^1 \frac{f(x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^1 f\left(\frac{1-y^2}{1-k^2y^2}\right) \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}.$$

$$23. \quad \int_a^b f(x, \sqrt{x-a}, \sqrt{b-x}) dx = \\ = \int_0^1 f\left[a + \frac{4(b-a)y^2}{(1+y^2)^2}, \frac{2\sqrt{b-a}y}{1+y^2}, \frac{\sqrt{b-a}(1-y^2)}{1+y^2}\right] \frac{8(b-a)y(1-y^2)}{(1+y^2)^3} dy.$$

$$24. \quad = (b-a) \int_0^{\pi/2} f[a + (b-a) \sin^2 \varphi, \sqrt{b-a} \sin \varphi, \sqrt{b-a} \cos \varphi] \sin 2\varphi d\varphi.$$

$$25. \quad = (b-a) \int_0^{\infty} f\left[\frac{a+by}{1+y}, \sqrt{\frac{(b-a)y}{1+y}}, \sqrt{\frac{b-a}{1+y}}\right] \frac{dy}{(1+y)^2}.$$

$$26. \quad \int_a^b f(x, \sqrt{(x-a)(b-x)}) dx = (b-a) \int_0^{\infty} f\left(\frac{a+by}{1+y}, \frac{(b-a)\sqrt{y}}{1+y}\right) \frac{dy}{(1+y)^2}.$$

$$27. \quad \int_a^b f(x, \sqrt{x^2-c^2}) dx = 4c \int_{\alpha}^{\beta} f\left(\frac{a(1+y^2)}{1-y^2}, \frac{2cy}{1-y^2}\right) \frac{y dy}{(1-y^2)^2} \\ \left[0 < c \leq a < b, \alpha = \frac{\sqrt{a^2-c^2}}{a+c}, \beta = \frac{\sqrt{b^2-c^2}}{b+c}\right].$$

$$28. \quad = c \int_u^v f\left(\frac{c}{\cos \varphi}, c \operatorname{tg} \varphi\right) \frac{\sin \varphi}{\cos^2 \varphi} d\varphi \\ [0 < c \leq a < b, \cos u = c/a, \cos v = c/b].$$

$$29. \quad \int_a^b f(x, \sqrt{c^2-x^2}) dx = 2c \int_{\alpha}^{\beta} f\left(\frac{2cy}{1+y^2}, \frac{c(1-y^2)}{1+y^2}\right) \frac{1-y^2}{(1+y^2)^2} dy \\ [-c \leq a < b \leq c; \alpha = a^{-1}(c - \sqrt{c^2-a^2}); \beta = b^{-1}(c - \sqrt{c^2-b^2})].$$

$$30. \quad = c \int_u^v f(c \sin \varphi, c \cos \varphi) \cos \varphi d\varphi \\ [-c \leq a < b \leq c; \sin u = a/c; \sin v = b/c].$$

$$31. \quad \int_0^{\infty} \frac{f(x, r)}{x^2+a^2} dx = \frac{\pi}{2a} F(re^{-a}) \\ [f(x, r) = \operatorname{Re} F(z), z = r(\cos x + i \sin x), F(z) \text{ аналитична при } |z| \leq r].$$

$$32. \quad \int_0^{\infty} \frac{xg(x, r)}{x^2+a^2} dx = \frac{\pi}{2} [F(re^{-a}) - F(0)] \quad [g(x, r) = \operatorname{Im} F(z); \text{ см. 2.1.2.31}].$$

$$33. \int_0^{\infty} \frac{g(x, r)}{x} dx = \frac{\pi}{2} [F(r) - F(0)] \quad [\text{см. 2.1.2.31}].$$

$$34. \int_0^{\infty} \frac{g(x, r)}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} [F(r) - F(re^{-a})] \quad [\text{см. 2.1.2.31}].$$

$$35. \int_0^{\infty} f(x) \left\{ \frac{2x+y+2\sqrt{x^2+xy}}{2y+x+2\sqrt{y^2+xy}} \right\}^{\sigma} dx = 2 \int_0^{\infty} x f(x^2) \left(\left\{ \frac{x}{\sqrt{y}} \right\} + \sqrt{x^2+y} \right)^{2\sigma} dx.$$

$$36. \int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\pi/2} f(x) \left\{ \frac{1}{\cos x} \right\} dx \quad [f(x+\pi) = \pm f(x), f(-x) = f(x)].$$

$$37. \int_0^{\infty} \frac{f(px) - f(qx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{q}{p} \\ \left[p, q > 0; f(x) \text{ непрерывна на } [0, \infty), f(+\infty) = \lim_{x \rightarrow +\infty} f(x) - \text{конечная величина} \right].$$

$$38. = f(0) \ln \frac{q}{p} \\ \left[p, q > 0; f(x) \text{ непрерывна на } [0, \infty), \text{ и существует } \int_R^{\infty} x^{-1} f(x) dx, R > 0 \right].$$

$$39. \int_0^{\pi/2} [f(x) - f(2x)] \operatorname{ctg} x dx = f(0) \ln 2 \quad [f(\pi-x) = -f(x)].$$

$$40. \int_0^{\infty} f(x) e^{-(ax^3+2bx+c)} dx = \frac{1}{V a} e^{(b^3-ac)/a} \int_{b/V a}^{\infty} f\left(\frac{y}{V a} - \frac{b}{a}\right) e^{-y^3} dy \\ [a > 0].$$

$$41. \int_{-\infty}^{\infty} f(x) e^{-\alpha(ax^3+2bx+c)} dx = \frac{1}{V a} e^{\alpha(b^3-ac)/a} \int_{-\infty}^{\infty} f\left(\frac{y}{V a} - \frac{b}{a}\right) e^{-\alpha y^3} dy \\ [\alpha = 1 \text{ или } i; a > 0].$$

$$42. \int_0^{\infty} x^{b-1} \left(ax + \frac{b}{x}\right) dx = \frac{2}{(2a)^{1/2}} \int_0^{\infty} \frac{x^{b-1} f(x)}{V x^2 - 4ab} dx \quad [b = 0 \text{ или } 1; a, b > 0].$$

$$43. \int_0^{\infty} x^{\pm 1/2} f\left(ax + \frac{b}{x}\right) dx = a^{-(2 \pm 1)/2} \int_0^{\infty} y^{-1/2} (y + V ab)^{(1 \pm 1)/2} f(y + 2 V ab) dy \\ [a, b > 0].$$

$$44. \int_0^{\infty} x^{\pm 3/2} f\left(ax + \frac{b}{x}\right) dx = \left\{ a^{-5/2} \right\} \int_0^{\infty} y^{-1/2} (y^2 + 3 V ab y + ab)^{\delta} f(y + 2 V ab) dy \\ [a, b > 0; \delta = \left\{ \frac{1}{6} \right\}].$$

$$45. \int_0^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right) \ln x dx = \ln a \int_0^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right) dx \quad [a > 0].$$

$$46. \int_0^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right) \operatorname{arctg} x dx = \frac{\pi}{4} \int_0^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right) dx.$$

$$47. \int_0^{\infty} \frac{1}{x} f\left(\frac{x^p}{a} + \frac{a}{x^p}\right) \ln x dx = \frac{\ln a}{p} \int_0^{\infty} \frac{1}{x} f\left(\frac{x^p}{a} + \frac{a}{x^p}\right) dx \quad [a, p > 0].$$

$$48. \int_0^{\infty} f(x^p + x^{-p}) \frac{\ln x}{1+x^2} dx = 0.$$

$$49. \int_0^{\infty} x^n f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \frac{1}{(2a)^{n+1}} \int_0^{\infty} f(y^2) \left[(V y^2 + 4ab - y)^n \left(1 - \frac{y}{V y^2 + 4ab}\right) + \right. \\ \left. + (V y^2 + 4ab + y)^n \left(1 + \frac{y}{V y^2 + 4ab}\right) \right] dy \\ \left[a, b > 0; ax - \frac{b}{x} = \mp y \text{ при } \begin{cases} 0 \leq x \leq V b/a \\ V b/a \leq x < \infty \end{cases} \right].$$

$$50. \int_0^{\infty} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \frac{1}{a} \int_0^{\infty} f(y^2) dy \quad [a, b > 0].$$

$$51. \int_0^{\infty} x^{\pm 2} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \left\{ \frac{a^{-3}}{b^{-1}} \right\} \int_0^{\infty} (y^2 + ab)^{\delta} f(y^2) dy \\ [a, b > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$52. \int_0^{\infty} x^{\pm 4} f\left(\left[ax - \frac{b}{x}\right]^2\right) dx = \left\{ \frac{a^{-5}}{b^{-3}} \right\} \int_0^{\infty} \left\{ \frac{y^4 + 3aby^2 + a^2b^3}{y^2 + ab} \right\} f(y^2) dy \\ [a, b > 0].$$

$$53. \int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f(x) dx \quad [ab > 0].$$

$$54. \int_{-\infty}^{\infty} f(a \operatorname{sh} x + b \operatorname{ch} x) \operatorname{sh} x dx = 2a \int_0^{\infty} f'(\operatorname{sgn} b \sqrt{b^2 - a^2} \operatorname{ch} x) \operatorname{sh}^2 x dx.$$

$$55. = \int_{-\infty}^{\infty} \frac{x}{V 1+x^2} f(ax + b \sqrt{1+x^2}) dx.$$

$$56. \int_a^b f(\sin x, \cos x) dx = \int_{a+2\pi n}^{b+2\pi n} f(\sin x, \cos x) dx.$$

$$57. = \int_{a+(2n+1)\pi}^{b+(2n+1)\pi} f(-\sin x, -\cos x) dx.$$

$$58. = \int_{a+(n+1/2)\pi}^{b+(n+1/2)\pi} f((-1)^{n+1} \cos x, (-1)^n \sin x) dx \quad [n=0, \pm 1, \pm 2, \dots].$$

$$59. \int_{(2n+1/2)\pi}^{(2n+1)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(\sin x, -\cos x) dx.$$

$$60. \int_{(2n+1)\pi}^{(2n+3/2)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(-\sin x, -\cos x) dx.$$

$$61. \int_{(2n+3/2)\pi}^{(2n+2)\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(-\sin x, \cos x) dx.$$

$$62. \int_0^{n\pi} xf(\sin x) dx = n\pi^2 \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = f(x)].$$

$$63. = n\pi (-1)^{n-1} \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = -f(x)].$$

$$64. \int_0^{\pi/2} f(\sin x, \cos x) dx = \int_0^{\pi/2} f(\cos x, \sin x) dx.$$

$$65. = \int_0^1 f(x, \sqrt{1-x^2}) \frac{dx}{\sqrt{1-x^2}}.$$

$$66. = \int_0^1 f(\sqrt{x}, \sqrt{1-x}) \frac{dx}{2\sqrt{x(1-x)}}.$$

$$67. \int_0^{\pi/2} f(\sin 2x) \cos x dx = \int_0^{\pi/2} f(\cos^2 x) \cos x dx.$$

$$68. \int_0^{\pi/2} f(\sin^2 x, \cos^2 x) dx = \frac{1}{2} \int_0^{\pi} f(\sin^2 x, \cos^2 x) dx.$$

$$69. = \frac{1}{2} \int_{-\infty}^{\infty} f\left(\frac{y^2}{1+y^2}, \frac{1}{1+y^2}\right) \frac{dy}{1+y^2}.$$

$$70. \int_0^{\pi/2} f(\sin^2 x, \cos^2 x) \cos^{\alpha-2} x \cos \alpha x dx = \frac{1}{2} \int_{-\infty}^{\infty} f\left(\frac{y^2}{1+y^2}, \frac{1}{1+y^2}\right) \frac{dy}{(1+iy)^{\alpha}}.$$

2.1.2.]

$$\begin{aligned}
 71. \int_0^{\pi/2} [f(e^{ix} \cos x) + f(e^{-ix} \cos x)] dx = \\
 = \frac{1}{2} \int_0^{\pi} [f(e^{ix} \cos x) + f(e^{-ix} \cos x)] dx = \frac{1}{2} \int_{-\pi}^{\pi} f(e^{ix} \cos x) dx = \pi f\left(\frac{1}{2}\right) \\
 \left[f\left(\frac{z+1}{2}\right) \text{ аналитична при } |z| \leq 1 \right].
 \end{aligned}$$

$$\begin{aligned}
 72. \int_0^{\pi/2} \frac{f(e^{ix} \cos x) + f(e^{-ix} \cos x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{1}{2} \int_0^{\pi} \frac{f(e^{ix} \cos x) + f(e^{-ix} \cos x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \\
 = \frac{1}{2} \int_{-\pi}^{\pi} \frac{f(e^{ix} \cos x) dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{ab} f\left(\frac{b}{a+b}\right) \\
 [a, b > 0; f((z+1)/2) \text{ аналитична при } |z| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 73. \int_0^{\pi} \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2b \cos x + b^2} dx = \frac{2\pi}{1-b^2} f(a+b) \\
 [b < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 74. \int_0^{\pi} \frac{f(a+e^{ix}) + f(a+e^{-ix})}{1-2b \cos x + b^2} (1-b \cos x) dx = \pi [f(a+b) + f(a)] \\
 [b < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$\begin{aligned}
 75. \int_0^{\pi} \frac{f(a+e^{ix}) - f(a+e^{-ix})}{1-2b \cos x + b^2} \sin x dx = \frac{\pi i}{b} [f(a+b) - f(a)] \\
 [b < 1; f(z) \text{ аналитична в } |z-a| \leq 1].
 \end{aligned}$$

$$76. \int_0^{\pi} f(\sin x, \cos x) dx = \int_0^{\pi/2} [f(\sin x, \cos x) + f(\sin x, -\cos x)] dx.$$

$$77. \int_0^{\pi} f(\sin x) dx = 2 \int_0^{\pi/2} f(\sin x) dx.$$

$$78. \int_0^{\pi} f^{(n)}(\cos x) \sin^{2n} x dx = (2n-1)!! \int_0^{\pi} f(\cos x) \cos nx dx.$$

$$\begin{aligned}
 79. \int_0^{\pi} f\left(\frac{\sin^2 x}{1+2b \cos x + b^2}\right) dx = \int_0^{\pi} f(\sin^2 x) dx \quad [b \geq 1] \\
 = \int_0^{\pi} f\left(\frac{\sin^2 x}{b^2}\right) dx \quad [0 < b < 1]
 \end{aligned}$$

$$80. \int_{-\pi/2}^{\pi/2} f(\sin x, \cos x) dx = \int_0^{\pi/2} [f(\sin x, \cos x) + f(-\sin x, \cos x)] dx.$$

$$81. \int_{-\pi/2}^{\pi/2} f(\cos x) dx = 2 \int_0^{\pi/2} f(\cos x) dx.$$

$$82. \int_0^{2\pi} f(\sin x, \cos x) dx = \int_0^{\pi} [f(\sin x, \cos x) + f(-\sin x, \cos x)] dx.$$

$$83. \begin{aligned} &= \int_0^{\pi/2} [f(\sin x, \cos x) + f(\sin x, -\cos x) + \\ &\quad + f(-\sin x, \cos x) + f(-\sin x, -\cos x)] dx. \end{aligned}$$

$$84. \int_0^{2\pi} f(a \sin x + b \cos x) dx = \int_0^{2\pi} f(\sqrt{a^2 + b^2} \sin x) dx.$$

$$85. \quad = 2 \int_0^{\pi} f(\sqrt{a^2 + b^2} \cos x) dx.$$

$$86. \int_0^{\infty} \frac{f(\sin x)}{x} dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin x} dx \quad [f(-x) = -f(x)].$$

$$87. \int_0^{\infty} \frac{f(\sin x)}{x^2} dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin^2 x} dx \quad [f(-x) = f(x)].$$

$$88. \int_0^{\infty} \frac{f(\sin x)}{x} \cos x dx = \int_0^{\pi/2} \frac{f(\sin x)}{\sin x} \cos^2 x dx \quad [f(-x) = -f(x)].$$

$$89. \int_0^{\infty} \frac{f(\sin x)}{x} \operatorname{tg} x dx = \int_0^{\pi/2} f(\sin x) dx \quad [f(-x) = f(x)].$$

$$90. \int_0^{\infty} \frac{f(\sin x)}{x^2 + z^2} dx = \frac{\operatorname{sh} 2z}{2z} \int_0^{\pi/2} \frac{f(\sin x)}{\operatorname{ch}^2 z - \cos^2 x} dx \quad [f(-x) = f(x)].$$

$$91. \int_0^{\infty} \frac{f(\cos x)}{x^2 + z^2} dx = \frac{\operatorname{sh} z}{z} \int_0^{\pi} \frac{f(\cos x) \cos x dx}{\operatorname{ch}^2 z - \cos^2 x} \quad [f(-x) = -f(x)].$$

$$92. \int_0^{\infty} \frac{x f(\sin x) \cos^{\delta} x}{x^2 + z^2} dx = \left\{ \frac{1}{\operatorname{ch} z} \right\} \int_0^{\pi/2} \frac{f(\sin x) \sin x \cos^{2\delta} x}{\operatorname{sh}^2 z + \sin^2 x} dx$$

$$[f(-x) = -f(x); \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$93. \int_a^b f(x) [g(x)]^{\lambda} \ln^n g(x) dx = \left(\frac{d}{d\lambda} \right)^n \int_a^b f(x) [g(x)]^{\lambda} dx.$$

$$94. \int_a^b f(x) \ln^n g(x) dx = \left(\frac{d}{d\lambda}\right)^n \int_a^b f(x) [g(x)]^\lambda dx \Big|_{\lambda=0}.$$

$$95. \int_a^b f(x) \ln^n x dx = \left(\frac{d}{d\lambda}\right)^n \int_a^b x^\lambda f(x) dx \Big|_{\lambda=0}.$$

$$96. \int_0^1 f(x) \ln(\sin \pi x) dx = (1+a) \int_0^{1/2} f(x) \ln(\sin \pi x) dx \quad [f(1-x) = af(x)].$$

2.2. СТЕПЕННАЯ И АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

2.2.1. Введение.

Интегралы от степенной и алгебраических функций представляют собой широкий класс интегралов, к которым с помощью замен переменных сводятся многие интегралы, содержащие различные элементарные и специальные функции.

2.2.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие степенную и алгебраические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций $V_j^\pm$, V_j , $j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. Значения интегралов этого пункта при $|y|=1$ получаются предельным переходом при $|y| \rightarrow 1$. Кроме того, $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$1. \int_0^{\min(1, y)} x^{\alpha-1} (1-x)^{\beta-1} (y-x)^{\gamma-1} dx = \begin{cases} V_1^+ \\ V_1^- \end{cases} \quad \begin{cases} [0 < y < 1], \\ [y > 1] \end{cases}$$

$$\{E_1^\pm = \Gamma \left[\begin{matrix} \beta, \gamma, rj+\alpha \\ \beta-j, \gamma+rj+\alpha \end{matrix} \right], E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, (\alpha+h)/r \\ \beta+(\alpha+h)/r, \gamma-h \end{matrix} \right], \\ s=rj+\alpha+\gamma-1, u=\gamma-h-1, a_1=rj+\alpha, b_1=1-\beta+j, \\ c_1=\gamma+rj+\alpha, a_3=1-\gamma+h, b_3=(\alpha+h)/r, c_3=\beta+(\alpha+h)/r, z=y^p \} \\ [\operatorname{Re} \alpha > 0 \text{ и } \operatorname{Re} \gamma > 0 \text{ (при } 0 < y < 1), \text{ или } \operatorname{Re} \beta > 0 \text{ (при } y > 1), \text{ или } \operatorname{Re}(\beta+\gamma) > 1 \\ \text{(при } y=1)].$$

$$2. \int_{\max(1, y)}^\infty x^{\alpha-1} (x-1)^{\beta-1} (x-y)^{\gamma-1} dx = \begin{cases} V_2^+ \\ V_2^- \end{cases} \quad \begin{cases} [0 < y < 1], \\ [y > 1] \end{cases}$$

$$\{E_2^\pm = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, 1-\beta+r^{-1}(1+h-\alpha-\gamma) \\ 1+r^{-1}(1+h-\alpha-\gamma), \gamma-h \end{matrix} \right], \\ E_2^- = \Gamma \left[\begin{matrix} \beta, \gamma, 1-\gamma-\alpha+r(1-\beta+i) \\ \beta-j, 1-\alpha+r(1-\beta+i) \end{matrix} \right], \\ t=h, v=\gamma+\alpha-1-r(1-\beta+i), a_2=1-\gamma+h, b_2=1-\beta+r^{-1}(1-\gamma-\alpha+h), \\ c_2=1+r^{-1}(1-\gamma-\alpha+h), a_3=1-\gamma-\alpha+r(1-\beta+i), b_3=1-\beta+j, \\ c_3=1-\alpha+r(1-\beta+i), z=y^p \} \\ [\operatorname{Re}(\alpha+\gamma+r\beta) < 1+r \text{ и } \operatorname{Re} \beta > 0 \text{ (при } 0 < y < 1), \text{ или } \operatorname{Re} \gamma > 0 \text{ (при } y > 1), \\ \text{или } \operatorname{Re}(\beta+\gamma) > 1 \text{ (при } y=1)].$$

$$3. \int_{\nu}^1 x^{\alpha-1} (1-x^r)^{\beta-1} (x-y)^{\gamma-1} dx = V_1^+ + V_2^+$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} \beta, \gamma, 1-\gamma-\alpha-rj \\ \beta-j, 1-\alpha-rj \end{matrix} \right], E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, r^{-1}(\gamma+\alpha-1-h) \\ \beta-r^{-1}(1-\gamma-\alpha+h), \gamma-h \end{matrix} \right], \\ s=rj+\alpha+\gamma-1, t=h, a_1=\alpha+rj, b_1=1-\beta+j, c_1=\gamma+\alpha+rj, \\ a_2=1-\gamma+h, b_2=1-\beta+r^{-1}(1-\gamma-\alpha+h), c_2=1+r^{-1}(1-\gamma-\alpha+h), z=y^p\} \\ [0 < \nu < 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$4. \int_1^{\nu} x^{\alpha-1} (x^r-1)^{\beta-1} (y-x)^{\gamma-1} dx = V_1^- + V_2^-$$

$$\{E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, \gamma, 1-\beta-(\alpha+h)/r \\ 1-(\alpha+h)/r, \gamma-h \end{matrix} \right], E_2^- = \Gamma \left[\begin{matrix} \beta, \gamma, \alpha-r(1-\beta+j) \\ \beta-j, \gamma+\alpha-r(1-\beta+j) \end{matrix} \right], \\ u=\gamma-h-1, v=\gamma+\alpha-r(1-\beta+j)-1, a_2=1-\gamma+h, b_2=(\alpha+h)/r, \\ c_2=\beta+(\alpha+h)/r, a_4=1-\alpha-\gamma+r(1-\beta+j), b_4=1-\beta+j, \\ c_4=1-\alpha+r(1-\beta+j), z=y^p\} \quad [\nu > 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$5. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} (y+x)^{\rho-1} dx = \{V_1^+ + V_2^+ \quad [|\nu| \leq 1], \\ V_1^- \quad [|\nu| > 1]\}$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} \beta, rj+\alpha, 1-\rho-\alpha-rj \\ 1-\rho, \beta-j \end{matrix} \right], E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, r^{-1}(\alpha+\rho-h-1), 1-\rho+h \\ 1-\rho, \beta+r^{-1}(\alpha+\rho-h-1) \end{matrix} \right], \\ E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} \beta, (\alpha+h)/r, 1-\rho+h \\ 1-\rho, \beta+(\alpha+h)/r \end{matrix} \right], s=rj+\alpha+\rho-1, \\ t=h, u=\rho-h-1, a_1=rj+\alpha, b_1=1-\beta+j, c_1=\alpha+rj+\rho, \\ a_2=1-\rho+h, b_2=1-\beta+r^{-1}(1-\rho-\alpha+h), c_2=1+r^{-1}(1-\rho-\alpha+h), \\ a_3=1-\rho+h, b_3=(\alpha+h)/r, c_3=\beta+(\alpha+h)/r, z=(-y)^p\} \\ [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$6. \int_0^{\infty} x^{\alpha-1} (1+x^r)^{\beta-1} (y+x)^{\rho-1} dx = \{V_1^+ + V_2^+ \quad [|\nu| \leq 1], \\ V_1^- + V_2^- \quad [|\nu| > 1]\}$$

$$\{E_1^+ = \Gamma \left[\begin{matrix} 1-\sigma+j, \alpha+rj, 1-\rho-\alpha-rj \\ 1-\sigma, 1-\rho \end{matrix} \right], \\ E_2^+ = \frac{1}{r} \Gamma \left[\begin{matrix} (\alpha+\rho-h-1)/r, 1-\sigma+r^{-1}(1-\alpha-\rho+h), 1-\rho+h \\ 1-\sigma, 1-\rho \end{matrix} \right], \\ E_1^- = \frac{1}{r} \Gamma \left[\begin{matrix} (\alpha+h)/r, 1-\sigma-(\alpha+h)/r, 1-\rho+h \\ 1-\sigma, 1-\rho \end{matrix} \right], \\ E_2^- = \Gamma \left[\begin{matrix} 1-\sigma+j, \alpha-r(1-\sigma+j), 1-\rho-\alpha+r(1-\sigma+j) \\ 1-\sigma, 1-\rho \end{matrix} \right], \\ s=rj+\alpha+\rho-1, t=h, u=\rho-1-h, v=\rho+\alpha-1-r(1-\sigma+j), \\ a_1=rj+\alpha, b_1=1-\sigma+j, c_1=\rho+\alpha+rj, a_2=1-\rho+h, \\ b_2=1-\sigma+r^{-1}(1+h-\alpha-\rho), c_2=1+r^{-1}(1+h-\alpha-\rho), a_3=1-\rho+h, \\ b_3=(\alpha+h)/r, c_3=\sigma+(\alpha+h)/r, a_4=1-\rho-\alpha+r(1-\sigma+j), b_4=1-\sigma+j, \\ c_4=1-\alpha+r(1-\sigma+j), z=(-1)^{\rho+j} y^p\} \\ [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha+\sigma+\rho) < 1+r].$$

$$7. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} \frac{dx}{y-x} = \begin{cases} V_1^+ + V_1^- \\ V_2^+ + V_2^- \end{cases} \quad \begin{cases} [0 < \nu \leq 1], \\ [\nu \geq 1] \end{cases}$$

$$\{E_3^+ = \pi \operatorname{ctg}[(\alpha + rj)\pi] \Gamma\left[\begin{matrix} \beta \\ \beta - j \end{matrix}\right], \quad E_4^+ = -\frac{1}{r} B\left(\beta, \frac{\alpha - h - 1}{r}\right), \\ E_3^- = \frac{1}{r} B\left(\beta, \frac{\alpha + h}{r}\right),$$

$$s = rj + \alpha - 1, \quad t = h, \quad u = -h - 1, \quad a_1 = 1 - \beta + j, \quad a_2 = 1 - \beta + (1 + h - \alpha)/r, \\ b_2 = 1 + (1 + h - \alpha)/r, \quad a_3 = (\alpha + h)/r, \quad b_3 = \beta + (\alpha + h)/r, \quad z = y^p \} \\ [\operatorname{Re} \alpha > 0 \text{ и } \operatorname{Re} \beta > 0 \text{ (при } \nu \neq 1), \text{ или } \operatorname{Re} \beta > 1 \text{ (при } \nu = 1)].$$

$$8. \int_0^\infty x^{\alpha-1} (1+x^r)^{\beta-1} \frac{dx}{y-x} = \begin{cases} V_1^+ + V_1^- \\ V_2^+ + V_2^- \end{cases} \quad \begin{cases} [0 < \nu \leq 1], \\ [\nu \geq 1] \end{cases}$$

$$\{E_3^+ = \pi \operatorname{ctg}[(rj + \alpha)\pi] \Gamma\left[\begin{matrix} 1 - \rho + j \\ 1 - \rho \end{matrix}\right],$$

$$E_4^+ = -\frac{1}{r} B\left(\frac{\alpha - 1 - h}{r}, 1 - \rho + \frac{1 + h - \alpha}{r}\right), \quad E_3^- = \frac{1}{r} B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right),$$

$$E_4^- = \pi \operatorname{ctg}[(\alpha - r + r\rho)\pi] \Gamma\left[\begin{matrix} 1 - \rho + j \\ 1 - \rho \end{matrix}\right],$$

$$s = \alpha + rj - 1, \quad t = h, \quad u = -h - 1, \quad v = \alpha - 1 - r(1 - \rho),$$

$$a_1 = 1 - \rho + j, \quad a_2 = 1 - \rho + \frac{1 - \alpha + h}{r}, \quad b_2 = 1 + \frac{1 - \alpha + h}{r}, \quad a_3 = \frac{\alpha + h}{r},$$

$$b_3 = \rho + \frac{\alpha + h}{r}, \quad a_4 = 1 - \rho + j, \quad z = (-1)^q y^p \}$$

$$[\nu > 0; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha + r\rho) < 1 + r].$$

$$9. \int_0^\infty x^{\alpha-1} \frac{dx}{(1-x^r)(y-x)} = \\ = \frac{\pi}{1-y^p} \left[\sum_{j=0}^{q-1} \operatorname{ctg}[(rj + \alpha)\pi] y^{rj+\alpha-1} + \frac{1}{r} \sum_{h=0}^{p-1} \operatorname{ctg}\left(\frac{1+h-\alpha}{r}\pi\right) y^h \right] \\ [0 < \nu \leq 1].$$

$$= \frac{\pi}{y^p - 1} \left[\sum_{j=0}^{q-1} \operatorname{ctg}[(r + rj - \alpha)\pi] y^{p+\alpha-r(1+j)-1} + \right. \\ \left. + \frac{1}{r} \sum_{h=0}^{p-1} \operatorname{ctg}\left(\frac{\alpha + h}{r}\pi\right) y^{p-h-1} \right] \quad [\nu > 1], \quad [0 < \operatorname{Re} \alpha < 1 + r].$$

$$10. \int_0^1 x^{\alpha-1} (1-x^r)^{\beta-1} (\sqrt{y} + \sqrt{y+x})^\sigma dx = \begin{cases} V_1^+ + V_2^+ + V_3^+ \\ V_1^- + V_2^- + V_3^- \end{cases} \quad \begin{cases} [|\nu| \leq 1], \\ [|\nu| \geq 1] \end{cases}$$

$$\{E_3^+ = -2^{\sigma+2\alpha+2jr} \sigma \Gamma\left[\begin{matrix} \beta, \quad jr + \alpha, \quad 1 - \sigma - 2\alpha - 2jr \\ \beta - j, \quad 1 - \sigma - \alpha - jr \end{matrix}\right],$$

$$\begin{aligned}
 E_0^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\frac{h-\sigma/2}{1-\sigma/2-h}, \frac{1/2-h}{1-\sigma/2-h}\right], \\
 E_1^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\sigma+2\alpha-2h-1}{2r}\right) \Gamma\left[\frac{(1-\sigma)/2+h}{(1-\sigma)/2-h}, \frac{-1/2-h}{(1-\sigma)/2-h}\right], \\
 E_0^- &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\beta, \frac{\alpha+h}{r}\right) \Gamma\left[\frac{2h-\sigma}{1+h-\sigma}\right], \quad s=\sigma/2+\alpha+rj, \\
 t=h, \quad t_1=1/2+h, \quad u=\sigma/2-h, \quad a_1=jr+\alpha, \quad b_1=\sigma+jr+\alpha, \\
 c_1=1-\beta+j, \quad d_1=1+\sigma/2+jr+\alpha, \quad e_1=(1+\sigma)/2+jr+\alpha, \\
 a_2=h-\sigma/2, \quad b_2=\sigma/2+h, \quad c_2=1-\beta+(2h-2\alpha-\sigma)/(2r), \quad d_2=1/2+h, \\
 e_2=1+(2h-2\alpha-\sigma)/(2r), \quad a_3=(1-\sigma)/2+h, \quad b_3=(1+\sigma)/2+h, \\
 c_3=1-\beta+(2h-2\alpha-\sigma+1)/(2r), \quad d_3=3/2+h, \quad e_3=1+(2h-2\alpha-\sigma+1)/(2r), \\
 a_4=h-\sigma/2, \quad b_4=(1-\sigma)/2+h, \quad c_4=(\alpha+h)/r, \quad d_4=1-\sigma+h, \\
 e_4=\beta+(\alpha+h)/r, \quad z=(-y)^p \quad [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$11. \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} (\sqrt{x} + \sqrt{y+x})^{\sigma} dx = \begin{cases} \frac{V_0^+ + V_0^-}{V_+^+ + V_+^-} & [|y| \leq 1], \\ \frac{V_0^+ + V_0^-}{V_-^+ + V_-^-} & [|y| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_0^+ &= -2^{-2\alpha-2jr} \sigma \Gamma\left[\beta, \frac{2\alpha+2jr}{\beta-j}, \frac{-\sigma/2-\alpha-jr}{1-\sigma/2+\alpha+jr}\right], \\
 E_0^+ &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\beta, \frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\frac{2h-\sigma}{1-\sigma+h}\right], \\
 E_0^- &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{\alpha+h}{r}\right) \Gamma\left[\frac{1/2-h}{1-\sigma/2-h}, \frac{h-\sigma/2}{1-\sigma/2-h}\right], \\
 E_1^- &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\beta, \frac{2\alpha+2h+1}{2r}\right) \Gamma\left[\frac{-1/2-h}{(1-\sigma)/2-h}, \frac{(1-\sigma)/2+h}{(1-\sigma)/2-h}\right], \\
 s &= \sigma/2+\alpha+jr, \quad t=h, \quad u=\sigma/2-h, \quad u_1=(\sigma-1)/2-h, \quad a_1=jr+\alpha, \\
 b_1 &= 1/2+jr+\alpha, \quad c_1=1-\beta+j, \quad d_1=1+\sigma/2+jr+\alpha, \quad e_1=1-\sigma/2+jr+\alpha, \\
 a_2 &= h-\sigma/2, \quad b_2=(1-\sigma)/2+h, \quad c_2=1-\beta+(2h-2\alpha-\sigma)/(2r), \quad d_2=1-\sigma+h, \\
 e_2 &= 1+(2h-2\alpha-\sigma)/(2r), \quad a_3=h-\sigma/2, \quad b_3=\sigma/2+h, \\
 c_3 &= (\alpha+h)/r, \quad d_3=1/2+h, \quad e_3=\beta+(\alpha+h)/r, \quad a_4=(1+\sigma)/2+h, \\
 b_4 &= (1-\sigma)/2+h, \quad c_4=(2\alpha+2h+1)/(2r), \quad d_4=3/2+h, \\
 e_4 &= \beta+(2\alpha+2h+1)/(2r), \quad z=(-y)^p \quad [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$12. \int_0^{\infty} x^{\alpha-1} (1+x)^{\rho-1} (\sqrt{y} + \sqrt{y+x})^{\sigma} dx = \begin{cases} \frac{V_0^+ + V_0^- + V_1^+}{V_+^+ + V_+^-} & [|y| \leq 1], \\ \frac{V_0^+ + V_0^-}{V_-^+ + V_-^-} & [|y| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_0^+ &= -2^{\sigma+2\alpha+2jr} \sigma \Gamma\left[\frac{1-\rho+i}{1-\rho}, \frac{jr+\alpha}{1-\sigma-\alpha-jr}, \frac{-2jr-2\alpha-\sigma}{1-\sigma-\alpha-jr}\right], \\
 E_0^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\frac{\sigma+2\alpha-2h}{2r}, 1-\rho-\frac{\sigma+2\alpha-2h}{2r}\right) \Gamma\left[\frac{h-\sigma/2}{1-\sigma/2-h}, \frac{1/2-h}{1-\sigma/2-h}\right], \\
 E_1^+ &= -\frac{\sigma}{2r\sqrt{\pi}} B\left(\frac{\sigma-1+2\alpha-2h}{2r}, 1-\rho-\frac{\sigma-1+2\alpha-2h}{2r}\right) \times \\
 &\quad \times \Gamma\left[\frac{(1-\sigma)/2+h}{(1-\sigma)/2-h}, \frac{-1/2-h}{(1-\sigma)/2-h}\right], \\
 E_0^- &= -\frac{\sigma}{r} 2^{\sigma-2h} B\left(\frac{\alpha+h}{r}, 1-\rho-\frac{\alpha+r}{r}\right) \Gamma\left[\frac{-\sigma+2h}{1-\sigma+h}\right],
 \end{aligned}$$

$$E_7^- = -2^{\sigma+2\alpha-2r(1-\rho+j)} \sigma \Gamma \left[\begin{matrix} 1-\rho+j, \alpha+r(\rho-j-1), 2r(1-\rho+j)-2\alpha-\sigma \\ 1-\rho, 1-\sigma-\alpha+r(1+j-\rho) \end{matrix} \right],$$

$$s = \sigma/2 + \alpha + jr, \quad t = h, \quad t_1 = 1/2 + h, \quad u = \sigma/2 - h, \quad v = \sigma/2 + \alpha - r(1-\rho+j),$$

$$a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 - \rho + j, \quad d_1 = 1 + \sigma/2 + \alpha + jr,$$

$$e_1 = (1 + \sigma)/2 + \alpha + jr, \quad a_2 = h - \sigma/2, \quad b_2 = \sigma/2 + h, \quad c_2 = 1 - \rho + (2h - \sigma - 2\alpha)/(2r),$$

$$d_2 = 1/2 + h, \quad e_2 = 1 + (2h - \sigma - 2\alpha)/(2r), \quad a_3 = (1 - \sigma)/2 + h, \quad b_3 = (1 + \sigma)/2 + h,$$

$$c_3 = 1 - \rho + (2h - \sigma - 2\alpha + 1)/(2r), \quad d_3 = 3/2 + h, \quad e_3 = 1 + (2h - \sigma - 2\alpha + 1)/(2r),$$

$$a_4 = h - \sigma/2, \quad b_4 = (1 - \sigma)/2 + h, \quad c_4 = (\alpha + h)/r, \quad d_4 = 1 - \sigma + h,$$

$$e_4 = \rho + (\alpha + h)/r, \quad a_5 = r(1 - \rho + j) - \alpha - \sigma/2, \quad b_5 = r(1 - \rho + j) - \alpha + (1 - \sigma)/2,$$

$$c_5 = 1 - \rho + j, \quad d_5 = 1 - \alpha + r(1 - \rho + j), \quad e_5 = 1 - \sigma - \alpha + r(1 - \rho + j),$$

$$z = (-1)^{q+p} y^p \quad \{ |\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (\alpha + \rho r + \sigma/2) < r \}.$$

$$13. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\sqrt{y+x})^\sigma}{1-x^r} dx = \begin{cases} V_5^+ + V_5^- + V_7^+ & \{0 < y \leq 1\}, \\ V_5^- + V_7^- & \{y \geq 1\} \end{cases}$$

$$\{E_5^+ = (-1)^{j+1} 2^{\sigma+2\alpha+2jr} \sigma j! \Gamma \left[\begin{matrix} \alpha + jr, -\sigma - 2\alpha - 2jr \\ 1 - \sigma - \alpha - jr \end{matrix} \right],$$

$$E_5^+ = \frac{\sigma \sqrt{\pi}}{2r} \operatorname{ctg} \left(\frac{2h - 2\alpha - \sigma}{2r} \pi \right) \Gamma \left[\begin{matrix} h - \sigma/2, 1/2 - h \\ 1 - \sigma/2 - h \end{matrix} \right],$$

$$E_7^+ = \frac{\sigma \sqrt{\pi}}{2r} \operatorname{ctg} \left(\frac{2h - 2\alpha - \sigma + 1}{2r} \pi \right) \Gamma \left[\begin{matrix} (1 - \sigma)/2 + h, -1/2 - h \\ (1 - \sigma)/2 - h \end{matrix} \right],$$

$$E_5^- = -\frac{\sigma \pi}{r} 2^{\sigma-2h} \operatorname{ctg} \left(\frac{\alpha + h}{r} \pi \right) \Gamma \left[\begin{matrix} 2h - \sigma \\ 1 - \sigma + h \end{matrix} \right],$$

$$E_7^- = (-1)^j j! \sigma \pi 2^{\sigma+2\alpha-2r(1+j)} \Gamma \left[\begin{matrix} -\sigma - 2\alpha + 2r(1+j), \alpha - r(1+j) \\ 1 - \sigma - \alpha + r(1+j) \end{matrix} \right],$$

$$s = \sigma/2 + \alpha + jr, \quad t = h, \quad t_1 = 1/2 + h, \quad u = \sigma/2 - h, \quad v = \sigma/2 + \alpha - r(1+j),$$

$$a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 + j, \quad d_1 = 1 + \sigma/2 + \alpha + jr,$$

$$e_1 = (1 + \sigma)/2 + \alpha + jr, \quad a_2 = h - \sigma/2, \quad b_2 = \sigma/2 + h, \quad c_2 = 1 + (2h - 2\alpha - \sigma)/(2r),$$

$$d_2 = 1/2 + h, \quad e_2 = c_2, \quad a_3 = (1 - \sigma)/2 + h, \quad b_3 = (1 + \sigma)/2 + h,$$

$$c_3 = 1 + (2h - 2\alpha - \sigma + 1)/(2r), \quad d_3 = 3/2 + h, \quad e_3 = c_3, \quad a_4 = h - \sigma/2, \quad b_4 = (1 - \sigma)/2 + h,$$

$$c_4 = (\alpha + h)/r, \quad d_4 = 1 - \sigma + h, \quad e_4 = c_4, \quad a_5 = r(1+j) - \alpha - \sigma/2,$$

$$b_5 = r(1+j) - \alpha + (1 - \sigma)/2, \quad c_5 = 1 + j, \quad d_5 = 1 - \alpha + r(1+j),$$

$$e_5 = 1 - \sigma - \alpha + r(1+j), \quad m = 2p \quad \text{или} \quad 2p+1, \quad z = (-y)^p \}$$

$$\{y, \operatorname{Re} \alpha > 0; \operatorname{Re} (z + \sigma/2) < r\}.$$

$$14. \int_0^\infty x^{\alpha-1} \frac{(V\bar{y} + V\sqrt{y+x})^\sigma}{(1 + V\sqrt{1+x})^\gamma} dx = \begin{cases} V_5^+ + V_5^- + V_{10}^+ & \{|y| \leq 1\}, \\ V_5^- + V_5^- + V_{10}^- & \{|y| \geq 1\} \end{cases}$$

$$\{E_5^+ = -2^{\sigma+2\alpha-\gamma} \sigma \Gamma \left[\begin{matrix} \alpha, -\sigma - 2\alpha \\ 1 - \sigma - \alpha \end{matrix} \right], \quad E_5^+ = \gamma \Gamma \left[\begin{matrix} \alpha + \sigma/2, \gamma - \sigma - 2\alpha \\ 1 + \gamma - \alpha - \sigma/2 \end{matrix} \right] 2^{2\alpha+\sigma-\gamma},$$

$$E_{10}^+ = 2^{\sigma+2\alpha-\gamma-1} \gamma \sigma \Gamma \left[\begin{matrix} \alpha + (\sigma-1)/2, 1 + \gamma - \sigma - 2\alpha \\ 1 + \gamma - \alpha + (1-\sigma)/2 \end{matrix} \right],$$

$$E_5^- = 2^{\sigma-\gamma+2\alpha} \gamma \Gamma \left[\begin{matrix} \alpha, \gamma - 2\alpha \\ 1 + \gamma - \alpha \end{matrix} \right], \quad E_5^- = -2^{\sigma+2\alpha-\gamma} \sigma \Gamma \left[\begin{matrix} \alpha - \gamma/2, \gamma - \sigma - 2\alpha \\ 1 - \alpha - \sigma + \gamma/2 \end{matrix} \right],$$

$$E_{10}^- = 2^{\sigma+2\alpha-\gamma-1} \gamma \sigma \Gamma \left[\begin{matrix} \alpha - (\gamma+1)/2, 1 + \gamma - \sigma - 2\alpha \\ 3/2 - \alpha - \sigma + \gamma/2 \end{matrix} \right],$$

$$s_1 = \alpha + \sigma/2, \quad s_2 = 0, \quad s_3 = 1/2, \quad s_4 = \alpha + (\sigma - \gamma)/2, \quad s_5 = \alpha + (\sigma - \gamma - 1)/2,$$

$$\begin{aligned}
s_1 &= \sigma/2, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = \gamma/2, \quad d_1 = (1 + \gamma)/2, \quad e_1 = 1 + \sigma/2 + \alpha, \\
f_1 &= (1 + \sigma)/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = -\sigma/2, \quad b_2 = \sigma/2, \quad c_2 = (\gamma - \sigma)/2 - \alpha, \\
d_2 &= (1 + \gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \quad f_2 = 1 + \gamma - \alpha - \sigma/2, \quad g_2 = 1 - \alpha - \sigma/2, \quad a_3 = (1 - \sigma)/2, \\
b_3 &= (1 + \sigma)/2, \quad c_3 = (1 + \gamma - \sigma)/2 - \alpha, \quad d_3 = 1 + (\gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \\
f_3 &= (3 - \sigma)/2 - \alpha + \gamma, \quad g_3 = (3 - \sigma)/2 - \alpha, \quad a_4 = (\gamma - \sigma)/2 - \alpha, \quad b_4 = (1 + \gamma - \sigma)/2 - \alpha, \\
c_4 &= \gamma/2, \quad d_4 = -\gamma/2, \quad e_4 = 1 - \alpha + \gamma/2, \quad f_4 = 1 - \sigma - \alpha + \gamma/2, \quad g_4 = 1/2, \\
a_5 &= (1 + \gamma - \sigma)/2 - \alpha, \quad b_5 = 1 - \alpha + (\gamma - \sigma)/2, \quad c_5 = (1 + \gamma)/2, \quad d_5 = (1 - \gamma)/2, \\
e_5 &= (3 + \gamma)/2 - \alpha, \quad f_5 = (3 + \gamma)/2 - \sigma - \alpha, \quad g_5 = 3/2, \quad a_7 = -\sigma/2, \quad b_7 = (1 - \sigma)/2, \\
c_7 &= \alpha, \quad d_7 = \alpha - \gamma, \quad e_7 = 1 - \sigma, \quad f_7 = 1 + \alpha - \gamma/2, \quad g_7 = (1 - \gamma)/2 + \alpha, \quad z = y\} \\
&\quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma) < 0].
\end{aligned}$$

$$15. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y+x})^\sigma}{(\sqrt{x} + \sqrt{1+x})^\gamma} dx = \begin{cases} V_5^+ + V_6^+ + V_{10}^+ + V_{11}^+ & [|\gamma| \leq 1] \\ V_5^- + V_6^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_5^+ = -2^{-2\alpha-2\sigma} \Gamma \left[\frac{\alpha}{1-\sigma-\alpha} \right], \quad E_6 = 2^{\sigma+2\alpha+1} \gamma \sigma \Gamma \left[\frac{-\sigma-2\alpha-1}{1/2-\alpha-\sigma}, \quad \alpha+1/2 \right],$$

$$E_{10}^+ = 2^{-2\sigma-\alpha} \gamma \Gamma \left[\frac{2\alpha + \sigma, (\gamma - \sigma)/2 - \alpha}{1 + \alpha + (\gamma + \sigma)/2} \right],$$

$$E_{11}^+ = 2^{1-2\sigma-\alpha} \gamma \sigma \Gamma \left[\frac{2\alpha + \sigma - 1, (\gamma - \sigma + 1)/2 - \alpha}{(1 + \gamma + \sigma)/2 + \alpha} \right],$$

$$E_5^- = -2^{\sigma+2\alpha-2\gamma} \sigma \Gamma \left[\frac{\alpha - \gamma/2, \gamma - \sigma - 2\alpha}{1 - \sigma - \alpha + \gamma/2} \right],$$

$$E_6^- = 2^{\sigma-2\alpha} \gamma \Gamma \left[\frac{2\alpha, \gamma/2 - \alpha}{1 + \gamma/2 + \alpha} \right],$$

$$s_1 = \alpha + \sigma/2, \quad s_2 = \alpha + (\sigma + 1)/2, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (-\gamma + \sigma)/2 + \alpha, \quad s_6 = \sigma/2,$$

$$a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = -\gamma/2, \quad d_1 = \gamma/2, \quad e_1 = 1 + \sigma/2 + \alpha, \quad f_1 = (1 + \sigma)/2 + \alpha,$$

$$g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \quad c_2 = (1 - \gamma)/2, \quad d_2 = (1 + \gamma)/2,$$

$$e_2 = (3 + \sigma)/2 + \alpha, \quad f_2 = 1 + \sigma/2 + \alpha, \quad g_2 = 3/2, \quad a_3 = -\sigma/2, \quad b_3 = \sigma/2, \quad c_3 = (\gamma - \sigma)/2 - \alpha,$$

$$d_3 = -(\gamma + \sigma)/2 - \alpha, \quad e_3 = 1/2, \quad f_3 = 1 - \alpha - \sigma/2, \quad g_3 = (1 - \sigma)/2 - \alpha, \quad a_4 = (1 - \sigma)/2,$$

$$b_4 = (1 + \sigma)/2, \quad c_4 = (1 + \gamma - \sigma)/2 - \alpha, \quad d_4 = (1 - \gamma - \sigma)/2 - \alpha, \quad e_4 = 3/2,$$

$$f_4 = (3 - \sigma)/2 - \alpha, \quad g_4 = 1 - \sigma/2 - \alpha, \quad a_5 = (\gamma - \sigma)/2 - \alpha, \quad b_5 = (1 + \gamma - \sigma)/2 - \alpha,$$

$$c_5 = \gamma/2, \quad d_5 = (1 + \gamma)/2, \quad e_5 = 1 + \gamma/2 - \alpha, \quad f_5 = 1 - \sigma - \alpha + \gamma/2, \quad g_5 = 1 + \gamma,$$

$$a_6 = -\sigma/2, \quad b_6 = (1 - \sigma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \quad e_6 = 1 - \sigma, \quad f_6 = 1 + \alpha - \gamma/2,$$

$$g_6 = 1 + \alpha + \gamma/2, \quad z = -y\} \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma) > 0].$$

$$16. \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \frac{(\sqrt{y} + \sqrt{y+x})^\sigma}{\sqrt{y+x}} dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [|\gamma| \leq 1] \\ V_5^- & [|\gamma| \geq 1] \end{cases}$$

$$\{E_5^+ = 2^{\sigma+2\alpha+2\gamma} \Gamma \left[\frac{\beta, \alpha + jr, 1 - \sigma - 2\alpha - 2jr}{\beta - j, 1 - \sigma - \alpha - jr} \right],$$

$$E_6^+ = \frac{1}{r \sqrt{\pi}} B \left(\beta, \frac{\sigma + 2\alpha - 2h - 1}{2r} \right) \Gamma \left[\frac{(1 - \sigma)/2 + h, 1/2 - h}{(1 - \sigma)/2 - h} \right],$$

$$E_7^+ = \frac{1}{r \sqrt{\pi}} B \left(\beta, \frac{\sigma + 2\alpha - 2h - 2}{2r} \right) \Gamma \left[\frac{1 - \sigma/2 + h, -1/2 - h}{-\sigma/2 - h} \right],$$

$$E_5^- = \frac{2^{\sigma+2\gamma}}{r} B \left(\beta, \frac{\alpha + h}{r} \right) \Gamma \left[\frac{1 - \sigma + 2h}{1 - \sigma + h} \right], \quad s = (\sigma - 1)/2 + \alpha + jr, \quad t = h,$$

$$t_1 = h + 1/2, \quad u = (\sigma - 1)/2 - h, \quad a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 - \beta + j,$$

$$\begin{aligned}
 d_1 &= (1+\sigma)/2 + \alpha + jr, \quad e_1 = \sigma/2 + \alpha + jr, \quad a_2 = (1-\sigma)/2 + h, \quad b_2 = (1+\sigma)/2 + h, \\
 c_2 &= 1 - \beta + (1-\sigma-2\alpha+2h)/(2r), \quad d_2 = 1/2 + h, \quad e_2 = 1 + (1-\sigma-2\alpha+2h)/(2r), \\
 a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = 1 - \beta + (2-\sigma-2\alpha+2h)/(2r), \quad d_3 = 3/2 + h, \\
 e_3 &= 1 + (2-\sigma-2\alpha+2h)/(2r), \quad a_4 = (1-\sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \\
 c_4 &= (\alpha+h)/r, \quad d_4 = 1 - \sigma + h, \quad e_4 = \beta + (\alpha+h)/r, \quad z = (-y)^p \\
 &\quad [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$17. \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \frac{(\sqrt{x} + \sqrt{y+x})^\sigma}{\sqrt{y+x}} dx = \begin{cases} V_0^+ + V_0^- & [|\nu| \leq 1], \\ V_0^+ + V_0^- & [|\nu| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_1^+ &= 2^{1-2\alpha-2jr} \Gamma \left[\beta, 2\alpha+2jr, (1-\sigma)/2 - \alpha - jr \right], \\
 E_0^+ &= \frac{2^{\alpha-2h}}{r} B \left(\beta, \frac{\sigma+2\alpha-2h-1}{2r} \right) \Gamma \left[1-\sigma+2h \right], \\
 E_0^- &= \frac{1}{r\sqrt{\pi}} B \left(\beta, \frac{\alpha+h}{r} \right) \Gamma \left[\frac{1}{2}-h, (1-\sigma)/2+h \right], \\
 E_0^- &= \frac{1}{r\sqrt{\pi}} B \left(\beta, \frac{2\alpha+2h+1}{2r} \right) \Gamma \left[-\frac{1}{2}-h, 1-\sigma/2+h \right], \\
 s &= (\sigma-1)/2 + \alpha + jr, \quad t = h, \quad u = (\sigma-1)/2 - h, \quad u_1 = \sigma/2 - 1 - h, \quad a_1 = \alpha + jr, \\
 b_1 &= 1/2 + \alpha + jr, \quad c_1 = 1 - \beta + j, \quad d_1 = (1-\sigma)/2 + \alpha + jr, \quad e_1 = (1+\sigma)/2 + \alpha + jr, \\
 a_2 &= (1-\sigma)/2 + h, \quad b_2 = 1 - \sigma/2 + h, \quad c_2 = 1 - \beta + (1-\sigma-2\alpha+2h)/(2r), \\
 d_2 &= 1 - \sigma + h, \quad e_2 = 1 + (1-\sigma-2\alpha+2h)/(2r), \quad a_3 = (1-\sigma)/2 + h, \quad b_3 = (1+\sigma)/2 + h, \\
 c_4 &= (\alpha+h)/r, \quad d_4 = 1/2 + h, \quad e_4 = \beta + (\alpha+h)/r, \quad a_5 = 1 - \sigma/2 + h, \\
 b_5 &= 1 + \sigma/2 + h, \quad c_5 = (2\alpha+2h+1)/(2r), \quad d_5 = 3/2 + h, \\
 e_5 &= \beta + (2\alpha+2h+1)/(2r), \quad z = (-y)^p \quad [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].
 \end{aligned}$$

$$18. \int_0^\infty x^{\alpha-1} (1+x)^{\rho-1} \frac{(\sqrt{y} + \sqrt{y+x})^\sigma}{\sqrt{y+x}} dx = \begin{cases} V_0^+ + V_0^- + V_0^+ & [|\nu| \leq 1], \\ V_0^+ + V_0^- & [|\nu| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_0^+ &= 2^{\alpha+2\alpha+2jr} \Gamma \left[1-\rho+j, 1-\sigma-2\alpha-2jr, \alpha+jr \right], \\
 E_0^+ &= \frac{1}{r\sqrt{\pi}} B \left(\frac{\sigma+2\alpha-2h-1}{2r}, 1-\rho-\frac{\sigma+2\alpha-2h-1}{2r} \right) \times \\
 &\quad \times \Gamma \left[\frac{(1-\sigma)/2+h, 1/2-h}{(1-\sigma)/2-h} \right], \\
 E_0^+ &= \frac{1}{r\sqrt{\pi}} B \left(\frac{\sigma+2\alpha-2h-2}{2r}, 1-\rho-\frac{\sigma+2\alpha-2h-2}{2r} \right) \times \\
 &\quad \times \Gamma \left[\frac{1-\sigma/2+h, -1/2-h}{-\sigma/2-h} \right], \\
 E_0^- &= \frac{2^{\alpha-2h}}{r} \Gamma \left[\frac{1-\sigma+2h}{1-\sigma+h} \right] B \left(\frac{\alpha+h}{r}, 1-\rho-\frac{\alpha+h}{r} \right), \\
 E_0^- &= 2^{\alpha+2\alpha-2r(1-\rho+j)} \Gamma \left[\frac{1-\rho+j, \alpha-r(1-\rho+j), 1-\sigma+2r(1-\rho+j)-2\alpha}{1-\rho, 1-\sigma-\alpha+r(1-\rho+j)} \right], \\
 s &= (\sigma-1)/2 + \alpha + jr, \quad t = h, \quad t_1 = 1/2 + h, \quad u = (\sigma-1)/2 - h, \\
 v &= (\sigma-1)/2 + \alpha - r(1-\rho+j), \quad a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 - \rho + j, \\
 d_1 &= (1+\sigma)/2 + \alpha + jr, \quad e_1 = \sigma/2 + \alpha + jr, \quad a_2 = (1-\sigma)/2 + h, \quad b_2 = \sigma + (1-\sigma)/2 + h,
 \end{aligned}$$

$$\begin{aligned}
 c_2 &= 1 - \rho + (1 - \sigma - 2\alpha + 2h)/(2r), \quad d_2 = 1/2 + h, \quad e_2 = 1 + (1 - \sigma - 2\alpha + 2h)/(2r), \\
 a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = 1 - \rho + (2 - \sigma - 2\alpha + 2h)/(2r), \quad d_3 = 3/2 + h, \\
 e_3 &= 1 + (2 - \sigma - 2\alpha + 2h)/(2r), \quad a_4 = (1 - \sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \quad c_4 = (\alpha + h)/r, \\
 d_4 &= 1 - \sigma + h, \quad e_4 = \rho + (\alpha + h)/r, \quad a_5 = (1 - \sigma)/2 - \alpha + r(1 - \rho + j), \\
 b_5 &= 1 - \sigma/2 - \alpha + r(1 - \rho + j), \quad c_5 = 1 - \rho + j, \quad d_5 = 1 - \alpha + r(1 - \rho + j), \\
 e_5 &= 1 - \sigma - \alpha + r(1 - \rho + j), \quad z = (-1)^{\sigma+p} y^p \{ \\
 &\quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha + 2\rho + \sigma - 1) < r].
 \end{aligned}$$

$$19. \int_0^{\infty} x^{\alpha-1} \frac{(V\bar{y} + \sqrt{y+x})^\sigma}{(1-x^r) V\bar{y}+x} dx = \begin{cases} V_5^+ + V_6^+ + V_7^+ & [|\nu| \leq 0, \\ V_5^- + V_7^- & [|\nu| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_5^+ &= (-1)^j j! \, 2^{\sigma+2\alpha+2r} \Gamma \left[\frac{1-\sigma-2\alpha-2rj}{1-\sigma-\alpha-rj}, \alpha+rj \right], \\
 E_5^- &= -\frac{V\pi}{r} \operatorname{ctg} \left(\frac{1-\sigma-2\alpha+2h}{2r} \pi \right) \Gamma \left[\frac{(1-\sigma)/2+h}{(1-\sigma)/2-h}, 1/2-h \right], \\
 E_7^+ &= -\frac{V\pi}{r} \operatorname{ctg} \left(\frac{2-\sigma-2\alpha+2h}{2r} \pi \right) \Gamma \left[\frac{1-\sigma/2+h}{-\sigma/2-h}, -1/2-h \right], \\
 E_7^- &= \frac{\pi}{r} 2^{\sigma-2h} \operatorname{ctg} \left(\frac{\alpha+h}{r} \pi \right) \Gamma \left[\frac{1-\sigma+2h}{1-\sigma+h} \right], \\
 E_7^- &= (-1)^{j+1} j! \, 2^{\sigma+2\alpha-2r(1+j)} \Gamma \left[\frac{1-\sigma-2\alpha+2r(1+j)}{1-\sigma-\alpha+r(1+j)}, \alpha-r(1+j) \right], \\
 s &= (\sigma-1)/2 + \alpha + jr, \quad t = h, \quad t_1 = h + 1/2, \quad u = (\sigma-1)/2 - h, \\
 v &= (\sigma-1)/2 + \alpha - r(1+j), \quad a_1 = \alpha + jr, \quad b_1 = \sigma + \alpha + jr, \quad c_1 = 1 + j, \\
 d_1 &= (1 + \sigma)/2 + \alpha + rj, \quad e_1 = \sigma/2 + \alpha + rj, \quad a_2 = (1 - \sigma)/2 + h, \\
 b_2 &= (\sigma + 1)/2 + h, \quad c_2 = e_2 = 1 + (1 - \sigma - 2\alpha + 2h)/(2r), \quad d_2 = 1/2 + h, \\
 a_3 &= 1 - \sigma/2 + h, \quad b_3 = 1 + \sigma/2 + h, \quad c_3 = e_3 = 1 + (2 - \sigma - 2\alpha + 2h)/(2r), \\
 d_3 &= 3/2 + h, \quad a_4 = (1 - \sigma)/2 + h, \quad b_4 = 1 - \sigma/2 + h, \quad c_4 = e_4 = (\alpha + h)/r, \\
 d_4 &= 1 - \sigma + h, \quad a_5 = (1 - \sigma)/2 - \alpha + r(1 + j), \quad b_5 = 1 - \sigma/2 - \alpha + r(1 + j), \\
 c_5 &= 1 + j, \quad d_5 = 1 - \alpha + r(1 + j), \quad e_5 = 1 - \sigma - \alpha + r(1 + j), \quad z = (-y)^p \{ \\
 &\quad [|\arg y| < \pi; 0 < \operatorname{Re} \alpha < r + (1 - \operatorname{Re} \sigma)/2].
 \end{aligned}$$

$$20. \int_0^{\infty} x^{\alpha-1} \frac{(V\bar{y} + \sqrt{y+x})^\sigma}{(1 + V\sqrt{1+x})^{\nu} V\bar{y}+x} dx = \begin{cases} V_5^+ + V_6^+ + V_{10}^+ & [|\nu| \leq 1, \\ V_5^- + V_6^- + V_{10}^- & [|\nu| \geq 1] \end{cases}$$

$$\begin{aligned}
 \{E_5^+ &= 2^{\sigma+2\alpha-\nu} \Gamma(\alpha, 1-\sigma-2\alpha), \\
 E_5^+ &= 2^{2\alpha+\sigma-\nu-1} \gamma \Gamma \left[\frac{\alpha + (\sigma-1)/2}{1 + \gamma - \alpha + (1-\sigma)/2}, \gamma - 2\alpha - \sigma + 1 \right], \\
 E_{10}^+ &= 2^{2\alpha+\sigma-\nu-2} \gamma \sigma \Gamma \left[\frac{\alpha + \sigma/2 - 1}{2 + \gamma - \alpha - \sigma/2}, \gamma + 2 - \sigma - 2\alpha \right], \\
 E_6^- &= 2^{\sigma+2\alpha-\nu} \gamma \Gamma \left[\frac{\alpha}{1 + \gamma - \alpha}, \gamma - 2\alpha \right], \quad E_6^- = 2^{\sigma+2\alpha-\nu} \Gamma \left(\alpha - \frac{\gamma}{2}, 1 + \gamma - \sigma - 2\alpha \right), \\
 E_{10}^- &= -2^{2\alpha+\sigma-\nu-1} \gamma \Gamma \left[\frac{\alpha - (\gamma+1)/2}{1 - \alpha - \sigma + (\gamma+1)/2}, 2 - 2\alpha + \gamma - \sigma \right], \\
 s_1 &= (\sigma-1)/2 + \alpha, \quad s_2 = 0, \quad s_3 = 1/2, \quad s_4 = (\sigma-1)/2, \quad s_5 = (\sigma-\gamma-1)/2 + \alpha, \\
 s_7 &= (\sigma-\gamma)/2 + \alpha - 1, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = \gamma/2, \quad d_1 = (1 + \gamma)/2, \\
 e_1 &= (1 + \sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = (1 - \sigma)/2,
 \end{aligned}$$

$$\begin{aligned}
 b_2 &= (1 + \sigma)/2, \quad c_3 = (1 - \sigma + \gamma)/2 - \alpha, \quad d_2 = 1 + (\gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \\
 f_2 &= (3 - \sigma)/2 - \alpha + \gamma, \quad g_2 = (3 - \sigma)/2 - \alpha, \quad a_3 = 1 - \sigma/2, \quad b_3 = 1 + \sigma/2, \\
 c_3 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad d_3 = (3 + \gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \quad f_3 = 2 + \gamma - \alpha - \sigma/2, \\
 g_3 &= 2 - \alpha - \sigma/2, \quad a_5 = (1 - \sigma)/2, \quad b_5 = 1 - \sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha - \gamma, \quad e_5 = 1 - \sigma, \\
 f_5 &= 1 + \alpha - \gamma/2, \quad g_5 = (1 - \gamma)/2 + \alpha, \quad a_6 = (1 - \sigma + \gamma)/2 - \alpha, \quad b_6 = 1 + (\gamma - \sigma)/2 - \alpha, \\
 c_6 &= \gamma/2, \quad d_6 = -\gamma/2, \quad e_6 = 1 + \gamma/2 - \alpha, \quad f_6 = 1 - \alpha - \sigma + \gamma/2, \quad g_6 = 1/2, \\
 a_7 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad b_7 = (3 + \gamma - \sigma)/2 - \alpha, \quad c_7 = (1 + \gamma)/2, \quad d_7 = (1 - \gamma)/2, \\
 e_7 &= (3 + \gamma)/2 - \sigma - \alpha, \quad f_7 = (3 + \gamma)/2 - \alpha, \quad g_7 = 3/2, \quad z = y \} \\
 & \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma - 1) < 0].
 \end{aligned}$$

$$21. \int_0^{\infty} x^{\alpha-1} \frac{(V\bar{y} + V\sqrt{y+x})^{\sigma}}{(Vx + V\sqrt{1+x})^{\gamma} V\sqrt{y+x}} dx = \begin{cases} V_0^+ + V_0^+ + V_{10}^+ + V_{11}^+ & [|\nu| \leq 1], \\ V_0^- + V_0^- & [|\nu| \geq 1] \end{cases}$$

$$\{E_0^+ = 2^{\sigma+2\alpha} B(\alpha, 1 - \sigma - 2\alpha), \quad E_0^- = -2^{\sigma+2\alpha+1} \gamma B\left(\alpha + \frac{1}{2}, -\sigma - 2\alpha\right),$$

$$E_{10}^+ = 2^{1-\sigma-2\alpha} \gamma \Gamma\left[\frac{2\alpha + \sigma - 1, (1 + \gamma - \sigma)/2 - \alpha}{\alpha + (\gamma + \sigma + 1)/2}\right],$$

$$E_{11}^+ = 2^{2-\sigma-2\alpha} \gamma \sigma \Gamma\left[\frac{2\alpha + \sigma - 2, (\gamma - \sigma)/2 + 1 - \alpha}{(\gamma + \sigma)/2 + \alpha}\right],$$

$$E_1^- = 2^{\sigma-2\alpha} \gamma \Gamma\left[\frac{2\alpha, \gamma/2 - \alpha}{1 + \gamma/2 + \alpha}\right],$$

$$E_0^- = 2^{\sigma+2\alpha-2\gamma} B(\alpha - \gamma/2, 1 - \sigma - 2\alpha + \gamma),$$

$$\begin{aligned}
 s_1 &= (\sigma - 1)/2 + \alpha, \quad s_2 = \sigma/2 + \alpha, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (\sigma - 1)/2, \\
 s_6 &= (\sigma - 1 - \gamma)/2 + \alpha, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad e_1 = \gamma/2, \quad d_1 = -\gamma/2, \quad e_1 = (1 + \sigma)/2 + \alpha, \\
 f_1 &= \sigma/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \quad c_2 = (1 - \gamma)/2, \\
 d_2 &= (\gamma + 1)/2, \quad e_2 = 1 + \sigma/2 + \alpha, \quad f_2 = (\sigma + 1)/2 + \alpha, \quad g_2 = 3/2, \quad a_3 = (1 - \sigma)/2, \\
 b_3 &= (1 + \sigma)/2, \quad c_3 = (1 - \sigma + \gamma)/2 - \alpha, \quad d_3 = (1 - \sigma - \gamma)/2 - \alpha, \quad e_3 = 1/2, \\
 f_3 &= (3 - \sigma)/2 - \alpha, \quad g_3 = 1 - \sigma/2 - \alpha, \quad a_4 = 1 - \sigma/2, \quad b_4 = 1 + \sigma/2, \\
 c_4 &= 1 + (\gamma - \sigma)/2 - \alpha, \quad d_4 = 1 - (\gamma + \sigma)/2 - \alpha, \quad e_4 = 3/2, \quad f_4 = 2 - \sigma/2 - \alpha, \\
 g_4 &= (3 - \sigma)/2 - \alpha, \quad a_5 = (1 - \sigma)/2, \quad b_5 = 1 - \sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha + 1/2, \\
 e_5 &= 1 + \gamma/2 + \alpha, \quad f_5 = 1 - \gamma/2 + \alpha, \quad g_5 = 1 - \sigma, \quad a_6 = (1 - \sigma + \gamma)/2 - \alpha, \\
 b_6 &= 1 - (\sigma - \gamma)/2 - \alpha, \quad c_6 = \gamma/2, \quad d_6 = (\gamma + 1)/2, \quad e_6 = 1 + \gamma, \quad f_6 = 1 + \gamma/2 - \alpha, \\
 g_6 &= 1 - \sigma - \alpha + \gamma/2, \quad z = y \} \quad [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma - 1) < 0].
 \end{aligned}$$

$$22. \int_0^{\infty} x^{\alpha-1} \frac{(V\bar{y} + V\sqrt{y+x})^{\sigma} dx}{(1 + V\sqrt{1+x})^{\gamma} V(1+x)(y+x)} = \begin{cases} V_0^+ + V_0^+ + V_{10}^+ & [|\nu| \leq 1], \\ V_0^- + V_0^- + V_{10}^- & [|\nu| \geq 1] \end{cases}$$

$$\{E_0^+ = 2^{\sigma+2\alpha} \gamma B(\alpha, 1 - \sigma - 2\alpha), \quad E_0^- = 2^{2\alpha+\sigma} \gamma^{-1} B(\alpha + (\sigma - 1)/2, 2 + \gamma - 2\alpha - \sigma), \\
 E_{10}^+ = 2^{-2-\gamma+2\alpha} \sigma B(\alpha + \sigma/2 - 1, 3 + \gamma - 2\alpha - \sigma), \quad E_1^- = 2^{\sigma+2\alpha} \gamma B(\alpha, 1 + \gamma - 2\alpha),$$

$$E_0^- = 2^{\sigma-2\alpha} \gamma^{-1} B(\alpha - (\gamma + 1)/2, 2 - \sigma - 2\alpha + \gamma),$$

$$E_{10}^- = -2^{\sigma+2\alpha-\gamma-3} \gamma B(\alpha - \gamma/2 - 1, 3 + \gamma - \sigma - 2\alpha),$$

$$s_1 = (\sigma - 1)/2 + \alpha, \quad s_2 = 0, \quad s_3 = 1/2, \quad s_5 = (\sigma - 1)/2, \quad s_6 = \alpha + (\sigma - \gamma)/2 - 1,$$

$$s_7 = \alpha + (\sigma - \gamma - 3)/2, \quad a_1 = \alpha, \quad b_1 = \sigma + \alpha, \quad c_1 = (1 + \gamma)/2, \quad d_1 = 1 + \gamma/2,$$

$$e_1 = (1 + \sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1 + \gamma, \quad a_2 = (1 - \sigma)/2, \quad b_2 = (1 + \sigma)/2,$$

$$c_2 = 1 + (\gamma - \sigma)/2 - \alpha, \quad d_2 = (3 + \gamma - \sigma)/2 - \alpha, \quad e_2 = 1/2, \quad f_2 = (3 - \sigma)/2 - \alpha,$$

$$g_2 = (3 - \sigma)/2 + \gamma - \alpha, \quad a_3 = 1 - \sigma/2, \quad b_3 = 1 + \sigma/2, \quad c_3 = (3 + \gamma - \sigma)/2 - \alpha,$$

$$d_3 = 2 + (\gamma - \sigma)/2 - \alpha, \quad e_3 = 3/2, \quad f_3 = 2 - \sigma/2 - \alpha, \quad g_3 = 2 + \gamma - \sigma/2 - \alpha,$$

$$\begin{aligned}
 a_5 &= (1-\sigma)/2, \quad b_5 = 1-\sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha-\gamma, \quad e_5 = 1-\sigma, \quad f_5 = (1-\gamma)/2 + \alpha, \\
 g_5 &= \alpha-\gamma/2, \quad a_6 = 1+(\gamma-\sigma)/2-\alpha, \quad b_6 = (3+\gamma-\sigma)/2-\alpha, \quad c_6 = (1+\gamma)/2, \\
 d_6 &= (1-\gamma)/2, \quad e_6 = (3+\gamma)/2-\alpha-\sigma, \quad f_6 = 1/2, \quad g_6 = (3+\gamma)/2-\alpha, \\
 a_7 &= (3-\sigma+\gamma)/2-\alpha, \quad b_7 = 2+(\gamma-\sigma)/2-\alpha, \quad c_7 = 1+\gamma/2, \quad d_7 = 1-\gamma/2, \\
 e_7 &= 2-\sigma-\alpha+\gamma/2, \quad f_7 = 3/2, \quad g_7 = 2+\gamma/2-\alpha, \quad z=y\} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (2\alpha + \sigma - \gamma) < 1].
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \int_0^\infty x^{2-1} \frac{(V\sqrt{y} + V\sqrt{y+x})^\sigma dx}{(V\sqrt{x} + V\sqrt{1+x})^\gamma V\sqrt{(1+x)(y+x)}} &= \begin{cases} V_{10}^+ + V_{10}^- + V_{10}^+ + V_{11}^+ & [|\gamma| \leq 1] \\ V_{10}^- + V_{10}^- & [|\gamma| \geq 1] \end{cases} \\
 \{ E_{10}^+ &= 2^{\sigma+2\alpha} B(\alpha, 1-\sigma-2\alpha), \quad E_{10}^- = -2^{\sigma+2\alpha+1} \gamma B(\alpha+1/2, -\sigma-2\alpha), \\
 E_{10}^+ &= 2^{2-\sigma-2\alpha} B(\sigma+2\alpha-1, 1-\alpha+(\gamma-\sigma)/2), \\
 E_{11}^+ &= 2^{2-2\alpha-\sigma} B(2\alpha+\sigma-2, (3+\gamma-\sigma)/2-\alpha), \quad E_{11}^- = 2^{\sigma-2\alpha+1} B(2\alpha, (1+\gamma)/2-\alpha), \\
 E_{11}^- &= 2^{2\alpha+1-\sigma-2\gamma-1} B(\alpha-(\gamma+1)/2, 2+\gamma-\sigma-2\alpha), \\
 s_1 &= (\sigma-1)/2 + \alpha, \quad s_2 = \sigma/2 + \alpha, \quad s_3 = 0, \quad s_4 = 1/2, \quad s_5 = (\sigma-1)/2, \\
 s_6 &= \alpha + (\sigma-\gamma)/2 - 1, \quad a_1 = \alpha, \quad b_1 = \alpha + \sigma, \quad c_1 = (1+\gamma)/2, \quad d_1 = (1-\gamma)/2, \\
 e_1 &= (1+\sigma)/2 + \alpha, \quad f_1 = \sigma/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \sigma + \alpha + 1/2, \\
 c_2 &= 1+\gamma/2, \quad d_2 = 1-\gamma/2, \quad e_2 = 1+\sigma/2 + \alpha, \quad f_2 = (\sigma+1)/2 + \alpha, \quad g_2 = 3/2, \\
 a_3 &= (1-\sigma)/2, \quad b_3 = (1+\sigma)/2, \quad c_3 = 1+(\gamma-\sigma)/2-\alpha, \quad d_3 = 1-(\gamma+\sigma)/2-\alpha, \\
 e_3 &= 1/2, \quad f_3 = (3-\sigma)/2-\alpha, \quad g_3 = 1-\sigma/2-\alpha, \quad a_4 = 1-\sigma/2, \quad b_4 = 1+\sigma/2, \\
 c_4 &= (3+\gamma-\sigma)/2-\alpha, \quad d_4 = (3-\gamma-\sigma)/2-\alpha, \quad e_4 = 3/2, \quad f_4 = 2-\sigma/2-\alpha, \\
 g_4 &= (3-\sigma)/2-\alpha, \quad a_5 = (1-\sigma)/2, \quad b_5 = 1-\sigma/2, \quad c_5 = \alpha, \quad d_5 = \alpha+1/2, \\
 e_5 &= 1-\sigma, \quad f_5 = (1-\gamma)/2 + \alpha, \quad g_5 = (1+\gamma)/2 + \alpha, \quad a_6 = 1+(\gamma-\sigma)/2-\alpha, \\
 b_6 &= (3+\gamma-\sigma)/2-\alpha, \quad c_6 = (1+\gamma)/2, \quad d_6 = 1+\gamma/2, \quad e_6 = (3+\gamma)/2-\alpha, \\
 f_6 &= (3+\gamma)/2-\alpha-\sigma, \quad g_6 = 1+\gamma, \quad z=y\} \\
 &[|\arg y| < \pi; \operatorname{Re} (2\alpha + \sigma - \gamma) < 1; \operatorname{Re} \alpha > 0].
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \int_0^1 x^{2-1} \frac{(1-x^r)^{\beta-1}}{y^2 + 2xy \cos \gamma + x^2} dx &= \begin{cases} V_{12}^+ + V_{12}^- & [|\gamma| \leq 1] \\ V_{12}^- & [|\gamma| \geq 1] \end{cases} \\
 \{ E_{12}^+ &= (-1)^{l+1} \frac{\pi}{\sin \gamma} (1-\beta)_l, \quad E_{12}^- = -\frac{1}{r \sin \gamma} \Gamma \left[\frac{\beta}{(\alpha-1-l)/r + \beta} \right], \\
 E_{12}^- &= \frac{1}{r \sin \gamma} \Gamma \left[\frac{\beta}{(\alpha+l)/r + \beta} \right] \} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\gamma| < \pi; r > 0 - \text{произвольное}].
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int_0^\infty x^{2-1} \frac{(1+x^r)^{\rho-1}}{y^2 + 2xy \cos \gamma + x^2} dx &= \begin{cases} V_{12}^+ + V_{12}^- & [|\gamma| \leq 1] \\ V_{12}^- + V_{12}^- & [|\gamma| \geq 1] \end{cases} \\
 \{ E_{12}^+ &= -\frac{\pi}{\sin \gamma} (1-\rho)_l, \quad E_{12}^- = -\frac{1}{r \sin \gamma} \Gamma \left[\frac{1-\rho+(1+l-\alpha)/r}{1-\rho} \right], \\
 E_{12}^- &= \frac{1}{r \sin \gamma} \Gamma \left[\frac{1-\rho-(l+\alpha)/r}{1-\rho} \right], \quad E_{12}^- = 1 \} \\
 &[|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re} (\alpha + r\rho) < r+2; |\gamma| < \pi; r > 0 - \text{произвольное}].
 \end{aligned}$$

$$26. \quad \int_0^\infty \frac{x^{\alpha-1}}{1-x^r} \frac{dx}{y^2 + 2xy \cos \gamma + x^2} = U,$$

$$U = -\frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \left\{ \frac{\sin [(\alpha + l r - 1) \gamma]}{\sin [(\alpha + l r) \pi]} y^{l r + \alpha - 2} + \right. \\ \left. + \frac{(-1)^l}{r} \operatorname{ctg} \left(\frac{\alpha - l - 1}{r} \pi \right) \sin (l \gamma) y^{l-1} \right\} \quad [0 < y \leq 1],$$

$$U = \frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \left\{ \frac{\sin [(\alpha - r - l r - 1) \gamma]}{\sin [(\alpha - r - l r) \pi]} y^{\alpha - r - l r - 2} + \right. \\ \left. + \frac{(-1)^l}{r} \operatorname{ctg} \left(\frac{\alpha + l}{r} \pi \right) \sin (l \gamma + \gamma) y^{l-2} \right\} \quad [y > 1]; \\ [0 < \operatorname{Re} \alpha < r + 2; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$27. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^2})^{\sigma}}{y^2 + 2xy \cos \gamma + x^2} dx = \begin{cases} V_{14}^{+} + V_{15}^{+} & [|\nu| \leq 1, \\ V_{14}^{-} + V_{15}^{-} & [|\nu| \geq 1] \end{cases}$$

$$\{E_{14}^{+} = -2^{\sigma+2}(\alpha-2)/r \pi (-\sigma)_{2l}, E_{15}^{+} = 2^{\sigma+2}(\alpha-2)/r \sigma r^{-1} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{14}^{-} = 2^{\sigma+2}(\alpha-2)/r^{-1} \sigma \pi, E_{15}^{-} = -2^{\sigma+2}(\alpha-2)/r \sigma r^{-1}, z = 2^{-2/r} y, h = -\sigma\} \\ [|\arg y| < \pi; \operatorname{Re} z > 0; \operatorname{Re}(2z + \sigma r) < 2; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$28. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^2})^{\sigma}}{\sqrt{1+x^2} (y^2 + 2xy \cos \gamma + x^2)} dx = \begin{cases} V_{14}^{+} + V_{15}^{+} & [|\nu| \leq 1, \\ V_{14}^{-} + V_{15}^{-} & [|\nu| \geq 1] \end{cases}$$

$$\{E_{14}^{+} = -2^{\sigma+2}(\alpha-2)/r \pi (1-\sigma)_{2l}, E_{15}^{+} = -2^{\sigma+2}(\alpha-2)/r r^{-1} \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{14}^{-} = -2^{\sigma+2}(\alpha-2)/r^{-1} \pi, E_{15}^{-} = 2^{\sigma+2}(\alpha-2)/r r^{-1}, z = 2^{-2/r} y, h = 1-\sigma\} \\ [|\arg y| < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha + \sigma r) < 2 + r/2; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$29. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1 + 2x^r \cos \lambda + x^{2r}) (y^2 + 2xy \cos \gamma + x^2)} = \begin{cases} V_{16}^{+} + V_{17}^{+} & [|\nu| \leq 1, \\ V_{16}^{-} + V_{17}^{-} & [|\nu| \geq 1] \end{cases} \\ [|\arg y| < \pi; 0 < \operatorname{Re} \alpha < 2r + 2; |\gamma|, |\lambda| < \pi; r > 0 - \text{произвольное}].$$

$$30. \int_0^1 x^{\alpha-1} \frac{(1-x^r)^{\beta-1}}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \begin{cases} V_{18}^{+} + V_{19}^{+} + V_{15}^{+} + V_{16}^{+} & [|\nu| \leq 1, \\ V_{18}^{-} + V_{19}^{-} & [|\nu| \geq 1] \end{cases}$$

$$\{E_{18}^{+} = (-1)^{l+1} \frac{\pi}{2 \sin^3 \gamma} (1-\beta)_l y^{-2}, E_{19}^{+} = -\frac{1}{2r \sin^3 \gamma} \Gamma\left[\beta + (\alpha - l - 1)/r\right] y^{-2}, \\ E_{18}^{-} = (-1)^l \frac{\pi}{2} (1-\beta)_l, E_{19}^{-} = \frac{1}{2r} \Gamma\left[\beta + (\alpha - l - 1)/r\right], \\ E_{15}^{-} = \frac{1}{2r \sin^3 \gamma} \Gamma\left[\beta + (\alpha + l)/r\right] y^{-2}, E_{16}^{-} = -\frac{1}{2r} \Gamma\left[\beta + (\alpha + l)/r\right]\} \\ [|\arg y| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$31. \int_0^{\infty} x^{\alpha-1} \frac{(1+x^r)^{\rho-1}}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \begin{cases} V_{18}^{+} + V_{19}^{+} + V_{15}^{+} + V_{16}^{+} & [|\nu| \leq 1, \\ V_{18}^{-} + V_{19}^{-} + V_{15}^{-} + V_{16}^{-} & [|\nu| \geq 1] \end{cases}$$

$$\{E_{18}^{+} = -\frac{\pi}{2 \sin^3 \gamma} (1-\rho)_l y^{-2}, E_{19}^{+} = -\frac{1}{2r \sin^3 \gamma} \Gamma\left[\frac{1-\rho - (\alpha - l - 1)/r}{1-\rho}\right] y^{-2}, \\ E_{18}^{-} = \frac{\pi}{2} (1-\rho)_l, E_{19}^{-} = \frac{1}{2r} \Gamma\left[\frac{1-\rho - (\alpha - l - 1)/r}{1-\rho}\right],$$

$$E_{13}^{-} = \frac{1}{2r \sin^3 \gamma} \Gamma \left[\frac{1 - \rho - (\alpha + l)/r}{1 - \rho} \right] y^{-2}, \quad E_{13}^{-} = \frac{y^{-2}}{2 \sin^2 \gamma},$$

$$E_{13}^{-} = -\frac{1}{2r} \Gamma \left[\frac{1 - \rho - (\alpha + l)/r}{1 - \rho} \right] \left\{ \begin{array}{l} [\arg y] < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha + \rho r) < r + 4; |\gamma| < \pi; r > 0 - \text{произвольное} \end{array} \right\}$$

$$32. \int_0^{\infty} \frac{x^{\alpha-1}}{1-x^r} \frac{dx}{(y^2 + 2xy \cos \gamma + x^2)^2} = \frac{1}{2} (y \sin \gamma)^{-2} U + U^* \quad [\text{см. 2.2.2.26}]$$

$$U^* = \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \left\{ \frac{(\alpha + lr - 1) \cos [\gamma (\alpha + lr - 2)]}{\sin [(\alpha + lr) \pi]} y^{\alpha + lr - 4} + \right. \\ \left. + \frac{(-1)^l}{r} l \cos (\gamma - l\gamma) \operatorname{ctg} \left(\frac{\alpha - l - 1}{r} \pi \right) y^{l-3} \right\} \quad [0 < \nu \leq 1,$$

$$U^* = \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \left\{ \frac{(1 - \alpha + r + rl) \cos [\gamma (\alpha - r - rl - 2)]}{\sin [(\alpha - r - rl) \pi]} y^{\alpha - r - rl - 4} + \right. \\ \left. + \frac{(-1)^{l+1}}{r} (l + 1) \cos (\gamma l + 2\gamma) \operatorname{ctg} \left(\frac{\alpha + l}{r} \pi \right) y^{l-4} \right\} \quad [\nu > 1,$$

$$[0 < \operatorname{Re} \alpha < r + 4; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$33. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^{\sigma}}{(y^2 + 2xy \cos \gamma + x^2)^2} dx = \left\{ \begin{array}{l} V_{13}^{+} + V_{13}^{-} + V_{20}^{+} + V_{21}^{+} \\ V_{13}^{-} + V_{13}^{+} + V_{20}^{-} + V_{21}^{-} \end{array} \right. \quad \left\{ \begin{array}{l} [\nu \leq 1], \\ [\nu \geq 1] \end{array} \right.$$

$$\left\{ \begin{array}{l} E_{13}^{+} = -\frac{2^{\sigma+2}(\alpha-2)/r-1}{\sin^2 \gamma} (-\sigma)_{2l} y^{-2}, \\ E_{13}^{+} = \frac{2^{\sigma+2}(\alpha-2)/r-1}{ry^2 \sin^2 \gamma} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{20}^{+} = 2^{\sigma+2}(\alpha-4)/r-1 \pi (-\sigma)_{2l}, \\ E_{21}^{+} = -2^{\sigma+2}(\alpha-4)/r-1 \sigma r^{-1} \Gamma[-\sigma + 2r^{-1}(l+1-\alpha)], \\ E_{13}^{-} = \frac{\pi \sigma}{\sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-2 y^{-2}, \quad E_{13}^{-} = -\frac{\sigma}{r \sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 y^{-2}, \\ E_{20}^{-} = -2^{\sigma+2}(\alpha-4)/r-2 \pi \sigma, \quad E_{21}^{-} = 2^{\sigma+2}(\alpha-4)/r-1 \sigma r^{-1}, \quad z = 2^{-2/r} y, \quad h = -\sigma \end{array} \right\}$$

$$[\arg y] < \pi; \operatorname{Re} \alpha > 0; \operatorname{Re}(2\alpha + \sigma r) < 4; |\gamma| < \pi; r > 0 - \text{произвольное}].$$

$$34. \int_0^{\infty} x^{\alpha-1} \frac{(1 + \sqrt{1+x^r})^{\sigma}}{\sqrt{1+x^r} (y^2 + 2xy \cos \gamma + x^2)^2} dx =$$

$$= \left\{ \begin{array}{l} V_{13}^{+} + V_{13}^{-} + V_{20}^{+} + V_{21}^{+} \\ V_{13}^{-} + V_{13}^{+} + V_{20}^{-} + V_{21}^{-} \end{array} \right. \quad \left\{ \begin{array}{l} [\nu \leq 1], \\ [\nu \geq 1] \end{array} \right.$$

$$\left\{ \begin{array}{l} E_{13}^{+} = -\frac{\pi}{\sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 (1-\sigma)_{2l} y^{-2}, \\ E_{13}^{+} = -\frac{1}{r \sin^2 \gamma} 2^{\sigma+2}(\alpha-2)/r-1 \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)] y^{-2}, \\ E_{20}^{+} = 2^{\sigma+2}(\alpha-4)/r-1 \pi (1-\sigma)_{2l}, \quad E_{21}^{+} = r^{-1} 2^{\sigma+2}(\alpha-4)/r-1 \Gamma[1-\sigma + 2r^{-1}(l+1-\alpha)], \end{array} \right.$$

$$E_{11}^- = -\frac{\pi}{\sin^2 \gamma} 2^{\sigma+2(\alpha-2)/r-2} y^{-2}, \quad E_{12}^- = \frac{1}{r \sin^2 \gamma} 2^{\sigma+2(\alpha-2)/r-1} y^{-2},$$

$$E_{21}^- = 2^{\sigma+2(\alpha-4)/r-2} \pi, \quad E_{22}^- = -r^{-1} 2^{\sigma+2(\alpha-4)/r-1}, \quad z = 2^{-2/r} y, \quad h = 1 - \sigma$$

[arg y] < π ; $\operatorname{Re} \alpha > 0$; $\operatorname{Re}(2\alpha + \sigma r) < r + 8$; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$35. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1+2x^r \cos \lambda + x^{2r})(y^2+2xy \cos \gamma + x^2)^2} = U_1,$$

$$U_1 = 2^{-1} (y \sin \gamma)^{-2} [V_{10}^+ + V_{17}^+] + V_{23}^+ + V_{28}^+ \quad \left[\begin{matrix} |y| \leq 1 \\ |y| \geq 1 \end{matrix} \right]$$

[arg y] < π ; $0 < \operatorname{Re} \alpha < 2r + 4$; $|\gamma|, |\lambda| < \pi$; $r > 0$ — произвольное].

$$36. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(1+2x^r \cos \lambda + x^{2r})^2 (y^2+2xy \cos \gamma + x^2)^2} =$$

$$= (2 \sin^2 \lambda)^{-1} U_1 + U_1^* \quad [\text{см. 2.2.2 35}]$$

$$U_1^* = \frac{\pi}{4 \sin^2 \lambda \sin^2 \gamma} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l (l+1) \cos (\lambda l + 2\lambda)}{\sin [(\alpha + lr) \pi]} \times \right.$$

$$\times \left[\frac{\sin [\gamma (\alpha + lr - 1)]}{\sin \gamma} - (\alpha + lr - 1) \cos [\gamma (\alpha + lr - 2)] \right] y^{\alpha+lr-4} +$$

$$+ \sum_{l=0}^{\infty} \frac{(-1)^l [(\alpha - l - 1)/r - 1]}{r \sin [(\alpha - l - 1) \pi / r]} \cos \left[\lambda \left(\frac{\alpha - l - 1}{r} - 2 \right) \right] \times$$

$$\times \left[l \cos (\gamma l - \gamma) - \frac{\sin l \gamma}{\sin \gamma} \right] y^{l-4} \left. \right\} \quad [|y| \leq 1],$$

$$U_1^* = \frac{\pi}{4 \sin^2 \lambda \sin^2 \gamma} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l l \cos (\lambda l - \lambda)}{\sin [(\alpha - lr - r) \pi]} \times \right.$$

$$\times \left[(\alpha - lr - r - 1) \cos [\gamma (\alpha - lr - r - 2)] - \frac{\sin [\gamma (\alpha - lr - r - 1)]}{\sin \gamma} \right] \times$$

$$\times y^{\alpha - r - lr - 4} + \sum_{l=0}^{\infty} \frac{(-1)^l [(\alpha + l)/r - 1] \cos [\lambda r^{-1} (\alpha + l - 2r)]}{r \sin [(\alpha + l) \pi / r]} \times$$

$$\times \left[(l - 1) \cos (\gamma l + 2\gamma) + \frac{\sin (\gamma l + \gamma)}{\sin \gamma} \right] y^{l-4} \left. \right\} \quad [|y| \geq 1],$$

[arg y] < π ; $0 < \operatorname{Re} \alpha < 4(r+1)$; $|\gamma|, |\lambda| < \pi$; $r > 0$ — произвольное].

$$37. \int_0^{\min(1, y)} x^{\alpha-1} (1-x^r)^{\beta-1} \frac{(V\bar{y} + V\bar{y}-x)^{\gamma} + (V\bar{y}-V\bar{y}-x)^{\gamma}}{V\bar{y}-x} dx =$$

$$= \begin{cases} V_5^+ & [0 < y \leq 1], \\ V_5^+ + V_6^+ & [y \geq 1] \end{cases}$$

$$\{E_1\} = (-1)^j 2^{\gamma+2h+2/r} (1-\beta)_j B(\alpha+jr, \gamma+\alpha+jr),$$

$$E_2^+ = 2^{\gamma-2h} r^{-1} (\gamma-2h)_h B(\beta, r^{-1}(\alpha+h)),$$

$$E_3^+ = 2^{-\gamma-2h} r^{-1} (-\gamma-2h)_h B(\beta, r^{-1}(\alpha+\gamma+h)),$$

$$s = \alpha + jr + (\gamma-1)/2, \quad u = (\gamma-1)/2 - h, \quad u_1 = -(1+\gamma)/2 - h,$$

$$\begin{aligned}
 a_1 &= \alpha + jr, \quad b_1 = \alpha + \gamma + jr, \quad c_1 = 1 - \beta + j, \quad d_1 = \alpha + \gamma/2 + jr, \\
 e_1 &= \alpha + (\gamma + 1)/2 + jr, \quad a_4 = 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h, \\
 c_4 &= (\alpha + h)/r, \quad d_4 = 1 - \gamma + h, \quad e_4 = \beta + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h, \\
 b_5 &= (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r, \quad d_5 = 1 + \gamma + h, \\
 e_5 &= \beta + (\alpha + \gamma + h)/r, \quad z = yp
 \end{aligned}$$

[$y, \operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0$ и $\operatorname{Re} \beta > 1/2$ (при $y=1$), или $\operatorname{Re} \beta > 0$ (при $y > 1$)].

$$38. \int_1^y x^{\alpha-1} (x^r - 1)^{\beta-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx = V_+^- + V_0^- + V_7^-$$

$$\{E_+^- = r^{-1} 2^{\gamma-2h} (\gamma - 2h)_h B(\beta, 1 - \beta - r^{-1}(\alpha + h)),$$

$$E_-^- = r^{-1} 2^{\gamma-2h} (-\gamma - 2h)_h B(\beta, 1 - \beta - r^{-1}(\alpha + \gamma + h)),$$

$$E_7^- = (-1)^j 2^{\gamma+2\alpha-2r} (1-\beta+j) (1-\beta)_j B(\alpha - r(1-\beta+j), \gamma + \alpha - r(1-\beta+j)),$$

$$u = (\gamma - 1)/2 - h, \quad u_1 = -(1 + \gamma)/2 - h, \quad v = \alpha + (\gamma - 1)/2 - r(1 - \beta + j),$$

$$a_4 = 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h, \quad c_4 = (\alpha + h)/r, \quad d_4 = 1 - \gamma + h,$$

$$e_4 = \beta + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h, \quad b_5 = (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r,$$

$$d_5 = 1 + \gamma + h, \quad e_5 = \beta + (\alpha + \gamma + h)/r, \quad a_6 = 1 - \gamma/2 - \alpha + r(1 - \beta + j),$$

$$b_6 = (1 - \gamma)/2 - \alpha + r(1 - \beta + j), \quad c_6 = 1 - \beta + j, \quad d_6 = 1 - \alpha + r(1 - \beta + j),$$

$$e_6 = 1 - \gamma - \alpha + r(1 - \beta + j), \quad z = yp \} \quad [y > 1, \operatorname{Re} \beta > 0].$$

$$39. \int_0^y x^{\alpha-1} (1+x^r)^{\rho-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx =$$

$$= \begin{cases} V_+^+ & [0 < y \leq 1], \\ V_5^+ + V_6^+ + V_7^+ & [y > 1] \end{cases}$$

$$\{E_+^+ = 2^{\gamma+2\alpha+2/r} (1-\rho)_j B(\alpha + jr, \gamma + \alpha + jr),$$

$$E_5^+ = \frac{2^{\gamma-2h}}{r} (\gamma - 2h)_h B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right),$$

$$E_6^+ = \frac{2^{\gamma-2h}}{r} (-\gamma - 2h)_h B\left(\frac{\alpha + \gamma + h}{r}, 1 - \rho - \frac{\alpha + \gamma + h}{r}\right),$$

$$E_7^+ = 2^{\gamma+2\alpha-2r} (1-\rho+j) (1-\rho)_j B(\alpha - r(1-\rho+j), \gamma + \alpha - r(1-\rho+j)),$$

$$s = (\gamma - 1)/2 + \alpha + jr, \quad u = (\gamma - 1)/2 - h, \quad u_1 = -(1 + \gamma)/2 - h,$$

$$v = \alpha + (\gamma - 1)/2 - r(1 - \rho + j), \quad a_1 = \alpha + jr, \quad b_1 = \gamma + \alpha + jr, \quad c_1 = 1 - \rho + j,$$

$$d_1 = \gamma/2 + \alpha + jr, \quad e_1 = (\gamma + 1)/2 + \alpha + jr, \quad a_4 = 1 - \gamma/2 + h, \quad b_4 = (1 - \gamma)/2 + h,$$

$$c_4 = (\alpha + h)/r, \quad d_4 = 1 - \gamma + h, \quad e_4 = \rho + (\alpha + h)/r, \quad a_5 = 1 + \gamma/2 + h,$$

$$b_5 = (1 + \gamma)/2 + h, \quad c_5 = (\alpha + \gamma + h)/r, \quad d_5 = 1 + \gamma + h, \quad e_5 = \rho + (\alpha + \gamma + h)/r,$$

$$a_6 = 1 - \gamma/2 - \alpha + r(1 - \rho + j), \quad b_6 = (1 - \gamma)/2 - \alpha + r(1 - \rho + j), \quad c_6 = 1 - \rho + j,$$

$$d_6 = 1 - \alpha + r(1 - \rho + j), \quad e_6 = 1 - \gamma - \alpha + r(1 - \rho + j), \quad z = (-1)^j yp \} \\ [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].$$

$$40. \int_0^y \frac{x^{\alpha-1}}{1-x^r} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{\sqrt{y-x}} dx =$$

$$= \begin{cases} V_+^+ & [0 < y \leq 1], \\ V_5^+ + V_6^+ + V_7^+ & [y > 1] \end{cases}$$

$$\{E_+^+ = (-1)^j j! 2^{\gamma+2\alpha+2r} B(\alpha + jr, \gamma + \alpha + jr),$$

$$E_5^+ = \frac{\pi}{r} \operatorname{ctg} \left(\frac{\alpha + h}{r} \pi \right) 2^{\gamma-2h} (\gamma - 2h)_h,$$

$$E_+ = \frac{\pi}{r} 2^{-\gamma-2h} (-\gamma-2h)_h \operatorname{ctg} \left(\frac{\alpha+\gamma+h}{r} \pi \right),$$

$$E_7 = (-1)^{j+1} j! 2^{\gamma+2\alpha-2r(j+1)} B(\alpha-r(j+1), \gamma+\alpha-r(j+1)),$$

$$s = \alpha + (\gamma-1)/2 + jr, u = (\gamma-1)/2 - h, u_1 = -(1+\gamma)/2 - h,$$

$$v = (\gamma-1)/2 + \alpha - r(j+1), a_1 = \alpha + jr, b_1 = \gamma + \alpha + jr, c_1 = 1 + j,$$

$$d_1 = \gamma/2 + \alpha + jr, e_1 = (\gamma+1)/2 + \alpha + jr, a_4 = 1 - \gamma/2 + h, b_4 = (1-\gamma)/2 + h,$$

$$c_4 = e_4 = 1/2, d_4 = 1 - \gamma + h, a_5 = 1 + \gamma/2 + h, b_5 = (1+\gamma)/2 + h,$$

$$c_5 = e_5 = 1/2, d_5 = 1 + \gamma + h, a_6 = 1 - \gamma/2 - \alpha + r(1+j),$$

$$b_6 = (1-\gamma)/2 - \alpha + r(1+j), c_6 = 1 + j, d_6 = 1 - \alpha + r(1+j),$$

$$e_6 = 1 - \gamma - \alpha + r(1+j), z = y^p \} \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha+\gamma) > 0].$$

$$41. \int_0^y x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{(1 + \sqrt{1+x})^{-\sigma} \sqrt{y-x}} dx =$$

$$= \begin{cases} V_+^+ + V_-^+ & [0 < \nu \leq 1], \\ V_+^- + V_-^- + V_{10}^- + V_{11}^- & [\nu > 1] \end{cases}$$

$$\{E_+ = 2^{2\alpha+\gamma+\sigma} B(\alpha, \gamma+\alpha), E_- = 2^{\gamma+\sigma+2\alpha} B(\alpha+\sigma/2, \alpha+\gamma+\sigma/2),$$

$$E_0 = 2^{\gamma+\sigma+2\alpha} \sigma B(\alpha+(\sigma-1)/2, \gamma+\alpha+(\sigma-1)/2),$$

$$E_{10} = -2^{\sigma+2\alpha+\gamma} \sigma \Gamma \left[\begin{matrix} \alpha, -\sigma-2\alpha \\ 1-\alpha-\sigma \end{matrix} \right], E_{11} = -2^{\sigma+\gamma+2\alpha} \sigma \Gamma \left[\begin{matrix} \alpha+\gamma, -\sigma-2\alpha-2\gamma \\ 1-\sigma-\alpha-\gamma \end{matrix} \right],$$

$$s_1 = \alpha + (\gamma-1)/2, s_5 = \alpha + (\gamma+\sigma-1)/2, s_8 = \alpha + (\gamma+\sigma)/2 - 1,$$

$$s_7 = (\gamma-1)/2, s_9 = -(1+\gamma)/2, a_1 = \alpha, b_1 = \gamma + \alpha, c_1 = -\sigma/2,$$

$$d_1 = (1-\sigma)/2, e_1 = \gamma/2 + \alpha, f_1 = (\gamma+1)/2 + \alpha, g_1 = 1 - \sigma,$$

$$a_5 = 1 - \alpha - (\gamma+\sigma)/2, b_5 = (1-\gamma-\sigma)/2 - \alpha, c_5 = \sigma/2, d_5 = -\sigma/2,$$

$$e_5 = 1 - \alpha - \sigma/2, f_5 = 1 - \gamma - \alpha - \sigma/2, g_5 = 1/2, a_6 = (3-\sigma-\gamma)/2 - \alpha,$$

$$b_6 = 1 - (\gamma+\sigma)/2 - \alpha, c_6 = (1-\sigma)/2, d_6 = (1+\sigma)/2, e_6 = (3-\sigma)/2 - \alpha,$$

$$f_6 = (3-\sigma)/2 - \gamma - \alpha, g_6 = 3/2, a_7 = 1 - \gamma/2, b_7 = (1-\gamma)/2, c_7 = \sigma + \alpha, d_7 = \alpha,$$

$$e_7 = 1 + \sigma/2 + \alpha, f_7 = (1+\sigma)/2 + \alpha, g_7 = 1 - \gamma, a_8 = 1 + \gamma/2,$$

$$b_8 = (1+\gamma)/2, c_8 = \sigma + \alpha + \gamma, d_8 = \alpha + \gamma, e_8 = 1 + \sigma/2 + \alpha + \gamma,$$

$$f_8 = (1+\sigma)/2 + \alpha + \gamma, g_8 = 1 + \gamma, z = -y \} \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha+\gamma) > 0].$$

$$42. \int_0^y x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{(\sqrt{x} + \sqrt{1+x})^{-\sigma} \sqrt{y-x}} dx =$$

$$= \begin{cases} V_+^+ + V_0^+ & [0 < \nu \leq 1], \\ V_+^- + V_0^- + V_{10}^- & [\nu > 1] \end{cases}$$

$$\{E_+ = 2^{\gamma+2\alpha} B(\alpha, \gamma+\alpha), E_- = 2^{\gamma+2\alpha+1} \sigma B(\alpha+1/2, \gamma+\alpha+1/2),$$

$$E_0 = 2^{\gamma+2\alpha} \sigma B(\alpha+\sigma/2, \gamma+\alpha+\sigma/2), E_8 = -2^{\gamma+2\alpha} \sigma \Gamma \left[\begin{matrix} 2\alpha, -\alpha-\sigma/2 \\ 1+\alpha-\sigma/2 \end{matrix} \right],$$

$$E_{10} = -2^{2\alpha+\gamma} \sigma \Gamma \left[\begin{matrix} 2\alpha+2\gamma, -\alpha-\gamma-\sigma/2 \\ 1+\alpha+\gamma-\sigma/2 \end{matrix} \right],$$

$$s_1 = \alpha + (\gamma-1)/2, s_2 = \alpha + \gamma/2, s_5 = \alpha + (\sigma+\gamma-1)/2, s_8 = (\gamma-1)/2,$$

$$s_7 = -(1+\gamma)/2, a_1 = \alpha, b_1 = \alpha + \gamma, c_1 = -\sigma/2, d_1 = \sigma/2,$$

$$e_1 = \gamma/2 + \alpha, f_1 = (\gamma+1)/2 + \alpha, g_1 = 1/2, a_2 = \alpha + 1/2,$$

$$b_2 = \alpha + \gamma + 1/2, c_2 = (1-\sigma)/2, d_2 = (\sigma+1)/2, e_2 = \alpha + (\gamma+1)/2,$$

$$f_2 = \alpha + 1 + \gamma/2, g_2 = 3/2, a_5 = 1 - (\gamma+\sigma)/2 - \alpha, b_5 = (1-\gamma-\sigma)/2 - \alpha,$$

$$c_5 = -\sigma/2, d_5 = (1-\sigma)/2, e_5 = 1 - \alpha - \sigma/2, f_5 = 1 - \gamma - \alpha - \sigma/2,$$

$$\begin{aligned}
 g_3 &= 1 - \sigma, \quad a_6 = 1 - \gamma/2, \quad b_6 = (1 - \gamma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \\
 e_6 &= 1 - \gamma, \quad f_6 = 1 + \sigma/2 + \alpha, \quad g_6 = 1 - \sigma/2 + \alpha, \quad a_7 = 1 + \gamma/2, \quad b_7 = (1 + \gamma)/2, \\
 c_7 &= \alpha + \gamma, \quad d_7 = \alpha + \gamma + 1/2, \quad e_7 = 1 + \sigma/2 + \alpha + \gamma, \quad f_7 = 1 - \sigma/2 + \alpha + \gamma, \\
 g_7 &= 1 + \gamma, \quad z = -y \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \int_0^y x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{(1 + \sqrt{1+x})^{-\sigma} \sqrt{(1+x)(y-x)}} dx = \\
 = \begin{cases} V_6^+ & [0 < \nu \leq 1], \\ V_6^+ + V_6^- + V_{10}^- + V_{11}^- & [\nu > 1] \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \{E_6^+ &= 2^{\gamma+\sigma+2\alpha} B(\alpha, \gamma + \alpha), \\
 E_6^- &= 2^{\gamma+2\alpha+\sigma-1} B(\alpha + (\sigma-1)/2, \gamma + \alpha + (\sigma-1)/2), \\
 E_6^0 &= 2^{\gamma+2\alpha+\sigma-2} \sigma B(\alpha + \sigma/2 - 1, \gamma + \alpha + \sigma/2 - 1), \\
 E_{10}^- &= 2^{\sigma+2\alpha+\gamma+1} B(\alpha, 1 - \sigma - 2\alpha), \quad E_{11}^- = 2^{\sigma+2\alpha+\gamma+1} B(\alpha + \gamma, 1 - \sigma - 2\alpha - 2\gamma), \\
 s_1 &= \alpha + (\gamma-1)/2, \quad s_5 = \alpha + (\gamma + \sigma)/2 - 1, \quad s_6 = \alpha + (\gamma + \sigma - 3)/2, \\
 s_7 &= (\gamma-1)/2, \quad s_8 = -(1 + \gamma)/2, \quad a_1 = \alpha, \quad b_1 = \alpha + \gamma, \quad c_1 = 1 - \sigma/2, \quad d_1 = (1 - \sigma)/2, \\
 e_1 &= \gamma/2 + \alpha, \quad f_1 = (\gamma + 1)/2 + \alpha, \quad g_1 = 1 - \sigma, \quad a_5 = (3 - \gamma - \sigma)/2 - \alpha, \\
 b_5 &= 1 - (\gamma + \sigma)/2 - \alpha, \quad c_5 = (1 + \sigma)/2, \quad d_5 = (1 - \sigma)/2, \quad e_5 = (3 - \sigma)/2 - \alpha, \\
 f_5 &= (3 - \sigma)/2 - \gamma - \alpha, \quad g_5 = 1/2, \quad a_6 = 2 - (\gamma + \sigma)/2 - \alpha, \quad b_6 = (3 - \gamma - \sigma)/2 - \alpha, \\
 c_6 &= 1 + \sigma/2, \quad d_6 = 1 - \sigma/2, \quad e_6 = 2 - \sigma/2 - \alpha, \quad f_6 = 2 - \sigma/2 - \alpha - \gamma, \quad g_6 = 3/2, \\
 a_7 &= 1 - \gamma/2, \quad b_7 = (1 - \gamma)/2, \quad c_7 = \sigma + \alpha, \quad d_7 = \alpha, \quad e_7 = (1 + \sigma)/2 + \alpha, \\
 f_7 &= \sigma/2 + \alpha, \quad g_7 = 1 - \gamma, \quad a_8 = 1 + \gamma/2, \quad b_8 = (1 + \gamma)/2, \quad c_8 = \sigma + \alpha + \gamma, \\
 d_8 &= \alpha + \gamma, \quad e_8 = 1 + \gamma, \quad f_8 = (1 + \sigma)/2 + \alpha + \gamma, \quad g_8 = \sigma/2 + \alpha + \gamma, \quad z = -y \quad \\
 &\quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \int_0^y x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y-x})^\gamma + (\sqrt{y} - \sqrt{y-x})^\gamma}{(\sqrt{x} + \sqrt{1+x})^{-\sigma} \sqrt{(1+x)(y-x)}} dx = \\
 = \begin{cases} V_6^+ + V_6^- & [0 < \nu \leq 1], \\ V_6^+ + V_6^- + V_{10}^- & [\nu > 1] \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \{E_6^+ &= 2^{\gamma+2\alpha} B(\alpha, \gamma + \alpha), \quad E_6^0 = 2^{\gamma+2\alpha+1} \sigma B(\alpha + 1/2, \gamma + \alpha + 1/2), \\
 E_6^- &= 2^{\gamma+2\alpha+2\alpha-1} B(\alpha + (\sigma-1)/2, \alpha + \gamma + (\sigma-1)/2), \\
 E_{10}^- &= 2^{\gamma+2\alpha+1} B(2\alpha, (1 - \sigma)/2 - \alpha), \quad E_{11}^- = 2^{\gamma+2\alpha+1} B(2\alpha + 2\gamma, (1 - \sigma)/2 - \alpha - \gamma), \\
 s_1 &= \alpha + (\gamma-1)/2, \quad s_2 = \alpha + \gamma/2, \quad s_3 = \alpha + (\gamma + \sigma)/2 - 1, \quad s_6 = (\gamma-1)/2, \\
 s_7 &= -(1 + \gamma)/2, \quad a_1 = \alpha, \quad b_1 = \alpha + \gamma, \quad c_1 = (1 - \sigma)/2, \quad d_1 = (1 + \sigma)/2, \\
 e_1 &= \alpha + \gamma/2, \quad f_1 = (\gamma + 1)/2 + \alpha, \quad g_1 = 1/2, \quad a_2 = \alpha + 1/2, \quad b_2 = \gamma + \alpha + 1/2, \\
 c_2 &= 1 - \sigma/2, \quad d_2 = 1 + \sigma/2, \quad e_2 = \alpha + (\gamma + 1)/2, \quad f_2 = \alpha + 1 + \gamma/2, \\
 g_2 &= 3/2, \quad a_5 = (3 - \gamma - \sigma)/2 - \alpha, \quad b_5 = 1 - (\gamma + \sigma)/2 - \alpha, \quad c_5 = (1 - \sigma)/2, \\
 d_5 &= 1 - \sigma/2, \quad e_5 = (3 - \sigma)/2 - \alpha, \quad f_5 = (3 - \sigma)/2 - \alpha - \gamma, \quad g_5 = 1 - \sigma, \quad a_6 = 1 - \gamma/2, \\
 b_6 &= (1 - \gamma)/2, \quad c_6 = \alpha, \quad d_6 = \alpha + 1/2, \quad e_6 = 1 - \gamma, \quad f_6 = (1 + \sigma)/2 + \alpha, \\
 g_6 &= (1 - \sigma)/2 + \alpha, \quad a_7 = 1 + \gamma/2, \quad b_7 = (1 + \gamma)/2, \quad c_7 = \alpha + \gamma, \\
 d_7 &= \alpha + \gamma + 1/2, \quad e_7 = 1 + \gamma, \quad f_7 = (1 + \sigma)/2 + \alpha + \gamma, \\
 g_7 &= (1 - \sigma)/2 + \alpha + \gamma, \quad z = -y \quad [\operatorname{Re} \alpha, \operatorname{Re}(\alpha + \gamma) > 0].
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \int_0^y x^{\alpha-1} \frac{(\sqrt{y} + \sqrt{y-x})^\lambda + (\sqrt{y} - \sqrt{y-x})^\lambda}{(1 + 2x \cos \gamma + x^2) \sqrt{y-x}} dx = U_4 = \\
 = \begin{cases} V_{12}^+ + V_{13}^+ + V_{24}^+ & [0 < \nu \leq 1], \\ V_{12}^+ & [\nu > 1] \end{cases}
 \end{aligned}$$

$$\{E_{11}^+ = -\frac{2^{\lambda-2l}}{\sin \gamma} \pi \Gamma \left[\frac{\lambda-l}{\lambda-2l} \right],$$

$$E_{12}^+ = -\frac{2^{\lambda+2(\alpha-l-1)/r}}{r \sin \gamma} \Gamma \left[\frac{\lambda+(\alpha-l-1)/r}{\lambda+2(\alpha-l-1)/r} \right],$$

$$E_{21}^+ = -\frac{2^{-\lambda-2l}\pi}{\sin \gamma} \Gamma \left[\frac{-l-\lambda}{-\lambda-2l} \right], \quad E_{22}^+ = \frac{2^{\lambda+2(\alpha+l)/r}}{r \sin \gamma} \Gamma \left[\frac{\lambda+(\alpha+l)/r}{\lambda+2(\alpha+l)/r} \right],$$

[Re α , Re $(\alpha+\lambda)$ > 0; $|\gamma| < \pi$; $r > 0$ — произвольное].

$$46. \int_0^y x^{2-1} \frac{(V\bar{y} + V\bar{y-x})^\lambda + (V\bar{y} - V\bar{y-x})^\lambda}{(1+2x \cos \gamma + x^2)^2 V\bar{y-x}} dx =$$

$$= 2^{-1} (y \sin \gamma)^{-2} U_3 + \left\{ \frac{V_{13}^+ + V_{13}^- + V_{23}^+}{V_{13}^-}, \quad \left[\begin{matrix} 0 \leq y \leq 1 \\ y \geq 1 \end{matrix} \right]; \text{ см. 2.22.45} \right\}$$

$$\{E_{13}^+ = 2^{\lambda-2l-1}\pi \Gamma \left[\frac{\lambda-l}{\lambda-2l} \right], \quad E_{13}^- = \frac{2^{\lambda+2(\alpha-l-1)/r-1}}{r} \Gamma \left[\frac{\lambda+(\alpha-l-1)/r}{\lambda+2(\alpha-l-1)/r} \right],$$

$$E_{23}^+ = 2^{-\lambda-2l-1}\pi \Gamma \left[\frac{-l-\lambda}{\gamma-2(\lambda+l)} \right], \quad E_{23}^- = -\frac{2^{\lambda+2(\alpha+l)/r-1}}{r} \Gamma \left[\frac{\lambda+(l+\alpha)/r}{\lambda+2(l+\alpha)/r} \right]$$

[Re α , Re $(\alpha+\lambda)$ > 0; $|\gamma| < \pi$; $r > 0$ — произвольное].

2.2.3. Интегралы от $(x^\mu \mp a^\mu)^\rho$ и $(a^\mu - x^\mu)^\rho$.

$$1. \int_0^a (a^\mu - x^\mu)^{\beta-1} dx = \mu^{-1} a^{\mu(\beta-1)+1} B(\beta, 1/\mu) \quad [a, \mu, \operatorname{Re} \beta > 0].$$

$$2. \int_0^a (a^2 - x^2)^{n-1/2} dx = \pi a^{2n} \frac{(2n-1)!!}{2(2n)!!} \quad [a > 0].$$

$$3. \int_0^a (a^2 - x^2)^n dx = a^{2n+1} \frac{(2n)!!}{(2n+1)!!} \quad [a > 0].$$

$$4. \int_a^\infty (x^\mu - a^\mu)^{\beta-1} dx = \mu^{-1} a^{\mu(\beta-1)+1} B(1-\beta-1/\mu, \beta)$$

[a, $\mu > 0$; $0 < \operatorname{Re} \beta < 1 - 1/\mu$].

$$5. \int_0^\infty (x^\mu + z^\mu)^{-\rho} dx = \mu^{-1} z^{1-\mu\rho} B(1/\mu, \rho - 1/\mu) \quad [\mu > 0; \mu |\arg z| < \pi; \mu \operatorname{Re} \rho > 1].$$

$$6. \int_0^\infty \frac{dx}{(x^2 + z^2)^n} = \frac{\pi}{2} z^{1-2n} \frac{(2n-3)!!}{(2n-2)!!} \quad [\operatorname{Re} z > 0; n = 1, 2, 3, \dots].$$

2.2.4. Интегралы от $x^\alpha (a^2 \pm x^2)^\rho$.

$$1. \int_0^a \frac{x^{\alpha-1}}{x+a} dx = a^{\alpha-1} \beta(\alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$2. \int_0^a \frac{x^n}{x+a} dx = (-a)^n \left\{ \ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right\} \quad [a > 0].$$

$$3. \int_0^a \frac{dx}{\sqrt{x(x+a)}} = \frac{\pi}{2\sqrt{a}} \quad [a > 0].$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(x+a)^2} dx = \frac{a^{\alpha-2}}{2} - (\alpha-1) a^{\alpha-2} \beta(\alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$5. \int_0^a \frac{x^n}{(x+a)^2} dx = n(-a)^{n-1} \left\{ \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{k} + \frac{(-1)^n}{2n} \right\} \quad [a > 0; n = 1, 2, 3, \dots].$$

$$6. \int_0^a \frac{x^{\alpha-1}}{(x+a)^{2\alpha}} dx = \frac{a^{-\alpha}}{2} B(\alpha, \alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$7. \int_0^a \frac{x^n dx}{\sqrt{x^2+a^2}} = I_n \quad [a > 0].$$

$$I_0 = \ln(1+\sqrt{2}), \quad I_1 = a(\sqrt{2}-1), \quad I_2 = \frac{a^2}{2} [\sqrt{2} - \ln(1+\sqrt{2})],$$

$$I_n = \frac{a^n \sqrt{2}}{n} \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{\left(\frac{1-n}{2}\right)_k}{(1-n/2)_k} +$$

$$+ (-1)^{[(n+1)/2]} a^n \left\{ \frac{1 - (-1)^n \frac{\left(\frac{1-n}{2}\right)_{[(n+1)/2]}}{(-n/2)_{[(n+1)/2]}} \ln(1+\sqrt{2}) - \right.$$

$$\left. - \frac{1 - (-1)^n}{2(n+1)} \frac{\left(\frac{1-n}{2}\right)_{[(n+1)/2]}}{(-n/2)_{[(n+1)/2]}} \right\}.$$

$$8. \int_0^a x^{\alpha-1} (a^2 - x^2)^{\beta-1} dx = a^{\mu} \beta^{-1} \mu^{-1} B\left(\frac{\alpha}{\mu}, \beta\right) \quad [a, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$9. \int_0^a x^{2m} (a^2 - x^2)^{n-1/2} dx = \frac{\pi a^{2m+2n} (2m-1)!! (2n-1)!!}{2^{m+n+1} (m+n)!} \quad [a > 0].$$

$$10. \int_0^a x^{2m+1} (a^2 - x^2)^{n-1/2} dx = \frac{a^{2m+2n+1} 2^m m! (2n-1)!!}{(2m+2n+1)!!} \quad [a > 0].$$

$$11. \int_0^a \frac{x^{2m+\delta}}{\sqrt{a^2 - x^2}} dx = a^{2m+\delta} \frac{(2m+\delta-1)!!}{(2m+\delta)!!} \left\{ \frac{1}{\pi/2} \right\} \quad [a > 0; \delta = \{ \frac{1}{2} \}].$$

$$12. \int_0^a x^{\alpha-1} (a-x)^{-\alpha} dx = \frac{\pi}{\sin \alpha \pi} \quad [a > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$13. \int_0^a x^{\alpha-1} (2a-x)^{\alpha-1} dx = \frac{1}{2} (2a)^{2\alpha-1} B(\alpha, \alpha) \quad [a, \operatorname{Re} \alpha > 0].$$

$$14. \int_0^1 \frac{x^\alpha + x^\beta}{(x+1)^{\alpha+\beta+2}} dx = B(\alpha+1, \beta+1) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$15. \int_0^1 \frac{x^{\alpha-1} + x^{-\alpha}}{x+1} dx = \frac{\pi}{\sin \alpha \pi} \quad [0 < \operatorname{Re} \alpha < 1].$$

$$16. \int_0^1 \frac{x^\alpha - x^{-\alpha}}{x+1} dx = \frac{1}{\alpha} - \frac{\pi}{\sin \alpha \pi} \quad [0 < |\operatorname{Re} \alpha| < 1].$$

$$17. \int_0^1 \frac{x^{p-1} + x^{mn-p-1}}{(x^n+1)^m} dx = \left(\frac{m-1-p/n}{m-1} \right) \frac{\pi}{n \sin(p\pi/n)} \\ [m, n, p = 1, 2, 3, \dots; mn > 1; p/n - \text{нечетное}].$$

$$18. = \frac{(p/n-1)! (m-1-p/n)!}{(m-1)! n} \quad [m, n, p = 1, 2, 3, \dots; mn > p; p/n - \text{целое}].$$

$$19. \int_0^1 \frac{x^\alpha - x^\beta}{(1-x)^{\alpha+\beta+2}} dx = \frac{\sin[(\alpha-\beta)\pi/2]}{\sin[(\alpha+\beta)\pi/2]} B(\alpha+1, \beta+1) \\ [\operatorname{Re}(\alpha+\beta) < 0; \operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$20. \int_0^1 \frac{x^\alpha - x^\beta}{1-x} dx = \psi(\beta+1) - \psi(\alpha+1) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > -1].$$

$$21. \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{1-x} dx = \pi \operatorname{ctg} \alpha \pi \quad [0 < \operatorname{Re} \alpha < 1].$$

$$22. \int_0^1 \frac{x^\alpha - x^{-\alpha}}{1-x} dx = \pi \operatorname{ctg} \alpha \pi - \frac{1}{\alpha} \quad [|\operatorname{Re} \alpha| < 1].$$

$$23. \int_0^1 \frac{x^\alpha - x^\beta}{x^\mu - x^\nu} dx = \frac{1}{\mu - \nu} \left[\psi \left(\frac{\mu - \alpha - 1}{\mu - \nu} \right) - \psi \left(\frac{\mu - \beta - 1}{\mu - \nu} \right) \right] \\ [\mu > \max(\nu, \alpha + 1, \beta + 1)].$$

$$24. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} dx = z^{\alpha-\rho} B(\alpha, \rho-\alpha) \quad [|\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho].$$

$$25. \int_0^\infty \frac{x^{\alpha-1}}{x+z} dx = \frac{\pi z^{\alpha-1}}{\sin \alpha \pi} \quad [|\arg z| < \pi; 0 < \operatorname{Re} \alpha < 1].$$

$$26. \int_0^\infty \frac{x^{\alpha-1} dx}{x-a} = -\pi a^{\alpha-1} \operatorname{ctg} \alpha \pi \quad [a > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$27. \int_0^\infty \frac{x^{\alpha-1}}{|x-a|^\rho} dx = \frac{a^{\alpha-\rho} \cos(\alpha-\rho/2)\pi}{\cos(\rho\pi/2)} B(\alpha, \rho-\alpha) \\ [a > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho < 1].$$

$$28. \int_0^\infty \frac{x^{\alpha-1}}{(x-a)^\pi} dx = \frac{(-1)^\pi \pi a^{\alpha-\pi}}{(n-1)! \sin \alpha \pi} \prod_{k=1}^{n-1} (\alpha - k) \\ [a > 0; 0 < \operatorname{Re} \alpha < \pi; n = 1, 3, 5, \dots].$$

$$29. \int_{-\infty}^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} dx = 0 \quad [0 < |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \rho].$$

$$30. \int_{-\infty}^\infty \frac{|x|^{\alpha-1}}{x-a} \left\{ \frac{1}{\operatorname{sgn} x} \right\} = \mp \pi \operatorname{tg}^{-1} \frac{\alpha \pi}{2} |a|^{\alpha-1} \left\{ \frac{\operatorname{sgn} a}{1} \right\} \quad [\operatorname{Im} a = 0; 0 < \operatorname{Re} \alpha < 1].$$

2.2.5. Интегралы от $(a^x \pm x^a)^\alpha (b^x \mp x^b)^\beta$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b (x-a)^{\alpha-1} (b-x)^{\beta-1} dx = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_a^b \frac{(x-a)^{\alpha-1}}{(cx+d)^{\alpha+n+1}} dx = \frac{(b-a)^\alpha}{(ac+d)(bc+d)^\alpha} \sum_{k=0}^n \binom{n}{k} \frac{B(\alpha+k, n-k+1)}{(bc+d)^k (ac+d)^{n-k}} \\ [(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0].$$

$$3. \int_a^b \frac{(x-a)^{\alpha-1}}{(cx+d)^{\alpha+1}} dx = \frac{(b-a)^\alpha}{\alpha (ac+d)(bc+d)^\alpha} \quad [(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0].$$

$$4. \int_a^b \frac{(b-x)^{\beta-1}}{(cx+d)^{\beta+n+1}} dx = \frac{(b-a)^\beta}{(bc+d)(ac+d)^\beta} \sum_{k=0}^n \binom{n}{k} \frac{B(k+1, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}} \\ [(ac+d)(bc+d) > 0; \operatorname{Re} \beta > 0].$$

$$5. \int_a^b \frac{(b-x)^{\beta-1}}{(cx+d)^{\beta+1}} dx = \frac{(b-a)^{\beta}}{\beta (bc+d)(ac+d)^{\beta}} \quad [(ac+d)(bc+d) > 0; \operatorname{Re} \beta > 0].$$

$$6. \int_{-a}^a \left(\frac{a-x}{a+x} \right)^{\beta-1} dx = \frac{2\pi(1-\beta)a}{\sin \beta\pi} \quad [0 < \operatorname{Re} \beta < 2, \beta \neq 1].$$

$$7. \int_{-a}^a \frac{(a^2-x^2)^m}{(cx+d)^{m+1/2}} dx = \frac{m!2^{m+1}}{(2m+1)!} (V_{ac+d} - V_{d-ac})^{2m+1} \quad [d > |ac| > 0].$$

$$8. \int_{-a}^a \frac{(a^2-x^2)^m}{(y-x)^{m+1}} dx = 2(2a)^m Q_m \left(\frac{y}{a} \right) \quad [|y| < a].$$

$$9. \int_{-a}^a \frac{(d+icx)^m}{V_{a^2-x^2}} dx = \pi (d^2+a^2c^2)^{m/2} P_m \left(\frac{d}{V_{d^2+a^2c^2}} \right) \quad [m=0, \pm 1, \pm 2, \dots].$$

$$10. \int_{-a}^a \frac{(a^2-x^2)^{\pm 1/2}}{x-y} dx = \pm \pi \operatorname{sgn} y (y^2-a^2)^{\pm 1/2} - \left\{ \frac{\pi y}{0} \right\}$$

$$[-\infty < y < -a \text{ или } a < y < \infty].$$

$$11. = - \left\{ \frac{\pi y}{0} \right\} \quad [-a < y < a].$$

$$12. \int_0^a \frac{dx}{V_{(a-x)(y-x)}} = \ln \frac{V_{y-a} + V_a}{V_y - V_a} \quad [0 < a < y].$$

$$13. \int_0^a \frac{dx}{V_{(a-x)(x+y)}} = \frac{2}{V_{a+y}} \ln \frac{V_{a+y} + V_{a+y}}{V_y} \quad [a > 0; |\arg y| < \pi].$$

$$14. \int_0^a \frac{dx}{V_{a-x}(x-y)} = -\frac{2}{V_{y-a}} \operatorname{arctg} V_{\frac{a}{y-a}} \quad [y > a > 0].$$

$$15. \int_0^a \frac{(a^2-x^2)^{\pm 1/2}}{x-y} dx = \pm \operatorname{sgn} y (y^2-a^2)^{\pm 1/2} \arccos \left(-\frac{a}{y} \right) - \left\{ \frac{a + \pi y/2}{0} \right\} \quad [-\infty < y < -a \text{ или } a < y < \infty].$$

$$16. = (a^2-y^2)^{\pm 1/2} \ln \left| \frac{a + V_{a^2-y^2}}{y} \right| - \left\{ \frac{a + \pi y/2}{0} \right\} \quad [-a < y < a].$$

$$17. \int_0^a \frac{a^\mu - x^\mu}{a-x} dx = a^\mu [C + \psi(\mu+1)] \quad [\operatorname{Re} \mu > -1].$$

$$18. \int_0^1 V_{\frac{(1-k^2x^2)^{\pm 1}}{1-x^2}} dx = \left\{ \frac{E(k)}{K(k)} \right\}, \quad [0 \leq k < 1].$$

$$19. \int_0^{\infty} \frac{(ax+b)^{\beta-1}}{(cx+d)^{\beta+1}} dx = \frac{(ad)^{\beta} - (bc)^{\beta}}{\beta (ad-bc) (cd)^{\beta}} \quad [cd > 0; ad \neq bc; \operatorname{Re} \beta \geq 1].$$

$$20. \int_0^{\infty} \frac{dx}{\sqrt{x+z} (x-a)} = \frac{2}{\sqrt{z+a}} \ln \left| \frac{\sqrt{z} + \sqrt{z+a}}{\sqrt{z} - \sqrt{z+a}} \right| \quad [a > 0; |\arg(1+a/z)| < \pi].$$

$$21. \int_0^{\infty} \frac{dx}{(x^2+z^2)^n (x-y)} = -\frac{1}{2(n-1)z^{2n-2}(z^2+y^2)} - \\ - \frac{(2n-3)!! y\pi}{(n-1)! 2^n z^{2n-1}(z^2+y^2)} + \frac{1}{z^2+y^2} \int_0^{\infty} \frac{dx}{(x^2+z^2)^{n-1} (x-y)} \\ [y, \operatorname{Re} z > 0; n = 2, 3, 4, \dots].$$

$$22. \int_0^{\infty} \frac{dx}{(x^2+z^2)(x \pm y)^n} = \frac{(-1)^{n-1}}{(n-1)! (z^2+y^2)} \frac{\partial^{n-1}}{\partial y^{n-1}} \left(\pm \frac{y\pi}{2z} - \ln \frac{y}{z} \right) \\ [\operatorname{Re} z > 0; \left\{ \begin{array}{l} \arg y < \pi, n = 1, 2, 3, \dots \\ y > 0, n = 1, 3, 5, \dots \end{array} \right\}].$$

$$23. \int_0^{\infty} \frac{dx}{\sqrt{x^2+z^2} (x+y)} = \frac{1}{\sqrt{z^2+y^2}} \ln \frac{z(\sqrt{z^2+y^2}+z)}{y(\sqrt{z^2+y^2}-y)} \\ [\operatorname{Re} z > 0; |\arg y| < \pi].$$

$$24. \int_0^{\infty} \frac{dx}{\sqrt{x^2+z^2} (x^2+y^2)} = \frac{1}{y \sqrt{z^2-y^2}} \operatorname{arctg} \frac{\sqrt{z^2-y^2}}{y} \\ [\operatorname{Re} z, \operatorname{Re} y > 0; |\arg(z^2-y^2)| < \pi; z \neq y].$$

$$25. = y^{-2} \quad [z = y; \operatorname{Re} y > 0].$$

$$26. = \frac{1}{y \sqrt{y^2-z^2}} \ln \frac{y + \sqrt{y^2-z^2}}{z} \\ [\operatorname{Re} z, \operatorname{Re} y > 0; |\arg(y^2-z^2)| < \pi; z \neq y].$$

$$27. \int_0^{\infty} \frac{dx}{(a^2x^2+b^2)(c^2x^4+d^2)} = \frac{\pi}{2(a^4d^2+b^4c^2)} \left[\frac{a^2}{b} - \left(a^2 - \frac{b^2c^2}{d} \right) \sqrt{\frac{c}{2d}} \right] \\ [a, b, c, d > 0].$$

$$28. \int_{-\infty}^{\infty} \frac{dx}{(x^2+z^2)^{\rho} (x-y)} = -2\sqrt{\pi} \Gamma \left[\frac{\rho+1/2}{\rho} \right] \frac{y^{2\rho-1}}{y^2+z^2} {}_2F_1 \left(1, 1-\rho; \frac{3}{2}; \frac{y^2}{y^2+z^2} \right) \\ [\operatorname{Re} z, \operatorname{Re} \rho > 0; \operatorname{Im} y = 0].$$

$$29. \int_{-\infty}^{\infty} \frac{(x-r_1 e^{i\theta_1})^{\beta-1}}{(x+r_2 e^{i\theta_2})^{\beta}} dx = 0 \quad [r_1, r_2 > 0; 0 < \theta_1 < \pi, 0 < \theta_2 < \pi; \theta_1 \theta_2 < 0; \rho > \beta].$$

$$30. = e \exp [e i \pi \beta - i (\rho - \beta) (\theta_1 + \theta_2)] \frac{2\pi i}{\beta r_2^{\rho - \beta} B(\rho, -\beta)}$$

$$[r_1, r_2 > 0; \rho > \beta; r_0 = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}, r_2 \sin \theta_2 = r_3 \sin(\theta_1 - \theta_2), \\ r_1 \cos \theta_2 = r_1 + r_2 \cos(\theta_1 - \theta_2); |\theta_1| < \pi; e = \begin{cases} -1, & \text{если } 0 < \theta_1 < \pi, 0 < \theta_2 < \pi, \\ 1, & \text{если } -\pi < \theta_1 < 0, -\pi < \theta_2 < 0. \end{cases}]$$

$$31. \int_{-\infty}^{\infty} \frac{dx}{(a+ix)^\mu (b-ix)^\nu} = 2\pi (a+b)^{1-\mu-\nu} \Gamma \left[\begin{matrix} \mu+\nu-1 \\ \mu, \nu \end{matrix} \right] \quad [a, b > 0; \mu+\nu > 1]$$

$$32. = 0 \quad [ab < 0; \mu+\nu > 1]$$

$$33. \int_{-\infty}^{\infty} \frac{dx}{(a+ix)^\mu (b+ix)^\nu} = 0 \quad [\operatorname{Re} a, \operatorname{Re} b > 0; \mu+\nu > 1]$$

2.2.6. Интегралы от $\prod_{k=1}^3 (a_k \pm x)^{\beta_k}$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b (x-a)^{\alpha-1} (b-x)^{\beta-1} (cx+d)^\gamma dx = \\ = (b-a)^{\alpha+\beta-1} (ac+d)^\gamma B(\alpha, \beta) {}_2F_1 \left(\alpha, -\gamma; \alpha+\beta; \frac{c(a-b)}{ac+d} \right) \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(d+cb)/(d+ca)| < \pi]$$

$$2. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{|x-y|^{\alpha+\beta}} dx = \frac{(b-a)^{\alpha+\beta-1}}{|y-a|^\beta |y-b|^\alpha} B(\alpha, \beta) \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; y < a < b \text{ или } a < b < y]$$

$$3. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{x-y} dx = \frac{(b-a)^{\alpha+\beta-1}}{b-y} B(\alpha, \beta) {}_2F_1 \left(1, \beta; \alpha+\beta; \frac{b-a}{b-y} \right) \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; y < a < b \text{ или } a < b < y]$$

$$4. = (y-a)^{\alpha-1} (b-y)^{\beta-1} \pi \operatorname{ctg} \alpha \pi - \\ - (b-a)^{\alpha+\beta-1} B(\alpha, \beta-1) {}_2F_1 \left(1, 2-\alpha-\beta; 2-\beta; \frac{b-y}{b-a} \right) \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; a < y < b]$$

$$5. \int_a^b \left(\frac{x-a}{b-x} \right)^{\alpha-1} \frac{dx}{x-y} = \frac{\pi}{\sin \alpha \pi} \left(\left| \frac{a-y}{b-y} \right|^{\alpha-1} - 1 \right) \\ [0 < \operatorname{Re} \alpha < 2; y < a < b \text{ или } a < b < y]$$

$$6. = -\frac{\pi}{\sin \alpha \pi} \left[\cos \alpha \pi \left(\frac{y-a}{b-y} \right)^{\alpha-1} + 1 \right] \quad [0 < \operatorname{Re} \alpha < 2; a < y < b]$$

$$7. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{-\alpha}}{x-y} dx = \frac{\pi}{\sin \alpha \pi} \left| \frac{a-y}{b-y} \right|^{\alpha-1} \\ [0 < \operatorname{Re} \alpha < 1; y < a < b \text{ или } a < b < y]$$

$$8. \quad = -\pi \operatorname{ctg} \alpha \pi \frac{(y-a)^{\alpha-1}}{(b-y)^{\alpha}} \quad [0 < \operatorname{Re} \alpha < 1; a < y < b].$$

$$9. \quad \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1}}{(cx+d)^{\alpha+\beta+n}} dx =$$

$$= \frac{(b-a)^{\alpha+\beta-1}}{(bc+d)^{\alpha} (ac+d)^{\beta}} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\alpha+k, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}}$$

$$[(ac+d)(bc+d) > 0; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$10. \quad \int_a^b \frac{\sqrt{(x-a)(b-x)}}{(cx+d)^n} dx = I_n \quad [c \neq 0; ac+d, bc+d > 0],$$

$$I_1 = \frac{\pi}{2c^2} (\sqrt{ac+d} - \sqrt{bc+d})^2,$$

$$I_2 = \frac{\pi}{2c^2} \frac{(\sqrt{ac+d} - \sqrt{bc+d})^2}{\sqrt{(ac+d)(bc+d)}}, \quad I_3 = \frac{\pi(b-a)^2}{8[(ac+d)(bc+d)]^{3/2}}.$$

$$11. \quad \int_a^b x^m (x-a)^n (b-x)^p dx = \frac{m! (b-a)^{n+p+1}}{(m+n+p+1)!} \sum_{k=0}^m \frac{(n+k)! (m+p-k)!}{k! (m-k)!} a^{m-k} b^k$$

$$[m, n, p = 0, 1, 2, \dots].$$

$$12. \quad \int_a^b x^m (x-a)^{n/2} (b-x)^{n/2} dx =$$

$$= \frac{(-1)^m m!}{(m+n+1)!} \Gamma\left(\frac{n}{2} + 1\right) (b-a)^{n+1} \sum_{k=0}^m \binom{-n/2-1}{k} \binom{-n/2-1}{m-k} a^k b^{m-k}$$

$$[m = 0, 1, 2, \dots; n = -1, 0, 1, 2, \dots].$$

$$13. \quad \int_a^b \frac{dx}{x^2 \sqrt{(x-a)(b-x)}} = \frac{\pi}{\sqrt{ab}} \left\{ 2^{-1} (a^{-1} + b^{-1}) \right\} \quad \left[z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

$$14. \quad \int_a^b \frac{x^m (x-a)^{\alpha-1}}{(cx+d)^{m+\alpha+1}} dx =$$

$$= \frac{(b-a)^{\alpha}}{(ac+d)(bc+d)^{\alpha}} \sum_{k=0}^m \binom{m}{k} b^k a^{m-k} \frac{\Gamma(\alpha+k, m-k+1)}{(bc+d)^k (ac+d)^{m-k}}$$

$$[(ac+d)(bc+d) > 0; \operatorname{Re} \alpha > 0].$$

$$15. \quad \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (x+z)^{-\rho} dx = a^{\alpha+\beta-1} z^{-\rho} B(\alpha, \beta) {}_2F_1\left(\alpha, \rho; \alpha+\beta; -\frac{a}{z}\right)$$

$$[|\arg z| < \pi; a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$16. \int_0^a \frac{x^{\alpha-1}}{[(a-x)(y-x)]^{\alpha-1/2}} dx = \frac{\Gamma(\alpha-1) \Gamma(3/2-\alpha)}{2 \sqrt{\pi}} \{(\sqrt{y}-\sqrt{a})^{2-2\alpha} - (\sqrt{y}+\sqrt{a})^{2-2\alpha}\} \\ [\nu > a; 0 < \operatorname{Re} \alpha < 3/2].$$

$$17. \int_0^a \frac{dx}{\sqrt{x(a-x)}(x \pm c)} = \pm \frac{\pi}{\sqrt{c(c \pm a)}} \quad \left[\begin{matrix} a, c > 0 \\ c > a > 0 \end{matrix} \right].$$

$$18. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} \frac{dx}{x+a} = I_{\alpha} \quad [a, \operatorname{Re} \alpha > 0]. \\ I_{1/4} = a^{-5/4} [\pi/2 + \sqrt{2} K(1/\sqrt{2})], \quad I_{3/4} = a^{-3/4} \pi/2, \\ I_{5/4} = -a^{-1/4} [\pi/2 - \sqrt{2} K(1/\sqrt{2})].$$

$$19. \int_0^a \sqrt{\frac{x}{a+x}} \frac{dx}{\sqrt{1+k^2x}} = \frac{1}{k^2} \left[\sqrt{2(1+k^2a)} - 2E\left(\frac{\pi}{4}, \sqrt{1-k^2a}\right) \right] \\ [a > 0; 0 < k \leq 1].$$

$$20. \int_0^a \left(\frac{a-x}{a+x} \right)^{\pm 1/2} \frac{dx}{x-y} = \mp \frac{\pi}{2} \pm \left| \frac{y-a}{y+a} \right|^{\pm 1/2} \arccos \left(-\frac{a}{y} \right) \\ [-\infty < y < -a \text{ или } a < y < \infty].$$

$$21. = \mp \frac{\pi}{2} + \left(\frac{a-y}{a+y} \right)^{\pm 1/2} \ln \left| \frac{a + \sqrt{a^2 - y^2}}{y} \right| \quad [-a < y < a].$$

$$22. \int_a^b \frac{x^m (b-x)^{\beta-1}}{(cx+d)^{m+\beta+1}} dx = \frac{(b-a)^{\beta}}{(bc+d)(ac+d)^{\beta}} \sum_{k=0}^m \binom{m}{k} b^k a^{m-k} \frac{B(k+1, \beta+m-k)}{(bc+d)^k (ac+d)^{m-k}} \\ [(ac+d)(bc+d), \operatorname{Re} \beta > 0].$$

$$23. \int_0^{\infty} \frac{x^{\alpha-1}}{[(x+z)(x+y)]^{\alpha-3/2}} dx = \frac{[y+z+2(\alpha-2)\sqrt{yz}]\sqrt{\pi}}{2(\sqrt{y}+\sqrt{z})^{2\alpha-4}} \Gamma\left[\frac{\alpha-3}{2}\right] \\ [|\arg z|, |\arg y| < \pi; \operatorname{Re} \alpha > 3].$$

$$24. \int_0^{\infty} x^{\alpha-1} (x+y)^{-\rho} (x+z)^{-\lambda} dx = z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho+\lambda-\alpha) {}_2F_1\left(\alpha, \lambda; \rho+\lambda; 1-\frac{y}{z}\right) \\ [|\arg y|, |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re}(\rho+\lambda)].$$

$$25. \int_0^{\infty} \frac{x^{\alpha-1} (x+y)^{\alpha-1}}{(2x+y)^{2\alpha}} dx = \frac{B(\alpha, \alpha)}{2y} \quad [\operatorname{Re} \alpha > 0; |\arg y| < \pi].$$

$$26. \int_0^{\infty} x^{\alpha-1} (x+z)^{-\lambda} \frac{dx}{x-y} = z^{-\lambda} (-y)^{\alpha-1} B(\alpha, 1-\alpha+\lambda) {}_2F_1\left(\alpha, \lambda; 1+\lambda; 1+\frac{y}{z}\right) \\ [|\arg(-y)| < \pi, 0 < \operatorname{Re} \alpha < 1 + \operatorname{Re} \lambda].$$

$$27. = -\pi \operatorname{ctg}(\alpha - \lambda) \pi y^{\alpha-1} (z+y)^{-\lambda} - \\ - \frac{z^{\alpha-\lambda}}{z+y} B(\alpha, \lambda - \alpha) {}_2F_1\left(1, 1-\lambda; 1+\alpha-\lambda; \frac{z}{z+y}\right) \\ [y > 0; 0 < \operatorname{Re} \alpha < 1 + \operatorname{Re} \lambda].$$

$$28. \int_0^{\infty} \frac{x^{\alpha-1} dx}{[(x+z)(x+y)]^{2-1/2}} = \frac{\sqrt{\pi}}{(\sqrt{z} + \sqrt{y})^{2\alpha-1}} \Gamma\left[\frac{\alpha-1}{2}\right] \\ [|\arg z|, |\arg y| < \pi, \operatorname{Re} \alpha > 1].$$

2.2.7. Интегралы от $\prod_{k=1}^3 (a_k^{\mu_k} \pm x^{\mu_k})^{\rho_k}$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b \frac{x^{2m}}{\sqrt{1-x^2} \sqrt{b^2-x^2}} dx = \theta_m(b, a) \quad [0 < a < b < 1; m = 0, \pm 1, \pm 2, \dots].$$

$$\theta_0(b, a) = F(\varphi, b), \theta_1(b, a) = F(\varphi, b) - E(\varphi, b) + a \sqrt{\frac{b^2-a^2}{1-a^2}} \\ \left[\varphi = \arcsin \frac{\sqrt{b^2-a^2}}{b\sqrt{1-a^2}} \right].$$

$$(2m+1) \theta_{m+1}(b, a) - 2m(b^2+1) \theta_m(b, a) - (2m-1)b^2 \theta_{m-1}(b, a) = \\ = -a^{2m+1} \sqrt{(1-a^2)(b^2-a^2)} \quad [m = 1, 2, 3, \dots].$$

$$\theta_{-1}(b, a) = b^{-2} [F(\varphi, b) - E(\varphi, b)] + b^{-2} a^{-1} \sqrt{(b^2-a^2)/(1-a^2)}, \\ (2m+1)b^2 \theta_{m-1}(b, a) - 2m(b^2+1) \theta_m(b, a) + (2m-1) \theta_{1-m}(b, a) = \\ = -a^{2m-1} \sqrt{(1-a^2)(b^2-a^2)} \quad [m = 1, 2, 3, \dots].$$

$$2. \int_{-a}^a \frac{(a+x)^{\alpha-1} (a-x)^{\beta-1}}{(a^2+x^2)^{(\alpha+\beta)/2}} dx = \frac{1}{2} (2a)^{(\alpha+\beta)/2-1} B\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) \\ [a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$3. \int_{-a}^a \left(\frac{a+x}{a-x}\right)^{\beta} \frac{dx}{x^2+z^2} = \frac{\pi}{2z \cos(\beta\pi/2)} \\ [|\operatorname{Re} \beta| < 1; a > 0].$$

$$4. \int_{-a}^a \frac{x(a^2-x^2)^m}{(cx+d)^{m+1/2}} dx = -\frac{m!2^m (\sqrt{d+ac} - \sqrt{d-ac})^{2m+3}}{(2m+3)(2m-1)!} \\ [d > |ac| > 0].$$

$$5. \int_0^a \frac{x^{\alpha-1} (a-x)^{\alpha-1}}{1-zx^2} dx = (n-1)! a^{\alpha+n-1} \sum_{k=0}^{\infty} \frac{(za^2)^k}{(\alpha+kr)_n} \\ [r, \operatorname{Re} \alpha > 0; |za^2| < 1; n = 1, 2, 3, \dots].$$

$$6. \int_0^a \frac{a^n - x^n}{a-x} \frac{dx}{(a+x)^{n+1}} = \frac{2 \cdot n-1}{a} \sum_{k=1}^n \frac{2^k}{k} \quad [n=1, 2, 3, \dots].$$

$$7. \int_0^{2\pi} \frac{x^{\alpha-1} dx}{(x^{2\mu} + 1)(x^{2\mu} + 1)} = \\ = \frac{\pi}{6\mu \sin(\alpha\pi/\mu)} \left[1 + 3 \sqrt{2} \sin\left(\frac{\pi\alpha}{2\mu} + \frac{\pi}{4}\right) - 4 \sin\left(\frac{2\pi\alpha}{3\mu} - \frac{\pi}{6}\right) \right] \\ [0 < \operatorname{Re} \alpha < 5\mu].$$

$$8. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(x+y)(x^{2/3} + y^{2/3})} = -\frac{3\pi}{8} y^{\alpha-5/3} \frac{1-4\cos^2 \alpha\pi}{\sin \alpha\pi} \\ [|\arg y| < \pi; 0 < \operatorname{Re} \alpha < 5/3].$$

$$9. \int_0^{\infty} \left(\frac{x^p}{1+x^{2p}} \right)^q \frac{dx}{1-x^2} = 0 \quad [p|q > -1].$$

$$10. \int_0^{\infty} \frac{x^m dx}{(x^2 + y^2)(x \pm y)^n} = \\ = \sum_{k=0}^m \binom{m}{k} y^{m-k} \frac{(\mp 1)^{m+n-1}}{(n-k-1)!} \frac{\partial^{n-k-1}}{\partial y^{n-k-1}} \left[\frac{1}{z^2 + y^2} \left(\pm \frac{y^n}{2z} - \ln \frac{y}{z} \right) \right] \\ [\operatorname{Re} z > 0; \{ |\arg y| < \pi; n=0, 1, 2, \dots \} \cup \{ y > 0; n=1, 3, 5, \dots \}].$$

$$11. \int_0^{\infty} x^{\alpha-1} \frac{x^\mu - y^\mu}{x-y} dx = y^{\alpha+\mu-1} \frac{\pi \sin \mu\pi}{\sin \alpha\pi \sin(\alpha+\mu)\pi} \\ [\mu > 0; 0 < \operatorname{Re} \alpha < 1; 0 < \operatorname{Re}(\alpha+\mu) < 1].$$

$$12. \int_0^{\infty} \left(\frac{x^\mu - x^{-\mu}}{x-1} \right)^2 dx = 2(1 - 2\pi\mu \operatorname{ctg} 2\mu\pi) \quad [|\operatorname{Re} \mu| < 1/2].$$

$$13. \int_0^{\infty} x^{\alpha-1} \left(\frac{x^\lambda - x^{-\mu}}{x^\lambda - x^{-\lambda}} \right) dx = \frac{\pi \sin(\mu\pi/\lambda)}{\lambda [\cos(\mu\pi/\lambda) + \cos(\alpha\pi/\lambda)]} \\ [\mu - \lambda < \operatorname{Re} \alpha < \mu + \operatorname{Re} \alpha < \lambda].$$

$$14. \int_0^{\infty} \left(\frac{x^\lambda - x^{-\mu}}{x^\lambda - x^{-\lambda}} \right)^n \frac{dx}{x} = I_n \quad [0 < \mu < \lambda].$$

$$I_1 = \frac{\pi}{\lambda} \operatorname{tg} \frac{\mu\pi}{2\lambda}, \quad I_2 = \frac{1}{\lambda} - \frac{\mu\pi}{\lambda^2} \operatorname{ctg} \frac{\mu\pi}{\lambda},$$

$$I_3 = \frac{\pi}{8\lambda^3} \left[3(\lambda^2 - \mu^2) \operatorname{tg} \frac{\mu\pi}{2\lambda} - (\lambda^2 - 9\mu^2) \operatorname{tg} \frac{3\mu\pi}{2\lambda} \right].$$

$$15. \int_0^{\infty} \frac{x^{\lambda-1}}{(x-1)^2} (x^{\lambda}-1) dx = \pi [\lambda \operatorname{ctg} \lambda \pi + \mu \operatorname{ctg} \mu \pi - (\mu + \lambda) \operatorname{ctg} (\mu + \lambda) \pi] - 1$$

$$[\lambda \neq -\mu; |\operatorname{Re}(\mu + \lambda)|, |\operatorname{Re} \lambda|, |\operatorname{Re} \mu| < 1].$$

$$16. = 2(\pi \mu \operatorname{ctg} \mu \pi - 1)$$

$$[\lambda = -\mu; |\operatorname{Re} \mu| < 1].$$

2.2.8. Интегралы от $\prod_{k=1}^n (a_k^{\mu_k} \pm x^{\mu_k})^{\rho_k}$, $n \geq 4$.

Условия: a, b, c, d — действительные числа.

$$1. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{\beta-1} (x+z)^m}{(cx+d)^{\alpha+\beta+n}} dx =$$

$$= \frac{(b-a)^{\alpha+\beta-1}}{(bc+d)^{\alpha} (ac+d)^{\beta}} \sum_{k=0}^n \frac{B(\alpha+k, \beta+n-k)}{(bc+d)^k (ac+d)^{n-k}} \sum_{l=0}^k \binom{m}{l} \binom{n-m}{k-l} (b+z)^l (a+z)^{n-l}$$

$$[(ac+d)(bc+d) > 0; m \leq n; \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_a^b \frac{(x-a)^{\alpha-1} (b-x)^{-\alpha}}{(c_1 x + d_1)(c_2 x + d_2)} dx =$$

$$= \frac{\pi}{(c_1 d_2 - c_2 d_1) \sin \alpha \pi} \left[\frac{c_1 (ac_1 + d_1)^{\alpha-1}}{(bc_1 + d_1)^{\alpha}} - \frac{c_2 (ac_2 + d_2)^{\alpha-1}}{(bc_2 + d_2)^{\alpha}} \right]$$

$$[c_1 d_2 \neq c_2 d_1; (ac_j + d_j)(bc_j + d_j) > 0, j=1, 2; 0 < \operatorname{Re} \alpha < 1].$$

$$3. \int_a^b \frac{|(x-a)(b-x)|^{\alpha-1}}{(x-y)|x-c|^{\alpha-r}} dx = \frac{\pi \operatorname{sgn}(y-c) \operatorname{ctg} \alpha \pi}{|y-c|^{\alpha-r} [(y-a)(b-y)]^{1-\alpha}}$$

$$[r=0 \text{ или } 1; a < y < b; a < c < b; 0 < 2 \operatorname{Re} \alpha < 1+r].$$

$$4. \int_a^b \frac{|(x-a)(b-x)|^{\alpha-1} \operatorname{sgn}(x-c)}{(x-y)|x-c|^{\alpha-r}} dx = \frac{(-1)^r \pi \operatorname{ctg} \alpha \pi}{|y-c|^{\alpha-r} [(y-a)(b-y)]^{1-\alpha}}$$

$$[r=0 \text{ или } 1; a < y < b; a < c < b; 0 < \operatorname{Re} \alpha < (1+r)/2].$$

$$5. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (1-ux)^{-\rho} (1-vx)^{-\lambda} dx =$$

$$= a^{\alpha+\beta-1} B(\alpha, \beta) F_1(\alpha, \rho, \lambda, \alpha+\beta; ua, va)$$

$$[|\arg u| < \pi \text{ или } a|u| < 1; |\arg v| < \pi \text{ или } a|v| < 1; a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$6. \int_0^1 \frac{a_0 x^{\alpha_0-1} - a_1 x^{\alpha_1-1} - \dots - a_n x^{\alpha_n-1}}{1-x^{\lambda}} dx =$$

$$= \lambda^{-1} [a_1 \psi(\alpha_1/\lambda) + \dots + a_n \psi(\alpha_n/\lambda) - a_0 \psi(\alpha_0/\lambda)]$$

$$[a_0 = a_1 + a_2 + \dots + a_n; \lambda, \operatorname{Re} \alpha_i > 0; i=0, 1, 2, \dots, n].$$

$$7. \int_0^1 \frac{x^{\alpha-1} + x^{\alpha-(n-1)/2} + \dots + x^{\alpha-1/n} - nx^{2n-1}}{1-x} dx = n \ln n$$

$$[\operatorname{Re} \alpha > 0; n=1, 2, 3, \dots].$$

$$8. \int_0^1 \frac{x^m}{1+x+x^2+\dots+x^{m+2}} dx = \frac{\pi}{2m+3} \operatorname{tg} \frac{\pi}{2(m+3)}.$$

$$9. \int_0^1 \frac{x^{2m} + x^{2m-2} + \dots + x^2 + 1}{x^{2m+2} + x^{2m} + \dots + x^2 + 1} dx = \frac{\pi}{2(m+2)} \operatorname{ctg} \frac{\pi}{2(m+2)}.$$

$$10. \int_0^1 \frac{x^{n-1} dx}{(x+1)(2x+1)\dots(nx+1)} = n! \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \ln(k+1)$$

[$n = 1, 2, 3, \dots$].

$$11. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} (x-u)^{-\rho} (x-v)^{-\lambda} dx = a^{\alpha+\beta-\lambda-\rho-1} \times \\ \times B(1+\lambda+\rho-\alpha-\beta, \beta) F_1(1+\lambda+\rho-\alpha-\beta, \rho, \lambda, 1+\lambda+\rho-\alpha; u/a, v/a) \\ [a > 0; |\arg u| < \pi \text{ или } |u| < a; |\arg v| < \pi \text{ или } |v| < a, 0 < \operatorname{Re} \beta < \operatorname{Re}(\lambda+\rho-\alpha)+1].$$

$$12. \int_0^\infty \frac{(Ax+B) dx}{(c_1x+d_1)(c_2x+d_2)(c_3x+d_3)} = \frac{Bc_1-Ad_1}{pq} \ln \frac{c_1d_2}{c_2d_1} + \frac{Bc_3-Ad_3}{qr} \ln \frac{c_3d_2}{c_2d_3} \\ [p = c_1d_2 - c_2d_1, \quad q = c_1d_3 - c_3d_1, \quad r = c_2d_3 - c_3d_2; \quad c_1/d_1 > c_2/d_2 > c_3/d_3 > 0].$$

$$13. \int_0^\infty \frac{y^\mu - y^\mu}{x-y} \frac{y^\nu - 1}{x-1} dx = \frac{\pi}{1-y} [(y^{1+\nu} - 1) \operatorname{ctg}(\mu+\nu)\pi - \\ - y^\mu (y^\nu - 1) \operatorname{ctg} \nu\pi - (y^\mu - 1) \operatorname{ctg} \mu\pi + y^\mu \frac{\ln y}{\pi}] \\ [\nu > 0; \mu \neq 1; \nu \neq -\mu; |\operatorname{Re} \mu|, |\operatorname{Re} \nu|, |\operatorname{Re}(\mu+\nu)| < 1].$$

$$14. \int_0^\infty \frac{y^\mu - y^\mu}{x-y} \frac{y^\nu - 1}{x-1} dx = \frac{\pi}{1-y} \left[\frac{y^\mu + 1}{\pi} \ln y - 2(y^\mu - 1) \operatorname{ctg} \mu\pi \right] \\ [\nu > 0; \mu \neq 1; |\operatorname{Re} \mu| < 1].$$

$$15. \int_0^\infty \frac{A_0 x^n + A_1 x^{n-1} + \dots + A_n}{(cx+d)^{n+2}} dx = \\ = \frac{1}{(n+1)cd} \left[\frac{A_0}{c^n} \binom{n}{0}^{-1} + \frac{A_1}{c^{n-1}d} \binom{n}{1}^{-1} + \dots + \frac{A_n}{d^n} \binom{n}{n}^{-1} \right] \quad [cd > 0].$$

$$16. \int_0^\infty \frac{A_0 x^{2n} + A_1 x^{2n-2} + \dots + A_{2n}}{(x^2+z^2)^{n+1}} dx = \\ = \frac{\pi}{n! 2^{n+1} z} \sum_{k=0}^n \frac{(2k-1)!! (2n-2k-1)!!}{z^{2k}} A_{2k} + \frac{1}{n! 2z^2} \sum_{k=0}^{n-1} k! (n-k-1)! \frac{A_{2k+1}}{z^{2k}} \\ [\operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} \frac{(A_1 x^2 + B_1)(A_2 x^2 + B_2) \dots (A_{n-1} x^2 + B_{n-1})}{(c_1 x^2 + d_1)(c_2 x^2 + d_2) \dots (c_n x^2 + d_n)} dx =$$

$$= \frac{\pi}{2} \left\{ \frac{(A_1 d_1 - B_1 c_1)(A_2 d_1 - B_2 c_1) \dots (A_{n-1} d_1 - B_{n-1} c_1)}{\sqrt{c_1 d_1} (c_2 d_1 - d_2 c_1)(c_3 d_1 - d_3 c_1) \dots (c_n d_1 - d_n c_1)} + \right.$$

$$+ \frac{(A_1 d_2 - B_1 c_2)(A_2 d_2 - B_2 c_2) \dots (A_{n-1} d_2 - B_{n-1} c_2)}{\sqrt{c_2 d_2} (c_1 d_2 - d_1 c_2)(c_3 d_2 - d_3 c_2) \dots (c_n d_2 - d_n c_2)} + \dots$$

$$\left. + \frac{(A_1 d_n - B_1 c_n)(A_2 d_n - B_2 c_n) \dots (A_{n-1} d_n - B_{n-1} c_n)}{\sqrt{c_n d_n} (c_1 d_n - d_1 c_n)(c_2 d_n - d_2 c_n) \dots (c_{n-1} d_n - d_{n-1} c_n)} \right\}$$

[$c_i, d_i > 0; c_i d_j \neq c_j d_i; i, j = 1, 2, 3, \dots, n; n = 2, 3, 4, \dots$].

$$18. \int_{-\infty}^{\infty} \frac{P_n(x)}{Q_n(x) Q_n(-x)} dx = I_n$$

[$P_n(x) = A_0 x^{2n-2} + A_1 x^{2n-4} + \dots + A_{n-1}$, $Q_n(x) = B_0 x^n + B_1 x^{n-1} + \dots + B_n$; все корни $Q_n(x)$ лежат в верхней полуплоскости].

$$I_n = \frac{\pi i}{B_0} \frac{M_n}{\Delta_n}, \quad M_n = \begin{vmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} \\ B_0 & B_1 & B_2 & \dots & 0 \\ 0 & B_1 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & B_n \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} B_1 & B_2 & B_3 & \dots & 0 \\ B_0 & B_2 & B_4 & \dots & 0 \\ 0 & B_1 & B_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & B_n \end{vmatrix}.$$

$$M_1 = A_0, \quad \Delta_1 = B_1; \quad M_2 = -A_0 + A_1 B_0 B_2^{-1}, \quad \Delta_2 = B_1;$$

$$M_3 = -A_0 B_2 + A_1 B_0 - B_0 B_1 A_2 B_2^{-1}, \quad \Delta_3 = B_0 B_2 - B_1 B_2;$$

$$M_4 = A_0 (-B_1 B_4 + B_2 B_3) - B_0 B_2 A_1 + B_0 B_1 A_2 + B_0 B_4^{-1} A_3 (B_0 B_2 - B_1 B_2),$$

$$\Delta_4 = B_0 B_2^2 + B_1^2 B_4 - B_1 B_2 B_3.$$

$$19. \int_0^{\infty} \frac{x^m}{1+x+x^2+\dots+x^n} dx = \frac{\pi}{n+1} \left[\operatorname{ctg} \frac{(m+1)\pi}{n+1} - \operatorname{ctg} \frac{(m+2)\pi}{n+1} \right]$$

[$n > m+1$].

2.2.9. Интегралы от $x^\alpha (ax^2 + bx + c)^\beta A(x)$.

Условия: a, b, c — действительные числа.

$$1. \int_0^1 \frac{x^{\alpha-1}}{x^2 + 2x \cos \gamma + 1} dx = \sum_{k=0}^{\infty} (-1)^k \frac{\sin \gamma k \operatorname{ctg} \gamma + \cos \gamma k}{\alpha + k} \quad [\operatorname{Re} \alpha > 0; |\gamma| < \pi]$$

$$2. \int_0^1 \frac{x^{\alpha-1} (1-x)^{\alpha-4+2\varepsilon}}{(ax^2 + 2bx + c)^{\alpha-3/2+\varepsilon}} dx =$$

$$= \frac{[b+c+(\alpha-2+\varepsilon)\sqrt{c}\sqrt{a+2b+c}] 2^{\frac{1}{2}} \alpha^{-\varepsilon} \sqrt{\pi} \Gamma(\alpha-3+\varepsilon)}{(b+c+\sqrt{c}\sqrt{a+2b+c})^{\alpha-3/2+\varepsilon} \Gamma(\alpha-3/2+\varepsilon)} \left\{ \frac{(a+2b+c)^{-3/2}}{c^{-3/2}} \right\}$$

[$c, a+2b+c > 0; a < (\sqrt{c} + \sqrt{a+2b+c})^2; \operatorname{Re} \alpha > 3-\varepsilon, \varepsilon = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix}$].

$$3. \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta}}{(ax^2+2bx+c)^{\alpha+1/2-\sigma}} dx =$$

$$= \frac{2^{-\alpha+\sigma} \sqrt{\pi} \Gamma(\alpha-\sigma)}{(b+c+\sqrt{c} \sqrt{a+2b+c})^{\alpha-\sigma} \Gamma(\alpha+1/2-\sigma)} \left\{ \frac{c^{-1/2}}{(a+2b+c)^{1/2}} \right\}$$

$$\left[c, a+2b+c > 0; a < (\sqrt{c} + \sqrt{a+2b+c})^2; \operatorname{Re} \alpha > \sigma; \beta = \left\{ \frac{\alpha}{a-2} \right\}, \sigma = \left\{ \frac{\alpha}{1} \right\} \right].$$

$$4. \int_0^1 x^{\alpha-1}(1-x)^{-\alpha} \left(x^2 + 2xy \left\{ \frac{\cos \gamma}{\operatorname{ch} \gamma} \right\} + y^2 \right)^{-1} dx =$$

$$= \frac{\pi y^{\alpha-2}}{\sin \alpha \pi} \left\{ \frac{\sin \gamma}{\operatorname{sh} \gamma} \right\}^{-1} \left(y^2 + 2y \left\{ \frac{\cos \gamma}{\operatorname{ch} \gamma} \right\} + 1 \right)^{-\alpha/2} \times$$

$$\times \left[\frac{\sin [\gamma - \alpha \operatorname{arctg} \{ \sin \gamma (y + \cos \gamma)^{-1} \}]}{\operatorname{sh} [\gamma - \alpha \operatorname{Arth} \{ \operatorname{sh} \gamma (y + \operatorname{ch} \gamma)^{-1} \}]} \right] \left[\left\{ \begin{array}{l} |\arg y| < \pi; 0 < \operatorname{Re} \alpha < 1; |\gamma| < \pi \end{array} \right\} \right].$$

$$5. \int_0^1 \frac{x^{\alpha-1}(1+x \cos \gamma) dx}{x^2+2x \cos \gamma+1} = \sum_{k=0}^{\infty} (-1)^k \frac{\cos k\gamma}{\alpha+k} \quad [|\gamma| < \pi; \operatorname{Re} \alpha > 0].$$

$$6. \int_0^1 \frac{x^{\alpha}+x^{-\alpha}}{x^2+2x \cos \gamma+1} dx = \frac{\pi \sin(\alpha \gamma)}{\sin \gamma \sin(\alpha \pi)} \quad [0 < |\gamma| < \pi; |\operatorname{Re} \alpha| < 1].$$

$$7. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(ax^2+2bx+c)^{\rho}} = a^{-\alpha/2} c^{\alpha/2-\rho} B(\alpha, 2\rho-\alpha) {}_2F_1\left(\frac{\alpha}{2}, \rho-\frac{\alpha}{2}; \rho+\frac{1}{2}; 1-\frac{b^2}{ac}\right)$$

$$[a > 0; b^2 < ac; 0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho].$$

$$8. \int_0^{\infty} \frac{x^{\alpha-1}}{[(x+a)^2-b^2]^{\rho}} dx =$$

$$= B(\alpha, 2\rho-\alpha) (a^2-b^2)^{\alpha/2-\rho} {}_2F_1\left(\alpha, 2\rho-\alpha; \rho+\frac{1}{2}; \frac{1}{2} \left(1-\frac{a}{\sqrt{a^2-b^2}}\right)\right)$$

$$[b < a; 0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho].$$

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(ax^2+2bx+c)^{\alpha+1/2-n}} = I_n \quad [a, c > 0; \operatorname{Re} \alpha > 2n-1; b > -\sqrt{ac}].$$

$$I_0 = \frac{\sqrt{\pi}}{\sqrt{c} (2\sqrt{ac}+2b)^{\alpha}} \Gamma\left[\frac{\alpha}{\alpha+1/2}\right], \quad I_1 = \frac{\sqrt{\pi}}{\sqrt{a} (2\sqrt{ac}+2b)^{\alpha-1}} \Gamma\left[\frac{\alpha-1}{\alpha-1/2}\right],$$

$$I_2 = \frac{[(\alpha-2)\sqrt{ac}+b]\sqrt{\pi}}{a^{3/2} (2\sqrt{ac}+2b)^{\alpha-2}} \Gamma\left[\frac{\alpha-3}{\alpha-3/2}\right],$$

$$I_3 = \frac{(\alpha-2)(\alpha-4)ac+3(\alpha-3)b\sqrt{ac}+3b^2}{a^{5/2} (2\sqrt{ac}+2b)^{\alpha-3}} \sqrt{\pi} \Gamma\left[\frac{\alpha-5}{\alpha-5/2}\right].$$

$$10. \int_0^{\infty} \frac{x^{m+2k+1/2}}{(ax^2+2bx+c)^n} dx =$$

$$= \frac{(-1)^{m+n-1} \pi (2m-1)!!}{V a 2^{2m+1/2} (n-1)!} \frac{\partial^{n-m-1}}{\partial a^k \partial c^{n-m-k-1}} [(b+V ac)^{-m-1/2}]$$

$[a, c > 0; 0 \leq m+k \leq n-1; n=1, 2, 3, \dots \text{ при } b^2 < ac \text{ и } n=1, 3, 5, \dots \text{ при } b^2 > ac].$

$$11. \int_0^{\infty} \frac{x^{m+2k}}{(ax^2+2bx+c)^n} dx =$$

$$= \frac{(-1)^{n-1}}{(n-1)!} 2^{-m} \frac{\partial^{n-1}}{\partial a^k \partial b^m \partial c^{n-m-k-1}} \left(\frac{1}{V ac-b^2} \operatorname{arccctg} \frac{b}{V ac-b^2} \right) \quad [b^2 < ac].$$

$$12. = \frac{(-1)^{n-1}}{(n-1)!} 2^{-m-1} \frac{\partial^{n-1}}{\partial a^k \partial b^m \partial c^{n-m-k-1}} \left(\frac{1}{V b^2-ac} \ln \frac{b+V b^2-ac}{b-V b^2-ac} \right)$$

$[ac < b^2].$

$$13. = \left(\frac{b}{a} \right)^{m+2k-n+1} b^{-n} \frac{(m+2k)! (2n-2k-m-2)!}{(2n-1)!}$$

$[ac = b^2],$
 $[a, b, c > 0; 0 \leq m+k \leq n-1; n=1, 2, 3, \dots].$

$$14. \int_0^{\infty} \frac{x dx}{(ax^2+2bx+c)^n} =$$

$$= \frac{(-1)^{n-1}}{2(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left(\frac{b}{(ac-b^2)^{3/2}} \operatorname{arccctg} \frac{b}{V ac-b^2} - \frac{1}{ac-b^2} \right) \quad [ac > b^2].$$

$$15. = \frac{(-1)^{n-1}}{2(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left(\frac{b}{(b^2-ac)^{3/2}} \ln \frac{V ac}{b+V b^2-ac} - \frac{1}{ac-b^2} \right) \quad [ac < b^2].$$

$$16. = \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}}$$

$[ac = b^2],$
 $[a, b, c > 0; n=2, 3, 4, \dots].$

$$17. \int_0^{\infty} \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{V ac-b^2} \operatorname{arccctg} \frac{b}{V ac-b^2} \right)$$

$[ac > b^2].$

$$18. = \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{V b^2-ac} \ln \frac{V ac}{b+V b^2-ac} \right) \quad [ac < b^2].$$

$$19. = \frac{a^{n-1}}{(2n-1)b^{2n-1}} \quad [ac = b^2], [a, b, c > 0; n=1, 2, 3, \dots].$$

$$20. \int_0^{\infty} \frac{x^{m+2k} dx}{(ax^2+2bx+c)^{n+1/2}} = \frac{(-1)^{n+m-1} 2^{n-k-1} m!}{(2n-1)!!} \frac{\partial^{n-m-1}}{\partial a^k \partial c^{n-m-k-1}} \left(\frac{1}{V c(V ac+b)^{m+1}} \right)$$

$[0 \leq m+k \leq n-1].$

$$21. = \frac{(-2)^{n-1}}{(2n-1)!!} \frac{\partial^{n-1}}{\partial a^{n-1}} \left(\frac{1}{V a(V ac+b)} \right)$$

$[m+2k=2n-1], [a, b, c > 0; n=1, 2, 3, \dots].$

$$22. \int_0^{\infty} \frac{dx}{(ax^2 + 2bx + c)^{n+1/2}} = \frac{(-2)^{n-1}}{(2n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left(\frac{1}{\sqrt{c} \sqrt{ac+b}} \right) \\ [a, b, c > 0; n = 1, 2, 3, \dots]$$

$$23. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2 + 2xy \cos \gamma + y^2)^{\rho}} dx = \\ = \frac{1}{2y^{2\rho-\alpha}} \left\{ B\left(\frac{\alpha}{2}, \rho - \frac{\alpha}{2}\right) {}_2F_1\left(\frac{\alpha}{2}, \rho - \frac{\alpha}{2}; \frac{1}{2}; \cos^2 \gamma\right) - \right. \\ \left. - (\alpha-1) \cos \gamma B\left(\frac{\alpha-1}{2}, \rho - \frac{\alpha-1}{2}\right) {}_2F_1\left(\frac{\alpha-1}{2}, \rho - \frac{\alpha-1}{2}; \frac{3}{2}; \cos^2 \gamma\right) \right\} \\ [|\arg y| < \pi; 0 < |\gamma| < \pi; 0 < \operatorname{Re} \alpha < 2 \operatorname{Re} \rho]$$

$$24. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2 + 2xy \cos \gamma + y^2} dx = \frac{\pi \sin [(1-\alpha) \gamma]}{y^2 {}^{\alpha} \sin \gamma \sin \alpha \pi} \\ [|\arg y| < \pi; 0 < |\gamma| < \pi; 0 < \operatorname{Re} \alpha < 2; \alpha \neq 1]$$

$$25. \int_0^{\infty} \frac{dx}{x^2 + 2xy \cos \gamma + y^2} = \frac{\gamma}{y \sin \gamma} \quad [0 < |\gamma| < \pi]$$

$$26. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2 + 2xy \cos \gamma + y^2)^2} dx = \frac{\pi \{ \sin [(1-\alpha) \gamma] - (1-\alpha) \sin \gamma \cos [(2-\alpha) \gamma] \}}{2y^4 {}^{\alpha} \sin^3 \gamma \sin \alpha \pi} \\ [|\arg y| < \pi; 0 < |\gamma| < \pi; 0 < \operatorname{Re} \alpha < 4; \alpha \neq 1, 2, 3]$$

$$27. \int_0^{\infty} \frac{dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{2\gamma - \sin 2\gamma}{4y^3 \sin^3 \gamma} \quad [0 < |\gamma| < \pi]$$

$$28. \int_0^{\infty} \frac{x dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{\sin \gamma - \gamma \cos \gamma}{2y^2 \sin^3 \gamma} \quad [0 < |\gamma| < \pi]$$

$$29. \int_0^{\infty} \frac{x^2 dx}{(x^2 + 2xy \cos \gamma + y^2)^2} = \frac{2\gamma - \sin 2\gamma}{4y \sin^3 \gamma} \quad [0 < |\gamma| < \pi]$$

$$30. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(x^2 + 2xy \cos \gamma + y^2)^3} = \\ = \frac{\pi \{ 3 \sin [(1-\alpha) \gamma] - 3(1-\alpha) \sin \gamma \cos [(2-\alpha) \gamma] - (1-\alpha)(2-\alpha) \sin^2 \gamma \sin [(3-\alpha) \gamma] \}}{8y^6 {}^{\alpha} \sin^5 \gamma \sin \alpha \pi} \\ [|\arg y| < \pi; 0 < |\gamma| < \pi; 0 < \operatorname{Re} \alpha < 6; \alpha \neq 1, 2, 3, 4, 5]$$

$$31. \int_0^{\infty} \frac{x^n dx}{(x^2 + 2xy \cos \gamma + y^2)^3} = I_n \quad [0 < |\gamma| < \pi]$$

$$I_0 = \frac{12\gamma - 8 \sin 2\gamma + \sin 4\gamma}{32y^6 \sin^5 \gamma}, \quad I_1 = \frac{9 \sin \gamma + \sin 3\gamma - 12\gamma \cos \gamma}{32y^4 \sin^5 \gamma}$$

$$I_2 = \frac{2\gamma(2 + \cos 2\gamma) - 3 \sin 2\gamma}{16\gamma^3 \sin^3 \gamma}, \quad I_3 = \frac{9 \sin \gamma + \sin 3\gamma - 12\gamma \cos \gamma}{32\gamma^2 \sin^5 \gamma},$$

$$I_4 = \frac{12\gamma - 8 \sin 2\gamma + \sin 4\gamma}{32\gamma \sin^5 \gamma}.$$

$$32. \int_0^\infty \frac{x^n dx}{(x^2 + 2xy \cos \gamma + y^2)^4} = J_n$$

$$[0 < |\gamma| < \pi]$$

$$J_0 = \frac{15\gamma - \sin \gamma \cos \gamma (15 + 10 \sin^2 \gamma + 8 \sin^4 \gamma)}{48\gamma^7 \sin^7 \gamma},$$

$$J_1 = \frac{15 \sin \gamma - 15\gamma \cos \gamma - 5 \sin^3 \gamma - 2 \sin^5 \gamma}{48\gamma^6 \sin^7 \gamma}.$$

$$33. \int_0^\infty \frac{x^{\alpha-1} |x-y|^{\beta-1}}{(x^2 + 2bx + y^2)^\rho} dx = y^{(\pm 1 - 1)/2} \frac{B\left(\rho - \alpha + \frac{1 \pm 1}{2}, \alpha - \frac{1 \pm 1}{2}\right)}{(2y + 2b)^\alpha - (1 \pm 1)/2}$$

$$[\beta = 2\rho - 2\alpha + 1 \pm 1; \operatorname{Re} \alpha > (1 \pm 1)/2; \operatorname{Re}(\rho - \alpha) > (\mp 1 - 1)/2; y > 0; b > -y].$$

$$34. = y^{(\pm 3 - 3)/2} \frac{\left[\left(2\rho - \alpha - \frac{1 \mp 3}{2}\right)y + (2\rho - 2\alpha + 1 \pm 3)b\right]}{(\rho - 1)(2y + 2b)^\alpha - (1 \pm 3)/2} \times$$

$$\times B\left(\rho - \alpha + \frac{1 \pm 3}{2}, \alpha - \frac{1 \pm 3}{2}\right)$$

$$[\beta = 2\rho - 2\alpha + 1 \pm 3; \operatorname{Re} \alpha > (1 \pm 3)/2; \operatorname{Re}(\rho - \alpha) > (\mp 3 - 1)/2; y, b + y > 0].$$

$$35. \int_0^\infty \frac{x^{\alpha-1}}{ax^2 + 2bx + c} dx = \frac{(y^{\alpha-1} - z^{\alpha-1})\pi}{a(z-y) \sin \alpha\pi} = \frac{\pi \operatorname{sh}[(1-\alpha)\varphi] c^{\alpha/2-1}}{a^{\alpha/2} \sin \alpha\pi \operatorname{sh} \varphi}$$

$$[0 < \operatorname{Re} \alpha < 1; 0 < ac < b^2; ab > 0; ay = b - \sqrt{b^2 - ac}, az = b + \sqrt{b^2 - ac}, \varphi = \operatorname{Arch}\left\{\frac{b}{\sqrt{ac}}\right\}].$$

$$36. = \frac{\pi \sin[(1-\alpha)\varphi] c^{\alpha/2-1}}{a^{\alpha/2} \sin \varphi \sin \alpha\pi} \quad [0 < \operatorname{Re} \alpha < 1; ac > b^2; \varphi = \arccos(b/\sqrt{ac})].$$

$$37. = -\frac{\pi(z^{\alpha-1} + y^{\alpha-1} \cos \alpha\pi)}{a(y+z) \sin \alpha\pi}$$

$$[0 < \operatorname{Re} \alpha < 1; ac < 0; az = b + \sqrt{b^2 - ac}, ay = -b + \sqrt{b^2 - ac}].$$

$$38. = \frac{\pi(y^{\alpha-1} - z^{\alpha-1})}{a(z-y)} \operatorname{ctg} \alpha\pi = \frac{\pi \operatorname{sh}[(1-\alpha)\varphi] c^{\alpha/2-1}}{a^{\alpha/2} \operatorname{sh} \varphi}$$

$$[0 < \operatorname{Re} \alpha < 1; b^2 > ac > 0; ab < 0; ay = -b - \sqrt{b^2 - ac}, az = -b + \sqrt{b^2 - ac}, \varphi = \operatorname{Arch}\left\{\frac{b}{\sqrt{ac}}\right\}].$$

$$39. \int_0^\infty \frac{dx}{Vx(ax^2 + 2bx + c)} = \frac{\pi}{V2c \sqrt{Vac + b}} \quad [b^2 < ac; a, b > 0]$$

$$40. \int_0^\infty \frac{Ax^3 + Bx^2 + Cx + D}{(ax^2 + 2bx + c)^{5/2}} dx = \frac{1}{3(Vac + b)^{3/2}} \times$$

$$\times \left[(2\sqrt{ac} + b) \left(\frac{A}{a^{3/2}} + \frac{D}{c^{3/2}} \right) + \frac{B}{\sqrt{a}} + \frac{C}{\sqrt{c}} \right] \quad [a, c > 0; b > -\sqrt{ac}].$$

$$41. \int_0^{\infty} \frac{dx}{(gx+h)\sqrt{ax^2+2bx+c}} = \frac{1}{r} \ln \frac{g(r\sqrt{c}+gc-hb)}{h(r\sqrt{a}+gb-ha)}$$

$$[g, h, a > 0; c \geq 0; b > -\sqrt{ac}; r = \sqrt{ah^2 - 2bgh + cg^2} > 0].$$

$$42. = \frac{1}{r} \arccos \left[1 - \frac{r^2}{gh(\sqrt{ac}+b)} \right]$$

$$[g, h, a > 0; c \geq 0; b > \sqrt{ac}; r = \sqrt{-ah^2 + 2bgh - cg^2} > 0].$$

$$43. = \frac{2}{h\sqrt{a}+g\sqrt{c}}$$

$$[ah^2 - 2bgh + cg^2 = 0; a, g, h > 0; c \geq 0; gb - ha \neq 0].$$

$$44. \int_{-\infty}^{\infty} \frac{x^m dx}{(ax^2+2bx+c)^n} =$$

$$= \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!! (ac-b^2)^{n-1/2}} \sum_{k=0}^{[m/2]} \binom{m}{2k} (2k-1)!! (2n-2k-3)!! (ac-b^2)^k b^{-2k}$$

$$[a > 0; ac > b^2; m=0, 1, 2, \dots, 2n-2; n=1, 2, 3, \dots].$$

$$45. \int_{-\infty}^{\infty} \frac{dx}{(ax^2+2bx+c)^n} = \frac{(2n-3)!! \pi a^{n-1}}{(2n-2)!! (ac-b^2)^{n-1/2}}$$

$$[a > 0; ac > b^2; n=1, 2, 3, \dots].$$

2.2.10. Интегралы от $x^a (ax^4+2bx^2+c)^p$.

Условия: a, b, c — действительные числа.

$$1. \int_0^1 \frac{x^2+1}{x^4 \pm x^2+1} dx = \frac{\pi}{2\sqrt{2 \pm 1}} \quad 2. \int_1^{\infty} \frac{x^2+1}{x^4 \pm x^2+1} dx = \frac{\pi}{2\sqrt{2 \pm 1}}.$$

$$3. \int_0^{\infty} \frac{x^{a-1} dx}{(ax^4+2bx^2+c)^p} = \frac{1}{2} \int_0^{\infty} \frac{x^{a/2-1} dx}{(ax^2+2bx+c)^p} \quad [\text{см. 2.2 9.7}].$$

$$4. \int_0^{\infty} \frac{x^k dx}{ax^4+2bx^2+c} = I_k$$

$$[a, c > 0; b^2 > ac].$$

$$I_0 = \frac{\pi}{2\sqrt{2c(b+\sqrt{ac})}}$$

$$[c > 0; b > -\sqrt{ac}].$$

$$I_1 = \frac{1}{2\sqrt{b^2-ac}} \ln \frac{b+\sqrt{b^2-ac}}{\sqrt{ac}}, \quad I_2 = \frac{\pi}{2\sqrt{2a(b+\sqrt{ac})}}.$$

$$5. \int_0^{\infty} x^{2\rho-1} [(ax^2+b)^2+cx^2]^{-\rho} dx = \frac{\sqrt{\pi}}{2a^{(1+1)/2} b^{(1+1)/2} c^{\rho-1/2}} \Gamma \left[\frac{\rho-1/2}{\rho} \right]$$

$$[c > 0; \operatorname{Re} \rho > 1/2].$$

2.2.11. Интегралы, содержащие $A(\sqrt{ax^2+bx+c})$.

$$1. \int_0^{\infty} \frac{x^{a-1} dx}{(\sqrt{x^2+z^2+x})^v} = (2z)^{a-v} \Gamma \left[\frac{a/2, v-a}{v+1-a/2} \right]$$

$$[\operatorname{Re} z > 0; 0 < \operatorname{Re} z < \operatorname{Re} v].$$

$$2. \int_0^{\infty} \frac{x^m dx}{(\sqrt{x^2+z^2}+z)^n} = \left(\frac{z}{2}\right)^{m-n+1} \frac{n(n-m-2)!}{2(m+1)(m+3)\dots(2n-m-1)}$$

[Re z > 0; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$3. \int_0^{\infty} \frac{x^{\alpha-1} dx}{(\sqrt{x^2+z^2} \pm x)^{\nu}} = 2^{-\alpha} z^{\alpha-\nu} \Gamma\left[\frac{\alpha, (\pm \nu - \alpha)/2}{(\pm \nu + \alpha)/2 + 1}\right]$$

[Re z > 0; 0 < Re α < ± Re ν].

$$4. \int_0^{\infty} \frac{x^m dx}{(\sqrt{x^2+z^2}+x)^n} = \frac{2^m z^{m-n+1} n \cdot m!}{(n-m-1)(n-m+1)\dots(n+m+1)}$$

[Re z > 0; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$5. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left(x+z+\sqrt{\left\{\frac{x^2}{z^2}\right\}+2xz}\right)^{\nu}} = 2^{1+\alpha} z^{\alpha-\nu} \left\{ \frac{\Gamma(2\alpha) \Gamma(\nu-\alpha) \Gamma^{-1}(1+\alpha+\nu)}{2^{-\nu} \Gamma(2\nu-2\alpha) \Gamma(\alpha) \Gamma^{-1}(1+2\nu-\alpha)} \right\}$$

[|arg z| < π; 0 < Re α < Re ν].

$$6. \int_0^{\infty} \frac{x^m dx}{\left(x+z+\sqrt{\left\{\frac{x^2}{z^2}\right\}+2xz}\right)^n} =$$

$$= \left\{ \frac{n z^{m-n+1} 2^{-m} (2m+1)! (n-m-2)! [(m+n+1)!]^{-1}}{2^n (2\alpha)^{m-n+1} m! (2n-2m-3)! [(2n-m-1)!]^{-1}} \right\}$$

[|arg z| < π; m = 0, 1, 2, ..., n-2; n = 2, 3, 4, ...].

$$7. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left[x+z+\sqrt{(x+z)^2-y^2\left\{\frac{x^2}{z^2}\right\}}\right]^{\nu}} =$$

$$= 2^{-\nu} z^{\alpha-\nu} B(\alpha, \nu-\alpha) {}_2F_1\left(\frac{1}{2} \pm \frac{\alpha}{2} + \frac{\nu}{2} \left\{\frac{0}{1}\right\}, \pm \frac{\alpha}{2} + \frac{\nu}{2} \left\{\frac{0}{1}\right\}; 1+\nu; y^2 \left\{\frac{1}{z^2}\right\}\right)$$

[|y| ≤ 1; |arg z| < π; 0 < Re α < Re ν].

$$8. \int_a^{\infty} \frac{dx}{\sqrt{x^2-b^2} (x+\sqrt{x^2-b^2})^{\nu}} = \frac{1}{\nu} (a+\sqrt{a^2-b^2})^{-\nu}$$

[|b| ≤ a; Re ν > 0].

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\left(\sqrt{x^2+z^2}+\sqrt{\left\{\frac{x^2+y^2}{x^2y^2+z^2}\right\}}\right)^{\nu}} =$$

$$= 2^{-\nu-1} z^{\alpha-\nu} B\left(\frac{\alpha}{2}; \frac{\nu-\alpha}{2}\right) {}_2F_1\left(\frac{1}{2} \left\{\frac{\nu-\alpha}{\alpha}\right\}, \frac{\nu+1}{2}; \nu+1; 1-y^2 \left\{\frac{z^2}{y^2}\right\}\right)$$

[Re y, Re z > 0; 0 < Re α < Re ν].

$$10. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x^2+z^2} (\sqrt{x^2+z^2}+y)^{\nu}} =$$

$$= (2z)^{\alpha-\nu-1} B\left(\frac{\alpha}{2}, 1-\alpha+\nu\right) {}_2F_1\left(1-\alpha+\nu, \nu; 1+\nu-\frac{\alpha}{2}; \frac{z-y}{2z}\right)$$

[Re z > 0; |arg(y/z+1)| < π; 0 < Re α < Re ν+1].

$$11. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x^2+z^2} (\sqrt{x^2+z^2}+z)^{\nu}} = (2z)^{\alpha-\nu-1} B\left(\frac{\alpha}{2}, 1+\nu-\alpha\right) \\ [\operatorname{Re} z > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1].$$

$$12. \int_0^{\infty} \frac{x^m dx}{\sqrt{x^2+z^2} (\sqrt{x^2+z^2}+z)^n} = \frac{z^{m-n} (n-m-1)!}{(2n-m-1)(2n-m+1)\dots(m+1)} \\ [\operatorname{Re} z > 0; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].$$

$$13. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} (\sqrt{x^2+z^2}+xy)^{-\nu} dx = \\ = 2^{-\alpha} z^{\alpha-\nu-1} B\left(\alpha, \frac{1-\alpha+\nu}{2}\right) {}_3F_1\left(\nu, \alpha; \frac{\alpha+\nu+1}{2}; \frac{1-y}{2}\right) \\ [\operatorname{Re} z > 0; |\arg(y+1)| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1].$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} (\sqrt{x^2+z^2}+x)^{-\nu}}{\sqrt{x^2+z^2}} dx = z^{\alpha-\nu-1/2} {}_2B\left(\alpha, \frac{1-\alpha+\nu}{2}\right) \\ [\operatorname{Re} z > 0; 0 < \operatorname{Re} \alpha < 1 \pm \operatorname{Re} \nu].$$

$$15. \int_0^{\infty} \frac{x^m (\sqrt{x^2+z^2}+x)^{-n}}{\sqrt{x^2+z^2}} dx = \frac{z^{m-n} m!}{(n-m)(n-m+2)\dots(n+m)} \\ [\operatorname{Re} z > 0; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{2x+z} (x+z+\sqrt{z^2+2xz})^{\nu}} = 2^{\alpha-\nu} z^{\alpha-\nu-1/2} B(\alpha, 2\nu-2\alpha+1) \\ [|\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2].$$

$$17. \int_0^{\infty} \frac{x^m dx}{\sqrt{2x+z} (x+z+\sqrt{z^2+2xz})^n} = \frac{\sqrt{2} m! (2n-2m-2)!}{(2z)^{n-m-1/2} (2n-m-1)!} \\ [|\arg z| < \pi; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].$$

$$18. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x+z}} [x+z+y+2\sqrt{y(x+z)}]^{-\nu} dx = \\ = 2^{-\alpha} 2^{\nu} z^{\alpha-\nu-1} \sqrt{y} B(2\nu-2\alpha+1, \alpha) {}_3F_1\left(\nu+1-\alpha, \nu+\frac{1}{2}; 2\nu+1-\alpha; 1-\frac{y}{z}\right) \\ [|\arg z|, |\arg y|, |\arg(y/z)| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2].$$

$$19. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{x+2z} (x+z+\sqrt{x^2+2xz})^{\nu}} = 2^{1/2-\alpha} z^{\alpha-\nu-1/2} B\left(2\alpha, \nu-\alpha+\frac{1}{2}\right) \\ [|\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2].$$

$$20. \int_0^{\infty} \frac{x^m dx}{\sqrt{x+2z} (x+z+\sqrt{x^2+2xz})^n} = \frac{(2z)^{m+1/2} z^{-n} (2m+1)!}{(2n-2m-1)(2n-2m+1)\dots(2n+2m+1)} \\ [|\arg z| < \pi; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].$$

$$\begin{aligned}
 21. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x+z}} [x+xy+z+2\sqrt{xy(x+z)}]^{-\nu} dx = \\
 = 2^{1-2\nu} z^{\alpha-\nu-1/2} \sqrt{y} B\left(2\alpha, \frac{1}{2} + \nu - \alpha\right) {}_2F_1\left(\alpha + \frac{1}{2}, \nu + \frac{1}{2}; \alpha + \nu + \frac{1}{2}; 1-y\right) \\
 [|\arg z|, |\arg y| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1/2]
 \end{aligned}$$

$$\begin{aligned}
 22. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{(x+z)^2 - y^2} [x+z+\sqrt{(x+z)^2 - y^2}]^{\nu}} = \\
 = 2^{-\nu} z^{\alpha-\nu-1} B\left(\alpha, 1 + \nu - \alpha\right) {}_2F_1\left(\frac{\nu-\alpha+1}{2}, 1 + \frac{\nu-\alpha}{2}; 1 + \nu; \frac{y^2}{z^2}\right) \\
 [|\nu| < |z|; |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1]
 \end{aligned}$$

$$\begin{aligned}
 23. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\sqrt{(x+z)^2 - x^2 y^2} [x+z+\sqrt{(x+z)^2 - x^2 y^2}]^{\nu}} = \\
 = 2^{-\nu} z^{\alpha-\nu-1} B\left(\alpha, \nu - \alpha + 1\right) {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; 1 + \nu; y^2\right) \\
 [|\nu| < 1; |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 1]
 \end{aligned}$$

$$\begin{aligned}
 24. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a-x}} [(z+\sqrt{a-x})^{\nu} + (z-\sqrt{a-x})^{\nu}] dx = \\
 = 2\sqrt{\pi} \Gamma\left[\frac{\alpha}{2} + 1/2\right] z^{\alpha-1/2} {}_2F_1\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2} + \alpha; \frac{a}{z^2}\right) \\
 [a, \operatorname{Re} \alpha > 0; z^2 \notin [0, a]]
 \end{aligned}$$

$$\begin{aligned}
 25. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a^2 - x^2}} [(a+\sqrt{a^2 - x^2})^{\nu} + (a-\sqrt{a^2 - x^2})^{\nu}] dx = (2a)^{\alpha+\nu-1} B\left(\frac{\alpha}{2}, \nu + \frac{\alpha}{2}\right) \\
 [a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+2\nu) > 0]
 \end{aligned}$$

$$\begin{aligned}
 26. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2+y^2)} (\sqrt{x^2+z^2} + \sqrt{x^2+y^2})^{\nu}} dx = \\
 = 2^{-\nu-1} z^{\alpha-\nu-2} B\left(\frac{\alpha}{2}, 1 + \frac{\nu-\alpha}{2}\right) {}_2F_1\left(1 + \frac{\nu-\alpha}{2}, \frac{\nu+1}{2}; \nu+1; 1 - \frac{y^2}{z^2}\right) \\
 [\operatorname{Re} z, \operatorname{Re} y > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 2]
 \end{aligned}$$

$$\begin{aligned}
 27. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{(x^2+z^2)(x^2 y^2 + z^2)} [\sqrt{x^2+z^2} + \sqrt{x^2 y^2 + z^2}]^{\nu}} dx = \\
 = 2^{-\nu-1} z^{\alpha-\nu-2} B\left(\frac{\nu-\alpha}{2} + 1, \frac{\alpha}{2}\right) {}_2F_1\left(\frac{\alpha}{2}, \frac{\nu+1}{2}; \nu+1; 1 - y^2\right) \\
 [\operatorname{Re} y, \operatorname{Re} z > 0; 0 < \operatorname{Re} \alpha < \operatorname{Re} \nu + 2]
 \end{aligned}$$

$$\begin{aligned}
 28. \int_0^{\infty} \frac{1}{\sqrt{x^2+z^2}} \left[\frac{1}{\sqrt{(x-c)^2 + y^2}} + \frac{1}{\sqrt{(x+c)^2 + y^2}} \right] dx = \frac{2\sqrt{u}}{\sqrt{yz}} K(\sqrt{1-u^2}) \\
 [u = c - \sqrt{c^2 - 1}, 2cyz = c^2 + y^2 + z^2; c, \operatorname{Re} z, \operatorname{Re} y > 0]
 \end{aligned}$$

2.2.12. Интегралы, содержащие разности алгебраических функций.

$$1. \int_0^1 \left(\frac{1}{1-x} - \frac{\lambda x^{\alpha-1}}{1-x^\lambda} \right) dx = C + \ln \lambda + \psi \left(\frac{\alpha}{\lambda} \right)$$

$$[|\arg \lambda| < \pi; \operatorname{Re} \alpha, \operatorname{Re}(\alpha - \lambda) > 0].$$

$$2. \int_0^\infty \left[\frac{x^{\alpha-1}}{(x+z)^\alpha} - \frac{x^{\sigma-1}}{(x+z)^\sigma} \right] dx = \psi(\sigma) - \psi(\alpha)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$3. \int_0^\infty \left[\frac{x^{\alpha-1}}{(x+z)^\alpha} - \frac{x^{-\alpha}}{(x+z)^{1-\alpha}} \right] dx = \pi \operatorname{ctg} \alpha \pi$$

$$[|\arg z| < \pi; 0 < \operatorname{Re} \alpha < 1].$$

$$4. \int_0^\infty \left[\frac{z^\alpha}{(x+z)^\alpha} - \frac{z^\sigma}{(x+z)^\sigma} \right] \frac{dx}{x} = \psi(\sigma) - \psi(\alpha)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$5. \int_0^\infty x^{\alpha-1} [(x+z)^{\rho-1} - x^{\rho-1}] dx = z^{\alpha+\rho-1} B(\alpha, 1-\alpha-\rho)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha > 0; 1 < \operatorname{Re}(\alpha + \rho) < 2].$$

$$6. \int_0^\infty \frac{(x+z)^\rho - z^\rho}{(x+z)^{\rho+1} x} dx = z^{\alpha-\rho} [\psi(\rho) - \psi(\rho-\sigma)]$$

$$[|\arg z| < \pi; \operatorname{Re} \rho > 0; \operatorname{Re} \rho > \operatorname{Re} \sigma].$$

$$7. \int_0^1 \left(\frac{x^{\alpha-1}}{1-x} - \frac{\lambda x^{\alpha\lambda-1}}{1-x^\lambda} \right) dx = \ln \lambda$$

$$[|\arg \lambda| < \pi; \operatorname{Re} \alpha, \operatorname{Re}(\lambda\alpha - \lambda) > 0].$$

$$8. \int_0^1 \left(\frac{x^{\alpha-1}}{1 \pm xy} \pm \frac{x^{-\alpha}}{y \pm x} \right) dx = \frac{\pi y^{-\alpha}}{\sin \alpha \pi} \left\{ \frac{1}{\cos \alpha \pi} \right\}$$

$$\left[0 < \operatorname{Re} \alpha < 1, \left\{ \begin{array}{l} y > 0 \text{ или } |\arg y| < \pi \\ y > 0 \end{array} \right\} \right].$$

$$9. \int_0^a [(x+a)^{\gamma-1} (a-x)^{\beta-1} + (x+a)^{\beta-1} (a-x)^{\gamma-1}] dx = (2a)^{\beta+\gamma-1} B(\beta, \gamma)$$

$$[a, b, \operatorname{Re} \beta, \operatorname{Re} \gamma > 0].$$

$$10. \int_0^1 [a^\alpha x^{\alpha-1} (a+b-ax)^{\beta-1} + b^\beta x^{\beta-1} (a+b-bx)^{\alpha-1}] dx = (a+b)^{\alpha+\beta-1} B(\alpha, \beta)$$

$$[a, b, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$11. \int_0^1 \left(\frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2} \right) \frac{dx}{x} = \frac{2}{\mu} \ln 2$$

$$[\mu > 0].$$

$$12. \int_0^\infty \frac{(a+bx)^\rho - (a+cx)^\rho}{x} dx = a^\rho \ln \frac{c}{b}$$

$$[a, b, c > 0; \operatorname{Re} \rho < 0]$$

$$13. \int_0^{\infty} \frac{a(x+a)^{\rho-1} x^{\alpha-1} + b(x+b)^{\alpha-1} x^{\rho-1}}{[(a+b)x+ab]^{\alpha+\rho}} dx = \frac{B(\alpha, \rho)}{a^{\alpha} b^{\rho}} \quad [a, b, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0].$$

$$14. \int_0^{\infty} \left[\left(1 + \frac{a}{\rho x}\right)^{\rho x} - \left(1 + \frac{a}{qx}\right)^{qx} \right] \frac{dx}{x} = (e^a - 1) \ln \frac{\rho}{q} \quad [a, \rho, q > 0].$$

$$15. \int_0^{\infty} \left(\frac{1}{1+x^{\alpha}} - \frac{1}{1+x^{\beta}} \right) \frac{dx}{x} = 0 \quad [\alpha, \beta - \text{произвольные}].$$

$$16. \int_0^{\infty} \left[\left(1 + \frac{x^{\alpha}}{y}\right)^{\mu} - \left(1 + \frac{x^{\beta}}{z}\right)^{\nu} \right] \frac{dx}{x} = \\ = -\frac{\ln y}{\alpha} - \frac{\ln z}{\beta} - \zeta \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) - \frac{\psi(-\mu)}{\alpha} + \frac{\psi(-\nu)}{\beta} \\ [\alpha, \beta, \operatorname{Re} \mu, \operatorname{Re} \nu > 0; \operatorname{Re} \mu, \operatorname{Re} \nu < 0].$$

2.3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

2.3.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входит показательная функция. Одним из способов вычисления таких интегралов является сведение их с помощью замены переменной к интегралам, содержащим степенную и алгебраические функции. Кроме того, некоторые интегралы, содержащие показательную функцию, могут быть получены из других интегралов с помощью предельного перехода по параметру; например, если

$$\int_0^{\infty} (a+x)^{\beta} \varphi(x) dx = f(a, \beta),$$

то при соответствующих условиях

$$\int_0^{\infty} \varphi(x) e^{-px} dx = \lim_{a \rightarrow +\infty} \int_0^{\infty} \left(1 + \frac{x}{a}\right)^{-a\beta} \varphi(x) dx = \lim_{a \rightarrow +\infty} [a^{\alpha\beta} f(a, -a\beta)].$$

2.3.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие степенную, алгебраические и показательную функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j = 1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. Кроме того, $r = p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$1. \int_0^y x^{r-1} (y-x)^{\beta-1} e^{-sx} dx = V_{20}$$

$$\{E_{21} = r^{-1} B(\beta, (\alpha+h)/r), u = \alpha + r(\beta-1) + h,$$

$$a = (\alpha+h)/r, b = \beta + (\alpha+h)/r, z = (-sy/p)^{\rho}\}$$

$$[y, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} (x^r - y^r)^{\beta-1} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = r^{-1} B(\beta, 1 - \beta - (\alpha + h)/r), E_{27} = (-1)^j / (1 - \beta)_j \Gamma[\alpha - r(1 - \beta + j)],$$

$$u = \alpha + r(\beta - 1) + h, u_1 = rj, v_1 = r(1 - \beta + j) - \alpha, a = (\alpha + h)/r,$$

$$b = \beta + (\alpha + h)/r, a_1 = 1 - \beta + j, b_1 = 1 - \alpha + r(1 - \beta + j), z = (-sy/p)^p\}$$

$$[y, \operatorname{Re} \beta > 0; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + r\beta) < r + 1)].$$

$$3. \int_0^{\infty} x^{\alpha-1} (y^r + x^r)^{\rho-1} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = \frac{1}{r} B\left(\frac{\alpha + h}{r}, 1 - \rho - \frac{\alpha + h}{r}\right), E_{27} = (1 - \rho)_j / \Gamma[\alpha - r(1 - \rho + j)],$$

$$u = \alpha + r(\rho - 1) + h, u_1 = rj, v_1 = r(1 - \rho + j) - \alpha, a = (\alpha + h)/r,$$

$$b = \rho + (\alpha + h)/r, a_1 = 1 - \rho + j, b_1 = 1 - \alpha + r(1 - \rho + j),$$

$$z = (-1)^{p+q} (sy/p)^p\}$$

$$[\operatorname{Re} \alpha > 0; r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + r\rho) < r + 1)].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{y^r - x^r} e^{-sx} dx = V_{26} + V_{27}$$

$$\{E_{26} = \frac{\pi}{r} \operatorname{ctg}\left(\frac{\alpha + h}{r}\right) \pi, E_{27} = (-1)^{j+1} j! \Gamma[\alpha - r(1 + j)],$$

$$u = \alpha - r + h, u_1 = rj, v_1 = r(1 + j) - \alpha, a = b = (\alpha + h)/r,$$

$$a_1 = 1 + j, b_1 = 1 - \alpha + r(1 + j), z = (-sy/p)^p\}$$

$$[y, \operatorname{Re} \alpha > 0; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} \alpha < r + 1)].$$

$$5. \int_0^{\infty} x^{\alpha-1} (y^{r/2} + \sqrt{y^r + x^r})^{\sigma} e^{-sx} dx = V_{28} + V_{29} + V_{30}$$

$$\{E_{28} = -\frac{\sigma}{2\sqrt{\pi}} \Gamma\left[\frac{j - \sigma/2}{1 - \sigma/2 - j}, \frac{1/2 - j}{1 - \sigma/2 - j}, \alpha - r(j - \sigma/2)\right],$$

$$E_{29} = -\frac{\sigma}{2\sqrt{\pi}} \Gamma\left[\frac{(1 - \sigma)/2 + j}{(1 - \sigma)/2 - j}, \frac{-1/2 - j}{(1 - \sigma)/2 - j}, \alpha - r(j + (1 - \sigma)/2)\right],$$

$$E_{30} = -\frac{\sigma}{\alpha + h} 2^{\sigma+2} (\alpha + h)/r B\left(1 + \frac{\alpha + h}{r}, -\sigma - \frac{2}{r}(\alpha + h)\right),$$

$$u_1 = jr, u_2 = (j + 1/2)r, u = \alpha + r\sigma/2 + h, v_1 = (j - \sigma/2)r - \alpha,$$

$$v_2 = [j + (1 - \sigma)/2]r - \alpha, a_1 = j - \sigma/2, b_1 = j + \sigma/2, c_1 = 1 - \alpha + (j - \sigma/2)r,$$

$$d_1 = j + 1/2, a_2 = (1 - \sigma)/2 + j, b_2 = (1 + \sigma)/2 + j,$$

$$c_2 = 1 - \alpha + [j + (1 - \sigma)/2]r, d_2 = 3/2 + j, a_3 = (\alpha + h)/r,$$

$$b_3 = \sigma + (\alpha + h)/r, c_3 = 1 + \sigma/2 + (\alpha + h)/r, d_3 = (1 + \sigma)/2 + (\alpha + h)/r,$$

$$z = (-1)^{p+q} (sy/p)^p\}$$

$$[\operatorname{Re} \alpha > 0; r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + \sigma r/2) < 1)].$$

$$6. \int_0^{\infty} x^{\alpha-1} (x^{r/2} + \sqrt{y^r + x^r})^{\sigma} e^{-sx} dx = V_{28} + V_{31}$$

$$\{E_{28} = -\sigma 2^{\sigma-3/2} \Gamma\left[\frac{2j - \sigma}{1 - \sigma + j}, \frac{\alpha - r(j - \sigma/2)}{1 - \sigma + j}\right],$$

$$E_{20} = -\frac{\sigma}{\alpha+h} 2^{-2(\alpha+h)/r-1} B\left(1+\frac{2}{r}(\alpha+h), -\frac{\sigma}{2}-\frac{\alpha+h}{r}\right),$$

$$u_1=rj, u=\alpha+r\sigma/2+h, v_1=r(j-\sigma/2)-\alpha, a_1=j-\sigma/2, \\ b_1=(1-\sigma)/2+j, c_1=1-\alpha+r(j-\sigma/2), d_1=1-\sigma+j, a_2=(\alpha+h)/r, \\ b_2=(\alpha+h)/r+1/2, c_2=1+\sigma/2+(\alpha+h)/r, d_2=1-\sigma/2+(\alpha+h)/r,$$

$$z=(-1)^{p+q}(sy/p)^p \\ [\operatorname{Re} \alpha > 0; r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + \sigma r/2) < 1)]$$

$$7. \int_0^\infty x^{\alpha-1} \frac{(y^{r/2} + \sqrt{y^r + x^r})^\sigma}{\sqrt{y^r + x^r}} e^{-sx} dx = V_{20} + V_{20} + V_{20}$$

$$\{E_{20} = \frac{1}{\sqrt{\pi}} \Gamma\left[\frac{(1-\sigma)/2+j}{(1-\sigma)/2-j}, \frac{1/2-j}{1}, \alpha-r(j+(1-\sigma)/2)\right],$$

$$E_{20} = \frac{1}{\sqrt{\pi}} \Gamma\left[\frac{1-\sigma/2+j}{-\sigma/2-j}, -\frac{1/2-j}{1}, \alpha-r(j+1-\sigma/2)\right],$$

$$E_{20} = \frac{2^{\sigma+2(\alpha+h)/r}}{r} B\left(\frac{\alpha+h}{r}, 1-\sigma-\frac{2}{r}(\alpha+h)\right),$$

$$u_1=rj, u_2=r(j+1/2), u=r(\sigma-1)/2+\alpha+h, v_1=[(1-\sigma)/2+j]r-\alpha, \\ v_2=(1-\sigma/2+j)r-\alpha, a_1=(1-\sigma)/2+j, b_1=(1+\sigma)/2+j, \\ c_1=1-\alpha+r[(1-\sigma)/2+j], d_1=1/2+j, a_2=1-\sigma/2+j, b_2=1+\sigma/2+j, \\ c_2=1-\alpha+r(1-\sigma/2+j), d_2=3/2+j, a_3=(\alpha+h)/r, \\ b_3=\sigma+(\alpha+h)/r, c_3=(1+\sigma)/2+(\alpha+h)/r, d_3=\sigma/2+(\alpha+h)/r,$$

$$z=(-1)^{p+q}(sy/p)^p \\ [\operatorname{Re} \alpha > 0, r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + \sigma r/2) < r/2 + 1)]$$

$$8. \int_0^\infty x^{\alpha-1} \frac{(x^{r/2} + \sqrt{y^r + x^r})^\sigma}{\sqrt{y^r + x^r}} e^{-sx} dx = V_{20} + V_{20}$$

$$\{E_{20} = 2^{\sigma-2}/\Gamma\left[\frac{1-\sigma+2j}{1-\sigma+j}, \alpha-r(1-\sigma+2j)/2\right],$$

$$E_{20} = \frac{2^{1-2(\alpha+h)/r}}{r} B\left(\frac{2}{r}(\alpha+h), \frac{1-\sigma}{2}-\frac{\alpha+h}{r}\right),$$

$$u_1=rj, u=r(\sigma-1)/2+\alpha+h, v_1=r[(1-\sigma)/2+j]-\alpha, a_1=(1-\sigma)/2+j, \\ b_1=1-\sigma/2+j, c_1=1-\alpha+r[(1-\sigma)/2+j], d_1=1-\sigma+j, a_2=(\alpha+h)/r, \\ b_2=1/2+(\alpha+h)/r, c_2=(1+\sigma)/2+(\alpha+h)/r, d_2=(1-\sigma)/2+(\alpha+h)/r,$$

$$z=(-1)^{p+q}(sy/p)^p \\ [\operatorname{Re} \alpha > 0; r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re}(\alpha + \sigma r/2) < r/2 + 1)]$$

$$9. \int_0^\infty \frac{x^{\alpha-1} e^{-sx}}{x^{2r} + 2(xy)^r \cos \gamma + y^{2r}} dx = U,$$

$$U = -\frac{\pi y^{2r}}{r \sin \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l \sin \left\{ \gamma \left((\alpha+l)/r - 1 \right) \right\}}{\sin [(\alpha+l)\pi/r]} (sy)^l -$$

$$- \sum_{l=0}^{\infty} (-1)^l \Gamma(\alpha-r-lr) \frac{\sin l\gamma}{\sin \gamma} s^{r(l+1)-\alpha} y^{r(l+1)-\alpha}$$

$$[\operatorname{Re} \alpha > 0; |\gamma|, 2r|\arg y| < \pi; \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0, \operatorname{Re} \alpha < 2r + 1)]$$

$$10. \int_0^{2\pi} \frac{x^{2l-1} e^{-sx} dx}{(x^{2l} + 2(xy)^l \cos \gamma + y^{2l})^{\frac{1}{2}}} =$$

$$= \frac{U}{2 \sin^2 \gamma} + \frac{\pi y^{2-2l}}{2l^2 \sin^2 \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l (\alpha + l - r) \cos [\gamma ((\alpha + l)/r - 2)]}{l! \sin [(\alpha + l) \pi / r]} (xy)^l +$$

$$+ \sum_{l=0}^{\infty} (-1)^l \Gamma(\alpha - r - lr) \frac{l \cos(\gamma l - \gamma)}{2 \sin^2 \gamma} s^{r(l+1)} a_{\gamma} s^{l(l-1)}$$

[$\operatorname{Re} \alpha > 0$, $|\gamma|$, $2r|\arg y| < \pi$ и $\operatorname{Re} s > 0$ (или $\operatorname{Re} s = 0$, $\operatorname{Re} \alpha < 4r + 1$); U см. 2.3.2.9].

$$11. \int_0^y \frac{x^{2-1}}{\sqrt{y^2 - x^2}} [(y'^2 + \sqrt{y'^2 - x'^2})^\sigma + (y'^2 - \sqrt{y'^2 - x'^2})^\sigma] e^{-sx} dx = V_{2\sigma}$$

$$\{E_{2\sigma} = \frac{y^{2+2(\alpha+h)/r}}{r} B\left(\frac{\alpha+h}{r}, \sigma + \frac{\alpha+h}{r}\right),$$

$$u = r(\sigma - 1)/2 + \alpha + h, \quad a_3 = (\alpha + h)/r, \quad h_3 = \sigma + (\alpha + h)/r,$$

$$c_3 = \sigma/2 + (\alpha + h)/r, \quad d_3 = (\sigma + 1)/2 + (\alpha + h)/r, \quad z = (-sy/p)^p\}$$

[y , $\operatorname{Re} \alpha$, $\operatorname{Re}(\alpha + \sigma r) > 0$].

$$12. \int_0^\infty \frac{x^{2-1}}{\sqrt{x'^2 - y'^2}} [(x'^2 + \sqrt{x'^2 - y'^2})^\sigma + (x'^2 - \sqrt{x'^2 - y'^2})^\sigma] e^{-sx} dx = V_{2\sigma} + V_{2\sigma} + V_{2\sigma}$$

$$\{E_{2\sigma} = (\sigma - 2)_l 2^{\sigma-2} \Gamma[\alpha - r(1 - \sigma + 2j)/2],$$

$$E_{2\sigma} = (-\sigma - 2)_l 2^{-\sigma-2} \Gamma[\alpha - r(1 + \sigma + 2j)/2],$$

$$E_{2\sigma} = \frac{2^{l-2} (\alpha + h)^{r/r}}{r} B\left(\frac{1-\sigma}{2} - \frac{\alpha+h}{r}, \frac{1+\sigma}{2} - \frac{\alpha+h}{r}\right),$$

$$u_1 = l, \quad v_2 = r(\sigma + j), \quad u = r(\sigma - 1)/2 + \alpha + h, \quad v_1 = r[(1 - \sigma)/2 + j] - \alpha,$$

$$v_2 = r[(1 + \sigma)/2 + j] - \alpha, \quad a_1 = (1 - \sigma)/2 + j, \quad b_1 = 1 - \sigma/2 + j,$$

$$c_1 = 1 - \alpha + r[(1 - \sigma)/2 + j], \quad d_1 = 1 - \sigma + j, \quad a_2 = (1 + \sigma)/2 + j, \quad b_2 = 1 + \sigma/2 + j,$$

$$c_2 = 1 - \alpha + r[(1 + \sigma)/2 + j], \quad d_2 = 1 + \sigma + j, \quad a_3 = 1/2 + (\alpha + h)/r,$$

$$b_3 = (\alpha + h)/r, \quad c_3 = (1 + \sigma)/2 + (\alpha + h)/r, \quad d_3 = (1 - \sigma)/2 + (\alpha + h)/r, \quad z = (-sy/p)^p\}$$

[$y > 0$; $\operatorname{Re} s > 0$ (или $\operatorname{Re} s = 0$, $\operatorname{Re}(\alpha \pm \sigma r/2) < 1 + r/2$)].

$$13. \int_0^\infty x^{\alpha-1} e^{-ax-r-sx} dx = U_1$$

$$\left[z = \left(\frac{p}{s}\right)^p \left(\frac{a}{q}\right)^q; \operatorname{Re} \alpha, \operatorname{Re} a, \operatorname{Re} s > 0 \right],$$

$$U_1 = \sum_{j=0}^{q-1} \frac{(-a)^j}{j!} \Gamma(\alpha + rj) s^{-\alpha-rj} {}_2F_1\left(1, \Delta(p, \alpha + rj); (-1)^q z\right)$$

[$\sigma < r < 1$].

$$U_1 = \sum_{h=0}^{p-1} \frac{(-1)^h}{h! r} \Gamma\left(\frac{\alpha + h}{r}\right) a^{-(\alpha + h)/r} z^{q+1} {}_2F_1\left(1, \Delta(q, (\alpha + h)/r); (-1)^p z^{-1}\right)$$

[$r > 1$].

$$\begin{aligned}
 14. \int_0^{\infty} x^{\alpha-1} e^{-ax-rx} dx &= \\
 &= \sum_{j=0}^{q-1} \frac{(-a)^j}{j!} \Gamma(\alpha-rj) s^{rj-a} {}_1F_{p+q} \left(\begin{matrix} 1; \\ \Delta(p, 1-\alpha+rj), \Delta(q, 1+j) \end{matrix} \right) + \\
 &+ \sum_{h=0}^{r-1} \frac{(-1)^h}{h! r} \Gamma\left(-\frac{\alpha+h}{r}\right) a^{(\alpha+h)/r} {}_1F_{p+q} \left(\begin{matrix} 1; \\ \Delta(q, 1+(\alpha+h)/r), \Delta(p, 1+h) \end{matrix} \right) \\
 &\quad \left[z = (-1)^{p+q} \left(\frac{a}{r}\right)^p \left(\frac{a}{q}\right)^q; \operatorname{Re} a, \operatorname{Re} s > 0 \right].
 \end{aligned}$$

2.3.3. Интегралы от $x^{\alpha} e^{-px}$.

$$\begin{aligned}
 1. \int_0^{\infty} x^{\alpha-1} e^{-px} dx &= \Gamma(\alpha) p^{-\alpha} \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0] \text{ или } [0 < \operatorname{Re} \alpha < 1, \operatorname{Re} p = 0], \\
 2. \int_0^{\infty} x^n e^{-px} dx &= n! p^{-n-1} \quad [\operatorname{Re} p > 0], \\
 3. \int_0^{\infty} x^{n-1/2} e^{-px} dx &= \frac{(2n-1)!! \sqrt{\pi}}{2^n p^{n+1/2}} \quad [\operatorname{Re} p > 0],
 \end{aligned}$$

$$4. \int_{\gamma-i\infty}^{\gamma+i\infty} z^{\alpha-1} e^{-pz} dz = \frac{2\pi i}{(-p)^{\alpha} \Gamma(1-\alpha)} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \left[\begin{matrix} p > 0 \\ p < 0 \end{matrix} \right]; \operatorname{Re} \alpha < 1; \gamma > 0.$$

2.3.4. Интегралы от $(x \pm a)^{\beta} e^{-px}$.

$$\begin{aligned}
 1. \int_a^{\infty} (x-a)^{\beta-1} e^{-px} dx &= p^{-\beta} e^{-pa} \Gamma(\beta) \\
 &\quad [a > 0; \operatorname{Re} \beta, \operatorname{Re} p > 0 \text{ (или } 0 < \operatorname{Re} \beta < 1, \operatorname{Re} p = 0)], \\
 2. \int_0^{\infty} \frac{e^{-px}}{(x+z)^{\rho}} dx &= p^{\rho-1} e^{pz} \Gamma(1-\rho, pz) \quad [\operatorname{Re} p > 0; |\arg z| < \pi], \\
 3. \int_0^{\infty} \frac{e^{-px}}{x+z} dx &= -e^{pz} \operatorname{Ei}(-pz) \quad [\operatorname{Re} p > 0], \\
 4. \int_0^{\infty} \frac{e^{-px}}{(x+z)^2} dx &= \frac{1}{z} + p e^{pz} \operatorname{Ei}(-pz) \quad [\operatorname{Re} p \geq 0; |\arg z| < \pi], \\
 5. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x+z}} dx &= \sqrt{\frac{\pi}{p}} e^{pz} \operatorname{erfc}(\sqrt{pz}) \quad [|\arg z| < \pi; \operatorname{Re} p > 0], \\
 6. \int_0^{\infty} \frac{e^{-px}}{(x+z)^{3/2}} dx &= \frac{2}{\sqrt{z}} - 2\sqrt{\pi p} e^{pz} \operatorname{erfc}(\sqrt{pz}) \quad [|\arg z| < \pi; \operatorname{Re} p \geq 0].
 \end{aligned}$$

$$7. \int_0^{\infty} \frac{e^{-px}}{x-y} dx = -e^{-py} \operatorname{Ei}(py) \quad [y, \operatorname{Re} p > 0].$$

$$8. \int_0^{\infty} \frac{e^{-p|x|}}{x-y} \left\{ \frac{1}{\operatorname{sgn} x} \right\} dx = [-e^{-p|y|} \operatorname{Ei}(p|y|) \pm e^{p|y|} \operatorname{Ei}(-p|y|)] \left\{ \frac{\operatorname{sgn} y}{1} \right\} \\ [\operatorname{Re} p \geq 0; y \neq 0, \operatorname{Im} y = 0].$$

$$9. \int_{-\infty}^{\infty} \frac{e^{\pm i\lambda x}}{(y \pm ix)^p} dx = \frac{2\pi}{\Gamma(p)} \lambda^{p-1} e^{-\lambda y} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \left[\begin{Bmatrix} \lambda > 0 \\ \lambda < 0 \end{Bmatrix}; \operatorname{Re} y, \operatorname{Re} p > 0 \right].$$

$$10. \int_{-\infty}^{\infty} \frac{e^{-px}}{(x \pm y)^p} dx = 2\pi i \frac{p-1}{\Gamma(p)} e^{-piy-py} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [p, y, \operatorname{Re} p > 0].$$

2.3.5. Интегралы от $(x^n \pm a^n)^p e^{-px}$.
Условие: $a > 0$.

$$1. \int_{-a}^a (a^2 - x^2)^{\beta-1} e^{-px} dx = 2^{\beta-1/2} \sqrt{\pi} \Gamma(\beta) (p/a)^{1/2-\beta} I_{\beta-1/2}(ap) \\ [\operatorname{Re} \beta > 0].$$

$$2. \int_0^a (a-x)^{\beta-1} e^{-px} dx = p^{-\beta} e^{-ap} \gamma(\beta, -ap) \\ [\operatorname{Re} \beta > 0].$$

$$3. \int_{-a}^a (a^2 - x^2)^{\beta-1} e^{i\lambda x} dx = \sqrt{\pi} \Gamma(\beta) (2a/\lambda)^{\beta-1/2} J_{\beta-1/2}(a\lambda) \\ [\operatorname{Re} \beta > 0].$$

$$4. \int_a^{\infty} (x^2 - a^2)^{\beta-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \Gamma(\beta) \left(\frac{2a}{p} \right)^{\beta-1/2} K_{\beta-1/2}(ap) \quad [\operatorname{Re} \beta, \operatorname{Re} p > 0].$$

$$5. \int_a^{\infty} \frac{e^{i\lambda x}}{(x^2 - a^2)^{\beta}} dx = \frac{e^{i\lambda a}}{2} \sqrt{\pi} \Gamma(\beta) 2^{\beta-3/2} \left(\frac{\lambda}{a} \right)^{1/2-\beta} H_{1/2-\beta}^{(2/2-\beta)}(\lambda a) \\ [e = \pm i; \operatorname{Re} \beta > 0, \pm \operatorname{Re}(\lambda a) < 0 \text{ или } 0 < \operatorname{Re} \beta < 1, \operatorname{Re}(\lambda a) = 0].$$

$$6. \int_0^{\infty} \frac{e^{-px}}{(x^2 + z^2)^p} dx = \\ = 2^{-p-1/2} \sqrt{\pi} \Gamma(1-p) \left(\frac{z}{p} \right)^{1/2-p} [H_{1/2-p}(pz) - Y_{1/2-p}(pz)] \\ [\operatorname{Re} z, \operatorname{Re} p > 0].$$

$$7. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x^2 + z^2)^p} dx = 2^{3/2-p} \sqrt{\pi} \Gamma(p) \left(\frac{\lambda}{z} \right)^{p-1/2} K_{p-1/2}(\lambda z) \\ d\lambda, \operatorname{Re} z > 0; 0 < \operatorname{Re} p < 1/2.$$

$$8. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + z^2} dx = \frac{\pi}{z} e^{-|\lambda|z} \\ [\operatorname{Re} z > 0; \operatorname{Im} \lambda = 0].$$

$$9. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + z^2} dx = \frac{\pi}{z^2} e^{-z|\lambda|/2} \sin\left(\frac{z|\lambda|}{\sqrt{2}} + \frac{\pi}{4}\right) \quad [\operatorname{Re} z > 0; \operatorname{Im} \lambda = 0].$$

$$10. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^3 + z^3} dx = \frac{\pi}{3z^3} e^{-z|\lambda|} + \frac{2\pi}{3z^3} e^{-z|\lambda|/2} \sin\left(z|\lambda| \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) \quad [\operatorname{Re} z > 0; \operatorname{Im} \lambda = 0].$$

2.3.6. Интегралы от $x^\alpha (x \pm a)^\rho e^{-px}$.

Условие: $a > 0$.

$$1. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} e^{-px} dx = B(\alpha, \beta) a^{\alpha+\beta-1} {}_1F_1(\alpha; \alpha+\beta; -ap) \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$2. \int_0^a x^{\alpha-1} (a-x)^{\alpha-1} e^{-px} dx = \sqrt{\pi} \Gamma(\alpha) (a/p)^{\alpha-1/2} e^{-ap/2} I_{\alpha-1/2}(ap/2) \quad [\operatorname{Re} \alpha > 0].$$

$$3. \int_0^a \sqrt{\frac{x}{a-x}} e^{-px} dx = \frac{\pi a}{2} e^{-ap/2} \left[I_0\left(\frac{ap}{2}\right) - I_1\left(\frac{ap}{2}\right) \right].$$

$$4. \int_0^a \frac{e^{-px}}{\sqrt{x(x+a)}} dx = \frac{\pi}{2\sqrt{a}} e^{ap} [1 - \operatorname{erfc}^2(\sqrt{ap})].$$

$$5. \int_0^1 \frac{xe^x}{(x+1)^2} dx = \frac{e}{2} - 1.$$

$$6. \int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} e^{-px} dx = \Gamma(\beta) a^{\alpha+\beta-1} e^{-ap} \Psi(\beta, \alpha+\beta; ap) \quad [\operatorname{Re} \beta, \operatorname{Re} p > 0].$$

$$7. \int_a^\infty x^{\alpha-1} (x-a)^{\alpha-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{p}\right)^{\alpha-1/2} \Gamma(\alpha) e^{-ap/2} K_{\alpha-1/2}\left(\frac{ap}{2}\right) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$8. \int_a^\infty \frac{e^{-px}}{\sqrt{x-a}} \frac{dx}{x} = \frac{\pi}{2\sqrt{a}} \operatorname{erfc}^2(\sqrt{ap}) \quad [\operatorname{Re} p > 0].$$

$$9. \int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} e^{-px} dx = \Gamma(\alpha) z^{\alpha-\rho} \Psi(\alpha, \alpha+1-\rho; pz) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$10. \int_0^\infty x^{\alpha-1} (x+z)^{\alpha-1} e^{-px} dx = \frac{1}{\sqrt{\pi}} \Gamma(\alpha) (p/z)^{1/2-\alpha} e^{p^2/2} K_{\alpha-1/2}(p^2/2) \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$11. \int_0^{\infty} \sqrt{\frac{x}{x+z}} e^{-px} dx = \frac{z}{2} e^{pz/2} \left[K_1\left(\frac{pz}{2}\right) - K_3\left(\frac{pz}{2}\right) \right] \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$12. \int_0^{\infty} x^{\alpha-1} (x+z)^{\pm 1/2 - \alpha} e^{-px} dx = 2^{\alpha} \left\{ \frac{(2\pi)^{-1/2}}{z^{-1/2}} \right\} \Gamma(\alpha) e^{pz/2} D_{\delta-2\alpha}(\sqrt{2pz})$$

$$[\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi; \delta = \left\{ \frac{1}{2} \right\}].$$

$$13. \int_0^{\infty} \frac{x^{\alpha-1}}{x+z} e^{-px} dx = \Gamma(\alpha) z^{\alpha-1} e^{pz} \Gamma(1-\alpha, pz)$$

$$[|\arg z| < \pi; \operatorname{Re} \alpha, \operatorname{Re} p > 0 \text{ (или } \operatorname{Re} p = 0, 0 < \operatorname{Re} \alpha < 2)].$$

$$14. \int_0^{\infty} \frac{x^n}{x+z} e^{-px} dx = (-1)^{n+1} z^n e^{pz} \operatorname{Ei}(-pz) + \sum_{k=1}^n (k-1)! (-z)^{n-k} p^{-k}$$

$$[\operatorname{Re} p > 0].$$

$$15. \int_0^{\infty} \frac{x^{n-1/2}}{x+z} e^{-px} dx = (-1)^n \pi z^{n-1/2} e^{pz} \operatorname{erfc}(\sqrt{pz}) +$$

$$+ 2^{1-n} \sqrt{\pi} p^{1/2-n} \sum_{k=0}^{n-1} (2n-2k-3)!! (-2pz)^k \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1}}{x-y} e^{-px} dx = \Gamma(\alpha-1) p^{1-\alpha} {}_1F_1(1-\alpha; -py) - \pi y^{\alpha-1} \operatorname{ctg} \alpha \pi e^{-py}$$

$$[y, \operatorname{Re} \alpha, \operatorname{Re} p > 0; \alpha \neq 1].$$

$$17. \int_0^{\infty} x^{\alpha-1} (x \mp iy)^{\alpha-1} e^{-px} dx = \pm \frac{i\sqrt{\pi}}{2} \Gamma(\alpha) \left(\frac{p}{y} \right)^{1/2-\alpha} e^{\mp i p y/2} \times$$

$$\times H_{\alpha-1/2}^{(1/2 \mp 1/2)} \left(\frac{p y}{2} \right) \quad [y, \operatorname{Re} \alpha, \operatorname{Re} p > 0].$$

$$18. \int_{-\infty}^{\infty} (z-ix)^{-\rho} (y-ix)^{-\sigma} e^{i\lambda x} dx =$$

$$= 2\pi \frac{(-\lambda)^{\rho+\sigma-1}}{\Gamma(\rho+\sigma)} e^{i\lambda z} {}_1F_1(\sigma; \rho+\sigma; -\lambda(z-y)) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left[\begin{matrix} \lambda > 0 \\ \lambda < 0 \end{matrix} \right]; \operatorname{Re} \rho, \operatorname{Re} z > 0; \operatorname{Re}(\rho+\sigma) > 1].$$

$$19. \int_{-\infty}^{\infty} (z+ix)^{-\rho} (y-ix)^{-\sigma} e^{i\lambda x} dx =$$

$$= \operatorname{sgn} \lambda \frac{2\pi}{\Gamma(\delta)} (y+z)^{1-\rho-\sigma} e^{-|\lambda|z} \Psi(1-\delta, 2-\rho-\sigma; |\lambda|(y+z))$$

$$[\operatorname{Re} \rho, \operatorname{Re} z > 0, \operatorname{Re}(\rho+\sigma) > 1; \delta = \sigma \text{ при } \lambda < 0, \delta = \rho \text{ при } \lambda > 0].$$

2.3.7. Интегралы от $x^\alpha (x^2 \pm a^2)^\beta e^{px}$ и $x^\alpha (a^2 - x^2)^\beta e^{px}$.

Условие: $a > 0$.

1. $\int_0^a x^{\alpha-1} (a^2 - x^2)^{\beta-1} e^{-px} dx =$

$$= \frac{1}{2} a^{\alpha+2\beta-2} B\left(\frac{\alpha}{2}, \beta\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, \frac{\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) -$$

$$- \frac{1}{2} p a^{\alpha+2\beta-1} B\left(\beta, \frac{\alpha+1}{2}\right) {}_1F_2\left(\frac{\alpha+1}{2}; \frac{3}{2}, \beta + \frac{\alpha+1}{2}; \frac{a^2 p^2}{4}\right) \quad |\operatorname{Re} \alpha, \operatorname{Re} \beta > 0|$$
2. $\int_0^a \frac{x^{\alpha-1/2}}{\sqrt{a^2 - x^2}} e^{-px} dx = \frac{(-1)^n \sqrt{\pi}}{2} \frac{\partial^n}{\partial p^n} \left\{ \sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) \left[I_{1/4}\left(\frac{ap}{2}\right) + I_{-1/4}\left(\frac{ap}{2}\right) \right] \right\},$
3. $\int_a^\infty x^{\alpha-1} (x^2 - a^2)^{\beta-1} e^{-px} dx =$

$$= \frac{1}{2} a^{\alpha+2\beta-2} B\left(\beta, 1 - \beta - \frac{\alpha}{2}\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, \frac{\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) -$$

$$- \frac{1}{2} p a^{\alpha+2\beta-1} B\left(\beta, \frac{1-\alpha}{2} - \beta\right) {}_1F_2\left(\frac{1+\alpha}{2}; \frac{3}{2}, \frac{1+\alpha}{2} + \beta; \frac{a^2 p^2}{4}\right) +$$

$$+ (2p)^{\frac{1}{2}-\alpha-2\beta} \Gamma(\alpha+2\beta-2) {}_1F_2\left(1-\beta; 2-\beta-\frac{\alpha}{2}, \frac{3-\alpha}{2}-\beta; \frac{a^2 p^2}{4}\right) \quad |\operatorname{Re} \beta, \operatorname{Re} p > 0|$$
4. $\int_a^\infty x^{\alpha+1} (x^2 - a^2)^{\beta-1} e^{-px} dx =$

$$= (-1)^n \frac{(2a)^{\beta+1/2}}{2\sqrt{\pi}} \Gamma(\beta) \frac{\partial^n}{\partial p^n} \left[p^{1/2-\beta} K_{\beta+1/2}(ap) \right] \quad |\operatorname{Re} \beta, \operatorname{Re} p > 0|$$
5. $\int_a^\infty \frac{x^{\alpha-3/2}}{\sqrt{x^2 - a^2}} e^{-px} dx = \frac{(-1)^n}{\sqrt{2\pi}} \frac{\partial^n}{\partial p^n} \left\{ p^{1/2} \left[K_{3/4}\left(\frac{ap}{2}\right) - K_{1/4}\left(\frac{ap}{2}\right) \right] \right\}$

$$|\operatorname{Re} p > 0|$$
6.
$$= \frac{(-1)^{n-1}}{\sqrt{2\pi}} \frac{\partial^{n-1}}{\partial p^{n-1}} \left[\sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) \right] \quad |\operatorname{Re} p > 0; n=1, 2, 3, \dots|$$
7.
$$= \frac{(-1)^n a}{\sqrt{2\pi}} \frac{\partial^{n-2}}{\partial p^{n-2}} \left[\sqrt{p} K_{1/4}\left(\frac{ap}{2}\right) K_{3/4}\left(\frac{ap}{2}\right) \right] \quad |\operatorname{Re} p > 0; n=2, 3, 4, \dots|$$
8. $\int_0^\infty \frac{x^{\alpha-1}}{(x^2 + z^2)^\mu} e^{-px} dx =$

$$= p^{2\mu-\alpha} \Gamma(\alpha-2\mu) {}_1F_2\left(\mu; 1+\mu-\frac{\alpha}{2}, \frac{1-\alpha}{2} + \mu; -\frac{p^2 z^2}{4}\right) +$$

$$+ \frac{z^{\alpha-2\mu}}{2} B\left(\mu-\frac{\alpha}{2}, \frac{\alpha}{2}\right) {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, 1+\frac{\alpha}{2}-\mu; -\frac{p^2 z^2}{4}\right) -$$

$$- \frac{p}{2} z^{1+\alpha-2\mu} B\left(\mu-\frac{\alpha+1}{2}, \frac{\alpha+1}{2}\right) {}_1F_2\left(\frac{\alpha+1}{2}; \frac{3}{2}, \frac{3+\alpha}{2}-\mu; -\frac{p^2 z^2}{4}\right)$$

$$|\operatorname{Re} z, \operatorname{Re} \alpha > 0, \operatorname{Re} \mu \geq 0; \text{ если } \operatorname{Re} p = 0, \text{ то добавляется условие } \operatorname{Re}(\alpha-2\mu) < 1|$$

$$9. \int_0^{\infty} \frac{x^{\alpha-1} e^{-\rho x}}{(x^2+z^2)^n} dx = \frac{(-2)^{-n}}{(n-1)!} \Gamma(\alpha-1) \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{\alpha-2} [e^{i\rho z} + \pi i^{1/2} \Gamma(2-\alpha, i\rho z) + e^{-i\rho z} - \pi i^{1/2} \Gamma(2-\alpha, -i\rho z)] \} \\ [\operatorname{Re} \alpha, \operatorname{Re} \rho, \operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

$$10. \int_0^{\infty} \frac{x^{k-1/2} e^{-\rho x}}{(x^2+z^2)^n} dx = \frac{(-1)^{k+n-1} 2^{3/2-n} \pi}{(n-1)!} \frac{\partial^k}{\partial \rho^k} \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{-3/2} [\cos(\rho z) (1/2 - S(\rho z)) - \sin(\rho z) (1/2 - C(\rho z))] \} \\ [\operatorname{Re} \rho, \operatorname{Re} z > 0; k = 0, 1, 2, \dots; n = 1, 2, 3, \dots]$$

$$11. \int_0^{\infty} \frac{x^m e^{-\rho x}}{(x^2+z^2)^n} dx = \frac{(-1)^{m+n-1} 2^{1-n}}{(n-1)!} \frac{\partial^m}{\partial \rho^m} \left(\frac{\partial}{z \partial z} \right)^{n-1} \times \\ \times \{ z^{-1} [\sin(\rho z) \operatorname{ci}(\rho z) - \cos(\rho z) \operatorname{si}(\rho z)] \} \quad [\operatorname{Re} \rho, \operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

$$12. \int_0^{\infty} \frac{x^{m-1/2} e^{-\rho x}}{(x^2+z^2)^{n+1/2}} dx = \frac{(-1)^{m+n} \pi^{3/2}}{4(2n-1)!} \frac{\partial^m}{\partial \rho^m} \left(\frac{\partial}{z \partial z} \right)^n \times \\ \times \left\{ V \rho \left[J_{1/4} \left(\frac{\rho z}{2} \right) + Y_{1/4} \left(\frac{\rho z}{2} \right) \right] \right\} \quad [\operatorname{Re} \rho, \operatorname{Re} z > 0]$$

$$13. \int_0^{\infty} \frac{x^m e^{-\rho x}}{x^2+z^2} dx = z^{m-1} \left[\sin \left(\rho z - \frac{\pi m}{2} \right) \operatorname{ci}(\rho z) - \right. \\ \left. - \cos \left(\rho z - \frac{\pi m}{2} \right) \operatorname{si}(\rho z) \right] + \rho^{1-m} \sum_{k=1}^{[m/2]} (m-2k)! (-\rho^2 z^2)^{k-1} \\ [\operatorname{Re} \rho, \operatorname{Re} z > 0]$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} e^{-\rho x}}{x^2-y^2} dx = \Gamma(\alpha-2) \rho^{2-\alpha} {}_1F_2 \left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}, \frac{\rho^2 y^2}{4} \right) - \\ - \frac{y^{\alpha-2} \pi}{2 \sin \alpha \pi} (e^{\rho y} + \cos \alpha \pi e^{-\rho y}) \quad [y, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0]$$

$$15. \int_0^{\infty} \frac{x^m e^{-\rho x}}{x^2-y^2} dx = \frac{y^{m-1}}{2} [(-1)^m e^{\rho y} \operatorname{Ei}(-\rho y) - e^{-\rho y} \operatorname{Ei}(\rho y)] + \\ + \rho^{1-m} \sum_{k=1}^{[m/2]} (m-2k)! (\rho y)^{2k-1} \quad [y, \operatorname{Re} \rho > 0]$$

$$16. \int_0^{\infty} \frac{x^m e^{-\rho x}}{x^4-y^4} dx = -\frac{y^{m-3}}{4} \left\{ e^{-\rho y} \operatorname{Ei}(\rho y) + (-1)^{m+1} e^{\rho y} \operatorname{Ei}(-\rho y) + \right. \\ \left. + 2 \operatorname{ci}(\rho y) \sin \left(\rho y - \frac{\pi m}{2} \right) - 2 \operatorname{si}(\rho y) \cos \left(\rho y - \frac{\pi m}{2} \right) \right\} + \\ + \rho^{3-m} \sum_{k=1}^{[m/4]} (m-4k)! (\rho y)^{4k-4} \quad [y, \operatorname{Re} \rho > 0]$$

$$17. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{x^{\alpha-1} e^{i\lambda x}}{x^2+z^2} dx = \pi i z^{\alpha-2} e^{\lambda z} - \pi \alpha i / 2 \quad [\gamma, \operatorname{Re} z > 0; \lambda < 0; 0 < \operatorname{Re} \alpha < 2]$$

2.3.8. Интегралы от $x^\alpha \prod_k (z_k + ix)^{\alpha_k} e^{-\lambda x}$.

$$1. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} (x+z)^{-\rho} e^{-\rho x} dx = B(\alpha, \beta) z^{-\rho} a^{\alpha+\beta-1} \times \\ \times \Phi_1(\alpha, \rho, \alpha+\beta, -a/z, \rho) \quad [\alpha, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(1+a/z)| < \pi]$$

$$2. \int_0^{\infty} x^{\alpha-1} (x+z)^{-\alpha} (2x+z)^{-1/2} e^{-\rho x} dx = z^{-1/2} e^{\rho z/2} \Gamma(\alpha) [D_\alpha(\sqrt{\rho z})]^2 \\ [\alpha, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0; |\arg z| < \pi]$$

$$3. \int_0^{\infty} \frac{x^m e^{-\rho x}}{y^3 + y^2 x + y x^2 + x^3} dx = \frac{y^{m-2}}{2} \left\{ \operatorname{ci}(\rho |y|) \left[\sin\left(\rho y - \frac{\pi m}{2}\right) + \cos\left(\rho y - \frac{\pi m}{2}\right) \right] + \right. \\ \left. + \operatorname{sgn} y \operatorname{si}(\rho |y|) \left[\sin\left(\rho y - \frac{\pi m}{2}\right) - \cos\left(\rho y - \frac{\pi m}{2}\right) \right] - \right. \\ \left. - (-1)^m e^{\rho y} \operatorname{Ei}(-\rho y) \right\} \quad [\operatorname{Re} \rho > 0; \operatorname{Im} y = 0]$$

$$4. \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(z^2+x^2)(y \pm ix)^\sigma} dx = \frac{\pi}{z} e^{\pm \lambda z} (y \pm z)^{-\sigma} \\ [\operatorname{Re} y, \operatorname{Re} z > 0; \operatorname{Re} \sigma > -1; \left\{ \begin{array}{l} \lambda > 0 \\ \lambda < 0, y \neq z \end{array} \right\}]$$

$$5. \int_{-\infty}^{\infty} \left(\frac{iy-x}{iy+x} \right)^n \frac{e^{-i\lambda x}}{x^2+y^2} dx = (-1)^{n-1} 2\pi \lambda e^{-\lambda y} L_{n-1}^{(1)}(2\lambda y) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \left[\left\{ \begin{array}{l} \lambda > 0 \\ \lambda < 0 \end{array} \right\}; y > 0; n = 1, 2, 3, \dots \right]$$

$$6. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{(ix)^\alpha}{ix-z} \prod_{k=1}^n (z_k+ix)^{\alpha_k} e^{-i\lambda x} dx = -2\pi e^{-\lambda z} z^\alpha \prod_{k=1}^n (z+z_k)^{\alpha_k} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \left[\left\{ \begin{array}{l} \operatorname{Re} z > 0 \\ \operatorname{Re} z < 0 \end{array} \right\}; \lambda, \gamma, \operatorname{Re} z_k > 0; k = 1, 2, \dots, n; \operatorname{Re} \alpha > -1; \operatorname{Re} \left(\alpha + \sum_{k=1}^n \alpha_k \right) < 1 \right]$$

$$7. \int_{-\infty-i\gamma}^{\infty-i\gamma} \frac{(ix)^\alpha}{x^2+z^2} \prod_{k=1}^n (z_k+ix)^{\alpha_k} e^{-i\lambda x} dx = \pi e^{-\lambda z} z^{\alpha-1} \prod_{k=1}^n (z+z_k)^{\alpha_k} \\ \left[\lambda, \gamma, \operatorname{Re} z, \operatorname{Re} z_k > 0; k = 1, 2, \dots, n; \operatorname{Re} \left(\alpha + \sum_{k=1}^n \alpha_k \right) < 2; \operatorname{Re} \alpha > -1 \right]$$

2.3.9. Интегралы от $x^\alpha (\sqrt{x \pm z} + ax^1 + bz^1)^p e^{-px}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x+z} \pm \sqrt{z})^v}{(x+z)^r} e^{-px} dx = 2^{2r-\alpha} z^{\alpha+v/2-r} (\mp v)^{1-2r} \times \\ \times \Gamma \left[\frac{\alpha}{1+\alpha-r-\frac{1}{4}v/2} \right] {}_2F_2 \left(\alpha, \alpha + \frac{1}{2}; 1+\alpha \pm \frac{v}{2}-r, 1+\alpha \mp \frac{v}{2}-r; pz \right) + \\ + 2^{v/2} (v \mp v)^{1/2} p^{r-\alpha \mp v/2} {}_2F_2 \left(r \mp \frac{v}{2}, \frac{1 \mp v}{2} + r, 1 \mp v, 1+r-\alpha \mp \frac{v}{2}; pz \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$2. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x+z} \pm \sqrt{z})^v}{(x+z)^r} e^{-px} dx = (\pm v)^{1-2r} p^{(1-v)/2-\alpha} z^{1/2-r} \Gamma \left(\alpha + \frac{v-1}{2} \right) \times \\ \times {}_2F_2 \left(\frac{1+v}{2}, \frac{1-v}{2}, \frac{3}{2}-2r, \frac{3-v}{2}-\alpha; pz \right) + (\pm v)^{2r} p^{2r-\alpha-v/2} z^r \Gamma \left(\alpha + \frac{v}{2}-2r \right) \times \\ \times {}_2F_2 \left(2r + \frac{v}{2}, 2r - \frac{v}{2}; \frac{1}{2} + 2r, 1+2r - \frac{v}{2}-\alpha; pz \right) + \\ + 2^{2\alpha+v/2+\alpha-r} (\mp v)^{1-2r} \Gamma \left[\frac{\alpha}{1-\alpha-(v \pm v)/2} \right] \times \\ \times {}_2F_2 \left(\frac{v \pm v}{2} + \alpha, \alpha; \frac{1+v}{2} + \alpha, 1+\alpha + \frac{v}{2}-2r; pz \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} p, \operatorname{Re}(2\alpha + v \mp v) > 0; |\arg z| < \pi].$$

$$3. \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x+z}} (\sqrt{x+z} + \sqrt{z})^{1-\alpha} e^{-px} dx = 2^{1-\alpha/2} p^{-\alpha/2} \Gamma(\alpha) e^{pz/2} D_{-\alpha}(\sqrt{2pz}) \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

$$4. \int_0^\infty \sqrt{\frac{\sqrt{x+z} + \sqrt{z}}{x(x+z)}} e^{-px} dx = \sqrt{2} z^{1/4} e^{pz/2} K_{1/4} \left(\frac{pz}{2} \right) [\operatorname{Re} p > 0; |\arg z| < \pi].$$

2.3.10. Интегралы от $x^\alpha (\sqrt{x^2+z^2} + ax + bz)^p e^{-px}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2+z^2} \pm x)^v}{(x^2+z^2)^r} e^{-px} dx = \\ = 2^{2r-\alpha-1} (\mp v)^{1-2r} z^{\alpha+v/2-r} \Gamma \left[\frac{\alpha}{1+(\alpha \mp v)/2-r} \right] \times \\ \times {}_2F_2 \left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}, 1+\frac{\alpha \pm v}{2}-r, 1+\frac{\alpha \mp v}{2}-r; -\frac{pz^2}{4} \right) - \\ - 2^{2r-\alpha-3} p^{2r+\alpha+v-2r} (\mp v)^{1-2r} \Gamma \left[\frac{\alpha+1}{(3+\alpha \mp v)/2-r} \right] \times \\ \times {}_2F_2 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2}+1; \frac{3}{2}, \frac{3+\alpha \pm v}{2}-r, \frac{3+\alpha \mp v}{2}-r; -\frac{pz^2}{4} \right) + \\ + 2^{\frac{1}{2}v} p^{2r+v-\alpha} z^{v \mp v} \Gamma(\alpha \pm v-2r) \times \\ \times {}_2F_2 \left(r \mp \frac{v}{2}, r + \frac{1 \mp v}{2}; 1 \mp v, 1+r - \frac{\alpha \pm v}{2}, \frac{1 \mp v - \alpha}{2} + r; -\frac{pz^2}{4} \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} \alpha, \operatorname{Re} p, \operatorname{Re} z > 0].$$

$$2. \int_0^{\infty} \frac{(\sqrt{x^2+z^2}+x)^{\nu}}{\sqrt{x^2+z^2}} e^{-px} dx = \frac{\pi z^{\nu}}{\operatorname{sh} \nu \pi} [J_{-\nu}(pz) - J_{\nu}(pz)] \quad [\operatorname{Re} p, \operatorname{Re} z > 0]$$

$$3. \int_0^{\infty} \frac{(\sqrt{x^2+z^2}+x)^{\nu}}{\sqrt{x} \sqrt{x^2+z^2}} e^{-px} dx = \\ = \left(\frac{\pi}{2}\right)^{3/2} \sqrt{p} z^{\nu} \left[J_{1/4-\nu/2}\left(\frac{pz}{2}\right) Y_{-1/4-\nu/2}\left(\frac{pz}{2}\right) - J_{-1/4-\nu/2}\left(\frac{pz}{2}\right) Y_{1/4-\nu/2}\left(\frac{pz}{2}\right) \right] \\ [\operatorname{Re} p, \operatorname{Re} z > 0]$$

$$4. \int_0^{\infty} x^{\alpha-1} \frac{(\sqrt{x^2+z^2}+x)^{\nu}}{(x^2+z^2)^r} e^{-px} dx = \\ = (\mp \nu)^{1-2r} 2^{\alpha+\nu-1} z^{\alpha+\nu-2r} \Gamma \left[\frac{\alpha/2, 2r-\nu-\alpha}{1-(\nu+\alpha \pm \nu)/2} \right] \times \\ \times {}_3F_2 \left(\frac{\nu \pm \nu + \alpha}{2}, \frac{\alpha}{2}; \frac{1}{2}, \frac{1+\nu+\alpha}{2}, 1+\frac{\nu+\alpha}{2}-2r, -\frac{p^2 z^2}{4} \right) - \\ - 2^{\nu+\alpha} (\mp \nu)^{1-2r} p z^{\alpha+\nu-2r+1} \Gamma \left[\frac{(\alpha+1)/2, 2r-\nu-\alpha-1}{(1-\alpha-\nu \mp \nu)/2} \right] \times \\ \times {}_3F_2 \left(\frac{\alpha+1}{2}, \frac{1+\alpha+\nu \pm \nu}{2}; \frac{3}{2}, 1+\frac{\nu+\alpha}{2}, \frac{3+\nu+\alpha}{2}-2r, -\frac{p^2 z^2}{4} \right) + \\ + p^{1-\alpha-\nu} (\pm z \nu)^{1-2r} \Gamma(\nu+\alpha-1) \times \\ \times {}_3F_2 \left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{3}{2}-2r, 1-\frac{\nu+\alpha}{2}, \frac{3-\alpha-\nu}{2}; -\frac{p^2 z^2}{4} \right) + \\ + p^{\nu-\nu-\alpha} (\pm z \nu)^{2r} \Gamma(\nu+\alpha-4r) \times \\ \times {}_3F_2 \left(\frac{\nu}{2}+2r, 2r-\frac{\nu}{2}; 1-\frac{\nu+\alpha}{2}+2r, \frac{1-\nu-\alpha}{2}+2r, \frac{1}{2}+2r, -\frac{p^2 z^2}{4} \right) \\ [r=0 \text{ или } 1/2; \operatorname{Re} p, \operatorname{Re} z, \operatorname{Re}(\alpha+\nu \mp \nu) > 0]$$

$$5. \int_{-\infty}^{\infty} \frac{(\sqrt{x^2+z^2}+x)^{\nu}}{\sqrt{x^2+z^2}} e^{\pm i \lambda x} dx = 4 z^{\nu} e^{\pm i \nu \pi/2} K_{\nu}(\lambda z) \quad [\lambda, \operatorname{Re} z > 0; |\operatorname{Re} \nu| < 1]$$

2.3.11. Интегралы от $x^{\alpha} [(\sqrt{ax^2+b} \pm \sqrt{cx^2+d})^{\rho} \pm$
 $\pm (\sqrt{ax^2+b} \mp \sqrt{cx^2+d})^{\rho}] e^{-px}$.

$$1. \int_0^a x^{\alpha-1} \frac{(\sqrt{a} + \sqrt{a-x})^{\nu} - (-1)^{2r} (\sqrt{a} - \sqrt{a-x})^{\nu}}{(a-x)^r} e^{-px} dx = \\ = 2^{2\alpha+\nu} a^{\nu/2+\alpha-r} \nu^{1-2r} \Gamma \left[\frac{\alpha, \nu+\alpha}{\nu+2\alpha-2r+1} \right] \times \\ \times {}_3F_2 \left(\alpha, \nu+\alpha; \frac{\nu+1}{2} + \alpha, \frac{\nu}{2} + \alpha + 1 - 2r, -a\rho \right) \\ [r=0 \text{ или } 1/2; a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+\nu) > 0]$$

$$\begin{aligned}
 2. \int_0^a x^{\alpha-1} \frac{(\sqrt{a+x} + \sqrt{a-x})^{\nu} - (-1)^{2r} (\sqrt{a+x} - \sqrt{a-x})^{\nu}}{(a^2 - x^2)^r} e^{-px} dx = \\
 = 2^{\nu/2} \int_0^a x^{\alpha-1} \frac{(a + \sqrt{a^2 - x^2})^{\nu/2} - (-1)^{2r} (a - \sqrt{a^2 - x^2})^{\nu/2}}{(a^2 - x^2)^r} e^{-px} dx \\
 [r=0 \text{ или } 1/2; \text{ см. 2.3.11.3}].
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^a x^{\alpha-1} \frac{(a + \sqrt{a^2 - x^2})^{\nu} - (-1)^{2r} (a - \sqrt{a^2 - x^2})^{\nu}}{(a^2 - x^2)^r} e^{-px} dx = \\
 = 2^{\alpha+\nu-1} \nu!^{-2r} a^{\alpha+\nu-2r} \Gamma \left[\frac{\alpha/2}{\nu+\alpha+1-2r} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha}{2}, \nu + \frac{\alpha}{2}; \frac{1}{2}, \frac{\nu+\alpha+1}{2}, \frac{\nu+\alpha}{2} + 1 - 2r; \frac{a^2 p^2}{4} \right) - \\
 - 2^{\nu+\alpha} p \nu!^{-2r} a^{1+\nu-2r} \Gamma \left[\frac{(\alpha+1)/2}{\alpha+\nu+2-2r} \right] \times \\
 \times {}_2F_3 \left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2} + \nu; \frac{3}{2}, \frac{\nu+\alpha}{2} + 1, \frac{\nu+\alpha+3}{2} - 2r; \frac{a^2 p^2}{4} \right) \\
 [r=0 \text{ или } 1/2; a, \operatorname{Re} \alpha, \operatorname{Re}(\alpha+\nu) > 0].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_a^{\infty} x^{\alpha-1} \frac{(\sqrt{x+a} + \sqrt{x-a})^{\nu} - (-1)^{2r} (\sqrt{x+a} - \sqrt{x-a})^{\nu}}{(x-a)^r} e^{-px} dx = \\
 = \nu!^{-2r} 2^{2r-2\alpha} a^{\nu/2+\alpha-r} \Gamma \left[\frac{r-\alpha+\nu/2}{1-2\alpha} \right] \times \\
 \times {}_2F_3 \left(\alpha, \frac{1}{2} + \alpha; 1 + \alpha - r - \frac{\nu}{2}, 1 + \alpha + \frac{\nu}{2} - r; -ap \right) + \\
 + (-1)^{1-2r} a^{\nu/2-\nu} p^{r+\nu/2-\alpha} \Gamma \left(\alpha - r - \frac{\nu}{2} \right) \times \\
 \times {}_2F_3 \left(r + \frac{\nu}{2}, \frac{1+\nu}{2} + r; 1 + \nu, 1 + r + \frac{\nu}{2} - \alpha; -ap \right) + \\
 + (-1)^{1-2r} a^{-\nu} 2^{\nu} p^{r-\nu/2-\alpha} \Gamma \left(\alpha + \frac{\nu}{2} - r \right) \times \\
 \times {}_2F_3 \left(r - \frac{\nu}{2}, \frac{1-\nu}{2} + r; 1 - \nu, 1 + r - \frac{\nu}{2} - \alpha; -ap \right) \\
 [r=0 \text{ или } 1/2; a, \operatorname{Re} p > 0].
 \end{aligned}$$

$$\begin{aligned}
 5. \int_a^{\infty} x^{\alpha-1} \frac{(\sqrt{x+a} + \sqrt{x-a})^{\nu} - (-1)^{2r} (\sqrt{x+a} - \sqrt{x-a})^{\nu}}{(x^2 - a^2)^r} e^{-px} dx = \\
 = 2^{\nu/2} \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu/2} - (-1)^{2r} (x - \sqrt{x^2 - a^2})^{\nu/2}}{(x^2 - a^2)^r} e^{-px} dx \\
 [r=0 \text{ или } 1/2; \text{ см. 2.3.11.7}].
 \end{aligned}$$

$$6. \int_a^{\infty} [(\sqrt{x+a} + \sqrt{x-a})^{\nu} - (\sqrt{x+a} - \sqrt{x-a})^{\nu}] e^{-px} dx = \\ = \left(\frac{\nu}{p}\right) (2a)^{\nu/2} K_{\nu/2}(ap) \quad [a, \operatorname{Re} p > 0]$$

$$7. \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu} - (-1)^{\nu} (x - \sqrt{x^2 - a^2})^{\nu}}{(x^2 - a^2)^r} e^{-px} dx = \\ = 2^{2\nu-1} a^{\alpha+\nu} 2^{\nu} \nu^{1-2r} \Gamma \left[\frac{r + (\nu - \alpha)/2, r - (\nu + \alpha)/2}{1-\alpha} \right] \times \\ \times {}_2F_2 \left(\frac{\alpha}{2}, \frac{1+\alpha}{2}; \frac{1}{2}, 1 + \frac{\alpha - \nu}{2} - r, 1 + \frac{\nu + \alpha}{2} - r; \frac{a^2 p^2}{4} \right) - \\ - 2^{2\nu-\alpha} a^{2\nu-2r} \rho a^{\alpha+\nu+1-2r} \Gamma \left[\frac{r + (\nu - \alpha - 1)/2, r - (\nu + \alpha + 1)/2}{-\alpha} \right] \times \\ \times {}_2F_2 \left(\frac{1+\alpha}{2}, 1 + \frac{\alpha}{2}; \frac{3}{2}, \frac{3+\alpha-\nu}{2} - r, \frac{3+\alpha+\nu}{2} - r; \frac{a^2 p^2}{4} \right) + \\ + (-1)^{1-2\nu} a^{2\nu} p^{2r+\nu} \alpha \Gamma(\alpha - \nu - 2r) \times \\ \times {}_2F_2 \left(r + \frac{\nu}{2}, \frac{1+\nu}{2} + r; 1 + \nu, 1 + r + \frac{\nu - \alpha}{2}, \frac{1+\nu - \alpha}{2} + r; \frac{a^2 p^2}{4} \right) + \\ + (-1)^{1-2\nu} a^{2\nu} p^{2r-\nu} \alpha \Gamma(\alpha + \nu - 2r) \times \\ \times {}_2F_2 \left(r - \frac{\nu}{2}, \frac{1-\nu}{2} + r; 1 - \nu, 1 + r - \frac{\nu + \alpha}{2}, \frac{1-\nu - \alpha}{2} + r; \frac{a^2 p^2}{4} \right) \\ [r=0 \text{ или } 1/2; a, \operatorname{Re} p > 0]$$

$$8. \int_a^{\infty} \frac{(x + \sqrt{x^2 - a^2})^{\nu} + (x - \sqrt{x^2 - a^2})^{\nu}}{\sqrt{x} \sqrt{x^2 - a^2}} e^{-px} dx = \\ = \sqrt{\frac{2p}{\pi}} a^{\nu} K_{\nu/2+1/4} \left(\frac{ap}{2} \right) K_{\nu/2-1/4} \left(\frac{ap}{2} \right) \quad [a, \operatorname{Re} p > 0]$$

$$9. \int_a^{\infty} \frac{(x + \sqrt{x^2 - a^2})^{\nu} + (x - \sqrt{x^2 - a^2})^{\nu}}{\sqrt{x^2 - a^2}} e^{-px} dx = 2a^{\nu} K_{\nu}(ap) \quad [a, \operatorname{Re} p > 0]$$

$$10. \int_a^{\infty} [(\sqrt{x + \sqrt{x^2 - a^2}})^{\nu} - (x - \sqrt{x^2 - a^2})^{\nu}] e^{-px} dx = 2p^{-1} \nu a^{\nu} K_{\nu}(ap) \\ [a, \operatorname{Re} p > 0]$$

$$11. \int_0^{\infty} x^{-r} \frac{(\sqrt{x+z} + \sqrt{x})^{\nu} - (-1)^{\nu} (\sqrt{x+z} - \sqrt{x})^{\nu}}{(x+z)^{1-\alpha}} e^{-px} dx = \\ = e^{pz} \int_a^{\infty} x^{\alpha-1} \frac{(\sqrt{x} + \sqrt{x-a})^{\nu} - (-1)^{\nu} (\sqrt{x} - \sqrt{x-a})^{\nu}}{(x-a)^r} e^{-px} dx \quad [r=0 \text{ или } 1/2; \operatorname{Re} p > 0; |\arg z| < \pi, \text{ см. 2.3.11.4}]$$

$$12. \int_0^{\infty} [(\sqrt{x+z} + \sqrt{x})^{\nu} - (\sqrt{x+z} - \sqrt{x})^{\nu}] e^{-px} dx = \\ = \nu p^{-1} z^{\nu/2} e^{p^2/2} K_{\nu/2}(p^2/2) \quad [\operatorname{Re} p > 0; |\arg z| < \pi].$$

$$13. \int_0^{\infty} x^{-r} \frac{(\sqrt{x+2z} + \sqrt{x})^{\nu} - (-1)^{2r} (\sqrt{x+2z} - \sqrt{x})^{\nu}}{(x+z)^{1-\alpha} (x+2z)^{-r}} e^{-px} dx = \\ = 2^{\nu/2} e^{p^2} \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu/2} - (-1)^{2r} (x - \sqrt{x^2 - a^2})^{\nu/2}}{(x^2 - a^2)^{-r}} e^{-px} dx \Big|_{a-z} \\ [r=0 \text{ или } 1/2; \operatorname{Re} p > 0; |\arg z| < \pi; \text{ см. 2.3.12.7}].$$

$$14. \int_0^{\infty} x^{-r} \frac{(x+z + \sqrt{x^2+2xz})^{\nu} - (-1)^{2r} (x+z - \sqrt{x^2+2xz})^{\nu}}{(x+z)^{1-\alpha} (x+2z)^{-r}} e^{-px} dx = \\ = e^{p^2} \int_a^{\infty} x^{\alpha-1} \frac{(x + \sqrt{x^2 - a^2})^{\nu} - (-1)^{2r} (x - \sqrt{x^2 - a^2})^{\nu}}{(x^2 - a^2)^{-r}} e^{-px} dx \Big|_{a-z} \\ [r=0 \text{ или } 1/2; \operatorname{Re} p > 0; |\arg z| < \pi; \text{ см. 2.3.12.7}].$$

$$15. \int_0^{\infty} [(\sqrt{x^2+z^2}+x)^{\nu} \pm (\sqrt{x^2+z^2}-x)^{\nu}] e^{-px} dx = \\ = \frac{(2\nu)^{(1\pm 1)/2}}{p} z^{\nu} S_{(1\pm 1)/2, \nu}(p^2) \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$16. \int_0^{\infty} \frac{(\sqrt{x^2+z^2}+x)^{\nu} \pm (\sqrt{x^2+z^2}-x)^{\nu}}{\sqrt{x^2+z^2}} e^{-px} dx = \\ = 2z^{\nu} \nu^{(1\pm 1)/2} S_{(\pm 1-1)/2, \nu}(p^2) \quad [\operatorname{Re} p, \operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} x^{\alpha-1} \frac{(2z^2 - c^2 + xz + c\sqrt{x^2+2zx+c^2})^{1/2} (c + \sqrt{x^2+2zx+c^2})^{1-\alpha}}{(x+2z)\sqrt{x^2+2zx+c^2}} e^{-px} dx = \\ = \sqrt{2} \Gamma(\alpha) e^{p^2} D_{-\alpha}(\sqrt{2p(z + \sqrt{z^2 - c^2})}) D_{-\alpha}(\sqrt{2p(z - \sqrt{z^2 - c^2})}) \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |\arg z| < \pi].$$

2.3.12. Интегралы от $x^{\alpha} (e^{qx} + z)^{\lambda} e^{-px}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(e^{qx} + z)^n} dx = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(k+1)_{n-1} (-z)^k}{(p+qk+qn)^{\alpha}} \\ [\operatorname{Re} q, \operatorname{Re}(p+qn) > 0; |z| \leq 1; z \neq -1, \operatorname{Re} \alpha > 0, \text{ или } z = -1, \operatorname{Re} \alpha > n; n = 1, 2, 3, \dots].$$

$$2. \int_0^{\infty} \frac{x e^{-px}}{(e^{-px} + 1)^2} dx = \frac{\ln 2}{p^2} \quad [p \neq 0].$$

$$3. \int_0^{\infty} \frac{x^{2n} e^{-px}}{(e^{-px} + 1)^2} dx = (-1)^{n-1} p^{-2n-1} (2^{2n-1} - 1) \pi^{2n} B_{2n} \quad [p \neq 0].$$

$$\begin{aligned}
 20. \int_{-\infty}^{\infty} \frac{x^{2m} e^{(m+1/2)x}}{(e^x + z)^{n+1/2}} dx &= \left(\frac{z}{2}\right)^{m-n} \frac{(2m-1)!! (n-m-1)!}{(2n-1)!!} \times \\
 &\times \left[\left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right)^3 - \right. \\
 &\quad \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{n-m-1} \left(\frac{1}{k^3} + \frac{2}{3} \pi^2 \right) \right] \\
 &\quad [|\arg z| < \pi; m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots]
 \end{aligned}$$

$$\begin{aligned}
 21. \int_{-\infty}^{\infty} \frac{x^{2m} e^{mx}}{(e^x + z)^n} dx &= z^{m-n} \frac{(m-1)! (n-m-1)!}{(n-1)!} \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right)^3 + \right. \\
 &\quad + 3 \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=1}^{n-m-1} \frac{1}{k} \right) \left(\frac{\pi^2}{3} - \sum_{k=1}^{m-1} \frac{1}{k^3} - \sum_{k=1}^{n-m-1} \frac{1}{k^3} \right) + \\
 &\quad \left. + 2 \left(\sum_{k=1}^{m-1} \frac{1}{k^3} - \sum_{k=1}^{n-m-1} \frac{1}{k^3} \right) \right] \\
 &\quad [|\arg z| < \pi; m=1, 2, \dots, n-1; n=2, 3, 4, \dots]
 \end{aligned}$$

$$\begin{aligned}
 22. \int_{-\infty}^{\infty} \frac{x^{2m} e^{mx}}{(e^x + z)^{n+1/2}} dx &= 2^m z^{m-n-1/2} \frac{(2n-2m-1)!! (m-1)!}{(2n-1)!!} \times \\
 &\times \left[\left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right)^3 + \right. \\
 &\quad + 3 \left(\ln z + \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-m} \frac{1}{2k-1} + 2 \ln 2 \right) \left(\frac{2}{3} \pi^2 - \sum_{k=1}^{m-1} \frac{1}{k^3} - 4 \sum_{k=1}^{n-m} \frac{1}{(2k-1)^3} \right) + \\
 &\quad \left. + 2 \left(\sum_{k=1}^{m-1} \frac{1}{k^3} - 8 \sum_{k=1}^{n-m} \frac{1}{(2k-1)^3} + 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} \right) \right] \\
 &\quad [|\arg z| < \pi; m=1, 2, \dots, n; n=1, 2, 3, \dots]
 \end{aligned}$$

$$\begin{aligned}
 23. \int_{-\infty}^{\infty} \frac{x^{2m} e^{(m+1/2)x}}{(e^x + z)^{n+1/2}} dx &= \\
 &= \left(\frac{z}{2}\right)^{m-n} \frac{(2m-1)!! (n-m-1)!}{(2n-1)!!} \left[\left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right)^3 + \right. \\
 &\quad \left. + 3 \left(\ln z + 2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{n-m-1} \frac{1}{k} - 2 \ln 2 \right) \times \right.
 \end{aligned}$$

$$\times \left(\frac{2}{3} \pi^2 - 4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{n-m-1} \frac{1}{k^2} \right) + \\ + 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{n-m-1} \frac{1}{k^3} - 16 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \right) \\ [|\arg z| < \pi, m=0, 1, 2, \dots, n-1; n=1, 2, 3, \dots].$$

2.3.13. Интегралы от $x^\alpha (e^x - z)^\lambda e^{-\rho x}$.

$$1. \int_0^{\ln 2} \frac{x^\alpha e^{-\rho x}}{(e^x - 1)^\rho} dx = \frac{1}{1-\rho} \left[\ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2-\rho+k} \right] \quad [\operatorname{Re} \rho < 2, \rho \neq 1].$$

$$2. \int_0^a \frac{x e^{-\rho x}}{e^x - 1} dx = \sum_{k=1}^{\infty} \frac{(1+\rho)_k}{k k!} (1-e^a)^k [\psi(1+\rho) - \psi(1+\rho+k)] \\ [0 < a < \ln 2].$$

$$3. = - (e^a - 1)^{-\rho-1} \sum_{k=0}^{\infty} \frac{(1+\rho)_k}{1+\rho+k} \frac{(1-e^a)^{-k}}{k!} [\psi(1+\rho) + \\ + \frac{1}{1+\rho+k} - \psi(1+\rho+k) + \ln(e^a - 1)] + \psi'(1+\rho) \quad [a > \ln 2].$$

$$4. \int_0^{\ln 2} \frac{x^\alpha e^{-x}}{e^x - 1} dx = \frac{\pi^2}{12}.$$

$$5. \int_a^{\infty} \frac{x^\alpha e^{-x}}{e^{2x} - 1} dx = I(a),$$

$$I[\ln(\sqrt{2}-1)] = \frac{\pi^2}{16} - \frac{1}{4} \ln^2(\sqrt{2}-1),$$

$$I\left(\ln \frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{24} - \frac{1}{12} \ln^2(\sqrt{5}-2).$$

$$6. \int_0^{\infty} \frac{x^n e^{-\rho x}}{(e^x - 1)^\rho} dx = (-1)^n \frac{\partial^n}{\partial \rho^n} B(1-\rho, \rho+\rho) = \\ = (-1)^{n-1} \frac{\partial^{n-1}}{\partial \rho^{n-1}} \{B(1-\rho, \rho+\rho) [\psi(1+\rho) - \psi(\rho+\rho)]\} \\ [\operatorname{Re}(\rho+\rho) > 0; \operatorname{Re} \rho < n+1; n=1, 2, 3, \dots].$$

$$7. \int_0^{\infty} \frac{x^{2n-m} e^{-x}}{(e^x - 1)^{1/2-\alpha}} dx = \frac{\pi}{2^{m-1} \Gamma(1-\alpha)} (2n-1)!! (2m-2n-1)!! \times \\ \times \left(\ln 2 - \sum_{k=1}^{m-n} \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^m \frac{1}{k} \right) \quad [n=0, 1, 2, \dots, m].$$

$$8. \int_0^{\infty} \frac{x e^{-(m-1/2)x}}{(e^x-1)^{1/2-n}} dx = -\frac{(2n-1)!! (m-n-1)!}{(2m-1)!! 2^{n-m-1}} \times$$

$$\times \left(\ln 2 - \sum_{k=1}^m \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^{m-n-1} \frac{1}{k} \right)$$

[$n=0, 1, 2, \dots, m-1$; $m=1, 2, 3, \dots$]

$$9. \int_0^{\infty} \frac{x e^x}{(e^x-1)^{3/2}} dx = 2\pi.$$

$$10. \int_0^{\infty} \frac{x e^x}{(e^x-1)^{(4+k)/2}} dx = 2^{1-k} \sqrt{3} \pi$$

[$k=0$ или 1].

$$11. \int_0^{\infty} \frac{x e^{-mx}}{(e^x-1)^{(4+\varepsilon)/2-n}} dx = \frac{3}{2(m!)} \Gamma\left(n - \frac{1+\varepsilon}{3}\right) \Gamma\left(m-n + \frac{4+\varepsilon}{3}\right) \times$$

$$\times \left[\ln 3 + \frac{(-1)^{\varepsilon} \pi}{3\sqrt{3}} - 2 \sum_{k=1}^{m-n+1} \frac{1}{3k-2+\varepsilon} + \frac{2}{3} \sum_{k=1}^m \frac{1}{k} \right]$$

[$n=0, 1, 2, \dots, m+1$; $\varepsilon=0$ или 1].

$$12. \int_0^{\infty} \frac{x e^{-(m-1/3+\varepsilon/3)x}}{(e^x-1)^{(4+\varepsilon)/2-n}} dx = -\frac{3}{2} \Gamma\left[n - \frac{1/3-\varepsilon/3}{m+2/3-\varepsilon/3}\right] (m-n)! \times$$

$$\times \left[\ln 3 - \frac{(-1)^{\varepsilon} \pi}{3\sqrt{3}} + \frac{2}{3} \sum_{k=1}^{m-n} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{3k-1-\varepsilon} \right]$$

[$n=0, 1, 2, \dots, m$; $\varepsilon=0$ или 1].

$$13. \int_0^{\infty} \frac{x e^{-(m-2/3+\varepsilon/3)x}}{(e^x-1)^{(4+\varepsilon)/2-n}} dx = -B\left(n - \frac{1+\varepsilon}{3}, m-n + \frac{2+2\varepsilon}{3}\right) \times$$

$$\times \left[\frac{(-1)^{\varepsilon} \pi}{\sqrt{3}} + 3(-1)^{\varepsilon} \sum_{k=1}^{m-n+\varepsilon} \frac{1}{3k-1-\varepsilon} - 3 \sum_{k=1}^m \frac{1}{3k-2+\varepsilon} \right]$$

[$n=0, 1, 2, \dots, m+\varepsilon$; $\varepsilon=0$ или 1].

$$14. \int_0^{\infty} \frac{x^{2n-m} dx}{(e^x-1)^{1/2-n}} = \frac{\pi (2n-1)!! (2m-2n-1)!!}{2^m m!} \times$$

$$\times \left[\frac{\pi^2}{3} + \sum_{k=1}^m \frac{1}{k^2} - 4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^2} + \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right]$$

[$n=0, 1, 2, \dots, m$].

$$15. \int_0^{\infty} \frac{x^{2e-(m-1/2)x}}{(e^x-1)^{1/2-n}} dx = -\frac{(2n-1)!!(m-n-1)!}{2^{n-m}(2m-1)!!} \left[\frac{\pi^2}{3} + \sum_{k=1}^{m-n-1} \frac{1}{k^3} - \right. \\ \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} \frac{x^{2e(m+1)x}}{(e^x-1)^{1/2+n}} dx = \\ = (-1)^{m+n-1} 2^{2+m} \pi m! \frac{(2e-2m-3)!!}{(2n-1)!!} \left[2 \ln 2 + \sum_{k=1}^m \frac{1}{k} - 2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} \right] \\ [m=0, n-1; n=1, 2].$$

$$17. \int_0^{\infty} \frac{x^{2e(m+1)x}}{(e^x-1)^{(1+e)/2+n}} dx = 2(-1)^{m-1} m! \Gamma\left(\frac{2-e}{3}-n\right) \Gamma\left(n-m-\frac{2-e}{3}\right) \times \\ \times \left[\frac{3}{2} \ln 3 + \frac{(-1)^e \pi}{2\sqrt{3}} - 3 \sum_{k=1}^{n-m-1} \frac{1}{3k-2+e} + \sum_{k=1}^m \frac{1}{k} \right] \\ [m=0, n-1; n=1, 2; e=0 \text{ или } 1].$$

$$18. \int_0^{\infty} \frac{x^{2e-mx}}{(e^x-1)^{1/2-n}} dx = \frac{\pi(2n-1)!!(2m-2n-1)!!}{m! 2^m} \times \\ \times \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^3} - \sum_{k=1}^m \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^2} - \sum_{k=1}^m \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right) - \right. \\ \left. - \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m].$$

$$19. \int_0^{\infty} \frac{x^{2e-(m-1/2)x}}{(e^x-1)^{1/2-n}} dx = \\ = -\frac{(2n-1)!!(m-n-1)!}{2^{n-m}(2m-1)!!} \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{m-n-1} \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{m-n-1} \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \ln 2 \right) + \right. \\ \left. + \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

2.3.13]

$$15. \int_0^{\infty} \frac{x^2 e^{-(m+1/2)x}}{(e^x - 1)^{1/2-n}} dx = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[\frac{\pi^2}{3} + \sum_{k=1}^{m-n-1} \frac{1}{k^3} - \right. \\ \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} \frac{x^2 e^{(m+1)x}}{(e^x - 1)^{1/2+n}} dx = \\ = (-1)^{m+n-1} \frac{2^{2(m+n)} (2n-1)!!}{(2n-1)!!} \left[2 \ln 2 + \sum_{k=1}^m \frac{1}{k} - 2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} \right] \\ [m=0, n-1; n=1, 2].$$

$$17. \int_0^{\infty} \frac{x^2 e^{(m+1)x}}{(e^x - 1)^{(1+\varepsilon)/2+n}} dx = 2 (-1)^{m-1} m! \Gamma\left(\frac{2-\varepsilon}{3} - n\right) \Gamma\left(n-m-\frac{2-\varepsilon}{3}\right) \times \\ \times \left[\frac{3}{2} \ln 3 + \frac{(-1)^{\varepsilon} \pi}{2\sqrt{3}} - 3 \sum_{k=1}^{n-m-1} \frac{1}{3k-2+\varepsilon} + \sum_{k=1}^m \frac{1}{k} \right] \\ [m=0, n-1; n=1, 2; \varepsilon=0 \text{ или } 1].$$

$$18. \int_0^{\infty} \frac{x^2 e^{-mx}}{(e^x - 1)^{1/2-n}} dx = \frac{\pi (2n-1)!! (2m-2n-1)!!}{m! 2^m} \times \\ \times \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^3} - \sum_{k=1}^m \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^3} - \sum_{k=1}^m \frac{1}{k^3} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right) - \right. \\ \left. - \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^3 \right] [n=0, 1, 2, \dots, m].$$

$$19. \int_0^{\infty} \frac{x^2 e^{-(m-1/2)x}}{(e^x - 1)^{1/2-n}} dx = \\ = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{m-n-1} \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{m-n-1} \frac{1}{k^3} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \ln 2 \right) + \right. \\ \left. + \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^3 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

2.3.13.1

$$15. \int_0^{\infty} \frac{x^{2n-(m-1/2)x}}{(e^x-1)^{1/2-n}} dx = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[\frac{\pi^2}{3} + \sum_{k=1}^{m-n-1} \frac{1}{k^2} - \right. \\ \left. - 4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} \frac{x^{2n-(m+1)x}}{(e^x-1)^{1/2+n}} dx = \\ = (-1)^{m+n-1} 2^{2+m} n! \frac{(2n-2m-3)!!}{(2n-1)!!} \left[2 \ln 2 + \sum_{k=1}^m \frac{1}{k} - 2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} \right] \\ [m=0, n-1; n=1, 2].$$

$$17. \int_0^{\infty} \frac{x^{2n-(m+1)x}}{(e^x-1)^{(1+\varepsilon)/2+n}} dx = 2 (-1)^{m-1} m! \Gamma\left(\frac{2-\varepsilon}{3} - n\right) \Gamma\left(n-m-\frac{2-\varepsilon}{3}\right) \times \\ \times \left[\frac{3}{2} \ln 3 + \frac{(-1)^{\varepsilon} \pi}{2 \sqrt{3}} - 3 \sum_{k=1}^{n-m-1} \frac{1}{3k-2+\varepsilon} + \sum_{k=1}^m \frac{1}{k} \right] \\ [m=0, n-1; n=1, 2; \varepsilon=0 \text{ или } 1].$$

$$18. \int_0^{\infty} \frac{x^{2n-mx}}{(e^x-1)^{1/2-n}} dx = \frac{\pi (2n-1)!! (2m-2n-1)!!}{m! 2^m} \times \\ \times \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^3} - \sum_{k=1}^m \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^{m-n} \frac{1}{(2k-1)^2} - \sum_{k=1}^m \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right) - \right. \\ \left. - \left(2 \sum_{k=1}^{m-n} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m].$$

$$19. \int_0^{\infty} \frac{x^3 e^{-(m-1/2)x}}{(e^x-1)^{1/2-n}} dx = \\ = -\frac{(2n-1)!! (m-n-1)!}{2^{n-m} (2m-1)!!} \left[16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - 2 \left(8 \sum_{k=1}^m \frac{1}{(2k-1)^3} - \sum_{k=1}^{m-n-1} \frac{1}{k^3} \right) + \right. \\ \left. + 3 \left(4 \sum_{k=1}^m \frac{1}{(2k-1)^2} - \sum_{k=1}^{m-n-1} \frac{1}{k^2} - \frac{\pi^2}{3} \right) \left(2 \sum_{k=1}^m \frac{1}{2k-1} - \sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \ln 2 \right) + \right. \\ \left. + \left(\sum_{k=1}^{m-n-1} \frac{1}{k} - 2 \sum_{k=1}^m \frac{1}{2k-1} + 2 \ln 2 \right)^2 \right] \\ [n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

20. $\int_0^{\infty} \frac{x^{2p(m+1)+x}}{(e^x-1)^{1/2+2x}} dx = \frac{(-1)^{m+1} 3\pi m! 2^{m+1}}{(2n-1)!!} (2n-2m-3)!! \times$
 $\times \left[\frac{\pi^2}{3} - 4 \sum_{k=1}^{n-m-1} \frac{1}{(2k-1)^2} + \sum_{j=1}^m \frac{1}{k^2} + \left(2 \sum_{k=1}^{n-m-1} \frac{1}{2k-1} - \sum_{k=1}^m \frac{1}{k} - 2 \ln 2 \right)^2 \right]$
 $[m=0, 1, 2, n-1; n=1, 2, 3].$
21. $\int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{e^{qx}-z} dx = \Gamma(\alpha) q^{-\alpha} \Phi\left(z, \alpha, \frac{p}{q} + 1\right)$
 $[\operatorname{Re}(p/q) > -1; |\arg(1-z)| < \pi, \operatorname{Re} \alpha > 0] \text{ или } |z| < 1, z \neq 1, \operatorname{Re} \alpha > 0 \text{ или } |z|=1, \operatorname{Re} \alpha > 1].$
22. $\int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{e^{qx}-1} dx = \Gamma(\alpha) q^{-\alpha} \zeta\left(\alpha, \frac{p}{q} + 1\right)$
 $[\operatorname{Re} \alpha > 1; \operatorname{Re}(p/q) > -1].$
23. $\int_0^{\infty} \frac{x^n e^{-px}}{e^{qx}-1} dx = (-q)^{-n-1} \psi^{(n)}\left(\frac{p}{q} + 1\right)$
 $[\operatorname{Re}(p/q) > -1; n=1, 2, 3, \dots].$
24. $\int_0^{\infty} \frac{x^{2n-1} e^{qx/2}}{e^{qx}-1} dx = \frac{2^{2n}-1}{4n} \left(\frac{2\pi}{q}\right)^{2n} |B_{2n}|$
 $[q \neq 0; n=1, 2, 3, \dots].$
25. $\int_0^{\infty} \frac{x^{2n-1} e^{-px}}{e^x-1} dx = \frac{\pi^2}{6} - \sum_{k=1}^m \frac{1}{k^2}.$
26. $\int_0^{\infty} \frac{x^{2n-1} e^{-px}}{e^x-1} dx = \frac{\pi^6}{15} - 6 \sum_{k=1}^m \frac{1}{k^4}.$
27. $\int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(e^x-z)^2} dx = \Gamma(\alpha) [\Phi(z, \alpha-1, p+1) - (p+1) \Phi(z, \alpha, p+1)]$
 $[\operatorname{Re} \alpha > 0; \operatorname{Re} p > -2; |\arg(1-z)| < \pi].$
28. $\int_0^{\infty} \frac{x^{2n-1} e^{-px}}{(e^x-1)^2} dx = \Gamma(\alpha) [\zeta(\alpha-1, p+1) - (p+1) \zeta(\alpha, p+1)]$
 $[\operatorname{Re} \alpha > 2; \operatorname{Re} p > -2].$
29. $\int_0^{\infty} \frac{x^{2n} e^{qx}}{(e^x-1)^2} dx = 2^{2n-1} \pi^{2n} |B_{2n}|$
 $[n=1, 2, 3, \dots].$
30. $\int_{-\infty}^{\infty} \frac{x^{2n} e^{-px}}{(e^x-u)^{2n}} dx = \frac{(-1)^{n+m-1} \pi}{(m-1)!} \frac{\partial^m}{\partial p^m} [(p+1)_{m-1} u^{-p-m} \operatorname{ctg} p\pi]$
 $[a > 0; -1 < \operatorname{Re} p < 0; m=1, 3, 5, \dots, n=0, 1, 2, \dots, a \neq 1 \text{ (или } m=1, 3, 5, \dots, m < n+1, n=1, 2, 3, \dots, a=1)]$

2.3.14. Интегралы от $(ae^{qx} + be^{qx} + c)^{-1} f(x)$.
 Условие: γ — действительное число.

1. $\int_0^{\infty} \frac{x dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2.$
2. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^x + 1} dx = (1-2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha)$
 $[\operatorname{Re} \alpha > 0].$
3. $\int_0^{\infty} \frac{dx}{e^x + 1} = \ln 2.$
4. $\int_0^{\infty} \frac{x^{2n-1}}{e^x + 1} dx = (-1)^{n+1} \frac{\pi^{2n}}{2^{2n}} (2^{2n}-1) B_{2n}$
 $[n=1, 2, 3, \dots].$
5. $\int_0^{\infty} \frac{x^{\alpha-1} dx}{e^x - z} = \frac{\Gamma(\alpha)}{z} \operatorname{Li}_{\alpha}(z)$
 $[\operatorname{Re} \alpha > 1; |\arg(1-z)| < \pi].$
6. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^x - 1} dx = \Gamma(\alpha) \zeta(\alpha)$
 $[\operatorname{Re} \alpha > 1].$
7. $\int_0^{\infty} \frac{x^{2n-1}}{e^x - 1} dx = (-1)^{n+1} \frac{(2\pi)^{2n}}{4n} B_{2n}$
 $[n=1, 2, 3, \dots].$
8. $\int_0^{\infty} \frac{(1+ix)^n - (1-ix)^n}{e^{ix} + 1} dx = \frac{i}{n+1} - \frac{2^{n+1} i}{n+1} B_{n+1}$
 $[n=1, 2, 3, \dots].$
9. $\int_0^{\infty} \frac{(1+ix)^{2n} - (1-ix)^{2n}}{e^{2ix} - 1} dx = \frac{i(2n-1)}{2(2n+1)}$
 $[n=1, 2, 3, \dots].$
10. $\int_0^{\infty} \frac{x dx}{(x^2 + z^2)^{\gamma} (e^{qx} - 1)} =$
 $= \frac{2\pi}{(n-1)!} \left\{ \left(-\frac{\partial}{\partial z} \right)^{n-1} \left[\ln \left(\frac{qz}{2\pi} \right) - \psi \left(\frac{qz}{2\pi} \right) \right] - \frac{\pi(2n-3)!!}{q^{2n-1} z^{2n-1}} \right\}$
 $[\operatorname{Re} q, \operatorname{Re} z > 0; n=1, 2, 3, \dots].$
11. $\int_0^{\infty} \frac{x dx}{(x^2 + z^2)^2 (e^{qx} - 1)} = \frac{\pi^2}{q^2 z^4} \sum_{k=0}^{\infty} |B_{2k+2}| \left(\frac{2\pi}{qz} \right)^{2k}$
 $[\operatorname{Re} q, \operatorname{Re} z > 0].$
12. $\int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{e^x + 1} \frac{dx}{x} = \ln \Gamma \left[\frac{(\mu+1)/2}{(\nu+1)/2}, \frac{1+\nu/2}{1+\mu/2} \right]$
 $[\operatorname{Re} \mu, \operatorname{Re} \nu > -1]$

$$13. \int_0^{\infty} \frac{e^{\mu x} - e^{(1+\mu)x}}{e^x + 1} \frac{dx}{x} = \ln \left(-\lg \frac{\mu}{2} \right)$$

$$[-1 < \operatorname{Re} \mu < 0]$$

$$14. \int_0^{\infty} x \frac{e^{2x/3} - e^{x/3}}{e^x + 1} dx = \frac{2}{3} \pi^2$$

$$15. \int_{-\infty}^{\infty} \frac{e^{\mu x} - e^{v x}}{e^x + 1} \frac{dx}{x} = \ln \left(\lg \frac{\mu \pi}{2q} \operatorname{ctg} \frac{v \pi}{2q} \right)$$

$$[0 < \operatorname{Re} (\mu/q) < 1; 0 < \operatorname{Re} (v/q) < 1]$$

$$16. \int_0^{\infty} \frac{x^p}{e^x - 1} \prod_{i=1}^n (e^{-v_i x} - 1) \frac{dx}{x} = \sum_{k=0}^n (-1)^k \ln \Gamma(1 + p + v_{i_1} + v_{i_2} + \dots + v_{i_k})$$

[$n = 1, 2, 3, \dots$; $\operatorname{Re} p > -1$; $\operatorname{Re} p > -1 - \operatorname{Re} (v_{i_1} + v_{i_2} + \dots + v_{i_k})$; символ $\sum_{k=0}^n$ означает, что k -й член содержит $\binom{n}{k}$ слагаемых, которые отличаются наборами индексов $i_1, i_2, \dots, i_k$ взятых из множеств $i = 1, 2, \dots, n$].

$$17. \int_0^{\infty} \frac{e^{\mu x} + e^{(1-\mu)x} - 2e^{x/2}}{e^x - 1} \frac{dx}{x} = -\ln(\sin \mu \pi)$$

$$[0 < \operatorname{Re} \mu < 1]$$

$$18. \int_0^{\infty} x^n \frac{e^{-\mu x} - 1}{e^x - 1} dx = -n! \sum_{k=1}^{\infty} \frac{1}{k^{n+1}}$$

$$19. \int_0^{\infty} x \frac{e^{\mu x} + e^{(1-\mu)x}}{e^x - 1} dx = \frac{\pi^2}{\sin^2 \mu \pi}$$

$$[0 < \operatorname{Re} \mu < 1]$$

$$20. \int_0^{\infty} x^{2n} \frac{e^{3x/2} + e^{x/2}}{(e^x - 1)^2} dx = (2^{2n} - 1)(2\pi)^{2n} |B_{2n}|$$

$$[n = 1, 2, 3, \dots]$$

$$21. \int_0^{\infty} x^2 \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} dx = \frac{2\pi^2}{3} - 2$$

$$22. \int_0^{\infty} \frac{x e^x}{a^2 e^{bx} - (a^2 \pm b^2) \sqrt{e^x - 1}} dx = \frac{2\pi}{ab} \left\{ \operatorname{arctg} \left(\frac{b}{a} \right) \right\}$$

$$[a, b > 0]$$

$$23. \int_0^{\infty} \frac{x^p - b^x}{x^q - a^x} dx = \frac{1}{\ln(c/d)} \left\{ \Psi \left[\frac{\ln(c/b)}{\ln(c/d)} \right] - \Psi \left[\frac{\ln(c/a)}{\ln(c/d)} \right] \right\}$$

$$[a, b, c, d > 0; \max(a, b) < \max(c, d)]$$

$$24. \int_0^{\infty} \frac{x dx}{e^x + e^{-x} - 1} = 1,1719536194 \dots$$

2.3.15.

$$25. \int_0^{\infty} \frac{x e^{-x} dx}{e^x + e^{-x} - 1} = 0,3118211319 \dots$$

$$26. \int_{-\infty}^{\infty} \frac{e^{\mu x} - e^{v x}}{e^x - 1} \frac{dx}{x} = i \pi \frac{\sin [v \pi / (2q)]}{\sin [\mu \pi / (2q)]}$$

$$[0 < \operatorname{Re} (v/q) < 1; 0 < \operatorname{Re} (\mu/q) < 1]$$

$$27. \int_{-\infty}^{\infty} \frac{x^{2n} dx}{e^x + e^{-x} + 2 \cos \gamma} = \frac{(-1)^n (2n)! (2\pi)^{2n+1}}{(2n+1)! \sin \gamma} B_{2n+1} \left(\frac{\gamma + \pi}{2\pi} \right)$$

$$[|\gamma| < \pi]$$

$$28. \int_{-\infty}^{\infty} \frac{x^2 dx}{e^x + e^{-x} + 2 \cos \gamma} = \frac{\gamma (\pi^2 - \gamma^2)}{3 \sin \gamma}$$

$$[|\gamma| < \pi]$$

$$29. \int_{-\infty}^{\infty} \frac{dx}{a^2 e^x + e^{-x} + 2a \cos \gamma} = \frac{\gamma}{a \sin \gamma}$$

$$[|\gamma| < \pi; a > 0]$$

$$30. \int_{-\infty}^{\infty} \frac{x dx}{a^2 e^x + e^{-x} + 2a \cos \gamma} = -\frac{\gamma \ln a}{a \sin \gamma}$$

$$[|\gamma| < \pi; a > 0]$$

2.3.15. Интегралы от $A(x) e^{-px^2 - qx}$.

$$1. \int_0^{\infty} (x-a)^{\beta-1} e^{-px^2 - qx} dx = \Gamma(\beta) (2p)^{-\beta/2} \exp \left[\frac{q^2}{8p} - \frac{a}{2} (q+ap) \right] D_{-\beta} \left(\frac{2ap+q}{\sqrt{2p}} \right)$$

$$[\operatorname{Re} \beta, \operatorname{Re} p > 0] \text{ или } [\operatorname{Re} \beta, \operatorname{Re} q > 0, \operatorname{Re} p = 0] \text{ или } [\operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0, 0 < \operatorname{Re} \beta < 2]$$

$$2. \int_0^{\infty} e^{-\lambda x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} e^{-\pi/4}$$

$$[\lambda > 0]$$

$$3. \int_0^{\infty} x^{2\alpha-1} e^{-px^2 - qx} dx = \Gamma(\alpha) (2p)^{-\alpha/2} \exp [q^2/(8p)] D_{-\alpha}(q/\sqrt{2p})$$

$$[\operatorname{Re} \alpha, \operatorname{Re} p > 0] \text{ или } [\operatorname{Re} \alpha, \operatorname{Re} q > 0, \operatorname{Re} p = 0] \text{ или } [0 < \operatorname{Re} \alpha < 2, \operatorname{Re} p = \operatorname{Re} q = 0, \operatorname{Im} p \neq 0]$$

$$4. \int_0^{\infty} x^{\beta-1} e^{-px^2 - qx} dx = \frac{1}{2} \sqrt{\frac{\pi}{p}} \exp \left(\frac{q^2}{4p} \right) \operatorname{erfc} \left(\frac{q}{2\sqrt{p}} \right)$$

$$[\operatorname{Re} p > 0] \text{ или } [\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$$

$$5. \int_0^{\infty} \frac{e^{-px^2 - qx}}{\sqrt{x}} dx = \frac{1}{2} \sqrt{\frac{q}{p}} \exp \left(\frac{q^2}{8p} \right) K_{1/4} \left(\frac{q^2}{8p} \right)$$

$$[\operatorname{Re} p > 0] \text{ или } [\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$$

$$6. \int_0^{\infty} \sqrt{x} e^{-px^2 - qx} dx = \frac{1}{8} \left(\frac{q}{p} \right)^{3/2} \exp \left(\frac{q^2}{8p} \right) \left\{ K_{3/4} \left(\frac{q^2}{8p} \right) - K_{1/4} \left(\frac{q^2}{8p} \right) \right\}$$

$$[\operatorname{Re} p > 0] \text{ или } [\operatorname{Re} p = 0, \operatorname{Im} p \neq 0, \operatorname{Re} q \geq 0]$$

$$7. \int_0^{\infty} x^n e^{-px^2-q/x} dx = \frac{(-1)^n}{2} \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \left[\exp\left(\frac{q^2}{4p}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right) \right] \\ | \operatorname{Re} p > 0 | \text{ или } | \operatorname{Re} p = 0, \operatorname{Re} q > 0 |$$

$$8. \int_{-\infty}^{\infty} x^n e^{-px^2-q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \exp\left(\frac{q^2}{4p}\right) =$$

$$9. = n! \sqrt{\frac{\pi}{p}} \left(-\frac{q}{2p}\right)^n \exp\left(\frac{q^2}{4p}\right) \sum_{k=0}^{[n/2]} \frac{p^k q^{2k}}{k! (n-2k)!} =$$

$$10. = \left(\frac{i}{2}\right)^n \sqrt{\pi} p^{-(n+1)/2} \exp\left(\frac{q^2}{4p}\right) H_n\left(\frac{iq}{2\sqrt{p}}\right) \quad | \operatorname{Re} p > 0 |$$

$$11. \int_{-\infty}^{\infty} e^{-px^2-q/x} dx = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right) \quad | \operatorname{Re} p > 0 |$$

$$12. \int_{-\infty}^{\infty} (ax^2+2bx+c) e^{-px^2-q/x} dx = \\ = \frac{\sqrt{\pi}}{4} p^{-b/2} [4cp^2-4pqb+a(q^2+2p)] \exp\left(\frac{q^2}{4p}\right) \quad | \operatorname{Re} p > 0 |$$

$$13. \int_{-\infty}^{\infty} e^{i(\lambda x^2+\mu x)} dx = \sqrt{\frac{\pi}{\lambda}} \exp\left[\frac{i(\lambda\pi-\mu^2)}{4\lambda}\right] \quad |\lambda| > 0$$

2.3.16. Интегралы от $A(x) e^{-px-q/x}$.

$$1. \int_0^{\infty} x^{n-1/2} e^{-px-q/x} dx = 2 \left(\frac{q}{p}\right)^{1/2} K_n(2\sqrt{pq}) \quad | \operatorname{Re} p, \operatorname{Re} q > 0 |$$

$$2. \int_0^{\infty} x^{n-1/2} e^{-px-q/x} dx = (-1)^n \sqrt{\pi} \frac{\partial^n}{\partial p^n} (p^{-1/2} e^{-2\sqrt{pq}}) \quad | \operatorname{Re} p, \operatorname{Re} q > 0 |$$

$$3. \int_0^{\infty} x^{n-1/2} e^{-px-q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}} \quad | \operatorname{Re} p, \operatorname{Re} q > 0 |$$

$$4. \int_0^{\infty} \frac{1}{\sqrt{x(x+z)}} e^{-px-q/x} dx = \frac{\pi}{\sqrt{z}} e^{pz+q/z} \operatorname{erfc}\left(\sqrt{\frac{q}{z}} + \sqrt{pz}\right) \\ | \operatorname{Re} p, \operatorname{Re} q \geq 0; | \arg z | < \pi/4$$

$$5. \int_{-\infty}^{\infty} e^{-p|x-q/x|^{\nu}} dx = \frac{2}{\nu} p^{-1/\nu} \Gamma\left(\frac{1}{\nu}\right) \quad [\nu, q, \operatorname{Re} p > 0]$$

$$6. \int_0^{\infty} e^{i(\lambda x-\mu/x)} \frac{dx}{x} = 2K_0(2\sqrt{\lambda\mu}) \quad [\mu, \lambda > 0]$$

$$7. \int_0^{\infty} e^{i(\lambda x-\mu/x)} \frac{dx}{\sqrt{x}} = (i+1) \sqrt{\frac{\pi}{2\lambda}} e^{-2\sqrt{\lambda\mu}} \quad [\lambda > 0, \mu \geq 0]$$

$$8. \int_{-\gamma-i\infty}^{\gamma+i\infty} x^{n-1} e^{px-q/x} dx = 2\pi i \left(\frac{q}{p}\right)^{n/2} J_n(2\sqrt{pq}) \quad [\rho, q, \gamma > 0; \operatorname{Re} \alpha < 1]$$

2.3.17. Интегралы от $A(x) e^{-b\sqrt{x^2+a^2}-px}$.

$$1. \int_a^{\infty} \frac{1}{\sqrt{x-a}} \exp(-b\sqrt{x^2+a^2}-px) dx = \\ = b \sqrt{\frac{2a}{p^2-b^2}} \left[\frac{1}{z_-} \exp(-a\sqrt{p^2-b^2}) \operatorname{erfc}(z_-) - \frac{1}{z_+} \exp(a\sqrt{p^2-b^2}) \operatorname{erfc}(z_+) \right] \\ [z_{\pm} = \sqrt{a(p \pm \sqrt{p^2-b^2})}; \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$2. \int_a^{\infty} \frac{1}{\sqrt{(x^2-a^2)^3}} \exp(-b\sqrt{x^2+a^2}-px) dx = \sqrt{\frac{b}{\pi}} K_{1/4}\left(\frac{z_+^2}{2}\right) K_{1/4}\left(\frac{z_-^2}{2}\right) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$3. \int_a^{\infty} \frac{1}{\sqrt{x-a}(x+a)} \exp(-b\sqrt{x^2+a^2}-px) dx = 2 \sqrt{\frac{2}{a}} e^{pa} \operatorname{erfc}(z_-) \operatorname{erfc}(z_+) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$4. \int_a^{\infty} \frac{(x+a)^{2\beta}}{(\lambda^2-x^2)^{\beta+1/4}} \exp(-b\sqrt{x^2+a^2}-px) dx = \\ = \sqrt{\frac{\pi}{a}} \Gamma\left(\frac{1}{2}-2\beta\right) D_{2\beta-1/2}(\sqrt{2}z_+) D_{-2\beta-1/2}(\sqrt{2}z_-) \\ [z_{\pm}^2 = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0; \operatorname{Re} \beta < 1/4]$$

$$5. \int_a^{\infty} \frac{[(x+\sqrt{x^2-a^2})^{\nu} \exp(b\sqrt{x^2-a^2}) + (x-\sqrt{x^2-a^2})^{\nu} \exp(-b\sqrt{x^2-a^2})]}{\sqrt{x^2-a^2}} \times \\ \times e^{-px} dx = 2a^{\nu} (p+b)^{\nu/2} (p-b)^{-\nu/2} K_{\nu}(a\sqrt{p^2-b^2}) \quad | \operatorname{Re}(p \pm b) > 0 |$$

$$6. \int_0^{\infty} \frac{1}{\sqrt{x}\sqrt{x^2+a^2}} \exp(-b\sqrt{x^2+a^2}-px) dx = \sqrt{\frac{p}{\pi}} K_{1/4}(z_+^2/2) K_{1/4}(z_-^2/2) \\ [z_{\pm}^2 = a(b \pm \sqrt{b^2-p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$7. \int_0^{\infty} x^n e^{-px^2-qx} dx = \frac{(-1)^n}{2} \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \left[\exp\left(\frac{q^2}{4p}\right) \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right) \right] \\ [\operatorname{Re} p > 0] \text{ или } [\operatorname{Re} p = 0, \operatorname{Re} q > 0]$$

$$8. \int_{-\infty}^{\infty} x^n e^{-px^2-qx} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} \operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right) =$$

$$= n! \sqrt{\frac{\pi}{p}} \left(-\frac{q}{2p}\right)^n \exp\left(\frac{q^2}{4p}\right) \sum_{k=0}^{[n/2]} \frac{p^k q^{n-2k}}{k! (n-2k)!} =$$

$$10. = \left(\frac{i}{2}\right)^n \sqrt{\pi p^{-(n+1)/2}} \exp\left(\frac{q^2}{4p}\right) H_n\left(\frac{iq}{2\sqrt{p}}\right) \quad [\operatorname{Re} p > 0]$$

$$11. \int_{-\infty}^{\infty} e^{-px^2-qx} dx = \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{4p}\right) \quad [\operatorname{Re} p > 0]$$

$$12. \int_{-\infty}^{\infty} (ax^2+2bx+c) e^{-px^2-qx} dx = \\ = \frac{1}{4} \sqrt{\frac{\pi}{p}} p^{-b/2} [4cp^2-4pqb+a(q^2+2p)] \exp\left(\frac{q^2}{4p}\right) \quad [\operatorname{Re} p > 0]$$

$$13. \int_{-\infty}^{\infty} e^{i(\lambda x^2+\mu x)} dx = \sqrt{\frac{\pi}{\lambda}} \exp\left[i\left(\frac{\lambda\pi-\mu^2}{4\lambda}\right)\right] \quad [\lambda > 0]$$

2.3.16. Интегралы от $A(x) e^{-px-q/x}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} dx = 2 \left(\frac{q}{p}\right)^{\alpha/2} K_{\alpha}(2\sqrt{pq}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$2. \int_0^{\infty} x^{\alpha-1/2} e^{-px-q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial p^n} (p^{-1/2} e^{-q^2/p}) \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$3. \int_0^{\infty} x^{\alpha-1/2} e^{-px-q/x} dx = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{pq}} \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$4. \int_0^{\infty} \frac{1}{\sqrt{x(x+z)}} e^{-px-q/x} dx = \frac{\pi}{\sqrt{z}} e^{pz/q/2} \operatorname{erfc}\left(\sqrt{\frac{q}{z}} + \sqrt{pz}\right) \\ [\operatorname{Re} p, \operatorname{Re} q \geq 0; |\arg z| < \pi]$$

$$5. \int_{-\infty}^{\infty} e^{-p|x-q/x|^{\alpha}} dx = \frac{2}{\alpha} p^{-1/\alpha} \Gamma\left(\frac{1}{\alpha}\right) \quad [\alpha, q, \operatorname{Re} p > 0]$$

$$6. \int_0^{\infty} e^{j(\lambda x - \mu/x)} \frac{dx}{x} = 2K_0(2\sqrt{\lambda\mu}) \quad [\mu, \lambda > 0]$$

$$7. \int_0^{\infty} e^{j(\lambda x - \mu/x)} \frac{dx}{\sqrt{x}} = (i+1) \sqrt{\frac{\pi}{2\lambda}} e^{-2\sqrt{\lambda\mu}} \quad [\lambda > 0, \mu \geq 0]$$

$$8. \int_{-\gamma-i\infty}^{\gamma+i\infty} x^{\alpha-1} e^{px-q/x} dx = 2\pi i \left(\frac{q}{p}\right)^{\alpha/2} J_{\alpha}(2\sqrt{pq}) \quad [\rho, q, \gamma > 0; \operatorname{Re} \alpha < 1]$$

2.3.17. Интегралы от $A(x) e^{-b\sqrt{x^2+a^2}-px}$.

$$1. \int_a^{\infty} \frac{1}{\sqrt{x-a}} \exp(-b\sqrt{x^2+a^2}-px) dx = \\ = b \sqrt{\frac{2a}{p^2-b^2}} \left[\frac{1}{z_+} \exp(-a\sqrt{p^2-b^2}) \operatorname{erfc}(z_+) - \frac{1}{z_-} \exp(a\sqrt{p^2-b^2}) \operatorname{erfc}(z_-) \right] \\ [z_{\pm} = \sqrt{a(p \pm \sqrt{p^2-b^2})}; \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$2. \int_a^{\infty} \frac{1}{\sqrt{(x^2-a^2)^3}} \exp(-b\sqrt{x^2+a^2}-px) dx = \sqrt{\frac{b}{\pi}} K_{1/4}\left(\frac{z_+}{2}\right) K_{1/4}\left(\frac{z_-}{2}\right) \\ [z_{\pm} = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$3. \int_a^{\infty} \frac{1}{\sqrt{x-a(x+a)}} \exp(-b\sqrt{x^2+a^2}-px) dx = 2 \sqrt{\frac{2}{a}} e^{pa} \operatorname{erfc}(z_+) \operatorname{erfc}(z_-) \\ [z_{\pm} = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$4. \int_a^{\infty} \frac{(x+a)^{2\beta}}{(x^2-a^2)^{\beta+1/4}} \exp(-b\sqrt{x^2+a^2}-px) dx = \\ = \sqrt{\frac{2}{a}} \Gamma\left(\frac{1}{2}-2\beta\right) D_{2\beta-1/2}(\sqrt{2}z_+) D_{2\beta-1/2}(\sqrt{2}z_-) \\ [z_{\pm} = a(p \pm \sqrt{p^2-b^2}); \operatorname{Re} b, \operatorname{Re}(p+b) > 0; \operatorname{Re} \beta < 1/4]$$

$$5. \int_a^{\infty} \frac{[x+\sqrt{x^2-a^2}]^{\nu} \exp(b\sqrt{x^2+a^2}) + (x-\sqrt{x^2-a^2})^{\nu} \exp(-b\sqrt{x^2+a^2})}{\sqrt{x^2-a^2}} \times \\ \times e^{-px} dx = 2a^{\nu} (p+b)^{\nu/2} (p-b)^{-\nu/2} K_{\nu}(a\sqrt{p^2-b^2}) \quad [\operatorname{Re}(p \pm b) > 0]$$

$$6. \int_0^{\infty} \frac{1}{\sqrt{x}\sqrt{x^2+a^2}} \exp(-b\sqrt{x^2+a^2}-px) dx = \sqrt{\frac{p}{\pi}} K_{1/4}(z_+/2) K_{1/4}(z_-/2) \\ [z_{\pm} = a(b \pm \sqrt{b^2-p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p+b) > 0]$$

$$7. \int_0^{\infty} \frac{(a + \sqrt{x^2 + a^2})^{1/2}}{\sqrt{x^2 + a^2}} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \frac{p \sqrt{2a}}{\sqrt{b^2 - p^2}} \left\{ \frac{1}{x_-} \exp(-a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_-) - \frac{1}{x_+} \exp(a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_+) \right\}$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p \pm b) > 0]$$

$$8. \int_0^{\infty} \frac{\exp(-b \sqrt{x^2 + a^2} - px)}{\sqrt{x^2 + a^2} (a + \sqrt{x^2 + a^2})^{1/2}} dx = 2 \sqrt{\frac{2}{a}} \exp(ab) \operatorname{erfc}(z_+) \operatorname{erfc}(z_-)$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p \pm b) > 0]$$

$$9. \int_0^{\infty} \frac{x^{-2\beta-1/2}}{\sqrt{x^2 + a^2}} (a + \sqrt{x^2 + a^2})^{2\beta} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \sqrt{\frac{2}{a}} \Gamma\left(\frac{1}{2} - 2\beta\right) D_{2\beta-1/2}(\sqrt{2} z_+) D_{2\beta-1/2}(\sqrt{2} z_-)$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(p \pm b) > 0; \operatorname{Re} \beta < 1/4]$$

$$13. \int_0^{\infty} \left[(\sqrt{x^2 + a^2} + x)^{\nu} e^{bx} + (\sqrt{x^2 + a^2} - x)^{\nu} e^{-bx} \right] e^{-c \sqrt{x^2 + a^2}} \frac{dx}{\sqrt{x^2 + a^2}} =$$

$$= 2a^{\nu} (c+b)^{\nu/2} (c-b)^{-\nu/2} K_{\nu}(a \sqrt{c^2 - b^2}) \quad [\operatorname{Re} a, \operatorname{Re}(c \pm b) > 0]$$

2.3.18. Интегралы от $e^{b(x)} f(x)$.

$$1. \int_0^1 x^{-p} dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k}$$

$$2. \int_0^{\infty} x^{\alpha} e^{-p x^{\mu}} dx = \frac{1}{\mu} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0]$$

$$3. \int_0^{\infty} e^{-p x^{\nu}} dx = -\operatorname{Ei}(-p) \quad [\operatorname{Re} p > 0]$$

$$4. \int_{-\infty}^{\infty} (a + b |x|) e^{-|x| + i \lambda x} dx = 2 \frac{\lambda^2 (a-b) + a + b}{(\lambda^2 + 1)^2} \quad [\lambda > 0]$$

$$5. \int_{-\infty}^{\infty} x e^{-p x^2 + i \lambda x} dx = \mu^{-1} \Gamma\left(\frac{1}{\mu}\right) p^{-1/\mu} \left[\Psi\left(\frac{1}{\mu}\right) - \ln p \right] \quad [\mu, \operatorname{Re} p > 0]$$

$$6. \int_{-\infty}^{\infty} x e^{x} e^{-x} dx = C.$$

2.3.19. Интегралы, содержащие разности алгебраических и показательных функций.

$$1. \int_0^1 \left(\frac{r e^{1-x} - r}{1-x} - \frac{e^x}{1-x} \right) \frac{dx}{x} = \psi(p) - \ln r \quad [r, \operatorname{Re} p > 0]$$

$$2. \int_0^1 (e^{1-1/x} - x^p) \frac{dx}{x(1-x)} = \psi(p) \quad [\operatorname{Re} p > 0]$$

$$3. \int_0^1 \left(\frac{r e^{1-x} - r}{1-x} - \frac{e^{1-x}}{1-x} \right) \frac{dx}{x} = -\ln r \quad [r > 0]$$

$$4. \int_0^1 \frac{1 - e^{-x} - e^{-1/x}}{x} dx = C.$$

$$5. \int_0^{\infty} x^{\alpha} \left[e^{-px} - \sum_{k=0}^{n-1} \frac{(-px)^k}{k!} \right] dx = \Gamma(\alpha) p^{-\alpha} \quad [\operatorname{Re} p \geq 0; -n < \operatorname{Re} \alpha < 1-n]$$

$$6. \int_0^{\infty} x^{-1} [e^{-x} - (x+1)^{-p}] dx = \psi(p) \quad [\operatorname{Re} p > 0]$$

$$7. \int_0^{\infty} \left(e^{-px} - \frac{1}{1+qx} \right) \frac{dx}{x} = \ln \frac{q}{p} - C \quad [\operatorname{Re} p > 0; |\arg q| < \pi]$$

$$8. \int_0^{\infty} \frac{e^{-px} - 1}{x(x+2)} dx = \frac{1}{2} [e^{pz} \operatorname{Ei}(-pz) - \ln(pz)] \quad [\operatorname{Re} p > 0; |\arg z| < \pi]$$

$$9. \int_0^{\infty} \left(e^{-px} - \frac{1}{1+q^{\mu} x^{\mu}} \right) \frac{dx}{x} = \ln \frac{q}{p} - C \quad [\mu, \operatorname{Re} p > 0; \mu |\arg q| < \pi]$$

$$10. \int_0^{\infty} \left[\frac{e^{-px} - 1}{p^{\mu}} + \frac{1}{(1+q^{\mu} x^{\mu})^{\mu}} \right] \frac{dx}{x} = \ln \frac{p}{q} + C \left(1 - \frac{1}{\mu} \right) - \frac{\psi(\sigma)}{\mu} - 1$$

$$[\mu, \operatorname{Re} \sigma, \operatorname{Re} p > 0; \mu |\arg q| < \pi]$$

$$11. \int_0^{\infty} \left(\frac{q}{e^{qx} - 1} - \frac{1}{x e^{qx}} \right) dx = C.$$

$$12. \int_0^{\infty} \left(\frac{e^{-2x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi.$$

$$13. \int_0^{\infty} \left(\frac{e^{-x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = \frac{1}{2} \ln \frac{\pi}{4}.$$

$$14. \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-px}}{x} dx = \ln \Gamma(p) - \left(p - \frac{1}{2} \right) \ln p + p - \frac{1}{2} \ln(2\pi) \quad [\operatorname{Re} p > 0]$$

$$15. \int_0^{\infty} \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-px}}{x} dx = \left(p - \frac{1}{2} \right) \ln p + \frac{1}{2} \ln(2\pi) - p - \ln \Gamma(p) \quad [\operatorname{Re} p > 0]$$

$$16. \int_0^{\infty} \left(\frac{p^2 e^{-x}}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right) \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3p^2}{4} \quad [\operatorname{Re} p > 0]$$

$$17. \int_0^{\infty} \frac{(e^{-px} - e^{-qx})^2}{x^2} dx = 2p \ln \frac{2p}{p+q} + 2q \ln \frac{2q}{p+q} \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$18. \int_0^{\infty} \left(\frac{e^{-px}}{x} - \frac{e^{-qx}}{1 - e^{-x}} \right) dx = \psi(p) \quad [\operatorname{Re} p > 0]$$

$$19. \int_0^{\infty} [(p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx}] \frac{dx}{x^2} = (r-q)p \ln p + (p-r)q \ln q + (q-p)r \ln r \quad [\operatorname{Re} p, \operatorname{Re} q, \operatorname{Re} r > 0]$$

$$20. \int_0^{\infty} \left[(p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right] \frac{dx}{x} = p \ln p - q \ln q - (p-q) \left(1 + \ln \frac{r}{m} \right) \quad [m, \operatorname{Re} p, \operatorname{Re} q, \operatorname{Re} r > 0]$$

$$21. \int_0^{\infty} \left(\frac{e^{-qx}}{e^x - 1} + \frac{e^{-rx}}{x} \right) e^{-px} dx = \psi(q+p) - \ln(r+p) \quad [\operatorname{Re}(q+p), \operatorname{Re}(r+p) > 0]$$

$$22. \int_0^{\infty} \left(qe^{-x} - \frac{1 - e^{-qx}}{e^x - 1} e^{-rx} \right) \frac{dx}{x} = \ln \Gamma \left[\frac{q+r+1}{r+1} \right] \quad [\operatorname{Re} r, \operatorname{Re}(q+r) > -1]$$

$$23. \int_0^{\infty} \left[1 - e^{-x} - \frac{(1 - e^{-qx})(1 - e^{-px})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(p, q) \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$24. \int_0^{\infty} \left(\frac{a + be^{-qx}}{ce^{lx} + g + he^{-qx}} - \frac{a + be^{-px}}{ce^{lx} + g + he^{-px}} \right) \frac{dx}{x} = \frac{a+b}{c+g+h} \ln \frac{q}{p} \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$25. \int_0^{\infty} \left(\frac{e^{-pnx} - e^{-qnz}}{n} - \frac{e^{-pmx} - e^{-qmz}}{m} \right) \frac{dx}{x^2} = (q-p) \ln \frac{m}{n} \quad [m, n, \operatorname{Re} p, \operatorname{Re} q > 0]$$

$$26. \int_0^{\infty} e^{-px} \prod_{i=1}^n (e^{-v_i x} - 1) \frac{dx}{x^{\alpha}} = \frac{1}{(n-1)!} \frac{\partial^{n-\alpha}}{\partial p^{n-\alpha}} \times \\ \times \sum_{k=0}^n (-1)^{k+\alpha} (p + v_{i_1} + v_{i_2} + \dots + v_{i_k})^{n-1} \ln(p + v_{i_1} + v_{i_2} + \dots + v_{i_k}) \\ [0 \leq m \leq n; \operatorname{Re} p > 0; \operatorname{Re} p > -\operatorname{Re}(v_{i_1} + v_{i_2} + \dots + v_{i_k}); \text{символ } \Sigma \text{ означает, что } k\text{-й член} \\ \text{суммы содержит } \binom{n}{k} \text{ слагаемых, которые стягиваются наборами индексов } i_1, i_2, \dots, i_k, \\ \text{взятых из множества } i = 1, 2, \dots, n]$$

$$27. \int_0^{\infty} \left[e^{-pxq} - \frac{y^{\sigma}}{(x^{\sigma} + y^{\sigma})^{\sigma}} \right] \frac{dx}{x} = \left(\frac{1}{r} - \frac{1}{q} \right) C - \ln y - \frac{\ln p}{q} + \frac{\psi(\sigma)}{r} \\ [r, q, \operatorname{Re} p, \operatorname{Re} \sigma > 0; r | \arg y| < \pi]$$

$$28. \int_0^{\infty} (e^{-pxq} - e^{-nqx}) \frac{dx}{x} = \frac{\ln h + C}{r} - \frac{\ln p + C}{q} \quad [r, q, \operatorname{Re} h, \operatorname{Re} p > 0]$$

$$29. \int_{-\infty}^{\infty} \frac{\alpha x}{(a^2 x^2 + b^2)^{\rho}} dx = \frac{a^{-\lambda} b^{-\lambda}}{\ln(\gamma/\beta)} \Gamma \left[\frac{\mu}{\rho} \right] \\ \left[\mu = \frac{\ln(\alpha/\beta)}{\ln(\gamma/\beta)}, \lambda = \frac{\ln(\gamma^{\rho}/\alpha)}{\ln(\gamma/\beta)}, a, b, \rho > 0; 0 < \beta < \alpha^{1/\rho} < \gamma \right]$$

2.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

2.4.1. Введение.

Этот раздел содержит интегралы, в подынтегральные выражения которых входят гиперболические функции. Для их вычисления можно использовать следующие свойства. Интегралы, содержащие гиперболические функции, могут быть преобразованы к интегралам от показательной или тригонометрических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\operatorname{sh}^n bx$ и $\operatorname{ch}^n bx$, сводятся к интегралам от функций $\operatorname{sh} bx$ и $\operatorname{ch} bx$, если использовать формулы для степеней гиперболических функций (см. приложение I). В ряде случаев интегралы от гиперболических функций с помощью замены переменной сводятся к интегралам от алгебраических и других функций. Например,

$$\int_0^a \varphi(x) \operatorname{ch}^2 bx dx = \frac{1}{2} \int_0^a \varphi(x) dx + \frac{1}{4} \int_0^a \varphi(x) e^{2bx} dx + \frac{1}{4} \int_0^a \varphi(x) e^{-2bx} dx = \\ = \int_0^a \varphi(x) \cos^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 + \operatorname{ch} 2bx) dx = \\ = \frac{1}{h} \int_1^{\operatorname{ch} ab} x^{\frac{ab}{h}} \left(\frac{\ln(x + \sqrt{x^2 - 1})}{b} \right) \frac{dx}{\sqrt{x^2 - 1}} \quad [a, b > 0]$$

В разделе «Гиперболические функции» a, b, c — комплексные числа.

2.4.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие гиперболические, а также степенную, алгебраические и показательную функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций $V_{\frac{r}{2}}, V_{\frac{r}{2}}, j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

В п. 2.4.2. приняты следующие обозначения: $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \operatorname{sh} bx; \\ 0, & \text{если взят } \operatorname{ch} bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2]+k}, & \text{если взят } \operatorname{sh} bx; \\ 1, & \text{если взят } \operatorname{ch} bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \operatorname{sh}^n bx, \quad n=1, 3, 5, \dots; \\ 0, & \text{если взят } \operatorname{sh}^n bx, \quad n=2, 4, 6, \dots; \\ 0, & \text{если взят } \operatorname{ch}^n bx, \quad n=1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{-1} \left(\frac{\alpha}{2} + \gamma + h \right), \quad \tilde{\delta} = \frac{by}{2} (n-2k),$$

$$1. \int_0^y x^{\alpha-1} (y^2 - x^2)^{\beta-1} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx = \\ = V_{\frac{r}{2}} + \frac{1+(-1)^n}{2} (-1)^{n/2} y^{2\alpha} (\beta-1)_{n/2} \frac{2^{-n/2} n!}{r [(n/2)!]^2} B\left(\frac{\alpha}{2r}, \beta\right)$$

$$\{E_{\frac{r}{2}} = (-1)^k (-1)^{[n/2]} \frac{\sqrt{\pi}}{2^n} \frac{B(\beta, \xi)}{\Gamma(1/2+2\gamma+h)}, \\ u = \alpha + 2r(\beta-1), \quad a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{\delta}/p)^{2p}\} \\ [y, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta n; |\arg b| < \pi]$$

$$2. \int_0^y x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} - x^{2r}})^v + (y^r - \sqrt{y^{2r} - x^{2r}})^v}{\sqrt{y^{2r} - x^{2r}}} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx = \\ = V_{\frac{r}{2}} + \frac{1+(-1)^n}{2} (-1)^{n/2} 2^{v-n+\alpha/r-1} y^{\alpha+r} \frac{n! \sqrt{\pi}}{r [(n/2)!]^2} \Gamma\left[\frac{\alpha}{(2r)}, \frac{v+\alpha/(2r)}{v+\alpha/r}\right]$$

$$\{E_{\frac{r}{2}} = (-1)^k (-1)^{[n/2]} \frac{\sqrt{\pi}}{2^n} \frac{\Gamma\left[\frac{\xi}{r}, \frac{v+\xi}{v+\alpha/r}\right]}{\Gamma\left[\frac{v+2\xi}{r}, \frac{1/2+2\gamma+h}{r}\right]}, \\ u = \alpha + r(v-1), \quad a_1 = \xi, \quad b_1 = v + \xi, \quad c_1 = v/2 + \xi, \quad d_1 = (v+1)/2 + \xi, \\ e_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{\delta}/p)^{2p}\} \\ [y > 0; \operatorname{Re} \alpha > -n\delta, \operatorname{Re} \alpha > -n\delta - 2r \operatorname{Re} v; |\arg b| < \pi]$$

$$3. \int_0^{\infty} x^{\alpha-1} e^{-sx} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx = \\ = U + \frac{1+(-1)^n}{2} (-1)^{n/2} 2^{-n-1} s^{-n/(2r)} \frac{n!}{r [(n/2)!]^2} \Gamma\left(\frac{\alpha}{2r}\right), \\ U = V_{\frac{r}{2}} \quad [r > 1/2]$$

$$U = 2^{1-n} \sqrt{\pi} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} \times \\ \times s^{-\alpha} \varphi^{2\gamma} \Gamma\left[\frac{\alpha+2\gamma}{1/2+2\gamma}\right] z F_3\left(\begin{matrix} \Delta(2, \alpha+2\lambda); \\ 1/2+2\gamma \end{matrix}; \frac{b^2(n-2k)^2}{s^2}\right) \quad [r=1/2] \\ \{E_{\frac{r}{2}} = (-1)^k (-1)^{[n/2]} \frac{\sqrt{\pi}}{2^n} \Gamma\left[\frac{\xi}{1/2+2\gamma+h}\right], \\ u = -\alpha/(2r), \quad v_1 = 2(h+\gamma), \quad \varphi = 2^{-1}(n-2k)bs^{-1/(2r)}, \\ a_1 = \xi, \quad b_1 = 1/2+2\gamma+h, \quad z = \left[\frac{\tilde{\delta}(n-2k)}{2p}\right]^{2p} q^{2\gamma-1}\} \\ [\operatorname{Re} s > 0; \operatorname{Re} \alpha > -\delta n; \arg b \text{ произволен при } r > 1/2 \text{ (при } |\operatorname{Re} \delta| < \operatorname{Re} s \text{ при } r=1/2)]$$

2.4.3. Интегралы от $A(x) \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}}$.

$$\text{Обозначение: } \delta = \begin{cases} 1 \\ 0 \end{cases}.$$

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{sh} cx} dx = I_{\alpha} \quad [\operatorname{Re} c > 0; \operatorname{Re} \alpha > 1]$$

$$I_{\alpha} = \frac{2^{\alpha-1}}{2^{\alpha} \Gamma(\alpha)} \Gamma(\alpha) \zeta(\alpha) = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\alpha}}, \\ I_{2n} = \frac{2^{2n-1}}{2^n} \left(\frac{\pi}{c}\right)^{2n} |B_{2n}|, \quad I_2 = \frac{\pi^2}{4c^2}, \quad I_4 = \frac{\pi^4}{8c^4} \quad [\text{см. 5.1.4}]$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{ch} cx} dx = J_{\alpha} \quad [\operatorname{Re} \alpha, \operatorname{Re} c > 0]$$

$$J_{\alpha} = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha}} = c^{-\alpha/2} \Gamma(\alpha) [\zeta(\alpha, 1/4) - \zeta(\alpha, 3/4)], \\ J_{2n+1} = \left(\frac{\pi}{2c}\right)^{2n+1} |E_{2n}|, \quad J_1 = \frac{\pi}{2c}, \quad J_3 = \left(\frac{\pi}{2c}\right)^3 \quad [\text{см. 5.1.4}]$$

$$3. \int_0^{\infty} x^{\alpha-1} \frac{\{\operatorname{sh} cx\}^{-2}}{\{\operatorname{ch} cx\}} dx = I_{\alpha} \quad [\operatorname{Re} c > 0; \operatorname{Re} \alpha > 2\delta]$$

$$I_{\alpha} = 2^{2-\alpha} c^{-\alpha} (1-2^{-2})^{\alpha-1} \delta \Gamma(\alpha) \zeta(\alpha-1), \quad I_{2n+1} = \pi^{2n} c^{-2n-1} (1-2^{1-\alpha})(1-\delta) |B_{2n}|.$$

$$4. \int_0^{\infty} \frac{x dx}{\operatorname{ch} cx} = \frac{\ln 2}{c^2} \quad [\operatorname{Re} c > 0]$$

$$5. \int_0^{\infty} \frac{dx}{\operatorname{ch}^n cx} = \frac{(2r-2)! (1-s)}{(2r-1)! c} + \frac{(2r-1)! \pi}{(2r)! 2c} \\ [r = [n/2], \quad s = n-2r; \operatorname{Re} c > 0; \quad n=2, 3, 4, \dots]$$

$$6. \int_0^a (a-x)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx = 2a^{\beta} \beta^{-1} [{}_1F_1(1; \beta+1; ab) - {}_1F_1(1; \beta+1; -ab)]$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$7. \int_0^a (a^2-x^2)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx = \frac{\sqrt{\pi}}{2} \left(\frac{a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{L_{\beta-1/2}(ab)}{I_{\beta-1/2}(ab)} \right\}$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$8. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx =$$

$$= \frac{1}{2} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_1F_1(\alpha; \alpha+\beta; ab) - {}_1F_1(\alpha; \alpha+\beta; -ab)]$$

$$[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi]$$

$$9. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx = \sqrt{\pi} \left(\frac{a}{b}\right)^{\alpha-1/2} \frac{\{sh(ab/2)\}}{\{ch(ab/2)\}} \Gamma(\alpha) I_{\alpha-1/2}\left(\frac{ab}{2}\right)$$

$$[a, \operatorname{Re} \alpha > 0; |\arg b| < \pi]$$

$$10. \int_0^a x^{\alpha-1} (a^2-x^2)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx =$$

$$= \frac{b^{\delta} a^{\alpha+\beta-1} \delta^{-1}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; \frac{a^2 b^2}{4}\right)$$

$$[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi]$$

$$11. \int_0^a x (a^2-x^2)^{\beta-1} \frac{\{sh bx\}}{\{ch bx\}} dx = \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{L_{\beta+1/2}(ab)}{I_{\beta+1/2}(ab)} \right\} + \left\{ \frac{0}{2-4\beta} \right\}$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$12. \int_0^a \frac{1}{\sqrt{x(a-x)}} \frac{\{sh bx\}}{\{ch bx\}} dx = \frac{\pi}{2} \sqrt{b} I_{1/2}^2\left(\frac{ab}{2}\right)$$

$$[a > 0; |\arg b| < \pi]$$

$$13. \int_0^{\infty} \frac{x}{(x^2+z^2)} \frac{dx}{sh bx} = \frac{\pi}{2az} - \beta \left(\frac{bz}{\pi} + 1\right)$$

$$[bz > 0]$$

$$14. = \ln 2 - \frac{1}{2}$$

$$[bz = \pi; \operatorname{Re} z > 0]$$

$$15. = \frac{\pi}{2} - 1$$

$$[2bz = \pi; \operatorname{Re} z > 0]$$

$$16. = 2^{-1/2} [\pi + 2 \ln(1 + \sqrt{2})] - 2$$

$$[4bz = \pi; \operatorname{Re} z > 0]$$

$$17. \int_0^{\infty} \frac{dx}{(x^2+z^2) ch bx} = \frac{1}{z} \beta \left(\frac{bz}{\pi} + \frac{1}{2}\right)$$

$$[\operatorname{Re} z > 0]$$

$$18. = \frac{b}{2\pi} (2 - \pi)$$

$$[bz = \pi; \operatorname{Re} z > 0]$$

$$19. = \frac{2b}{\pi} \ln 2$$

$$[2bz = \pi; \operatorname{Re} z > 0]$$

$$20. = 2^{3/2} b [1 - 2\pi^{-1} \ln(1 + \sqrt{2})]$$

$$[4bz = \pi; \operatorname{Re} z > 0]$$

$$21. \int_0^{\infty} \frac{x^3 dx}{(x^6+z^6) sh bx} = \frac{1}{3} \left(\ln 2 - 1 + \frac{\pi}{ch(\pi\sqrt{3}/2)} \right)$$

$$[bz = \pi; 0 < \arg z < \pi]$$

$$22. \int_0^{\infty} \frac{(1+x)^n - (1-x)^n}{sh(x\pi/2)} dx = 2i - 2i(-1)^{n/2} \frac{1+(-1)^n}{2} |E_n|$$

2.4.4. Интегралы от $\prod_k \frac{\{sh b_k x\}}{\{ch b_k x\}}$.

Обозначение: $\delta = \left\{ \frac{1}{0} \right\}$.

$$1. \int_0^{\infty} \frac{sh bx}{sh cx} dx = \frac{\pi}{2c} \lg \frac{b\pi}{2c}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$2. \int_0^{\infty} \frac{sh^2 bx}{sh^2 cx} dx = \frac{c - \pi b \operatorname{ctg}(b\pi/c)}{2c^2}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$3. \int_0^{\infty} \frac{1}{sh cx} \frac{\{sh bx\}}{\{ch bx\}} dx = \frac{\pi}{2c \cos[b\pi/(2c)]} - \frac{1}{c} \beta \left(\frac{b+c}{2c} \right) \left\{ \frac{1}{0} \right\}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$4. \int_0^{\infty} \frac{sh bx}{ch^v cx} dx = \frac{2^{v-1}}{c \Gamma(v)} \Gamma\left(\frac{vc+b}{2c}\right) \Gamma\left(\frac{vc-b}{2c}\right)$$

$$[\operatorname{Re}(vc) > |\operatorname{Re} b|]$$

$$5. \int_0^{\infty} \frac{sh bx}{ch^{n-1} cx} dx = \frac{2^{n-1} \pi^{-1} \Gamma(n-1)}{(n-1)!} b^{1-2\lambda} c^{2\lambda-2} \left\{ \sec[b\pi/(2c)] \right\} \prod_{k=1}^{n-1} \left[(k-\lambda)^2 - \frac{b^2}{4c^2} \right]$$

$$[n \operatorname{Re} c > |\operatorname{Re} b|; \lambda = \frac{1+(-1)^n}{4}, n \geq 1]$$

$$6. \int_0^{\infty} sh^{u-1} bx ch^{v-1} bx dx = \frac{1}{2b} \Pi\left(\frac{\mu}{2}, 1 - \frac{\mu+v}{2}\right)$$

$$[b > 0; 0 < \operatorname{Re} \mu < 2 - \operatorname{Re} v]$$

$$7. \int_0^{\infty} sh^{v-1} ax \frac{\{sh bx\}}{\{ch bx\}} dx =$$

$$= \frac{1}{2a \sqrt{\pi}} \frac{\sin[b\pi/(2a)]}{\cos[b\pi/(2a)]} \Gamma\left[\frac{(v+\delta)/2}{1-v/2}\right] \Gamma\left[\frac{1-v}{2} + \frac{b}{2a}\right] \Gamma\left[\frac{1-v}{2} - \frac{b}{2a}\right]$$

$$[a > 0; -\delta < \operatorname{Re} v < 1 - |\operatorname{Re} b|/a]$$

8. $\int_0^{\infty} \text{sh}^{v-1} ax \text{ch} ax \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx =$
 $= \frac{b}{4a^2} \frac{1}{\pi} \left\{ \frac{\cos [b\pi/(2a)]}{\sin [b\pi/(2a)]} \right\} \Gamma \left[\frac{(v+\delta)/2}{(1-v)/2} \right] \Gamma \left(\frac{b-va}{2a} \right) \Gamma \left(-\frac{b+va}{2a} \right)$
 $|a| > 0; -\delta < \text{Re } v < -|\text{Im } v|$
9. $\int_0^{\infty} \frac{\text{sh} ax}{\text{sh} cx} \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx =$
 $= \frac{\pi}{2c} \left(\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c} \right)^{-1} \left\{ \frac{\sin (b\pi/c)}{\sin (a\pi/c)} \right\} + \left\{ \frac{(2c)^{-1}}{0} \right\} \left[\psi \left(\frac{a+c-b}{2c} \right) - \psi \left(\frac{a+c+b}{2c} \right) \right]$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$
10. $\int_0^{\infty} \frac{\text{sh} ax \text{sh} bx}{\text{ch} ax \text{ch} bx} \frac{dx}{\text{ch} cx} =$
 $= \frac{\pi}{c} \left\{ \frac{\sin [a\pi/(2c)] \sin [b\pi/(2c)]}{\cos [a\pi/(2c)] \cos [b\pi/(2c)]} \right\} \left\{ \cos (a\pi/c) + \cos (b\pi/c) \right\}^{-1}$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$
11. $\int_0^{\infty} \frac{\text{sh} ax \text{ch} bx}{\text{ch} cx} dx = \frac{1}{4c} \left[\psi \left(\frac{3c-a+b}{4c} \right) - \psi \left(\frac{3c+a-b}{4c} \right) \right] +$
 $+ \psi \left(\frac{3c-a-b}{4c} \right) - \psi \left(\frac{3c+a+b}{4c} \right) + \frac{2\pi \sin (a\pi/c)}{\cos (a\pi/c) + \cos (b\pi/c)}$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$
12. $\int_0^{\infty} \frac{\text{sh} ax \text{sh} cx}{\text{ch}^2 cx} dx = \frac{\pi a}{2c^2 \cos [a\pi/(2c)]}$
 $[\text{Re } c > | \text{Re } a |]$

2.4.5. Интегралы от $A(x) \prod_k \left\{ \frac{\text{sh} b_k x}{\text{ch} b_k x} \right\}$.

Обозначение: $\delta = \left\{ \frac{1}{0} \right\}$.

1. $\int_0^{\infty} \frac{x^{\alpha-1}}{\text{sh} cx} \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx = \frac{\Gamma(\alpha)}{(2c)^\alpha} \left[\zeta \left(\alpha, \frac{1}{2} - \frac{b}{2c} \right) - \zeta \left(\alpha, \frac{1}{2} + \frac{b}{2c} \right) \right]$
 $[\text{Re } \alpha > 1 - \delta; \text{Re } c > | \text{Re } b |]$
2. $\int_0^{\infty} \frac{x^{\alpha+\sigma}}{\text{sh} cx} \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx = \frac{\pi}{2c} \frac{\zeta^{\alpha+\sigma}}{c^{b\pi/\sigma}} \lg \frac{b\pi}{2c}$
 $[\text{Re } c > | \text{Re } b |; \sigma = \left\{ \frac{0}{1} \right\}]$
3. $\int_0^{\infty} \frac{\text{sh}^2 bx}{x \text{sh} cx} dx = -\frac{1}{2} \ln \cos \frac{b\pi}{c}$
 $[\text{Re } c > 2; | \text{Re } b |]$

4. $\int_0^{\infty} \frac{x^{\alpha-1}}{\text{ch} cx} \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx =$
 $= 2^{-\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta \left(\alpha, \frac{c-b}{4c} \right) + \zeta \left(\alpha, \frac{c+b}{4c} \right) - \zeta \left(\alpha, \frac{3c+b}{4c} \right) - \zeta \left(\alpha, \frac{3c-b}{4c} \right) \right]$
 $[\text{Re } \alpha > -\delta; \text{Re } c > | \text{Re } b |]$
5. $\int_0^{\infty} \frac{\text{sh} bx}{x \text{ch} cx} dx = \ln \lg \left(\frac{b+c}{4c} \pi \right)$
 $[\text{Re } c > | \text{Re } b |]$
6. $\int_0^{\infty} \frac{x^{\alpha+\delta}}{\text{sh} cx} \left\{ \frac{\text{sh} bx}{\text{ch} bx} \right\} dx = \frac{\pi}{2c} \frac{\zeta^{\alpha+\delta}}{\partial b^{\alpha+\delta}} \sec \frac{b\pi}{2c}$
 $[\text{Re } c > | \text{Re } b |; \delta = \left\{ \frac{1}{0} \right\}]$
7. $\int_0^{\infty} x^{\alpha-1} \frac{\text{ch} bx}{\text{ch} cx} dx = \Gamma(\alpha) \sum_{k=0}^{\infty} (-1)^k \{ [c(2k+1)-b]^{-\alpha} + [c(2k+1)+b]^{-\alpha} \}$
 $[\text{Re } \alpha > 0; \text{Re } c > | \text{Re } b |]$
8. $\int_0^{\infty} x \frac{\text{sh} bx}{\text{ch}^2 cx} dx = \frac{\pi}{4c} \sin^2 \frac{b\pi}{2c} \left(2c \sin \frac{b\pi}{2c} - b\pi \cos \frac{b\pi}{2c} \right)$
 $[2 \text{Re } c > | \text{Re } b |]$
9. $\int_0^{\infty} \frac{x^{\alpha-1} dx}{\text{sh} ax \text{ch} bx} = \frac{\Gamma(\alpha)}{(2b)^\alpha} \left[{}_2F_1 \left(-1; \alpha; \frac{1}{2} + \frac{a}{2b} \right) + {}_2F_1 \left(-1; \alpha; \frac{1}{2} - \frac{a}{2b} \right) \right]$
 $[\text{Re } a, \text{Re } b > 0; \text{Re } \alpha > 1]$
10. $\int_0^{\infty} x^{\alpha-1} \frac{\text{sh} cx}{\text{ch}^2 cx} dx = \frac{2\Gamma(\alpha)}{c^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha-1}}$
 $[\text{Re } c > 0; \text{Re } \alpha > -1; \text{см. 5.1.4}]$
11. $\int_0^{\infty} x \text{ch}^{\mu-1} ax \text{sh} ax dx = -\frac{\sqrt{\pi}}{2\mu a^2} \Gamma \left[\frac{-\mu/2}{(1-\mu)/2} \right]$
 $[\mu > 0; \text{Re } \mu < 0]$
12. $\int_0^{\infty} x^{\alpha-1} \frac{\text{ch} cx}{\text{sh}^2 cx} dx = \frac{2\Gamma(\alpha)}{c^\alpha} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\alpha-1}}$
 $[\text{Re } \alpha > 2; \text{см. 5.1.4}]$
13. $\int_0^{\infty} \frac{\text{sh} ax \text{sh} bx}{x \text{sh} cx} dx = \frac{1}{2} \ln \frac{\cos [(a-b)\pi/(2c)]}{\cos [(a+b)\pi/(2c)]}$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$
14. $\int_0^{\infty} \frac{\text{sh}^2 ax \text{ch} bx}{x \text{sh} cx} dx = \frac{1}{4} \ln \frac{1 + \cos (b\pi/c)}{\cos (2a\pi/c) + \cos (b\pi/c)}$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$
15. $\int_0^{\infty} \frac{\text{sh} ax \text{ch} bx}{x \text{ch} cx} dx = \frac{1}{2} \ln \frac{\cos [b\pi/(2c)] + \sin [a\pi/(2c)]}{\cos [b\pi/(2c)] - \sin [a\pi/(2c)]}$
 $[\text{Re } c > | \text{Re } a | + | \text{Re } b |]$

$$16. \int_0^{\infty} \frac{1}{x^2+z^2} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{\pi}{z} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(bk\pi/c)}{cz + k\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$17. = (-1)^n b \frac{e^{ibn\pi/c}}{2n} + \frac{c}{2\pi n} \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} e^{ibk\pi/c} +$$

$$+ (-1)^n \frac{c}{2\pi n} e^{ibn\pi/c} \ln(1 + e^{ib\pi/c}) + \frac{c}{2\pi n} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{(1+z)^{n-1}}{z} \ln(1+z) \right]_{z=e^{ib\pi/c}}$$

$$[cz = \pi n; \operatorname{Re} c > |\operatorname{Re} b|; \operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

$$18. = -\frac{b}{2} \cos \frac{b\pi}{c} + \frac{c}{\pi} \sin \frac{b\pi}{c} \ln \left(2 \cos \frac{b\pi}{2c} \right) \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$19. = c \sin \frac{b\pi}{2c} - \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \operatorname{ctg} \left(\frac{c-b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$20. \int_0^{\infty} \frac{x}{x^2+z^2} \frac{\operatorname{sh} bx}{\operatorname{ch}(\pi x/z)} dx = -2 \sin \frac{bz}{2} + \frac{\pi}{2} \sin bz - \operatorname{ch} bz \ln \operatorname{tg} \frac{bz+\pi}{4}$$

$$[z > 0; z |\operatorname{Re} b| < \pi]$$

$$21. \int_0^{\infty} \frac{x}{x^2+z^2} \frac{\operatorname{ch} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2cz} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k \cos(bk\pi/c)}{cz + k\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$22. = \frac{b\pi}{2c} \sin \frac{b\pi}{c} - \frac{1}{2} + \frac{1}{2} \cos \frac{b\pi}{c} \ln \left[2 \left(1 + \cos \frac{b\pi}{c} \right) \right]$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$23. = \frac{\pi}{2} \cos \frac{b\pi}{2c} - 1 + \sin \frac{b\pi}{2c} \ln \operatorname{tg} \left(\frac{c+b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$24. \int_0^{\infty} \frac{1}{x^2+z^2} \frac{\operatorname{ch} bx}{\operatorname{ch} cx} dx = \frac{(-1)^n}{2n+1} \exp \left[\frac{ib}{c} \left(n + \frac{1}{2} \right) \pi \right] \left[-ib + \right.$$

$$\left. + \frac{c}{\pi} \ln(1 + e^{ib\pi/c}) \right] + \frac{c \exp[ib\pi/(2c)]}{\pi(2n+1)} \sum_{k=0}^{n-1} \frac{(-1)^k}{k-n} \exp(ibk\pi/c) +$$

$$+ \frac{c \exp[ib\pi/(2c)]}{\pi(2n+1)(n+1)} {}_2F_1(n+1, 1; n+2; -e^{ib\pi/c})$$

$$[cz = \pi(n+1/2); \operatorname{Re} c > |\operatorname{Re} b|]$$

$$25. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} - \frac{c}{2} \cos \frac{b\pi}{c} + \frac{2c}{\pi} \sin \frac{b\pi}{c} \ln \operatorname{tg} \left(\frac{b+c}{4c} \pi \right)$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$26. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \left(2 \cos \frac{b\pi}{2c} \right) + b \sin \frac{b\pi}{2c} \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

2.4.6. Интегралы от $(a+b \operatorname{ch}^n bx + c \operatorname{sh}^n bx)^p \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.

$$1. \int_0^{\infty} \frac{\operatorname{ch} bx dx}{(\operatorname{ch} a - \operatorname{ch} x)^v} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-v)}{\operatorname{sh}^{v-1/2} a} P_{v-1/2}^{v-1/2}(\operatorname{ch} a)$$

$$[a > 0; \operatorname{Re} v < 1; |\arg b| < \pi]$$

$$2. \int_0^{\infty} \frac{dx}{a+b \operatorname{ch} x} = \frac{2}{\sqrt{b^2-a^2}} \operatorname{arctg} \frac{\sqrt{b^2-a^2}}{a+b} \quad [|b| > |a|]$$

$$3. = \frac{1}{\sqrt{a^2-b^2}} \ln \left| \frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \right| \quad [0 < |b| < |a|]$$

$$= a^{-1} \quad [b = a]$$

$$4. \int_0^{\infty} \frac{dx}{a+b \operatorname{sh} x} = \frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \right| \quad [ab \neq 0]$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} bx}{a+\operatorname{ch} cx} dx = \frac{\pi}{c\sqrt{a^2-1}} \operatorname{cosec} \frac{b\pi}{c} \operatorname{sh} \left(\frac{b}{c} \operatorname{Arch} a \right) \quad [a > 1; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$7. = \frac{\pi}{c\sqrt{1-a^2}} \operatorname{cosec} \frac{b\pi}{c} \sin \left(\frac{b}{c} \arccos a \right) \quad [-1 < a < 1; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{1+\operatorname{ch} cx} dx = \frac{\pi b}{c^2 \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$9. \int_0^{\infty} \frac{\operatorname{ch} bx}{(1+\operatorname{ch} cx)^v} dx = \frac{1}{c} \sqrt{\frac{\pi}{2}} (a^2-1)^{1/4-v/2} \Gamma \left[\frac{v+b/c}{2}, \frac{v-b/c}{2} \right] P_{b/c-1/2}^{1/2-v} (a)$$

$$[a \notin [-1, 1]; \operatorname{Re}(v \pm b/c) > 0]$$

$$10. \int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{a+\operatorname{ch} cx}} dx = \frac{\pi}{c\sqrt{2}} \sec \frac{b\pi}{c} P_{b/c-1/2}^{1/2} (a)$$

$$[-1 < a < 1; \operatorname{Re} c > 2|\operatorname{Re} b|]$$

$$11. \int_0^{\infty} \frac{\operatorname{ch} bx dx}{(\cos t + \operatorname{ch} cx)^2} = \frac{\pi [b \sin t \cos(bt/c) - c \cos t \sin(bt/c)]}{c^2 \sin^2 t \sin(b\pi/c)}$$

$$[2\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi]$$

$$12. \int_0^{\infty} \frac{\operatorname{ch} b - \operatorname{ch} cx}{(1+\operatorname{ch} cx)^v} dx = \frac{2^{v-1}}{c} B \left(v + \frac{b}{c}, v - \frac{b}{c} \right)$$

$$[\operatorname{Re} v > 0; \operatorname{Re}(v) > \operatorname{Re} c + |\operatorname{Re} b|]$$

$$13. \int_0^{\infty} \frac{\cos t - \operatorname{ch} bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi}{c} \frac{\sin [b(\pi-t)/c]}{\sin t \sin(b\pi/c)} - \frac{\pi-t}{c \sin t} \cot t$$

$$[\operatorname{Re} c > |\operatorname{Re} b|; 0 < t < \pi; 0 < \pi < \pi]$$

$$14. \int_0^{\infty} \frac{(\operatorname{sh} ax \operatorname{sh} bx)}{(\operatorname{ch} ax \operatorname{ch} bx)} \frac{dx}{\cos t + \operatorname{ch} cx} = \frac{\pi}{c \sin t} \left(\cos \frac{2b\pi}{c} - \cos \frac{2a\pi}{c} \right)^{-1} \times$$

$$\times \left\{ \sin [a(\pi-t)/c] \sin [b(\pi+t)/c] - \sin [a(\pi+t)/c] \sin [b(\pi-t)/c] \right\}$$

$$\times \left\{ \cos [a(\pi-t)/c] \cos [b(\pi+t)/c] - \cos [a(\pi+t)/c] \cos [b(\pi-t)/c] \right\}$$

$$[0 < |t| < \pi; \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|; \pi \operatorname{Re} c > |\operatorname{Re}(a)|]$$

$$15. \int_0^{\infty} \frac{\{ \operatorname{sh} ax \operatorname{sh} bx \}}{\{ \operatorname{ch} ax \operatorname{ch} bx \}} \frac{dx}{1 + \operatorname{ch} cx} = \frac{\pi}{c^2} \frac{2\pi}{\{ \cos(2b\pi/c) - \cos(2a\pi/c) \}} \times$$

$$\times \left[b \begin{cases} \sin(a\pi/c) \cos(b\pi/c) \\ \cos(a\pi/c) \sin(b\pi/c) \end{cases} - a \begin{cases} \cos(a\pi/c) \sin(b\pi/c) \\ \sin(a\pi/c) \cos(b\pi/c) \end{cases} \right] \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$16. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi \sin(bt/c)}{2c \operatorname{ch}(b\pi/c) \sin(t/2)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|; 0 < |t| < \pi]$$

$$17. \int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \cos(b\pi/c)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|]$$

$$18. \int_0^{\infty} \frac{\operatorname{sh} cx \operatorname{sh} bx}{(\cos t + \operatorname{ch} cx)^2} dx = \frac{b\pi \sin(bt/c)}{c^2 \sin t \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi]$$

$$19. \int_0^{\infty} \frac{dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{\sqrt{b^2 - c^2}}{b + c} \quad [|b| > |c|]$$

$$20. = \frac{1}{\sqrt{c^2 - b^2}} \ln \left| \frac{b + c + \sqrt{c^2 - b^2}}{b + c - \sqrt{c^2 - b^2}} \right| \quad [0 < |b| < |c|]$$

$$21. = c^{-1} \quad [b = c]$$

$$22. \int_0^{\infty} \frac{\{ \operatorname{sh} ax \operatorname{sh} bx \}}{\{ \operatorname{ch} ax \operatorname{ch} bx \}} \frac{dx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} = \frac{\pi}{4(a^2 - b^2)} \left\{ \frac{b}{a} \right\} \quad [\operatorname{Re} a, \operatorname{Re} b > 0]$$

$$23. \int_0^{\infty} \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \left| \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} \right|$$

$$[b^2 < a^2 + c^2; |a| \neq |b|]$$

$$24. = \frac{2(a-b)}{c(a-b-c)} \quad [b^2 = a^2 + c^2; c(a-b-c) < 0]$$

$$25. = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left(\operatorname{arctg} \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + e\pi \right)$$

$[b^2 > a^2 + c^2, e = 0 \text{ при } (b-a)(a+b+c) > 0; e = 1 \text{ при } (b-a)(a+b+c) < 0, a < b+c; e = -1 \text{ при } (b-a)(a+b+c) < 0, a > b+c]$

$$26. \int_0^{\infty} \frac{\operatorname{ch} cx \operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{arctg} \frac{b\pi}{2c} \operatorname{ch} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$27. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{sh}^2 t + \operatorname{ch}^2 cx} dx = \frac{\pi}{c} \left(\operatorname{sh} 2t \sin \frac{b\pi}{2c} \right)^{-1} \operatorname{sh} \frac{bt}{c} \quad [2\operatorname{Re} c > |\operatorname{Re} b|]$$

$$28. \int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{\operatorname{sh}^2 t + \operatorname{ch}^2 cx}} dx = \frac{\pi}{2c} \operatorname{arctg} \frac{b\pi}{2c} P_{b/(2a)-1/2}(\operatorname{ch} 2t) \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$29. \int_0^{\infty} \frac{\operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{sh}^{-1} \frac{b\pi}{2c} \operatorname{sh} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [2\operatorname{Re} c > |\operatorname{Re} b|]$$

$$30. \int_0^{\infty} \frac{dx}{b \operatorname{ch}^2 x + c \operatorname{sh}^2 x} = \frac{1}{\sqrt{bc}} \operatorname{arctg} \frac{c}{\sqrt{bc}} \quad [bc > 0]$$

$$31. = \frac{1}{2\sqrt{-bc}} \ln \left| \frac{c - \sqrt{-bc}}{b + \sqrt{-bc}} \right| \quad [bc < 0]$$

2.4.7. Интегралы от $A(x) (a + b \operatorname{ch}^2 bx + c \operatorname{sh}^2 bx)^{-1} \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.

$$1. \int_0^{\infty} \frac{x dx}{\operatorname{sh}(x/2) \sqrt{\operatorname{ch} a - \operatorname{ch} x}} = \frac{\pi \sqrt{2}}{\operatorname{sh}(a/2)} \arcsin \operatorname{th} \frac{a}{2} \quad [a > 0]$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1} dx}{\cos t + \operatorname{ch} x} = f_{\alpha} \quad [\operatorname{Re} \alpha > 0; |t| < \pi]$$

$$f_{\alpha} = \frac{\pi^{2\alpha-1}}{\sin t \sin(\alpha\pi/2)} \left[\zeta \left(1-\alpha, \frac{1}{2} - \frac{t}{2\pi} \right) - \zeta \left(1-\alpha, \frac{1}{2} + \frac{t}{2\pi} \right) \right] =$$

$$= \frac{2\Gamma(\alpha)}{\sin t} \sum_{k=1}^{\infty} \frac{\sin kt}{k^{\alpha}}$$

$$J_1 = \frac{4}{\sin t} \left[t \ln 2 - 2L \left(\frac{t}{2} \right) \right], \quad J_2 = \frac{t}{3 \sin t} (\pi^2 - t^2),$$

$$J_3 = \frac{t}{15 \sin t} (\pi^2 - t^2) (7\pi^2 - 3t^2).$$

$$3. \int_0^{\infty} \frac{x^{\alpha-1}}{\cos t + \operatorname{ch} x} \left\{ \frac{\operatorname{sh}(x/2)}{\operatorname{ch}(x/2)} \right\} dx = \pi^{2\alpha-1} \frac{\left\{ \sin(t/2) \cos(\alpha\pi/2) \right\}^{-1}}{\left\{ \cos(t/2) \sin(\alpha\pi/2) \right\}} \times$$

$$\times \left[\zeta \left(1-\alpha, \frac{1}{4} - \frac{t}{4\pi} \right) - \zeta \left(1-\alpha, \frac{1}{4} + \frac{t}{4\pi} \right) \pm \zeta \left(1-\alpha, \frac{3}{4} + \frac{t}{4\pi} \right) - \right.$$

$$\left. - \zeta \left(1-\alpha, \frac{3}{4} - \frac{t}{4\pi} \right) \right] \quad [\operatorname{Re} \alpha > 0; |t| < \pi]$$

$$4. \int_0^{\infty} \frac{x \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi}{c^2} \operatorname{sh}^{-2} \frac{b\pi}{c} \left(c \sin \frac{b\pi}{c} - b\pi \cos \frac{b\pi}{c} \right) \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$5. \int_0^{\infty} \frac{x^{2v+1} \operatorname{sh} bx}{(1 + \operatorname{ch} cx)^v} dx = -\frac{\pi}{c^2} \Gamma \left[\frac{2-v}{v-2}, 2v \right] \frac{\partial^{2v+1}}{\partial b^{2v+1}} \frac{b}{\sin(b\pi/c)} \quad [\operatorname{Re}(v) > |\operatorname{Re} b|]$$

$$6. \int_0^{\infty} \frac{x \operatorname{ch}(x/2) dx}{\cos t + \operatorname{ch} x} = \frac{4}{\cos(t/2)} \left[\frac{\pi}{2} \ln 2 - L \left(\frac{\pi+t}{4} \right) - L \left(\frac{\pi-t}{4} \right) \right]$$

$[t \neq \pi(2k+1); k=0, \pm 1, \pm 2, \dots]$

$$7. \int_0^{\pi} \frac{x^{2n+1} \operatorname{sh} cx \, dx}{(\cos t + \operatorname{ch} cx)^2} = I_n$$

$$[\operatorname{Re} c > 0; |t| < \pi]$$

$$I_n = \frac{2(2n+1)!}{c^{2n+2} \sin t} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos kt}{k^{2n+1}},$$

$$I_0 = -\frac{t+\pi}{c^2 \sin t}, \quad I_1 = \frac{t(\pi^2 - t^2)}{c^4 \sin t}.$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{(x^2 + z^2)(\cos t + \operatorname{ch} cx)} dx = \frac{\pi \operatorname{ch} ibx}{2z(\cos cz + \cos t)} +$$

$$+ \frac{\pi}{\operatorname{sh} t} \sum_{k=0}^{\infty} \left\{ \frac{\exp[-ib\pi[(2k+1)\pi - t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) - t]^2} - \frac{\exp[-ib\pi[(2k+1)\pi + t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) + t]^2} \right\}$$

$$[\operatorname{Re} c > \operatorname{Re} b; |t| < \pi; \operatorname{Re} z > 0]$$

$$9. \int_0^{\infty} \frac{x^{-1/2}}{1 + 2 \operatorname{ch}^2 cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \sqrt{\frac{2\pi}{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{kc + \sqrt{k^2 c^2 - b^2}}{k^2 c^2 - b^2} \right)^{1/2} \sin \frac{k\pi}{3}$$

$$[2 \operatorname{Re} c > \operatorname{Re} b]$$

$$10. \int_0^{\infty} \frac{x^{2n+1} \operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \operatorname{sh} bx \, dx =$$

$$= \frac{\pi}{2\sqrt{2}} \frac{\partial^{2n+1}}{\partial b^{2n+1}} \left[\sec \frac{b\pi}{2c} \operatorname{ch} \left(\frac{b}{2c} \ln(1 + \sqrt{2}) \right) \right] \quad [\operatorname{Re} c > \operatorname{Re} b]$$

$$11. \int_0^{\infty} \frac{x^{\alpha-1} \operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} dx = \frac{\Gamma(\alpha)}{2(a^2 - b^2)^{\alpha/2}} \left[\left(\frac{a+b}{a-b} \right)^{\alpha/2} - \left(\frac{a-b}{a+b} \right)^{\alpha/2} \right] \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

$$[\operatorname{Re} a, \operatorname{Re} b > 0; \operatorname{Re} \alpha > -2; \text{см. 5.1.4}]$$

$$12. \int_0^{\infty} \frac{x^n dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{(2n)!}{(b+c) 2^{n+1}} \Phi \left(\frac{c-b}{c+b}, n+1, \frac{1}{2} \right) \quad [b, c > 0]$$

2.4.8. Интегралы от $A(x) \operatorname{ch} bx \left\{ \frac{\operatorname{sh} \sqrt{a^2 - x^2}}{\operatorname{ch} \sqrt{a^2 - x^2}} \right\}$.

Условие: $a > 0$.

$$1. \int_0^a (a^2 - x^2)^{(\delta-1)/2} \operatorname{ch} bx \left\{ \frac{\operatorname{sh}(c \sqrt{a^2 - x^2})}{\operatorname{ch}(c \sqrt{a^2 - x^2})} \right\} dx = \frac{\pi a^{\delta} c^{\delta}}{2(b^2 + c^2)^{\delta/2}} I_{\delta}(a \sqrt{b^2 + c^2})$$

$$[\delta = \{1\}].$$

$$2. \int_0^a \frac{\operatorname{ch} bx}{(a^2 - x^2)^{3/4}} \left\{ \frac{\operatorname{sh}(c \sqrt{a^2 - x^2})}{\operatorname{ch}(c \sqrt{a^2 - x^2})} \right\} dx =$$

$$= \left(\frac{\pi}{2} \right)^{3/2} \sqrt{c} I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - b) \right] I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + b) \right].$$

2.4.10]

$$3. \int_0^a \sqrt{a^2 - x^2} \operatorname{ch}(c \sqrt{a^2 - x^2}) \operatorname{ch} bx \, dx =$$

$$= \frac{\pi a^2 c^3}{2(b^2 + c^2)} \left[I_0(a \sqrt{b^2 + c^2}) + \frac{b^2 - c^2}{ac^2 \sqrt{b^2 + c^2}} I_1(a \sqrt{b^2 + c^2}) \right].$$

$$4. \int_0^a \frac{\operatorname{sh}(c \sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \operatorname{ch} bx \, dx =$$

$$= \left(\frac{\pi}{2} \right)^{3/2} \sqrt{b} I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right].$$

2.4.9. Интегралы, содержащие $\operatorname{th} ax$, $\operatorname{cth} ax$ и разности гиперболических и алгебраических функций.

$$1. \int_0^{\infty} x^{\alpha-1} \operatorname{th} ax \, dx = 2^{1-\alpha} a^{-\alpha} (2^{1-\alpha} - 1) \Gamma(\alpha) \zeta(\alpha) \quad [\operatorname{Re} a > 0; -1 < \operatorname{Re} \alpha < 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\operatorname{th} ax - 1}{\operatorname{cth} ax - 1} \right\} dx = 2^{1-\alpha} a^{-\alpha} \Gamma(\alpha) \zeta(\alpha) \left\{ \frac{2^{1-\alpha} - 1}{1} \right\} \quad [a > 0; \operatorname{Re} \alpha > \{0\}].$$

$$3. \int_0^{\infty} \frac{x}{(x^2 + 1)^{\rho}} \operatorname{th} \frac{\pi x}{2} \, dx = 2^{-2\rho-2} \zeta(2\rho-3) \quad [\operatorname{Re} \rho > 1].$$

$$4. \int_0^{\infty} \frac{\operatorname{th}(cx/2)}{x \operatorname{ch} cx} dx = \ln 2 \quad [c > 0].$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} bx - 1}{x \operatorname{sh} cx} dx = -\ln \cos \frac{b\pi}{2c} \quad [\operatorname{Re} c > \operatorname{Re} b].$$

$$6. \int_0^{\infty} \left(\frac{\pi}{\operatorname{sh} ax} - \frac{b}{\operatorname{sh} bx} \right) \frac{dx}{x} = (b-a) \ln 2 \quad [\operatorname{Re} a, \operatorname{Re} b > 0].$$

$$7. \int_0^{\infty} x^{2\rho-1} \left(\frac{1}{\operatorname{sh} x} - \frac{1}{x} \right) dx = 2(1-2^{-\rho}) \Gamma(\rho) \zeta(\rho) \quad [|\operatorname{Re} \rho| < 1].$$

2.4.10. Интегралы от $x^{\alpha} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^{\alpha}$.

$$1. \int_0^{\infty} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{p^2 - b^2} \left\{ \frac{b}{p} \right\} \quad [\operatorname{Re} p > \operatorname{Re} b].$$

$$2. \int_0^{\infty} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^2 dx = \frac{1}{p(p^2 - 4b^2)} \left\{ \frac{2b^2}{p^2 - 2b^2} \right\} \quad [\operatorname{Re} p > 2 \operatorname{Re} b].$$

$$3. \int_0^{\infty} e^{-px} \operatorname{sh}^n bx dx = b^n p^{1-n} n! \prod_{k=0}^{[n/2]} [p^2 - b^2(2k+1)^2]^{-1} \\ [2n=1 \rightarrow (-1)^n; \operatorname{Re} p > n | \operatorname{Re} b|].$$

$$4. \int_0^{\infty} e^{-px} \operatorname{sh}^{n-1} bx dx = 2^{-n} b^{n-1} \Gamma\left(n, \frac{p}{2b} + \frac{1-n}{2}\right) \\ [\operatorname{Re} v, \operatorname{Re} b > 0; \operatorname{Re} p > \operatorname{Re}(nb - b_1)].$$

$$5. \int_0^{\infty} e^{-px} \operatorname{ch}^n bx dx = 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \frac{p}{p^2 - (n-2k)^2 b^2} + \frac{1 + (-1)^n}{2} \left(\frac{n}{2}\right) \frac{1}{2^n p} \\ [\operatorname{Re} p > n | \operatorname{Re} b|; n=1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch}^n cx} dx = I_n \\ [\operatorname{Re} p > -n | \operatorname{Re} c|].$$

$$I_1 = \frac{1}{c} \beta\left(\frac{p+c}{2c}\right), \quad I_2 = \frac{p}{c^2} \beta\left(\frac{p}{2c}\right) - \frac{1}{c}, \\ I_{2m+1} = \frac{2^{2m-1}}{c} B(m, m) - \frac{1}{2mc}.$$

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\Gamma(\alpha)}{2} [(p-b)^{-\alpha} - (p+b)^{-\alpha}] \\ [\operatorname{Re} p > | \operatorname{Re} b|; \operatorname{Re} \alpha > -\frac{1}{2}].$$

$$8. \int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh} bx dx = \frac{1}{2} \ln \frac{p+b}{p-b} \\ [\operatorname{Re} p > | \operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh}^2 bx dx = \frac{1}{2} \ln \sqrt{1 - \frac{4b^2}{p^2}} \\ [\operatorname{Re} p > 2 | \operatorname{Re} b|].$$

$$10. \int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^m dx = \\ = 2^{1-m} \Gamma(\alpha) \sum_{k=0}^{[(m-1)/2]} (-1)^{[m/2]-k} \binom{m}{k} [(p-b(m-2k))^{-\alpha} - (p+b(m-2k))^{-\alpha}] + \\ + \frac{1 + (-1)^m}{2} 2^{-m} p^{-\alpha} \Gamma(\alpha) \left(\frac{n}{2}\right) \\ [\operatorname{Re} \alpha > -\frac{1}{2}; \operatorname{Re} p > m | \operatorname{Re} b|].$$

$$11. \int_0^{\infty} x^m e^{-px} \operatorname{sh}^n bx dx = 2^{-n} m! \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(p-bn+2kb)^{m+1}} \\ [\operatorname{Re} p > n | \operatorname{Re} b|].$$

$$12. \int_0^{\infty} \frac{e^{-px}}{x^n} \operatorname{sh}^n bx dx = I_n^n \\ [\operatorname{Re} p > n | \operatorname{Re} b|; n \leq m; m, n=1, 2, 3, \dots].$$

$$I_1^{2n} = 2^{-2n-1} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \ln [p^2 - 4(n-k)^2 b^2] - (-1)^n 2^{-2n} \binom{2n}{n} \ln p,$$

$$I_1^{2n+1} = 2^{-2n-1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \ln \frac{p+b(2n-2k+1)}{p-b(2n-2k+1)}, \quad I_1^1 = \frac{1}{2} \ln \frac{p+b}{p-b},$$

$$I_2^2 = \frac{b}{2} \ln \frac{p+2b}{p-2b} + \frac{p}{4} \ln \left(1 - \frac{4b^2}{p^2}\right),$$

$$I_4^4 = \frac{p}{8} \ln \frac{p-3b}{p+3b} - \frac{3p}{4} \ln \frac{p-b}{p+b} + \frac{3b}{8} \ln \frac{p^2-b^2}{p^2-3b^2}.$$

$$13. \int_0^{\infty} x^{\alpha-1} \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{2^{1-\alpha}}{c^{\alpha}} \Gamma(\alpha) \zeta\left(\alpha, \frac{p+c}{2c}\right) \\ [\operatorname{Re} \alpha > 1; \operatorname{Re} p > - | \operatorname{Re} c|].$$

$$14. \int_0^{\infty} x^{2n+1} \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{\pi^{2n+2}}{2(n+1)c^{2n+2}} |B_{2n+1}| \\ [\operatorname{Re} c > 0].$$

$$15. \int_0^{\infty} x^2 \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{1}{8} \left(\frac{\pi}{c}\right)^4 - \frac{12}{c^4} \sum_{k=1}^n \frac{1}{(2k-1)^4} \\ [\operatorname{Re} c > 0; n=1, 2, 3, \dots].$$

$$16. \int_0^{\infty} x^{\alpha-1} \frac{e^{-px}}{\operatorname{ch} cx} dx = 2^{1-2\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{1}{4} + \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{3}{4} + \frac{p}{4c}\right) \right] \\ [\operatorname{Re} \alpha > 0; \operatorname{Re} p > - | \operatorname{Re} c|].$$

$$17. \int_0^{\infty} x^{\alpha-1} \frac{e^{-cx}}{\operatorname{ch} cx} dx = \frac{2^{1-\alpha}}{c^{\alpha}} (1 - 2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha) \\ [\operatorname{Re} \alpha, \operatorname{Re} c > 0].$$

$$18. \int_0^{\infty} x^{2n+1} \frac{e^{-cx}}{\operatorname{ch} cx} dx = \frac{1 - 2^{1-2n}}{2(n+1)c^{2n+2}} \pi^{2n+2} |B_{2n+1}| \\ [\operatorname{Re} c > 0].$$

$$19. \int_0^{\infty} \frac{e^{iax}}{\operatorname{ch} cx} dx = \frac{\pi}{c \operatorname{ch} [a\pi/(2c)]} \\ [a, \operatorname{Re} c > 0].$$

2.4.11. Интегралы от e^{-px} и гиперболических функций.

$$1. \int_0^{\infty} e^{-px} |\operatorname{sh} bx| dx = \frac{b}{p^2 - b^2} \operatorname{ctg} \frac{p\pi}{2b} \\ [\operatorname{Re} p > | \operatorname{Re} b|].$$

$$2. \int_0^{\infty} e^{-px} |\operatorname{ch} bx| dx = \frac{1}{p^2 - b^2} \left(p - b \operatorname{cosech} \frac{p\pi}{2b} \right) \\ [\operatorname{Re} p > | \operatorname{Re} b|].$$

$$3. \int_0^{\infty} e^{-px} |\operatorname{sh} bx|^v dx = - \frac{\pi}{(v+1) 2^{v+1} b} \left[\operatorname{sh} \frac{p\pi}{2b} \Gamma\left(1 + \frac{v}{2} + \frac{p}{2b}, 1 + \frac{v}{2} - \frac{p}{2b}\right) \right]^{-1} \\ [\operatorname{Re} p > \operatorname{Re} v | \operatorname{Re} b|].$$

$$4. \int_0^{\infty} e^{-px} \operatorname{th} cx dx = \frac{1}{c} \beta\left(\frac{p}{2c}\right) - \frac{1}{p}$$

[c, Re p > 0].

$$5. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{1}{2c} \left[\psi\left(\frac{c+p+b}{2c}\right) - \psi\left(\frac{c+p-b}{2c}\right) \right]$$

[Re p > |Re b| - |Re c|].

$$6. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{1}{b} - \frac{\pi}{2c} \operatorname{ctg} \frac{b\pi}{2c}$$

[2 Re c > |Re b|].

$$7. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} px}{\operatorname{ch}^2 px} dx = \frac{1 - \ln 2}{p}$$

[Re p > 0].

2.4.12. Интегралы от $A(x)$, e^{-px} и гиперболических функ.

$$1. \int_{-a}^a x^k \sqrt{a^2 - x^2} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{a^{2k+1}}{2} [I_k(ap+ab) \mp I_k(ap-ab)]$$

[k = 0 или 1; a, Re p > 0; |arg b| < π].

$$2. \int_0^{\infty} \frac{(x+z)^n}{x} e^{-px} \operatorname{sh} bx dx = (-1)^n b p^{-n-1} \sum_{k=0}^{\infty} \left(\frac{b}{2p}\right)^{2k} L_{n-n-2k-1}(pz)$$

[Re p > |Re b|; |arg z| < π].

$$3. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{th} cx dx = 2^{1-\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{2c+p}{4c}\right) \right] - p^{-\alpha} \Gamma(\alpha)$$

[Re p > 0; Re α > -1].

$$4. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{th} cx dx = \ln \frac{p}{4c} + 2 \ln \Gamma\left(\frac{p}{4c}\right) - 2 \ln \Gamma\left(\frac{p+2c}{4c}\right)$$

[c, Re p > 0].

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{cth} cx dx = \Gamma(\alpha) \left[2^{1-\alpha} c^{-\alpha} \zeta\left(\alpha, \frac{p}{2c}\right) - p^{-\alpha} \right]$$

[c, Re p > 0; Re α > 1].

$$6. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{cth} px dx = \frac{1}{p^{\alpha}} \left(\frac{\pi^2}{3} - 1 \right)$$

[Re p > 0].

$$7. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{sh} bx dx = \frac{1}{4} \ln \frac{p^2 - (a-b)^2}{p^2 - (a+b)^2}$$

[Re p > |Re a| + |Re b|].

$$8. \int_0^{\infty} \frac{1}{x^3} e^{-px} \operatorname{sh} ax \operatorname{sh} bx dx =$$

$$= \frac{a}{4} \ln \frac{(p+b)^2 - a^2}{(p-b)^2 - a^2} + \frac{b}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2} + \frac{p}{4} \ln \frac{p^2 - (a+b)^2}{p^2 - (a-b)^2}$$

[Re p > |Re a| + |Re b|].

$$9. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{ch} bx dx = \frac{1}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2}$$

[Re p > |Re a| + |Re b|; p^2 + a^2 > b^2].

$$10. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{sh} px} dx = \frac{1}{2} \ln \frac{b\pi}{p \sin(b\pi/p)}$$

[Re p > |Re b|].

$$11. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{ch} 2bx} dx = \ln \Gamma\left(\frac{p+b}{2b}\right) - \ln \Gamma\left(\frac{p+2b}{4b}\right) - \frac{1}{2} \ln \frac{p}{2b}$$

[Re p > 0].

$$12. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{th} \frac{px}{2} dx = 2 \ln \frac{\pi}{2\sqrt{2}}$$

[Re p > 0].

2.4.13. Интегралы от $x^{\alpha} (e^x + a)^{\rho} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{bx} + 1} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \Gamma(\alpha) \left\{ b^{-\alpha} i^{-\alpha-\delta} \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} + \right.$$

$$\left. + \frac{(2a)^{-\alpha}}{2} \left[\zeta\left(\alpha, \frac{a-b}{2a}\right) \mp \zeta\left(\alpha, \frac{a+b}{2a}\right) - \zeta\left(\alpha, -\frac{b}{2a}\right) \pm \zeta\left(\alpha, \frac{b}{2a}\right) \right] \right\}$$

[Re α > -δ; Re a > |Re b|; δ = {0}].

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{2bx} - 1} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{-\alpha}}{2} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{a-b}{a}\right) \mp \zeta\left(\alpha, \frac{a+b}{a}\right) \right]$$

[Re α > {0}]; Re α > |Re b|].

$$3. \int_0^{\infty} \frac{\operatorname{sh} bx}{e^{bx} + 1} dx = \frac{1}{2b} \pm \frac{\pi}{2a \sin(b\pi/a)} \left\{ \cos\left(\frac{b\pi}{2}\right) \right\}$$

[Re a > |Re b|].

$$4. \int_0^{\infty} \frac{x \operatorname{ch} bx}{e^{bx} - 1} dx = \frac{\pi^2}{2a^2 \sin^2(b\pi/a)} - \frac{1}{2b^2}$$

[Re a > |Re b|].

$$5. \int_0^{\infty} \frac{\operatorname{sh}^2 bx}{e^{bx} - 1} dx = -\frac{1}{2a} \left[C + \operatorname{Re} \psi\left(-\frac{2b}{a}\right) \right]$$

[Re a > 2 |Re b|].

$$6. \int_0^{\infty} \frac{x^{\alpha-1} \operatorname{sh}^2 bx}{e^{bx} + 1} dx = \frac{1}{4} \ln \left(\frac{a}{2\pi} \operatorname{tg} \frac{b\pi}{a} \right)$$

[Re a > 2 |Re b|].

$$4. \int_0^{\infty} x^2 e^{-ax} \operatorname{sh} bx dx = \frac{\sqrt{\pi}}{8a^{5/2}} (2a+b^2) \exp\left(\frac{b^2}{4a}\right) \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) - \frac{b}{4a^2} \quad [\operatorname{Re} a > 0]$$

$$5. \int_0^{\infty} \frac{1}{x} e^{-ax} \operatorname{sh} bx dx = \frac{\pi}{2} \operatorname{erf}\left(\frac{ib}{2\sqrt{a}}\right) \quad [\operatorname{Re} a > 0]$$

$$6. \int_0^{\infty} x^{1/2} e^{-ax} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{\pi}{4} \left(\frac{b}{2a}\right)^{3/2} \exp\left(\frac{b^2}{8a}\right) \left[I_{\mp 1/4}\left(\frac{b^2}{8a}\right) + I_{\pm 3/4}\left(\frac{b^2}{8a}\right)\right] \quad [\operatorname{Re} a > 0]$$

$$7. \int_0^{\infty} x^{-1/2} e^{-ax} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{\pi}{2} \sqrt{\frac{b}{2a}} \exp\left(\frac{b^2}{8a}\right) I_{\pm 1/4}\left(\frac{b^2}{8a}\right) \quad [\operatorname{Re} a > 0]$$

$$8. \int_0^{\infty} e^{-ax-cx} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{1}{4} \sqrt{\frac{\pi}{a}} \left\{ \exp\left[\frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp \exp\left[\frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\} \quad [\operatorname{Re} a > 0]$$

$$9. \int_0^{\infty} x^{\alpha-1} e^{-ax-cx} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{\Gamma(\alpha)}{2(2a)^{\alpha/2}} \exp\left(\frac{c^2+b^2}{8a}\right) \left[\exp\left(-\frac{bc}{4a}\right) D_{-\alpha}\left(\frac{c-b}{\sqrt{2a}}\right) \mp \exp\left(\frac{bc}{4a}\right) D_{-\alpha}\left(\frac{c+b}{\sqrt{2a}}\right) \right] \quad [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\{1\}]$$

$$10. \int_0^{\infty} x^{\alpha-1} e^{-ax-cx} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = -\frac{1}{8a} \sqrt{\frac{\pi}{a}} \left\{ (c-b) \exp\left[\mp \frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp (c+b) \exp\left[\mp \frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\} + \left\{ \frac{0}{(2a)^{1/2}} \right\} \quad [\operatorname{Re} a > 0]$$

$$11. \int_0^{\infty} x^{\delta} \frac{e^{-ax}}{x^2+z^2} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{\pi}{4} \left\{ \frac{i}{z^2} \right\} e^{ax^2} \left[2 \left\{ \frac{i \sin bz}{\cos bz} \right\} - \operatorname{erf}\left(z\sqrt{a} + \frac{ib}{2\sqrt{a}}\right) \pm \operatorname{erf}\left(z\sqrt{a} - \frac{ib}{2\sqrt{a}}\right) \right] \quad [\operatorname{Re} a, \operatorname{Re} z > 0; \delta = \{1\}]$$

$$12. \int_0^{\infty} e^{-ax} \operatorname{sh}[(2n+1)x] \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^n (\pm 1)^k \exp\left(\pm \frac{k^2}{a}\right) \right] \quad [\operatorname{Re} a > 0]$$

$$13. \int_{-\infty}^{\infty} e^{-ax-cx} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2+c^2}{4a}\right) \frac{\operatorname{sh}[b/(2a)]}{\operatorname{ch}[bc/(2a)]} \quad [\operatorname{Re} a > 0]$$

2.4.16. Интегралы от $A(x) \exp(-p\sqrt{x^2+z^2}) \frac{\operatorname{sh} ax}{\operatorname{ch} ax}$.

$$1. \int_0^{\infty} \exp(-p\sqrt{x^2+z^2}) \operatorname{ch} bx dx = \frac{pz}{\sqrt{p^2-b^2}} K_1(z\sqrt{p^2-b^2}) \quad [\operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} z > 0]$$

$$2. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{sh} bx dx = J_{\alpha} \quad [\operatorname{Re} \alpha > -1; \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|]$$

$$J_{\alpha} = z \frac{\partial^{\alpha-1}}{\partial b^{\alpha-1}} \left[\frac{b}{\sqrt{p^2-b^2}} K_1(z\sqrt{p^2-b^2}) \right],$$

$$J_1 = \frac{bz^2}{(p^2-b^2)^{3/2}} (3p^2+b^2) K_2(z\sqrt{p^2-b^2}) + \frac{b^2 p^2}{(p^2-b^2)^{3/2}} K_1(z\sqrt{p^2-b^2}).$$

$$3. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{ch} bx dx = I_{\alpha} \quad [\operatorname{Re} z, \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|]$$

$$I_{\alpha+1} = \frac{\partial^{\alpha}}{\partial b^{\alpha}} K_0(z\sqrt{p^2-b^2}),$$

$$I_1 = \frac{z(p^2+b^2)}{(p^2-b^2)^{3/2}} K_1(z\sqrt{p^2-b^2}) + \frac{b^2 z^2}{p^2-b^2} K_0(z\sqrt{p^2-b^2}).$$

$$4. \int_0^{\infty} x^{1-\delta} \left(\sqrt{\frac{x^2+z^2}{x^2+z^2}} \pm 1/2 \exp(-p\sqrt{x^2+z^2}) \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right) dx = b^{\delta} \sqrt{\frac{\pi}{2}} \frac{(\sqrt{p^2-b^2}+p)^{\delta/2}}{\sqrt{p^2-b^2}} \exp(-z\sqrt{p^2-b^2}) \quad [\operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|; \delta = \{1\}]$$

2.4.17. Интегралы от $x^{\alpha} f(x) \frac{\operatorname{sh} bx}{\operatorname{ch} bx}$.

$$1. \int_0^{\infty} e^{-ax} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{z}{2b} \left[e^{z^2/2} S_{\delta, 1/3}(z^{3/2}/2) \mp S_{\delta, 1/3}(z) \right] \quad [z = 2(b/3)^{2/3} a^{-1/2}; \operatorname{Re} a > 0]$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = q^{\alpha/2} [(p-b)^{-\alpha/2} K_{\alpha}(2\sqrt{q}\sqrt{p-b}) \mp (p+b)^{-\alpha/2} K_{\alpha}(2\sqrt{q}\sqrt{p+b})] \quad [\operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} q > 0]$$

$$3. \int_0^{\infty} x^{-k/2} e^{-px-q/x} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \frac{\sqrt{\pi}}{2^{1/2}(k-1)^{1/2}} [(p-b)^{(k-3)/4} \exp(-2\sqrt{q}\sqrt{p-b}) \mp (p+b)^{(k-3)/4} \exp(-2\sqrt{q}\sqrt{p+b})] \quad [k=1 \text{ или } 3; \operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} q > 0]$$

$$4. \int_{-\infty}^{\infty} \exp(ax - e^x) \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \mp |\Gamma(a-b)| \left\{ \frac{\sin t}{\cos t} \right\} \\ \left[t = \ln \frac{\Gamma(a-b)}{\Gamma(a-b)}; \operatorname{Re} a > |\operatorname{Re} b| \right]$$

2.4.18. Интегралы от $f(x) \exp \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} e^{-a \operatorname{sh} x} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} dx = \\ = \frac{\pi}{2} \left(\operatorname{tg} \frac{b\pi}{2} \right)^{-1} [J_b(a) - J_b(a)] - \frac{\pi}{2} [E_b(a) + Y_b(a)] = b^{\delta} S_{1-\delta, \delta}(a) \\ [\operatorname{Re} a > 0; b - \text{нечетное}].$$

$$2. \int_0^{\infty} e^{-a \operatorname{sh} x} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \operatorname{ch} x dx = \frac{b^{\delta}}{a} S_{1-\delta, \delta}(a) \quad [\operatorname{Re} a > 0].$$

$$3. \int_0^{\infty} e^{-a \operatorname{sh} x} \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \operatorname{sh}^{-1/2} x dx = \\ = \frac{\pi \sqrt{\pi a}}{8i^{\delta}} \left[e^{\pi b i/2} H_{(1+b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1-b)/2}^{(2)} \left(\frac{a}{2} \right) + e^{-\pi b i/2} H_{(1-b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1+b)/2}^{(2)} \left(\frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0].$$

$$4. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{ch} bx dx = K_b(a) \quad [\operatorname{Re} a > 0].$$

$$5. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} bx \operatorname{sh} x dx = \frac{b}{a} K_b(a) \quad [\operatorname{Re} a > 0].$$

$$6. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{sh}(x/2) \operatorname{sh} bx}{\operatorname{ch}(x/2) \operatorname{ch} bx} dx = \begin{cases} i \operatorname{Im} K_{1/2-b}(a) \\ \operatorname{Re} K_{1/2-b}(a) \end{cases} \quad [a, b > 0].$$

$$7. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx}{\operatorname{ch} x} dx = \sqrt{\frac{a}{2\pi}} K_{b/2+1/4} \left(\frac{a}{2} \right) K_{b/2-1/4} \left(\frac{a}{2} \right) \quad [\operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx \operatorname{sh} x}{\operatorname{ch} x \operatorname{ch} x - 1} dx = K_b(a) \quad [\operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} e^{-a \operatorname{sh} x - p x} dx = \frac{\pi}{\operatorname{sh} p \pi} [J_p(a) - J_p(a)] \quad [\operatorname{Re} a > 0].$$

$$10. \int_0^{\infty} e^{-a \operatorname{sh} x - p x} \frac{dx}{V \operatorname{sh} x} = \\ = \frac{\pi}{2} \sqrt{\frac{\pi a}{2}} \left[J_{1/4-p/2} \left(\frac{a}{2} \right) J_{-1/4-p/2} \left(\frac{a}{2} \right) + Y_{1/4-p/2} \left(\frac{a}{2} \right) Y_{-1/4-p/2} \left(\frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0].$$

$$11. \int_0^{\infty} e^{-a \operatorname{ch} x - p x} dx = \frac{1}{\operatorname{sh} p \pi} \left[\int_0^{\pi} e^{a \cos x} \cos p x dx - \pi I_p(a) \right] \quad [\operatorname{Re} a > 0].$$

$$12. \int_{-\infty}^{\infty} e^{-a \operatorname{ch} x - p x} dx = 2K_p(a) \quad [\operatorname{Re} a > 0].$$

$$13. \int_{-\infty}^{\infty} e^{i a \operatorname{sh} x - p x} dx = 2e^{-p \pi i/2} K_p(a) \quad [|\operatorname{Re} p| < 1; a > 0].$$

$$14. \int_{-\infty}^{\infty} e^{i a \operatorname{ch} x - p x} dx = \pm \pi i e^{\pm p \pi i/2} H_p^{(3/2 \mp 1/2)}(a) \quad [\pm \operatorname{Re}(ia) < 0].$$

2.4.19. Интегралы от $A(x) e^{-p x} \frac{\operatorname{sh} \sqrt{ax^2 + bx + c}}{\operatorname{ch} \sqrt{ax^2 + bx + c}}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^a \frac{e^{-p x}}{V x \sqrt{a^2 - x^2}} \operatorname{ch}(c \sqrt{ax^2 + bx + c}) dx = \\ = -\frac{\pi}{2} \sqrt{\frac{a}{p}} [I_{1/4}(z_+) I_{1/4}(z_-) - I_{-1/4}(z_+) I_{-1/4}(z_-)] \\ [z_{\pm} = \sqrt{c^2 + p^2} \pm c; a, c > 0].$$

$$2. \int_0^a \frac{e^{-p x}}{V \frac{ax}{x^2 - a^2}} \operatorname{ch}(c \sqrt{ax^2 + bx + c}) dx = \pi e^{-a p/2} I_0 \left(\frac{a}{2} \sqrt{p^2 + c^2} \right) \quad [a, c > 0].$$

$$3. \int_a^{\infty} x^2 e^{-p x} \operatorname{sh}(c \sqrt{x^2 - a^2}) dx = \frac{a^{1+\delta} c p^{\delta}}{(p^2 - c^2)^{(1+\delta)/2}} K_{1+\delta}(a \sqrt{p^2 - c^2}) \\ [c = 0 \text{ или } 1; a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

$$4. \int_a^{\infty} \frac{e^{-p x}}{V x + a} \frac{\operatorname{sh}(c \sqrt{x^2 - a^2})}{\operatorname{ch}(c \sqrt{x^2 - a^2})} dx = \\ = \sqrt{\frac{\pi}{2}} \frac{c^{\delta}}{V p^2 - c^2} (V p^2 - c^2 + p)^{-1/2} \exp(-a \sqrt{p^2 - c^2}) \\ [a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

$$5. \int_a^{\infty} \frac{x^{-p x}}{V x - a} \operatorname{ch}(c \sqrt{x^2 - a^2}) dx = \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2 - c^2}}{p^2 - c^2}} \exp(-a \sqrt{p^2 - c^2}) \\ [a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

6. $\int_0^{\infty} (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \frac{\text{sh}(c\sqrt{x^2-a^2})}{\text{ch}(c\sqrt{x^2-a^2})} \right\} dx =$
 $= \frac{2^{\alpha+1/2}}{a\sqrt{c}} \Gamma\left(\alpha + \frac{\delta}{2}\right) W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-)$
 $[z_{\pm} = a(p \pm \sqrt{p^2-c^2}); a > 0; \text{Re } p > |\text{Re } c|].$
7. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \frac{\text{sh}(c\sqrt{x^2-a^2})}{\text{ch}(c\sqrt{x^2-a^2})} \right\} dx = \frac{\pi}{\sqrt{2a}} e^{ap} \text{erfc}(z_+) \left\{ \frac{\text{erf}(z_-)}{1} \right\}$
 $[z_{\pm} = a(p \pm \sqrt{p^2-c^2}); a > 0; \text{Re } p > |\text{Re } c|].$
8. $\int_0^{\infty} \frac{e^{-px}}{(x^2-a^2)^{3/4}} \left\{ \frac{\text{sh}(c\sqrt{x^2-a^2})}{\text{ch}(c\sqrt{x^2-a^2})} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[2z_{\pm} = a(p \pm \sqrt{p^2-c^2}); a > 0; \text{Re } p > |\text{Re } c|].$
9. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x^2-a^2}} \text{ch}(c\sqrt{x^2-a^2}) dx = \frac{1}{\sqrt{2\pi p}} K_{1/4}(z_+) K_{1/4}(z_-)$
 $[2z_{\pm} = a(p \pm \sqrt{p^2-c^2}); a > 0; \text{Re } p > |\text{Re } c|].$
10. $\int_0^{\infty} \frac{x^2 e^{-px}}{\sqrt{x^2-a^2}} \text{ch}(c\sqrt{x^2-a^2}) dx = \frac{a^2 p^2}{(p^2-c^2)^{3/2}} K_2(a\sqrt{p^2-c^2})$
 $[c = 0 \text{ или } 1; a > 0; \text{Re } p > |\text{Re } c|].$
11. $\int_0^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} \mp (x-\sqrt{x^2-a^2})^{\nu}}{\sqrt{x^2-a^2}} e^{-px} \left\{ \frac{\text{sh}(c\sqrt{x^2-a^2})}{\text{ch}(c\sqrt{x^2-a^2})} \right\} dx =$
 $= a^{\nu} \left[\left(\frac{p+c}{p-c} \right)^{\nu/2} \mp \left(\frac{p+c}{p-c} \right)^{-\nu/2} \right] K_{\nu}(a\sqrt{p^2-c^2})$
 $[a > 0; \text{Re } p > |\text{Re } c|].$
12. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x^2+z^2}} \left\{ \frac{\text{sh}(b\sqrt{x^2+z^2})}{\text{sh}(b\sqrt{x^2+z^2})} \right\} dx =$
 $= \frac{\pi i^{\delta}}{4} \sqrt{\frac{p}{\pi p}} \left[\left\{ \frac{J_{1/4}(z_-)}{Y_{1/4}(z_-)} \right\} Y_{1/4}(z_+) \mp \left\{ \frac{Y_{1/4}(z_-)}{J_{1/4}(z_-)} \right\} J_{1/4}(z_+) \right]$
 $[2z_{\pm} = z(\sqrt{p^2-b^2} \pm ib); \text{Re } z > 0; \text{Re } p > |\text{Re } b|].$
13. $\int_0^{\infty} e^{-px} \text{sh}(b\sqrt{x^2+xz}) dx = \frac{bz}{2\sqrt{p^2-b^2}} e^{pz/2} K_1\left(\frac{z}{2}\sqrt{p^2-b^2}\right)$
 $[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
14. $\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-px} \text{ch}(b\sqrt{x^2+xz}) dx =$
 $= \sqrt{\frac{\pi}{2}} \sqrt{\frac{p+\sqrt{p^2-b^2}}{p^2-b^2}} \exp\left[\frac{z}{2}(p-\sqrt{p^2-b^2})\right]$
 $[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$

15. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x+z}} \left\{ \frac{\text{sh}(b\sqrt{x^2+xz})}{\text{ch}(b\sqrt{x^2+xz})} \right\} dx =$
 $= \sqrt{\frac{\pi}{2}} b^{\delta} (p+\sqrt{p^2-b^2})^{\pm 1/2} (p^2-b^2)^{-1/2} \exp\left[\frac{z}{2}(p-\sqrt{p^2-b^2})\right]$
 $[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
16. $\int_0^{\infty} x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \frac{\text{sh}(b\sqrt{x^2+xz})}{\text{ch}(b\sqrt{x^2+xz})} \right\} dx =$
 $= \frac{2^{\alpha+1/2}}{z\sqrt{b}} \Gamma\left(\alpha + \frac{\delta}{2}\right) e^{pz/2} W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-)$
 $[2z_{\pm} = z(p \pm \sqrt{p^2-b^2}); 2\text{Re } \alpha > -\delta; \text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
17. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x}(x+z)} \left\{ \frac{\text{sh}(b\sqrt{x^2+xz})}{\text{ch}(b\sqrt{x^2+xz})} \right\} dx = \frac{\pi}{\sqrt{z}} e^{pz} \text{erfc}(z_+) \left\{ \frac{\text{erf}(z_-)}{1} \right\}$
 $[2z_{\pm} = z(p \pm \sqrt{p^2-b^2}); \text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
18. $\int_0^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x+z}} \text{ch}(b\sqrt{x^2+xz}) dx = e^{pz/2} K_0\left(\frac{z}{2}\sqrt{p^2-b^2}\right)$
 $[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
19. $\int_0^{\infty} (x^2+xz)^{-3/4} e^{-px} \left\{ \frac{\text{sh}(b\sqrt{x^2+xz})}{\text{ch}(b\sqrt{x^2+xz})} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{pz/2} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[4z_{\pm} = z(p \pm \sqrt{p^2-b^2}); \text{Re } p > |\text{Re } b|; |\arg z| < \pi].$
20. $\int_0^{\infty} \frac{1}{\sqrt{x^2+xz}} \left[\left(\frac{z}{2} + x + \sqrt{x^2+xz} \right)^{\nu} \mp \left(\frac{z}{2} + x - \sqrt{x^2+xz} \right)^{\nu} \right] e^{-px} \times$
 $\times \left\{ \frac{\text{sh}(b\sqrt{x^2+xz})}{\text{ch}(b\sqrt{x^2+xz})} \right\} dx = \left(\frac{z}{2} \right)^{\nu} \left[\left(\frac{p+b}{p-b} \right)^{\nu/2} \mp \left(\frac{p+b}{p-b} \right)^{-\nu/2} \right] e^{pz/2} K_{\nu}\left(\frac{z}{2}\sqrt{p^2-b^2}\right)$
 $[\text{Re } p > |\text{Re } b|; |\arg z| < \pi].$

2.4.20. Интегралы от $A(x) e^{f(x)} \left\{ \frac{\text{sh}[cx/(a^2-x^2)]}{\text{ch}[cx/(a^2-x^2)]} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

1. $\int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{a^2-x^2}} \exp\left(-b\frac{a^2+x^2}{a^2-x^2}\right) \left\{ \frac{\text{sh}[cx/(a^2-x^2)]}{\text{ch}[cx/(a^2-x^2)]} \right\} dx =$
 $= 2^{\delta+1} (2\alpha-3)/4 \sqrt{\frac{a}{c^{\delta}}} \Gamma\left(\frac{\alpha+\delta}{2}\right) (V\sqrt{4a^2b^2-c^2+2ab})^{1/4} \times$
 $\times D_{-\alpha}\left(\sqrt{\frac{V\sqrt{4a^2b^2-c^2+2ab}}{a}}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{V\sqrt{4a^2b^2-c^2+2ab}}{2a}\right)$
 $[\text{Re } \alpha > -\delta; 2ab > c > 0; a > 0].$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-\frac{b}{x^2-a^2}\right) \left\{ \frac{\operatorname{sh} [cx/(x^2-a^2)]}{\operatorname{ch} [cx/(x^2-a^2)]} \right\} dx =$$

$$= \frac{a^{\alpha-1}}{\sqrt{\pi} 2^{1/2} \Gamma(\frac{1}{2})} \exp\left(\frac{b}{2a^2}\right) \Gamma\left(\frac{1+\delta-\alpha}{2}\right) [V b^2-a^2 c^2+b]^{1/4} \times$$

$$\times D_{\alpha-1}\left(\frac{V b^2-a^2 c^2+b}{a}\right) M_{1/4-\alpha/2, 1/4}\left(\frac{b-V b^2-a^2 c^2}{2a^2}\right)$$

$$[\operatorname{Re} \alpha < 1+b; b > a c > 0; a > 0]$$

$$3. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-b \frac{x^2+a^2}{x^2-a^2}\right) \left\{ \frac{\operatorname{sh} [cx/(x^2-a^2)]}{\operatorname{ch} [cx/(x^2-a^2)]} \right\} dx =$$

$$= 2^{\delta-(2\alpha+1)/4} a^{\alpha-3/4} \Gamma\left(\frac{1+\delta-\alpha}{2}\right) (2ab+V 4a^2b^2-c^2)^{1/4} \times$$

$$\times D_{\alpha-1}\left(\frac{2ab+V 4a^2b^2-c^2}{a}\right) M_{\alpha/2-1/4, 1/4}\left(\frac{2ab-V 4a^2b^2-c^2}{2a}\right)$$

$$[\operatorname{Re} \alpha < 1+b; 2ab > c > 0; a > 0]$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-\frac{b}{x^2+z^2}\right) \left\{ \frac{\operatorname{sh} [cx/(x^2+z^2)]}{\operatorname{ch} [cx/(x^2+z^2)]} \right\} dx =$$

$$= \frac{2^{\delta-1/2}}{\sqrt{\pi c}} z^{\alpha-1/2} \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \Gamma\left(\frac{\alpha+\delta}{2}\right) \exp\left(-\frac{b}{2z^2}\right) \times$$

$$\times M_{1/4-\alpha/2, 1/4}\left(\frac{b+V b^2+c^2 z^2}{2z^2}\right) M_{\alpha/2-1/4, 1/4}\left(\frac{V b^2+c^2 z^2-b}{2z^2}\right)$$

$$[-\delta < \operatorname{Re} \alpha < 1+b; b, c, \operatorname{Re} z > 0]$$

$$5. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-b \frac{x^2-x^2}{z^2+x^2}\right) \left\{ \frac{\operatorname{sh} [cx/(x^2+z^2)]}{\operatorname{ch} [cx/(x^2+z^2)]} \right\} dx =$$

$$= \frac{2^{\delta-1/2} z^{\alpha-1/2}}{\sqrt{\pi c}} \Gamma\left(\frac{\alpha+\delta}{2}\right) \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \times$$

$$\times M_{1/4-\alpha/2, 1/4}\left(\frac{V 4b^2 z^2+c^2+2bz}{2z}\right) M_{\alpha/2-1/4, 1/4}\left(\frac{V 4b^2 z^2+c^2-2bz}{2z}\right)$$

$$[-\delta < \operatorname{Re} \alpha < 1+b; b, c, \operatorname{Re} z > 0]$$

2.4.21. Интегралы, содержащие $A(x)$, e^{cx} , $\operatorname{sh} g(x)$, $\operatorname{ch} g(x)$.

$$1. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} a [x]}{\operatorname{ch} a [x]} dx = \frac{e^p-1}{p(e^{2p}-2e^p \operatorname{ch} a+1)} \left\{ \frac{\operatorname{sh} a}{e^p-\operatorname{ch} a} \right\}$$

$$[\operatorname{Re} p > |\operatorname{Re} a|]$$

$$2. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} a [x]}{\operatorname{ch} a [x]} dx = \frac{e^p-1}{p(e^{2p}-2e^p \operatorname{ch} a+1)} \left\{ \frac{b \operatorname{sh} a}{e^p-b \operatorname{ch} a} \right\}$$

$$[\operatorname{Re} p > \ln b + |\operatorname{Re} a|; b \geq 1]$$

$$3. \int_0^{\infty} e^{-ax} \left\{ \frac{\operatorname{sh} cx^2}{\operatorname{ch} cx^2} \right\} \operatorname{ch} bx dx =$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{a^2-c^2}} e^{-aA} \left(B_+ \left\{ \frac{\operatorname{sh} cA}{\operatorname{ch} cA} \right\} - B_- \left\{ \frac{\operatorname{ch} cA}{\operatorname{sh} cA} \right\} \right) = \frac{V \pi}{2 \sqrt{a^2-c^2}} e^{-aA} \left\{ \frac{\operatorname{sh} C}{\operatorname{ch} C} \right\}$$

$$\left[A = \frac{b}{2} \sqrt{\frac{a^2-c^2}{a^2-c^2}}; B_{\pm} = \left\{ \frac{1}{2} (a \pm \sqrt{a^2-c^2}) \right\}^{1/2}; 2C = \frac{1}{2} \ln \frac{a+c}{a-c}; \operatorname{Re} a > |\operatorname{Re} c| \right].$$

$$4. \int_0^{\infty} x^{-1/2} \frac{\operatorname{sh} (bx+c/x)}{\operatorname{ch} (bx+c/x)} dx =$$

$$= \frac{V \pi}{r} \exp\left(-rs \sqrt{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - rs \sqrt{\frac{(p-b)(q-c)}{(p+b)(q+c)}}\right) \left\{ \frac{\operatorname{sh} d}{\operatorname{ch} d} \right\}$$

$$\left[d = \frac{1}{4} \ln \frac{p+b}{p-b} + rs \left(\sqrt{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - \sqrt{\frac{(p-b)(q-c)}{(p+b)(q+c)}} \right), \right.$$

$$\left. r = \frac{1}{2} \sqrt{p^2-b^2}, s = \frac{1}{2} \sqrt{q^2-c^2}; \operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} q > |\operatorname{Re} c| \right].$$

2.4.22. Интегралы, содержащие разности $A(x)$, e^{-px} , $\frac{\operatorname{sh} bx}{\operatorname{ch} bx}$.

$$1. \int_0^{\infty} e^{-px} \frac{1-\operatorname{ch} ax}{x} dx = \frac{1}{2} \ln \left(1 - \frac{a^2}{p^2} \right)$$

$$[\operatorname{Re} p > |\operatorname{Re} a|].$$

$$2. \int_0^{\infty} e^{-px} \frac{1-\operatorname{ch} ax}{x^2} dx = -p \ln \frac{V p^2-a^2}{p} - \frac{a}{2} \ln \frac{p+a}{p-a}$$

$$[\operatorname{Re} p > |\operatorname{Re} a|].$$

$$3. \int_0^{\infty} e^{-px} \frac{e^{-\operatorname{sh} x}}{x^2} dx = I_n$$

$$[\operatorname{Re} p > 1]$$

$$I_1 = \frac{1}{p} - \frac{1}{2} \ln \frac{p+1}{p-1}, \quad I_2 = \frac{1}{2} \ln \left(1 - \frac{1}{p^2} \right) + \frac{p}{2} \ln \frac{p+1}{p-1} - 1,$$

$$I_3 = \frac{1}{2} \left[p \ln \left(1 - \frac{1}{p^2} \right) + \frac{p^2-1}{2} \ln \frac{p+1}{p-1} - p \right].$$

$$4. \int_0^{\infty} \frac{e^{-px}}{x} \left(1 - \frac{1}{\operatorname{ch} x} \right) dx = 2 \ln \Gamma\left(\frac{p+3}{4}\right) - 2 \ln \Gamma\left(\frac{p+1}{4}\right) - \ln \frac{p}{4}$$

$$[\operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \frac{1}{\operatorname{sh} x} \right) dx = \psi\left(\frac{p+1}{2}\right) - \ln \frac{p}{2}$$

$$[\operatorname{Re} p > 0].$$

$$6. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \operatorname{ch} x \right) dx = \psi\left(\frac{p}{2}\right) - \ln \frac{p}{2} + \frac{1}{p}$$

$$[\operatorname{Re} p > 0].$$

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px} (\operatorname{ch} ax - 1) dx = 2^{1-\alpha} \Gamma(\alpha) \zeta\left(\alpha, 1 + \frac{p}{2a}\right)$$

$$[\operatorname{Re} \alpha > 1; \operatorname{Re} p > -2].$$

$$8. \int_0^{\infty} x^{-1} \frac{(1-e^{-x})^2}{\operatorname{ch} x + \cos(m\pi/n)} dx = \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \times \\ \times \sin \frac{k m \pi}{n} \ln \Gamma \left[\frac{(n+(-1)^k k+1)/(rn), (n+(-1)^k k+1)/(rn), (k+2)/(rn), k/(rn))}{(n+(-1)^k k)/(rn), (n+(-1)^k k+2)/(rn), (k+1)/(rn), (k+1)/(rn))} \right] \\ \left[\begin{matrix} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{matrix} \right].$$

$$9. \int_0^{\infty} x^n \frac{1-e^{-2mx}}{\operatorname{sh}^2 x} dx = J_n,$$

$$J_1 = \frac{m\pi^2}{6} - \sum_{k=1}^{m-1} \frac{m-k}{k^3}, \quad J_2 = m\zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3},$$

$$J_3 = \frac{1}{60} m\pi^4 - \frac{3}{2} \sum_{k=1}^{m-1} \frac{m-k}{k^4}.$$

$$10. \int_0^{\infty} x \frac{1-(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = \frac{m}{12} \pi^2 + \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^3}.$$

$$11. \int_0^{\infty} x^n \frac{1+(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = I_n,$$

$$I_1 = \frac{3m}{4} \zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3}, \quad I_2 = \frac{7m\pi^4}{480} + \frac{3}{2} \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^4}.$$

$$12. \int_{-\infty}^{\infty} x \frac{1-e^{ax}}{\operatorname{sh} x} dx = -\frac{\pi^2}{2} \operatorname{tg}^2 \frac{a\pi}{2} \quad [| \operatorname{Re} a | < 1].$$

$$13. \int_0^{\infty} \frac{e^{-ax} - e^{-cx} \operatorname{ch} bx}{x} dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2} \quad [\operatorname{Re} a > 0; \operatorname{Re} c > | \operatorname{Re} b |].$$

$$14. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x} \operatorname{ch} bx dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} \quad [\operatorname{Re} a, \operatorname{Re} c > | \operatorname{Re} b |].$$

$$15. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x^2} \operatorname{sh} bx dx = \frac{b}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} + \frac{c}{2} \ln \frac{c+b}{c-b} - \frac{a}{2} \ln \frac{a+b}{a-b} \\ [\operatorname{Re} a, \operatorname{Re} c > | \operatorname{Re} b |].$$

$$16. \int_0^{\infty} \frac{1+x-e^x}{x \operatorname{sh} x} e^{-px} dx = 2 \ln \Gamma \left(\frac{p+1}{2} \right) - 2 \ln \Gamma \left(\frac{p}{2} \right) - \psi \left(\frac{p+1}{2} \right) \\ [\operatorname{Re} p > 0].$$

$$17. \int_0^{\infty} \frac{(1-e^{-ax})(1-e^{-(a+1)x})}{x \operatorname{sh} x} dx = 2a \ln 2 \quad [\operatorname{Re} a > -1].$$

$$18. \int_0^{\infty} \frac{1}{x} (ae^{-bx} - e^{-cx} \frac{\operatorname{sh} ax}{x}) dx = \frac{a}{2} \ln \frac{c^2 - a^2}{b^2} + \frac{c}{2} \ln \frac{c+a}{c-a} - a \\ [\operatorname{Re} b > 0; \operatorname{Re} c > | \operatorname{Re} a |].$$

$$19. \int_0^{\infty} \frac{1}{x} \frac{1 - \operatorname{ch} ax}{e^{\operatorname{sh} cx} - e^{a(n+1)cx}} dx = -\frac{1}{2} \ln \frac{2c \sin [a\pi/(2c)]}{a\pi} + \frac{1}{2} \sum_{k=1}^n \ln \left(1 - \frac{a^2}{4k^2 c^2} \right) \\ [a < 2(n+1)c; a < 2nc].$$

$$20. \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{\operatorname{ch} x - \cos(m\pi/n)} \frac{dx}{x} = \\ = \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{k m \pi}{n} \ln \Gamma \left[\frac{(n+b+(-1)^k k)/(rn), (a+k)/(rn)}{(n+a+(-1)^k k)/(rn), (b+k)/(rn)} \right] \\ [\operatorname{Re} a, \operatorname{Re} b > -1; \begin{matrix} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{matrix}].$$

$$21. \int_0^{\infty} \left[e^{-x} \operatorname{tg} \frac{m\pi}{2n} - \frac{e^{-ax} \sin(m\pi/n)}{\operatorname{ch} x + \cos(m\pi/n)} \right] \frac{dx}{x} = \\ = \operatorname{tg} \frac{m\pi}{2n} \ln(rn) + 2 \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{k m \pi}{n} \pi \ln \Gamma \left[\frac{(a+n+(-1)^k k)/(rn)}{(a+k)/(rn)} \right] \\ [\operatorname{Re} a > -1; \begin{matrix} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{matrix}].$$

$$22. \int_0^{\infty} e^{-px} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{x^n} dx = I_n \quad [\operatorname{Re} p > | \operatorname{Re} a |, | \operatorname{Re} b |].$$

$$I_1 = \frac{1}{2} \ln \frac{p^2 - b^2}{p^2 - a^2}, \quad I_2 = \frac{p}{2} \ln \frac{p^2 - a^2}{p^2 - b^2} + \frac{b}{2} \ln \frac{p-b}{p+b} + \frac{a}{2} \ln \frac{p+a}{p-a}.$$

$$23. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - x \operatorname{ch} x}{x^n} dx = J_n \quad [\operatorname{Re} p > 1].$$

$$J_1 = \frac{1}{2} \ln \frac{p+1}{p-1} - \frac{p}{p^2-1}, \quad J_2 = 1 - \frac{p}{2} \ln \frac{p+1}{p-1}.$$

$$J_3 = \frac{1}{4} \left[(p^2-1) \ln \frac{p+1}{p-1} - 2p \right].$$

$$24. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - 2x \operatorname{ch} x}{x^2} \operatorname{sh} x dx = \frac{p}{4} \ln \frac{p^2-4}{p^2} \quad [\operatorname{Re} p > 2].$$

$$25. \int_0^{\infty} \frac{1}{x} \left(\frac{\operatorname{sh} ax}{\operatorname{sh} x} - ae^{-ax} \right) dx = \ln \left(\pi \cos \frac{a\pi}{2} \right) - 2 \ln \Gamma \left(\frac{a+1}{2} \right) \quad [| \operatorname{Re} a | < 1].$$

$$26. \int_0^{\infty} \frac{ae^{bx} \operatorname{sh} cx - ce^{hx} \operatorname{sh} ax}{x^2} dx = \frac{ac}{2} \ln \frac{b^2 - a^2}{b^2 - c^2} + \frac{ch}{2} \ln \frac{h+a}{h-a} - \frac{ab}{2} \ln \frac{b+a}{b-a}$$

$$[\operatorname{Re} b > \operatorname{Re} c; \operatorname{Re} h > \operatorname{Re} a]$$

$$27. \int_0^{\infty} \frac{1}{x} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{e^{mx} - e^{(m-2i)x}} dx = \frac{1}{2} \left\{ \ln \cos \left[\frac{b\pi}{2c} \right] - \ln \cos \left[\frac{a\pi}{2c} \right] \right\} -$$

$$- \sum_{k=1}^{(m-1)/2} \ln \frac{c^2 (2k-2\lambda)^2 - b^2}{c^2 (2k-2\lambda)^2 - a^2} + \begin{cases} 0 \\ \ln(a/b) \end{cases}$$

$$\left\{ \begin{array}{l} a < (m-2)c; \quad b < (m-2)c; \quad \lambda = 1/2, \quad m = 1, 3, 5, \dots \\ \lambda = 0, \quad m = 2, 4, 6, \dots \end{array} \right\}$$

$$28. \int_0^{\infty} (e^{-ax} - e^{-bx}) \operatorname{ch} cx dx = \frac{1}{b} - \frac{1}{a} + \frac{1}{c} \left[\psi \left(\frac{b}{2c} \right) - \psi \left(\frac{a}{2c} \right) \right]$$

$$[\operatorname{Re} a, \operatorname{Re} b > 0]$$

2.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

2.5.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входят тригонометрические функции. Для их вычисления можно использовать следующие свойства.

Интегралы, содержащие тригонометрические функции, могут быть преобразованы к интегралам от показательной или гиперболических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\sin^a bx$ и $\cos^a bx$, сводятся к интегралам от функций $\sin kbx$ и $\cos kbx$, если использовать формулы для степеней тригонометрических функций (см. приложение I). В ряде случаев интегралы от тригонометрических функций с помощью замены переменной сводятся к интегралам от алгебраических и других функций. Например,

$$\int_0^a \varphi(x) \sin^2 bx dx = \frac{1}{2} \int_0^a \varphi(x) dx - \frac{1}{4} \int_0^a \varphi(x) e^{i2bx} dx - \frac{1}{4} \int_0^a \varphi(x) e^{-i2bx} dx =$$

$$= - \int_0^a \varphi(x) \operatorname{sh}^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 - \cos 2bx) dx =$$

$$= \frac{1}{b} \int_0^{\sin ab} x^2 \varphi \left(\frac{\arcsin x}{b} \right) \frac{dx}{\sqrt{1-x^2}} \quad [ab \leq \pi/2]$$

2.5.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие тригонометрические, а также степенную, алгебраические, показательную и гиперболические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j=1, 2, 3, \dots$, помещенных в приложения III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

2.5.2.]

В п. 2.5.2 приняты следующие обозначения: $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \sin bx, \\ 0, & \text{если взят } \cos bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2]+k}, & \text{если взят } \sin bx, \\ 1, & \text{если взят } \cos bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \sin^n bx, \quad n=1, 3, 5, \dots, \\ 0, & \text{если взят } \sin^n bx, \quad n=2, 4, 6, \dots, \\ 0, & \text{если взят } \cos^n bx, \quad n=1, 2, 3, \dots \end{cases}$$

$$\xi = r^{-1}(\alpha/2 + \gamma + h), \quad \tilde{b} = by(n-2k)/2.$$

$$1. \int_0^{\tilde{b}} x^{\alpha-1} (y^{2r} - x^{2r})^{\beta-1} \frac{\{\sin bx\}^n}{\{\cos bx\}^n} dx = V_{21} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r} \times$$

$$\times y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, \beta\right)$$

$$\left\{ E_{21} = V_{21} 2^{-n} \frac{B(\beta, \xi)}{r\Gamma(1/2+2\gamma+h)}, \quad u = \alpha + 2r(\beta-1), \right.$$

$$a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (-\tilde{b}^2/p^2)^p \}$$

$$[y, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta n; |\arg b| < \pi]$$

$$2. \int_0^{\tilde{b}} x^{\alpha-1} (y^{2r} - y^{2r})^{\beta-1} \frac{\{\sin bx\}^n}{\{\cos bx\}^n} dx =$$

$$= V_{21} + V_{22} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\beta, 1 - \frac{\alpha}{2r} - \beta\right)$$

$$\left\{ E_{21} = V_{21} 2^{-n} \frac{B(\beta, 1-\beta-\xi)}{r\Gamma(1/2+2\gamma+h)}, \quad E_{22} = V_{22} 2^{-n} \frac{B(\beta, 1-b_2)}{r\Gamma(1/2+2\gamma+h)}, \right.$$

$$u = \alpha + 2r(\beta-1), \quad v_1 = 2r(1-\beta+\gamma) - \alpha, \quad a_1 = \xi, \quad b_1 = \beta + \xi,$$

$$c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \beta + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1-\beta+j),$$

$$c_2 = (1-\alpha)/2 + \gamma + r(1-\beta+j), \quad z = (-\tilde{b}^2/p^2)^p \}$$

$$[y, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha + 2r\beta) < (1-(-1)^n)/2 + 2r]$$

$$3. \int_0^{\tilde{b}} x^{\alpha-1} (y^{2r} + y^{2r})^{\beta-1} \frac{\{\sin bx\}^n}{\{\cos bx\}^n} dx =$$

$$= V_{21} + V_{22} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, 1 - \rho - \frac{\alpha}{2r}\right)$$

$$\left\{ E_{21} = V_{21} 2^{-n} \frac{B(\beta, 1-\rho-\xi)}{r\Gamma(1/2+2\gamma+h)}, \quad E_{22} = V_{22} 2^{-n} \frac{B(1-\rho+h, 1-b_1)}{r\Gamma(1/2+2\gamma+h)}, \right.$$

$$u = 2r(\rho-1) + \alpha, \quad v_1 = 2r(1-\rho+j) - \alpha, \quad a_1 = \xi, \quad b_1 = \rho + \xi,$$

$$c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \rho + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1-\rho+j),$$

$$c_2 = (1-\alpha)/2 + \gamma + r(1-\rho+j), \quad z = (-1)^{\rho+j} \tilde{b}^2 p^{-2\rho} \}$$

$$[b > 0; 2r|\arg y| < \pi; \operatorname{Re} \alpha > -\delta n; \operatorname{Re}(\alpha + 2r\rho) < (1-(-1)^n)/2 + 2r]$$

$$4. \int_0^{\tilde{b}} \frac{x^{\alpha-1}}{2y^{2r} - y^{2r}} \frac{\{\sin bx\}^n}{\{\cos bx\}^n} dx = V_{23} + V_{24} - \frac{1+(-1)^n}{2} 2^{-n-1} y^{\alpha-2r} \frac{n! \pi}{r[(n/2)!]^2} \operatorname{ctg} \frac{\alpha \pi}{2r}$$

$$\{E_{31} = -\frac{\pi^{3/2}}{2^2 r} \frac{\operatorname{ctg} \frac{\pi}{2}}{\Gamma(1/2 + 2\gamma + h)}, \quad E_{31} = \frac{\sqrt{\pi}}{2^n} \Gamma\left[\frac{1-\alpha_1}{b_1}\right],$$

$$u = \alpha - 2r, \quad v_1 = 2r(1+j) - \alpha, \quad a_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \gamma + r(1+j) - \alpha/2, \\ b_2 = (1-\alpha)/2 + \gamma + r(1+j), \quad z = (-\tilde{b}^2/p^2)^p \\ [y, b > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re} \alpha < (1 - (-1)^n)/2 + 2r]$$

$$5. \int_0^\infty x^{\alpha-1} |x^{2r} - y^{2r}|^{\sigma-1} \frac{(\sin bx)^n}{(\cos bx)} dx = \\ = V_{31} + V_{32} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r-1} \frac{n!}{r[(n/2)!]^2} \frac{\sin[(\alpha+r\sigma)/(2r)]}{\sin(\pi\sigma/2)} \times \\ \times B\left(\frac{\alpha}{2r}, 1-\sigma-\frac{\alpha}{2r}\right)$$

$$\{E_{31} = \frac{\sqrt{\pi}}{2^n} \frac{\sin[(\sigma/2 + \xi)\pi]}{r \sin(\pi\sigma/2)} \Gamma\left[\frac{1}{2} + 2\gamma + h\right] B(\xi, 1-\sigma-\xi), \quad E_{31} = (-1)^n (1-\sigma) \sqrt{\pi} 2^{-n}, \\ u = \alpha + 2r(\sigma-1), \quad v_1 = 2r(1-\sigma+j) - \alpha, \quad a_1 = \xi, \quad b_1 = \sigma + \xi, \\ c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \sigma + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1-\sigma+j), \\ c_2 = (1-\alpha)/2 + \gamma + (1-\sigma+j)r, \quad z = (-\tilde{b}^2/p^2)^p \\ [y, b, \operatorname{Re} \sigma > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re}(\alpha + 2r\sigma) < (1 - (-1)^n)/2 + 2r]$$

$$6. \int_0^\infty x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} + x^{2r}})^v}{(x^{2r} + y^{2r})^p} \frac{(\sin bx)^n}{(\cos bx)} dx = J_\rho \quad [\rho = 0, 1/2] \\ J_\rho = V_{35} + V_{36} + V_{37} + \\ + \frac{1+(-1)^n}{2} 2^{v-n-1+\alpha/r} (-v)^{1-2\rho} \frac{n!}{r[(n/2)!]^2} y^{r\nu+p\alpha/r} \Gamma\left[\frac{\alpha/(2r)}{1-v-\alpha/(2r)}\right]$$

$$\{E_{35} = \frac{(-v)^{1-2\rho}}{r} \sqrt{\pi} 2^{v-n-1+\alpha/r} \Gamma\left[\frac{\xi}{1-v-\xi}, \frac{2\rho-v-2\xi}{1/2+2\gamma+h}\right], \\ E_{35} = (-v)^{1-2\rho} 2^{2\rho-1-n} \Gamma\left[\frac{\rho+j-v/2, 1/2-j, 1-d_2}{1-v/2-\rho-j, e_2}\right], \\ E_{37} = (-v)^{-2\rho+1} 2^{2\rho-1-n} \Gamma\left[\frac{\rho+j+(1-v)/2, -1/2-j, 1-d_2}{(1-v)/2-\rho-j, e_2}\right], \\ u = \alpha + r(v-2\rho), \quad v_1 = r(2\rho-v+2j) - \alpha, \quad v_2 = r(2\rho+1-v+2j) - \alpha, \\ a_1 = v + \xi, \quad b_1 = \xi, \quad c_1 = 1 - \rho + v/2 + \xi, \quad d_1 = (1+v)/2 - \rho + \xi, \\ e_1 = 1/2 + 2\gamma + h, \quad a_2 = \rho - v/2 + j, \quad b_2 = v/2 + \rho + j, \quad c_2 = 1/2 + j, \\ d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, \quad e_2 = (1-\alpha)/2 + \gamma + r(\rho - v/2 + j), \\ a_3 = \rho + (1-v)/2 + j, \quad b_3 = (1+v)/2 + \rho + j, \quad c_3 = 3/2 + j, \quad d_3 = d_2 + r/2, \\ e_3 = e_2 + r/2, \quad z = (-1)^{p+q} (\tilde{b}/p)^{2p} \\ [b > 0; 2r | \arg y| < \pi; -\delta n < \operatorname{Re} \alpha < 2\rho - r \operatorname{Re} v + (1 - (-1)^n)/2]$$

$$7. \int_0^\infty x^{\alpha-1} \frac{(y^r + \sqrt{x^{2r} + y^{2r}})^v}{(x^{2r} + y^{2r})^p} \frac{(\sin bx)^n}{(\cos bx)} dx = J_\rho \quad [\rho = 0, 1/2] \\ J_\rho = V_{38} + V_{39} + \\ + \frac{1+(-1)^n}{2} 2^{2\rho-n-1-\alpha/r} (-v)^{1-2\rho} \frac{n!}{r[(n/2)!]^2} y^{r\nu-2\rho+\alpha/r} \Gamma\left[\frac{\rho-v/2-\alpha/(2r)}{1-\rho-v/2+\alpha/(2r)}\right]$$

$$\{E_{35} = (-v)^{1-2\rho} \frac{\sqrt{\pi}}{r} 2^{2\rho-2j-n} \Gamma\left[\frac{2\xi, \rho-v/2-\xi}{1-\rho-v/2+\xi, 1/2+2\gamma+h}\right], \\ E_{35} = (-v)^{1-2\rho} \sqrt{\pi} 2^{2j-n} \Gamma\left[\frac{2\rho-v+2j, 1-d_2}{1-v+j, e_2}\right], \\ u = \alpha + r(v-2\rho), \quad v_1 = r(2\rho-v+2j) - \alpha, \quad a_1 = \xi, \quad b_1 = \xi + 1/2, \\ c_1 = 1 - \rho - v/2 + \xi, \quad d_1 = 1 - \rho + v/2 - \xi, \quad e_1 = 1/2 + 2\gamma + h, \\ a_2 = \rho - v/2 + j, \quad b_2 = \rho + (1-v)/2 + j, \quad c_2 = 1 - v + j, \\ d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, \quad e_2 = (1-\alpha)/2 + \gamma + r(\rho - v/2 + j), \\ z = (-1)^{p+q} (\tilde{b}/p)^{2p} \\ [b > 0; 2r | \arg y| < \pi; -\delta n < \operatorname{Re} \alpha < 2\rho - r \operatorname{Re} v + (1 - (-1)^n)/2]$$

$$8. \int_0^\infty x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} - x^{2r}})^v + (y^r - \sqrt{y^{2r} - x^{2r}})^v}{y^{2r} - x^{2r}} \frac{(\sin bx)^n}{(\cos bx)} dx = \\ = V_{38} + \frac{1+(-1)^n}{2} 2^{v-n-1+\alpha/r} y^{r\nu+\alpha} \frac{n!}{r[(n/2)!]^2} \Gamma\left[\frac{\alpha/(2r)}{v+\alpha/r}\right] \\ \{E_{38} = \sqrt{\pi} \frac{2^{v+\xi-n}}{r} \Gamma\left[\frac{\xi, v+\xi}{v+2\xi, 1/2+2\gamma+h}\right], \\ u = \alpha + r(v-1), \quad a_1 = \xi, \quad b_1 = v + \xi, \quad c_1 = v/2 + \xi, \quad d_1 = (v+1)/2 + \xi, \\ e_1 = 1/2 + 2\gamma + h, \quad z = (-\tilde{b}^2/p^2)^p \\ [y > 0; \operatorname{Re} \alpha > -n\delta; \operatorname{Re} \alpha > -n\delta - 2r \operatorname{Re} v; | \arg b| < \pi]$$

$$9. \int_0^\infty x^{\alpha-1} \frac{(x^r + \sqrt{x^{2r} - y^{2r}})^v + (x^r - \sqrt{x^{2r} - y^{2r}})^v}{x^{2r} - y^{2r}} \frac{(\sin bx)^n}{(\cos bx)} dx = \\ = V_{35} + V_{36} + V_{37} + \frac{1+(-1)^n}{2} 2^{-n-\alpha/r} y^{r(v-1)+\alpha} \times \\ \times \frac{n!}{r[(n/2)!]^2} \Gamma\left[\frac{(1+v)/2-\alpha/(2r), (1-v)/2-\alpha/(2r)}{1-\alpha/r}\right] \\ \{E_{35} = \frac{2^{-n-\alpha/r+1}}{r} \sqrt{\pi} \Gamma\left[\frac{(1+v)/2-\xi, (1-v)/2-\xi}{1-2\xi, 1/2+2\gamma+h}\right], \\ E_{36} = 2^{-n-v/2} \sqrt{\pi} \Gamma\left[\frac{-v-j, 1-d_2}{-v-2j, e_2}\right], \quad E_{37} = 2^{-n+v/2} \sqrt{\pi} \Gamma\left[\frac{v-j, 1-d_2}{v-2j, e_2}\right], \\ u = \alpha + r(v-1), \quad v_1 = r(1+v+2j) - \alpha, \quad v_2 = r(1-v+2j) - \alpha, \quad a_1 = \xi, \\ b_1 = \xi + 1/2, \quad c_1 = (1-v)/2 + \xi, \quad d_1 = (1+v)/2 + \xi, \quad e_1 = 1/2 + 2\gamma + h, \\ a_2 = (1+v)/2 + j, \quad b_2 = 1 + v/2 + j, \quad c_2 = 1 + v + j, \\ d_2 = 1 - \gamma + r[(1+v)/2 + j] - \alpha/2, \quad e_2 = (1-\alpha)/2 + \gamma + r[(1+v)/2 + j], \\ a_3 = (1-v)/2 + j, \quad b_3 = 1 - v/2 + j, \quad c_3 = 1 - v + j, \\ d_3 = 1 - \gamma + r[(1-v)/2 + j] - \alpha/2, \quad e_3 = (1-\alpha)/2 + \gamma + r[(1-v)/2 + j], \\ z = (-1)^p \tilde{b}^{2r} p^{-2\rho} \\ [y, b > 0; \operatorname{Re} \alpha < r(1 - |\operatorname{Re} v|) + (1 - (-1)^n)/2]$$

$$10. \int_0^\infty \frac{x^{\alpha-1}}{x^{2r} + 2(xy)^r \cos \beta + y^{2r}} \frac{(\sin bx)^n}{(\cos bx)} dx = \\ = V_{38} + V_{39} - \frac{1+(-1)^n}{2} 2^{-n-3} y^{-2r+\alpha} \frac{\pi n!}{r[(n/2)!]^2} \frac{\sin[\beta(\alpha-r)/r]}{\sin \beta \sin(\pi\alpha/r)} \\ \{E_{38} = -\frac{2^{1-n}}{r} \frac{1}{\sin \beta}, \quad E_{39} = -2^{-n} \frac{\sqrt{\pi}}{\sin \beta}, \quad u = \alpha - 2r,$$

$$\begin{aligned} v_1 &= 2\gamma + 2l, \quad v_2 = r(1+l) - \alpha, \quad a_l = \beta(2\gamma + 2l + \alpha - r) r^{-1}, \\ b_l &= (2\gamma + 2l + \alpha) n/r, \quad c_l = 1/2 + 2\gamma + l, \quad d_l = \beta l, \quad e_l = 1 + l - \alpha/r, \\ f_l &= \gamma + \alpha/2 - r(1+l)/2, \quad g_l = (1-\alpha)/2 + \gamma + r(1+l)/2, \\ [v, b > 0; |\beta| < \pi; -n\delta < \operatorname{Re} \alpha < 2r + (1-(-1)^n)/2; r > 0 - \text{произвольные}]. \end{aligned}$$

$$11. \int_0^\infty x^{\alpha-1} e^{-sx} \frac{\sin bx}{\cos bx} dx = U + \frac{1+(-1)^n}{2} 2^{-n-1} s^{-\alpha/2} \frac{n!}{r[(n/2)!]^2} \Gamma\left(\frac{\alpha}{2r}\right),$$

$$U = V_{43}$$

$$U = V_{43}$$

$$[r > 1/2]$$

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$$U = 2^{1-n} \pi \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} \sum_{l=0}^{[(m-1)/2]} \tau\left(\frac{m}{l}\right) A_{kl} \left[\frac{n-2k}{2} b\right]^{-ik},$$

$$A_{kl} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(j+\rho)} \Gamma\left[\frac{a_1}{c_1}, \frac{a_1}{1-b_1}\right] {}_2F_2\left(\frac{\Delta(\rho, a_1), \Delta(\rho, b_1), 1; z}{\Delta(q, c_1), \Delta(q, 1+i)}\right)$$

$$[r < 1],$$

$$A_{kl} = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \varphi^{-2\xi} \Gamma\left[1/2+\rho-\xi, 1/2+2\gamma+h\right] \times$$

$$\times {}_3F_3\left(\frac{\Delta(q, \xi+\rho), \Delta(q, 1/2-\rho+\xi), 1; z}{\Delta(\rho, 1/2+2\gamma+h), \Delta(\rho, 1+h)}\right) \quad [r > 1]$$

$$\{a_1 = \gamma + \alpha/2 + r(j+\rho), \quad b_1 = (1+\alpha)/2 - \gamma + r(j+\rho),$$

$$c_1 = 1/2 + 2\rho + j, \quad \varphi = \frac{a(m-2l)}{2} \left(\frac{2}{(n-2k)b}\right)^r, \quad z = (-1)^{\rho+q} e^{2\rho q} e^{2i(q+\gamma)}$$

$$[a, b > 0; -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < 0 \text{ (при } m+n=2, 4, 6, \dots), \text{ ил} \\ -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < \max(1, r) \text{ (при } m+n=1, 3, 5, \dots)].$$

$$14. \int_0^\infty x^{2-1} \cos^m ax \left\{ \frac{\sin^m bx}{\cos^m bx - c} \right\} dx = I(0, 0, 1) \quad [\text{см. 2.5.2.13}]$$

$$15. \int_0^\infty x^{2-1} \sin^m ax \left\{ \frac{\sin^m bx}{\cos^m bx - c} \right\} dx = J\left[\frac{1-(-1)^m}{4}, 1, (-1)^{[m/2]+1}\right],$$

$$J(a, \sigma, \tau) = 2^{1-n} \pi \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} \sum_{l=0}^{[(m-1)/2]} \tau\left(\frac{m}{l}\right) \times$$

$$\times \left[\frac{b(n-2k)}{2}\right]^{-a} \left\{ \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(j+\rho)} \Gamma\left[\frac{1-b_1}{c_1}\right] \times \right.$$

$$\times {}_1F_3\left(\frac{1}{\Delta(q, a_1), \Delta(\rho, b_1), \Delta(\rho, c_1), \Delta(q, 1+i)}\right) +$$

$$+ \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \varphi^{-2\xi} \Gamma\left[1/2+\rho+\xi, 1/2+2\gamma+h\right] \times$$

$$\times {}_1F_3\left(\frac{1}{\Delta(q, 1-\rho+\xi), \Delta(q, 1/2+\rho+\xi), \Delta(\rho, 1+h), \Delta(\rho, 1/2+2\gamma+h)}\right) +$$

$$+ \left[\frac{1+(-1)^n}{2} \frac{n}{2^n [(n/2)!]^2} - \frac{[0]}{[c]} \right] \frac{a^{\alpha/r}}{r} \frac{\sqrt{\pi}}{2^m} \sum_{l=0}^{[(m-1)/2]} \tau\left(\frac{m}{l}\right) \times$$

$$\times \left(\frac{m-2l}{2}\right)^{\alpha/r} \Gamma\left[\frac{\rho-\alpha/(2r)}{1/2+\rho+\alpha/(2r)}\right] + \frac{1+(-1)^m}{2} 2^{-m-n} b^{-a} \sqrt{\pi} \times$$

$$\times \frac{m!}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} \left(\frac{2}{n-2k}\right)^a \Gamma\left[\frac{\gamma+\alpha/2}{\gamma+(1-\alpha)/2}\right]$$

$$v_1 = 2\gamma + 2l, v_2 = r(1+l) - \alpha, a_l = \beta(2\gamma + 2l + \alpha - r)r^{-1}, \\ b_l = (2\gamma + 2l + \alpha)n/r, c_l = 1/2 + 2\gamma + l, d_l = \beta l, e_l = 1 + l - \alpha/r, \\ f_l = \gamma + \alpha/2 - r(1+l)/2, g_l = (1-\alpha)/2 + \gamma + r(1+l)/2, \\ [\nu, b > 0; |\beta| < \pi; -n\delta < \operatorname{Re} \alpha < 2r + (1-(-1)^n)/2; r > 0 - \text{произвольные}]$$

$$11. \int_0^\infty x^{b-1} e^{-sx} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = U + \frac{1+(-1)^n}{2} 2^{-n-1} s^{-\alpha/(2r)} \frac{n!}{r[(n/2)!]^2} \Gamma\left(\frac{\alpha}{2r}\right),$$

$$U = V_{43}$$

$$U = V_{41}$$

$$U = \sqrt{\pi} 2^{1-n} \sum_{k=0}^{[(n-1)/2]} s \binom{n}{k} s^{-\alpha/2} \Gamma\left[\frac{\alpha+2\gamma}{1/2+2\gamma}\right] {}_2F_1\left(\Delta(2, \alpha+2\gamma); -\frac{b^2(n-2\gamma)^2}{1/2+2\gamma}\right) \quad [r > 1/2]$$

$$\{E_{43} = \frac{\sqrt{\pi}}{2^{\alpha} r} \Gamma\left[\frac{\xi}{1/2+2\gamma+h}\right], u = -\frac{\alpha}{2r}, v_1 = 2(h+\gamma),$$

$$\varphi = 2^{-1}(n-2k)bs^{-1/(2r)}, a_1 = \xi, b_1 = 1/2+2\gamma+h,$$

$$z = (-1)^{\rho} \left\{ \frac{(n-2k)b}{2\rho} \right\}^{2\rho} q^{\rho} s^{-\rho}; E_{41} = \frac{\sqrt{\pi}}{2^{\alpha} r} \Gamma\left[\frac{\gamma+rj+\alpha/2}{1/2+\gamma-rj-\alpha/2}\right],$$

$$v_1 = -2rj - \alpha, a_2 = \gamma + rj + \alpha/2, b_2 = (1+\alpha)/2 - \gamma + rj,$$

$$z_2 = (-1)^{\rho+j} \left\{ \frac{(n-2k)b}{2\rho} \right\}^{-2\rho} q^{-\rho} s^{\rho}$$

$$[\operatorname{Re} s > 0; \operatorname{Re} \alpha > -\delta n; \arg b \text{ произволен (при } r > 1/2); b > 0 \text{ (при } r < 1/2); n | \operatorname{Im} b| < \operatorname{Re} s \text{ (при } r = 1/2)]$$

$$12. \int_0^\infty x^{b-1} e^{-sx} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{42} + V_{43} + \frac{1+(-1)^n}{2} 2^{-n-1} s^{-\alpha/(2r)} \frac{n!}{r[(n/2)!]^2} \Gamma\left(-\frac{\alpha}{2r}\right)$$

$$\{E_{42} = \frac{\sqrt{\pi}}{2^{\alpha} r} \Gamma\left[\frac{-\xi}{1/2+2\gamma+h}\right], E_{43} = \frac{\sqrt{\pi}}{2^{\alpha} r} \Gamma\left[\frac{\gamma-rj+\alpha/2}{(1-\alpha)/2+\gamma+rj}\right],$$

$$u = \alpha/(2r), v_1 = 2(\gamma+h), v_2 = 2rj - \alpha, \varphi = 2^{-1}b(n-2k)s^{1/(2r)},$$

$$a_1 = 1/2+2\gamma+h, b_1 = 1+\xi, a_2 = 1-\gamma+rj-\alpha/2, b_2 = (1-\alpha)/2+\gamma+rj,$$

$$z = (-1)^{\rho+j} \left\{ \frac{(n-2k)b}{2\rho} \right\}^{2\rho} s^{\rho} q^{-\rho} \quad [b, \operatorname{Re} s > 0; \operatorname{Re} \alpha < (1-(-1)^n)/2]$$

$$13. \int_0^\infty x^{\alpha-1} \sin^m ax \left\{ \frac{\sin bx}{\cos bx - c} \right\} dx = I \left[\frac{1-(-1)^m}{4}, 1, (-1)^{[m/2]+l} \right],$$

$$I(\rho, \sigma, \tau) = U + \left[\frac{1+(-1)^n}{2} \frac{n!}{2^{\alpha} [(n/2)!]^2} - \left\{ \frac{0}{c} \right\} \right] \frac{a^{-\alpha/r}}{r} 2^{-m} \sqrt{\pi} \times$$

$$\times \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \left(\frac{2}{m-2l} \right)^{\alpha/r} \Gamma\left[\frac{\rho+\alpha/(2r)}{\rho+1/2-\alpha/(2r)}\right] +$$

$$+ \frac{1+(-1)^m}{2} 2^{-m-n} \frac{b^{\alpha m} \sqrt{\pi}}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \left(\frac{2}{n-2k} \right)^{\alpha} \Gamma\left[\frac{\gamma+\alpha/2}{\gamma+(1-\alpha)/2}\right]$$

$$[r \neq 1 \text{ (} r = 1 \text{ см. 2.5.13.20-21); } c = 1 \text{ при } n = 2, 4, 6, \dots; c = 0 \text{ при } n = 1, 3, 5, \dots]$$

$$U = 2^{1-m-n} \pi \sum_{k=0}^{[(m-1)/2]} e \binom{m}{k} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} A_{kl} \left[\frac{n-2k}{2} b \right]^{-\alpha},$$

$$A_{kl} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2j/\rho} \Gamma\left[\frac{a_1}{c_1, 1-b_1}\right] {}_2F_2\left(\Delta(\rho, a_1), \Delta(\rho, b_1), 1; z\right)$$

$$[r < 1],$$

$$A_{kl} = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \varphi^{2h/\xi} \Gamma\left[1/2+\rho-\frac{\rho+\xi}{\xi}, 1/2+2\gamma+h\right] \times$$

$$\times {}_2F_2\left(\Delta(q, \xi+\rho), \Delta(q, 1/2-\rho+\xi), 1; z\right) \quad [r > 1]$$

$$[a_1 = \gamma + \alpha/2 + r(j+\rho), b_1 = (1+\alpha)/2 - \gamma + r(j+\rho),$$

$$c_1 = 1/2+2\rho+j, \varphi = \frac{a(m-2l)}{(n-2k)b} \left(\frac{2}{(n-2k)b} \right)^{\gamma}, z = (-1)^{\rho+j} q^{\rho} s^{\rho} q^{-2\rho} \varphi^{2\rho}]$$

$$[a, b > 0; -\delta n - \alpha r m - 2c < \operatorname{Re} \alpha < 0 \text{ (при } m+n=2, 4, 6, \dots), \text{ или } -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < \max(1, r) \text{ (при } m+n=1, 3, 5, \dots)]$$

$$14. \int_0^\infty x^{b-1} \cos^m ax \left\{ \frac{\sin bx}{\cos bx - c} \right\} dx = I(0, 0, 1) \quad [\text{см. 2.5.2.13}]$$

$$15. \int_0^\infty x^{b-1} \sin^m ax \left\{ \frac{\sin bx}{\cos bx - c} \right\} dx = J \left[\frac{1-(-1)^m}{4}, 1, (-1)^{[m/2]+l} \right],$$

$$J(\rho, \sigma, \tau) = 2^{1-m-n} \pi \sum_{k=0}^{[(m-1)/2]} e \binom{n}{k} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \times$$

$$\times \left[\frac{b(n-2k)}{2} \right]^{-\alpha} \left\{ \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2j/\rho} \Gamma\left[\frac{1-b_1}{a_1, c_1}\right] \times \right.$$

$$\times {}_2F_2\left(\Delta(q, a_1), \Delta(\rho, b_1), \Delta(\rho, c_1), \Delta(q, 1+j)\right) +$$

$$+ \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \varphi^{2h/\xi} \Gamma\left[1/2+\rho+\frac{\rho-\xi}{\xi}, 1/2+2\gamma+h\right] \times$$

$$\times {}_2F_2\left(\Delta(q, 1-\rho+\xi), \Delta(q, 1/2+\rho+\xi), \Delta(\rho, 1+h), \Delta(\rho, 1/2+2\gamma+h)\right) \Big\} +$$

$$+ \left[\frac{1+(-1)^n}{2} \frac{n}{2^{\alpha} [(n/2)!]^2} - \left\{ \frac{0}{c} \right\} \right] \frac{a^{\alpha/r} \sqrt{\pi}}{r 2^m} \sum_{l=0}^{[(m-1)/2]} \tau \binom{m}{l} \times$$

$$\times \left(\frac{m-2l}{2} \right)^{\alpha/r} \Gamma\left[\frac{\rho-\alpha/(2r)}{1/2+\rho+\alpha/(2r)}\right] + \frac{1+(-1)^m}{2} 2^{-m-n} b^{-\alpha} \sqrt{\pi} \times$$

$$\times \frac{n!}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \left(\frac{2}{n-2k} \right)^{\alpha} \Gamma\left[\frac{\gamma+\alpha/2}{\gamma+(1-\alpha)/2}\right]$$

$$(a_1 = 1/2 + 2\rho + j, \quad b_1 = 1 - \gamma - \alpha/2 + r(\rho + j), \quad c_1 = (1 - \alpha)/2 + \gamma + r(\rho + j),$$

$$\psi = a \frac{n-2j}{2} \left(b \frac{n-2k}{2} \right)^r, \quad z = (-1)^{p+q} \psi^{2j} \rho^{-2\rho} \gamma^{2q}$$

$$\left[a, b > 0; -2\epsilon - \delta n - \sigma r m < \operatorname{Re} \alpha < \left(\frac{1 - (-1)^n}{2} + \frac{1 - (-1)^m}{2} \right) r, \right. \\ \left. \epsilon = 1 \text{ при } n = 2, 4, 6, \dots; \epsilon = 0 \text{ при } n = 1, 3, 5, \dots \right]$$

$$16. \int_0^\infty x^{\alpha-1} \cos^\eta ax \cdot r \left\{ \frac{\sin bx}{\cos bx - c} \right\} dx = J(0, 0, 1) \quad [\text{см. 2.5.2 15}]$$

$$17. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(\sigma) \Big|_{\sigma=1} \quad [r > 0],$$

$$I(\sigma) = U + \frac{1 + (-1)^n 2^{-n}}{2} \frac{\pi}{r} (s^2 + a^2)^{-\alpha/(2r)} \frac{\pi}{[(n/2)!]^2} \Gamma\left(\frac{\alpha}{r}\right) \cos\left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right),$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\sigma\pi}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (r+l+\alpha)/2}{\gamma + (1-r-\alpha)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{-\alpha-2r} \quad [r < 1],$$

$$U = \frac{\sqrt{\pi}}{2^n 1^r} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left(\frac{\pi\sigma}{2} - 2\gamma \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{2\gamma}{1/2 + h + 2\gamma}\right] (s^2 + a^2)^{-\frac{1}{2}} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad [r > 1],$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) A_k(z) \quad [r = 1],$$

$$z = \left(b \frac{n-2k}{\sqrt{s^2 + a^2}}\right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s}\right] \times \\ \times \Gamma\left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad \text{при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} + l \operatorname{arctg} \frac{a}{s}\right) \Gamma\left[\frac{\gamma + (\alpha+l)/2}{\gamma + (1-\alpha-l)/2}\right] \times \\ \times (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{-\alpha-l} \quad \text{при } |z| > 1$$

$$[a, \operatorname{Re} s > 0; \operatorname{Re} \alpha > -\sigma r - \delta n; b - \text{произвольное (при } r > 1), |1mb|n < \operatorname{Re} s \text{ (при } r = 1) \\ \text{и } b > 0 \text{ (при } r < 1)] \text{ или } [\operatorname{Re} s = 0; a, b > 0; -\sigma r - \delta n < \operatorname{Re} \alpha < \max(r, 1) \text{ (при } r \neq 1) \\ \text{или } -\sigma r - \delta n < \operatorname{Re} \alpha < 0 \text{ (при } r = 1)]$$

$$18. \int_0^\infty x^{\alpha-1} e^{-sx} \cos ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(\sigma) \Big|_{\sigma=0} \quad [I(\sigma) \text{ см. 2.5.2 17}]$$

$$19. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J(\sigma) \Big|_{\sigma=1}$$

$$J(\sigma) = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\sigma\pi}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (\alpha-lr)/2}{\gamma + (1-\alpha+lr)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{r-l-\alpha} + \\ + \frac{\sqrt{\pi}}{r 2^{n-1}} \sum_{h=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left(\frac{\sigma\pi}{2} + 2\gamma \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{-2\gamma}{2\gamma + 1/2 + h}\right] (s^2 + a^2)^{-\frac{1}{2}} \left(b \frac{n-2k}{2}\right)^{2\gamma+2h} + \\ + \frac{1 + (-1)^n (s^2 + a^2)^{\alpha/(2r)} n!}{2^{2n} [(n/2)!]^2} \cos\left(\frac{\pi\sigma}{2} + \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right) \Gamma\left(-\frac{\alpha}{r}\right) \\ [a, b > 0; \operatorname{Re} \alpha < r\sigma + (1 - (-1)^n)/2 \text{ и } \operatorname{Re} s > 0 \text{ или } \operatorname{Re} s = 0 \text{ и } \operatorname{Re} \alpha > -r - n\delta].$$

$$20. \int_0^\infty x^{\alpha-1} e^{-sx} \cos ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2.5.2 19}].$$

$$21. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = K(\sigma) \Big|_{\sigma=1} \quad [r > 0],$$

$$K(\sigma) = U + \frac{1 + (-1)^n}{2} (-1)^{n/2} \frac{2^{-n} \pi}{r [(n/2)!]^2} \Gamma\left(\frac{\alpha}{r}\right) (s^2 + a^2)^{-\alpha/(2r)} \times \\ \times \cos\left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right),$$

$$U = \frac{\sqrt{\pi}}{2^n 1^r} \sum_{k=0}^{[(n-1)/2]} (-1)^k e\left(\frac{n}{k}\right) \sum_{h=0}^\infty \frac{1}{h!} \cos\left(\frac{\pi\sigma}{2} - 2\gamma \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{2\gamma}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\frac{1}{2}} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad [r > 1],$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} (-1)^k e\left(\frac{n}{k}\right) A_k(z) \quad [r = 1],$$

$$z = \left(b \frac{n-2k}{\sqrt{s^2 + a^2}}\right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^\infty \frac{1}{h!} \cos\left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s}\right] \times \\ \times \Gamma\left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad \text{при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^{\infty} \frac{(-1)^l \sin[c - \gamma + (\alpha + l)/2] \pi}{l! \sin c \pi} \cos\left(\frac{\alpha \pi}{2} + l \arctg \frac{a}{y}\right) \times$$

$$\times \Gamma\left[\frac{\gamma + (\alpha + l)/2}{\gamma + (1 - \alpha - l)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n - 2k}{2}\right)^{-\alpha - l}$$

при $|z| > 1$

$[c - \text{произвольное вещелое}; a, \operatorname{Re} s > 0; \operatorname{Re} \alpha > -\sigma r - \delta n; b - \text{произвольное (при } r > 1, \text{ или } n; \operatorname{Re} b < \operatorname{Re} s \text{ (при } r = 1))}]$

$$22. \int_0^{\infty} x^{n-1} e^{-\lambda x} \cos ax \left(\frac{\sin bx}{\cos bx}\right)^n dx = K(0) \Big|_{\sigma=0} \quad [\text{см. 2.5.2.1}]$$

2.5.3. Интегралы от $x^a \left(\frac{\sin bx}{\cos bx}\right)^n$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} \left(\frac{\sin x}{\cos x}\right)^n dx = \frac{\sqrt{\pi}}{2} \Gamma\left[\frac{(\alpha+1)/2}{\alpha/2+1}\right] \quad [\operatorname{Re} \alpha > -1]$$

$$2. \int_0^{\pi/2} x \left(\frac{\sin x}{\cos x}\right)^n dx =$$

$$= \left(1 + 2 \left[\frac{n+1}{2}\right] + n\right) A \frac{(n-1)! \pi}{4n!} + \sum_{k=0}^{[(n-1)/2]} \frac{((1-n)/2)_k}{(-n/2)_k (n-2k)} \cdot$$

[$A=1$ при $n=1, 3, 5, \dots$; $A=\pi/2$ при $n=2, 4, 6, \dots$]

$$3. \int_0^{\pi/2} x^m \left(\frac{\sin x}{\cos x}\right)^n dx =$$

$$= \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k-\delta)!} \left(\frac{\pi}{2}\right)^{m-2k-\delta} + (-1)^{[m/2]} m! \left(2 \left[\frac{m}{2}\right] + \delta - m\right) \cdot$$

$$4. \int_0^{\pi/2} \frac{\sin^2 x}{x} dx = \frac{1}{2} [C + \ln \pi - c_1(\pi)].$$

$$5. \int_0^{\pi} x^m \left(\frac{\sin nx}{\cos nx}\right)^n dx = \frac{(-1)^n \pi^{\delta+1}}{n^{m+1}} \sum_{k=0}^{[(m+\delta-1)/2]} (-1)^k \frac{m!}{(m-2k+\delta-1)!} \times$$

$$\times (\pi n)^{m-2k+\delta-1} + (-1)^{[(m+\delta+1)/2]} m! n^{-m-1} \left(2 \left[\frac{m-\delta+1}{2}\right] - m + \frac{1}{2}\right).$$

$$6. \int_0^{\pi} x^{\alpha-1} \sin^n x dx = -\frac{(\alpha-1)(\alpha-2)}{n^2} \int_0^{\pi} x^{\alpha-3} \sin^n x dx + \frac{n-1}{n} \int_0^{\pi} x^{\alpha-1} \sin^{n-2} x dx$$

[$\operatorname{Re} \alpha > 2-n$; $n=2, 3, 4, \dots$]

$$7. \int_0^{\pi} x \cos^{2n} x dx = \frac{(2n-1)! \pi^2}{2(2n)!}.$$

$$8. \int_0^{\pi/2} x \sin^n x dx = \frac{\pi(n-1)!}{2(n!)} \begin{cases} \pi k^2, & n=0, 2, 4, \dots \\ 2k, & n=1, 3, 5, \dots \end{cases}$$

$$9. \int_0^{\pi/2} x^m \cos nx dx = - \sum_{k=0}^{m-1} \frac{k!}{n^{k+1}} \left(\frac{m}{k}\right) (2\pi)^{m-k} \cos \frac{k+1}{2} \pi$$

[$n, l=1, 2, 3, \dots$].

$$10. \int_0^{\infty} x^{\alpha-1} \left(\frac{\sin bx}{\cos bx}\right) dx = \frac{\Gamma(\alpha)}{b^{\alpha}} \left(\frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)}\right) \quad [b > 0; -\delta < \operatorname{Re} \alpha < 1]$$

$$11. \int_0^{\infty} x^{-1/2} \left(\frac{\sin bx}{\cos bx}\right) dx = \sqrt{\frac{\pi}{2b}} \quad [b > 0].$$

$$12. \int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2} \operatorname{sgn} b \quad [m=0].$$

$$13. \int_0^{\infty} x^{\alpha-1} \sin^n bx dx = I_{\alpha, n}$$

[$\operatorname{Im} b=0$; $-n < \operatorname{Re} \alpha < 2\lambda$; $n=1, 2, 3, \dots$; $\lambda=(1-(-1)^n)/4$].

$$I_{\alpha, n} = 2^{2\lambda-n} \sqrt{\pi} |b|^{-\alpha} (\operatorname{sgn} b)^n \Gamma\left[\frac{\lambda+\alpha/2}{\lambda+(1-\alpha)/2}\right] \sum_{j=0}^{[(n-1)/2]} (-1)^{[n/2]+j} \frac{n!(n-2j)^{-\alpha}}{j!(n-j)!}$$

при $\alpha \neq -2(\lambda+k)$, $k=0, 1, 2, \dots$.

$I_{-2\lambda-2k, n} = \lim_{\alpha \rightarrow -2\lambda-2k} I_{\alpha, n}$ при $\alpha \rightarrow -2\lambda-2k$.

$$I_{1/2-n, n-1} = (-1)^{[m/2]} \sqrt{\pi} 2^{m-2l+3/2} \frac{|b|^{m-1/2}}{(2m-1)!} \operatorname{sgn} b \times$$

$$\times \sum_{j=1}^l (-1)^{j-1} \binom{2l-1}{l+j-1} (2l-1)^{m-1/2} \quad [0 \leq m \leq 2l-1].$$

$$I_{1/2-n, n} = (-1)^{[(n+1)/2]} \sqrt{\pi} 2^{2n-2l} \frac{|b|^{m-1/2}}{(2m-1)!} \sum_{j=1}^l (-1)^j \binom{2l}{l+j} j^{m-1/2}$$

[$m=1, 2, \dots, 2l$].

$$I_{-1/2, 1} = \sqrt{2\pi|b|} \operatorname{sgn} b, \quad I_{-3/2, 2} = \frac{4}{3} b \sqrt{\pi|b|}.$$

$$I_{-1/2, 3} = \frac{3-\sqrt{3}}{4} \sqrt{\pi|b|} \operatorname{sgn} b.$$

$$I_{3/2, 1} = 2\pi \sum_{j=0}^{l-1} (-1)^{l+j+1} \frac{(2l-1)!}{j!(2l-j-1)!} \operatorname{sgn} b.$$

$$I_{0, (2m+1)/(2n+1)} = 2^{(2m-4n-1)/(2n+1)} B((2m+1)/(4n+2), (2m+1)/(4n+2))$$

[b > 0]

$$I_{-m, n} = \frac{2^{-n} b^m \pi}{m! \Gamma(m\pi/2)} \sum_{j=0}^{(n-1)/2} (-1)^{(n-1)/2+j} \frac{n! (n-2j)^m}{j! (n-j)!}$$

[b > 0, T(x) = cos x при n = 2l - 1; T(x) = sin x при n = 2l, l = 1, 2, 3, ...]

$$I_{-1, 2l} = \frac{\pi |b| (2l-3)!!}{2 (2l-2)!!}, \quad I_{-1, 2l-1} = \frac{\pi |b| (2l-1)!!}{4 (2l-2)!!},$$

$$I_{\alpha, 1} = -2^{-\alpha-1} \Gamma(\alpha) |b|^{-\alpha} \cos(\alpha\pi/2)$$

$$I_{\alpha, 2} = 2^{-2} |b|^{-\alpha} (3-3^{-\alpha}) \sin(\alpha\pi/2) \Gamma(\alpha) \operatorname{sgn} b$$

[-2 < Re α < 0]

[-3 < Re α < 1]

$$I_{1, 2} = \frac{\pi}{4} \operatorname{sgn} b, \quad I_{-1, 2} = \frac{3}{4} b \ln 3, \quad I_{-2, 2} = \frac{3}{8} \pi |b|.$$

$$14. \int_0^{\pi} x^{\alpha-1} \cos^{2n+1} bx \, dx =$$

$$= 2^{\alpha-2n-1} \sqrt{\pi} (2n+1)! b^{-\alpha} \Gamma\left[\frac{\alpha/2}{(1-\alpha)/2}\right] \sum_{k=0}^n \frac{(2n-2k+1)^{-\alpha}}{k! (2n-k+1)!}$$

[b > 0; 0 < Re α < 1]

2.5.4. Интегралы от $x^m \left\{ \frac{\sin x}{\cos x} \right\}^{-n}$.

$$1. \int_0^{\pi/4} \frac{x^m}{\sin^2 x} \, dx = -\left(\frac{\pi}{4}\right)^m \left[1 - \frac{4m}{\pi(m-1)} + \frac{4m}{\pi} \sum_{k=1}^{\infty} \frac{\pi^{2k} B_{2k}}{2^{2k} (2k)! (m+2k-1)} \right]$$

[m = 2, 3, 4, ...]

$$2. \int_0^{\pi/4} \frac{x^2}{\{\sin x\}^2} \, dx = \frac{\pi}{4} \ln 2 + \frac{\pi^3}{16} \pm G.$$

$$3. \int_0^{\pi/4} \frac{dx}{\cos^n x} =$$

$$= -\sum_{k=1}^{[n/2]} \frac{(1-n/2)_{k-1}}{((1-n/2)/2)_k} 2^{n/2-k} + \left(n-2\left[\frac{n}{2}\right]\right) \frac{(2[n/2]-1)!!}{(2[n/2])!!} \ln(1+\sqrt{2})$$

[n = 1, 2, 3, ...]

$$4. \int_0^{\pi/2} \frac{x^m}{\sin x} \, dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{4^{2k-1} (m+2k)} \zeta(2k) \right]$$

[m = 1, 2, 3, ...]

$$5. \int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2G.$$

$$6. \int_0^{\pi/2} \frac{x^m}{\sin^2 x} \, dx = m \left(\frac{\pi}{2}\right)^{m-1} \left[\frac{1}{m-1} + \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k} \pi^{2k}}{(m+2k-1)(2k)!} \right]$$

[m = 2, 3, 4, ...]

[2.5.6]

$$7. \int_0^{\pi/2} \frac{x^{2\sigma+1}}{\sin^2 x} \, dx = \frac{\pi^{2\sigma+1}}{2^{\sigma+1}} \ln 2 - \frac{9\pi^{\sigma}}{4} \zeta(3)$$

[σ = 0 или 1]

$$8. \int_0^{\pi} \frac{x}{\cos x} \, dx = -4G.$$

2.5.5. Интегралы от $(x+z)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\}$.Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi} (x-z)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \frac{b^{\delta} a^{\beta-1} \delta}{\beta(\beta+1)^{\delta}} {}_2F_2\left(1; 1+\frac{\beta}{2}, \frac{1+\beta}{2}+\delta; -\frac{a^2 b^2}{4}\right)$$

[a, Re β > 0; |arg b| < π]

$$2. \int_a^{\infty} (x-a)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \frac{\Gamma(\beta)}{b^{\beta}} \left\{ \frac{\sin(ab+\beta\pi/2)}{\cos(ab+\beta\pi/2)} \right\}$$

[a, b > 0; 0 < Re β < 1]

$$3. \int_0^{\infty} \frac{1}{(x+z)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx =$$

$$= \frac{b^{\delta} z^{1+\delta-\rho}}{(\rho-1)(\rho-2)^{\delta}} {}_2F_2\left(1; \frac{3-\rho}{2}, 1+\delta-\frac{\rho}{2}; -\frac{b^2 z^2}{4}\right) +$$

$$+ \Gamma(1-\rho) b^{\delta-1} \left\{ \frac{\cos(\rho\pi/2+bz)}{\sin(\rho\pi/2+bz)} \right\}$$

[b, Re ρ > 0; |arg z| < π]

$$4. \int_0^{\infty} \frac{1}{\sqrt{x+z}} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx =$$

$$= \sqrt{\frac{\pi}{2b}} \left[\cos bz \mp \sin bz \pm 2C(bz) \left\{ \frac{\sin bz}{\cos bz} \right\} - 2S(bz) \left\{ \frac{\cos bz}{\sin bz} \right\} \right]$$

[b > 0; |arg z| < π]

$$5. \int_0^{\infty} \frac{1}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \pm \operatorname{ci}(bz) \left\{ \frac{\sin bz}{\cos bz} \right\} - \operatorname{si}(bz) \left\{ \frac{\cos bz}{\sin bz} \right\}$$

[b > 0; |arg z| < π]

$$6. \int_0^{\infty} \frac{1}{x-y} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \pm \left\{ \frac{\cos by}{\sin by} \right\} [\operatorname{si}(by) + \pi] - \left\{ \frac{\sin by}{\cos by} \right\} \operatorname{ci}(by)$$

[b, y > 0]

$$7. \int_{-\infty}^{\infty} \frac{1}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \pi \left\{ \frac{\cos bz}{\sin bz} \right\}$$

[b > 0, Im z ≠ 0]

2.5.6. Интегралы от $(x^2 \pm a^2)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$, $(a^2 - x^2)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$.

$$1. \int_0^{\infty} (a^2 - x^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{H_{\beta-1/2}(ab)}{J_{\beta-1/2}(ab)} \right\}$$

[a, Re β > 0; |arg b| < π]

2. $\int_a^\infty (x^2 - a^2)^{\beta-1} \frac{\sin bx}{\cos bx} dx = \pm \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{J_{1/2-\beta}(ab)}{V_{1/2-\beta}(ab)} \right\}$
[a, b > 0; 0 < Re β < 1]
3. $\int_0^\infty \frac{\sin bx}{(x^2 + z^2)^\rho} dx = \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} b^{\rho-1/2} [J_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)]$
[b, Re ρ, Re z > 0]
4. $\int_0^\infty \frac{\cos bx}{(x^2 + z^2)^\rho} dx = \left(\frac{2z}{b}\right)^{1/2-\rho} \frac{\sqrt{\pi}}{\Gamma(\rho)} K_{1/2-\rho}(bz)$
[b, Re ρ, Re z > 0]
5. $\int_0^\infty \frac{\sin bx}{x^2 + z^2} dx = \frac{1}{2z} [e^{-bz} \operatorname{Ei}(bz) - e^{bz} \operatorname{Ei}(-bz)]$
[b, Re z > 0]
6. $\int_0^\infty \frac{\cos bx}{(x^2 + z^2)^{m+1}} dx = \frac{\pi e^{-bz}}{(2z)^{2m+1} m!} \sum_{k=0}^m \frac{(2m-k)! (2bz)^k}{k! (m-k)!}$
[b, Re z > 0]
7. $\int_0^\infty \frac{\sin bx}{x^2 - y^2} dx = \frac{1}{y} [\cos by \operatorname{Si}(by) - \sin by \operatorname{Ci}(by)]$
[b, y > 0]
8. $\int_0^\infty \frac{\cos by}{x^2 - y^2} dx = -\frac{\pi}{2y} \sin by$
[b, y > 0]
9. $\int_0^\infty \frac{1}{x^2 - y^2} \left\{ \frac{\sin bx}{\cos bx} \right\}^2 dx = \pm \frac{\pi}{4y} \sin 2by$
[b, y > 0]
10. $\int_0^\infty \frac{\sin^{2n} bx}{x^2 + z^2} dx = \frac{(-1)^n \pi}{2^{2n+1} z} \left\{ 2^{2n} \sin^{2n} bz - 2 \sum_{k=0}^n (-1)^k \binom{2n}{k} z^{2(n-k)} [2(n-k)bz] \right\}$
[b, Re z > 0]
11. $\int_0^\infty \frac{\sin^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 - e^{-2bz})$
[b, Re z > 0]
12. $\int_0^\infty \frac{\sin^{2n+1} bx}{x^2 + z^2} dx =$

$$= \frac{(-1)^{n+1}}{2^{2n+1} z} \left\{ e^{i(2n+1)bz} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{-2kz} \operatorname{Ei}[(2k-2n-1)bz] + \right.$$

$$\left. + e^{-i(2n+1)bz} \sum_{k=1}^{2n+1} (-1)^{k-1} \binom{2n+1}{k} e^{2kz} \operatorname{Ei}[(2n+1-2k)bz] \right\}$$

[b, Re z > 0]

13. $\int_0^\infty \frac{\cos^{2n} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1} z} \binom{2n}{n} + \frac{\pi}{2^{2n}} \sum_{k=1}^n \binom{2n}{n+k} e^{-2kz}$
[b, Re z > 0]
 14. $\int_0^\infty \frac{\cos^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 + e^{-2bz})$
[b, Re z > 0]
 15. $\int_0^\infty \frac{\cos^{2n+1} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1} z} \sum_{k=0}^n \binom{2n+1}{n+k+1} e^{-i(2k+1)bz}$
[b, Re z > 0]
- 2.5.7. Интегралы от $x^\alpha (x \pm a)^\mu \left\{ \frac{\sin bx}{\cos bx} \right\}$, $x^\alpha (a-x)^\mu \left\{ \frac{\sin bx}{\cos bx} \right\}$.
 Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.
1. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$

$$= (-i)^\delta 2^{-1} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_1F_1(\alpha; \alpha+\beta; iab) \mp {}_1F_1(\alpha; \alpha+\beta; -iab)]$$

[a, Re β > 0; Re α > -δ; |arg b| < π].
 2. $\int_0^a x^{\alpha-1} (a-x)^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\pi} \left(\frac{a}{b}\right)^{\alpha-1/2} \left\{ \frac{\sin(ab/2)}{\cos(ab/2)} \right\} \Gamma(\alpha) J_{\alpha-1/2}\left(\frac{ab}{2}\right)$
[a > 0; Re α > -δ; |arg b| < π].
 3. $\int_0^a \frac{1}{x(a-x)} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2} \sqrt{\delta} J_{3/2}^2\left(\frac{ab}{2}\right)$
[a > 0; |arg b| < π].
 4. $\int_0^a x^{\alpha-1} (x-a)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = B(\beta, 1-\alpha-\beta-\delta) b^\delta a^{\alpha+\beta+\delta-1} \times$

$$\times {}_2F_3\left(\frac{\alpha+1}{2}, \frac{1+\delta}{2}; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+\delta+1}{2}, \frac{1}{2}+\delta; -\frac{a^2 b^2}{4}\right) +$$

$$+ (\beta-1) \Gamma(\alpha+\beta-2) \left\{ \frac{\sin[(\alpha+\beta)\pi/2]}{\cos[(\alpha+\beta)\pi/2]} \right\} a b^{\alpha+\beta} \times$$

$$\times {}_2F_3\left(1-\frac{\beta}{2}, \frac{3-\beta}{2}, \frac{3}{2}; 2-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}, -\frac{a^2 b^2}{4}\right) \mp$$

$$\mp \Gamma(\alpha+\beta-1) \left\{ \frac{\cos[(\alpha+\beta)\pi/2]}{\sin[(\alpha+\beta)\pi/2]} \right\} b^{1-\alpha-\beta} \times$$

$$\times {}_2F_3\left(1-\frac{\beta}{2}, \frac{1-\beta}{2}, \frac{1}{2}; 1-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}, -\frac{a^2 b^2}{4}\right)$$

[a, b, Re β > 0; Re(α+β) < 2].
 5. $\int_a^\infty x^{\alpha-1} (x-a)^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$

$$= \pm \frac{\sqrt{\pi}}{2} \left(\frac{a}{b}\right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha}\left(\frac{ab}{2}\right) \left\{ \frac{\cos(ab/2)}{\sin(ab/2)} \right\} \mp Y_{1/2-\alpha}\left(\frac{ab}{2}\right) \left\{ \frac{\sin(ab/2)}{\cos(ab/2)} \right\} \right]$$

[a, b > 0; 0 < Re α < 1].

$$\begin{aligned}
 6. \int_0^{\infty} \frac{x^{2\alpha-1} \{\sin bx\}}{(x+z)^{\alpha}} \cos bx \, dx = & B(\alpha+\delta, \rho-\alpha-\delta) b^{\delta} z^{\alpha-\delta} \times \\
 & \times {}_2F_3\left(\frac{\alpha+2\delta}{2}, \frac{\alpha+1}{2}; \delta+\frac{1}{2}, \delta+\frac{\alpha+1-\rho}{2}, \frac{2+\alpha-\rho}{2}; -\frac{b^2 z^2}{4}\right) + \\
 & + \Gamma(\alpha-\rho) \frac{\{\sin[(\alpha-\rho)\pi/2]\}}{\{\cos[(\alpha-\rho)\pi/2]\}} b^{\rho-\alpha} \times \\
 & \times {}_2F_3\left(\frac{\rho}{2}, \frac{\rho+1}{2}; \frac{1}{2}, 1+\frac{\rho-\alpha}{2}, \frac{1-\alpha+\rho}{2}; -\frac{b^2 z^2}{4}\right) + \\
 & + \rho \Gamma(\alpha-\rho-1) \frac{\{\cos[(\rho-\alpha)\pi/2]\}}{\{\sin[(\rho-\alpha)\pi/2]\}} z b^{\rho+1} \times \\
 & \times {}_2F_3\left(\frac{\rho+1}{2}, 1+\frac{\rho}{2}; \frac{3}{2}, \frac{3+\rho-\alpha}{2}, 1+\frac{\rho-\alpha}{2}; -\frac{b^2 z^2}{4}\right) \\
 & [b > 0; \operatorname{Re} \alpha > -\delta; \operatorname{Re}(\alpha-\rho) < 1; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^{\infty} \frac{x^{2\alpha-1} \{\sin bx\}}{x+z} \cos bx \, dx = \\
 = i b^{\delta} \Gamma(\alpha) z^{\alpha-1} [e^{-ibz} \Gamma(1-\alpha, -ibz) \mp e^{ibz} \Gamma(1-\alpha, ibz)] \\
 [b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 2].
 \end{aligned}$$

$$\begin{aligned}
 8. \int_0^{\infty} \frac{x^{2\alpha-1} \{\sin bx\}}{(x+z)^{\alpha}} \cos bx \, dx = \\
 = \frac{\sqrt{\pi}}{2} \left(\frac{z}{b}\right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha}\left(\frac{bz}{2}\right) \frac{\{\cos(bz/2)\}}{\{\sin(bz/2)\}} \pm Y_{1/2-\alpha}\left(\frac{bz}{2}\right) \frac{\{\sin(bz/2)\}}{\{\cos(bz/2)\}} \right] \\
 [b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 1].
 \end{aligned}$$

$$\begin{aligned}
 9. \int_0^{\infty} \frac{x^{2\alpha-1/2} \{\sin bx\}}{x+z} \cos bx \, dx = 2\pi \left(\sigma - \frac{1}{2}\right) z^{\alpha-1/2} \left[\sqrt{2} S(bz) \frac{\{\cos(bz+\pi/4)\}}{\{\sin(bz+\pi/4)\}} \mp \right. \\
 \left. \mp \sqrt{2} C(bz) \frac{\{\sin(bz+\pi/4)\}}{\{\cos(bz+\pi/4)\}} \pm \frac{\{\sin bz\}}{\{\cos bz\}} \right] \mp \frac{1-(-1)^{\sigma}}{2} \sqrt{\frac{\pi}{2b}} \\
 [\sigma = 0 \text{ или } 1; b > 0; |\arg z| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 10. \int_0^{\infty} \frac{x^{2\alpha-1} \{\sin bx\}}{x-y} \cos bx \, dx = -\pi y^{\alpha-1} \operatorname{ctg} \alpha \pi \frac{\{\sin by\}}{\{\cos by\}} - \\
 - b^{2-\alpha} \Gamma(\alpha-2) \frac{\{\sin(\alpha\pi/2)\}}{\{\cos(\alpha\pi/2)\}} {}_1F_2\left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4}\right) \mp \\
 \mp b^{1-\alpha} \Gamma(\alpha-1) \frac{\{\cos(\alpha\pi/2)\}}{\{\sin(\alpha\pi/2)\}} {}_1F_2\left(1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{b^2 y^2}{4}\right) \\
 [b, y > 0; -\delta < \operatorname{Re} \alpha < 2].
 \end{aligned}$$

$$\begin{aligned}
 11. \int_0^{\infty} x^{2n} [(\beta+ix)^{-\nu} \mp (\beta-ix)^{-\nu}] \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \\
 = \frac{(-1)^n}{\Gamma(\nu)} i^{\delta} (2n)! \pi b^{\nu-2n-1} e^{-\delta\beta} L_{2n}^{\nu-2n-1}(b\beta) \\
 [b, \operatorname{Re} \beta > 0; 0 \leq 2n < \operatorname{Re} \nu].
 \end{aligned}$$

$$\begin{aligned}
 12. \int_0^{\infty} x^{2n+1} [(\beta+ix)^{-\nu} \pm (\beta-ix)^{-\nu}] \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \\
 = \frac{(-1)^{n+1}}{\Gamma(\nu)} i^{1-\delta} (2n+1)! \pi b^{\nu-2n-2} e^{-\delta\beta} L_{2n+1}^{\nu-2n-2}(b\beta) \\
 [b, \operatorname{Re} \beta > 0; -2-\delta < 2n < \operatorname{Re} \nu-1].
 \end{aligned}$$

2.5.8. Интегралы от $x^2(x^2-a^2)^{\mu} \frac{\{\sin bx\}}{\{\cos bx\}}$, $x^2(a^2-x^2)^{\mu} \frac{\{\sin bx\}}{\{\cos bx\}}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$\begin{aligned}
 1. \int_0^a x^{2\alpha-1} (a^2-x^2)^{\beta-1} \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \\
 = \frac{b^{\delta} a^{2\beta+\alpha+1/2}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; -\frac{a^2 b^2}{4}\right) \\
 [a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 2. \int_0^a x (a^2-x^2)^{\beta-1} \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \\
 = \pm \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ H_{\beta+1/2}^{(ab)} \right\} + (1-\delta) \frac{a^{2\beta}}{2} \\
 [a, \operatorname{Re} \beta > 0; |\arg b| < \pi].
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^a x^{2\alpha-1} (x^2-a^2)^{\beta-1} \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \\
 = \frac{b^{\delta}}{2} a^{\delta+1/2+\alpha-2} B\left(\beta, 1-\beta-\frac{\alpha+\delta}{2}\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \beta+\frac{\alpha+\delta}{2}, \delta+\frac{1}{2}; -\frac{a^2 b^2}{4}\right) + \\
 + \frac{\sqrt{\pi}}{2} \left(\frac{b}{2}\right)^{2-2\beta-\alpha} \Gamma\left[(\alpha+\delta)/2+\beta-1\right] {}_1F_2\left(1-\beta; 2-\beta-\frac{\alpha}{2}, \frac{3-\alpha}{2}-\beta; -\frac{a^2 b^2}{4}\right) \\
 [a, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+2\beta) < 3].
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^{\infty} x (x^2-a^2)^{\beta-1} \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b}\right)^{\beta-1/2} a \Gamma(\beta) \left\{ H_{-\beta-1/2}^{(ab)} \right\} \\
 [a, b > 0; 0 < \operatorname{Re} \beta < 1/2].
 \end{aligned}$$

$$5. \int_a^{\infty} \frac{1}{\sqrt{x(x^2-a^2)}} \frac{\{\sin bx\}}{\{\cos bx\}} \, dx = -\frac{\pi}{2} \sqrt{b} J_{\pm 1/4}\left(\frac{ab}{2}\right) Y_{\pm 1/4}\left(\frac{ab}{2}\right) \quad [a, b > 0].$$

$$\begin{aligned}
 6. \int_a^{\infty} \frac{(x^2-a^2)^{\beta-1}}{x} \cos bx \, dx = \\
 = \frac{\pi a^{2\beta}}{2 \sin \beta\pi} \left\{ 1 - \frac{\pi ab}{2} \left[J_{1/2-\beta}(ab) H_{-1/2-\beta}(ab) - J_{-1/2-\beta}(ab) H_{1/2-\beta}(ab) \right] \right\} \\
 [a, b > 0; 0 < \operatorname{Re} \beta < 3/2].
 \end{aligned}$$

$$7. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+y^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2} y^{\alpha-2} \left\{ \begin{aligned} &\operatorname{tg}(\alpha\pi/2) \sin by \\ &\operatorname{ctg}(\alpha\pi/2) \cos by \end{aligned} \right\} - \\ - \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} b^{\alpha-2} {}_2F_2 \left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4} \right) \\ [b, y > 0; -\delta < \operatorname{Re} \alpha < 3]$$

$$8. \int_0^{\infty} \frac{x}{x^2+y^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \left\{ \begin{aligned} &(\pi/2) \cos by \\ &\sin by \operatorname{Si}(by) - \cos by \operatorname{ci}(by) \end{aligned} \right\} [b, y > 0]$$

2.5.9. Интегралы от $\frac{x^{\alpha}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{b^{\delta}}{2} z^{\delta+\alpha-2\rho} \Gamma\left(\frac{\alpha+\delta}{2}, \rho - \frac{\alpha+\delta}{2}\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \delta+\frac{1}{2}, 1-\rho+\frac{\alpha+\delta}{2}; -\frac{b^2 z^2}{4}\right) + \\ + b^{\delta+\alpha-2\rho} \Gamma(\alpha-2\rho) \left\{ \frac{\sin[(\alpha/2-\rho)\pi]}{\cos[(\alpha/2-\rho)\pi]} \right\} {}_1F_2\left(\rho; \frac{1-\alpha}{2}+\rho, 1-\frac{\alpha}{2}+\rho; -\frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1+2\operatorname{Re} \rho]$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+z^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi z^{\alpha-2}}{2} \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} \left\{ \begin{aligned} &\operatorname{sh} bz \\ &\operatorname{ch} bz \end{aligned} \right\} - \\ - \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} b^{\alpha-2} {}_1F_2\left(1; \frac{3-\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 3]$$

$$3. \int_0^{\infty} \frac{x^{2n} \sin bx}{(x^2+z^2)^{\rho}} dx = \\ = (-1)^n \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} \frac{\partial^{2n}}{\partial b^{2n}} \left\{ b^{\rho-1/2} [I_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)] \right\} \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n]$$

$$4. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^{n+\delta} \frac{\sqrt{\pi}}{\Gamma(\rho)} (2z)^{1/2-\rho} \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} [b^{\rho-1/2} K_{\rho-1/2}(bz)] \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n+\delta/2; \delta = \begin{cases} 1 \\ 0 \end{cases}]$$

$$5. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^m} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{(-1)^{m+n+\delta}}{(m-1)!} 2^{-m} \pi \left(\frac{\partial}{\partial z} \right)^{m-1} (z^{2n+\delta+1} e^{-bz}) \\ [b, \operatorname{Re} z > 0; n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots]$$

$$6. \int_0^{\infty} \frac{x^{2+\delta}}{(x^2+z^2)^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{1+\delta-bz}{4z^{1-\delta}} \pi e^{-bz} [b, \operatorname{Re} z > 0; \delta = \begin{cases} 1 \\ 0 \end{cases}]$$

$$7. \int_0^{\infty} \frac{1}{\sqrt{x(x^2+z^2)}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{\pi b}{2}} I_{\pm 1/4}\left(\frac{bz}{2}\right) K_{3/4}\left(\frac{bz}{2}\right) [b, \operatorname{Re} z > 0]$$

$$8. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^m} dx = \frac{\pi b e^{-bz}}{2^{2m-2} (m-1)!} z^{2-2m} \sum_{k=0}^{m-2} \frac{(2m-k-4)! (2bz)^k}{k! (m-2-k)!} [b, \operatorname{Re} z > 0; m=1, 2, 3, \dots]$$

$$9. \int_0^{\infty} \frac{x \sin bx}{x^2+z^2} dx = \frac{\pi}{2} e^{-bz} [b, \operatorname{Re} z > 0]$$

$$10. \int_0^{\infty} \frac{\sin bx}{x(x^2+z^2)^2} dx = \frac{\pi}{2z^4} \left[1 - \frac{e^{-bz}}{2} (2+bz) \right] [b, \operatorname{Re} z > 0]$$

$$11. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^{\rho}} dx = \frac{\sqrt{\pi}}{\Gamma(\rho)} \left(\frac{2}{b} \right)^{1/2-\rho} z^{\rho-1/2} K_{3/2-\rho}(bz) [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > 1/2]$$

$$12. \int_0^{\infty} \frac{x \cos bx}{x^2+z^2} dx = -\frac{1}{2} [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] [b, \operatorname{Re} z > 0]$$

$$13. \int_0^{\infty} \frac{x \cos bx}{(x^2+z^2)^{\rho}} dx = \frac{z^{3/2-\rho} b^{\rho-1/2}}{\sqrt{2}} [2^{-\rho} \Gamma(1-\rho) I_{\rho-3/2}(bz) + \sqrt{2} e^{-\pi i(1-2\rho)/4} S_{\rho-1/2, 3/2-\rho}(ibz)] \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > 1/2]$$

$$14. \int_0^{\infty} \frac{x \sin^{2n+1} bx}{x^2+z^2} dx = \frac{(-1)^{n+1}}{2^{2n+2}} e^{-(2n+1)bz} \times \\ \times \left[(1 - e^{2(2n+1)bz}) (1 - e^{-2bz})^{2n+1} - 2 \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{2kbz} \right] \\ [b, \operatorname{Re} z > 0]$$

$$15. \int_0^{\infty} \frac{\sin^2 bx}{x(x^2+z^2)} dx = \frac{1}{4z^2} \{ 2[C + \ln(bz)] - [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] \} [b, \operatorname{Re} z > 0]$$

$$16. \int_0^{\infty} \frac{\sin^2 bx}{x^3(x^2+z^2)} dx = \frac{\pi}{4z^3} (e^{-2bz} + 2bz - 1) [b, \operatorname{Re} z > 0]$$

$$17. \int_0^{\infty} \frac{x dx}{(x^2+z^2) \sin bx} = \frac{\pi}{2 \operatorname{sh} bz} [b, \operatorname{Re} z > 0]$$

$$15. \int_0^{\infty} \frac{x \cos^{2n+1} bx}{x^4 + z^4} dx = -\frac{e^{(2n+1)bz}}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{2kbz} \operatorname{Ei}[(2n-2k+1)bz] - \\ - \frac{e^{-2n(1-i)bz}}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{-2kbz} \operatorname{Ei}[(2k-2n-1)bz] \quad [b, \operatorname{Re} z > 0].$$

2.5.10. Интегралы от $\frac{x^m}{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0} \left\{ \frac{\sin bx}{\cos bx} \right\}^l$

$$1. \int_0^{\infty} \frac{x^{2l+1} \sin bx}{x^{2n} + z^{2n}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 0 \\ [b > 0; -\delta < k < n + \frac{1-\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$2. \int_0^{\infty} \frac{x^{2k+\delta} \sin bx}{x^{2n} + z^{2n}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2n} z^{2k+1-2n-\delta} \sum_{l=0}^{n-1} \exp\left(-bz \sin \frac{2l+1}{2n} \pi\right) \left\{ \frac{\cos l}{\sin l} \right\} \\ [l = \frac{2l+1}{2n} (2k+1-\delta)\pi + bz \cos \frac{2l+1}{2n} \pi; b > 0; 0 \leq k < n + \frac{\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$3. \int_0^{\infty} \frac{x^k \sin bx}{x^4 + z^4} dx = I_k \quad [b > 0; |\arg z| < \frac{\pi}{4}],$$

$$I_{-1} = \frac{\pi}{2z^3} \left[1 - \exp\left(-\frac{bz}{\sqrt{2}}\right) \cos\left(\frac{bz}{\sqrt{2}}\right) \right],$$

$$I_{2 \pm 1} = \frac{\pi}{2z^{1 \mp 1}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left\{ \frac{\sin(bz/\sqrt{2})}{\cos(bz/\sqrt{2})} \right\}.$$

$$4. \int_0^{\infty} \frac{x^k \cos bx}{x^4 + z^4} dx = J_k \quad [b > 0; |\arg z| < \frac{\pi}{4}],$$

$$J_{1 \pm 1} = \frac{\pi}{4z^{3 \mp 1}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left\{ \cos \frac{bz}{\sqrt{2}} \mp \sin \frac{bz}{\sqrt{2}} \right\}.$$

$$5. \int_0^{\infty} \frac{x^{2k} \cos bx}{x^{2n} + y^{2n}} dx = -\frac{1}{y^{2k+1-2n}} \sum_{l=0}^{n-1} \exp\left(-by \sin \frac{l\pi}{n}\right) \sin\left(\frac{2k+1}{n} l\pi + by \cos \frac{l\pi}{n}\right) \\ [b, y > 0; 0 \leq k < n].$$

$$6. \int_0^{\infty} \frac{x^k \sin bx}{x^4 - y^4} dx = I_k \quad [b, y > 0; k = -1, 0, 1, 2, 3],$$

$$I_{-1} = \frac{\pi}{4y^3} (e^{-by} + \cos by - 2),$$

$$I_{1 \pm 1} = \frac{1}{4y^{2 \mp 1}} [2 \cos by \operatorname{Si}(by) - 2 \sin by \operatorname{ci}(by) \mp e^{by} \operatorname{Ei}(-by) \pm e^{-by} \operatorname{Ei}(by)],$$

$$I_{2 \pm 1} = \frac{\pi}{4y^{1 \mp 1}} (\cos by \pm e^{-by}).$$

$$7. \int_0^{\infty} \frac{x^k \cos bx}{x^4 - y^4} dx = J_k \quad [b, y > 0; k = 0, 1, 2, 3].$$

$$J_{1 \pm 1} = -\frac{\pi}{4y^{2 \mp 1}} (\sin by \mp e^{-by}),$$

$$J_{2 \pm 1} = -\frac{1}{4y^{1 \mp 1}} [2 \cos by \operatorname{ci}(by) + 2 \sin by \operatorname{Si}(by) \pm e^{-by} \operatorname{Ei}(by) \pm e^{by} \operatorname{Ei}(-by)].$$

$$8. \int_{-\infty}^{\infty} \frac{\sin x}{x(x-z)} dx = \frac{\pi}{z} (e^{\pm iz} - 1) \quad [z \neq 0; \operatorname{Im} z > 0].$$

$$9. = \frac{\pi}{y} (\cos y - 1) \quad [z = y; \operatorname{Im} y = 0].$$

$$10. \int_0^{\infty} \frac{x^{2l+1} \sin bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_1 b} \left(\pm z^{1 \mp 1} \sin A_1 b + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y \cos A_1 b \right) \\ [2A_1^2 = \sqrt{z^2 + y^2} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$11. \int_0^{\infty} \frac{x^{1 \mp 1} \cos bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_1 b} (z^4 + y^2)^{-1/2} \times \\ \times \left[\pm z^{1 \mp 1} (A_1 \sin A_1 b + A_1 \cos A_1 b) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y (A_1 \cos A_1 b - A_1 \sin A_1 b) \right] \\ [2A_1^2 = \sqrt{z^2 + y^2} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$12. \int_{-\infty}^{\infty} \frac{cx + d}{x^2 + 2qx + p} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pi \exp(-b \sqrt{p-q^2}) \left[c \left\{ \frac{\cos bq}{\sin bq} \right\} \pm \frac{cq-d}{\sqrt{p-q^2}} \right] \\ [b > 0; p > q^2].$$

$$13. \int_0^{\infty} \frac{\sin^{2l} mx \pi}{(1-x^2/l^2)(1-x^2/2^2) \dots (1-x^2/n^2)} dx = 0 \quad [l, m, n = 1, 2, 3, \dots].$$

$$14. \int_0^{\infty} \frac{\sin^{2l+1} mx \pi}{x(1-x^2/l^2)(1-x^2/2^2) \dots (1-x^2/n^2)} dx = \frac{1-(-1)^m}{2} \frac{2^{2l} n^{l-1} (2l)!}{(2n)!} \frac{(n!)^2}{(l!)^2} \pi \\ [l, m, n = 1, 2, 3, \dots].$$

$$15. \int_{-\infty}^{\infty} \frac{x^m \sin bx}{c^4 \pm c^2 x + cx^2 \pm x^3} dx = I_m \quad [b, c > 0],$$

$$I_1 = (4c)^{-1} \{ \pi (e^{-bc} - \cos bc) \pm [e^{-bc} \operatorname{Ei}(bc) - e^{bc} \operatorname{Ei}(-bc) - \\ - 2 \sin bc \operatorname{ci}(bc) + 2 \cos bc \operatorname{Si}(bc)] \},$$

$$I_2 = 2^{-2} [e^{bc} \operatorname{Ei}(-bc) - e^{-bc} \operatorname{Ei}(bc) + 2 \operatorname{ci}(bc) \sin bc - \\ - 2 \operatorname{Si}(bc) \cos bc] \pm \pi (e^{-bc} + \cos bc).$$

$$16. \int_{-\infty}^{\infty} \frac{x^m \sin bx}{x^4 + 2a^2 x^2 + c^4} dx = I_m \quad [a, b, c > 0].$$

$$I_{-1} = \frac{\pi}{2c^2} \left\{ 1 - \exp \left(-b \sqrt{\frac{a^2+c^2}{2}} \right) \times \right. \\ \left. \times \left[\cos \left(b \sqrt{\frac{c^2-a^2}{2}} \right) + \frac{a^2}{\sqrt{c^2-a^2}} \sin \left(b \sqrt{\frac{c^2-a^2}{2}} \right) \right] \right\}, \\ I_1 = \frac{\pi}{2\sqrt{c^2-a^2}} \exp \left(-b \sqrt{\frac{a^2+c^2}{2}} \right) \sin \left(b \sqrt{\frac{c^2-a^2}{2}} \right).$$

$$I_3 = \frac{\pi}{2} \exp \left(-b \sqrt{\frac{a^2+c^2}{2}} \right) \left[\cos \left(b \sqrt{\frac{c^2-a^2}{2}} \right) - \frac{a^2}{\sqrt{c^2-a^2}} \sin \left(b \sqrt{\frac{c^2-a^2}{2}} \right) \right].$$

$$17. \int_0^\infty \frac{x^m \cos bx}{x^3 + 2a^2x^2 + c^2} dx = J_m \quad [a, b, c > 0]$$

$$J_0 = \frac{\pi}{2\sqrt{2}c^2} \exp \left(-b \sqrt{\frac{c^2+a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2+a^2}} \cos \left(b \sqrt{\frac{c^2-a^2}{2}} \right) + \frac{1}{\sqrt{c^2-a^2}} \sin \left(b \sqrt{\frac{c^2-a^2}{2}} \right) \right], \\ J_2 = \frac{\pi}{2\sqrt{2}c^2} \exp \left(-b \sqrt{\frac{c^2+a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2+a^2}} \cos \left(b \sqrt{\frac{c^2-a^2}{2}} \right) - \frac{1}{\sqrt{c^2-a^2}} \sin \left(b \sqrt{\frac{c^2-a^2}{2}} \right) \right].$$

$$18. \int_0^\infty \left[\frac{1}{(x-a)^2+z^2} + \frac{1}{(x+a)^2+z^2} \right] \frac{\sin bx}{\cos bx} dx = \frac{\pi}{z} e^{-bz} \frac{\sin ab}{\cos ab} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|]$$

$$19. \int_0^\infty \left[\frac{x-a}{(x-a)^2+z^2} + \frac{x+a}{(x+a)^2+z^2} \right] \sin bx dx = \pi e^{-bz} \frac{\sin ab}{\cos ab} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|]$$

$$20. \int_0^\infty \frac{\sin bx dx}{x(x^2+2^2) \dots (x^2+4n^2)} = \\ = \frac{\pi}{(2n)!} \frac{(-1)^n}{2^{2n+1}} \left[2 \sum_{k=1}^{n-1} (-1)^k \binom{2n}{k} e^{i(2k-n)\pi} + (-1)^n \binom{2n}{n} \right] \\ [b > 0; n = 1, 2, 3, \dots]$$

$$21. \int_0^\infty \frac{\cos bx dx}{(x^2+1^2)(x^2+3^2) \dots (x^2+(2n+1)^2)} = \\ = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{i(2k-2n-1)\pi} \quad [b > 0]$$

2.5.11. Интегралы от $x^2 (\sqrt{x^2 \pm z^2} + ax + bz)^v A(x) \frac{\sin bx}{\cos bx}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2+z^2}+z)^v}{(x^2+z^2)^r} \frac{\sin bx}{\cos bx} dx = \\ = \frac{1}{2} (vz)^{1-2r} b^{1-\alpha-v} \Gamma(\alpha+v-1) \frac{\cos[(\alpha+v)\pi/2]}{\sin[(\alpha+v)\pi/2]} \times \\ \times {}_2F_3 \left(\frac{1+v}{2}, \frac{1-v}{2}; \frac{3}{2}-2r, 1-\frac{\alpha+v}{2}, \frac{3-\alpha-v}{2}; \frac{b^2 z^2}{4} \right) + \\ + (vz)^{2r} b^{1-\alpha-v} \Gamma(\alpha+v-4r) \frac{\sin[(\alpha+v)/2-2r]\pi}{\cos[(\alpha+v)/2-2r]\pi} \times \\ \times {}_2F_3 \left(2r-\frac{v}{2}, 2r+\frac{v}{2}; \frac{1}{2}+2r, 2r+\frac{1-\alpha-v}{2}, 1+2r-\frac{\alpha+v}{2}; \frac{b^2 z^2}{4} \right) + \\ + b^2 z^{2r+v+\delta-2r-2r+1} (-v)^{1-2r} \Gamma \left[\frac{(\alpha+\delta)/2, 2r-v-\alpha-\delta}{1-v-(\alpha+\delta)/2} \right] \times \\ \times {}_2F_3 \left(\frac{\alpha+\delta}{2}, v+\frac{\alpha+\delta}{2}; \delta+\frac{1}{2}, \frac{1+v+\alpha+\delta}{2}, 1+\frac{v+\alpha+\delta}{2}-2r; \frac{b^2 z^2}{4} \right) \\ [r=0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} v+2r+1].$$

$$2. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2+z^2}+z)^{1/2-\alpha}}{\sqrt{x^2+z^2}} \frac{\sin bx}{\cos bx} dx = \\ = \frac{1}{z} \sqrt{\frac{2}{b}} \Gamma \left(\frac{\alpha+\delta}{2} \right) W_{1/4-\alpha/2, \pm 1/4} (bz) M_{\alpha/2-1/4, \pm 1/4} (bz) \\ [\delta, \operatorname{Re} z > 0; \operatorname{Re} \alpha > -\delta].$$

$$3. \int_0^\infty x^\delta \frac{(\sqrt{x^2+z^2}+z)^{1/2-\delta}}{\sqrt{x^2+z^2}} \frac{\sin bx}{\cos bx} dx = \sqrt{\frac{\pi}{2b}} e^{-bz} \quad [\delta, \operatorname{Re} z > 0; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$4. \int_0^\infty x^{2-1} \frac{(\sqrt{x^2+z^2}+x)^v}{(x^2+z^2)^r} \frac{\sin bx}{\cos bx} dx = \\ = \frac{b^\delta}{2} \left(-\frac{v}{2} \right)^{1-2r} z^{2r-v-2r+1} \delta \Gamma \left[\frac{\alpha+\delta, r-(\alpha+v+\delta)/2}{1+(\alpha+\delta-v)/2-r} \right] \times \\ \times {}_2F_3 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2}+\delta; \delta+\frac{1}{2}, 1+\frac{\alpha+\delta-v}{2}-r, 1+\frac{\alpha+v+\delta}{2}-r; \frac{b^2 z^2}{4} \right) + \\ + 2^{2r} b^{2r-\alpha-v} \Gamma(\alpha+v-2r) \frac{\sin[(\alpha+v)/2-r]\pi}{\cos[(\alpha+v)/2-r]\pi} \times \\ \times {}_2F_3 \left(r-\frac{v}{2}, \frac{1-v}{2}+r; 1-v, r+\frac{1-v-\alpha}{2}, r+1-\frac{v+\alpha}{2}; \frac{b^2 z^2}{4} \right) \\ [r=0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} v+2r+1].$$

$$5. \int_0^\infty \frac{(\sqrt{x^2+z^2}+x)^v}{\sqrt{x(x^2+z^2)}} \frac{\sin bx}{\cos bx} dx = \sqrt{\frac{\pi b}{2}} z^{v/2} \pm 1/4 - v/2 \left(\frac{bz}{2} \right) K_{\pm 1/4+v/2} \left(\frac{bz}{2} \right) \\ [\delta, \operatorname{Re} z > 0; \operatorname{Re} v < 3/2].$$

$$6. \int_0^{\infty} \frac{(1-x^2+z^2+x)^{\nu} (1-x^2+z^2-x)^{\nu}}{1-x^2+z^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2z^{\nu} \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} K_{\nu}(bz) \\ [b > 0; \operatorname{Re} z > 0; |\operatorname{Re} \nu| < 1]$$

$$7. \int_0^{\infty} \frac{(x+z+\sqrt{x^2+2xz})^{\nu} (x+z-\sqrt{x^2+2xz})^{\nu}}{1-x^2+2xz} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \pi z^{\nu} \left[\left\{ \frac{\cos(bz-\nu\pi/2)}{\sin(bz-\nu\pi/2)} \right\} J_{\nu}(bz) + \left\{ \frac{\sin(bz-\nu\pi/2)}{\cos(bz-\nu\pi/2)} \right\} Y_{\nu}(bz) \right] \\ [b > 0; |\operatorname{Re} \nu| < 1; |\arg z| < \pi]$$

$$8. \int_0^{\infty} \frac{(x+i\sqrt{a^2-x^2})^{\nu} (x-i\sqrt{a^2-x^2})^{\nu}}{1-a^2-x^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{\pi a^{\nu}}{2} \left\{ \frac{\operatorname{cosec}(\nu\pi/2)}{\sec(\nu\pi/2)} \right\} [J_{\nu}(ab) \mp J_{-\nu}(ab)] \quad [a > 0; |\arg b| < \pi]$$

$$9. \int_a^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} (x-\sqrt{x^2-a^2})^{\nu}}{1-x^2-a^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \pm \pi a^{\nu} \left[\left\{ \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} \right\} J_{\nu}(ab) \mp \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} Y_{\nu}(ab) \right] \quad [a, b > 0; |\operatorname{Re} \nu| < 1]$$

$$10. \int_a^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} (x-\sqrt{x^2-a^2})^{\nu}}{1-x(x^2-a^2)} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -\frac{\pi}{2} \sqrt{b} a^{\nu} \times \\ \times \left[J_{\pm 1/4+\nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4-\nu/2} \left(\frac{ab}{2} \right) + J_{\pm 1/4-\nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4+\nu/2} \left(\frac{ab}{2} \right) \right] \\ [a, b > 0; |\operatorname{Re} \nu| < 3/4]$$

2.5.12. Интегралы от $\prod_k \left\{ \frac{\sin a_k x}{\cos a_k x} \right\}^{\mu_k} \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^{\nu_k}$,

$$\prod_k \sin^{\mu_k} a_k x \cos^{\nu_k} b_k x.$$

$$1. \int_0^{\pi/4} \frac{\sin^{2\mu} x}{\cos^{2\mu+1} x} dx = 2^{-2\mu-1} B(2\mu+1, -\mu) \quad [-1/2 < \operatorname{Re} \mu < 0]$$

$$2. \int_0^{\pi/4} \frac{\cos^{2\mu} x}{\cos^{2\mu+1} x} dx = 2^{2\mu} B(\mu+1, \mu+1) \quad [\operatorname{Re} \mu > -1]$$

$$3. \int_0^{\pi/4} \frac{\cos^{n-1/2} x}{\cos^{2n+1} x} dx = \frac{(2n-1)!!}{2(2n)!!} \pi.$$

$$4. \int_0^{\pi/4} \frac{\sin^m x \cos^{\nu} 2x}{\cos^{2m+2\nu+1} x} dx = \frac{1}{2} B\left(\frac{m+1}{2}, \nu+1\right) \quad [\operatorname{Re} \nu > -1]$$

$$5. \int_0^{\pi/4} \frac{\sin^m x \cos^{n-1/2} 2x}{\cos^{2m+2n+1} x} dx = \frac{(m-1)!!(2n-1)!!}{(m+2n)!!} \begin{cases} \pi/2, & m=0, 2, 4, \dots \\ 1, & m=1, 3, 5, \dots \end{cases} \\ [n=1, 2, 3, \dots]$$

$$6. \int_0^{\pi/2} \sin^n x dx = \frac{(n+1)!!}{n!!} \begin{cases} 1, & n=1, 3, 5, \dots \\ \pi/2, & n=0, 2, 4, \dots \end{cases}$$

$$7. \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx = 2 \sum_{k=1}^{n/2} \frac{(-1)^{k-1}}{2k-1} \quad [n=2, 4, \dots]$$

$$8. \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx = \pi/2 \quad [n=1, 3, 5, \dots]$$

$$9. \int_0^{\pi/2} \frac{\sin 2nx}{\sin x} dx = (-1)^n 2 \sum_{k=1}^n \frac{(-1)^k}{2k-1}$$

$$10. \int_0^{\pi/2} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \frac{\pi}{2}$$

$$11. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2^{-\mu} \pi \left\{ \Gamma\left[\frac{\mu}{2}\right] \Gamma\left[\frac{\mu}{2}\right] \right\} \times \\ \times \left\{ \frac{\sin(b\pi/2)}{\cos(b\pi/2)} \right\} \mp \frac{b}{2} \Gamma\left[\frac{\mu}{2}\right] \Gamma\left[\frac{\mu}{2}\right] \left\{ \frac{\cos(b\pi/2)}{\sin(b\pi/2)} \right\} \times \\ \times \left\{ \frac{\mu}{2}, \frac{\mu+1}{2}, \frac{1}{2}, 1+\frac{\mu+b}{2}, 1+\frac{\mu-b}{2}, 1 \right\} \quad [\operatorname{Re} \mu > -1; |\arg b| < \pi]$$

$$12. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin(\mu+1)x}{\cos(\mu+1)x} \right\} dx = \frac{1}{\mu} \left\{ \frac{\sin(\mu\pi/2)}{\cos(\mu\pi/2)} \right\} \quad [\operatorname{Re} \mu > -1]$$

$$13. \int_0^{\pi/2} \sin^{2n+\delta} x \left\{ \frac{\sin(2m+1)x}{\cos 2mx} \right\} dx = \frac{(-1)^m (2n+\delta)!}{2^{2n+1} \delta!} \left\{ \frac{\pi}{1} \right\} \quad [m \leq n; \delta = \{1\}]$$

$$14. = 0 \quad [m > n; \delta = \{1\}]$$

$$15. \int_0^{\pi/2} \sin^{2n+\delta-1} x \left\{ \frac{\sin(2m+1)x}{\cos 2mx} \right\} dx = \\ = \frac{(-1)^m (2n-1)!! (n-1)!! 2^{n-1}}{(2n-2m-1)!! (2m+2n+1)!!} \left\{ \frac{\pi}{2m+2n+1} \right\} \quad [m \leq n; \delta = \{1\}]$$

$$16. = \frac{(-1)^m (n-1)!! 2^{n-1} (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \left\{ \frac{2n}{2m+2n+1} \right\} \quad [m > n; \delta = \{1\}]$$

17. $\int_0^{\pi/2} \cos^{v-1} x \sin bx dx = \pi b 2^{-1-v} \Gamma \left[1 + (v+b)/2, 1 + (v-b)/2 \right] \times$
 $\times {}_3F_2 \left(\frac{v}{2}, \frac{1+v}{2}, \frac{1}{2}; 1 + \frac{v+b}{2}, 1 + \frac{v-b}{2}; 1 \right) \quad [\operatorname{Re} v > 0; |\arg b| < \pi]$
18. $\int_0^{\pi/2} \cos^{v-1} x \sin (v+1) x dx = \frac{1}{v}$
19. $\int_0^{\pi/2} \cos^n x \sin mx dx = \sum_{j=0}^{k-1} \frac{n! (m+n-2j-2)!}{(n-j)! (m+n)!} + \theta \frac{n! (m-n-2j)!}{(m+n)!}$
 $[k = \min(m, n), \theta = 2 \text{ при } m-n=4l+2 > 0; \theta = 1 \text{ при } m-n=2l+1 > 0; \theta = 0 \text{ при } m-n=4l \geq 0 \text{ или } m-n < 0]$
20. $\int_0^{\pi/2} \cos^n x \sin nx dx = \frac{1}{2^{n+1}} \sum_{j=1}^n \frac{2^j}{j}$
21. $\int_0^{\pi/2} \cos^{v-1} x \sin (2m+1) x dx = (-1)^m \sum_{j=0}^m \frac{(-1)^{j/2} (m)}{v+j} \binom{m}{j}$
22. $\int_0^{\pi/2} \cos^{v-1} x \cos bx dx = 2^{-v} \pi \Gamma \left[(1+v+b)/2, (1+v-b)/2 \right]$
 $[\operatorname{Re} v > 0; |\arg b| < \pi]$
23. $\int_0^{\pi/2} \cos^{v-1} x \cos (v+1) x dx = 0$
 $[\operatorname{Re} v > 0]$
24. $\int_0^{\pi/2} \cos^n x \cos mx dx = \frac{n!}{(n-m)! (n+m)!} \begin{cases} 1, & n-m=1, 3, 5, \dots \\ \pi/2, & n-m=0, 2, 4, \dots \end{cases}$
25. $= B \frac{n!}{(m-n)(m-n+2) \dots (m+n)}$
 $[n < m; B = 0, \text{ если } m-n \text{ четно; } B = \pm 1, \text{ если } m-n=4l \pm 1, l=0, 1, 2, \dots]$
26. $\int_0^{\pi/2} \sin^{\mu-1} x \cos^{v-1} x dx = \frac{1}{2} B \left(\frac{\mu}{2}, \frac{v}{2} \right)$
 $[\operatorname{Re} \mu, \operatorname{Re} v > 0]$
27. $\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{\pi a}{2} - \frac{\sin a\pi}{2} [2a\beta(a)-1]$
 $[\mu > 0]$
28. $\int_0^{\pi/2} \sin^{\mu-1} x \cos^{v-1} x \left\{ \frac{\sin (\mu+v) x}{\cos (\mu+v) x} \right\} dx =$
 $= \frac{\pi}{2} \left\{ \sec (\mu\pi/2) \right\} \Gamma \left[\frac{v}{\mu+v}, 1-\mu \right] \quad [\operatorname{Re} v > 0; \operatorname{Re} \mu > -\frac{1}{2}]$

29. $\int_0^{\pi} \sin^{\mu-1} x \cos^{v-1} x \left\{ \frac{\sin (\mu+v) x}{\cos (\mu+v) x} \right\} dx = \frac{\sin (\mu\pi/2)}{\cos (\mu\pi/2)} B(\mu, v)$
 $[\operatorname{Re} v > 0; \operatorname{Re} \mu > -\frac{1}{2}]$
30. $\int_0^{\pi} \frac{\sin nx \cos mx}{\sin x} dx = 0$
 $[n \leq m \text{ или } n > m; m+n=2k]$
31. $= \pi$
 $[n > m; m+n=2k-1, k=1, 2, 3, \dots]$
32. $\int_0^{\pi} \sin^m x \cos^n x dx = \frac{1+(-1)^n}{2} B \left(\frac{m+1}{2}, \frac{n+1}{2} \right)$
 $[m+n=0, 2, 4, \dots]$
33. $\int_0^{\pi} \frac{\sin nx}{\sin x} dx = \frac{1-(-1)^n}{2} \pi$
34. $\int_0^{\pi} \frac{\sin 2nx}{\cos x} dx = (-1)^n 4 \sum_{k=1}^n \frac{(-1)^k}{2k-1}$
35. $\int_0^{\pi} \frac{\cos (2n+1) x}{\cos x} dx = (-1)^n \pi$
36. $\int_0^{\pi} \sin^{\mu-1} x \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{2^{1-\mu} \pi}{\mu B((\mu+b+1)/2, (\mu-b+1)/2)} \frac{\sin (b\pi/2)}{\cos (b\pi/2)}$
 $[\operatorname{Re} \mu > 0; |\arg b| < \pi]$
37. $\int_0^{\pi} \sin^m x \left\{ \frac{\sin (2m+1) x}{\cos 2mx} \right\} dx = 2 \int_0^{\pi/2} \sin^m x \left\{ \frac{\sin (2m+1) x}{\cos 2mx} \right\} dx$
 $[\text{см. 2.5.12.13-16}]$
38. $\int_0^{\pi} \sin^{2n} x \cos 2mx dx = \frac{(-1)^m \pi}{2^{2n}} \binom{2n}{n-m}$
 $[n \geq m]$
39. $= 0$
 $[n < m]$
40. $\int_0^{\pi} \cos^b x \left\{ \frac{\sin mx}{\cos mx} \right\} dx = [1 + (-1)^{m+n}] \int_0^{\pi/2} \cos^n x \left\{ \frac{\sin mx}{\cos mx} \right\} dx$
 $[\text{см. 2.5.12.19-25}]$
41. $\int_0^{\pi} \cos^n x \cos bx dx = \frac{(-1)^n \sin b\pi}{2^n (n+b)} {}_2F_1 \left(-n, \frac{b+n}{2}; 1 + \frac{b+n}{2}; -1 \right)$
 $[b \neq 0, \pm 1, \pm 2, \dots]$
42. $\int_0^{\pi} \cos^n x \cos mx dx = 0$
 $[m+n=1, 3, 5, \dots \text{ или } m > n, m+n=2, 4, 6, \dots]$

$$43. = \frac{(-1)^{m+n-1} 2^{1-n} \pi (-n)_l (l)_l}{(m+n) (l-1)!} \quad [m \leq n; m+n=2l, l=1, 2, 3, \dots]$$

$$44. \int_a^{a+2\pi} \sin^m x \cos^n x dx = \frac{[1+(-1)^m][1+(-1)^n]}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

2.5.13. Интегралы от $x^2 \sin^{\mu} ax \cos^{\nu} bx$.

$$1. \int_0^{\pi/4} x \frac{\sin^{\mu-1} x}{\cos^{\mu+1} x} dx = \frac{\pi}{4\mu} - \frac{1}{2\mu} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1]$$

$$2. \int_0^{\pi/4} x \frac{\sin^{\mu} x}{\cos^{\mu+2} x} dx =$$

$$= \frac{1}{\mu+1} \left\{ \frac{\pi}{4} - (-1)^{(\mu+1)/2} \left(\frac{1-(-1)^{\mu}}{2} \frac{\pi}{4} + \frac{1+(-1)^{\mu}}{2} \ln 2 \right) - \sum_{k=0}^{[(\mu-1)/2]} \frac{(-1)^k}{\mu-2k} \right\}.$$

$$3. \int_0^{\pi/4} x^2 \frac{\sin x}{\cos^3 x} dx = \frac{\ln 2}{2} - \frac{\pi}{4} + \frac{\pi^2}{16}.$$

$$4. \int_0^{\pi/4} x^2 \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + G \right).$$

$$5. \int_0^{\pi/4} \frac{x dx}{\sin x \sqrt{\cos 2x}} = \frac{\pi}{2} \ln(1 + \sqrt{2}).$$

$$6. \int_{\pi/4}^{\pi/2} x \frac{\cos^{\mu-1} x}{\sin^{\mu+1} x} dx = \frac{1}{2\mu} \left[\frac{\pi}{2} + \beta\left(\frac{\mu+1}{2}\right) \right] \quad [\operatorname{Re} \mu > 0]$$

$$7. \int_0^{\pi/2} x \left(\frac{\sin^{\mu-1} x \cos x}{\cos^{\mu-1} x \sin x} \right) dx = \frac{\pi}{2\mu} \left\{ \frac{1}{0} \right\} + \frac{\sqrt{\pi}}{2\mu} \Gamma\left[\frac{(\mu+1)/2}{\mu/2+1}\right] \quad [\operatorname{Re} \mu > -\frac{1}{2}]$$

$$8. \int_0^{\pi/2} x \sin^{2n} x \cos x dx = \frac{1}{2(2n+1)} \left[\pi - \frac{2(2n)!}{(2n+1)!} \right].$$

$$9. \int_0^{\pi/2} x \sin^{2n+1} x \cos x dx = \frac{1}{4(n+1)} \left[1 - \frac{(2n+1)!}{(2n+2)!} \right].$$

$$10. \int_0^{\pi/2} x^2 \frac{\cos x}{\sin^3 x} dx = 4G - \frac{\pi^2}{4}.$$

$$11. \int_0^{\pi/2} x^2 \frac{\cos x}{\sin^3 x} dx = \frac{3\pi}{2} \ln 2 - \frac{\pi^2}{16}.$$

2.5.13.1

$$12. \int_0^{\pi/2} x \cos^n x \sin x dx = \frac{n!}{(n+1)!} \begin{cases} 1, & n=0, 2, 4, \dots \\ \pi, & n=1, 3, 5, \dots \end{cases}$$

$$13. \int_0^{\pi/2} x \left(\frac{\cos x}{\sin x} \right)^{\mu-1} \left(\frac{\sin x}{\cos x} \right)^{\nu-1} dx = \pm \frac{\pi}{2\mu \cos(\mu\pi/2)} \quad [0 < \pm \operatorname{Re} \mu < 1].$$

$$14. \int_0^{\pi/2} x \sin^n x \cos^{2m-1} x dx =$$

$$= \frac{B(n/2, m+1/2)}{2(2m+n-1)} + \frac{n-1}{2m+n-1} \int_0^{\pi/2} x \sin^{n-2} x \cos^{2m-1} x dx \quad [m, n=1, 2, 3, \dots]$$

$$15. \int_0^{\pi/2} x \cos^{\nu-1} x \sin^{\mu} x dx = \frac{\pi}{2^{\nu+1} \Gamma(\nu)} \frac{\Psi((\nu+b+1)/2) - \Psi((\nu-b+1)/2)}{\Gamma((\nu+b+1)/2) \Gamma((\nu-b+1)/2)}$$

$$[\operatorname{Re} \nu > 0; -1 - \operatorname{Re} \nu < \operatorname{Re} b < 1 + \operatorname{Re} \nu].$$

$$16. \int_0^{\pi} x \sin^{\mu-1} x \cos x dx = -\frac{\sqrt{\pi}}{\mu} \Gamma\left[\frac{(\mu+1)/2}{\mu/2+1}\right] \quad [\operatorname{Re} \mu > 0]$$

$$17. \int_0^{\pi} x \cos^n x \sin x dx = \frac{\pi}{n+1} \left\{ (-1)^n + \frac{1-(-1)^n}{2} \frac{(2[(n+1)/2]-1)!}{(2[(n+1)/2])!} \right\}.$$

$$18. \int_0^{\pi} x \frac{\sin x}{\cos x} dx = -\pi \ln 2.$$

$$19. \int_0^{\pi} x \sin^{2n-1} x \cos^m x dx =$$

$$= \frac{(-1)^{m-1}}{2(2n+m-1)} B\left(\frac{n}{2}, n+\frac{1}{2}\right) + \frac{m-1}{2n+m-1} \int_0^{\pi} x \sin^{2n-1} x \cos^{m-2} x dx$$

$$[m, n=1, 2, 3, \dots]$$

$$20. \int_0^{\pi} x^{2-1} \sin^n ax \sin^m bx dx = I_{\alpha}^{n, m},$$

$$I_{\alpha}^{n, m} = \sqrt{\pi} 2^{1-n-m} \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} \times$$

$$\times \sum_{l=0}^{[(m-1)/2]} (-1)^{[m/2]+l} \frac{m!}{l! (m-l)!} A_{kl}(l), \quad l = \frac{(n-2k)^2 a^2}{(m-2l)^2 b^2}.$$

$$7. \int_0^{\infty} \frac{(a + \sqrt{x^2 + a^2})^{1/2}}{\sqrt{x^2 + a^2}} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \frac{p \sqrt{2a}}{\sqrt{b^2 - p^2}} \left\{ \frac{1}{z_-} \exp(-a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_-) - \frac{1}{z_+} \exp(a \sqrt{b^2 - p^2}) \operatorname{erfc}(z_+) \right\}$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(\rho + b) > 0]$$

$$8. \int_0^{\infty} \frac{\exp(-b \sqrt{x^2 + a^2} - px)}{\sqrt{x^2 + a^2} (a + \sqrt{x^2 + a^2})^{1/2}} dx = 2 \sqrt{\frac{2}{a}} e^{ab} \operatorname{erfc}(z_+) \operatorname{erfc}(z_-)$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(\rho + b) > 0]$$

$$9. \int_0^{\infty} \frac{x^{2\beta-1/2}}{\sqrt{x^2 + a^2}} (a + \sqrt{x^2 + a^2})^{2\beta} \exp(-b \sqrt{x^2 + a^2} - px) dx =$$

$$= \sqrt{\frac{2}{a}} \Gamma\left(\frac{1}{2} - 2\beta\right) D_{2\beta-1/2}(\sqrt{2} z_+) D_{2\beta-1/2}(\sqrt{2} z_-)$$

$$[z_{\pm} = a(b \pm \sqrt{b^2 - p^2}); \operatorname{Re} a, \operatorname{Re} b, \operatorname{Re}(\rho + b) > 0; \operatorname{Re} \beta < 1/4]$$

$$13. \int_0^{\infty} \left[(\sqrt{x^2 + a^2} + x)^{\nu} e^{bx} + (\sqrt{x^2 + a^2} - x)^{\nu} e^{-bx} \right] e^{-c \sqrt{x^2 + a^2}} \frac{dx}{\sqrt{x^2 + a^2}} =$$

$$= 2a^{\nu} (c+b)^{\nu/2} (c-b)^{-\nu/2} K_{\nu}(a \sqrt{c^2 - b^2})$$

$$[\operatorname{Re} a, \operatorname{Re}(c \pm b) > 0]$$

2.3.18. Интегралы от $e^{ax} f(x)$.

$$1. \int_0^1 x^{\rho} dx = \sum_{k=1}^{\infty} \frac{\rho^k - 1}{k^k}$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-\mu x} dx = \frac{1}{\mu} \rho^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right)$$

$$[\mu, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0]$$

$$3. \int_0^{\infty} e^{-\rho x^2} dx = -\operatorname{Ei}(-\rho)$$

$$[\operatorname{Re} \rho > 0]$$

$$4. \int_{-\infty}^{\infty} (a + b|x|) e^{-|x| + i\lambda x} dx = 2 \frac{\lambda^2(a-b) + a+b}{(\lambda^2 + 1)^2}$$

$$[\lambda > 0]$$

$$5. \int_{-\infty}^{\infty} x e^{-\mu x^2} dx = \mu^{-2} \Gamma\left(\frac{1}{\mu}\right) \rho^{-1/\mu} \left[\psi\left(\frac{1}{\mu}\right) - \ln \rho \right]$$

$$[\mu, \operatorname{Re} \rho > 0]$$

$$6. \int_{-\infty}^{\infty} x e^{\mu x} dx = C.$$

2.3.19. Интегралы, содержащие разности алгебраических и показательной функций.

$$1. \int_0^1 \left(\frac{r e^{1-x-r}}{1-x} - \frac{e^{\rho}}{1-x} \right) \frac{dx}{x} = \psi(\rho) - \ln r$$

$$[r, \operatorname{Re} \rho > 0]$$

$$2. \int_0^1 \left(e^{1-x} - e^{\rho} \right) \frac{dx}{x(1-x)} = \psi(\rho)$$

$$[\operatorname{Re} \rho > 0]$$

$$3. \int_0^1 \left(\frac{r e^{1-x-r}}{1-x} - \frac{e^{\rho}}{1-x} \right) \frac{dx}{x} = -\ln r$$

$$[r > 0]$$

$$4. \int_0^1 \frac{1 - e^{-x} - e^{-1/x}}{x} dx = C.$$

$$5. \int_0^{\infty} x^{\alpha-1} \left[e^{-px} - \sum_{k=0}^{n-1} \frac{(-px)^k}{k!} \right] dx = \Gamma(\alpha) p^{-\alpha}$$

$$[\operatorname{Re} \rho \geq 0; -n < \operatorname{Re} \alpha < 1-n]$$

$$6. \int_0^{\infty} x^{-1} [e^{-x} - (x+1)^{-\rho}] dx = \psi(\rho)$$

$$[\operatorname{Re} \rho > 0]$$

$$7. \int_0^{\infty} \left(e^{-px} - \frac{1}{1+qx} \right) \frac{dx}{x} = \ln \frac{q}{p} - C$$

$$[\operatorname{Re} \rho > 0; |\arg q| < \pi]$$

$$8. \int_0^{\infty} \frac{e^{-px} - 1}{x(x+2)} dx = \frac{1}{2} [e^{\rho z} \operatorname{Ei}(-\rho z) - \ln(\rho z)]$$

$$[\operatorname{Re} \rho > 0; |\arg z| < \pi]$$

$$9. \int_0^{\infty} \left(e^{-px} - \frac{1}{1+q^{\mu} x^{\mu}} \right) \frac{dx}{x} = \ln \frac{q}{p} - C$$

$$[\mu, \operatorname{Re} \rho > 0; \mu |\arg q| < \pi]$$

$$10. \int_0^{\infty} \left[\frac{e^{-px} - 1}{p^2} + \frac{1}{(1+q^{\mu} x^{\mu})^2} \right] \frac{dx}{x} = \ln \frac{p}{q} + C \left(1 - \frac{1}{\mu} \right) - \frac{\psi(\sigma)}{\mu}$$

$$[\mu, \operatorname{Re} \sigma, \operatorname{Re} \rho > 0; \mu |\arg q| < \pi]$$

$$11. \int_0^{\infty} \left(\frac{q}{e^{qx} - 1} - \frac{1}{x e^{qx}} \right) dx = C.$$

$$12. \int_0^{\infty} \left(\frac{e^{-2x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi.$$

$$13. \int_0^{\infty} \left(\frac{e^{-x}}{2} - \frac{1}{1+e^x} \right) \frac{dx}{x} = \frac{1}{2} \ln \frac{\pi}{4}.$$

$$14. \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-px}}{x} dx = \ln \Gamma(p) - \left(p - \frac{1}{2} \right) \ln p + p - \frac{1}{2} \ln(2\pi) \quad [\operatorname{Re} p > 0]$$

$$15. \int_0^{\infty} \left(\frac{1}{2} + \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-px}}{x} dx = \left(p - \frac{1}{2} \right) \ln p + \frac{1}{2} \ln(2\pi) - p - \ln \Gamma(p) \quad [\operatorname{Re} p > 0]$$

$$16. \int_0^{\infty} \left(\frac{p^2 e^{-x}}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right) \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3p^2}{4} \quad [\operatorname{Re} p > 0]$$

$$17. \int_0^{\infty} \frac{(e^{-px} - e^{-qx})^2}{x^3} dx = 2p \ln \frac{2p}{p+q} + 2q \ln \frac{2q}{p+q} \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$18. \int_0^{\infty} \left(\frac{e^{-px}}{x} - \frac{e^{-qx}}{1 - e^{-x}} \right) dx = \psi(p) \quad [\operatorname{Re} p > 0]$$

$$19. \int_0^{\infty} [(p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx}] \frac{dx}{x^2} = (r-q)p \ln p + (p-r)q \ln q + (q-p)r \ln r \quad [\operatorname{Re} p, \operatorname{Re} q, \operatorname{Re} r > 0]$$

$$20. \int_0^{\infty} \left[(p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right] \frac{dx}{x} = p \ln p - q \ln q - (p-q) \left(1 + \ln \frac{r}{m} \right) \quad [m, \operatorname{Re} p, \operatorname{Re} q, \operatorname{Re} r > 0]$$

$$21. \int_0^{\infty} \left(\frac{e^{-qx}}{e^x - 1} + \frac{e^{-rx}}{x} \right) e^{-px} dx = \psi(q+p) - \ln(r+p) \quad [\operatorname{Re}(q+p), \operatorname{Re}(r+p) > 0]$$

$$22. \int_0^{\infty} \left(qe^{-x} - \frac{1 - e^{-qx}}{e^x - 1} e^{-rx} \right) \frac{dx}{x} = \ln \Gamma \left[\frac{q+r+1}{r+1} \right] \quad [\operatorname{Re} r, \operatorname{Re}(q+r) > -1]$$

$$23. \int_0^{\infty} \left[1 - e^{-x} - \frac{(1 - e^{-qx})(1 - e^{-px})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(p, q) \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$24. \int_0^{\infty} \left(\frac{a + be^{-qx}}{ce^{fx} + g + he^{-qx}} - \frac{a + be^{-px}}{ce^{px} + g + he^{-px}} \right) \frac{dx}{x} = \frac{a+b}{c+g+h} \ln \frac{q}{p} \quad [\operatorname{Re} p, \operatorname{Re} q > 0]$$

$$25. \int_0^{\infty} \left(\frac{e^{-pmx} - e^{-qnx}}{n} - \frac{e^{-pmx} - e^{-qmx}}{m} \right) \frac{dx}{x^2} = (q-p) \ln \frac{m}{n} \quad [m, n, \operatorname{Re} p, \operatorname{Re} q > 0]$$

$$26. \int_0^{\infty} e^{-px} \prod_{i=1}^n (e^{-v_i x} - 1) \frac{dx}{x^m} = \frac{1}{(n-1)!} \frac{\partial^{n-m}}{\partial p^{n-m}} \times \\ \times \sum_{k=0}^n (-1)^{k+m} (p + v_1 + v_2 + \dots + v_k)^{n-1} \ln(p + v_1 + v_2 + \dots + v_k) \\ [0 \leq m \leq n; \operatorname{Re} p > -\operatorname{Re}(v_1 + v_2 + \dots + v_k); \text{символ } \sum \text{ означает, что } k\text{-й член} \\ \text{суммы содержит } \binom{n}{k} \text{ слагаемых, которые отличаются наборами индексов } i_1, i_2, \dots, i_k, \\ \text{взятых из множества } i = 1, 2, \dots, n].$$

$$27. \int_0^{\infty} \left[e^{-rx^q} - \frac{y^{q^2}}{(x^q + y^q)^q} \right] \frac{dx}{x} = \left(\frac{1}{r} - \frac{1}{q} \right) C - \ln y - \frac{\ln p}{q} + \frac{\psi(0)}{r} \quad [r, q, \operatorname{Re} p, \operatorname{Re} q > 0; r | \arg y| < \pi]$$

$$28. \int_0^{\infty} (e^{-pxq} - e^{-nxr}) \frac{dx}{x} = \frac{\ln h + C^*}{r} - \frac{\ln p + C}{q} \quad [r, q, \operatorname{Re} h, \operatorname{Re} p > 0]$$

$$29. \int_{-\infty}^{\infty} \frac{\alpha x}{(a\beta^x + b\gamma^x)^\mu} dx = \frac{a^{-\lambda} b^{-\mu}}{\ln(\gamma/\beta)} \Gamma \left[\frac{\mu}{\rho} \right] \\ \left[\mu = \frac{\ln(\alpha/\beta^0)}{\ln(\gamma/\beta)}, \lambda = \frac{\ln(\gamma^0/\alpha)}{\ln(\gamma/\beta)}; a, b, \rho > 0; 0 < \beta < \alpha^{1/\rho} < \gamma \right]$$

2.4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

2.4.1. Введение.

Этот раздел содержит интегралы, в подынтегральные выражения которых входят гиперболические функции. Для их вычисления можно использовать следующие свойства. Интегралы, содержащие гиперболические функции, могут быть преобразованы к интегралам от показательной или тригонометрических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\operatorname{sh}^n bx$ и $\operatorname{ch}^n bx$, сводятся к интегралам от функций $\operatorname{sh} bx$ и $\operatorname{ch} bx$, если использовать формулы для степеней гиперболических функций (см. приложение I). В ряде случаев интегралы от гиперболических функций с помощью замены переменных сводятся к интегралам от алгебраических и других функций. Например,

$$\int_0^a \varphi(x) \operatorname{ch}^2 bx dx = \frac{1}{2} \int_0^a \varphi(x) dx + \frac{1}{4} \int_0^a \varphi(x) e^{2bx} dx + \frac{1}{4} \int_0^a \varphi(x) e^{-2bx} dx = \\ = \int_0^a \varphi(x) \cos^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 + \operatorname{ch} 2ix) dx = \\ = \frac{1}{b} \int_1^{\operatorname{ch} ab} x^2 \varphi \left(\frac{\ln(x + \sqrt{x^2 - 1})}{b} \right) \frac{dx}{\sqrt{x^2 - 1}} \quad [a, b > 0].$$

В разделе «Гиперболические функции» a, b, c — комплексные числа.

2.4.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие гиперболические, а также степенную, алгебраические и показательную функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций $V_j^{\pm}$, V_j , $j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

В п. 2.4.2. приняты следующие обозначения: $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \operatorname{sh} bx, \\ 0, & \text{если взят } \operatorname{ch} bx; \end{cases} \quad \varepsilon = \begin{cases} (-1)^{[n/2]+k}, & \text{если взят } \operatorname{sh} bx, \\ 1, & \text{если взят } \operatorname{ch} bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \operatorname{sh}^n bx, n=1, 3, 5, \dots, \\ 0, & \text{если взят } \operatorname{sh}^n bx, n=2, 4, 6, \dots, \\ 0, & \text{если взят } \operatorname{ch}^n bx, n=1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{-1} \left(\frac{\alpha}{2} + \gamma + h \right), \quad \tilde{b} = \frac{by}{2} (n-2k).$$

$$1. \int_0^{\tilde{b}} x^{\alpha-1} (y^{2r} - x^{2r})^{\beta-1} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx =$$

$$= V_{21} + \frac{1+(-1)^n}{2} (-1)^{n/2} y^{2r} (\beta-1) \frac{2^{-n-1} n!}{r [(n/2)!]^2} B \left(\frac{\alpha}{2r}, \beta \right)$$

$$\{E_{21} = (-1)^k (-1)^{[n/2]} \frac{V_{-n}}{2^n} \frac{B(\beta, \xi)}{r \Gamma(1/2+2\gamma+h)},$$

$$u = \alpha + 2r(\beta-1), \quad a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{b}/p)^{2p}\}$$

$$[\gamma, \operatorname{Re} \beta > 0, \operatorname{Re} \alpha > -\delta n; |\arg b| < \pi].$$

$$2. \int_0^{\tilde{b}} x^{\alpha-1} \frac{(y^r + \sqrt{y^{2r} - x^{2r}})^v + (y^r - \sqrt{y^{2r} - x^{2r}})^v}{y^{v/r} \sqrt{y^{2r} - x^{2r}}} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx =$$

$$= V_{22} + \frac{1+(-1)^n}{2} (-1)^n 2^{v-n+\alpha/r-1} y^{v/r} \frac{n! \sqrt{\pi}}{r [(n/2)!]^2} \Gamma \left[\frac{\alpha}{r}(2r), \frac{v+\alpha/r}{v+\alpha/r} \right]$$

$$\{E_{22} = (-1)^k (-1)^{[n/2]} \frac{V_{-n}}{2^n} \frac{\Gamma \left[\frac{\xi}{v+2\xi}, \frac{v+\xi}{1/2+2\gamma+h} \right]}{r},$$

$$u = \alpha + r(v-1), \quad a_1 = \xi, \quad b_1 = v + \xi, \quad c_1 = v/2 + \xi, \quad d_1 = (v+1)/2 + \xi,$$

$$e_1 = 1/2 + 2\gamma + h, \quad z = (\tilde{b}/p)^{2p}\}$$

$$[\gamma > 0; \operatorname{Re} \alpha > -n\delta, \operatorname{Re} \alpha > -n\delta - 2r \operatorname{Re} v; |\arg b| < \pi].$$

$$3. \int_0^{\tilde{b}} x^{2s-1} x^{-2s-2r} \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}} dx =$$

$$= U + \frac{1+(-1)^n}{2} (-1)^{n/2} 2^{-n-1} s^{-\alpha/(2r)} \frac{n!}{r [(n/2)!]^2} \Gamma \left(\frac{\alpha}{2r} \right),$$

$$U = V_{20}$$

$$[r > 1/2].$$

$$U = 2^{1-n} V_{20} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} \times$$

$$\times s^{-\alpha/2r} \Gamma \left[\frac{\alpha+2\gamma}{1/2+2\gamma} \right] {}_2F_1 \left(\begin{matrix} \Lambda(2, \alpha+2\lambda); \\ 1/2+2\gamma \end{matrix}; \frac{b^2 (s-2k)^2}{s^2} \right) \quad [r=1/2]$$

$$\{E_{20} = (-1)^k (-1)^{[n/2]} \frac{V_{-n}}{2^n} \Gamma \left[\frac{\xi}{1/2+2\gamma+h} \right],$$

$$u = -\alpha/(2r), \quad v_1 = 2(h+\gamma), \quad \varphi = 2^{-1} (n-2k) \frac{b^2-1/4r^2}{s^2},$$

$$a_1 = \xi, \quad b_1 = 1/2 + 2\gamma + h, \quad z = \left[\frac{b(n-2k)}{2p} \right]^{2p} \frac{q^{2s-\varphi}}{q^{2s-\varphi}}\}$$

$$[\operatorname{Re} s > 0; \operatorname{Re} \alpha > -\delta n; \arg b \text{ произволен при } r > 1/2 (n | \operatorname{Re} \delta | < \operatorname{Re} s \text{ при } r=1/2)].$$

2.4.3. Интегралы от $A(x) \frac{\{\operatorname{sh} bx\}^\sigma}{\{\operatorname{ch} bx\}}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{sh} cx} dx = I_\alpha \quad [\operatorname{Re} c > 0; \operatorname{Re} \alpha > 1]$$

$$I_\alpha = \frac{2^{\alpha-1}}{2^{\alpha-1} c^\alpha} \Gamma(\alpha) \zeta(\alpha) = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^\alpha},$$

$$I_{2n} = \frac{2^{2n-1}}{2n} \left(\frac{\pi}{c} \right)^{2n} |B_{2n}|, \quad I_2 = \frac{\pi^2}{4c^2}, \quad I_4 = \frac{\pi^4}{8c^4} \quad [\text{см. 5.1.4}].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{\operatorname{ch} cx} dx = J_\alpha \quad [\operatorname{Re} \alpha, \operatorname{Re} c > 0].$$

$$J_\alpha = 2c^{-\alpha} \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^\alpha} = c^{-\alpha} 2^{1-\alpha} \Gamma(\alpha) [\zeta(\alpha, 1/4) - \zeta(\alpha, 3/4)],$$

$$J_{2n+1} = \left(\frac{\pi}{2c} \right)^{2n+1} |E_{2n+1}|, \quad J_1 = \frac{\pi}{2c}, \quad J_3 = \left(\frac{\pi}{2c} \right)^3 \quad [\text{см. 5.1.4}].$$

$$3. \int_0^{\infty} x^{\alpha-1} \frac{\{\operatorname{sh} cx\}^{-2}}{\{\operatorname{ch} cx\}} dx = I_\alpha \quad [\operatorname{Re} c > 0; \operatorname{Re} \alpha > 2],$$

$$I_\alpha = 2^{1-\alpha} c^{-\alpha} (1-2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha-1), \quad I_{2n+1} = \pi^{2n} c^{-2n-1} (1-2^{1-\alpha}) |B_{2n}|.$$

$$4. \int_0^{\infty} \frac{x dx}{\operatorname{ch} cx} = \frac{\ln 2}{c^2} \quad [\operatorname{Re} c > 0].$$

$$5. \int_0^{\infty} \frac{dx}{\operatorname{ch}^n cx} = \frac{(2r-2)! (1-s)}{(2r-1)! c} + \frac{(2r-1)! s \pi}{(2r)! 2c}$$

$$[r = [n/2], \quad s = n-2r; \operatorname{Re} c > 0; n=2, 3, 4, \dots].$$

$$6. \int_0^a (a-x)^{\beta-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = 2a^{\beta+1} {}_2F_1(1; \beta+1; ab) - {}_2F_1(1; \beta+1; -ab)$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$7. \int_0^a (a^2-x^2)^{\beta-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{L_{\beta-1/2}(ab)}{\Gamma(\beta-1/2(ab))} \right\}$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$8. \int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{1}{2} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_2F_1(\alpha; \alpha+\beta; ab) - {}_2F_1(\alpha; \alpha+\beta; -ab)]$$

$$[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi]$$

$$9. \int_0^a x^{\alpha-1} (a-x)^{\alpha-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \sqrt{\pi} \left(\frac{a}{b}\right)^{2-1/2} \frac{\{\operatorname{sh}(ab/2)\}}{\{\operatorname{ch}(ab/2)\}} \Gamma(\alpha) {}_2F_1(\alpha-1/2; \frac{ab}{2})$$

$$[a, \operatorname{Re} \alpha > 0; |\arg b| < \pi]$$

$$10. \int_0^a x^{\alpha-1} (a^2-x^2)^{\beta-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{b^{\delta} a^{2\beta+1+\delta-1}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_2F_1\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; \frac{a^2 b^2}{4}\right)$$

$$[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi]$$

$$11. \int_0^a x (a^2-x^2)^{\beta-1} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{L_{\beta+1/2}(ab)}{\Gamma(\beta+1/2(ab))} + \frac{U_{\beta+1/2}(ab)}{2^{-1/2} \Gamma(\beta+1/2)} \right\}$$

$$[a, \operatorname{Re} \beta > 0; |\arg b| < \pi]$$

$$12. \int_0^a \frac{1}{\sqrt{x(a-x)}} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{\pi}{2} \sqrt{b^2} {}_2F_1(1/2; \frac{ab}{2})$$

$$[a > 0; |\arg b| < \pi]$$

$$13. \int_0^{\infty} \frac{x}{(x^2+z^2)} \frac{dx}{\operatorname{sh} bx} = \frac{\pi}{2bz} - \beta \left(\frac{bz}{\pi} + 1\right)$$

$$[\operatorname{Re} z > 0]$$

$$14. = \ln 2 - \frac{1}{2}$$

$$[bz = \pi; \operatorname{Re} z > 0]$$

$$15. = \frac{\pi}{2} - 1$$

$$[2bz = \pi; \operatorname{Re} z > 0]$$

$$16. = 2^{-1/2} [\pi + 2 \ln(1 + \sqrt{2})] - 2$$

$$[4bz = \pi; \operatorname{Re} z > 0]$$

$$17. \int_0^{\infty} \frac{dx}{(x^2+z^2) \operatorname{ch} bx} = \frac{1}{z} \beta \left(\frac{bz}{\pi} + \frac{1}{2}\right)$$

$$[\operatorname{Re} z > 0]$$

$$18. = \frac{b}{2\pi} (2-\pi)$$

$$[bz = \pi; \operatorname{Re} z > 0]$$

$$19. = \frac{2b}{\pi} \ln 2$$

$$[2bz = \pi; \operatorname{Re} z > 0]$$

$$20. = 2^{3/2} b [1 - 2\pi^{-1} \ln(1 + \sqrt{2})]$$

$$[4bz = \pi; \operatorname{Re} z > 0]$$

$$21. \int_0^{\infty} \frac{x^{\delta} dx}{(x^2+z^2) \operatorname{sh} bx} = \frac{1}{3} \left(\ln 2 - 1 + \frac{\pi}{\operatorname{ch}(\pi \sqrt{3/2})} \right)$$

$$[bz = \pi; 0 < |\arg z| < \pi]$$

$$22. \int_0^{\infty} \frac{(1+ix)^{\alpha} - (1-ix)^{\alpha}}{\operatorname{sh}(\pi \sqrt{2})} dx = 2i - 2i(-1)^{\alpha/2} \frac{1+(-1)^{\alpha}}{2} |E_{\pi}|$$

2.4.4. Интегралы от $\prod_k \frac{\{\operatorname{sh} b_k x\}}{\{\operatorname{ch} b_k x\}}$

Обозначение: $\delta = \left\{ \frac{1}{0} \right\}$.

$$1. \int_0^{\infty} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \operatorname{tg} \frac{b\pi}{2a}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$2. \int_0^{\infty} \frac{\operatorname{sh}^2 bx}{\operatorname{sh}^2 cx} dx = \frac{c - \pi b \operatorname{ctg}(b\pi/c)}{2c^2}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$3. \int_0^{\infty} \frac{1}{\operatorname{ch} cx} \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{\pi}{2c \cos[b\pi/(2c)]} - \frac{1}{c} \beta \left(\frac{b+c}{2c}\right) \left\{ \frac{1}{0} \right\}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$4. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{ch}^v cx} dx = \frac{2^{v-1}}{c \Gamma(v)} \Gamma\left(\frac{vc+b}{2c}\right) \Gamma\left(\frac{vc-b}{2c}\right)$$

$$[\operatorname{Re}(vc) > |\operatorname{Re} b|]$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{ch}^n cx} dx = \frac{2\pi^{1+\lambda-2}\pi}{(n-1)!} b^{1-2\lambda} c^{2\lambda-2} \left\{ \frac{\sec[b\pi/(2c)]}{\operatorname{cosec}[b\pi/(2c)]} \right\} \prod_{k=1}^{[(n-1)/2]} \left\{ (b-\lambda)^2 - \frac{b^2}{4c^2} \right\}$$

$$[\pi \operatorname{Re} c > |\operatorname{Re} b|; \lambda = \frac{1-(-1)^n}{4}, n \geq 1]$$

$$6. \int_0^{\infty} \operatorname{sh}^{1-\nu} bx \operatorname{ch}^{1-\nu} bx dx = \frac{1}{2b} B\left(\frac{\nu}{2}, 1 - \frac{\nu+\nu}{2}\right)$$

$$[b > 0; 0 < \operatorname{Re} \nu < 2 - \operatorname{Re} \nu]$$

$$7. \int_0^{\infty} \operatorname{sh}^{\nu-1} ax \frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}} dx = \frac{1}{2a \sqrt{\pi}} \left\{ \frac{\sin[b\pi/(2a)]}{\cos[b\pi/(2a)]} \right\} \Gamma\left(\frac{\nu+\delta}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2} + \frac{b}{2a}\right) \Gamma\left(\frac{1-\nu}{2} - \frac{b}{2a}\right)$$

$$[a > 0; -\delta < \operatorname{Re} \nu < 1 - |\operatorname{Re} b|/a]$$

$$8. \int_0^{\infty} \frac{ch^{v-1} ax \, ch \, bx}{ch \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx =$$

$$= \pm \frac{b}{4a^2} \frac{\{\cos [b\pi/(2a)]\}}{\{\sin [b\pi/(2a)]\}} \Gamma \left[\frac{(v+\delta)/2}{(1-v)/2} \right] \Gamma \left(\frac{b-va}{2a} \right) \Gamma \left(-\frac{b+va}{2a} \right)$$

$$[a > 0; -\delta < \operatorname{Re} v < -\operatorname{Re} b/a]$$

$$9. \int_0^{\infty} \frac{sh \, ax}{sh \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx =$$

$$= \frac{\pi}{2c} \left(\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c} \right)^{-1} \frac{\{\sin (b\pi/c)\}}{\{\sin (a\pi/c)\}} + \left\{ \frac{(2c)^{-1}}{0} \right\} \left[\psi \left(\frac{a+c-\delta}{2c} \right) - \psi \left(\frac{a+c+b}{2c} \right) \right]$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$10. \int_0^{\infty} \frac{\{sh \, ax \, sh \, bx\}}{\{ch \, ax \, ch \, bx\}} \frac{dx}{ch \, cx} =$$

$$= \frac{\pi}{c} \frac{\{\sin [a\pi/(2c)] \sin [b\pi/(2c)]\}}{\{\cos [a\pi/(2c)] \cos [b\pi/(2c)]\}} \{\cos (a\pi/c) + \cos (b\pi/c)\}^{-1}$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$11. \int_0^{\infty} \frac{sh \, ax \, ch \, bx}{ch \, cx} dx = \frac{1}{4c} \left[\psi \left(\frac{3c-a+b}{4c} \right) - \psi \left(\frac{3c+a-b}{4c} \right) \right] +$$

$$+ \psi \left(\frac{3c-a-b}{4c} \right) - \psi \left(\frac{3c+a+b}{4c} \right) + \frac{2\pi \sin (a\pi/c)}{c \pi \{\cos (a\pi/c) + \cos (b\pi/c)\}}$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$12. \int_0^{\infty} \frac{sh \, ax \, sh \, cx}{ch^2 \, cx} dx = \frac{\pi a}{2c^2 \cos [a\pi/(2c)]}$$

$$[\operatorname{Re} c > |\operatorname{Re} a|]$$

2.4.5. Интегралы от $A(x) \prod_h \frac{\{sh \, b_h x\}}{\{ch \, b_h x\}}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{sh \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx = \frac{\Gamma(\alpha)}{(2c)^{\alpha}} \left[\zeta \left(\alpha, \frac{1}{2} - \frac{b}{2c} \right) - \zeta \left(\alpha, \frac{1}{2} + \frac{b}{2c} \right) \right]$$

$$[\operatorname{Re} \alpha > 1 - \delta; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$2. \int_0^{\infty} \frac{x^{2n+\delta}}{sh \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx = \frac{\pi}{2c} \frac{\partial^{2n+\delta}}{\partial b \cdot \partial^{1+\delta}} \lg \frac{b\pi}{2c}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|; \sigma = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}]$$

$$3. \int_0^{\infty} \frac{sh^2 \, bx}{x \, sh \, cx} dx = -\frac{1}{2} \ln \cos \frac{b\pi}{c}$$

$$[\operatorname{Re} c > 2|\operatorname{Re} b|]$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{ch \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx =$$

$$= 2^{-\pi i \alpha} \Gamma(\alpha) \left[\zeta \left(\alpha, \frac{c-b}{4c} \right) + \zeta \left(\alpha, \frac{c+b}{4c} \right) \pm \zeta \left(\alpha, \frac{3c+b}{4c} \right) - \zeta \left(\alpha, \frac{3c-b}{4c} \right) \right]$$

$$[\operatorname{Re} \alpha > -\delta; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$5. \int_0^{\infty} \frac{sh \, bx}{x \, ch \, cx} dx = \ln \left(\frac{b+\pi}{4c} \pi \right)$$

$$[\operatorname{Re} c > |\operatorname{Re} b|]$$

$$6. \int_0^{\infty} \frac{x^{2n+\delta}}{ch \, cx} \frac{\{sh \, bx\}}{\{ch \, bx\}} dx = \frac{\pi}{2c} \frac{\partial^{2n+\delta}}{\partial b \cdot \partial^{1+\delta}} \sec \frac{b\pi}{2c}$$

$$[\operatorname{Re} c > |\operatorname{Re} b|; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}]$$

$$7. \int_0^{\infty} x^{\alpha-1} \frac{ch \, bx}{ch \, cx} dx = \Gamma(\alpha) \sum_{k=0}^{\infty} (-1)^k \{ [c(2k+1)-b]^{-\alpha} + [c(2k+1)+b]^{-\alpha} \}$$

$$[\operatorname{Re} \alpha > 0; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$8. \int_0^{\infty} x \frac{sh \, bx}{ch^2 \, cx} dx = \frac{\pi}{4c^2} \sin^2 \frac{b\pi}{2c} \left(2c \sin \frac{b\pi}{2c} - b\pi \cos \frac{b\pi}{2c} \right)$$

$$[2 \operatorname{Re} c > |\operatorname{Re} b|]$$

$$9. \int_0^{\infty} \frac{x^{\alpha-1} dx}{sh \, ax \, ch \, bx} = \frac{\Gamma(\alpha)}{(2b)^{\alpha}} \left[{}_1F_1 \left(-1; \alpha; \frac{1}{2} + \frac{a}{2b} \right) + {}_1F_1 \left(-1; \alpha; \frac{1}{2} - \frac{a}{2b} \right) \right]$$

$$[\operatorname{Re} a, \operatorname{Re} b > 0; \operatorname{Re} \alpha > \frac{1}{2}]$$

$$10. \int_0^{\infty} x^{\alpha-1} \frac{sh \, cx}{ch^2 \, cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha-1}}$$

$$[\operatorname{Re} c > 0; \operatorname{Re} \alpha > -1; \operatorname{cm} 5.1.9]$$

$$11. \int_0^{\infty} c \, ch^{\mu-1} ax \, sh \, ax \, dx = -\frac{\sqrt{\pi}}{2\mu a^2} \Gamma \left[\frac{-\mu/2}{(1-\mu)/2} \right]$$

$$[a > 0; \operatorname{Re} \mu < 0]$$

$$12. \int_0^{\infty} x^{\alpha-1} \frac{ch \, cx}{sh^2 \, cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\alpha-1}}$$

$$[\operatorname{Re} \alpha > 2; \operatorname{cm} 5.1.4]$$

$$13. \int_0^{\infty} \frac{sh \, ax \, sh \, bx}{x \, sh \, cx} dx = \frac{1}{2} \ln \frac{\cos [(a-b)\pi/(2c)]}{\cos [(a+b)\pi/(2c)]}$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$14. \int_0^{\infty} \frac{sh^2 \, ax \, ch \, bx}{x \, sh \, cx} dx = \frac{1}{4} \ln \frac{1 + \cos (b\pi/c)}{\cos (2a\pi/c) + \cos (b\pi/c)}$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

$$15. \int_0^{\infty} \frac{sh \, ax \, ch \, bx}{x \, ch \, cx} dx = \frac{1}{2} \ln \frac{\cos [b\pi/(2c)] + \sin [a\pi/(2c)]}{\cos [b\pi/(2c)] - \sin [a\pi/(2c)]}$$

$$[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|]$$

8. $\int_0^{\infty} \frac{sh \nu x}{sh \nu x} \frac{sh bx}{ch bx} dx =$
 $= \pm \frac{b}{4a^2 \sqrt{\pi}} \left\{ \frac{\cos [b\pi/(2a)]}{\sin [b\pi/(2a)]} \right\} \Gamma \left[\frac{(1+\delta)/2}{(1-\nu)/2} \right] \Gamma \left(\frac{b-\nu a}{2a} \right) \Gamma \left(-\frac{b+\nu a}{2a} \right)$
 $[a > 0; -\delta < \operatorname{Re} \nu < -|\operatorname{Re} \nu|]$
9. $\int_0^{\infty} \frac{sh ax}{sh cx} \frac{sh bx}{ch bx} dx =$
 $= \frac{\pi}{2c} \left\{ \cos \frac{a\pi}{c} + \cos \frac{b\pi}{c} \right\}^{-1} \left\{ \frac{\sin (b\pi/c)}{\sin (a\pi/c)} \right\} + \left\{ \frac{(2c)^{-1}}{0} \right\} \left[\psi \left(\frac{a+c-b}{2c} \right) - \psi \left(\frac{a+c+b}{2c} \right) \right]$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$
10. $\int_0^{\infty} \frac{sh ax sh bx}{ch ax ch bx} \frac{dx}{ch cx} =$
 $= \frac{\pi}{c} \left\{ \frac{\sin [a\pi/(2c)] \sin [b\pi/(2c)]}{\cos [a\pi/(2c)] \cos [b\pi/(2c)]} \right\} \left\{ \cos (a\pi/c) + \cos (b\pi/c) \right\}^{-1}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$
11. $\int_0^{\infty} \frac{sh ax ch bx}{ch cx} dx = \frac{1}{4c} \left[\psi \left(\frac{3c-a+b}{4c} \right) - \psi \left(\frac{3c+a-b}{4c} \right) + \right.$
 $\left. + \psi \left(\frac{3c-a-b}{4c} \right) - \psi \left(\frac{3c+a+b}{4c} \right) \right] + \frac{2\pi \sin (a\pi/c)}{c \sin (a\pi/c) + \cos (b\pi/c)}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$
12. $\int_0^{\infty} \frac{sh ax sh cx}{ch^2 cx} dx = \frac{\pi a}{2c^2 \cos [a\pi/(2c)]}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a|]$

2.4.5. Интегралы от $A(x) \prod_k \frac{sh b_k x}{ch b_k x}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

1. $\int_0^{\infty} \frac{x^{\alpha-1}}{sh cx} \frac{sh bx}{ch bx} dx = \frac{\Gamma(\alpha)}{(2c)^{\alpha}} \left[\zeta \left(\alpha, \frac{1}{2} - \frac{b}{2c} \right) - \zeta \left(\alpha, \frac{1}{2} + \frac{b}{2c} \right) \right]$
 $[|\operatorname{Re} \alpha| > 1 - \delta; |\operatorname{Re} c| > |\operatorname{Re} b|]$
2. $\int_0^{\infty} \frac{x^{2\pi+1}}{sh cx} \frac{sh bx}{ch bx} dx = \frac{\pi}{2c} \frac{\partial^{2\pi+1}}{\partial b^{2\pi+1}} \frac{b\pi}{2c}$
 $[|\operatorname{Re} c| > |\operatorname{Re} b|; \sigma = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}]$
3. $\int_0^{\infty} \frac{sh^2 bx}{x sh cx} dx = -\frac{1}{2} \ln \cos \frac{b\pi}{c}$
 $[|\operatorname{Re} c| > 2|\operatorname{Re} b|]$

4. $\int_0^{\infty} \frac{x^{\alpha-1}}{ch cx} \frac{sh bx}{ch bx} dx =$
 $= 2^{-2\alpha} c^{\alpha} \Gamma(\alpha) \left[\zeta \left(\alpha, \frac{c-b}{4c} \right) + \zeta \left(\alpha, \frac{c+b}{4c} \right) - \zeta \left(\alpha, \frac{3c+b}{4c} \right) - \zeta \left(\alpha, \frac{3c-b}{4c} \right) \right]$
 $[|\operatorname{Re} \alpha| > -\delta; |\operatorname{Re} c| > |\operatorname{Re} b|]$
5. $\int_0^{\infty} \frac{sh bx}{x ch cx} dx = \ln \lg \left(\frac{b+c}{4c} \pi \right)$
 $[|\operatorname{Re} c| > |\operatorname{Re} b|]$
6. $\int_0^{\infty} \frac{x^{2\pi+1}}{ch cx} \frac{sh bx}{ch bx} dx = \frac{\pi}{2c} \frac{\partial^{2\pi+1}}{\partial b^{2\pi+1}} \frac{b\pi}{2c}$
 $[|\operatorname{Re} c| > |\operatorname{Re} b|; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}]$
7. $\int_0^{\infty} x^{\alpha-1} \frac{ch bx}{ch cx} dx = \Gamma(\alpha) \sum_{k=0}^{\infty} (-1)^k \{ (c(2k+1)-b)^{-\alpha} + (c(2k+1)+b)^{-\alpha} \}$
 $[|\operatorname{Re} \alpha| > 0; |\operatorname{Re} c| > |\operatorname{Re} b|]$
8. $\int_0^{\infty} x \frac{sh bx}{ch^2 cx} dx = \frac{\pi}{4c^2} \sin^2 \frac{b\pi}{2c} \left(2c \sin \frac{b\pi}{2c} - b\pi \cos \frac{b\pi}{2c} \right)$
 $[2|\operatorname{Re} c| > |\operatorname{Re} b|]$
9. $\int_0^{\infty} \frac{x^{\alpha-1}}{sh ax ch bx} dx = \frac{\Gamma(\alpha)}{(2b)^{\alpha}} \left[{}_1F_1 \left(-1; \alpha; \frac{1}{2} + \frac{a}{2b} \right) + {}_1F_1 \left(-1; \alpha; \frac{1}{2} - \frac{a}{2b} \right) \right]$
 $[|\operatorname{Re} a|, |\operatorname{Re} b| > 0; |\operatorname{Re} \alpha| > 1]$
10. $\int_0^{\infty} x^{\alpha-1} \frac{sh cx}{ch^2 cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha-1}}$
 $[|\operatorname{Re} c| > 0; |\operatorname{Re} \alpha| > -1; \text{см. 5.1.4}]$
11. $\int_0^{\infty} x ch^{\mu-1} ax sh ax dx = -\frac{\sqrt{\pi}}{2\mu a^2} \Gamma \left[\frac{-\mu/2}{(1-\mu)/2} \right]$
 $[a > 0; \operatorname{Re} \mu < 0]$
12. $\int_0^{\infty} \frac{x^{\alpha-1}}{sh^2 cx} \frac{ch cx}{ch^2 cx} dx = \frac{2\Gamma(\alpha)}{c^{\alpha}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\alpha-1}}$
 $[|\operatorname{Re} \alpha| > 2; \text{см. 5.1.4}]$
13. $\int_0^{\infty} \frac{sh ax sh bx}{x sh cx} dx = \frac{1}{2} \ln \frac{\cos [(a-b)\pi/(2c)]}{\cos [(a+b)\pi/(2c)]}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$
14. $\int_0^{\infty} \frac{sh^2 ax ch bx}{x sh cx} dx = \frac{1}{4} \ln \frac{1 + \cos (b\pi/c)}{\cos (2a\pi/c) + \cos (b\pi/c)}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$
15. $\int_0^{\infty} \frac{sh ax ch bx}{x ch cx} dx = \frac{1}{2} \ln \frac{\cos [b\pi/(2c)] + \sin [a\pi/(2c)]}{\cos [b\pi/(2c)] - \sin [a\pi/(2c)]}$
 $[|\operatorname{Re} c| > |\operatorname{Re} a| + |\operatorname{Re} b|]$

$$16. \int_0^{\infty} \frac{1}{x^2+z^2} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{\pi}{z} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(bk\pi/c)}{cz + i\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Im} b|]$$

$$17. = (-1)^n b \frac{e^{ibn\pi/c}}{2n} + \frac{c}{2\pi n} \sum_{k=1}^{n-1} \frac{(-1)^k e^{ibk\pi/c}}{n-k} +$$

$$+ (-1)^n \frac{c}{2\pi n} e^{ibn\pi/c} \ln(1 + e^{ib\pi/c}) + \frac{c}{2\pi n} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{(1+z)^{n-1}}{z} \ln(1+z) \right] \Big|_{z=e^{ib\pi/c}}$$

$$[cz = \pi n; \operatorname{Re} c > |\operatorname{Re} b|; \operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

$$18. = -\frac{b}{2} \cos \frac{b\pi}{c} + \frac{c}{\pi} \sin \frac{b\pi}{c} \ln \left(2 \cos \frac{b\pi}{2c} \right) \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$19. = c \sin \frac{b\pi}{2c} - \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \operatorname{ctg} \left(\frac{c-b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$20. \int_0^{\infty} \frac{x}{x^2+z^2} \frac{\operatorname{sh} bx}{\operatorname{ch}(\pi x/z)} dx = -2 \sin \frac{bz}{2} + \frac{\pi}{2} \sin bz - \operatorname{ch} bz \ln \operatorname{tg} \frac{bz+\pi}{4}$$

$$[z > 0; z |\operatorname{Re} b| < \pi]$$

$$21. \int_0^{\infty} \frac{x}{x^2+z^2} \frac{\operatorname{ch} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2cz} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k \cos(bk\pi/c)}{cz + k\pi} \quad [\operatorname{Re} z > 0; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$22. = \frac{b\pi}{2c} \sin \frac{b\pi}{c} - \frac{1}{2} + \frac{1}{2} \cos \frac{b\pi}{c} \ln \left[2 \left(1 + \cos \frac{b\pi}{c} \right) \right]$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$23. = \frac{\pi}{2} \cos \frac{b\pi}{2c} - 1 + \sin \frac{b\pi}{2c} \ln \operatorname{tg} \left(\frac{c+b}{4c} \pi \right) \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$24. \int_0^{\infty} \frac{1}{x^2+z^2} \frac{\operatorname{ch} bx}{\operatorname{ch} cx} dx = \frac{(-1)^n}{2n+1} \exp \left[\frac{ib}{c} \left(n + \frac{1}{2} \right) \pi \right] \left[-ib + \right.$$

$$\left. + \frac{c}{\pi} \ln(1 + e^{ib\pi/c}) \right] + \frac{c \exp \{ ib\pi/(2c) \}}{\pi(2n+1)} \sum_{k=0}^{n-1} \frac{(-1)^k}{k-n} \exp \{ ibk\pi/c \} +$$

$$+ \frac{c \exp \{ ib\pi/(2c) \}}{\pi(2n+1)(n+1)} {}_2F_1 \left(n+1, 1; n+2; -e^{ib\pi/c} \right)$$

$$[cz = \pi(n+1/2); \operatorname{Re} c > |\operatorname{Re} b|]$$

$$25. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} - \frac{c}{2} \cos \frac{b\pi}{c} + \frac{2c}{\pi} \sin \frac{b\pi}{c} \ln \operatorname{tg} \left(\frac{b+c}{4c} \pi \right)$$

$$[cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$26. = \frac{2c}{\pi} \cos \frac{b\pi}{2c} \ln \left(2 \cos \frac{b\pi}{2c} \right) + b \sin \frac{b\pi}{2c}$$

$$[2cz = \pi; \operatorname{Re} c > |\operatorname{Re} b|]$$

2.4.6. Интегралы от $(a+b \operatorname{ch}^n bx + c \operatorname{sh}^n bx)^p \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.

$$1. \int_0^a \frac{\operatorname{ch} bx dx}{(\operatorname{ch} a - \operatorname{ch} x)^v} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-v)}{\operatorname{sh}^{v-1/2} a} P_{b-1/2}^{v-1/2}(\operatorname{ch} a)$$

$$[a > 0; \operatorname{Re} v < 1; |\arg b| < \pi]$$

$$2. \int_0^{\infty} \frac{dx}{a+b \operatorname{ch} x} = \frac{2}{\sqrt{b^2-a^2}} \operatorname{arctg} \frac{\sqrt{b^2-a^2}}{a+b} \quad [|b| > |a|]$$

$$3. = \frac{1}{\sqrt{a^2-b^2}} \ln \left| \frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \right| \quad [0 < |b| < |a|]$$

$$= a^{-1} \quad [b = a]$$

$$4. \int_0^{\infty} \frac{dx}{a+b \operatorname{sh} x} = \frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \right| \quad [ab \neq 0]$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} bx}{a+\operatorname{ch} cx} dx = \frac{\pi}{c\sqrt{a^2-1}} \operatorname{cosec} \frac{b\pi}{c} \operatorname{sh} \left(\frac{b}{c} \operatorname{Arch} a \right) \quad [a > 1; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$6. = \frac{\pi}{c\sqrt{1-a^2}} \operatorname{cosec} \frac{b\pi}{c} \sin \left(\frac{b}{c} \arccos a \right) \quad [-1 < a < 1; \operatorname{Re} c > |\operatorname{Re} b|]$$

$$7. \int_0^{\infty} \frac{\operatorname{ch} bx}{1+\operatorname{ch} cx} dx = \frac{\pi b}{c^2 \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|]$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{(a+\operatorname{ch} cx)^v} dx = \frac{1}{c} \sqrt{\frac{\pi}{2}} (a^2-1)^{1/4-v/2} \Gamma \left[\frac{v+b/c}{v}, \frac{v-b/c}{v} \right] P_{b/c-1/2}^{v-1/2}(a)$$

$$[a \notin [-1, 1]; \operatorname{Re}(v \pm b/c) > 0]$$

$$9. \int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{a+\operatorname{ch} cx}} dx = \frac{\pi}{c\sqrt{2}} \sec \frac{b\pi}{c} P_{b/c-1/2}(a)$$

$$[-1 < a < 1; \operatorname{Re} c > 2|\operatorname{Re} b|]$$

$$10. \int_0^{\infty} \frac{\operatorname{ch} bx dx}{(\cos t + \operatorname{ch} cx)^2} = \frac{\pi [b \sin t \cos(bt/c) - c \cos t \sin(b/c)]}{c^2 \sin^3 t \sin(b\pi/c)}$$

$$[2\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi]$$

$$11. \int_0^{\infty} \frac{\operatorname{ch} b^* \operatorname{ch} cx}{(1+\operatorname{ch} cx)^v} dx = \frac{2^{v-1}}{c} B \left(v + \frac{b}{c}, v - \frac{b}{c} \right)$$

$$[\operatorname{Re} v > 0; \operatorname{Re}(v) > \operatorname{Re} c + |\operatorname{Re} b|]$$

$$12. \int_0^{\infty} \frac{\cos t - \operatorname{ch} bx}{\cos t - \operatorname{ch} cx} dx = \frac{\pi}{c} \frac{\sin [b(\pi-t)/c]}{\sin t \sin(b\pi/c)} - \frac{\pi-t}{c \sin t} \cos v$$

$$[\operatorname{Re} c > |\operatorname{Re} b|; 0 < t < \pi; 0 < v < \pi]$$

$$13. \int_0^{\infty} \left\{ \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} ax \operatorname{ch} bx} \right\} \frac{dx}{\cos t + \operatorname{ch} cx} = \frac{\pi}{c \sin t} \left(\cos \frac{2b\pi}{c} - \cos \frac{2a\pi}{c} \right)^{-1} \times$$

$$\times \left\{ \sin [a(\pi-t)/c] \sin [b(\pi+t)/c] - \sin [a(\pi+t)/c] \sin [b(\pi-t)/c] \right\}$$

$$\times \left\{ \cos [a(\pi-t)/c] \cos [b(\pi+t)/c] - \cos [a(\pi+t)/c] \cos [b(\pi-t)/c] \right\}$$

$$[0 < |t| < \pi; \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Re} b|; \pi \operatorname{Re} c > |\operatorname{Re}(at)|]$$

15. $\int_0^{\infty} \frac{\{ \operatorname{sh} ax \operatorname{sh} bx \}}{\{ \operatorname{ch} ax \operatorname{ch} bx \}} \frac{dx}{1 + \operatorname{ch} cx} = \pm \frac{2\pi}{c^2 [\cos(2b\pi/c) - \cos(2a\pi/c)]} \times$
 $\times \left[b \left\{ \frac{\sin(a\pi/c) \cos(b\pi/c)}{\cos(a\pi/c) \sin(b\pi/c)} \right\} - a \left\{ \frac{\cos(a\pi/c) \sin(b\pi/c)}{\sin(a\pi/c) \cos(b\pi/c)} \right\} \right] \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$
16. $\int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi \sin(bt/c)}{2c \operatorname{ch}(b\pi/c) \sin(t/2)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|; 0 < |t| < \pi].$
17. $\int_0^{\infty} \frac{\operatorname{sh}(cx/2) \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \cos(b\pi/c)} \quad [\operatorname{Re} c > 2|\operatorname{Re} b|].$
18. $\int_0^{\infty} \frac{\operatorname{sh} cx \operatorname{sh} bx}{(\cos t + \operatorname{ch} cx)^2} dx = \frac{b\pi \sin(bt/c)}{c^2 \sin t \sin(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} b|; |t| < \pi].$
19. $\int_0^{\infty} \frac{dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{\sqrt{b^2 - c^2}}{b + c} \quad [|b| > |c|].$
20. $= \frac{1}{\sqrt{c^2 - b^2}} \ln \left| \frac{b + c + \sqrt{c^2 - b^2}}{b + c - \sqrt{c^2 - b^2}} \right| \quad [0 < |b| < |c|].$
21. $= c^{-1} \quad [b = c].$
22. $\int_0^{\infty} \frac{\{ \operatorname{sh} ax \operatorname{sh} bx \}}{\{ \operatorname{ch} ax \operatorname{ch} bx \}} \frac{dx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} = \frac{\pi}{4(a^2 - b^2)} \left\{ \frac{b}{a} \right\} \quad [\operatorname{Re} a, \operatorname{Re} b > 0].$
23. $\int_0^{\infty} \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \left| \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} \right|$
 $[b^2 < a^2 + c^2; |a| \neq |b|].$
24. $= \frac{2(a-b)}{c(a-b-c)} \quad [b^2 = a^2 + c^2; c(a-b-c) < 0].$
25. $= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left(\operatorname{arctg} \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + e\pi \right)$
 $[b^2 > a^2 + c^2, e = 0 \text{ при } (b-a)(a+b+c) > 0; e = 1 \text{ при } (b-a)(a+b+c) < 0, a < b+c; e = -1 \text{ при } (b-a)(a+b+c) < 0, a > b+c].$
26. $\int_0^{\infty} \frac{\operatorname{ch} cx \operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{sc} \frac{b\pi}{2c} \operatorname{ch} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [\operatorname{Re} c > |\operatorname{Re} b|].$
27. $\int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{sh}^2 t + \operatorname{ch}^2 cx} dx = \frac{\pi}{c} \left(\operatorname{sh} 2t \sin \frac{b\pi}{2c} \right)^{-1} \operatorname{sh} \frac{bt}{c} \quad [2\operatorname{Re} c > |\operatorname{Re} b|].$
28. $\int_0^{\infty} \frac{\operatorname{ch} bx}{\sqrt{\operatorname{sh}^2 t + \operatorname{ch}^2 cx}} dx = \frac{\pi}{2c} \operatorname{sc} \frac{b\pi}{2c} P_{b/(2a) - 1/2}(\operatorname{ch} 2t) \quad [\operatorname{Re} c > |\operatorname{Re} b|].$

29. $\int_0^{\infty} \frac{\operatorname{ch} bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c\sqrt{2}} \operatorname{sh}^{-1} \frac{b\pi}{2c} \operatorname{sh} \left[\frac{b}{2c} \ln(1 + \sqrt{2}) \right] \quad [2\operatorname{Re} c > |\operatorname{Re} b|].$
30. $\int_0^{\infty} \frac{dx}{b \operatorname{ch}^2 x + c \operatorname{sh}^2 x} = \frac{1}{\sqrt{bc}} \operatorname{arctg} \frac{c}{\sqrt{bc}} \quad [bc > 0].$
31. $= \frac{1}{2\sqrt{-bc}} \ln \left| \frac{c - \sqrt{-bc}}{b + \sqrt{-bc}} \right| \quad [bc < 0].$
- 2.4.7. Интегралы от $A(x)(a + b \operatorname{ch}^2 x + c \operatorname{sh}^2 x)^{-1} \prod_k \left\{ \frac{\operatorname{sh} c_k x}{\operatorname{ch} c_k x} \right\}$.
1. $\int_0^{\frac{\pi}{2}} \frac{x dx}{\operatorname{sh}(x/2) \sqrt{\operatorname{ch} a - \operatorname{ch} x}} = \frac{\pi \sqrt{2}}{\operatorname{sh}(a/2)} \arcsin \operatorname{th} \frac{a}{2} \quad [a > 0].$
2. $\int_0^{\frac{\pi}{2}} \frac{x^{\alpha-1} dx}{\cos t + \operatorname{ch} x} = J_{\alpha} \quad [\operatorname{Re} x > 0; |t| < \pi].$
 $J_{\alpha} = \frac{\pi^{2\alpha-1}}{\sin t \sin(\alpha\pi/2)} \left[\zeta \left(1-\alpha, \frac{1}{2} - \frac{t}{2\pi} \right) - \zeta \left(1-\alpha, \frac{1}{2} + \frac{t}{2\pi} \right) \right] =$
 $= \frac{2\Gamma(\alpha)}{\sin t} \sum_{k=1}^{\infty} \frac{\sin kt}{k^{\alpha}},$
 $J_1 = \frac{4}{\sin t} \left[t \ln 2 - 2L \left(\frac{t}{2} \right) \right], \quad J_2 = \frac{t}{3 \sin t} (\pi^2 - t^2),$
 $J_3 = \frac{t}{15 \sin t} (\pi^2 - t^2) (7\pi^2 - 3t^2).$
3. $\int_0^{\infty} \frac{x^{\alpha-1}}{\cos t + \operatorname{ch} x} \left\{ \frac{\operatorname{sh}(x/2)}{\operatorname{ch}(x/2)} \right\} dx = \pi^{2\alpha-1} \left\{ \frac{\sin(t/2) \cos(\alpha\pi/2)}{\cos(t/2) \sin(\alpha\pi/2)} \right\}^{-1} \times$
 $\times \left[\zeta \left(1-\alpha, \frac{1}{4} - \frac{t}{4\pi} \right) - \zeta \left(1-\alpha, \frac{1}{4} + \frac{t}{4\pi} \right) \pm \zeta \left(1-\alpha, \frac{3}{4} + \frac{t}{4\pi} \right) - \right.$
 $\left. - \zeta \left(1-\alpha, \frac{3}{4} - \frac{t}{4\pi} \right) \right] \quad [\operatorname{Re} x > 0; |t| < \pi].$
4. $\int_0^{\infty} \frac{x \operatorname{sh} bx}{1 + \operatorname{ch} cx} dx = \frac{\pi}{c^2} \operatorname{sh}^{-2} \frac{b\pi}{c} \left(c \sin \frac{b\pi}{c} - b\pi \cos \frac{b\pi}{c} \right) \quad [\operatorname{Re} c > |\operatorname{Re} b|].$
5. $\int_0^{\infty} \frac{x^{2v+1} \operatorname{sh} bx}{(1 + \operatorname{ch} cx)^v} dx = -\frac{\pi}{c^2} \Gamma \left[\frac{2-v}{2} \right] \frac{\partial^{2v+1}}{\partial b^{2v+1}} \frac{b}{\sin(b\pi/c)} \quad [\operatorname{Re}(vc) > |\operatorname{Re} b|].$
6. $\int_0^{\infty} \frac{x \operatorname{ch}(x/2) \operatorname{th} x}{\cos t + \operatorname{ch} x} dx = \frac{4}{\cos(t/2)} \left[\frac{\pi}{2} \ln 2 - L \left(\frac{\pi+t}{4} \right) - L \left(\frac{\pi-t}{4} \right) \right]$
 $[t \neq \pi(2k-1); k=0, \pm 1, \pm 2, \dots].$

$$7. \int_0^{\infty} \frac{x^{2n+1} \operatorname{sh} cx \, dx}{(\cos t + \operatorname{ch} cx)^2} = I_n \quad [\operatorname{Re} c > 0; |t| < \pi]$$

$$I_n = \frac{2(2n+1)!}{c^{2n+2} \sin t} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos kt}{k^{2n+2}},$$

$$I_0 = -\frac{t+\pi}{c^2 \sin t}, \quad I_1 = \frac{t(\pi^2 - t^2)}{c^4 \sin t}.$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{(x^2 + z^2)(\cos t + \operatorname{ch} cx)} \, dx = \frac{\pi z - ibx}{2z(\cos cz + \cos t)} +$$

$$+ \frac{c\pi}{\operatorname{sh} t} \sum_{k=1}^{\infty} \left\{ \frac{\exp[-ib\pi[(2k+1)\pi - t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) - t]^2} - \frac{\exp[-ib\pi[(2k+1)\pi + t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) + t]^2} \right\}$$

$$[\operatorname{Re} c > \operatorname{Re} b; |t| < \pi; \operatorname{Re} z > 0].$$

$$9. \int_0^{\infty} \frac{x^{-1/2}}{1 + 2\operatorname{ch}^2 cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \sqrt{\frac{2\pi}{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{kc + \sqrt{k^2 c^2 - b^2}}{k^2 c^2 - b^2} \right)^{1/2} \sin \frac{k\pi}{3}$$

$$[2\operatorname{Re} c > |\operatorname{Re} b|].$$

$$10. \int_0^{\infty} \frac{x^{2n+1} \operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \operatorname{sh} bx \, dx =$$

$$= \frac{\pi}{2\sqrt{2}} \frac{\partial^{2n+1}}{\partial b^{2n+1}} \left[\sec \frac{b\pi}{2c} \operatorname{ch} \left(\frac{b}{2c} \ln(1 + \sqrt{2}) \right) \right] \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$11. \int_0^{\infty} \frac{x^{\alpha-1} \operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} 2ax + \operatorname{ch} 2bx} \, dx = \frac{\Gamma(\alpha)}{2(a^2 - b^2)^{\alpha/2}} \left[\left(\frac{a+b}{a-b} \right)^{\alpha/2} - \left(\frac{a-b}{a+b} \right)^{\alpha/2} \right] \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

$$[\operatorname{Re} a, \operatorname{Re} b > 0, \operatorname{Re} \alpha > -2; \text{см. 5.1.4}].$$

$$12. \int_0^{\infty} \frac{x^n \, dx}{b \operatorname{ch} x + c \operatorname{sh} x} = \frac{(2n)!}{(b+c) 2^{n+1}} \Phi \left(\frac{c-b}{c+b}, n+1, \frac{1}{2} \right) \quad [b, c > 0].$$

2.4.8. Интегралы от $A(x) \operatorname{ch} bx \left\{ \frac{\operatorname{sh} \sqrt{a^2 - x^2}}{\operatorname{ch} \sqrt{a^2 - x^2}} \right\}$.

Условие: $a > 0$.

$$1. \int_0^a \frac{(a^2 - x^2)^{(\delta-1)/2} \operatorname{ch} bx \left\{ \frac{\operatorname{sh}(c\sqrt{a^2 - x^2})}{\operatorname{ch}(c\sqrt{a^2 - x^2})} \right\}}{\operatorname{ch}(c\sqrt{a^2 - x^2})} \, dx = \frac{\pi a^{\delta} c^{\delta}}{2(b^2 + c^2)^{\delta/2}} I_{\delta}(a\sqrt{b^2 + c^2})$$

$$[\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$2. \int_0^a \frac{\operatorname{ch} bx}{(a^2 - x^2)^{3/4}} \left\{ \frac{\operatorname{sh}(c\sqrt{a^2 - x^2})}{\operatorname{ch}(c\sqrt{a^2 - x^2})} \right\} \, dx =$$

$$= \left(\frac{\pi}{2} \right)^{3/2} \sqrt{c} I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} - b) \right] I_{\pm 1/4} \left[\frac{a}{2} (\sqrt{c^2 + b^2} + b) \right].$$

$$3. \int_0^a \sqrt{a^2 - x^2} \operatorname{ch}(c\sqrt{a^2 - x^2}) \operatorname{ch} bx \, dx =$$

$$= \frac{\pi a^2 c^2}{2(b^2 + c^2)} \left[I_0(a\sqrt{b^2 + c^2}) + \frac{b^2 - c^2}{ac^2 \sqrt{b^2 + c^2}} I_1(a\sqrt{b^2 + c^2}) \right].$$

$$4. \int_0^a \frac{\operatorname{ch}(c\sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \operatorname{ch} bx \, dx =$$

$$= \left(\frac{\pi}{2} \right)^{3/2} \sqrt{b} I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] I_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right].$$

2.4.9. Интегралы, содержащие $\operatorname{th} ax$, $\operatorname{cth} ax$ и разности гиперболических и алгебраических функций.

$$1. \int_0^{\infty} x^{\alpha-1} \operatorname{th} ax \, dx = 2^{1-\alpha} a^{-\alpha} (2^{1-\alpha} - 1) \Gamma(\alpha) \zeta(\alpha) \quad [\operatorname{Re} a > 0; -1 < \operatorname{Re} \alpha < 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\operatorname{th} ax - 1}{\operatorname{cth} ax - 1} \right\} \, dx = 2^{1-\alpha} a^{-\alpha} \Gamma(\alpha) \zeta(\alpha) \left\{ \frac{2^{1-\alpha} - 1}{1} \right\} \quad [a > 0; \operatorname{Re} \alpha > \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}].$$

$$3. \int_0^{\infty} \frac{x}{(x^2 + 1)^{\rho}} \operatorname{th} \frac{\pi x}{2} \, dx = 2^{-2\rho-2} \zeta(2\rho-3) \quad [\operatorname{Re} \rho > 1].$$

$$4. \int_0^{\infty} \frac{\operatorname{th}(cx/2)}{x \operatorname{ch} cx} \, dx = \ln 2 \quad [c > 0].$$

$$5. \int_0^{\infty} \frac{\operatorname{ch} bx - 1}{x \operatorname{sh} cx} \, dx = -\ln \cos \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Re} b|].$$

$$6. \int_0^{\infty} \left(\frac{a}{\operatorname{sh} ax} - \frac{b}{\operatorname{sh} bx} \right) \frac{dx}{x} = (b-a) \ln 2 \quad [\operatorname{Re} a, \operatorname{Re} b > 0].$$

$$7. \int_0^{\infty} x^{\alpha-1} \left(\frac{1}{\operatorname{sh} x} - \frac{1}{x} \right) \, dx = 2(1-2^{-\alpha}) \Gamma(\alpha) \zeta(\alpha) \quad [|\operatorname{Re} \alpha| < 1].$$

2.4.10. Интегралы от $x^{\sigma} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^{\sigma}$.

$$1. \int_0^{\infty} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} \, dx = \frac{1}{p^2 - b^2} \left\{ \frac{b}{p} \right\} \quad [\operatorname{Re} p > \operatorname{Re} b].$$

$$2. \int_0^{\infty} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^2 \, dx = \frac{1}{p(p^2 - 4b^2)} \left\{ \frac{2b^2}{p^2 - 2b^2} \right\} \quad [\operatorname{Re} p > 2|\operatorname{Re} b|].$$

3. $\int_0^{\infty} e^{-px} \operatorname{sh}^n bx dx = b^n p^{1-n} n! \prod_{k=0}^{[n/2]} [p^2 - b^2(2k+1)^2]^{-1/2}$
 $[2\sigma+1-(-1)^n; \operatorname{Re} p > n | \operatorname{Re} b|].$
4. $\int_0^{\infty} e^{-px} \operatorname{sh}^{v-1} bx dx = 2^{-v} b^{-1} \Gamma\left(v, \frac{p}{2b} + \frac{1-v}{2}\right)$
 $[\operatorname{Re} v, \operatorname{Re} b > 0; \operatorname{Re} p > \operatorname{Re}(vb - \frac{1}{2})].$
5. $\int_0^{\infty} e^{-px} \operatorname{ch}^n bx dx = 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \frac{p}{p^2 - (n-2k)^2 b^2} + \frac{1+(-1)^n}{2} \left(\frac{n+1}{2}\right) \frac{1}{2^n p}$
 $[\operatorname{Re} p > n | \operatorname{Re} b|; n=1, 2, 3, \dots].$
6. $\int_0^{\infty} \frac{e^{-px}}{\operatorname{ch}^m cx} dx = I_n$
 $[\operatorname{Re} p > -n | \operatorname{Re} c|].$
 $I_1 = \frac{1}{c} \beta\left(\frac{p+c}{2c}\right), \quad I_2 = \frac{p}{c^2} \beta\left(\frac{p}{2c}\right) - \frac{1}{c},$
 $I_{2m+1} = \frac{2^{2m-2}}{c} B(m, m) - \frac{1}{2mc}.$
7. $\int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\Gamma(\alpha)}{2} [(p-b)^{-\alpha} - (p+b)^{-\alpha}]$
 $[\operatorname{Re} p > | \operatorname{Re} b|; \operatorname{Re} \alpha > -\frac{1}{2}].$
8. $\int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh} bx dx = \frac{1}{2} \ln \frac{p+b}{p-b}$
 $[\operatorname{Re} p > | \operatorname{Re} b|].$
9. $\int_0^{\infty} \frac{e^{-px}}{x} \operatorname{sh}^2 bx dx = \frac{1}{2} \ln \sqrt{1 - \frac{4b^2}{p^2}}$
 $[\operatorname{Re} p > 2 | \operatorname{Re} b|].$
10. $\int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^m dx =$
 $= 2^{\alpha} m \Gamma(\alpha) \sum_{k=0}^{[(m-1)/2]} (-1)^{[m/2]-k} \binom{m}{k} \{ [p-b(m-2k)]^{-\alpha} - [p+b(m-2k)]^{-\alpha} \} +$
 $+ \frac{1+(-1)^m}{2} 2^{-m} p^{-\alpha} \Gamma(\alpha) \left(\frac{n}{2}\right) \quad [\operatorname{Re} \alpha > -\frac{1}{2}; \operatorname{Re} p > m | \operatorname{Re} b|].$
11. $\int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{sh}^n bx dx = 2^{-n} n! \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(p-bn+2kb)^{n+1}}$
 $[\operatorname{Re} p > n | \operatorname{Re} b|].$
12. $\int_0^{\infty} \frac{e^{-px}}{x^{n+1}} \operatorname{sh}^n bx dx = I_n^*$
 $[\operatorname{Re} p > n | \operatorname{Re} b|; m \leq n; m, n=1, 2, 3, \dots].$

- $I_1^{2n} = 2^{-2n} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \ln [p^2 - 4(n-k)^2 b^2] - (-1)^n 2^{-2n} \binom{2n}{n} \ln p,$
 $I_1^{2n+1} = 2^{-2n-1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \ln \frac{p+b(2n-2k+1)}{p-b(2n-2k+1)}, \quad I_1^1 = \frac{1}{2} \ln \frac{p+b}{p-b},$
 $I_2^1 = \frac{b}{2} \ln \frac{p+2b}{p-2b} + \frac{p}{4} \ln \left(1 - \frac{4b^2}{p^2}\right),$
 $I_3^1 = \frac{p}{8} \ln \frac{p-3b}{p+3b} - \frac{3p}{4} \ln \frac{p-b}{p+b} + \frac{3b}{8} \ln \frac{p^2-b^2}{p^2-3b^2}.$
12. $\int_0^{\infty} x^{\alpha-1} \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{2^{1-\alpha}}{c^{\alpha}} \Gamma(\alpha) \zeta\left(\alpha, \frac{p+c}{2c}\right)$
 $[\operatorname{Re} \alpha > 1; \operatorname{Re} p > - | \operatorname{Re} c|].$
14. $\int_0^{\infty} x^{2n+1} \frac{e^{-cx}}{\operatorname{sh} cx} dx = \frac{\pi^{2n+2}}{2(n+1)! c^{2n+2}} |B_{2n+1}|$
 $[\operatorname{Re} c > 0].$
15. $\int_0^{\infty} x^{\alpha-1} \frac{e^{-px}}{\operatorname{sh} cx} dx = \frac{1}{8} \left(\frac{\pi}{c}\right)^{\alpha} - \frac{12}{c^{\alpha}} \sum_{k=1}^n \frac{1}{(2k-1)^{\alpha}}$
 $[\operatorname{Re} c > 0; n=1, 2, 3, \dots].$
16. $\int_0^{\infty} x^{\alpha-1} \frac{e^{-px}}{\operatorname{ch} cx} dx = 2^{1-2\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{1}{4} + \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{3}{4} + \frac{p}{4c}\right) \right]$
 $[\operatorname{Re} \alpha > 0; \operatorname{Re} p > - | \operatorname{Re} c|].$
17. $\int_0^{\infty} x^{2\alpha-1} \frac{e^{-px}}{\operatorname{ch} cx} dx = \frac{2^{1-\alpha}}{c^{2\alpha}} (1-2^{1-\alpha}) \Gamma(\alpha) \zeta(\alpha)$
 $[\operatorname{Re} \alpha, \operatorname{Re} c > 0].$
18. $\int_0^{\infty} x^{2n+1} \frac{e^{-cx}}{\operatorname{ch} cx} dx = \frac{1-2^{1-2n}}{2(n+1)! c^{2n+2}} \pi^{2n+2} |B_{2n+1}|$
 $[\operatorname{Re} c > 0].$
19. $\int_0^{\infty} \frac{e^{i\alpha x}}{\operatorname{ch} cx} dx = \frac{\pi}{c \operatorname{ch} [\alpha \pi / (2c)]}$
 $[\alpha, \operatorname{Re} c > 0].$
- 2.4.11. Интегралы от e^{-px} и гиперболических функций.
1. $\int_0^{\infty} e^{-px} |\operatorname{sh} bx| dx = \frac{b}{p^2 - b^2} \operatorname{ctg} \frac{p\pi}{2b}$
 $[\operatorname{Re} p > | \operatorname{Re} b|].$
2. $\int_0^{\infty} e^{-px} |\operatorname{ch} bx| dx = \frac{1}{p^2 - b^2} \left(p - b \operatorname{cosech} \frac{p\pi}{2b} \right)$
 $[\operatorname{Re} p > | \operatorname{Re} b|].$
3. $\int_0^{\infty} e^{-px} |\operatorname{sh} bx|^v dx = - \frac{\pi}{(v+1) 2^{v+1} b} \left[\operatorname{sh} \frac{p\pi}{2b} \beta \left(1 + \frac{v}{2} + \frac{p}{2b}, 1 + \frac{v}{2} - \frac{p}{2b} \right) \right]^{-1}$
 $[\operatorname{Re} p > \operatorname{Re} v | \operatorname{Re} b|].$

$$4. \int_0^{\infty} e^{-px} \operatorname{th} cx dx = \frac{1}{c} \beta\left(\frac{p}{2c}\right) - \frac{1}{p}$$

[c, Re p > 0]

$$5. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{1}{2c} \left[\psi\left(\frac{c+p+b}{2c}\right) - \psi\left(\frac{c+p-b}{2c}\right) \right]$$

[Re p > |Re b| - |Re c|]

$$6. \int_0^{\infty} e^{-cx} \frac{\operatorname{sh} bx}{\operatorname{sh} cx} dx = \frac{1}{b} - \frac{\pi}{2c} \operatorname{ctg} \frac{b\pi}{2c}$$

[2 Re c > |Re b|]

$$7. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} px}{\operatorname{ch}^2 px} dx = \frac{1 - \ln 2}{p}$$

[Re p > 0]

2.4.12. Интегралы от $A(x)$, e^{-px} и гиперболических функ.

$$1. \int_{-a}^a x^k \sqrt{a^2 - x^2} e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{a^{2k+1}}{2} [I_k(ap+ab) \mp I_k(ap-ab)]$$

[k = 0 или 1; a, Re p > 0; |arg b| < π]

$$2. \int_0^{\infty} \frac{(x+z)^n}{x} e^{-px} \operatorname{sh} bx dx = (-1)^n b p^{n-1} \sum_{k=0}^{\infty} \left(\frac{b}{2p}\right)^{2k} L_{n-2k-1}(\rho z)$$

[Re p > |Re b|; |arg z| < π]

$$3. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{th} cx dx = 2^{1-\alpha} c^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{p}{4c}\right) - \zeta\left(\alpha, \frac{2c+p}{4c}\right) \right] - p^{-\alpha} \Gamma(\alpha)$$

[Re p > 0; Re α > -1]

$$4. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{th} cx dx = \ln \frac{p}{4c} + 2 \ln \Gamma\left(\frac{p}{4c}\right) - 2 \ln \Gamma\left(\frac{p+2c}{4c}\right)$$

[c, Re p > 0]

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{cth} cx dx = \Gamma(\alpha) \left[2^{1-\alpha} c^{-\alpha} \zeta\left(\alpha, \frac{p}{2c}\right) - p^{-\alpha} \right]$$

[c, Re p > 0; Re α > 1]

$$6. \int_0^{\infty} x^{\alpha-1} e^{-px} \operatorname{cth} px dx = \frac{1}{p^{\alpha}} \left(\frac{\pi^2}{3} - 1 \right)$$

[Re p > 0]

$$7. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{sh} bx dx = \frac{1}{4} \ln \frac{p^2 - (a-b)^2}{p^2 - (a+b)^2}$$

[Re p > |Re a| + |Re b|]

$$8. \int_0^{\infty} \frac{1}{x^2} e^{-px} \operatorname{sh} ax \operatorname{sh} bx dx =$$

$$= \frac{a}{4} \ln \frac{(p+b)^2 - a^2}{(p-b)^2 - a^2} + \frac{b}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2} + \frac{p}{4} \ln \frac{p^2 - (a+b)^2}{p^2 - (a-b)^2}$$

[Re p > |Re a| + |Re b|]

$$9. \int_0^{\infty} \frac{1}{x} e^{-px} \operatorname{sh} ax \operatorname{ch} bx dx = \frac{1}{4} \ln \frac{(p+a)^2 - b^2}{(p-a)^2 - b^2}$$

[Re p > |Re a| + |Re b|; p^2 + a^2 > b^2]

$$10. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{sh} px} dx = \frac{1}{2} \ln \frac{b\pi}{p \sin(b\pi/p)}$$

[Re p > |Re b|]

$$11. \int_0^{\infty} \frac{1}{x} e^{-px} \frac{\operatorname{sh}^2 bx}{\operatorname{ch} 2bx} dx = \ln \Gamma\left(\frac{p+6b}{8b}\right) - \ln \Gamma\left(\frac{p+2b}{4b}\right) - \frac{1}{2} \ln \frac{p}{8b}$$

[Re p > 0]

$$12. \int_0^{\infty} \frac{1}{x} e^{-2px} \operatorname{th} \frac{px}{2} dx = 2 \ln \frac{\pi}{2\sqrt{2}}$$

[Re p > 0]

2.4.13. Интегралы от $x^{\alpha}(e^x+a)^{\rho}e^{-px}\left\{\frac{\operatorname{sh} bx}{\operatorname{ch} bx}\right\}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax}+1} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \Gamma(\alpha) \left\{ b^{-\alpha} i^{-\alpha-\delta} \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} + \right.$$

$$\left. + \frac{(2a)^{-\alpha}}{2} \left[\zeta\left(\alpha, \frac{a-b}{2a}\right) \mp \zeta\left(\alpha, \frac{a+b}{2a}\right) - \zeta\left(\alpha, -\frac{b}{2a}\right) \pm \zeta\left(\alpha, \frac{b}{2a}\right) \right] \right\}$$

[Re α > -δ; Re a > |Re b|; δ = {1/2}].

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax}-1} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{e^{-\alpha}}{2} \Gamma(\alpha) \left[\zeta\left(\alpha, \frac{a-b}{a}\right) \mp \zeta\left(\alpha, \frac{a+b}{a}\right) \right]$$

[Re α > {0/1}; Re a > |Re b|]

$$3. \int_0^{\infty} \frac{\operatorname{sh} bx}{e^{ax} \pm 1} dx = \frac{1}{2b} \pm \frac{\pi}{2a \sin(b\pi/a)} \left\{ \frac{1}{\cos(b\pi/2)} \right\}$$

[Re a > |Re b|]

$$4. \int_0^{\infty} \frac{x \operatorname{ch} bx}{e^{ax}-1} dx = \frac{\pi^2}{2a^2 \sin^2(b\pi/a)} - \frac{1}{2b^2}$$

[Re a > |Re b|]

$$5. \int_0^{\infty} \frac{\operatorname{sh}^2 bx}{e^{ax}-1} dx = -\frac{1}{2a} \left[C + \operatorname{Re} \psi\left(-\frac{2b}{a}\right) \right]$$

[Re a > 2|Re b|]

$$6. \int_0^{\infty} \frac{x^{-1} \operatorname{sh}^2 bx}{e^{ax}-1} dx = \frac{1}{4} \ln \left(\frac{a}{b\pi} \operatorname{tg} \frac{b\pi}{a} \right)$$

[Re a > 2|Re b|]

$$7. \int_0^{\infty} \frac{x^{-1} \operatorname{sh}^2 bx}{e^{ax}-1} dx = \frac{1}{4} \ln \left(\frac{a}{2b\pi} \sin \frac{2b\pi}{a} \right) \quad [\operatorname{Re} a > 2|\operatorname{Re} b|],$$

$$8. \int_0^{\infty} \frac{e^{-ax}}{(e^{-ax}+1)^2} \operatorname{ch} bx dx = \frac{\pi b}{2a^2 \sin(b\pi/a)} \quad [\operatorname{Re} a > |\operatorname{Re} b|],$$

$$9. \int_0^{\infty} \frac{\operatorname{sh} bx}{e^{ax}-e^{-ax}} dx = \frac{1}{2(c-a)} \left[\psi \left(\frac{a-b}{a-c} \right) - \psi \left(\frac{a+b}{a-c} \right) \right] \quad (\max(\operatorname{Re} a, \operatorname{Re} c) > |\operatorname{Re} b|),$$

$$10. \int_0^{\infty} \frac{x^{2n} \operatorname{sh} bx}{e^{mx}-e^{-(m-2)c}x} dx = \frac{\partial^{2n}}{\partial b^{2n}} \left[\pm \frac{\pi}{4c} \left\{ \lg \{b\pi/(2c)\} \right\} + \right. \\ \left. + \frac{1}{2b} \left\{ \frac{0}{1} \right\} - \sum_{k=1}^{(m-1)/2} \frac{b}{c^2 (2k-2\lambda)^2 - b^2} \right] \\ \left[\operatorname{Re}(mc-b) > 0; \operatorname{Re}[(m-2)c-b] > 0; \begin{cases} \lambda = 1/2, m=1, 3, 5, \dots \\ \lambda = 0, m=2, 4, 6, \dots \end{cases} \right]$$

$$11. \int_{-\infty}^{\infty} \frac{e^{2ix}}{(e^x+1)^{\nu}} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = I_{\nu}^{\pm} \quad [0 < \operatorname{Re} a + |\operatorname{Re} b| < \rho],$$

$$I_{\nu}^{\pm} = \frac{1}{\Gamma(\nu)} \left\{ -i \operatorname{Im} z \right\}, \quad z = \Gamma(a-b) \Gamma(\rho-a+b),$$

$$I_1^{\pm} = \mp \frac{\pi}{\sin(a-b)\pi \sin(a+b)\pi} \left\{ \frac{\cos a\pi \sin b\pi}{\sin a\pi \cos b\pi} \right\},$$

$$I_2^{\pm} = \mp \frac{\pi}{\sin(a-b)\pi \sin(a+b)\pi} \left[(1-a) \left\{ \frac{\cos a\pi \sin b\pi}{\sin a\pi \cos b\pi} \right\} + b \left\{ \frac{\sin a\pi \cos b\pi}{\cos a\pi \sin b\pi} \right\} \right],$$

$$I_n^{\pm} = \frac{1}{n-1} \left[(n-1-a) I_{n-1}^{\pm} - b I_{n-1}^{\mp} \right] \quad [n=2, 3, 4, \dots].$$

2.4.14. Интегралы от $x^a e^{-px} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}^{\sigma}$

$$1. \int_a^{\infty} e^{-px} (\operatorname{ch} x - \operatorname{ch} a)^{\nu-1} dx = -i \sqrt{\frac{2}{\pi}} \frac{e^{va}}{e^{va}} \Gamma(\nu) \operatorname{sh}^{\nu-1/2} a Q_{\nu-1/2}^{\nu}(\operatorname{ch} a) \\ [a, \operatorname{Re} \nu > 0; \operatorname{Re} p > |\operatorname{Re} \nu - 1|].$$

$$2. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} x + \operatorname{ch} a} dx = \frac{1}{\operatorname{sh} a} \left[\frac{1}{\rho} + 2\rho \sum_{k=1}^{\infty} \frac{(-1)^k e^{-ak}}{\rho^2 - k^2} \right] \\ [a > 0; \operatorname{Re} p > -1; \text{см. 5.2.5}].$$

$$3. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} x - \cos a} dx = -\frac{2}{\sin a} \sum_{k=1}^{\infty} \frac{\sin ak}{\rho + ak} \quad [\operatorname{Re} p > -1; a \neq \pm 2\pi n; \text{см. 5.1.3}].$$

$$4. \int_0^{\infty} \frac{x^{2n-1} e^{-px}}{\operatorname{ch} px + \cos a} dx = \left(\frac{2\pi}{p} \right)^{2n} (\sin a \sin a\pi)^{-1} \left[\cos \left(\frac{2a + a\pi}{2} \right) \times \right. \\ \left. \times \zeta \left(1 - \alpha, \frac{\pi + a}{2\pi} \right) - \cos \left(\frac{2a - a\pi}{2} \right) \zeta \left(1 - \alpha, \frac{\pi - a}{2\pi} \right) \right] \quad [\operatorname{Re} \alpha, \operatorname{Re} p > 0; |a| < \pi].$$

$$5. \int_0^{\infty} \frac{x^{2n-1} e^{-px}}{\operatorname{ch} px - \cos a} dx = \frac{2\Gamma(a)}{p^a \sin a} \sum_{k=1}^{\infty} \frac{\sin ak}{(k+1)^a} \\ [\operatorname{Re} \alpha, \operatorname{Re} p > 0; a \neq \pm 2\pi n; \text{см. 5.4.3}].$$

$$6. \int_0^{\infty} x^{2n+1} \frac{e^{-px} - \cos a}{\operatorname{ch} px - \cos a} dx = I_n \quad [\operatorname{Re} p > 0],$$

$$I_n = p^{-2} \left(\frac{a\pi}{2} - \frac{a^2}{2} - \frac{\pi^2}{3} \right), \quad I_n = 2(2n+1)! p^{-2n-2} \sum_{k=1}^{\infty} \frac{\cos ak}{k^{2n+2}} \quad [\text{см. 5.4.2}].$$

$$7. \int_0^{\infty} \frac{e^{px} + \cos a}{(\operatorname{ch} px + \cos a)^2} dx = \frac{1}{p} \left(\frac{a}{\sin a} + \frac{1}{1 + \cos a} \right) \quad [\operatorname{Re} p > 0; |a| < \pi].$$

$$8. \int_0^{\infty} e^{-px} (\operatorname{ch} cx - 1)^{\nu} dx = 2^{-\nu-1} \Gamma \left(\frac{\rho}{c} - \nu, 2\nu + 1 \right) \\ [\operatorname{Re} c > 0, 2 \operatorname{Re} \nu > -1; \operatorname{Re} p > \operatorname{Re}(c\nu)].$$

$$9. \int_{-\infty}^{\infty} \frac{e^{-ibx}}{\operatorname{sh} x + \operatorname{sh} a} dx = -\frac{i\pi e^{iab}}{\operatorname{sh} b\pi \operatorname{ch} a} (\operatorname{ch} b\pi - e^{2iab}) \quad [a, b > 0].$$

2.4.15. Интегралы от $x^a e^{-ax-cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}$

$$1. \int_0^{\infty} x^{2n-1} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \\ = \frac{(2n)^{-1/2}}{2} \Gamma(a) \exp \left(\frac{b^2}{8a} \right) \left[D_{-a} \left(-\frac{b}{\sqrt{2a}} \right) \mp D_{-a} \left(\frac{b}{\sqrt{2a}} \right) \right] \quad [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\left\{ \frac{1}{0} \right\}]$$

$$2. \int_0^{\infty} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp \left(\frac{b^2}{4a} \right) \left\{ \frac{\operatorname{erf}(-b/(2\sqrt{a}))}{1} \right\} \quad [\operatorname{Re} a > 0].$$

$$3. \int_0^{\infty} x^{2n+\delta} e^{-ax} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = (-1)^{n+\delta} \sqrt{\frac{\pi}{2}} 2^{-2n-1-\delta} a^{-n-1/2-\delta/2} \times \\ \times \exp \left(\frac{b^2}{4a} \right) H_{2n+\delta} \left(\frac{i\delta b}{2\sqrt{a}} \right) \quad [\operatorname{Re} a > 0, \delta = \left\{ \frac{1}{0} \right\}].$$

$$4. \int_0^{\infty} x^2 e^{-ax^2} \operatorname{sh} bx dx = \frac{\sqrt{\pi}}{8a^{3/2}} (2a + b^2) \exp\left(\frac{b^2}{4a}\right) \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) - \frac{b}{4a^2} \quad [\operatorname{Re} a > 0].$$

$$5. \int_0^{\infty} \frac{1}{x} e^{-ax^2} \operatorname{sh} bx dx = \frac{\pi}{2} \operatorname{erf}\left(\frac{ib}{2\sqrt{a}}\right) \quad [\operatorname{Re} a > 0].$$

$$6. \int_0^{\infty} x^{1/2} e^{-ax^2} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\pi}{4} \left(\frac{b}{2a}\right)^{3/2} \exp\left(\frac{b^2}{8a}\right) \left[I_{-1/4}\left(\frac{b^2}{8a}\right) + I_{\pm 3/4}\left(\frac{b^2}{8a}\right) \right] \quad [\operatorname{Re} a > 0].$$

$$7. \int_0^{\infty} x^{-1/2} e^{-ax^2} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\pi}{2} \sqrt{\frac{b}{2a}} \exp\left(\frac{b^2}{8a}\right) I_{\pm 1/4}\left(\frac{b^2}{8a}\right) \quad [\operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} e^{-ax^2 - cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{1}{4} \sqrt{\frac{\pi}{a}} \left\{ \exp\left[\frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp \exp\left[\frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\} \quad [\operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} x^{2n-1} e^{-ax^2 - cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\Gamma(n)}{2(2n)^{n/2}} \exp\left(\frac{b^2 + c^2}{8a}\right) \left[\exp\left(-\frac{bc}{4a}\right) D_{-n}\left(\frac{c-b}{\sqrt{2a}}\right) \mp \exp\left(\frac{bc}{4a}\right) D_{-n}\left(\frac{c+b}{\sqrt{2a}}\right) \right] \quad [\operatorname{Re} a > 0; \operatorname{Re} z > -\frac{1}{2}].$$

$$10. \int_0^{\infty} x e^{-ax^2 - cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = -\frac{1}{8a} \sqrt{\frac{\pi}{a}} \left\{ (c-b) \exp\left[\mp \frac{(c-b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c-b}{2\sqrt{a}}\right) \mp (c+b) \exp\left[\mp \frac{(c+b)^2}{4a}\right] \operatorname{erfc}\left(\frac{c+b}{2\sqrt{a}}\right) \right\} + \left\{ \frac{0}{(2a) \cdot 1} \right\} \quad [\operatorname{Re} a > 0].$$

$$11. \int_0^{\infty} x^{\delta} \frac{e^{-ax^2}}{x^2 + z^2} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\pi}{4} \left(\frac{1}{x^2}\right) e^{az^2} \left[2 \left\{ \frac{\operatorname{sh} bz}{\operatorname{ch} bz} \right\} - \operatorname{erf}\left(z\sqrt{a} + \frac{ib}{2\sqrt{a}}\right) \pm \operatorname{erf}\left(z\sqrt{a} - \frac{ib}{2\sqrt{a}}\right) \right] \quad [\operatorname{Re} a, \operatorname{Re} z > 0; \delta = \left\{ \frac{1}{2} \right\}].$$

$$12. \int_0^{\infty} e^{-ax^2} \operatorname{sh}((2n+1)x) \left\{ \frac{\operatorname{sh} x}{\operatorname{ch} x} \right\} dx = \left\{ \frac{1}{-i} \right\} \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^n (\pm 1)^k \exp\left(\pm \frac{b^2}{a}\right) \right] \quad [\operatorname{Re} a > 0].$$

$$13. \int_0^{\infty} e^{-ax^2 - cx} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 + c^2}{4a}\right) \left\{ \frac{\operatorname{sh}(b/2a)}{\operatorname{ch}(b/2a)} \right\} \quad [\operatorname{Re} a > 0].$$

2.4.16. Интегралы от $A(x) \exp(-p\sqrt{x^2+z^2}) \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}$.

$$1. \int_0^{\infty} \exp(-p\sqrt{x^2+z^2}) \operatorname{ch} bx dx = \frac{p\pi}{\sqrt{p^2-b^2}} K_1(z\sqrt{p^2-b^2}) \quad [\operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} z > 0].$$

$$2. \int_0^{\infty} x^{2n-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{sh} bx dx = J_{2n} \quad [\operatorname{Re} z > -1; \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|].$$

$$J_{2n} = z \frac{\partial^{2n-2}}{\partial b^{2n-2}} \left[\frac{b}{\sqrt{p^2-b^2}} K_1(z\sqrt{p^2-b^2}) \right].$$

$$J_4 = \frac{b^2 z^3}{(p^2-b^2)^2} (3p^2+b^2) K_2(z\sqrt{p^2-b^2}) + \frac{b^2 z^3}{(p^2-b^2)^{3/2}} K_1(z\sqrt{p^2-b^2}).$$

$$3. \int_0^{\infty} x^{2n-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \operatorname{ch} bx dx = I_{2n} \quad [\operatorname{Re} z, \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|].$$

$$I_{2n+1} = \frac{\partial^{2n}}{\partial b^{2n}} K_0(z\sqrt{p^2-b^2}).$$

$$I_3 = \frac{z(p^2+b^2)}{(p^2-b^2)^{3/2}} K_1(z\sqrt{p^2-b^2}) + \frac{b^2 z^3}{p^2-b^2} K_0(z\sqrt{p^2-b^2}).$$

$$4. \int_0^{\infty} x^{1-\delta} (\sqrt{x^2+z^2}-z) \pm 1/2 \exp(-p\sqrt{x^2+z^2}) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = b^{\delta} \sqrt{\frac{\pi}{2}} \frac{(\sqrt{p^2-b^2}+p)^{\mp 1/2}}{\sqrt{p^2-b^2}} \exp(-\pi\sqrt{p^2-b^2}) \quad [\operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Re} b|; \delta = \left\{ \frac{1}{2} \right\}].$$

2.4.17. Интегралы от $x^{\alpha} f(x) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}$.

$$1. \int_0^{\infty} e^{-ax^2} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{z}{2\alpha} [e^{az^2/2} S_0, 1/2(z\sqrt{2\alpha}/4) \mp S_0, 1/2(z)] \quad [z = 2(b/3)^{2/3} a^{-1/3}; \operatorname{Re} a > 0].$$

$$2. \int_0^{\infty} x^{2-1} e^{-px^2 - q/x} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = -q^{\alpha/2} [(p-b)^{-\alpha/2} K_{\alpha}(2\sqrt{q}\sqrt{p-b}) \mp (p+b)^{-\alpha/2} K_{\alpha}(2\sqrt{q}\sqrt{p+b})] \quad [\operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} q > 0].$$

$$3. \int_0^{\infty} x^{-k/2} e^{-px^2 - q/x} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \frac{\sqrt{\pi}}{2q^{(k-1)/4}} [(p-b)^{(k-3)/4} \exp(-2\sqrt{q}\sqrt{p-b}) \mp (p+b)^{(k-3)/4} \exp(-2\sqrt{q}\sqrt{p+b})] \quad [k=1 \text{ или } 3; \operatorname{Re} p > |\operatorname{Re} b|; \operatorname{Re} q > 0].$$

$$4. \int_{-\infty}^{\infty} \exp(ax - e^x) \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \mp \frac{1}{\Gamma(a-b)} \left\{ \frac{\sin t}{\cos t} \right\} \\ \left[t = \ln \frac{\Gamma(a-b)}{\Gamma(a-b)}, \operatorname{Re} a > |\operatorname{Re} b| \right].$$

2.4.18. Интегралы от $f(x) \exp \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} dx = \\ = \frac{\pi}{2} \left(\operatorname{tg} \frac{b\pi}{2} \right)^{-1} \left[J_b(a) - J_b(a) \right] - \frac{\pi}{2} [E_b(a) + Y_b(a)] = b^{\delta} S_{-\delta, b}(a) \\ [\operatorname{Re} a > 0; b - \text{вещное}].$$

$$2. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} \operatorname{ch} x dx = \frac{b^{\delta}}{a} S_{1-\delta, b}(a) \quad [\operatorname{Re} a > 0].$$

$$3. \int_0^{\infty} e^{-a \operatorname{sh} x} \left\{ \frac{\operatorname{sh} bx}{\operatorname{ch} bx} \right\} \operatorname{sh}^{-1/2} x dx = \\ = \frac{\pi \sqrt{a}}{8i^{\delta}} \left[e^{\pi b i/2} H_{(1+b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1-b)/2}^{(2)} \left(\frac{a}{2} \right) - e^{-\pi b i/2} H_{(1-b)/2}^{(1)} \left(\frac{a}{2} \right) H_{(1+b)/2}^{(2)} \left(\frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0].$$

$$4. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{ch} bx dx = K_b(a) \quad [\operatorname{Re} a > 0].$$

$$5. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} bx \operatorname{sh} x dx = \frac{b}{a} K_b(a) \quad [\operatorname{Re} a > 0].$$

$$6. \int_0^{\infty} e^{-a \operatorname{ch} x} \left\{ \frac{\operatorname{sh}(x/2) \operatorname{sh} bx}{\operatorname{ch}(x/2) \operatorname{ch} bx} \right\} dx = \mp \left\{ \begin{matrix} i \operatorname{Im} K_{1/2-b}(a) \\ \operatorname{Re} K_{1/2-b}(a) \end{matrix} \right\} \\ [a, b > 0].$$

$$7. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx}{\operatorname{ch} x} dx = \sqrt{\frac{a}{2\pi}} K_{b/2+1/4} \left(\frac{a}{2} \right) K_{b/2-1/4} \left(\frac{a}{2} \right) \quad [\operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} e^{-a \operatorname{ch} x} \frac{\operatorname{ch} bx \operatorname{sh} x}{\sqrt{\operatorname{ch} x} \sqrt{\operatorname{ch} x - 1}} dx = K_b(a) \quad [\operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} e^{-a \operatorname{sh} x - p x} dx = \frac{\pi}{\operatorname{sh} p \pi} [J_p(a) - J_p(a)] \quad [\operatorname{Re} a > 0].$$

$$10. \int_0^{\infty} e^{-a \operatorname{sh} x - p x} \frac{dx}{\operatorname{sh} x} = \\ = \frac{\pi}{2} \sqrt{\frac{a}{2}} \left[J_{1/4-p/2} \left(\frac{a}{2} \right) J_{-1/4-p/2} \left(\frac{a}{2} \right) + Y_{1/4-p/2} \left(\frac{a}{2} \right) Y_{-1/4-p/2} \left(\frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0].$$

$$11. \int_0^{\infty} e^{-a \operatorname{ch} x - p x} dx = \frac{1}{\operatorname{sh} p \pi} \left[\int_0^{\pi} e^{a \cos \theta} \cos p \theta d\theta - \pi J_p(a) \right] \quad [\operatorname{Re} a > 0].$$

$$12. \int_{-\infty}^{\infty} e^{-a \operatorname{ch} x - p x} dx = 2K_p(a) \quad [\operatorname{Re} a > 0].$$

$$13. \int_{-\infty}^{\infty} e^{i a \operatorname{sh} x - p x} dx = 2e^{-p \pi i/2} K_p(a) \quad [|\operatorname{Re} p| < 1; a > 0].$$

$$14. \int_{-\infty}^{\infty} e^{i a \operatorname{ch} x - p x} dx = \pm \pi (e^{\pm p \pi i/2} H_{p/2}^{(3/2 \mp 1/2)}(a)) \quad [\pm \operatorname{Re}(i a) < 0].$$

2.4.19. Интегралы от $A(x) e^{-p x} \left\{ \frac{\operatorname{sh} \sqrt{ax^2 + bx + c}}{\operatorname{ch} \sqrt{ax^2 + bx + c}} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{e^{-p x}}{\sqrt{x} \sqrt{a^2 - x^2}} \operatorname{ch}(c \sqrt{ax^2 + bx + c}) dx = \\ = -\frac{\pi}{2} \sqrt{a p} [I_{1/4}(z_+) I_{1/4}(z_-) - I_{-1/4}(z_+) I_{-1/4}(z_-)] \\ [2z_{\pm} = \sqrt{c^2 + p^2} \pm c; a, c > 0].$$

$$2. \int_0^a \frac{e^{-p x}}{\sqrt{ax - x^2}} \operatorname{ch}(c \sqrt{ax - x^2}) dx = \pi e^{-a p/2} I_0 \left(\frac{a}{2} \sqrt{p^2 + c^2} \right) \quad [a, c > 0].$$

$$3. \int_a^{\infty} x^{\delta} e^{-p x} \operatorname{sh}(c \sqrt{x^2 - a^2}) dx = \frac{a^{1+\delta} c p^{\delta}}{(p^2 - c^2)^{(1+\delta)/2}} K_{1+\delta}(a \sqrt{p^2 - c^2}) \\ [\delta = 0 \text{ или } 1; a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

$$4. \int_a^{\infty} \frac{e^{-p x}}{\sqrt{x+a}} \left\{ \frac{\operatorname{sh}(c \sqrt{x^2 - a^2})}{\operatorname{ch}(c \sqrt{x^2 - a^2})} \right\} dx = \\ = \sqrt{\frac{\pi}{2}} \frac{c^{\delta}}{\sqrt{p^2 - c^2}} (V \sqrt{p^2 - c^2} + p)^{-1/2} \exp(-a \sqrt{p^2 - c^2}) \\ [a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

$$5. \int_a^{\infty} \frac{e^{-p x}}{\sqrt{x-a}} \operatorname{ch}(c \sqrt{x^2 - a^2}) dx = \sqrt{\frac{\pi}{2}} \sqrt{\frac{p + \sqrt{p^2 - c^2}}{p^2 - c^2}} \exp(-a \sqrt{p^2 - c^2}) \\ [a > 0; \operatorname{Re} p > |\operatorname{Re} c|].$$

6. $\int_a^\infty (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx =$
 $= \frac{2^{\alpha+1/2}}{a\sqrt{c}} \Gamma\left(\alpha + \frac{\delta}{2}\right) W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-)$
 $[z_\pm = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
7. $\int_a^\infty \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \frac{\pi}{V 2a} e^{ap} \operatorname{erfc}(z_+) \left\{ \frac{\operatorname{erf}(z_-)}{1} \right\}$
 $[z_\pm = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
8. $\int_a^\infty \frac{e^{-px}}{(x^2-a^2)^{3/4}} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[2z_\pm = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
9. $\int_a^\infty \frac{e^{-px}}{\sqrt{x}\sqrt{x^2-a^2}} \operatorname{ch}(c\sqrt{x^2-a^2}) dx = \frac{1}{\sqrt{2\pi\rho}} K_{1/4}(z_+) K_{1/4}(z_-)$
 $[2z_\pm = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
10. $\int_a^\infty \frac{x e^{-px}}{\sqrt{x^2-a^2}} \operatorname{ch}(c\sqrt{x^2-a^2}) dx = \frac{a^2 \rho^2}{(\rho^2 - c^2)^{3/2}} K_0(a\sqrt{\rho^2 - c^2})$
 $[a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
11. $\int_a^\infty \frac{(x+\sqrt{x^2-a^2})^\nu \mp (x-\sqrt{x^2-a^2})^\nu}{\sqrt{x^2-a^2}} e^{-px} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx =$
 $= a^\nu \left[\left(\frac{\rho+c}{\rho-c} \right)^{\nu/2} \mp \left(\frac{\rho+c}{\rho-c} \right)^{-\nu/2} \right] K_\nu(a\sqrt{\rho^2 - c^2})$
 $[a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$
12. $\int_0^\infty \frac{e^{-px}}{\sqrt{x}\sqrt{x^2+z^2}} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+z^2})}{\operatorname{sh}(b\sqrt{x^2+z^2})} \right\} dx =$
 $= \frac{\pi i^\delta}{4} V \pi p \left[\left\{ \frac{J_{1/4}(z_+)}{Y_{1/4}(z_+)} \right\} Y_{1/4}(z_-) \mp \left\{ \frac{Y_{1/4}(z_+)}{J_{1/4}(z_+)} \right\} J_{1/4}(z_-) \right]$
 $[2z_\pm = z(\rho \pm \sqrt{\rho^2 - b^2} \pm b); \operatorname{Re} z > 0; \operatorname{Re} \rho > |\operatorname{Re} b|].$
13. $\int_0^\infty e^{-px} \operatorname{sh}(b\sqrt{x^2+xz}) dx = \frac{bz}{2\sqrt{\rho^2 - b^2}} e^{pz/2} K_1\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right)$
 $[\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
14. $\int_0^\infty \frac{1}{\sqrt{x}} e^{-px} \operatorname{ch}(b\sqrt{x^2+xz}) dx =$
 $= \sqrt{\frac{\pi}{2}} \sqrt{\frac{\rho + \sqrt{\rho^2 - b^2}}{\rho^2 - b^2}} \exp\left[\frac{z}{2}(\rho - \sqrt{\rho^2 - b^2})\right]$
 $[\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$

15. $\int_0^\infty \frac{e^{-px}}{\sqrt{x+z}} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx =$
 $= \sqrt{\frac{\pi}{2}} b^\delta (\rho + \sqrt{\rho^2 - b^2})^{\pm 1/2} (\rho^2 - b^2)^{-1/2} \exp\left[\frac{z}{2}(\rho - \sqrt{\rho^2 - b^2})\right]$
 $[\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
16. $\int_0^\infty x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx =$
 $= \frac{2^{\alpha+1/2}}{z\sqrt{b}} \Gamma\left(\alpha + \frac{\delta}{2}\right) e^{pz/2} W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-)$
 $[2z_\pm = z(\rho \pm \sqrt{\rho^2 - b^2}); 2\operatorname{Re} \alpha > -\delta; \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
17. $\int_0^\infty \frac{e^{-px}}{\sqrt{x}(x+z)} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \frac{\pi}{V z} e^{pz/2} \operatorname{erfc}(z_+) \left\{ \frac{\operatorname{erf}(z_-)}{1} \right\}$
 $[2z_\pm = z(\rho \pm \sqrt{\rho^2 - b^2}); \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
18. $\int_0^\infty \frac{e^{-px}}{\sqrt{x}\sqrt{x+z}} \operatorname{ch}(b\sqrt{x^2+xz}) dx = e^{pz/2} K_0\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right)$
 $[\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
19. $\int_0^\infty (x^2+xz)^{-3/4} e^{-px} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{pz/2} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[2z_\pm = z(\rho \pm \sqrt{\rho^2 - b^2}); \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
20. $\int_0^\infty \frac{1}{\sqrt{x^2+xz}} \left[\left(\frac{z}{2} + x + \sqrt{x^2+xz} \right)^\nu \mp \left(\frac{z}{2} + x - \sqrt{x^2+xz} \right)^\nu \right] e^{-px} \times$
 $\times \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \left(\frac{z}{2} \right)^\nu \left[\left(\frac{\rho+b}{\rho-b} \right)^{\nu/2} \mp \left(\frac{\rho+b}{\rho-b} \right)^{-\nu/2} \right] e^{pz/2} K_\nu\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right)$
 $[\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$
- 2.4.20. Интегралы от $A(x) e^{ix} \left\{ \frac{\operatorname{sh}[cx/(a^2-x^2)]}{\operatorname{ch}[cx/(a^2-x^2)]} \right\}$.
- Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.
1. $\int_0^\infty \frac{x^{\alpha-1}}{\sqrt{a^2-x^2}} \exp\left(-b\frac{a^2+x^2}{a^2-x^2}\right) \left\{ \frac{\operatorname{sh}[cx/(a^2-x^2)]}{\operatorname{ch}[cx/(a^2-x^2)]} \right\} dx =$
 $= 2^{\delta+1} (2z-3)^{1/4} \sqrt{\frac{a}{c^2}} \Gamma\left(\frac{\alpha+\delta}{2}\right) (V 4a^2b^2-c^2+2ab)^{1/4} \times$
 $\times D_{-\alpha}\left(\sqrt{\frac{4a^2b^2-c^2+2ab}{a}}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{V 4a^2b^2-c^2-2ab}{2a}\right)$
 $[\operatorname{Re} \alpha > -\delta; 2ab > c > 0; a > 0].$

$$6. \int_0^{\infty} (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \\ = \frac{2^{\alpha+1/2}}{a\sqrt{c}} \Gamma\left(\alpha + \frac{\delta}{2}\right) W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-) \\ [z_{\pm} = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$7. \int_a^{\infty} \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \frac{\pi}{\sqrt{2a}} e^{\rho z} \operatorname{erfc}(z_+) \left\{ \frac{\operatorname{erf}(z_-)}{1} \right\} \\ [z_{\pm} = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$8. \int_a^{\infty} \frac{e^{-px}}{(x^2-a^2)^{3/4}} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4}(z_+) I_{\pm 1/4}(z_-) \\ [2z_{\pm} = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$9. \int_a^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x^2-a^2}} \operatorname{ch}(c\sqrt{x^2-a^2}) dx = \frac{1}{\sqrt{2\pi\rho}} K_{1/4}(z_+) K_{1/4}(z_-) \\ [2z_{\pm} = a(\rho \pm \sqrt{\rho^2 - c^2}); a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$10. \int_a^{\infty} \frac{x e^{-px}}{\sqrt{x^2-a^2}} \operatorname{ch}(c\sqrt{x^2-a^2}) dx = \frac{a^2 p^2}{(\rho^2 - c^2)^{3/2}} K_e(a\sqrt{\rho^2 - c^2}) \\ [e = 0 \text{ или } 1; a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$11. \int_a^{\infty} \frac{(x + \sqrt{x^2-a^2})^{\nu} \mp (x - \sqrt{x^2-a^2})^{\nu}}{\sqrt{x^2-a^2}} e^{-px} \left\{ \frac{\operatorname{sh}(c\sqrt{x^2-a^2})}{\operatorname{ch}(c\sqrt{x^2-a^2})} \right\} dx = \\ = a^{\nu} \left[\left(\frac{\rho+c}{\rho-c} \right)^{\nu/2} \mp \left(\frac{\rho+c}{\rho-c} \right)^{-\nu/2} \right] K_{\nu}(a\sqrt{\rho^2 - c^2}) \quad [a > 0; \operatorname{Re} \rho > |\operatorname{Re} c|].$$

$$12. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x^2+z^2}} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+z^2})}{\operatorname{sh}(b\sqrt{x^2+z^2})} \right\} dx = \\ = \frac{\pi i^{\delta}}{4} \sqrt{\pi p} \left[\left\{ \frac{J_{1/4}(z_+)}{Y_{1/4}(z_+)} \right\} Y_{1/4}(z_-) \mp \left\{ \frac{Y_{1/4}(z_+)}{J_{1/4}(z_+)} \right\} J_{1/4}(z_-) \right] \\ [2z_{\pm} = z(\rho \pm \sqrt{\rho^2 - b^2}); \operatorname{Re} z > 0; \operatorname{Re} \rho > |\operatorname{Re} b|].$$

$$13. \int_0^{\infty} e^{-px} \operatorname{sh}(b\sqrt{x^2+xz}) dx = \frac{bz}{2\sqrt{\rho^2 - b^2}} e^{\rho z/2} K_1\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right) \\ [\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$14. \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-px} \operatorname{ch}(b\sqrt{x^2+xz}) dx = \\ = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\rho + \sqrt{\rho^2 - b^2}}{\rho^2 - b^2}} \exp\left[\frac{z}{2}(\rho - \sqrt{\rho^2 - b^2})\right] \\ [\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$15. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x+z}} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \\ = \sqrt{\frac{\pi}{2}} b^{\delta} (\rho + \sqrt{\rho^2 - b^2})^{1/2} (\rho^2 - b^2)^{-1/2} \exp\left[\frac{z}{2}(\rho - \sqrt{\rho^2 - b^2})\right] \\ [\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$16. \int_0^{\infty} x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \\ = \frac{2^{\alpha+1/2}}{z\sqrt{b}} \Gamma\left(\alpha + \frac{\delta}{2}\right) e^{\rho z/2} W_{1/4-\alpha, 1/4}(z_+) M_{1/4-\alpha, \pm 1/4}(z_-) \\ [2z_{\pm} = z(\rho \pm \sqrt{\rho^2 - b^2}); 2\operatorname{Re} \alpha > -\delta; \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$17. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}(x+z)} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \frac{\pi}{\sqrt{z}} e^{\rho z} \operatorname{erfc}(z_+) \left\{ \frac{\operatorname{erf}(z_-)}{1} \right\} \\ [2z_{\pm} = z(\rho \pm \sqrt{\rho^2 - b^2}); \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$18. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}\sqrt{x+z}} \operatorname{ch}(b\sqrt{x^2+xz}) dx = e^{\rho z/2} K_0\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right) \\ [\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$19. \int_0^{\infty} (x^2+xz)^{-3/4} e^{-px} \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{\rho z/2} K_{1/4}(z_+) I_{\pm 1/4}(z_-) \\ [2z_{\pm} = z(\rho \pm \sqrt{\rho^2 - b^2}); \operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

$$20. \int_0^{\infty} \frac{1}{\sqrt{x^2+xz}} \left[\left(\frac{z}{2} + x + \sqrt{x^2+xz} \right)^{\nu} \mp \left(\frac{z}{2} + x - \sqrt{x^2+xz} \right)^{\nu} \right] e^{-px} \times \\ \times \left\{ \frac{\operatorname{sh}(b\sqrt{x^2+xz})}{\operatorname{ch}(b\sqrt{x^2+xz})} \right\} dx = \left(\frac{z}{2} \right)^{\nu} \left[\left(\frac{\rho+b}{\rho-b} \right)^{\nu/2} \mp \left(\frac{\rho+b}{\rho-b} \right)^{-\nu/2} \right] e^{\rho z/2} K_{\nu}\left(\frac{z}{2}\sqrt{\rho^2 - b^2}\right) \\ [\operatorname{Re} \rho > |\operatorname{Re} b|; |\arg z| < \pi].$$

2.4.20. Интегралы от $A(x) \operatorname{erf}(x) \left\{ \frac{\operatorname{sh}[cx/(a^2-x^2)]}{\operatorname{ch}[cx/(a^2-x^2)]} \right\}^{\pm 1}$

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{a^2-x^2}} \exp\left(-\delta \frac{a^2+x^2}{a^2-x^2}\right) \left\{ \frac{\operatorname{sh}[cx/(a^2-x^2)]}{\operatorname{ch}[cx/(a^2-x^2)]} \right\} dx = \\ = 2^{\delta+1} (\pi/2)^{1/4} \sqrt{\frac{a}{c^3}} \Gamma\left(\frac{\alpha+\delta}{2}\right) (V_{4a^2b^2-c^2+2ab})^{1/4} \times \\ \times I_{-\alpha}\left(\sqrt{\frac{V_{4a^2b^2-c^2+2ab}}{4a}}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{V_{4a^2b^2-c^2-2ab}}{2a}\right) \\ [\operatorname{Re} \alpha > -\delta; 2ab > c > 0; a > 0].$$

$$2. \int_a^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-\frac{b}{x^2-a^2}\right) \frac{\{\operatorname{sh}[cx/(x^2-a^2)]\}}{\{\operatorname{ch}[cx/(x^2-a^2)]\}} dx =$$

$$= \frac{a^{\alpha-1}}{\sqrt{c}} \frac{a^{\alpha-1}}{2^{1/2} \Gamma(1/2)} \exp\left(\frac{b}{2a^2}\right) \Gamma\left(\frac{1+\delta-\alpha}{2}\right) [V\sqrt{b^2-a^2c^2}+b]^{1/4} \times$$

$$\times D_{\alpha-1}\left(\frac{V\sqrt{b^2-a^2c^2}+b}{a}\right) M_{1/4-\alpha/2, \pm 1/4}\left(\frac{b-V\sqrt{b^2-a^2c^2}}{2a^2}\right)$$

[Re $\alpha < 1+\delta$; $b > ac > 0$; $a > 0$].

$$3. \int_a^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp\left(-\frac{b}{x^2-a^2}\right) \frac{\{\operatorname{sh}[cx/(x^2-a^2)]\}}{\{\operatorname{ch}[cx/(x^2-a^2)]\}} dx =$$

$$= 2^{\delta-(3\alpha+1)/4} a^{\alpha-3/4} c^{-1/2} \Gamma\left(\frac{1+\delta-\alpha}{2}\right) (2ab + V\sqrt{4a^2b^2-c^2})^{1/4} \times$$

$$\times D_{\alpha-1}\left(\sqrt{\frac{2ab + V\sqrt{4a^2b^2-c^2}}{a}}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{2ab - V\sqrt{4a^2b^2-c^2}}{2a}\right)$$

[Re $\alpha < 1+\delta$; $2ab > c > 0$; $a > 0$].

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-\frac{b}{x^2+z^2}\right) \frac{\{\operatorname{sh}[cx/(x^2+z^2)]\}}{\{\operatorname{ch}[cx/(x^2+z^2)]\}} dx =$$

$$= \frac{2^{2\delta-1/2}}{\sqrt{\pi c}} z^{\alpha-1/2} \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \Gamma\left(\frac{\alpha+\delta}{2}\right) \exp\left(-\frac{b}{2z^2}\right) \times$$

$$\times M_{1/4-\alpha/2, \pm 1/4}\left(\frac{b+V\sqrt{b^2+c^2z^2}}{2z^2}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{V\sqrt{b^2+c^2z^2}-b}{2z^2}\right)$$

[$-\delta < \operatorname{Re} \alpha < 1+\delta$; $b, c, \operatorname{Re} z > 0$].

$$5. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp\left(-\frac{b}{x^2+z^2}\right) \frac{\{\operatorname{sh}[cx/(x^2+z^2)]\}}{\{\operatorname{ch}[cx/(x^2+z^2)]\}} dx =$$

$$= \frac{2^{2\delta-1/2}}{\sqrt{\pi c}} z^{\alpha-1/2} \Gamma\left(\frac{\alpha+\delta}{2}\right) \Gamma\left(\frac{1-\alpha+\delta}{2}\right) \times$$

$$\times M_{1/4-\alpha/2, \pm 1/4}\left(\frac{V\sqrt{4b^2z^2+c^2+2bz}}{2z}\right) M_{\alpha/2-1/4, \pm 1/4}\left(\frac{V\sqrt{4b^2z^2+c^2-2bz}}{2z}\right)$$

[$-\delta < \operatorname{Re} \alpha < 1+\delta$; $b, c, \operatorname{Re} z > 0$].

2.4.21. Интегралы, содержащие $A(x)$, e^{jcx} , $\operatorname{sh} g(x)$, $\operatorname{ch} g(x)$.

$$1. \int_0^{\infty} e^{-px} \frac{\{\operatorname{sh} a[x]\}}{\{\operatorname{ch} a[x]\}} dx = \frac{e^p-1}{p(e^{2p}-2e^p \operatorname{ch} a+1)} \frac{\{\operatorname{sh} a\}}{e^p-\operatorname{ch} a} \quad [\operatorname{Re} p > |\operatorname{Re} a|]$$

$$2. \int_0^{\infty} e^{-px} \frac{\{\operatorname{sh} a[x]\}}{\{\operatorname{ch} a[x]\}} dx = \frac{e^p-1}{p(e^{2p}-2e^p \operatorname{ch} a+b^2)} \frac{\{b \operatorname{sh} a\}}{e^p-b \operatorname{ch} a}$$

[Re $p > \ln b + |\operatorname{Re} a|$; $b > 0$].

$$2. \int_0^{\infty} e^{-qx} \frac{\{\operatorname{sh} cx^2\}}{\{\operatorname{ch} cx^2\}} \operatorname{ch} bx dx =$$

$$= \frac{1}{2} \sqrt{\frac{1}{a^2-c^2}} e^{-aA} \left(B_+ \frac{\{\operatorname{sh} cA\}}{\{\operatorname{ch} cA\}} - B_- \frac{\{\operatorname{ch} cA\}}{\{\operatorname{sh} cA\}} \right) = \frac{V\pi}{2\sqrt{a^2-c^2}} e^{-aA} \frac{\{\operatorname{sh} C\}}{\{\operatorname{ch} C\}}$$

[$A = \frac{b^2}{4(c^2-a^2)}$; $B_{\pm} = \left\{ \frac{1}{2} (a \pm \sqrt{a^2-c^2}) \right\}^{1/2}$; $2C = \frac{1}{2} \ln \frac{a+c}{a-c} - A$; Re $a > |\operatorname{Re} c|$].

$$4. \int_0^{\infty} x^{-1/2} e^{-p \sqrt{x^2+q^2}} \frac{\{\operatorname{sh}(bx+c/x)\}}{\{\operatorname{ch}(bx+c/x)\}} dx =$$

$$= \frac{V\pi}{r} \exp\left(-rs \sqrt{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - rs \sqrt{\frac{(p-b)(q-c)}{(p+b)(q+c)}}\right) \frac{\{\operatorname{sh} d\}}{\{\operatorname{ch} d\}}$$

[$d = \frac{1}{4} \ln \frac{p+b}{p-b} + rs \left(\sqrt{\frac{(p+b)(q+c)}{(p-b)(q-c)}} - \sqrt{\frac{(p-b)(q-c)}{(p+b)(q+c)}} \right)$,
 $r = \sqrt{p^2-b^2}$, $s = \sqrt{q^2-c^2}$; Re $p > |\operatorname{Re} b|$; Re $q > |\operatorname{Re} c|$].

2.4.22. Интегралы, содержащие разности $A(x)$, e^{-px} , $\frac{\{\operatorname{sh} bx\}}{\{\operatorname{ch} bx\}}$.

$$1. \int_0^{\infty} e^{-px} \frac{1-\operatorname{ch} ax}{x} dx = \frac{1}{2} \ln\left(1-\frac{a^2}{p^2}\right) \quad [\operatorname{Re} p > |\operatorname{Re} a|]$$

$$2. \int_0^{\infty} e^{-px} \frac{1-\operatorname{ch} ax}{x^2} dx = -p \ln \frac{V\sqrt{p^2-a^2}}{p} - \frac{a}{2} \ln \frac{p+a}{p-a} \quad [\operatorname{Re} p > |\operatorname{Re} a|]$$

$$3. \int_0^{\infty} e^{-px} \frac{x-\operatorname{sh} x}{x^n} dx = I_n \quad [\operatorname{Re} p > 1]$$

$$I_1 = \frac{1}{p} - \frac{1}{2} \ln \frac{p+1}{p-1}, \quad I_2 = \frac{1}{2} \ln\left(1-\frac{1}{p^2}\right) + \frac{p}{2} \ln \frac{p+1}{p-1} - 1,$$

$$I_3 = \frac{1}{2} \left[p \ln\left(1-\frac{1}{p^2}\right) + \frac{p^2-1}{2} \ln \frac{p+1}{p-1} - p \right].$$

$$4. \int_0^{\infty} \frac{e^{-px}}{x} \left(1 - \frac{1}{\operatorname{ch} x}\right) dx = 2 \ln \Gamma\left(\frac{p+3}{4}\right) - 2 \ln \Gamma\left(\frac{p+1}{4}\right) - \ln \frac{p}{4} \quad [\operatorname{Re} p > 0]$$

$$5. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \frac{1}{\operatorname{sh} x}\right) dx = \psi\left(\frac{p+1}{2}\right) - \ln \frac{p}{2} \quad [\operatorname{Re} p > 0]$$

$$6. \int_0^{\infty} e^{-px} \left(\frac{1}{x} - \operatorname{cth} x\right) dx = \psi\left(\frac{p}{2}\right) - \ln \frac{p}{2} + \frac{1}{p} \quad [\operatorname{Re} p > 0]$$

$$7. \int_0^{\infty} x^{\alpha} e^{-px} (\operatorname{cth} cx - 1) dx = 2^{1-\alpha} c^{-\alpha} \Gamma(\alpha) \zeta\left(\alpha, 1 + \frac{p}{2c}\right)$$

[Re $\alpha > 1$; Re $p > -2$].

$$8. \int_0^{\infty} x^{-1} \frac{(1-e^{-x})^2}{\operatorname{ch} x + \cos(m\pi/n)} dx = \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \times \\ \times \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(n+(-1)^k k+1)/(rn), (n+(-1)^k k+1)/(rn), (k+2)/(rn), k/(rn))}{(n+(-1)^k k)/(rn), (n+(-1)^k k+2)/(rn), (k+1)/(rn), (k+1)/(rn))} \right] \\ \left[\begin{matrix} r=2, m+n=1, 3, 5, \\ r=1, m+n=2, 4, 6, \dots \end{matrix} \right].$$

$$9. \int_0^{\infty} x^n \frac{1-e^{-2mx}}{\operatorname{sh}^2 x} dx = J_n,$$

$$J_1 = \frac{m\pi^2}{6} - \sum_{k=1}^{m-1} \frac{m-k}{k^2}, \quad J_2 = m\zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3},$$

$$J_3 = \frac{1}{60} m\pi^4 - \frac{3}{2} \sum_{k=1}^{m-1} \frac{m-k}{k^4}.$$

$$10. \int_0^{\infty} x \frac{1-(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = \frac{m}{12} \pi^2 + \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^3}.$$

$$11. \int_0^{\infty} x^n \frac{1+(-1)^m e^{-2mx}}{\operatorname{ch}^2 x} dx = J_n,$$

$$J_2 = \frac{3m}{4} \zeta(3) - \sum_{k=1}^{m-1} \frac{m-k}{k^3}, \quad J_3 = \frac{7m\pi^4}{480} + \frac{3}{2} \sum_{k=1}^{m-1} (-1)^k \frac{m-k}{k^4}.$$

$$12. \int_{-\infty}^{\infty} x \frac{1-e^{ax}}{\operatorname{sh} x} dx = -\frac{\pi^2}{2} \operatorname{tg}^2 \frac{a\pi}{2} \quad [| \operatorname{Re} a | < 1].$$

$$13. \int_0^{\infty} \frac{e^{-ax} - e^{-cx} \operatorname{ch} bx}{x} dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2} \quad [\operatorname{Re} a > 0; \operatorname{Re} c > |\operatorname{Re} b|].$$

$$14. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x} \operatorname{ch} bx dx = \frac{1}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Re} b|].$$

$$15. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x^2} \operatorname{sh} bx dx = \frac{b}{2} \ln \frac{c^2 - b^2}{a^2 - b^2} + \frac{c}{2} \ln \frac{c+b}{c-b} - \frac{a}{2} \ln \frac{a+b}{a-b} \\ [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Re} b|].$$

$$16. \int_0^{\infty} \frac{1+x-e^x}{x \operatorname{sh} x} e^{-px} dx = 2 \ln \Gamma\left(\frac{p+1}{2}\right) - 2 \ln \Gamma\left(\frac{p}{2}\right) - \psi\left(\frac{p+1}{2}\right) \\ [\operatorname{Re} p > 0].$$

$$17. \int_0^{\infty} \frac{(1-e^{-ax})(1-e^{-(a+b)x})}{x \operatorname{sh} x} dx = 2a \ln 2 \quad [\operatorname{Re} a > -1].$$

$$18. \int_0^{\infty} \frac{1}{x} \left(ae^{-bx} - e^{-cx} \frac{\operatorname{sh} ax}{x} \right) dx = \frac{a}{2} \ln \frac{c^2 - a^2}{b^2} + \frac{c}{2} \ln \frac{c+a}{c-a} - a \\ [\operatorname{Re} b > 0; \operatorname{Re} c > |\operatorname{Re} a|].$$

$$19. \int_0^{\infty} \frac{1}{x} \frac{1 - \operatorname{ch} ax}{e^{2\pi cx} - e^{2\pi(n+1)cx}} dx = -\frac{1}{2} \ln \frac{2c \sin[a\pi/(2c)]}{a\pi} + \frac{1}{2} \sum_{k=1}^n \ln \left(1 - \frac{a^2}{4k^2 c^2} \right) \\ [a < 2(n+1)c; a < 2nc].$$

$$20. \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{\operatorname{ch} x - \cos(m\pi/n)} \frac{dx}{x} = \\ = \frac{2}{\sin(m\pi/n)} \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(n+b+(-1)^k k)/(rn), (a+k)/(rn)}{(n+a+(-1)^k k)/(rn), (b+k)/(rn)} \right] \\ [\operatorname{Re} a, \operatorname{Re} b > -1; \begin{matrix} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{matrix}].$$

$$21. \int_0^{\infty} \left[e^{-x} \operatorname{tg} \frac{m\pi}{2n} - \frac{e^{-ax} \sin(m\pi/n)}{\operatorname{ch} x + \cos(m\pi/n)} \right] \frac{dx}{x} = \\ = \operatorname{tg} \frac{m\pi}{2n} \ln(rn) + 2 \sum_{k=1}^{r(n-1)/2} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma \left[\frac{(a+n+(-1)^k k)/(rn)}{(a+k)/(rn)} \right] \\ [\operatorname{Re} a > -1; \begin{matrix} r=2, m+n=1, 3, 5, \dots \\ r=1, m+n=2, 4, 6, \dots \end{matrix}].$$

$$22. \int_0^{\infty} e^{-px} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{x^2} dx = J_n \quad [\operatorname{Re} p > |\operatorname{Re} a|, |\operatorname{Re} b|], \\ J_1 = \frac{1}{2} \ln \frac{p^2 - b^2}{p^2 - a^2}, \quad J_2 = \frac{p}{2} \ln \frac{p^2 - a^2}{p^2 - b^2} + \frac{b}{2} \ln \frac{p-b}{p+b} + \frac{a}{2} \ln \frac{p+a}{p-a}.$$

$$23. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - x \operatorname{ch} x}{x^3} dx = J_n \quad [\operatorname{Re} p > 1], \\ J_1 = \frac{1}{2} \ln \frac{p+1}{p-1} - \frac{p}{p^2-1}, \quad J_2 = 1 - \frac{p}{2} \ln \frac{p+1}{p-1}, \\ J_3 = \frac{1}{4} \left[(p^2-1) \ln \frac{p+1}{p-1} - 2p \right].$$

$$24. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} x - 2x \operatorname{ch} x}{x^3} \operatorname{sh} x dx = \frac{p}{4} \ln \frac{p^2-4}{p^2} \quad [\operatorname{Re} p > 2].$$

$$25. \int_0^{\infty} \frac{1}{x} \left(\frac{\operatorname{sh} ax}{\operatorname{sh} x} - ae^{-2x} \right) dx = \ln \left(\pi \cos \frac{a\pi}{2} \right) - 2 \ln \Gamma\left(\frac{a+1}{2}\right) \quad [| \operatorname{Re} a | < 1].$$

$$26. \int_0^{\infty} \frac{ae^{-bx} \operatorname{sh} cx - ce^{-hx} \operatorname{sh} ax}{x^2} dx = \frac{ac}{2} \ln \frac{b^2 - a^2}{b^2 - c^2} + \frac{ch}{2} \ln \frac{h+a}{h-a} - \frac{ab}{2} \ln \frac{b+a}{b-a}$$

[Re b > |Re c|; Re h > |Re a|]

$$27. \int_0^{\infty} \frac{1}{x} \frac{\operatorname{ch} ax - \operatorname{ch} bx}{e^{mx} - e^{-mx}} dx = \frac{1}{2} \left[\frac{\ln \cos [b\pi/(2c)] - \ln \cos [a\pi/(2c)]}{\ln \sin [b\pi/(2c)] - \ln \sin [a\pi/(2c)]} - \sum_{k=1}^{(m-1)/2} \ln \frac{c^2(2k-2\lambda)^2 - b^2}{c^2(2k-2\lambda)^2 - a^2} + \frac{0}{\ln(a/b)} \right]$$

[a < (m-2)c; b < (m-2)c; {λ = 1/2, m = 1, 3, 5, ...}, {λ = 0, m = 2, 4, 6, ...}]

$$28. \int_0^{\infty} (e^{-ax} - e^{-bx}) \operatorname{ch} cx dx = \frac{1}{b} - \frac{1}{a} + \frac{1}{c} \left[\psi\left(\frac{b}{2c}\right) - \psi\left(\frac{a}{2c}\right) \right]$$

[Re a, Re b > 0]

2.5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

2.5.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входят тригонометрические функции. Для их вычисления можно использовать следующие свойства.

Интегралы, содержащие тригонометрические функции, могут быть преобразованы к интегралам от показательной или гиперболических функций, если воспользоваться соответствующими формулами, приведенными в приложении I. Интегралы, содержащие функции $\sin^m bx$ и $\cos^m bx$, сводятся к интегралам от функций $\sin kbx$ и $\cos kbx$, если использовать формулы для степеней тригонометрических функций (см. приложение I). В ряде случаев интегралы от тригонометрических функций с помощью замены переменной сводятся к интегралам от алгебраических и других функций. Например,

$$\begin{aligned} \int_0^a \varphi(x) \sin^2 bx dx &= \frac{1}{2} \int_0^a \varphi(x) dx - \frac{1}{4} \int_0^a \varphi(x) e^{2ibx} dx - \frac{1}{4} \int_0^a \varphi(x) e^{-2ibx} dx = \\ &= - \int_0^a \varphi(x) \operatorname{sh}^2 ibx dx = \frac{1}{2} \int_0^a \varphi(x) (1 - \cos 2bx) dx = \\ &= \frac{1}{b} \int_0^{\sin ab} x^2 \varphi\left(\frac{\arcsin x}{b}\right) \frac{dx}{\sqrt{1-x^2}} \quad [ab < \pi/2] \end{aligned}$$

2.5.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие тригонометрические, а также степенную, алгебраические, показательную и гиперболические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j=1, 2, 3, \dots$, помещенных в приложение III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла.

2.5.2.1

В п. 2.5.2 приняты следующие обозначения: $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\begin{aligned} \delta &= \begin{cases} 1, & \text{если взят } \sin bx, \\ 0, & \text{если взят } \cos bx; \end{cases} \quad \eta = \begin{cases} (-1)^{n/2+1}, & \text{если взят } \sin bx, \\ 1, & \text{если взят } \cos bx; \end{cases} \\ \epsilon &= \begin{cases} 1/2, & \text{если взят } \sin^n bx, \quad n=1, 3, 5, \dots, \\ 0, & \text{если взят } \sin^n bx, \quad n=2, 4, 6, \dots, \\ 0, & \text{если взят } \cos^n bx, \quad n=1, 2, 3, \dots; \end{cases} \\ \xi &= r^{-1}(\alpha/2 + \gamma + h), \quad \bar{b} = by(n-2k)/2. \end{aligned}$$

$$1. \int_0^{\infty} x^{\alpha-1} (x^{2r} - y^{2r})^{\beta-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} dx = V_{31} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r(\beta-1)+\alpha} \times$$

$$\times y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, \beta\right)$$

$$\{E_{31} = \sqrt{\pi} 2^{-n} \frac{B(\beta, \xi)}{\Gamma(1/2+2\gamma+h)}, \quad u = \alpha + 2r(\beta-1),$$

$$a_1 = \xi, \quad b_1 = \beta + \xi, \quad c_1 = 1/2 + 2\gamma + h, \quad z = (-\bar{b}^2/p^2)\rho\}$$

[ν , Re $\beta > 0$; Re $\alpha > -\delta n$; $|\arg b| < \pi$]

$$2. \int_0^{\infty} x^{\alpha-1} (x^{2r} - y^{2r})^{\beta-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} dx =$$

$$= V_{31} + V_{32} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r(\beta-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, 1 - \frac{\alpha}{2r} - \beta\right)$$

$$\{E_{31} = \sqrt{\pi} 2^{-n} \frac{B(\beta, 1-\beta-\xi)}{\Gamma(1/2+2\gamma+h)}, \quad E_{32} = \sqrt{\pi} 2^{-n} \Gamma\left[\frac{\beta}{\beta-j}, \frac{1-b_1}{c_1}\right],$$

$$u = \alpha + 2r(\beta-1), \quad v_2 = 2r(1-\beta+j) - \alpha, \quad a_1 = \xi, \quad b_1 = \beta + \xi,$$

$$c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \beta + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \beta + j),$$

$$c_2 = (1 - \alpha)/2 + \gamma + r(1 - \beta + j), \quad z = (-\bar{b}^2/p^2)\rho\}$$

[ν , b , Re $\beta > 0$; Re $(\alpha + 2r\beta) < (1 - (-1)^n)/2 + 2r$]

$$3. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{\rho-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} dx =$$

$$= V_{31} + V_{32} + \frac{1+(-1)^n}{2} 2^{-n-1} y^{2r(\rho-1)+\alpha} \frac{n!}{r[(n/2)!]^2} B\left(\frac{\alpha}{2r}, 1 - \rho - \frac{\alpha}{2r}\right)$$

$$\{E_{31} = \sqrt{\pi} 2^{-n} \frac{B(\xi, 1-\rho-\xi)}{\Gamma(1/2+2\gamma+h)}, \quad E_{32} = \sqrt{\pi} 2^{-n} \Gamma\left[\frac{1-\rho+h}{1-\rho}, \frac{1-b_1}{c_1}\right],$$

$$u = 2r(\rho-1) + \alpha, \quad v_2 = 2r(1-\rho+j) - \alpha, \quad a_1 = \xi, \quad b_1 = \rho + \xi,$$

$$c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \rho + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \rho + j),$$

$$c_2 = (1 - \alpha)/2 + \gamma + r(1 - \rho + j), \quad z = (-1)^{\rho} q \bar{b}^{2\rho-2r}\rho\}$$

[$b > 0$; $2r|\arg y| < \pi$; Re $\alpha > -\delta n$; Re $(\alpha + 2r\rho) < (1 - (-1)^n)/2 + 2r$]

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{x^{2r} + y^{2r}} \frac{\{\sin bx\}^n}{\{\cos bx\}} dx = V_{31} + V_{32} - \frac{1+(-1)^n}{2} 2^{-n-1} y^{\alpha-2r} \frac{n! \pi}{r[(n/2)!]^2} \operatorname{ctg} \frac{\alpha \pi}{2r}$$

$$\{E_{31} = \frac{\pi^{3/2}}{2^{\alpha} r} \frac{\operatorname{ctg} \xi \pi}{\Gamma(1/2 + 2\gamma + h)}, \quad E_{32} = \frac{\sqrt{\pi}}{2^{\alpha}} \Gamma\left[\frac{1-a_2}{b_2}\right],$$

$$u = \alpha - 2r, \quad v_2 = 2r(1+j) - \alpha, \quad a_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \gamma + r(1+j) - \alpha/2, \\ b_2 = (1-\alpha)/2 + \gamma + r(1+j), \quad z = (-\tilde{b}^2/p^2)^p \\ [\nu, b > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re} \alpha < (1 - (-1)^n)/2 + 2\gamma]$$

$$5. \int_0^{\infty} x^{2\gamma-1} |x^{2\nu} - y^{2\nu}|^{\sigma-1} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{31} + V_{32} + \frac{1 + (-1)^n}{2} 2^{-n-1} y^{2\nu(\sigma-1)+\alpha} \frac{n!}{r[(n/2)!]^2} \frac{\sin[(\alpha + r\sigma)/(2r)]}{\sin(\pi\sigma/2)} \times \\ \times B\left(\frac{\alpha}{2r}, 1 - \sigma - \frac{\alpha}{2r}\right)$$

$$\{E_{31} = \frac{\sqrt{\pi} 2^{-n} \sin[(\sigma/2 + \xi)\pi] \Gamma(\xi, 1 - \sigma - \xi)}{r \sin(\pi\sigma/2) \Gamma(1/2 + 2\gamma + h)}, \quad E_{32} = (-1)^{\nu} (1 - \sigma) \sqrt{\pi} 2^{-n}, \\ u = \alpha + 2r(\sigma - 1), \quad v_2 = 2r(1 - \sigma + j) - \alpha, \quad a_1 = \xi, \quad b_1 = \sigma + \xi, \\ c_1 = 1/2 + 2\gamma + h, \quad a_2 = 1 - \sigma + j, \quad b_2 = 1 - \gamma - \alpha/2 + r(1 - \sigma + j), \\ c_2 = (1 - \alpha)/2 + \gamma + (1 - \sigma + j)r, \quad z = (-\tilde{b}^2/p^2)^p \\ [\nu, b, \operatorname{Re} \sigma > 0; \operatorname{Re} \alpha > -\delta n; \operatorname{Re}(\alpha + 2r\sigma) < (1 - (-1)^n)/2 + 2\gamma]$$

$$6. \int_0^{\infty} x^{2\gamma-1} \frac{(y^2 + \sqrt{x^{2\nu} + y^{2\nu}})^{\nu}}{(x^{2\nu} + y^{2\nu})^{\nu}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J_p \quad [p = 1, 1/2] \\ J_p = V_{33} + V_{34} + V_{37} + \\ + \frac{1 + (-1)^n}{2} 2^{2\nu-n-1} y^{2\nu} (-v)^{1-2\nu} \frac{n!}{r[(n/2)!]^2} y^{2\nu-2\nu+1+\alpha} \Gamma\left[\frac{\alpha/(2r), 2\gamma - v - \alpha/r}{1 - v - \alpha/(2r)}\right]$$

$$\{E_{31} = \frac{(-v)^{1-2\nu}}{r} \sqrt{\pi} 2^{2\nu-n-1} \Gamma\left[\frac{\xi, 2\gamma - v - 2\xi}{1 - v - \xi, 1/2 + 2\gamma + h}\right], \\ E_{32} = (-v)^{1-2\nu} 2^{2\nu-1-n} \Gamma\left[\frac{\rho + j + (1-v)/2, 1/2 - j, 1 - d_2}{1 - v/2 - \rho - j, e_2}\right], \\ E_{37} = (-v)^{-2\nu+1} 2^{2\nu-1-n} \Gamma\left[\frac{\rho + j + (1-v)/2, -1/2 - j, 1 - d_2}{(1-v)/2 - \rho - j, e_2}\right], \\ u = \alpha + r(v - 2\rho), \quad v_2 = r(2\rho - v + 2j) - \alpha, \quad v_3 = r(2\rho + 1 - v + 2j) - \alpha, \\ a_1 = v + \xi, \quad b_1 = \xi, \quad c_1 = 1 - \rho + v/2 + \xi, \quad d_1 = (1 + v)/2 - \rho + \xi, \\ e_1 = 1/2 + 2\gamma + h, \quad a_2 = \rho - v/2 + j, \quad b_2 = v/2 + \rho + j, \quad c_2 = 1/2 + j, \\ d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, \quad e_2 = (1 - \alpha)/2 + \gamma + r(\rho - v/2 + j), \\ a_3 = \rho + (1 - v)/2 + j, \quad b_3 = (1 + v)/2 + \rho + j, \quad c_3 = 3/2 + j, \quad d_3 = d_2 + r/2, \\ e_3 = e_2 + r/2, \quad z = (-1)^{p+j} (\tilde{b}/p)^{2p} \\ [b > 0; 2r \operatorname{arg} y < \pi; -\delta n < \operatorname{Re} \alpha < 2r\rho - r \operatorname{Re} v + (1 - (-1)^n)/2]$$

$$7. \int_0^{\infty} x^{2\gamma-1} \frac{(x^2 + \sqrt{x^{2\nu} + y^{2\nu}})^{\nu}}{(x^{2\nu} + y^{2\nu})^{\nu}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J_p \quad [p = 1, 1/2] \\ J_p = V_{31} + V_{32} + \\ + \frac{1 + (-1)^n}{2} 2^{2\nu-n-1} y^{2\nu} (-v)^{1-2\nu} \frac{n!}{r[(n/2)!]^2} y^{2\nu-2\nu+1+\alpha} \Gamma\left[\frac{\rho - v/2 - \alpha/(2r), \alpha/r}{1 - \rho - v/2 + \alpha/(2r)}\right]$$

$$\{E_{31} = (-v)^{1-2\nu} \frac{\sqrt{\pi}}{r} 2^{2\nu-n-1} \Gamma\left[\frac{2\xi, \rho - v/2 - \xi}{1 - \rho - v/2 + \xi, 1/2 + 2\gamma + h}\right], \\ E_{32} = (-v)^{1-2\nu} \sqrt{\pi} 2^{2\nu-n-1} \Gamma\left[\frac{2\rho - v + 2j, 1 - d_2}{1 - v + j, e_2}\right], \\ u = \alpha + r(v - 2\rho), \quad v_2 = r(2\rho - v + 2j) - \alpha, \quad a_1 = \xi, \quad b_1 = \xi + 1/2, \\ c_1 = 1 - \rho - v/2 + \xi, \quad d_1 = 1 - \rho + v/2 - \xi, \quad e_1 = 1/2 + 2\gamma + h, \\ a_2 = \rho - v/2 + j, \quad b_2 = \rho + (1 - v)/2 + j, \quad c_2 = 1 - v + j, \\ d_2 = 1 - \gamma + r(\rho - v/2 + j) - \alpha/2, \quad e_2 = (1 - \alpha)/2 + \gamma + r(\rho - v/2 + j), \\ z = (-1)^{p+j} (\tilde{b}/p)^{2p} \\ [b > 0; 2r \operatorname{arg} y < \pi; -\delta n < \operatorname{Re} \alpha < 2r\rho - r \operatorname{Re} v + (1 - (-1)^n)/2]$$

$$8. \int_0^{\infty} x^{2\gamma-1} \frac{(y^2 + \sqrt{y^{2\nu} - x^{2\nu}})^{\nu} + (y^2 - \sqrt{y^{2\nu} - x^{2\nu}})^{\nu}}{\sqrt{y^{2\nu} - x^{2\nu}}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{33} + \frac{1 + (-1)^n}{2} 2^{2\nu-n-1} y^{2\nu} y^{2\nu+1+\alpha} \frac{n!}{r[(n/2)!]^2} \Gamma\left[\frac{\alpha/(2r), v + \alpha/(2r)}{v + \alpha/r}\right] \\ \{E_{31} = V \sqrt{\pi} \frac{2^{2\nu-n}}{r} \Gamma\left[\frac{\xi, v + \xi}{v + 2\xi, 1/2 + 2\gamma + h}\right], \\ u = \alpha + r(v - 1), \quad a_1 = \xi, \quad b_1 = v + \xi, \quad c_1 = v/2 + \xi, \quad d_1 = (v + 1)/2 + \xi, \\ e_1 = 1/2 + 2\gamma + h, \quad z = (-\tilde{b}^2/p^2)^p \\ [b > 0; \operatorname{Re} \alpha > -n\delta; \operatorname{Re} \alpha > -n\delta - 2r \operatorname{Re} v; |\arg b| < \pi]$$

$$9. \int_0^{\infty} x^{2\gamma-1} \frac{(x^2 + \sqrt{x^{2\nu} - y^{2\nu}})^{\nu} + (x^2 - \sqrt{x^{2\nu} - y^{2\nu}})^{\nu}}{\sqrt{x^{2\nu} - y^{2\nu}}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{33} + V_{34} + V_{37} + \frac{1 + (-1)^n}{2} 2^{-n-\alpha/r} y^{2\nu} (v-1)+\alpha \times \\ \times \frac{n!}{r[(n/2)!]^2} \Gamma\left[\frac{(1+v)/2 - \alpha/(2r), (1-v)/2 - \alpha/(2r)}{1 - \alpha/r}\right] \\ \{E_{31} = \frac{2^{-n-2\xi+1}}{r} \sqrt{\pi} \Gamma\left[\frac{(1+v)/2 - \xi, (1-v)/2 - \xi}{1 - 2\xi, 1/2 + 2\gamma + h}\right], \\ E_{32} = 2^{-n-2\xi} \sqrt{\pi} \Gamma\left[\frac{-v-j, 1-d_2}{-v-2j, e_2}\right], \quad E_{37} = 2^{-n-2\xi} \sqrt{\pi} \Gamma\left[\frac{v-j, 1-d_2}{v-2j, e_2}\right], \\ u = \alpha + r(v - 1), \quad v_2 = r(1 + v + 2j) - \alpha, \quad v_3 = r(1 - v + 2j) - \alpha, \quad a_1 = \xi, \\ b_1 = \xi + 1/2, \quad c_1 = (1 - v)/2 + \xi, \quad d_1 = (1 + v)/2 + \xi, \quad e_1 = 1/2 + 2\gamma + h, \\ a_2 = (1 + v)/2 + j, \quad b_2 = 1 + v/2 + j, \quad c_2 = 1 + v + j, \\ d_2 = 1 - \gamma + r[(1 + v)/2 + j] - \alpha/2, \quad e_2 = (1 - \alpha)/2 + \gamma + r[(1 + v)/2 + j], \\ a_3 = (1 - v)/2 + j, \quad b_3 = 1 - v/2 + j, \quad c_3 = 1 - v + j, \\ d_3 = 1 - \gamma + r[(1 - v)/2 + j] - \alpha/2, \quad e_3 = (1 - \alpha)/2 + \gamma + r[(1 - v)/2 + j], \\ z = (-1)^p \tilde{b}^{2r} p^{-2p} \quad [\nu, b > 0; \operatorname{Re} \alpha < r(1 - \operatorname{Re} v) + (1 - (-1)^n)/2]$$

$$10. \int_0^{\infty} \frac{x^{2\gamma-1}}{x^{2\nu} + 2(xy)^r \cos \beta + y^{2\nu}} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = \\ = V_{33} + V_{34} - \frac{1 + (-1)^n}{2} 2^{-n-1} y^{2\nu+2} \frac{\pi n!}{r[(n/2)!]^2} \frac{\sin[\beta(\alpha - r)/r]}{\sin \beta \sin(\pi\alpha/r)} \\ \{E_{31} = -\frac{2^{1-n}}{r} x^{1/2} \frac{1}{\sin \beta}, \quad E_{32} = -2^{-n} \frac{\sqrt{\pi}}{\sin \beta}, \quad u = \alpha - 2r,$$

$$\begin{aligned} v_1 &= 2\gamma + 2l, \quad v_2 = r(1+l) - \alpha, \quad a_l = \beta(2\gamma + 2l + \alpha - r)r^{-1}, \\ b_l &= (2\gamma + 2l + \alpha)r/r, \quad c_l = 1/2 + 2\gamma + l, \quad d_l = \beta l, \quad e_l = 1 + l - \alpha/r, \\ f_l &= \gamma + \alpha/2 - r(1+l)/2, \quad g_l = (1-\alpha)/2 + \gamma + r(1+l)/2 \\ &[\gamma, b > 0; |\beta| < \pi; -n\delta < \operatorname{Re} \alpha < 2r + (1-(-1)^n)/2; r > 0 - \text{произвольное}]. \end{aligned}$$

$$11. \int_0^\infty x^{\alpha-1} e^{-sx-2r} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = U + \frac{1+(-1)^n}{2} 2^{-n-1} s^{-\alpha/2} \frac{n!}{r[(n/2)!]^2} \Gamma\left(-\frac{\alpha}{2r}\right),$$

$$U = V_{43}$$

$$U = V_{41}$$

$$U = V_{41} 2^{1-n} \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} x^{-\alpha} q^{2k} \Gamma\left[\frac{\alpha+2\gamma}{1/2+2\gamma}\right] e^{\rho} \left(\frac{\Delta(2-\alpha+2\gamma)}{1/2+2\gamma} - \frac{b^2(n-2k)^2}{s^2}\right)$$

$$\{E_{43} = \frac{\sqrt{\pi}}{2^n r} \Gamma\left[\frac{\xi}{1/2+2\gamma+h}\right], \quad u = -\frac{\alpha}{2r}, \quad v_1 = 2(h+\gamma),$$

$$\varphi = 2^{-1}(n-2k)bs^{-1/(2r)}, \quad a_1 = \xi, \quad b_1 = 1/2+2\gamma+h,$$

$$z = (-1)^p \left[\frac{(n-2k)b}{2\rho}\right]^{2\rho} q^{\alpha-q}; \quad E_{41} = \frac{\sqrt{\pi}}{2^n} \Gamma\left[\frac{\gamma+rj+\alpha/2}{1/2+\gamma-rj-\alpha/2}\right],$$

$$v_2 = -2rj - \alpha, \quad a_2 = \gamma + rj + \alpha/2, \quad b_2 = (1+\alpha)/2 - \gamma + rj,$$

$$z_2 = (-1)^{p+q} \left[\frac{(n-2k)b}{2\rho}\right]^{-2\rho} q^{-q} s^q$$

$$[\operatorname{Re} s > 0; \operatorname{Re} \alpha > -\delta n; \arg b \text{ произволен (при } r > 1/2); b > 0 \text{ (при } r < 1/2); n | \operatorname{Im} b| < \operatorname{Re} s \text{ (при } r = 1/2)].$$

$$12. \int_0^\infty x^{\alpha-1} e^{-sx-2r} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx =$$

$$= V_{41} + V_{43} + \frac{1+(-1)^n}{2} 2^{-n-1} s^{-\alpha/2} \frac{n!}{r[(n/2)!]^2} \Gamma\left(-\frac{\alpha}{2r}\right)$$

$$\{E_{43} = \frac{\sqrt{\pi}}{2^n r} \Gamma\left[\frac{-\xi}{1/2+2\gamma+h}\right], \quad E_{41} = \frac{\sqrt{\pi}}{2^n} \Gamma\left[\frac{\gamma-rj+\alpha/2}{(1-\alpha)/2+\gamma+rj}\right],$$

$$u = \alpha/(2r), \quad v_1 = 2(\gamma+h), \quad v_2 = 2rj - \alpha, \quad \varphi = 2^{-1}b(n-2k)s^{1/(2r)},$$

$$a_1 = 1/2+2\gamma+h, \quad b_1 = 1+\xi, \quad a_2 = 1-\gamma+rj-\alpha/2, \quad b_2 = (1-\alpha)/2+\gamma+rj,$$

$$z = (-1)^{p+q} \left[\frac{(n-2k)b}{2\rho}\right]^{2\rho} s^q q^{-q} \quad [\delta, \operatorname{Re} s > 0; \operatorname{Re} \alpha < (1-(-1)^n)/2].$$

$$13. \int_0^\infty x^{\alpha-1} \sin^m ax \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I \left[\frac{1-(-1)^n}{4}, 1, (-1)^{[m/2]+1} \right],$$

$$I(p, \sigma, \tau) = U + \left[\frac{1+(-1)^n}{2} \frac{n!}{2^n [(n/2)!]^2} - \{0\} \right] \frac{\alpha^{-\alpha/r}}{r} 2^{-m} \sqrt{\pi} \times$$

$$\times \sum_{l=0}^{[(m-1)/2]} \tau \left(\frac{m}{l} \right) \left(\frac{2}{m-2l} \right)^{\alpha/r} \Gamma\left[\frac{\rho+\alpha/(2r)}{\rho+1/2-\alpha/(2r)}\right] +$$

$$+ \frac{1+(-1)^m}{2} 2^{-m-n} \frac{b^{\alpha m} \sqrt{\pi}}{[(m/2)!]^2} \sum_{k=0}^{[(m-1)/2]} e^{\left(\frac{n}{k}\right)} \left(\frac{2}{n-2k} \right)^{\alpha} \Gamma\left[\frac{\gamma+\alpha/2}{\gamma+(1-\alpha)/2}\right]$$

$$[r \neq 1 \text{ (} r = 1 \text{ см. 2.5.13.20-21)}; c = 1 \text{ при } n = 2, 4, 6, \dots; c = 0 \text{ при } n = 1, 3, 5, \dots]$$

$$U = 2^{1-n} s^{-\alpha/2} \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} \sum_{l=0}^{[(m-1)/2]} \tau \left(\frac{m}{l} \right) A_{kl} \left[\frac{n-2k}{2} b \right]^{-\alpha},$$

$$A_{kl} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(j+p)} \Gamma\left[\frac{a_1}{c_1}, 1-b_1\right] {}_2F_2 \left(\frac{\Delta(p, a_1), \Delta(p, b_1)}{\Delta(q, c_1), \Delta(q, 1+l)}; z \right) \quad [r < 1],$$

$$A_{kl} = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \varphi^{-2h} \Gamma\left[1/2+\rho-\xi, 1/2+2\gamma+h\right] \times \\ \times {}_2F_2 \left(\frac{\Delta(q, \xi+\rho), \Delta(q, 1/2-\rho+\xi)}{\Delta(p, 1/2+2\gamma+h), \Delta(p, 1+h)}; z \right) \quad [r > 1]$$

$$\{a_1 = \gamma + \alpha/2 + r(j+p), \quad b_1 = (1+\alpha)/2 - \gamma + r(j+p),$$

$$c_1 = 1/2+2\rho+j, \quad \varphi = \frac{a(m-2l)}{2} \left(\frac{2}{(n-2k)b} \right)^r, \quad z = (-1)^{p+q} r^{2\rho} q^{-2q} \varphi^{2q}\}$$

$$[a, b > 0; -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < 0 \text{ (при } m+n=2, 4, 6, \dots), \text{ или } -\delta n - \sigma r m - 2c < \operatorname{Re} \alpha < \max(1, r) \text{ (при } m+n=1, 3, 5, \dots)].$$

$$14. \int_0^\infty x^{\alpha-1} \cos^m ax \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(0, 0, 1) \quad [\text{см. 2.5.2.13}],$$

$$15. \int_0^\infty x^{\alpha-1} \sin^m ax \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J \left[\frac{1-(-1)^n}{4}, 1, (-1)^{[m/2]+1} \right],$$

$$J(p, \sigma, \tau) = 2^{1-n} s^{-\alpha/2} \sum_{k=0}^{[(n-1)/2]} e^{\left(\frac{n}{k}\right)} \sum_{l=0}^{[(m-1)/2]} \tau \left(\frac{m}{l} \right) \times \\ \times \left[\frac{b(n-2k)}{2} \right]^{-\alpha} \left\{ \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \varphi^{2(j+p)} \Gamma\left[\frac{1-b_1}{a_1}, c_1\right] \times \right. \\ \times {}_2F_2 \left(\frac{1; \Delta(q, a_1), \Delta(p, b_1), \Delta(p, c_1), \Delta(q, 1+j)}{\Delta(q, 1-\rho+\xi), \Delta(q, 1/2+\rho+\xi), \Delta(p, 1+h), \Delta(p, 1/2+2\gamma+h)}; z \right) \} +$$

$$+ \left[\frac{1+(-1)^n}{2} \frac{n}{2^n [(n/2)!]^2} - \{0\} \right] \frac{\alpha^{-\alpha/r}}{r} \frac{\sqrt{\pi}}{2^m} \sum_{l=0}^{[(m-1)/2]} \tau \left(\frac{m}{l} \right) \times$$

$$\times \left(\frac{m-2l}{2} \right)^{\alpha/r} \Gamma\left[\frac{\rho-\alpha/(2r)}{1/2+\rho+\alpha/(2r)}\right] + \frac{1+(-1)^m}{2} 2^{-m-n} \frac{b^{\alpha m} \sqrt{\pi}}{[(m/2)!]^2} \sum_{k=0}^{[(m-1)/2]} e^{\left(\frac{n}{k}\right)} \left(\frac{2}{n-2k} \right)^{\alpha} \Gamma\left[\frac{\gamma+\alpha/2}{\gamma+(1-\alpha)/2}\right]$$

$$[a_1 = 1/2 + 2\rho + i, \quad b_1 = 1 - \gamma - \alpha/2 + r(\rho + i), \quad c_1 = (1 - \alpha)/2 + \gamma + r(\rho + i), \\ \psi = a \frac{m-2l}{2} \left(b \frac{n-2k}{2}\right)^r, \quad z = (-1)^{p+q} \psi^{2q} \rho^{-2p} q^{2q} \\ [a, b > 0; -2c - \delta n - \sigma r m < \operatorname{Re} \alpha < \left(\frac{1 - (-1)^n}{2} + \frac{1 - (-1)^m}{2} r\right) \\ c = 1 \text{ при } n = 2, 4, 6, \dots; c = 0 \text{ при } n = 1, 3, 5, \dots]$$

$$16. \int_0^\infty x^{\alpha-1} \cos^m ax \cdot r \left\{ \frac{\sin^m bx}{\cos^m bx - c} \right\} dx = J(0, 0, 1) \quad [\text{см. 2.5.2.16}]$$

$$17. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin^m bx}{\cos^m bx} \right\}^n dx = I(\sigma) |_{\sigma=1} \quad [r > 0]$$

$$I(\sigma) = U + \frac{1 + (-1)^n}{2} \frac{2^{-n}}{r} (s^2 + a^2)^{-\alpha/(2r)} \frac{n}{[(n/2)!]^2} \Gamma\left(\frac{\alpha}{r}\right) \cos\left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right),$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (r+l+\alpha)/2}{\gamma + (1-r-l-\alpha)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{-\alpha-2l} \quad [r < 1]$$

$$U = \frac{\sqrt{\pi}}{2^{n-1}r} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} - 2l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{2\gamma}{1/2 + h + 2\gamma}\right] (s^2 + a^2)^{-1/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad [r > 1]$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) A_k(z) \quad [r = 1] \\ z = \left(b \frac{n-2k}{\sqrt{s^2 + a^2}}\right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^\infty \frac{(-1)^h}{h!} \cos\left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s}\right] \times \\ \times \Gamma\left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad \text{при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} + l \operatorname{arctg} \frac{a}{s}\right) \Gamma\left[\frac{\gamma + (\alpha+l)/2}{\gamma + (1-\alpha-l)/2}\right] \times \\ \times (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{-\alpha-2l} \quad \text{при } |z| > 1$$

$$[a, \operatorname{Re} s > 0; \operatorname{Re} \alpha > -\sigma r - \delta n; b - \text{произвольное (при } r > 1), \quad |\operatorname{Im} b| n < \operatorname{Re} s \quad (\text{при } r = 1) \\ \text{и } b > 0 \quad (\text{при } r < 1)] \text{ или } [-\sigma r - \delta n < \operatorname{Re} \alpha < \max(r, 1) \quad (\text{при } r \neq 1) \\ \text{или } -\sigma r - \delta n < \operatorname{Re} \alpha < 0 \quad (\text{при } r = 1)]$$

$$18. \int_0^\infty x^{\alpha-1} e^{-sx} \cos ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = I(\sigma) |_{\sigma=0} \quad [I(\sigma) \text{ см. 2.5.2.17}]$$

$$19. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J(\sigma) |_{\sigma=1}$$

$$J(\sigma) = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (\alpha-lr)/2}{\gamma + (1-\alpha+lr)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n-2k}{2}\right)^{r-l-\alpha} + \\ + \frac{\sqrt{\pi}}{r 2^{n-1}} \sum_{k=0}^{[(n-1)/2]} e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{(-1)^l}{l!} \cos\left(\frac{\pi\sigma}{2} + 2l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{-2\gamma}{2\gamma + 1/2 + h}\right] (s^2 + a^2)^{-1/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} + \\ + \frac{1 + (-1)^n}{2} \frac{(s^2 + a^2)^{\alpha/(2r)} n!}{2^n [(n/2)!]^2} \cos\left(\frac{\pi\sigma}{2} + \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right) \Gamma\left(-\frac{\alpha}{r}\right) \\ [a, b > 0; \operatorname{Re} \alpha < r\sigma + (1 - (-1)^n)/2 \text{ и } \operatorname{Re} s > 0 \text{ (или } \operatorname{Re} s = 0 \text{ и } \operatorname{Re} \alpha > -r - n\delta)]$$

$$20. \int_0^\infty x^{\alpha-1} e^{-sx} \cos ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = J(\sigma) |_{\sigma=0} \quad [\text{см. 2.5.2.19}]$$

$$21. \int_0^\infty x^{\alpha-1} e^{-sx} \sin ax \cdot r \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = K(\sigma) |_{\sigma=1} \quad [r > 0]$$

$$K(\sigma) = U + \frac{1 + (-1)^n}{2} \frac{(-1)^{n/2}}{r} \frac{2^{-n}}{[(n/2)!]^2} \Gamma\left(\frac{\alpha}{r}\right) (s^2 + a^2)^{-\alpha/(2r)} \times \\ \times \cos\left(\frac{\pi\sigma}{2} - \frac{\alpha}{r} \operatorname{arctg} \frac{a}{s}\right),$$

$$U = \frac{\sqrt{\pi}}{2^{n-1}r} \sum_{k=0}^{[(n-1)/2]} (-1)^k e\left(\frac{n}{k}\right) \sum_{l=0}^\infty \frac{1}{l!} \cos\left(\frac{\pi\sigma}{2} - 2l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{2\gamma}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-1/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad [r > 1]$$

$$U = \frac{\sqrt{\pi}}{2^n} \sum_{k=0}^{[(n-1)/2]} (-1)^k e\left(\frac{n}{k}\right) A_k(z) \quad [r = 1] \\ z = \left(b \frac{n-2k}{\sqrt{s^2 + a^2}}\right)^2,$$

$$A_k(z) = 2 \sum_{h=0}^\infty \frac{1}{h!} \cos\left[\frac{\pi\sigma}{2} - (2\gamma + 2h + \alpha) \operatorname{arctg} \frac{a}{s}\right] \times \\ \times \Gamma\left[\frac{2\gamma + 2h + \alpha}{1/2 + 2\gamma + h}\right] (s^2 + a^2)^{-\gamma-h-\alpha/2} \left(b \frac{n-2k}{2}\right)^{2h+2\gamma} \quad \text{при } |z| < 1,$$

$$A_k(z) = \sum_{l=0}^{\infty} \frac{(-1)^l \sin[c - \gamma + (\alpha + l)/2] \pi}{l! \sin c \pi} \cos\left(\frac{\alpha \pi}{2} + l \operatorname{arctg} \frac{a}{s}\right) \times \\ \times \Gamma\left[\frac{\gamma + (\alpha + l)/2}{\gamma + (1 - \alpha - l)/2}\right] (s^2 + a^2)^{l/2} \left(b \frac{n - 2k}{2}\right)^{-\alpha - l} \quad \text{при } |z| > 1$$

$\{c - \text{произвольно; } a, \operatorname{Re} s > 0; \operatorname{Re} \alpha > -\sigma - \delta n; b - \text{произвольное (при } r > 1, \text{ или } n | \operatorname{Re} b| < \operatorname{Re} s \text{ (при } r = 1))\}.$

$$22. \int_0^{\infty} x^{\alpha-1} e^{-sx} \cos ax \frac{\{\operatorname{sh} bx\}^n}{\{\operatorname{ch} bx\}^n} dx = K(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2.5.2.21}]$$

2.5.3. Интегралы от $x^{\alpha} \left\{ \frac{\sin bx}{\cos bx} \right\}^{\sigma}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^{\alpha} dx = \frac{\sqrt{\pi}}{2} \Gamma\left[\frac{(\alpha+1)/2}{\alpha/2+1}\right] \quad [\operatorname{Re} \alpha > -1]$$

$$2. \int_0^{\pi/2} x \left\{ \frac{\sin x}{\cos x} \right\}^{\alpha} dx = \\ = \left(1 + 2 \left[\frac{n+1}{2}\right] \pm n\right) A \frac{(n-1)! n}{4n!} \pm \sum_{k=0}^{[(n-1)/2]} \frac{((1-n)/2)_k}{(-n/2)_k (n-2k)!} \\ [A = 1 \text{ при } n = 1, 3, 5, \dots; A = \pi/2 \text{ при } n = 2, 4, 6, \dots]$$

$$3. \int_0^{\pi/2} x^m \left\{ \frac{\sin x}{\cos x} \right\}^{\alpha} dx = \\ = \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k-\delta)!} \left(\frac{\pi}{2}\right)^{m-2k-\delta} + (-1)^{[m/2]-m} \left(2 \left[\frac{m}{2}\right] + \delta - m\right)$$

$$4. \int_0^{\pi/2} \frac{\sin^2 x}{x} dx = \frac{1}{2} (C + \ln \pi - \operatorname{ci}(\pi)).$$

$$5. \int_0^{\pi} x^m \left\{ \frac{\sin nx}{\cos nx} \right\}^{\alpha} dx = \frac{(-1)^{\alpha} \delta + 1}{n^{m+1}} \sum_{k=0}^{[(m+\delta-1)/2]} (-1)^k \frac{m!}{(m-2k+\delta-1)!} \times \\ \times (\pi n)^{m-2k+\delta-1} + (-1)^{[(m+\delta+1)/2]} m! n^{-m-1} \left(2 \left[\frac{m-\delta+1}{2}\right] - m + \begin{cases} 1 \\ 0 \end{cases}\right)$$

$$6. \int_0^{\pi} x^{\alpha-1} \sin^{\alpha} x dx = -\frac{(\alpha-1)(\alpha-2)}{n^{\alpha}} \int_0^{\pi} x^{\alpha-3} \sin^{\alpha} x dx + \frac{n-1}{n} \int_0^{\pi} x^{\alpha-2} \sin^{\alpha-2} x dx \\ [\operatorname{Re} \alpha > 2-n; n = 2, 3, 4, \dots]$$

$$7. \int_0^{\pi} x \cos^{2n} x dx = \frac{(2n-1)! \pi^3}{2(2n)!}.$$

$$8. \int_0^{\pi/2} x \sin^{\alpha} x dx = \frac{\pi(n-1)!}{2(n!)} \begin{cases} \pi k^2, & n=0, 2, 4, \dots \\ 2k, & n=1, 3, 5, \dots \end{cases}$$

$$9. \int_0^{2\pi l} x^m \cos nx dx = -\sum_{k=0}^{m-1} \frac{k!}{n^{k+1}} \binom{m}{k} (2\pi l)^{m-k} \cos \frac{k+1}{2} \pi \\ [n, l = 1, 2, 3, \dots]$$

$$10. \int_0^{\pi} x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\Gamma(\alpha)}{b^{\alpha}} \left\{ \frac{\sin(2\pi/2)}{\cos(2\pi/2)} \right\} \\ [b > 0; -\delta < \operatorname{Re} \alpha < 1]$$

$$11. \int_0^{\infty} x^{-1/2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{\pi}{2b}} \quad [b > 0]$$

$$12. \int_0^{\infty} \frac{\sin bx}{x} dx = \frac{\pi}{2} \operatorname{sgn} b \quad [1m b = 0]$$

$$13. \int_0^{\infty} x^{\alpha-1} \sin^{\alpha} bx dx = I_{\alpha, n} \\ [1m b = 0; -\pi < \operatorname{Re} \alpha < 2\lambda; n = 1, 2, 3, \dots; \lambda = (1 - (-1)^n)/4]$$

$$I_{\alpha, n} = 2^{\alpha-1} \sqrt{\pi} |b|^{-\alpha} (\operatorname{sgn} b)^n \Gamma\left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2}\right] \sum_{j=0}^{[(n-1)/2]} (-1)^j \frac{j! (n-2j)^{-\alpha}}{j! (n-j)!} \\ \text{при } \alpha \neq -2(\lambda + k), k = 0, 1, 2, \dots \\ I_{-2\lambda-2k, n} = \lim_{\alpha \rightarrow -2\lambda-2k} I_{\alpha, n} \text{ при } \alpha \rightarrow -2\lambda-2k,$$

$$I_{1/2-m, 2l-1} = (-1)^{[m/2]} \sqrt{\pi} 2^{m-2l+3/2} \frac{|b|^{m-1/2}}{(2m-1)!} \operatorname{sgn} b \times \\ \times \sum_{j=1}^l (-1)^{j-1} \binom{2l-1}{l+j-1} (2l-1)^{m-1/2} \\ [0 \leq m \leq 2l-1]$$

$$I_{1/2-m, 2l} = (-1)^{[(m+1)/2]} \sqrt{\pi} 2^{m-2l} \frac{|b|^{m-1/2}}{(2m-1)!} \sum_{j=1}^l (-1)^j \binom{2l}{l+j} j^{m-1/2} \\ [m = 1, 2, \dots, 2l]$$

$$I_{-1/2, 1} = \sqrt{2\pi} |b| \operatorname{sgn} b, \quad I_{-3/2, 2} = \frac{4}{3} b \sqrt{\pi} |b|,$$

$$I_{-1/2, 3} = \frac{3-\sqrt{3}}{4} \sqrt{\pi} |b| \operatorname{sgn} b,$$

$$I_{0, 2l-1} = 2^{1/2} \pi \sum_{j=0}^{l-1} (-1)^{l+j+1} \frac{(2l-1)!}{j! (2l-j-1)!} \operatorname{sgn} b,$$

$$I_{0, (2m+1)/(2n+1)} = 2^{(2m-4n-1)/(2n+1)} B((2m+1)/(4n+2), (2m+1)/(4n+2))$$

[b > 0]

$$I_{-m, n} = \frac{2^{-n} b^m \pi}{m! \Gamma(m\pi/2)} \sum_{j=0}^{[(n-1)/2]} (-1)^{[(n-1)/2]+j} \frac{n! (n-2j)^m}{j! (n-j)!}$$

[b > 0, T(x) = cos x при n = 2l - 1; T(x) = sin x при n = 2l, l = 1, 2, 3, ...]

$$I_{-1, 2l} = \frac{\pi |b| (2l-3)!}{2 (2l-2)!}, \quad I_{-2, 2l-1} = \frac{\pi b \cdot b! (2l-1)}{4 (2l-2)!}, \quad I_{-2, 2l} = \frac{\pi (2l-5)!}{(2l-2)!},$$

$$I_{\alpha, 1} = -2^{-\alpha} \Gamma(\alpha) |b|^{-\alpha} \cos(\alpha\pi/2), \quad [-2 < \operatorname{Re} \alpha < 0],$$

$$I_{\alpha, 2} = 2^{-2} b^{-\alpha} (3-3^{-\alpha}) \sin(\alpha\pi/2) \Gamma(\alpha) \operatorname{sgn} b, \quad [-3 < \operatorname{Re} \alpha < 1],$$

$$I_{0, 3} = \frac{\pi}{4} \operatorname{sgn} b, \quad I_{-1, 3} = \frac{3}{4} b \ln 3, \quad I_{-2, 3} = \frac{3}{8} \pi b |b|.$$

$$14. \int_0^{\infty} x^{\alpha-1} \cos^{2n+1} bx \, dx =$$

$$= 2^{\alpha-2n-1} \sqrt{\pi} (2n+1)! b^{-\alpha} \Gamma\left[\frac{\alpha}{2}\right] \sum_{k=0}^n \frac{(2n-2k+1)^{-\alpha}}{k! (2n-k+1)!} \quad [b > 0; 0 < \operatorname{Re} \alpha < 1].$$

2.5.4. Интегралы от $x^m \left\{ \frac{\sin x}{\cos x} \right\}^{-n}$.

$$1. \int_0^{\pi/4} \frac{x^m}{\sin^2 x} \, dx = -\left(\frac{\pi}{4}\right)^m \left[1 - \frac{4m}{\pi(m-1)} + \frac{4m}{\pi} \sum_{k=1}^{\infty} \frac{\pi^{2k} |B_{2k}|}{2^{2k} (2k)! (m+2k-1)} \right] \quad [m=2, 3, 4, \dots]$$

$$2. \int_0^{\pi/4} \frac{x^2}{\left\{ \frac{\sin x}{\cos x} \right\}^2} \, dx = \frac{\pi}{4} \ln 2 + \frac{\pi^3}{16} \pm G_4.$$

$$3. \int_0^{\pi/4} \frac{dx}{\cos^n x} = -\sum_{k=1}^{[n/2]} \frac{(1-n/2)_{k-1}}{((1-n)/2)_k} 2^{n/2-k} + \left(n-2 \left[\frac{n}{2} \right] \right) \frac{(2[n/2]-1)!}{(2[n/2])!} \ln(1+\sqrt{2}) \quad [n=1, 2, 3, \dots]$$

$$4. \int_0^{\pi/2} \frac{x^m}{\sin x} \, dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1}}{4^{2k-1} (m+2k)} \zeta(2k) \right] \quad [m=1, 2, 3, \dots]$$

$$5. \int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2G.$$

$$6. \int_0^{\pi/2} \frac{x^m}{\sin^2 x} \, dx = m \left(\frac{\pi}{2}\right)^{m-1} \left[\frac{1}{m-1} + \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k} \pi^{2k}}{(m+2k-1)(2k)!} \right] \quad [m=2, 3, 4, \dots]$$

[2.5.6.]

$$7. \int_0^{\pi/2} \frac{x^{2l+1} \sigma}{\sin^2 x} \, dx = \frac{\pi^{2\sigma+1}}{2^{\sigma}} \ln 2 - \frac{9\pi\sigma}{4} \zeta(3) \quad [\sigma=0 \text{ или } 1].$$

$$8. \int_0^{\pi} \frac{x}{\cos x} \, dx = -4G.$$

2.5.5. Интегралы от $(x+z)^{\beta} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^a (x-z)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{b^{\delta} a^{\beta+\delta}}{\beta(\beta+1)^{\delta}} {}_1F_2\left(1; 1+\frac{\beta}{2}, \frac{1+\beta}{2}+\delta; -\frac{a^2 b^2}{4}\right) \quad [a, \operatorname{Re} \beta > 0; |\arg b| < \pi].$$

$$2. \int_{-\infty}^{\infty} (x-a)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\Gamma(\beta)}{b^{\beta}} \left\{ \frac{\sin(ab+\beta\pi/2)}{\cos(ab+\beta\pi/2)} \right\} \quad [a, b > 0; 0 < \operatorname{Re} \beta < 1].$$

$$3. \int_0^{\infty} \frac{1}{(x+z)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{b^{\delta} z^{1+\delta-\rho}}{(\rho-1)(\rho-2)^{\delta}} {}_1F_2\left(1; \frac{3-\rho}{2}, 1+\delta-\frac{\rho}{2}; -\frac{b^2 z^2}{4}\right) + \Gamma(1-\rho) b^{\delta-1} \left\{ \frac{\cos(\rho\pi/2+bz)}{\sin(\rho\pi/2+bz)} \right\} \quad [b, \operatorname{Re} \rho > 0; |\arg z| < \pi].$$

$$4. \int_0^{\infty} \frac{1}{\sqrt{x+z}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{\pi}{2b}} \left[\cos bz \mp \sin bz \pm 2C(bz) \left\{ \frac{\sin bz}{\cos bz} \right\} - 2S(bz) \left\{ \frac{\cos bz}{\sin bz} \right\} \right] \quad [b > 0; |\arg z| < \pi].$$

$$5. \int_0^{\infty} \frac{1}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pm \operatorname{ci}(bz) \left\{ \frac{\sin bz}{\cos bz} \right\} - \operatorname{si}(bz) \left\{ \frac{\cos bz}{\sin bz} \right\} \quad [b > 0; |\arg z| < \pi].$$

$$6. \int_0^{\infty} \frac{1}{x-y} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pm \left\{ \frac{\cos by}{\sin by} \right\} [\operatorname{si}(by) + \pi] - \left\{ \frac{\sin by}{\cos by} \right\} \operatorname{ci}(by) \quad [y, y > 0].$$

$$7. \int_{-\infty}^{\infty} \frac{1}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pi \left\{ \frac{\cos bz}{\sin bz} \right\} \quad [b > 0, \operatorname{Im} z \neq 0].$$

2.5.6. Интегралы от $(x^2 \pm a^2)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$; $(a^2 - x^2)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$.

$$1. \int_0^a (a^2 - x^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b}\right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{H_{\beta-1/2}(ab)}{J_{\beta-1/2}(ab)} \right\} \quad [a, \operatorname{Re} \beta > 0; |\arg b| < \pi].$$

2. $\int_0^{\infty} \frac{(x^2 - a^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pm \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ Y_{1/2-\beta}(ab) \right\}$
 $[a, b > 0; 0 < \operatorname{Re} \beta < 1].$
3. $\int_0^{\infty} \frac{\sin bx}{(x^2 + z^2)^{\rho}} dx = \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} b^{\rho-1/2} [J_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)]$
 $[b, \operatorname{Re} \rho, \operatorname{Re} z > 0].$
4. $\int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)^{\rho}} dx = \left(\frac{2z}{b} \right)^{1/2-\rho} \frac{\sqrt{\pi}}{\Gamma(\rho)} K_{1/2-\rho}(bz)$
 $[b, \operatorname{Re} \rho, \operatorname{Re} z > 0].$
5. $\int_0^{\infty} \frac{\sin bx}{x^2 + z^2} dx = \frac{1}{2z} [e^{-bz} \operatorname{Ei}(bz) - e^{bz} \operatorname{Ei}(-bz)]$
 $[b, \operatorname{Re} z > 0].$
6. $\int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)^{m+1/2}} dx = \frac{\pi e^{-bz}}{(2z)^{2m+1} m!} \sum_{k=0}^m \frac{(2m-k)! (2bz)^k}{k! (m-k)!}$
 $[b, \operatorname{Re} z > 0].$
7. $\int_0^{\infty} \frac{\sin bx}{x^2 - y^2} dx = \frac{1}{y} [\cos by \operatorname{Si}(by) - \sin by \operatorname{Ci}(by)]$
 $[b, y > 0].$
8. $\int_0^{\infty} \frac{\cos by}{x^2 - y^2} dx = -\frac{\pi}{2y} \sin by$
 $[b, y > 0].$
9. $\int_0^{\infty} \frac{1}{x^2 - y^2} \left\{ \frac{\sin bx}{\cos bx} \right\}^2 dx = \pm \frac{\pi}{4y} \sin 2by$
 $[b, y > 0].$
10. $\int_0^{\infty} \frac{\sin^{2n} bx}{x^2 + z^2} dx = \frac{(-1)^n \pi}{2^{2n+1} z} \left\{ 2^{-n} \operatorname{sh}^{2n} bz - 2 \sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{Si} [2(n-k)bz] \right\}$
 $[b, \operatorname{Re} z > 0].$
11. $\int_0^{\infty} \frac{\sin^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 - e^{-2bz})$
 $[b, \operatorname{Re} z > 0].$
12. $\int_0^{\infty} \frac{\sin^{2n+1} bx}{x^2 + z^2} dx =$
 $= \frac{(-1)^n \pi}{2^{2n+1} z} \left\{ e^{(2n+1)bz} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{-2kbz} \operatorname{Ei} [(2k-2n-1)bz] + \right.$
 $\left. + e^{-(2n+1)bz} \sum_{k=1}^{2n+1} (-1)^{k-1} \binom{2n+1}{k} e^{2kbz} \operatorname{Ei} [(2n+1-2k)bz] \right\}$
 $[b, \operatorname{Re} z > 0].$

11. $\int_0^{\infty} \frac{\cos^{2n} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1} z} \binom{2n}{n} + \frac{\pi}{2^{2n}} \sum_{k=1}^n \binom{2n}{n+k} e^{-2kbz}$
 $[b, \operatorname{Re} z > 0].$
14. $\int_0^{\infty} \frac{\cos^2 bx}{x^2 + z^2} dx = \frac{\pi}{4z} (1 + e^{-2bz})$
 $[b, \operatorname{Re} z > 0].$
15. $\int_0^{\infty} \frac{\cos^{2n+1} bx}{x^2 + z^2} dx = \frac{\pi}{2^{2n+1} z} \sum_{k=0}^n \binom{2n+1}{n+k+1} e^{-(2k+1)bz}$
 $[b, \operatorname{Re} z > 0].$

2.5.7. Интегралы от $x^{\alpha} (x \pm a)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}$, $x^{\alpha} (a-x)^{\mu} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

1. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= (-i)^{\delta} 2^{-1} a^{\alpha+\beta-1} B(\alpha, \beta) [{}_1F_1(\alpha; \alpha+\beta; iab) - {}_1F_1(\alpha; \alpha+\beta; -iab)]$
 $[a, \operatorname{Re} \alpha > 0; \operatorname{Re} \beta > -\delta; |\arg b| < \pi].$
2. $\int_0^{\infty} x^{\alpha-1} (a-x)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\pi} \left(\frac{a}{b} \right)^{\alpha-1/2} \left\{ \frac{\sin(ab/2)}{\cos(ab/2)} \right\} \Gamma(\alpha) J_{\alpha-1/2} \left(\frac{ab}{2} \right)$
 $[a > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi].$
3. $\int_0^a \frac{1}{\sqrt{x(a-x)}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2} \sqrt{\delta} J_{1/2+\delta}^2 \left(\frac{ab}{2} \right)$
 $[a > 0; |\arg b| < \pi].$
4. $\int_0^a x^{\alpha-1} (x-a)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = B(\beta, 1-\alpha-\beta-\delta) b^{\delta} a^{\alpha+\beta+\delta-1} \times$
 $\times {}_3F_3 \left(\frac{\alpha+1}{2}, \frac{1+2\delta}{2}; \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2\delta}{2}, \frac{1}{2}+\delta; -\frac{a^2 b^2}{4} \right) +$
 $+ (\beta-1) \Gamma(\alpha+\beta-2) \left\{ \frac{\sin [(\alpha+\beta)\pi/2]}{\cos [(\alpha+\beta)\pi/2]} \right\} ab^{2-\alpha-\beta} \times$
 $\times {}_2F_3 \left(1-\frac{\beta}{2}, \frac{3-\beta}{2}; \frac{3}{2}, 2-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}; -\frac{a^2 b^2}{4} \right) \mp$
 $\mp \Gamma(\alpha+\beta-1) \left\{ \frac{\cos [(\alpha+\beta)\pi/2]}{\sin [(\alpha+\beta)\pi/2]} \right\} b^{1-\alpha-\beta} \times$
 $\times {}_2F_3 \left(1-\frac{\beta}{2}, \frac{1-\beta}{2}, \frac{1}{2}; 1-\frac{\alpha+\beta}{2}, \frac{3-\alpha-\beta}{2}, \frac{1}{2}; -\frac{a^2 b^2}{4} \right)$
 $[a, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 2].$
5. $\int_0^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \sqrt{\pi} \left(\frac{a}{b} \right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha} \left(\frac{ab}{2} \right) \left\{ \frac{\cos(ab/2)}{\sin(ab/2)} \right\} \mp Y_{1/2-\alpha} \left(\frac{ab}{2} \right) \left\{ \frac{\sin(ab/2)}{\cos(ab/2)} \right\} \right]$
 $[a, b > 0; 0 < \operatorname{Re} \alpha < 1].$

6. $\int_0^{\infty} \frac{x^{\alpha-1}}{(x+z)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = B(\alpha+\delta, \rho-\alpha-\delta) b^{\delta} z^{\alpha+\delta} \rho \times$
 $\times {}_2F_3\left(\frac{\alpha+2\delta}{2}, \frac{\alpha+1}{2}; \delta+\frac{1}{2}, \delta+\frac{\alpha+1-\rho}{2}, \frac{2+\alpha-\rho}{2}; -\frac{b^2 z^2}{4}\right) +$
 $+ \Gamma(\alpha-\rho) \left\{ \frac{\sin[(\alpha-\rho)\pi/2]}{\cos[(\alpha-\rho)\pi/2]} \right\} b^{\rho-\alpha} \times$
 $\times {}_2F_3\left(\frac{\rho}{2}, \frac{\rho+1}{2}; \frac{1}{2}, 1+\frac{\rho-\alpha}{2}, \frac{1-\alpha+\rho}{2}; -\frac{b^2 z^2}{4}\right) +$
 $+ \rho \Gamma(\alpha-\rho-1) \left\{ \frac{\cos[(\rho-\alpha)\pi/2]}{\sin[(\rho-\alpha)\pi/2]} \right\} z b^{\rho+1-\alpha} \times$
 $\times {}_2F_3\left(\frac{\rho+1}{2}, 1+\frac{\rho}{2}; \frac{3}{2}, \frac{3+\rho-\alpha}{2}, 1+\frac{\rho-\alpha}{2}; -\frac{b^2 z^2}{4}\right)$
 $[b > 0; \operatorname{Re} \alpha > -\delta; \operatorname{Re}(\alpha-\rho) < 1; |\arg z| < \pi].$
7. $\int_0^{\infty} \frac{x^{\alpha-1}}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= i^{\delta} 2^{-1} \Gamma(\alpha) z^{\alpha-1} [e^{-i\delta} \Gamma(1-\alpha, -ibz) \mp e^{i\delta} \Gamma(1-\alpha, ibz)]$
 $[b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 2].$
8. $\int_0^{\infty} \frac{x^{\alpha-1}}{(x+z)^{1-\alpha}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \frac{\sqrt{\pi}}{2} \left(\frac{z}{b} \right)^{\alpha-1/2} \Gamma(\alpha) \left[J_{1/2-\alpha}\left(\frac{bz}{2}\right) \left\{ \frac{\cos(bz/2)}{\sin(bz/2)} \right\} \pm Y_{1/2-\alpha}\left(\frac{bz}{2}\right) \left\{ \frac{\sin(bz/2)}{\cos(bz/2)} \right\} \right]$
 $[b > 0; |\arg z| < \pi; -\delta < \operatorname{Re} \alpha < 1].$
9. $\int_0^{\infty} \frac{x^{\alpha-1/2}}{x+z} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2\pi \left(\sigma - \frac{1}{2} \right) z^{\sigma-1/2} \left[\sqrt{2} S(bz) \left\{ \frac{\cos(bz+\pi/4)}{\sin(bz+\pi/4)} \right\} \mp \right.$
 $\left. \mp \sqrt{2} C(bz) \left\{ \frac{\sin(bz+\pi/4)}{\cos(bz+\pi/4)} \right\} \right] \pm \frac{1-(-1)^{\sigma}}{2} \sqrt{\frac{\pi}{2b}}$
 $[\sigma = 0 \text{ или } 1; b > 0; |\arg z| < \pi].$
10. $\int_0^{\infty} \frac{x^{\alpha-1}}{x-y} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -\pi y^{\alpha-1} \operatorname{ctg} \alpha \pi \left\{ \frac{\sin by}{\cos by} \right\} -$
 $- b^{\alpha-2} \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} {}_1F_2\left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4}\right) \mp$
 $\mp b^{1-\alpha} \Gamma(\alpha-1) \left\{ \frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi/2)} \right\} {}_1F_2\left(1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{b^2 y^2}{4}\right)$
 $[b, y > 0; -\delta < \operatorname{Re} \alpha < 2].$
11. $\int_0^{\infty} x^{2n} [(\beta+ix)^{-\nu} \mp (\beta-ix)^{-\nu}] \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \frac{(-1)^n}{\Gamma(\nu)} i^{\delta} (2n)! \pi b^{\nu-2n-1} e^{-b\beta} L_{2n}^{\nu-2n-1}(b\beta)$
 $[b, \operatorname{Re} \beta > 0; 0 \leq 2n < \operatorname{Re} \nu].$

$$12. \int_0^{\infty} x^{2n+1} [(\beta+ix)^{-\nu} \pm (\beta-ix)^{-\nu}] \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= \frac{(-1)^{n+1}}{\Gamma(\nu)} i^{1-\delta} (2n+1)! \pi b^{\nu-2n-2} e^{-b\beta} L_{2n+1}^{\nu-2n-2}(b\beta)$$

$$[b, \operatorname{Re} \beta > 0; -2-\delta < 2n < \operatorname{Re} \nu-1].$$

2.5.8. Интегралы от $x^{\alpha} (x^2 - a^2)^{\beta} \left\{ \frac{\sin bx}{\cos bx} \right\}$, $x^{\alpha} (a^2 - x^2)^{\beta} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

1. $\int_0^a x^{\alpha-1} (a^2 - x^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \frac{b^{\delta} a^{2\beta+\alpha+\delta-2}}{2} B\left(\frac{\alpha+\delta}{2}, \beta\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \frac{1}{2}+\delta, \beta+\frac{\alpha+\delta}{2}; -\frac{a^2 b^2}{4}\right)$
 $[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > -\delta; |\arg b| < \pi].$
2. $\int_0^a x (a^2 - x^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \pm \frac{\sqrt{\pi}}{2} a \left(\frac{2a}{b} \right)^{\beta-1/2} \Gamma(\beta) \left\{ \frac{J_{\beta+1/2}(ab)}{H_{\beta+1/2}(ab)} \right\} + (1-\delta) \frac{a^{2\beta}\beta}{2}$
 $[a, \operatorname{Re} \beta > 0, |\arg b| < \pi].$
3. $\int_a^{\infty} x^{\alpha-1} (x^2 - a^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$
 $= \frac{b^{\delta}}{2} a^{\delta+\alpha+\beta-2} B\left(\beta, 1-\beta-\frac{\alpha+\delta}{2}\right) {}_1F_2\left(\frac{\alpha+\delta}{2}; \beta+\frac{\alpha+\delta}{2}, \delta+\frac{1}{2}; -\frac{a^2 b^2}{4}\right) +$
 $+ \frac{\sqrt{\pi}}{2} \left(\frac{b}{2} \right)^{2-2\beta-2} \Gamma\left[\frac{(\alpha+\delta)/2+\beta-1}{(3+\delta-\alpha)/2-\beta}\right] {}_1F_2\left(1-\beta; 2-\beta-\frac{\alpha}{2}, \frac{3-\alpha}{2}-\beta; -\frac{a^2 b^2}{4}\right)$
 $[a, b, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+2\beta) < 3].$
4. $\int_a^{\infty} x (x^2 - a^2)^{\beta-1} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2a}{b} \right)^{\beta-1/2} a \Gamma(\beta) \left\{ \frac{Y_{\beta-1/2}(ab)}{J_{\beta-1/2}(ab)} \right\}$
 $[a, b > 0; 0 < \operatorname{Re} \beta < 1/2].$
5. $\int_a^{\infty} \frac{1}{x(x^2 - a^2)} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -\frac{\pi}{2} \sqrt{b} J_{\pm 1/4}\left(\frac{ab}{2}\right) Y_{\pm 1/4}\left(\frac{ab}{2}\right)$
 $[a, b > 0].$
6. $\int_0^{\infty} \frac{(x^2 - a^2)^{\beta-1}}{x} \cos bx dx =$
 $= \frac{\pi a^{2\beta}}{2 \sin \beta \pi} \left\{ 1 - \frac{\pi ab}{2} \left[J_{1/2-\beta}(ab) H_{-1/2-\beta}(ab) - J_{-1/2-\beta}(ab) H_{1/2-\beta}(ab) \right] \right\}$
 $[a, b > 0; 0 < \operatorname{Re} \beta < 3/2].$

$$7. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+y^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pm \frac{\pi}{2} y^{\alpha-2} \left\{ \operatorname{tg}(\alpha\pi/2) \sin by \right\} - \\ - \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} b^{2-\alpha} {}_2F_2 \left(1; 2-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{b^2 y^2}{4} \right) \\ [b, y > 0; -\delta < \operatorname{Re} \alpha < 3].$$

$$8. \int_0^{\infty} \frac{x}{x^2+y^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \left\{ \frac{\pi/2}{\sin by} \operatorname{Si}(by) - \cos by \operatorname{Ci}(by) \right\} [b, y > 0].$$

2.5.9. Интегралы от $\frac{x^{\alpha}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{b^{\delta}}{2} z^{\delta+\alpha-2\rho} \Gamma\left(\frac{\alpha+\delta}{2}, \rho-\frac{\alpha+\delta}{2}\right) {}_2F_2\left(\frac{\alpha+\delta}{2}; \delta+\frac{1}{2}, 1-\rho+\frac{\alpha+\delta}{2}; \frac{b^2 z^2}{4}\right) + \\ + b^{2\rho-\alpha} \Gamma(\alpha-2\rho) \left\{ \frac{\sin[(\alpha/2-\rho)\pi]}{\cos[(\alpha/2-\rho)\pi]} \right\} {}_2F_2\left(\rho; \frac{1-\alpha}{2}+\rho, 1-\frac{\alpha}{2}+\rho; \frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1+2\operatorname{Re} \rho].$$

$$2. \int_0^{\infty} \frac{x^{\alpha-1}}{x^2+z^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi z^{\alpha-2}}{2} \left\{ \frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi/2)} \right\}^{-1} \left\{ \frac{\operatorname{sh} bz}{\operatorname{ch} bz} \right\} - \\ - \Gamma(\alpha-2) \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} b^{2-\alpha} {}_2F_2\left(1; \frac{3-\alpha}{2}, 2-\frac{\alpha}{2}; \frac{b^2 z^2}{4}\right) \\ [b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 3].$$

$$3. \int_0^{\infty} \frac{x^{2n} \sin bx}{(x^2+z^2)^{\rho}} dx = \\ = (-1)^n \frac{\sqrt{\pi}}{2} \Gamma(1-\rho) (2z)^{1/2-\rho} \frac{\partial^{2n}}{\partial b^{2n}} \left\{ b^{\rho-1/2} [I_{\rho-1/2}(bz) - L_{1/2-\rho}(bz)] \right\} \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n].$$

$$4. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^{\rho}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^{n+\delta} \frac{\sqrt{\pi}}{\Gamma(\rho)} (2z)^{1/2-\rho} \frac{\partial^{2n+1}}{\partial b^{2n+1}} \left\{ b^{\rho-1/2} K_{\rho-1/2}(bz) \right\} \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > n+\delta/2; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$5. \int_0^{\infty} \frac{x^{2n+\delta}}{(x^2+z^2)^m} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{(-1)^{n+\delta}}{(m-1)!} 2^{-m} \pi \left(\frac{\partial}{\partial z} \right)^{m-1} \left(\frac{z^{2m-1+\delta} e^{-bz}}{z^{2m-1+\delta} e^{-bz}} \right) \\ [b, \operatorname{Re} z > 0; n=0, 1, 2, \dots, m-1; m=1, 2, 3, \dots].$$

$$6. \int_0^{\infty} \frac{x^{n+\delta}}{(x^2+z^2)^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{1+\delta-bz}{4z^{1-\delta}} \pi e^{-bz} [b, \operatorname{Re} z > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$7. \int_0^{\infty} \frac{1}{\sqrt{x(x^2+z^2)}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{\pi b}{2}} I_{\pm 1/2}\left(\frac{bz}{2}\right) K_{1/4}\left(\frac{bz}{2}\right) [b, \operatorname{Re} z > 0].$$

$$8. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^m} dx = \frac{\pi b e^{-bz}}{2^{2m-2} (m-1)!} z^{2-2m} \sum_{k=0}^{m-2} \frac{(2m-k-4)! (2bz)^k}{k! (m-2-k)!} \\ [b, \operatorname{Re} z > 0; m=1, 2, 3, \dots].$$

$$9. \int_0^{\infty} \frac{x \sin bx}{x^2+z^2} dx = \frac{\pi}{2} e^{-bz} [b, \operatorname{Re} z > 0].$$

$$10. \int_0^{\infty} \frac{\sin bx}{x(x^2+z^2)^2} dx = \frac{\pi}{2z^4} \left[1 - \frac{e^{-bz}}{2} (2+bz) \right] [b, \operatorname{Re} z > 0].$$

$$11. \int_0^{\infty} \frac{x \sin bx}{(x^2+z^2)^{\rho}} dx = \frac{\sqrt{\pi}}{\Gamma(\rho)} \left(\frac{2}{b} \right)^{1/2-\rho} z^{3/2-\rho} K_{3/2-\rho}(bz) [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > 1/2].$$

$$12. \int_0^{\infty} \frac{x \cos bx}{x^2+z^2} dx = -\frac{1}{2} [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] [b, \operatorname{Re} z > 0].$$

$$13. \int_0^{\infty} \frac{x \cos bx}{(x^2+z^2)^{\rho}} dx = \\ = \frac{z^{3/2-\rho} b^{\rho-1/2}}{\sqrt{2}} [2^{-\rho} \Gamma(1-\rho) I_{\rho-3/2}(bz) + \sqrt{2} e^{-\pi i(1-2\rho)/4} S_{\rho-1/2, 3/2-\rho}(ibz)] \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \rho > 1/2].$$

$$14. \int_0^{\infty} \frac{x \sin^{2n+1} bx}{x^2+z^2} dx = \frac{(-1)^{n+1}}{2^{2n+3}} e^{-(2n+1)bz} \times \\ \times \left[(1 - e^{(2n+1)bz}) (1 - e^{-2bz})^{2n+1} - 2 \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{2kbz} \right] \\ [b, \operatorname{Re} z > 0].$$

$$15. \int_0^{\infty} \frac{\sin^2 bx}{x(x^2+z^2)} dx = \frac{1}{4z^2} \{ 2[C + \ln(bz)] - [e^{-bz} \operatorname{Ei}(bz) + e^{bz} \operatorname{Ei}(-bz)] \} \\ [b, \operatorname{Re} z > 0].$$

$$16. \int_0^{\infty} \frac{\sin^2 bx}{x^2(x^2+z^2)} dx = \frac{\pi}{4z^3} (e^{-2bz} + 2bz - 1) [b, \operatorname{Re} z > 0].$$

$$17. \int_0^{\infty} \frac{x dx}{(x^2+z^2) \sin bx} = \frac{\pi}{2 \operatorname{sh} bz} [b, \operatorname{Re} z > 0].$$

$$13. \int_0^{\infty} \frac{x \cos^{2n+1} bx}{x^2 + z^2} dx = -\frac{e^{-i(2n+1)bz}}{2^{2n+1}z} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{ikbz} \operatorname{Ei}[(2n-2k+1)bz] - \\ - \frac{e^{i(2n+1)bz}}{2^{2n+1}z} \sum_{k=0}^{2n+1} \binom{2n+1}{k} e^{-ikbz} \operatorname{Ei}[(2k-2n-1)bz] \quad (b, \operatorname{Re} z > 0).$$

2.5.10. Интегралы от $\frac{x^n}{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0} \left\{ \frac{\sin bx}{\cos bx} \right\}^l$.

$$1. \int_0^{\infty} \frac{x^{2k+1} \delta}{x^{2n} + z^{2n}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 0 \\ [b > 0; -\delta < k < n + \frac{1-\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$2. \int_0^{\infty} \frac{x^{2k+\delta}}{x^{2n} + z^{2n}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2n} z^{2k+1-\delta} \sum_{l=0}^{n-1} \exp\left(-bz \sin \frac{2l+1}{2n} \pi\right) \left\{ \frac{\cos l}{\sin l} \right\} \\ [l = \frac{2l+1}{2n} (\pi k + 1 - \delta) \pi + bz \cos \frac{2l+1}{2n} \pi; b > 0; 0 \leq k < n + \frac{\delta}{2}; |\arg z| < \frac{\pi}{2n}; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$3. \int_0^{\infty} \frac{x^k \sin bx}{x^4 + z^4} dx = J_k \quad [b > 0; |\arg z| < \frac{\pi}{4}].$$

$$I_{-1} = \frac{\pi}{2z^4} \left[1 - \exp\left(-\frac{bz}{\sqrt{2}}\right) \cos\left(\frac{bz}{\sqrt{2}}\right) \right],$$

$$I_{2 \pm 1} = \frac{\pi}{2z^{1/4}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left\{ \frac{\sin(bz/\sqrt{2})}{\cos(bz/\sqrt{2})} \right\}.$$

$$4. \int_0^{\infty} \frac{x^k \cos bx}{x^4 + z^4} dx = J_k \quad [b > 0; |\arg z| < \frac{\pi}{4}].$$

$$J_{1 \pm 1} = \frac{\pi \sqrt{2}}{4z^{3/4}} \exp\left(-\frac{bz}{\sqrt{2}}\right) \left\{ \frac{\cos(bz/\sqrt{2})}{\sin(bz/\sqrt{2})} \mp \sin\left(\frac{bz}{\sqrt{2}}\right) \right\}.$$

$$5. \int_0^{\infty} \frac{x^{2k} \cos bx}{x^{2n} - y^{2n}} dx = -y^{2k+1-2n} \sum_{l=0}^{n-1} \exp\left(-by \sin \frac{l\pi}{n}\right) \sin\left(\frac{2k+1}{n} l\pi + by \cos \frac{l\pi}{n}\right) \\ [b, y > 0; 0 \leq k < n].$$

$$6. \int_0^{\infty} \frac{x^k \sin bx}{x^4 - y^4} dx = J_k \quad [b, y > 0; k = -1, 0, 1, 2, 3].$$

$$I_{-1} = \frac{\pi}{4y^4} (e^{-by} + \cos by - 2),$$

$$I_{1 \pm 1} = \frac{1}{4y^{3/4}} [2 \cos by \operatorname{Si}(by) - 2 \sin by \operatorname{ci}(by) \mp e^{by} \operatorname{Ei}(-by) \pm e^{-by} \operatorname{Ei}(by)],$$

$$I_{2 \pm 1} = \frac{\pi}{4y^{1/4}} (\cos by \pm e^{-by}).$$

$$7. \int_0^{\infty} \frac{x^k \cos bx}{x^4 - y^4} dx = J_k \quad [b, y > 0; k = 0, 1, 2, 3].$$

$$J_{1 \pm 1} = -\frac{\pi}{4y^{3/4}} (\sin by \mp e^{-by}).$$

$$J_{2 \pm 1} = -\frac{1}{4y^{1/4}} [2 \cos by \operatorname{ci}(by) + 2 \sin by \operatorname{Si}(by) \pm e^{-by} \operatorname{Ei}(by) \mp e^{by} \operatorname{Ei}(-by)].$$

$$8. \int_{-\infty}^{\infty} \frac{\sin x}{x(x-z)} dx = \frac{\pi}{z} (e^{\pm iz} - 1) \quad [\pm \operatorname{Im} z > 0].$$

$$9. \int_{-\infty}^{\infty} \frac{\sin x}{x} (\cos y - 1) dx = \frac{\pi}{y} (\cos y - 1) \quad [z = y; \operatorname{Im} y = 0].$$

$$10. \int_0^{\infty} \frac{x^{2l+1} \sin bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_+ b} \left(\pm z^{1/2+1} \sin A_+ b + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y \cos A_+ b \right) \\ [2A_{\pm}^2 = \sqrt{z^4 + y^4} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$11. \int_0^{\infty} \frac{x^{1/2+1} \cos bx}{(x^2 + z^2)^2 + y^2} dx = \frac{\pi}{2y} e^{-A_+ b} (x^4 + y^2)^{-1/2} \times \\ \times \left[\pm z^{1/2+1} (A_+ \sin A_+ b + A_- \cos A_+ b) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} y (A_+ \cos A_+ b - A_- \sin A_+ b) \right] \\ [2A_{\pm}^2 = \sqrt{z^4 + y^4} \pm z^2; b, \operatorname{Re} z, \operatorname{Re} y > 0].$$

$$12. \int_{-\infty}^{\infty} \frac{cx + d}{x^2 + 2qx + p} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \pi \exp(-b\sqrt{p-q^2}) \left[c \left\{ \frac{\cos bq}{\sin bq} \right\} \pm \frac{cq-d}{\sqrt{p-q^2}} \right] \\ [b > 0; p > q^2].$$

$$13. \int_0^{\infty} \frac{\sin^{2l} m x \pi}{-x^2/l^2 (1-x^2/2^2) \dots (1-x^2/n^2)} dx = 0 \quad [l, m, n = 1, 2, 3, \dots].$$

$$14. \int_0^{\infty} \frac{\sin^{2l+1} m x \pi}{x(1-x^2/l^2)(1-x^2/2^2) \dots (1-x^2/n^2)} dx = \frac{1-(-1)^m}{2} 2^{2l+n-l-1} \frac{(2l)!}{(2n)!} \frac{(n!)^2}{(l!)^2} \pi \\ [l, m, n = 1, 2, 3, \dots].$$

$$15. \int_0^{\infty} \frac{x^m \sin bx}{c^3 \pm c^2 x + cx^2 \pm x^3} dx = J_m \quad [b, c > 0].$$

$$I_{-1} = (4c)^{-1} \{ \pi (e^{-bc} - \cos bc) \pm [e^{-bc} \operatorname{Ei}(bc) - e^{bc} \operatorname{Ei}(-bc) - \\ - 2 \sin bc \operatorname{ci}(bc) + 2 \cos bc \operatorname{Si}(bc)] \},$$

$$I_{1 \pm 1} = 2^{-1} [e^{bc} \operatorname{Ei}(-bc) - e^{-bc} \operatorname{Ei}(bc) + 2 \operatorname{ci}(bc) \sin bc - \\ - 2 \operatorname{Si}(bc) \cos bc] \pm \pi (e^{-bc} + \cos bc).$$

$$16. \int_0^{\infty} \frac{x^m \sin bx}{x^4 + 2a^2 x^2 + c^4} dx = J_m \quad [a, b, c > 0].$$

$$I_{-1} = \frac{\pi}{2c} \left\{ 1 - \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \times \right. \\ \left. \times \left[\cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) + \frac{a^2}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right] \right\},$$

$$I_1 = \frac{\pi}{2\sqrt{c^2 - a^2}} \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right),$$

$$I_2 = \frac{\pi}{2} \exp \left(-b \sqrt{\frac{a^2 + c^2}{2}} \right) \left[\cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) - \frac{a^2}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right].$$

$$17. \int_0^\infty \frac{x^m \cos bx}{x^4 + 2a^2x^2 + c^4} dx = J_m \quad [a, b, c > 0].$$

$$J_0 = \frac{\pi}{2\sqrt{2}c^2} \exp \left(-b \sqrt{\frac{c^2 + a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2 + a^2}} \cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) + \frac{1}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right],$$

$$J_2 = \frac{\pi}{2\sqrt{2}} \exp \left(-b \sqrt{\frac{c^2 + a^2}{2}} \right) \times \\ \times \left[\frac{1}{\sqrt{c^2 + a^2}} \cos \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) - \frac{1}{\sqrt{c^2 - a^2}} \sin \left(b \sqrt{\frac{c^2 - a^2}{2}} \right) \right].$$

$$18. \int_0^\infty \left[\frac{1}{(x-a)^2 + z^2} \mp \frac{1}{(x+a)^2 + z^2} \right] \frac{\sin bx}{\cos bx} dx = \frac{\pi}{z} e^{\mp bz} \begin{cases} \sin ab \\ \cos ab \end{cases} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|].$$

$$19. \int_0^\infty \left[\frac{x-a}{(x-a)^2 + z^2} \mp \frac{x+a}{(x+a)^2 + z^2} \right] \sin bx dx = \mp \pi e^{\mp bz} \begin{cases} \sin ab \\ \cos ab \end{cases} \\ [b > 0; \operatorname{Re} z > |\operatorname{Im} a|].$$

$$20. \int_0^\infty \frac{\sin bx dx}{x(x^2 + 2^2) \dots (x^2 + 4n^2)} = \\ = \frac{\pi}{(2n)!} \frac{(-1)^n}{2^{2n+1}} \left[2 \sum_{k=1}^{n-1} (-1)^k \binom{2n}{k} e^{i(2k-n)\pi/2} + (-1)^n \binom{2n}{n} \right] \\ [b > 0; n = 1, 2, 3, \dots].$$

$$21. \int_0^\infty \frac{\cos bx dx}{(x^2 + 1^2)(x^2 + 3^2) \dots (x^2 + (2n+1)^2)} = \\ = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{i(2k-n-1)\pi/2} \quad [b > 0].$$

2.5.11. Интегралы от $x^\alpha (\sqrt{x^2 \pm z^2} + ax + bz)^\nu A(x) \begin{cases} \sin bx \\ \cos bx \end{cases}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + z)^\nu}{(x^2 + z^2)^r} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \\ = \mp (vz)^{1-2r} b^{1-\alpha-\nu} \Gamma(\alpha + v - 1) \begin{cases} \cos [(\alpha + v)\pi/2] \\ \sin [(\alpha + v)\pi/2] \end{cases} \times \\ \times {}_2F_2 \left(\frac{1+v}{2}, \frac{1-v}{2}; \frac{3}{2} - 2r, 1 - \frac{\alpha+v}{2}, \frac{3-\alpha-v}{2}; \frac{b^2 z^2}{4} \right) + \\ + (vz)^{2r} b^{4r-\alpha-\nu} \Gamma(\alpha + v - 4r) \begin{cases} \sin [(\alpha + v)/2 - 2r] \pi \\ \cos [(\alpha + v)/2 - 2r] \pi \end{cases} \times \\ \times {}_2F_2 \left(2r - \frac{v}{2}, 2r + \frac{v}{2}; \frac{1}{2} + 2r, 2r + \frac{1-\alpha-v}{2}, 1 + 2r - \frac{\alpha+v}{2}; \frac{b^2 z^2}{4} \right) + \\ + b^{\delta} z^{1+v+\delta-2r} 2^{1+v+\delta-1} (-v)^{1-2r} \Gamma \left[\frac{(\alpha+\delta)/2, 2r-v-\alpha-\delta}{1-v-(\alpha+\delta)/2} \right] \times \\ \times {}_2F_2 \left(\frac{\alpha+\delta}{2}, v + \frac{\alpha+\delta}{2}; \delta + \frac{1}{2}, \frac{1+v+\alpha+\delta}{2}, 1 + \frac{v+\alpha+\delta}{2} - 2r; \frac{b^2 z^2}{4} \right) \\ [r = 0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} v + 2r + 1].$$

$$2. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + z)^{1/2-\alpha}}{\sqrt{x^2 + z^2}} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \\ = \frac{1}{z} \sqrt{\frac{2}{b}} \Gamma \left(\frac{\alpha+\delta}{2} \right) W_{1/4-\alpha/2, \pm 1/4} (bz) M_{\alpha/2-1/4, \pm 1/4} (bz) \\ [b, \operatorname{Re} z > 0; \operatorname{Re} \alpha > -\delta].$$

$$3. \int_0^\infty x^{\delta} \frac{(\sqrt{x^2 + z^2} + z)^{1/2-\delta}}{\sqrt{x^2 + z^2}} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \sqrt{\frac{\pi}{2b}} e^{\mp bz} \quad [b, \operatorname{Re} z > 0; \delta = \begin{cases} 1 \\ 0 \end{cases}].$$

$$4. \int_0^\infty x^{\alpha-1} \frac{(\sqrt{x^2 + z^2} + x)^\nu}{(x^2 + z^2)^r} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \\ = \frac{b^{\delta}}{2} \left(-\frac{v}{2} \right)^{1-2r} z^{2(1+v-2r)+\delta} \Gamma \left[\frac{\alpha+\delta, r-(\alpha+v+\delta)/2}{1+(\alpha+\delta-v)/2-r} \right] \times \\ \times {}_2F_2 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2} + \delta; \delta + \frac{1}{2}, 1 + \frac{\alpha+\delta-v}{2} - r, 1 + \frac{\alpha+v+\delta}{2} - r; \frac{b^2 z^2}{4} \right) + \\ + 2^v b^{2r-\alpha-\nu} \Gamma(\alpha + v - 2r) \begin{cases} \sin [(\alpha + v)/2 - r] \pi \\ \cos [(\alpha + v)/2 - r] \pi \end{cases} \times \\ \times {}_2F_2 \left(r - \frac{v}{2}, \frac{1-v}{2} + r; 1 - v, r + \frac{1-v-\alpha}{2}, r + 1 - \frac{v+\alpha}{2}; \frac{b^2 z^2}{4} \right) \\ [r = 0 \text{ или } 1/2; b, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < -\operatorname{Re} v + 2r + 1].$$

$$5. \int_0^\infty \frac{(\sqrt{x^2 + z^2} + x)^\nu}{\sqrt{x(x^2 + z^2)}} \begin{cases} \sin bx \\ \cos bx \end{cases} dx = \sqrt{\frac{\pi b}{2}} z^{v/2} \pm 1/4 - v/2 \left(\frac{bz}{2} \right) K_{\pm 1/4 + v/2} \left(\frac{bz}{2} \right) \\ [b, \operatorname{Re} z > 0; \operatorname{Re} v < 3/2].$$

$$6. \int_0^{\infty} \frac{(1-x^2+z^2+x)^{\nu} (1-x^2+z^2-x)^{\nu}}{1-x^2+z^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2z^{\nu} \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} K_{\nu}(bz) \\ \{b, \operatorname{Re} z > 0; |\operatorname{Re} \nu| < 1\}$$

$$7. \int_0^{\infty} \frac{(x+z+\sqrt{x^2+2xz})^{\nu} + (x+z-\sqrt{x^2+2xz})^{\nu}}{\sqrt{x^2+2xz}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \pi z^{\nu} \left[\left\{ \frac{\cos(bz-\nu\pi/2)}{\sin(bz-\nu\pi/2)} \right\} J_{\nu}(bz) \pm \left\{ \frac{\sin(bz-\nu\pi/2)}{\cos(bz-\nu\pi/2)} \right\} Y_{\nu}(bz) \right] \\ \{b > 0; |\operatorname{Re} \nu| < 1; |\arg z| < \pi\}$$

$$8. \int_0^{\infty} \frac{(x+i\sqrt{a^2-x^2})^{\nu} + (x-i\sqrt{a^2-x^2})^{\nu}}{\sqrt{a^2-x^2}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{\pi a^{\nu}}{2} \left\{ \frac{\operatorname{cosec}(\nu\pi/2)}{\sec(\nu\pi/2)} \right\} [J_{\nu}(ab) \mp J_{-\nu}(ab)] \quad \{a > 0; |\arg b| < \pi\}$$

$$9. \int_a^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} + (x-\sqrt{x^2-a^2})^{\nu}}{\sqrt{x^2-a^2}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \pm \pi a^{\nu} \left[\left\{ \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} \right\} J_{\nu}(ab) \mp \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} Y_{\nu}(ab) \right] \quad \{a, b > 0; |\operatorname{Re} \nu| < 1\}$$

$$10. \int_0^{\infty} \frac{(x+\sqrt{x^2-a^2})^{\nu} + (x-\sqrt{x^2-a^2})^{\nu}}{\sqrt{x(x^2-a^2)}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -\frac{\pi}{2} \sqrt{b/a}^{\nu} \times \\ \times \left[J_{\pm 1/4+\nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4-\nu/2} \left(\frac{ab}{2} \right) + J_{\pm 1/4-\nu/2} \left(\frac{ab}{2} \right) Y_{\pm 1/4+\nu/2} \left(\frac{ab}{2} \right) \right] \\ \{a, b > 0; |\operatorname{Re} \nu| < 3/2\}$$

2.5.12. Интегралы от $\prod_k \left\{ \frac{\sin a_k x}{\cos a_k x} \right\}^{\mu_k} \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^{\nu_k}$.

$$\prod_k \sin^{\mu_k} a_k x \cos^{\nu_k} b_k x.$$

$$1. \int_0^{\pi/4} \frac{\sin^{2\mu} x}{\cos^{2\mu+1/2} x} dx = 2^{-\mu-1/2} B(2\mu+1, -\mu) \quad \{-1/2 < \operatorname{Re} \mu < 0\}$$

$$2. \int_0^{\pi/4} \frac{\cos^{2\mu} 2x}{\cos^{2\mu+1/2} x} dx = 2^{2\mu+1} B(\mu+1, \mu+1) \quad \{\operatorname{Re} \mu > -1\}$$

$$3. \int_0^{\pi/4} \frac{\cos^{n-1/2} 2x}{\cos^{2n+1} x} dx = \frac{(2n-1)!!}{2(2n)!!} \pi \quad \left[m \leq n; \delta = \left\{ \frac{1}{0} \right\} \right]$$

$$4. \int_0^{\pi/4} \frac{\sin^m x \cos^{\nu} 2x}{\cos^{2\nu+1/2} x} dx = \frac{1}{2} B\left(\frac{m+1}{2}, \nu+1\right) \quad \{\operatorname{Re} \nu > -1\}$$

$$5. \int_0^{\pi/4} \frac{\sin^m x \cos^{n-1/2} 2x}{\cos^{2n+1/2} x} dx = \frac{(m-1)!! (2n-1)!!}{(m+2n)!!} \left\{ \frac{\pi/2}{1}, \quad m=0, 2, 4, \dots \right. \\ \left. \left\{ \frac{\pi/2}{1}, \quad m=1, 3, 5, \dots \right\} \right\} \quad \{n=1, 2, 3, \dots\}$$

$$6. \int_0^{\pi/2} \sin^{\nu} x dx = \frac{(n+1)!!}{n!!} \left\{ \frac{\pi/2}{1}, \quad n=1, 3, 5, \dots \right. \\ \left. \left\{ \frac{\pi/2}{1}, \quad n=0, 2, 4, \dots \right\} \right\}$$

$$7. \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx = 2 \sum_{k=1}^{n/2} \frac{(-1)^{k-1}}{2k-1} \quad \{n=2, 4, \dots\}$$

$$8. = \pi/2 \quad \{n=1, 3, 5, \dots\}$$

$$9. \int_0^{\pi/2} \frac{\sin 2nx}{\cos x} dx = (-1)^n 2 \sum_{k=1}^n \frac{(-1)^k}{2k-1}$$

$$10. \int_0^{\pi/2} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \frac{\pi}{2}$$

$$11. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2^{-\mu} \pi \left\{ \Gamma \left[\frac{\mu}{2} \right] \Gamma \left[\frac{\mu}{2} \right] \right\} \times \\ \times \left\{ \frac{\sin(b\pi/2)}{\cos(b\pi/2)} \right\} \mp \frac{b}{2} \Gamma \left[\frac{\mu}{2} \right] \Gamma \left[\frac{\mu}{2} \right] \left\{ \frac{\cos(b\pi/2)}{\sin(b\pi/2)} \right\} \times \\ \times {}_2F_1 \left(\frac{\mu}{2}, \frac{\mu+1}{2}, \frac{1}{2}; 1 + \frac{\mu+b}{2}, 1 + \frac{\mu-b}{2}; 1 \right) \quad \left\{ \operatorname{Re} \mu > -\left\{ \frac{1}{0} \right\}; |\arg b| < \pi \right\}$$

$$12. \int_0^{\pi/2} \sin^{\mu-1} x \left\{ \frac{\sin(\mu+1)x}{\cos(\mu+1)x} \right\} dx = \frac{1}{\mu} \left\{ \frac{\sin(\mu\pi/2)}{\cos(\mu\pi/2)} \right\} \quad \left[\operatorname{Re} \mu > -\left\{ \frac{1}{0} \right\} \right]$$

$$13. \int_0^{\pi/2} \sin^{2m+\delta} x \left\{ \frac{\sin(2m+1)x}{\cos 2mx} \right\} dx = \frac{(-1)^m (2n-1)!!}{2^{2m+1/2} (2n-1)!!} \left\{ \frac{\pi}{1} \right\} \quad \left[m \leq n; \delta = \left\{ \frac{1}{0} \right\} \right]$$

$$14. = 0 \quad \left[m > n; \delta = \left\{ \frac{1}{0} \right\} \right]$$

$$15. \int_0^{\pi/2} \sin^{2n+\delta-1} x \left\{ \frac{\sin(2m+1)x}{\cos 2mx} \right\} dx = \\ = \frac{(-1)^m (2n-1)!! (n-1)! 2^{n-1} \int 2n}{(2n-2m-1)!! (2m+2n+1)!! (2m+2n+1)} \quad \left[m \leq n; \delta = \left\{ \frac{1}{0} \right\} \right]$$

$$16. = \frac{(-1)^n (n-1)! 2^{n-1} (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \left\{ \frac{2n}{2m+2n+1} \right\} \quad \left[m > n; \delta = \left\{ \frac{1}{0} \right\} \right]$$

$$17. \int_0^{\pi/2} \cos^{\nu-1} x \sin bx dx = \pi b 2^{-1-\nu} \left[1 + \frac{\nu}{2} \left(\frac{1+\nu}{2}, \frac{1}{2}; 1 + \frac{\nu+b}{2}, 1 + \frac{\nu-b}{2}; 1 \right) \right] \times$$

$$\times {}_3F_2 \left(\frac{\nu}{2}, \frac{1+\nu}{2}, \frac{1}{2}; 1 + \frac{\nu+b}{2}, 1 + \frac{\nu-b}{2}; 1 \right) \quad [\operatorname{Re} \nu > 0; |\arg b| < \pi].$$

$$18. \int_0^{\pi/2} \cos^{\nu-1} x \sin (\nu+1) x dx = \frac{1}{\nu} \quad [\operatorname{Re} \nu > 0].$$

$$19. \int_0^{\pi/2} \cos^{\nu} x \sin mx dx = \sum_{j=0}^{k-1} \frac{n! (m+n-2j-2)!}{(n-j)! (m+n)!} + \theta \frac{n! (m-n-2j)!}{(m+n)!}$$

$[k = \min(m, n), \theta = 2 \text{ при } m-n=4l+2 > 0; \theta = 1 \text{ при } m-n=2l+1 > 0; \theta = 0 \text{ при } m-n=4l \geq 0 \text{ или } m-n < 0].$

$$20. \int_0^{\pi/2} \cos^{\nu} x \sin nx dx = \frac{1}{2^{\nu+1}} \sum_{j=1}^n \frac{2^j}{j}.$$

$$21. \int_0^{\pi/2} \cos^{\nu-1} x \sin (2m+1) x dx = (-1)^m \sum_{j=0}^m \frac{(-1)^j 2^j}{\nu+j} \binom{m}{j} \quad [\operatorname{Re} \nu > 0].$$

$$22. \int_0^{\pi/2} \cos^{\nu-1} x \cos bx dx = 2^{-\nu} \pi \Gamma \left[\frac{\nu}{2} \right] \left[\frac{1+\nu+b}{2}, \frac{1+\nu-b}{2} \right]$$

$[\operatorname{Re} \nu > 0; |\arg b| < \pi].$

$$23. \int_0^{\pi/2} \cos^{\nu-1} x \cos (\nu+1) x dx = 0 \quad [\operatorname{Re} \nu > 0].$$

$$24. \int_0^{\pi/2} \cos^{\nu} x \cos mx dx = \frac{n!}{(n-m)! (n+m)!} \left\{ \begin{array}{l} 1, \quad n-m=1, 3, 5, \dots \\ \pi/2, \quad n-m=0, 2, 4, \dots \end{array} \right.$$

$$25. = B \frac{n!}{(m-n)(m-n+2) \dots (m+n)}$$

$[n < m; B = 0, \text{ если } m-n \text{ четно}; B = \pm 1, \text{ если } m-n=4l \pm 1, l=0, 1, 2, \dots]$

$$26. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{1}{2} B \left(\frac{\mu}{2}, \frac{\nu}{2} \right) \quad [\operatorname{Re} \mu, \operatorname{Re} \nu > 0].$$

$$27. \int_0^{\pi/2} \left(\frac{\sin ax}{\sin x} \right)^2 dx = \frac{\pi a}{2} - \frac{\sin a\pi}{2} [2a\beta(a)-1] \quad [a > 0].$$

$$28. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \left\{ \frac{\sin (\mu+\nu) x}{\cos (\mu+\nu) x} \right\} dx =$$

$$= \frac{\pi}{2} \left\{ \sec (\mu\pi/2) \right\} \Gamma \left[\frac{\nu}{2} \right] \left[\mu+\nu, 1-\mu \right] \quad [\operatorname{Re} \nu > 0; \operatorname{Re} \mu > -\frac{1}{2}].$$

$$29. \int_0^{\pi} \sin^{\mu-1} x \cos^{\nu-1} x \left\{ \frac{\sin (\mu+\nu) x}{\cos (\mu+\nu) x} \right\} dx = \frac{\sin (\mu\pi/2)}{\cos (\mu\pi/2)} B(\mu, \nu)$$

$[\operatorname{Re} \nu > 0; \operatorname{Re} \mu > -\frac{1}{2}].$

$$30. \int_0^{\pi} \frac{\sin nx \cos mx}{\sin x} dx = 0 \quad [n \leq m \text{ или } n > m; m+n=2k].$$

$$31. = \pi \quad [n > m; m+n=2k-1, k=1, 2, 3, \dots].$$

$$32. \int_a^{a+\pi} \sin^m x \cos^n x dx = \frac{1+(-1)^n}{2} B \left(\frac{m+1}{2}, \frac{n+1}{2} \right) \quad [m+n=0, 2, 4, \dots].$$

$$33. \int_0^{\pi} \frac{\sin nx}{\sin x} dx = \frac{1-(-1)^n}{2} \pi.$$

$$34. \int_0^{\pi} \frac{\sin 2nx}{\cos x} dx = (-1)^n \pi \sum_{k=1}^n \frac{(-1)^k}{2k-1}.$$

$$35. \int_0^{\pi} \frac{\cos (2n+1) x}{\cos x} dx = (-1)^n \pi.$$

$$36. \int_0^{\pi} \sin^{\mu-1} x \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{2^{1-\mu} \pi}{\mu B((\mu+b+1)/2, (\mu-b+1)/2)} \left\{ \frac{\sin (b\pi/2)}{\cos (b\pi/2)} \right\}$$

$[\operatorname{Re} \mu > 0; |\arg b| < \pi].$

$$37. \int_0^{\pi} \sin^{\mu} x \left\{ \frac{\sin (2m+1) x}{\cos 2mx} \right\} dx = 2 \int_0^{\pi/2} \sin^{\mu} x \left\{ \frac{\sin (2m+1) x}{\cos 2mx} \right\} dx$$

[см. 2.5.12.13–16].

$$38. \int_0^{\pi} \sin^{\mu} x \cos 2mx dx = \frac{(-1)^n \pi}{2^{2n}} \binom{2n}{n-m} \quad [n \geq m].$$

$$39. = 0 \quad [n < m].$$

$$40. \int_0^{\pi} \cos^{\nu} x \left\{ \frac{\sin mx}{\cos mx} \right\} dx = (1+(-1)^{m+\nu}) \int_0^{\pi/2} \cos^{\nu} x \left\{ \frac{\sin mx}{\cos mx} \right\} dx$$

[см. 2.5.12.19–25].

$$41. \int_0^{\pi} \cos^{\nu} x \cos bx dx = \frac{(-1)^n \sin b\pi}{2^n (n+b)} {}_2F_1 \left(-n, \frac{b+n}{2}, 1 - \frac{b+n}{2}, -1 \right)$$

$[b \neq 0, \pm 1, \pm 2, \dots].$

$$42. \int_0^{\pi} \cos^{\nu} x \cos mx dx = 0 \quad [m+n=1, 3, 5, \dots \text{ или } m > n, m+n=2, 4, 6, \dots].$$

$$43. \quad \frac{(-1)^{m+n-1} 2^{1-n} \pi (-n)_l (l)_l}{(m+n)(l-1)!} \quad [m \leq n; m+n=2l, l=1, 2, 3, \dots]$$

$$44. \quad \int_0^{2\pi} \sin^m x \cos^n x dx = \frac{[1+(-1)^m][1+(-1)^n]}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

2.5.13. Интегралы от $x^2 \sin^{\mu} x \cos^{\nu} x$.

$$1. \quad \int_0^{\pi/4} x \frac{\sin^{\mu-1} x}{\cos^{\mu+1} x} dx = \frac{\pi}{4\mu} - \frac{1}{2\mu} \psi\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1].$$

$$2. \quad \int_0^{\pi/4} x \frac{\sin^{\mu} x}{\cos^{\mu+2} x} dx = \\ = \frac{1}{n+1} \left\{ \frac{\pi}{4} - (-1)^{(n+1)/2} \left(\frac{1-(-1)^n}{2} \frac{\pi}{4} + \frac{1+(-1)^n}{2} \ln 2 \right) - \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{n-2k} \right\}.$$

$$3. \quad \int_0^{\pi/4} x^3 \frac{\sin x}{\cos^3 x} dx = \frac{\ln 2}{2} - \frac{\pi}{4} + \frac{\pi^3}{16}.$$

$$4. \quad \int_0^{\pi/4} x^2 \frac{\sin^2 x}{\cos^3 x} dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + G \right).$$

$$5. \quad \int_0^{\pi/4} \frac{x dx}{\sin x \sqrt{\cos 2x}} = \frac{\pi}{2} \ln(1+\sqrt{2}).$$

$$6. \quad \int_0^{\pi/4} x \frac{\cos^{\mu-1} x}{\sin^{\mu+1} x} dx = \frac{1}{2\mu} \left[\frac{\pi}{2} + \psi\left(\frac{\mu+1}{2}\right) \right] \quad [\operatorname{Re} \mu > 0].$$

$$7. \quad \int_0^{\pi/2} x \frac{\sin^{\mu-1} x \cos x}{\cos^{\mu-1} x \sin x} dx = \frac{\pi}{2\mu} \left\{ \frac{1}{0} \right\} + \frac{\sqrt{\pi}}{2\mu} \Gamma\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -\frac{1}{2}].$$

$$8. \quad \int_0^{\pi/2} x \sin^{2n} x \cos x dx = \frac{1}{2(2n+1)} \left[\pi - \frac{2(2n)!}{(2n+1)!} \right].$$

$$9. \quad \int_0^{\pi/2} x \sin^{2n+1} x \cos x dx = \frac{1}{4(n+1)} \left[1 - \frac{(2n+1)!}{(2n+2)!} \right].$$

$$10. \quad \int_0^{\pi/2} x^2 \frac{\cos x}{\sin^3 x} dx = 4G - \frac{\pi^3}{4}.$$

$$11. \quad \int_0^{\pi/2} x^3 \frac{\cos x}{\sin^3 x} dx = \frac{3\pi}{2} \ln 2 - \frac{\pi^3}{16}.$$

$$12. \quad \int_0^{\pi/2} x \cos^n x \sin x dx = \frac{n!}{(n+1)!} \begin{cases} 1, & n=0, 2, 4, \dots \\ \pi, & n=1, 3, 5, \dots \end{cases}$$

$$13. \quad \int_0^{\pi/2} x \frac{(\cos x)^{\mu-1}}{(\sin x)^{\mu-1}} \frac{(\sin x)^{\nu-1}}{(\cos x)^{\nu-1}} dx = \frac{\pi}{2\mu \cos(\mu\pi/2)} \quad [0 < \pm \operatorname{Re} \mu < 1].$$

$$14. \quad \int_0^{\pi/2} x \sin^n x \cos^{2m-1} x dx = \\ = \frac{B(n/2, m+1/2)}{2(2m+n-1)} + \frac{n-1}{2m+n-1} \int_0^{\pi/2} x \sin^{n-2} x \cos^{2m-1} x dx \quad [m, n=1, 2, 3, \dots].$$

$$15. \quad \int_0^{\pi/2} x \cos^{\nu-1} x \sin^{\mu} x dx = \frac{\pi}{2^{\nu+1} \Gamma(\nu)} \frac{\Psi((\nu+b+1)/2) - \Psi((\nu-b+1)/2)}{\Gamma((\nu+b+1)/2) \Gamma((\nu-b+1)/2)} \\ [\operatorname{Re} \nu > 0; -1 - \operatorname{Re} \nu < \operatorname{Re} b < 1 + \operatorname{Re} \nu].$$

$$16. \quad \int_0^{\pi} x \sin^{\mu-1} x \cos x dx = -\frac{1}{\mu} \Gamma\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > 0].$$

$$17. \quad \int_0^{\pi} x \cos^n x \sin x dx = \frac{\pi}{n+1} \left\{ (-1)^n + \frac{1-(-1)^n}{2} \frac{(2[(n+1)/2]-1)!}{(2[(n+1)/2])!} \right\}.$$

$$18. \quad \int_0^{\pi} x \frac{\sin x}{\cos x} dx = -\pi \ln 2.$$

$$19. \quad \int_0^{\pi} x \sin^{2n-1} x \cos^m x dx = \\ = \frac{(-1)^{m-1}}{2(2n+m-1)} B\left(\frac{m}{2}, \frac{n+1}{2}\right) + \frac{m-1}{2n+m-1} \int_0^{\pi} x \sin^{2n-1} x \cos^{m-2} x dx \\ [m, n=1, 2, 3, \dots].$$

$$20. \quad \int_0^{\pi} x^{2-1} \sin^n x \sin^m x dx = I_{\alpha}^{n, m}.$$

$$I_{\alpha}^{n, m} = \sqrt{\pi} 2^{1-m-n+\alpha} \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k!(n-k)!} \times \\ \times \sum_{l=0}^{[(m-1)/2]} (-1)^{[m/2]+l} \frac{m!}{l!(m-l)!} A_{kl}(\alpha), \quad \alpha = \frac{(a-2k)^2 \pi^2}{(a-2l)^2 b^2},$$

$$A_{kl}(t) = (4t)^{\lambda} b^{-\alpha} (m-2t)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] \times \\ \times \{ {}_2F_1(\lambda + \gamma + \alpha/2, \lambda - \gamma + (1+\alpha)/2; 2\lambda + 1/2; t) + 2\lambda - 1 \} + \\ + (2\gamma - 1) 4t^{\gamma} \gamma a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \text{ при } 0 < t < 1,$$

$$A_{kl}(t) = 4t^{\gamma} \gamma a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \times \\ \times \{ {}_2F_1(\lambda + \gamma + \alpha/2, \gamma - \lambda + (1+\alpha)/2; 2\gamma + 1/2; t^{-1}) + 2\gamma - 1 \} + \\ + (2\lambda - 1) (4t)^{\lambda} b^{-\alpha} (m-2t)^{-\alpha} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] \text{ при } t > 1,$$

$$A_{kl}(1) = a^{-\alpha} (n-2k)^{-\alpha} \left\{ \sqrt{\pi} \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2}, \gamma - \lambda + (1-\alpha)/2, \lambda + \gamma - \alpha/2 \right] + \right. \\ \left. + 4^{\lambda} (2\lambda - 1) \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\gamma - \lambda + (1-\alpha)/2} \right] + 4\gamma (2\gamma - 1) \Gamma \left[\frac{\lambda + \gamma + \alpha/2}{\lambda - \gamma + (1-\alpha)/2} \right] \right\} \\ \left[\lambda = \frac{1-(-1)^m}{4}, \gamma = \frac{1-(-1)^n}{4}; a, b > 0; -m-n < \operatorname{Re} \alpha < 0 \leftarrow m-n < \operatorname{Re} \alpha < 1, \text{ если}$$

одно из чисел m, n нечетно или если m, n нечетны и $a \neq b$; $m, n = 1, 2, 3, \dots$;

$$J_{0+}^{1,1} = \frac{1}{2} \ln \frac{a+b}{a-b}; \quad J_{0+}^{2,1} = \frac{\pi}{2} \min(a, b);$$

$$J_{0+}^{3,1} = \frac{\pi}{4} \quad [b < 2a], \quad J_{0+}^{3,1} = \frac{\pi}{8} \quad [b = 2a], \quad J_{0+}^{3,1} = 0 \quad [b > 2a];$$

$$J_{0+}^{2,2} = \frac{a}{2} \ln \frac{2a+b}{|2a-b|} + \frac{b}{4} \ln \frac{|4a^2-b^2|}{b^2} \quad [b \neq 2a],$$

$$J_{0+}^{2,2} = a \ln 2 \quad [b = 2a]; \quad J_{0+}^{2,2} = \frac{\pi}{8} b (4a-b) \quad [b \leq 2a],$$

$$J_{0+}^{2,2} = \frac{\pi}{2} a^2 \quad [b > 2a]; \quad J_{0+}^{2,2} = \frac{\pi}{4} \min(a, b);$$

$$J_{0+}^{2,3} = \frac{a^2}{4} \ln \frac{|a^2-b^2|}{a^2} + \frac{b^2}{4} \ln \frac{|a^2-b^2|}{b^2} - \frac{ab}{2} \ln \frac{a+b}{|a-b|} \quad [a \neq b],$$

$$J_{0+}^{2,3} = a^2 \ln 2 \quad [b = a]; \quad J_{0+}^{2,3} = \frac{1}{6} \pi a^2 (3b-a) \quad [a \leq b],$$

$$J_{0+}^{2,3} = \frac{1}{6} \pi b^2 (3a-b) \quad [b \leq a]; \quad J_{0+}^{2,3} = \frac{1}{8} \ln \left| \frac{(a+b)^2 (3a-b)}{(a-b)^2 (3a+b)} \right| \quad [b \neq a, b \neq 3a];$$

$$J_{0+}^{2,3} = \frac{\pi}{4} b \quad [b \leq a], \quad J_{0+}^{2,3} = \frac{\pi}{8} (3a-b) \quad [a \leq b \leq 3a], \quad J_{0+}^{2,3} = 0 \quad [b > 3a];$$

$$J_{0+}^{n,1} = \frac{\pi}{2} \left\{ a^n - \frac{2^{1-n}}{\pi!} \sum_{k=0}^{[(na-b)/(2a)]} (-1)^k \binom{n}{k} [a(n-2k)-b]^n \right\} \quad [b \leq na],$$

$$J_{0+}^{n,1} = \frac{\pi}{2} a^n \quad [b = an, m \geq n, m = 1, 2, 3, \dots];$$

$$21. \int_0^{\infty} x^{2-1} \sin^n ax \cos^m bx \, dx = J_{\alpha}^{n,m},$$

$$J_{\alpha}^{n,m} = \sqrt{\pi} 2^{1-m-n+\alpha} \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} \times \\ \times \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} B_{kl}(t) + \frac{1+(-1)^m}{2} \sqrt{\pi} 2^{\alpha-m-n} a^{-\alpha} \frac{m!}{[(m/2)!]} \times \\ \times \sum_{k=0}^{[(n-1)/2]} (-1)^{[n/2]+k} \frac{n!}{k! (n-k)!} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2} \right], \quad t = \frac{(n-2k)^2}{(m-2l)^2 b^2},$$

$$B_{kl}(t) = (4t)^{\lambda} b^{-\alpha} (m-2l)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2} \right] {}_2F_1(\lambda + \alpha/2, \\ \lambda + (1+\alpha)/2; 2\lambda + 1/2; t) - 1 + 2\lambda \quad \text{при } 0 < t < 1,$$

$$B_{kl}(t) = a^{-\alpha} (n-2k)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{\lambda + (1-\alpha)/2} \right] {}_2F_1(\lambda + \alpha/2, \\ (1+\alpha)/2 - \lambda; 1/2; t^{-1}) + (2\lambda - 1) (4t)^{\lambda} [b(m-2l)]^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{(1-\alpha)/2 - \lambda} \right] \quad \text{при } t > 1,$$

$$B_{kl}(1) = -4^{\lambda} a^{-\alpha} (n-2l)^{-\alpha} \Gamma \left[\frac{\lambda + \alpha/2}{(1-\alpha)/2 - \lambda} \right] (1-2\lambda - \alpha)^{-1} \\ \left[\lambda = \frac{1-(-1)^n}{4}; a, b > 0; -n < \operatorname{Re} \alpha < 0 \leftarrow -n < \operatorname{Re} \alpha < 1, \text{ если } m \text{ или } n \text{ нечетно}; m, \right. \\ \left. n = 1, 2, 3, \dots \right];$$

$$J_{1/2}^{1,1} = \frac{\sqrt{\pi}}{2} \sqrt{\frac{a + \sqrt{a^2 - b^2}}{a^2 - b^2}} \quad [a > b], \quad J_{1/2}^{1,1} = \frac{1}{4} \sqrt{\frac{\pi}{a}} \quad [b = a],$$

$$J_{1/2}^{1,1} = -\frac{\sqrt{\pi}}{2} \sqrt{\frac{b - \sqrt{b^2 - a^2}}{b^2 - a^2}} \quad [b > a]; \quad J_{1/2}^{1,1} = \frac{\pi}{2} \quad [a > b],$$

$$J_{1/2}^{1,1} = \frac{\pi}{4} \quad [a = b], \quad J_{1/2}^{1,1} = 0 \quad [b > a];$$

$$J_{1/2}^{2,1} = \frac{1}{4} \ln \frac{|4a^2 - b^2|}{b^2} \quad [b \neq 2a]; \quad J_{1/2}^{2,1} = \frac{\pi}{4} (2a-b) \quad [b \leq 2a],$$

$$J_{1/2}^{2,1} = 0 \quad [b > 2a]; \quad J_{1/2}^{2,1} = \frac{\pi}{4} \quad [a < 2b], \quad J_{1/2}^{2,1} = \frac{3}{8} \pi \quad [a = 2b],$$

$$J_{1/2}^{2,1} = \frac{\pi}{2} \quad [a > 2b]; \quad J_{1/2}^{2,1} = \frac{2a-b}{4} \pi \quad [a > b], \quad J_{1/2}^{2,1} = \frac{\pi a}{4} \quad [a \leq b];$$

$$J_{1/2}^{2,1} = \frac{\pi}{4} \quad [b < a], \quad J_{1/2}^{2,1} = \frac{\pi}{16} \quad [b = a],$$

$$J_{1/2}^{2,1} = -\frac{\pi}{8} \quad [a < b < 3a], \quad J_{1/2}^{2,1} = -\frac{\pi}{16} \quad [b = 3a], \quad J_{1/2}^{2,1} = 0 \quad [b > 3a];$$

$$J_{1/2}^{2,2} = \frac{3a}{8} \ln \left| \frac{9a^2 - b^2}{a^2 - b^2} \right| + \frac{b}{8} \ln \left| \frac{(a-b)^2 (3a+b)}{(a+b)^2 (3a-b)} \right| \quad [b \neq a, b \neq 3a],$$

$$J_{1/2}^{2,2} = \frac{\ln 2}{2} a \quad [b = a], \quad J_{1/2}^{2,2} = \frac{3}{4} (\ln 3 - 2 \ln 2) a \quad [b = 3a];$$

$$J_{1/2}^{2,2} = \frac{\pi}{8} (3a^2 - b^2) \quad [b \leq a], \quad J_{1/2}^{2,2} = \frac{\pi}{16} (3a-b)^2 \quad [a \leq b \leq 3a],$$

$$J_{-1}^{1,2} = 0 \quad [b > 3a]; \quad J_{-1}^{1,2} = \frac{\pi}{4} \quad [2b < a],$$

$$J_{-1}^{1,2} = \frac{5}{32}\pi \quad [2b = a]; \quad J_{-1}^{1,2} = \frac{\pi}{16} \quad [a < 2b < 3a],$$

$$J_{-1}^{1,2} = \frac{3}{32}\pi \quad [2b = 3a]; \quad J_{-1}^{1,2} = \frac{\pi}{8} \quad [2b > 3a];$$

$$J_{-1}^{1,2} = \frac{3 \ln 3}{8} a + \frac{3}{16} a \ln \left| \frac{9a^2 - 4b^2}{a^2 - 4b^2} \right| + \frac{b}{8} \ln \left| \frac{(a-2b)^2 (3a+2b)}{(a+2b)^2 (3a-2b)} \right| \quad [2b \neq a, 2b \neq 3a],$$

$$J_{-1}^{1,2} = \frac{3 \ln 3 + 2 \ln 2}{8} a \quad [2b = a]; \quad J_{-1}^{1,2} = \frac{3}{4} (\ln 3 - \ln 2) a \quad [2b = 3a];$$

$$J_{-1}^{1,2} = \frac{\pi}{8} (3a^2 - 2b^2) \quad [2b \leq a]; \quad J_{-1}^{1,2} = \frac{\pi}{32} [6a^2 + (3a-2b)^2] \quad [a \leq 2b \leq 3a],$$

$$J_{-1}^{1,2} = \frac{3}{16} \pi a^2 \quad [2b \geq 3a].$$

$$J_{-1}^{2j+1, m} = \frac{1}{2} B \left(j + \frac{1}{2}, \left[\frac{m+1}{2} \right] + \frac{1}{2} \right) = \\ = \frac{(2j-1)!! (2[(m+1)/2] - 1)!!}{2^{j+1} + [(m+1)/2]! (j + [(m+1)/2])!} \pi \quad [b=a],$$

$$J_{-1}^{2j+1, -1} = \frac{\pi}{2} \frac{(2j-1)!!}{(2j)!!} \quad [b=a]; \quad J_{-1}^{2j+1, 2k} = \frac{1}{2} B \left(j + \frac{1}{2}, k + \frac{1}{2} \right) \quad [b=a],$$

$$J_{-1}^{2j+1, 1} = \frac{(2j-3)!!}{(2j)!!} \frac{\pi a}{2} \quad [b=a];$$

$$J_{1-n}^{n, 1} = \frac{\pi}{2^n} \sum_{k=0}^{[(an-b)/(2a)]} \frac{(-1)^k}{k! (n-k)!} [a(n-2k)-b]^{n-1} \quad [b < na],$$

$$J_{1-n}^{n, 1} = 0 \quad [b \geq na]; \quad J_{1-k}^{2j+1, 1} = \frac{(2j-k)!!}{(2j-k+3)!!} \frac{\pi}{2} a^{k-1} \quad [b=a; k=1, 2, 3],$$

$$22. \int_0^\infty \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{x} dx = \frac{1}{2} B \left(\frac{\mu-1}{2}, \frac{\nu}{2} \right)$$

$$[\mu > 1, \nu > 0; \mu = 2k/(2l+1), \nu = 2m/(2n+1)]$$

$$23. \int_0^\infty x^{\alpha-1} \cos^m ax \cos^m bx dx = \sqrt{\pi} 2^{1-m} n^{\alpha-1} \Gamma \left[\frac{\alpha+2}{2} \right] \times \\ \times \left\{ \sum_{k=0}^{[(n-1)/2]} \frac{n!}{k! (n-k)!} \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} C_{kl}(l) + \right. \\ \left. + \frac{1+(-1)^n}{4} \frac{n! b^\alpha}{[(n/2)!]^2} \sum_{l=0}^{[(m-1)/2]} \frac{m!}{l! (m-l)!} (m-2l)^{-\alpha} + \right. \\ \left. + \frac{1+(-1)^m}{4} \frac{m! a^\alpha}{[(m/2)!]^2} \sum_{k=0}^{[(n-1)/2]} \frac{n!}{k! (n-k)!} (n-2k)^{-\alpha} \right\},$$

$$t = \frac{(n-2k)^2 a^2}{(m-2l)^2 b^2},$$

$$C_{kl}(l) = (m-2l)^{-\alpha} b^{-\alpha} {}_2F_1 \left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}; t \right) \text{ при } 0 < l < 1,$$

$$C_{kl}(l) = (n-2k)^{-\alpha} a^{-\alpha} {}_2F_1 \left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{1}{2}; t^{-1} \right) \text{ при } l > 1$$

[a, b > 0; 0 < Re α < 1; m, n = 1, 2, 3, ..., кроме случая, когда m и n четны одновременно].

$$2.5.14. \text{ Интегралы от } \frac{x^2}{x^2 \pm z^2} \sin^m ax \cos^m bx.$$

$$1. \int_0^\infty \frac{x^{\alpha-1}}{x^2+z^2} \sin^m ax \sin^m bx dx = J_{\alpha, b}^n \quad [\alpha=1, 2; a, b, \operatorname{Re} z > 0],$$

$$J_{1, (2k-2)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k-1} 2} [(1-e^{-2az})^{2k-2} - 1] \operatorname{sh} az,$$

$$J_{1, (2k-1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k-1} 2} (1-e^{-2az})^{2k-1},$$

$$J_{1, (2k+1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2^{2k-1} 2} (1-e^{-2az})^{2k-1} e^{-2az},$$

$$J_{1, 3(2k-1)a}^{2k-1} = \frac{(-1)^{k-1} \pi}{2z} \operatorname{sh}^{2k-1} az e^{-3(2k-1)az},$$

$$J_{2, (2k-1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} e^{az} [(1-e^{-2az})^{2k} - 1],$$

$$J_{2, 2ka}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} [(1-e^{-2az})^{2k} - 1],$$

$$J_{2, 2(k+1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} e^{-2az} (1-e^{-2az})^{2k},$$

$$J_{2, 4ka}^{2k} = \frac{(-1)^k \pi}{2} e^{-4kaz} \operatorname{sh}^{2k} az.$$

$$2. \int_0^\infty \frac{x^{\alpha-1}}{x^2+z^2} \sin^m ax \cos^m bx dx = J_{\alpha, b}^n \quad [\alpha=1, 2; a, b, \operatorname{Re} z > 0],$$

$$J_{2, (2k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} [(1-e^{-2az})^{2k-1} - 1],$$

$$J_{2, 2(k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} [(1-e^{-2az})^{2k-1} - 1] e^{-az},$$

$$J_{2, (2k+1)a}^{2k-1} = \frac{(-1)^k \pi}{2^{2k}} (1-e^{-2az})^{2k-1} e^{-2az},$$

$$J_{2, 2(2k-1)a}^{2k-1} = \frac{(-1)^k \pi}{2} \operatorname{sh}^{2k-1} az e^{-2(2k-1)az},$$

$$J_{1, (2k-1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1}} [(1-e^{-2az})^{2k-1} - 1] \operatorname{sh} az,$$

$$J_{1, 2ka}^{2k} = \frac{(-1)^k \pi}{2^{2k+1} z} (1 - e^{-2az})^{2k},$$

$$J_{1, 2(k+1)a}^{2k} = \frac{(-1)^k \pi}{2^{2k+1} z} (1 - e^{-2az})^{2k} e^{-2az},$$

$$J_{1, 4ka}^{2k} = \frac{(-1)^k \pi}{2z} e^{-4kaz} \sin \frac{2k}{az}.$$

$$3. \int_0^{\infty} \frac{x^{a-1}}{x^2 + z^2} \cos^a ax \sin bx dx = I_{a, b} \quad [a, b, \operatorname{Re} z > 0],$$

$$I_{1, na} = \frac{1}{2^{n+1} z} \sum_{l=1}^n \binom{n}{l} [e^{-2laz} \operatorname{Ei}(2laz) - e^{2laz} \operatorname{Ei}(-2laz)],$$

$$I_{1, (n+1)a} = \frac{\pi}{2^{n+1} z} e^{-2az} (1 + e^{-2az})^n,$$

$$I_{2, (n+1)a} = \frac{\pi}{2^{n+1} z} e^{-az} (1 + e^{-2az})^n, \quad I_{2, na} = \frac{\pi}{2^{n+1} z} [(1 + e^{-2az})^n - 1],$$

$$I_{2, (n-1)a} = \frac{\pi}{2^{n+1} z} (1 + e^{-2az})^n e^{az}, \quad I_{2, 3na} = \frac{\pi}{2} e^{-3az} \operatorname{ch}^n az.$$

$$4. \int_0^{\infty} \frac{1}{x^2 + z^2} \cos^a ax \cos bx dx = I_b \quad [a, b, \operatorname{Re} z > 0],$$

$$I_{1, (n+1)a} = \frac{\pi}{2^{n+1} z} e^{-2az} (1 + e^{-2az})^n, \quad I_{1, n+1a} = \frac{\pi}{2^{n+1} z} e^{-az} (1 + e^{-2az})^n,$$

$$I_{1, na} = \frac{\pi}{2^{n+1} z} (1 + e^{-2az})^n, \quad I_{1, (n-1)a} = \frac{\pi}{2^{n+1} z} e^{az} (1 + e^{-2az})^n.$$

$$5. \int_0^{\infty} \frac{x^{\delta} \cos^a ax \{\sin any\}}{x^2 + y^2 \{\cos any\}} dx = \frac{\pi}{2} (-y)^{\delta-1} \cos^a ay \frac{\{\cos any\}}{\{\sin any\}} - 2^{-n-1} \delta \pi \quad \left[n=1, 2, 3, \dots; a>0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right].$$

$$6. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\sin bx}{\sin ax} dx = \frac{\pi}{2z} \frac{\operatorname{sh} bz}{\operatorname{sh} az} \quad [0 < b < a; \operatorname{Re} z > 0],$$

$$7. \int_0^{\infty} \frac{x^{\pm 1}}{x^2 + z^2} \frac{\sin bx}{\cos ax} dx = \frac{\pi}{2} z^{-1 \pm 1} \frac{\operatorname{sh} bz}{\operatorname{ch} az} \quad [0 < b < a; \operatorname{Re} z > 0],$$

$$8. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\cos bx}{\sin ax} dx = \frac{\pi}{2} \frac{\operatorname{ch} bz}{\operatorname{sh} az} \quad [0 < b < a; \operatorname{Re} z > 0],$$

$$9. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\cos bx}{\cos ax} dx = \frac{\pi}{2z} \frac{\operatorname{ch} bz}{\operatorname{ch} az} \quad [0 < b < a; \operatorname{Re} z > 0].$$

$$10. \int_0^{\infty} \frac{1}{x^2 + y^2} \frac{\sin bx}{\sin ax} dx = -\frac{\pi}{y} \frac{\sin^2 (by/2)}{\sin ay} \quad [0 < b < a; y > 0].$$

$$11. \int_0^{\infty} \frac{x^{-1}}{x^2 + y^2} \frac{\sin bx}{\cos ax} dx = 0 \quad [0 < b < a; y > 0].$$

2.5.15. Интегралы от $x^a \prod_k \sin^{\mu_k} a_k x \cos^{\nu_k} b_k x$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} x^{(1-n)/2} \prod_{k=0}^n \sin b_k x dx = 0 \quad \left[b_k > 0; b_0 > \sum_{k=1}^n b_k \right].$$

$$2. \int_0^{\infty} x^{n-1} \prod_{k=0}^n \sin b_k x dx = \frac{\pi}{2} \prod_{k=0}^n b_k \quad \left[b_k > 0; b_0 > \sum_{k=1}^n b_k \right].$$

$$3. \int_0^{\infty} x^{a-1} \sin ax \sin bx \sin cx dx = I_a \quad [-3 < \operatorname{Re} z < 1; a \geq b \geq c \geq 0, \text{ исключая } a=b=c=0].$$

$$I_a = \frac{\Gamma(\alpha)}{4} \sin \left(\frac{\alpha\pi}{4} \right) [-(a+b+c)^{-\alpha} + (a-b+c)^{-\alpha} + (a+b-c)^{-\alpha} - e |a-b-c|^{-\alpha}]$$

[$\alpha \neq 0, -1, -2$; $e = \operatorname{sgn}(a-b-c)$, $e=0$ при $a=b+c$]

$$I_0 = \frac{\pi}{8} (1-e),$$

$$I_{-1} = \frac{a+b+c}{4} \ln(a+b+c) - \frac{a-b+c}{4} \ln(a-b+c) - \frac{a+b-c}{4} \ln(a+b-c) + \frac{a-b-e}{4} \ln|a-b-c|,$$

$$I_{-2} = \frac{\pi}{16} [(a+b+c)^2 - (a+b-c)^2 - (a-b+c)^2 + e(a-b-c)^2].$$

$$4. \int_0^{\infty} x^{\alpha-1} \prod_{j=1}^m \sin a_j x \prod_{k=1}^n \cos b_k x \left\{ \frac{\sin \xi x}{\cos \xi x} \right\} dx =$$

$$= 2^{m+n-1} \sqrt{\pi} a_1 \dots a_m e^{-\alpha-m-1} \Gamma \left\{ \frac{(\alpha+m+\delta)/2}{(1+\delta-\alpha-m)/2} \right\} \times$$

$$\times F_0^{(m+n)} \left[\frac{\alpha+m}{2}, \frac{\alpha+m+1}{2}, \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}; \right.$$

$$\left. \frac{a_1^2}{\xi^2}, \dots, \frac{a_m^2}{\xi^2}, \frac{b_1^2}{\xi^2}, \dots, \frac{b_n^2}{\xi^2} \right]$$

[$a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0$; $\xi > a_1 + \dots + a_m + b_1 + \dots + b_n$; $-m-\delta < \operatorname{Re} z < 1$].

$$5. \int_0^{\infty} x^{m+2l} \prod_{i=1}^m \sin a_i x \prod_{k=1}^n \cos b_k x \left(\frac{\sin \xi x}{\cos \xi x} \right) dx = 0$$

$$\{a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0; \xi > a_1 + \dots + a_m + b_1 + \dots + b_n;$$

$$l = 0, 1, 2, \dots, \left[\frac{m-\delta}{2} \right] - 1; \delta = \begin{cases} 1 \\ 0 \end{cases} \}$$

$$6. \int_0^{\infty} x^{m-1} \prod_{i=1}^m \sin a_i x \prod_{k=1}^n \cos b_k x \sin \xi x dx = \frac{1}{2} \pi a_1 \dots a_m \xi^{-1/2}$$

$$\{a_1, \dots, a_m, b_1, \dots, b_n, \xi > 0; \xi > a_1 + \dots + a_m + b_1 + \dots + b_n\}$$

$$7. \int_0^{\infty} \frac{1}{x^2} \prod_{k=1}^n \cos^p a_k x \sin b x \sin x dx = \frac{\pi}{2}$$

$$\left[b > \sum_{k=1}^n a_k^p; a_k, b, x > 0 \right]$$

2.5.16. Интегралы, содержащие $\left(a + b \left(\frac{\sin cx}{\cos cx} \right)^p \right)$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi} (\cos x - \cos a)^{\nu-1} \cos bx dx = \sqrt{\pi/2} \sin^{\nu-1/2} a \Gamma(\nu) P_{\frac{1}{2}(\nu-1/2)}^1(\cos a)$$

$$\{b, \operatorname{Re} \nu > 0; 0 < a < \pi\}$$

$$2. \int_0^{\pi} \frac{\cos(n+1/2)x}{b \cos x - \cos a} dx = \frac{\pi}{2} P_n(\cos a)$$

$$\{a > 0\}$$

$$3. \int_0^{\pi/2} \frac{dx}{a + b \sin x} = \frac{1}{\sqrt{a^2 - b^2}} \arccos \frac{b}{a}$$

$$\{|b| < a\}$$

$$4. = \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \frac{b + \sqrt{b^2 - a^2}}{a} \right|$$

$$\{0 < |a| < |b|\}$$

$$5. = 1/a$$

$$\{b = a\}$$

$$6. \int_0^{\pi/2} \frac{\sin x dx}{1 + \sin x} = \int_0^{\pi/2} \frac{\cos x dx}{1 + \cos x} = \frac{\pi}{2} - 1.$$

$$7. \int_0^{\pi/2} \frac{1 - a \cos 2nx}{1 - 2a \cos 2nx + a^2} \cos^m x \cos mx dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + 2^{m-1} \pi$$

$$\{|a| < 1; n = 1, 2, 3, \dots\}$$

$$8. \int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a \cos 2x + a^2} dx = \frac{\pi}{4a} \left[\left(\frac{1+a^2}{2} \right)^n - 2^n \right]$$

$$\left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases}; n = 1, 2, 3, \dots \right]$$

$$9. \int_0^{\pi/2} \frac{\sin^{2n} x \cos^{2\mu} x \cos \beta x}{1 - 2a \cos 2x + a^2} dx = \frac{(-1)^n \pi (1-a)^{2n+1}}{2^{2n} \beta (1+a)^{2n+1}} \begin{cases} 1 \\ a\beta \end{cases}$$

$$\left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases}; \beta < 1 - 2n \right]$$

$$10. \int_0^{\pi/2} \frac{dx}{(a + b \cos x)^2} = \frac{1}{a^2 - b^2} \left(\frac{a}{\sqrt{a^2 - b^2}} \arccos \frac{b}{a} - \frac{b}{a} \right)$$

$$\{a > |b|\}$$

$$11. = \frac{1}{b^2 - a^2} \left(\frac{b}{a} - \frac{a}{\sqrt{b^2 - a^2}} \ln \frac{b + \sqrt{b^2 - a^2}}{a} \right)$$

$$\{b > a > 0\}$$

$$12. \int_0^{\pi/2} \frac{\cos^{2n} x \cos \beta x}{(1 - 2a \cos 2x + a^2)^{n+1}} dx = \frac{\pi (1-a)^{2n+1}}{2^{2n} \beta^{n+1} (a+1)^{n+1}} \sum_{k=0}^n \binom{\beta}{k} \binom{2n-k}{n} (a-1)^k$$

$$\{|a| < 1; \beta < 2n+1\}$$

$$13. \int_0^{\pi/2} \frac{\sin^{2m} x \cos^{2\mu} x \cos \beta x}{(1 - 2a \cos 2x + a^2)^n} dx =$$

$$= \frac{(-1)^m \pi (1-a)^{2m+2n+1}}{2^{2n} \beta^{n+1} (1+a)^{2n+1}} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} \binom{\beta}{k} \binom{2m}{l} \binom{2n-k-l-2}{n-1} (-2)^l (a-1)^k$$

$$\{\beta = 2n - 2m - \mu - 2; |a| < 1; \mu > -1\}$$

$$14. \int_0^{\pi/2} \frac{x^{n+2}}{1 - \cos x} dx = -\left(\frac{\pi}{2}\right)^{n+2} + \left(\frac{\pi}{2}\right)^{n+1} (n+2) \left[\frac{2}{n+1} - \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (n+2k+1)} \right]$$

$$15. \int_0^{\pi/2} \frac{x - \sin x}{1 - \cos x} dx = 2 - \frac{\pi}{2}.$$

$$16. \int_0^{\pi/2} \frac{x \cos x dx}{1 + 2a \sin x + a^2} = \frac{\pi}{2a} \ln(1 + a^2) - a^{-1} \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k+1)^2}$$

$$\left[\begin{cases} |a| < 1 \\ |a| > 1 \end{cases} \right]$$

$$17. \int_0^{\pi/2} \frac{x dx}{1 + \sin x} = \ln 2. \quad 18. \int_0^{\pi/2} \frac{x dx}{1 + \cos x} = \frac{\pi}{2} - \ln 2.$$

$$19. \int_0^{\pi/2} \frac{x \cos x dx}{1 + \sin x} = \pi \ln 2 - 2G, \quad 20. \int_0^{\pi/2} \frac{x \sin x dx}{1 + \cos x} = \pi \ln 2 + 2G.$$

$$21. \int_0^{\pi/2} \frac{x^2 dx}{1 - \cos x} = 4G + \pi \ln 2 - \frac{\pi^3}{4}.$$

$$22. \int_0^{\pi} \frac{\cos nx}{a+b \cos x} dx = \frac{\pi}{\sqrt{a^2-b^2}} \left(\frac{\sqrt{a^2-b^2}-a}{b} \right)^n \quad [a > b > 0]$$

$$23. \int_0^{\pi} \frac{\cos nx}{\cos a - \cos x} dx = -\frac{\pi \sin an}{\sin a} \quad [a \neq 0], \\ = -\pi n \quad [a=0]$$

$$24. \int_0^{\pi} \frac{\sin^n x}{1 \pm \cos x} dx = 2^{n-1} B\left(\frac{n-1}{2}, \frac{n+1}{2}\right) \quad [n=2, 3, 4, \dots]$$

$$25. \int_0^{\pi} \frac{\sin^n x}{a+b \cos x} dx = \\ = 2^{n-2} \frac{a}{b^2} \sum_{l=1}^k \left(\frac{b^2-a^2}{4b^2} \right)^{l-1} B\left(\frac{n+1-2l}{2}, \frac{n+1-2l}{2}\right) + A \left(\frac{b^2-a^2}{b^2} \right)^k \\ [A = \pi b^{-1}(a - \sqrt{a^2-b^2}) \text{ при } n=2k+2, A = \frac{1}{b} \ln \frac{a+b}{a-b} \text{ при } n=2k+1, 0 < b \leq a]$$

$$26. \int_0^{\pi} \frac{\sin nx \sin x}{1-2a \cos x + a^2} dx = \frac{\pi}{2} a^{\frac{1}{2}n-1} \quad \left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; n=1, 2, 3, \dots \right]$$

$$27. \int_0^{\pi} \frac{\cos nx \cos x}{1-2a \cos x + a^2} dx = \frac{\pi}{2a} \left(1 - \frac{1+a^2}{|a^2-1|} a^n \right) \quad \left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; n=1, 2, 3, \dots \right]$$

$$28. \int_0^{\pi} \frac{\cos nx}{(a+b \cos x)^\mu} dx = \frac{\pi (a^2-b^2)^{-\mu/2}}{(1-\mu)_n} P_n^{-\mu} \left(\frac{a}{\sqrt{a^2-b^2}} \right) \quad [a > |b|]$$

$$29. \int_0^{\pi} \frac{\cos mx}{(1-2a \cos x + a^2)^n} dx = \frac{\pi a^m}{|1-a^2|^{n-1}} \sum_{k=0}^{n-1} \binom{m+n-1}{k} \times \\ \times \binom{2n-k-2}{n-1} (a^{\pm 2}-1)^k \quad \left[\left\{ \begin{array}{l} l=m, |a| > 1 \\ l=2n+m-2, |a| < 1 \end{array} \right\} \right]$$

$$30. \int_0^{\pi} \frac{\cos nx}{(1-2a \cos x + a^2)^{n+1}} dx = \binom{2n}{n} \frac{a^n \pi}{|1-a^2|^{n+1/2}} \quad [|a| \neq 1]$$

$$31. \int_0^{\pi} (1-2a \cos x + a^2)^n \cos mx dx = 0 \quad [n < m]$$

$$32. = n(-a)^m (1+a^2)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k} \binom{n-k}{m+k} \left(\frac{a}{1+a^2} \right)^{2k} \quad [n > m]$$

2.5.11

$$33. \int_0^{\pi} \frac{\cos nx}{1-2a \cos x + a^2} dx = \frac{\pi a^{\pm n}}{|1-a^2|} \quad \left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\} \right]$$

$$34. \int_0^{\pi} \frac{\sin nx - a \sin (n+1)x}{1-2a \cos x + a^2} dx = 0 \quad [|a| < 1]$$

$$35. \int_0^{\pi} \frac{dx}{(a+b \cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \quad [|b| < a]$$

$$36. \int_0^{\pi} (1-2a \cos x + a^2)^n dx = \pi \sum_{k=0}^n \binom{n}{k} a^{2k}$$

$$37. \int_0^{\pi} \frac{dx}{(1-2a \cos x + a^2)^n} = \\ = \frac{\pi}{|1-a^2|^{n-1/2}} \left\{ \begin{array}{l} a^{2n-1} \\ 1 \end{array} \right\} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{2n-k-2}{n-1} |1-a^{\pm 2}|^k \\ \left[\left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}; n=1, 2, 3, \dots \right]$$

$$38. \int_0^{\pi} \frac{dx}{(a+b \cos x)^{\pm m}} = \pi (a^2-b^2)^{\mp m/2} P_n \left(\frac{a}{\sqrt{a^2-b^2}} \right) \quad \left[\left\{ \begin{array}{l} n=m-1 \\ n=m \end{array} \right\}; a > |b| \right]$$

$$39. \int_0^{\pi} \frac{\cos x}{1+(a+b \sin x)^2} \left\{ \frac{\sin 2nx}{\cos (2n+1)x} \right\} dx = \\ = \mp \frac{\pi}{b} \sin \left[\left(2n + \left\{ \begin{array}{l} 0 \\ 1 \end{array} \right\} \right) \arctg \sqrt{\frac{s}{2}} \right] \left[\lg \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right) \right]^{2n+1-\delta} \\ [s=a^2-b^2-1+\sqrt{(a^2-b^2-1)^2+4a^2}, a>0; n=1, 2, 3, \dots]$$

$$40. \int_0^{\pi} \frac{\sin x \sin px}{(a^2-2ab \cos x + b^2)(a^{2p}-2a^p c^p \cos px + c^{2p})} dx = \frac{\pi b^{p-1} c^{-p}}{2a^{p+1}(c^p-b^p)} \\ [0 < a < c, 0 < a < b, p > 0]$$

$$41. \int_0^{\pi} \frac{\sin^{2\nu-1} x}{(a+b \cos x)^\mu} dx = \\ = 2^{\nu-1/2} \sqrt{\pi} (-b)^{1/2-\nu} \Gamma(\nu) (\sqrt{a^2-b^2})^{\nu-\mu-1/2} P_{\nu-\mu-1/2}^{\nu} \left(\frac{a}{\sqrt{a^2-b^2}} \right) \\ [a > |b|; \operatorname{Re} \nu > 0]$$

$$42. \int_0^\pi \frac{\cos^n x \, dx}{(a+b \cos x)^{n+1}} = \frac{\pi}{2^n (a+b)^n \sqrt{a^2-b^2}} \sum_{k=0}^n (-1)^k \frac{(2n-2k-1)!! (2k-1)!!}{(n-k)! k!} \left(\frac{a+b}{a-b} \right)^k$$

[a > b > 0].

$$43. \int_0^\pi \frac{\sin^{\mu-1} x}{(1-2a \cos x + a^2)^\nu} dx = \pi \left(\frac{\mu}{2}, \frac{1}{2} \right) {}_2F_1 \left(\nu, \nu + \frac{1-\mu}{2}; \frac{1+\mu}{2}; a^2 \right)$$

[|a| < 1; Re μ > 0].

$$44. \int_0^\pi \frac{\sin(n+1/2)x}{\sqrt{\cos a - \cos x}} dx = \frac{\pi}{\sqrt{2}} P_n(\cos a)$$

[a > 0].

$$45. \int_0^\pi \frac{dx}{\sqrt{a \pm b \cos x}} = \frac{2}{\sqrt{a+b}} K \left(\sqrt{\frac{2b}{a+b}} \right)$$

[a > b > 0].

$$46. \int_0^\pi \frac{dx}{\sqrt{3-\cos x}} = \frac{1}{4} \sqrt{\pi} \Gamma^2 \left(\frac{1}{4} \right).$$

$$47. \int_0^\pi \frac{dx}{\sqrt{1 \pm 2a \cos x + a^2}} = 2K(a)$$

[|a| < 1].

$$48. \int_0^\pi \frac{\sin x \, dx}{\sqrt{1 \pm 2a \cos x + a^2}} = \begin{cases} 2|a|^{-1} \\ 2 \end{cases}$$

[|a| > 1].

$$49. \int_0^\pi \frac{\cos x \, dx}{\sqrt{1-2a \cos x + a^2}} = \frac{2}{a} [K(a) - E(a)]$$

[|a| < 1].

$$50. \int_0^\pi \frac{\cos(n+1/2)x}{\sqrt{\cos x + 1}} dx = \frac{(-1)^n \pi}{2}.$$

$$51. \int_0^\pi \frac{x \, dx}{a \pm b \cos x} = \frac{\pi^2}{2 \sqrt{a^2-b^2}} + \frac{4}{\sqrt{a^2-b^2}} \sum_{k=1}^\infty \frac{(a - \sqrt{a^2-b^2})^{2k+1}}{(2k+1)^2 b^{2k+2}}$$

[a > b > 0].

$$52. \int_0^\pi \frac{x \sin x \, dx}{a+b \cos x} = \frac{\pi}{b} \ln \frac{a+\sqrt{a^2-b^2}}{2(a-b)}$$

[a > |b| > 0].

$$53. \int_0^\pi \frac{x - \sin x}{1 - \cos x} dx = 2. \quad 54. \int_0^\pi \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 4G.$$

$$55. \int_0^\pi \frac{x \sin x \, dx}{1-2a \cos x + a^2} = \frac{\pi}{a} \ln(1+a^2)$$

[|a| < 1].

$$56. \int_0^\pi \frac{x \sin x}{(a+b \cos x)^2} dx = \frac{\pi}{b} \left(\frac{1}{a-b} - \frac{1}{\sqrt{a^2-b^2}} \right)$$

[0 < |b| < a].

$$57. \int_0^\pi \frac{x^2 \, dx}{1 - \cos x} = 4\pi \ln 2.$$

$$58. \int_0^\pi \frac{x^2 \cos x + b}{(a+b \cos x)^2} dx = \frac{2\pi}{b} \ln \frac{2(a-b)}{a+\sqrt{a^2-b^2}}$$

[0 < |b| < a].

$$59. \int_0^{2\pi} (1 - \cos x)^n \begin{Bmatrix} \sin nx \\ \cos nx \end{Bmatrix} dx = (-1)^n \frac{\pi}{2^{n-1}} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

$$60. \int_0^{2\pi} (a+b \cos x)^m \sin nx \, dx = 0$$

[a > |b|].

$$61. \int_0^{2\pi} \frac{dx}{(1-2a \cos x + a^2)^{n+1}} = \frac{2\pi}{|1-a^2|^{n+1}} \sum_{k=0}^n \frac{(n+k)!}{(k!)^2 (n-k)!} |a|^{2n-2k-1}$$

[|a| < 1].

$$62. \int_0^{2\pi} \frac{\sin(x/2) \, dx}{(1-2a \cos x + a^2)^{n+1}} = \pm \frac{1}{2na} [(1-a)^{-2n} - (1+a)^{-2n}]$$

[|a| < 1].

$$63. \int_0^{2\pi} \frac{\cos(n-1/2)x}{1-2a \cos x + a^2} dx = 0$$

[|a| ≠ 1; n=0, ±1, ±2, ...].

$$64. \int_0^{2\pi} \begin{Bmatrix} \sin 2nx \\ \cos(2n+1)x \end{Bmatrix} \frac{dx}{a+b \sin x} = 0$$

[|a| > |b| > 0].

$$65. \int_0^{2\pi} \begin{Bmatrix} \sin(2n+1)x \\ \cos 2nx \end{Bmatrix} \frac{dx}{a+b \sin x} = \frac{(-1)^{n+1} 2\pi (a - \sqrt{a^2-b^2})^{2n+1}}{b^{2n+1} \sqrt{a^2-b^2}}$$

[|a| > |b| > 0].

$$66. \int_0^{2\pi} \begin{Bmatrix} \sin(nx/2) \sin(x/2) \\ \cos(nx/2) \cos(x/2) \end{Bmatrix} \frac{dx}{1-2a \cos x + a^2} = 0$$

[|a| ≠ 1].

$$67. \int_0^{2\pi} \frac{\sin(n-1/2)x \sin(x/2)}{1-2a \cos x + a^2} dx = \frac{\pi a^n}{1+a}$$

(m=n-1 при |a|<1; m=-n при |a|>1; n=1, 2, 3, ...).

$$68. \int_0^{2\pi} \frac{\cos(n-1/2)x \cos(x/2)}{1-2a \cos x + a^2} dx = \frac{\pi a^n}{|1-a|}$$

[$m=n-1$ при $|a|<1$; $m=-n$ при $|a|>1$; $n=1, 2, 3, \dots$]

$$69. \int_0^{2\pi} \frac{c + \cos x}{(a-b \cos x)^{3/2}} dx = -\frac{4}{b} \frac{1}{a+b} \left[K \left(\sqrt{\frac{2b}{a+b}} \right) + \frac{a+bc}{a-b} E \left(\sqrt{\frac{2b}{a+b}} \right) \right]$$

[$a > b > 0$]

$$70. \int_0^{2\pi} \frac{x \sin x}{a \pm b \cos x} dx = \pm \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a \pm b)}$$

[$a > |b| > 0$]

$$71. \int_0^{2\pi} \frac{x \sin nx}{a \pm b \cos x} dx =$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}} \left\{ (\mp b)^n [(a + \sqrt{a^2 - b^2})^n - (a - \sqrt{a^2 - b^2})^n] \ln \frac{2\sqrt{a \pm b}}{\sqrt{a+b} + \sqrt{a-b}} + \right.$$

$$\left. + \sum_{k=1}^{n-1} \frac{(\mp b)^k}{(n-k)} [(a + \sqrt{a^2 - b^2})^k - (a - \sqrt{a^2 - b^2})^k] \right\}$$

[$a > |b| > 0$]

$$72. \int_0^{2\pi} \frac{x \cos nx}{a \pm b \cos x} dx = \frac{2\pi^2}{\sqrt{a^2 - b^2}} \left(\frac{a - \sqrt{a^2 - b^2}}{\pm b} \right)^n$$

[$a > |b| > 0$]

$$73. \int_0^{2\pi} \frac{x \sin x}{1-2a \cos x + a^2} dx = \frac{2\pi}{a} \ln(1-a^2)$$

[$|a| < 1$]

$$74. \int_0^{2\pi} \frac{x \sin nx}{1-2a \cos x + a^2} dx = \frac{2\pi}{1-a^2} \left[(a^{-n} - a^n) \ln(1-a^2) + \sum_{k=1}^{n-1} \frac{a^k - a^{-k}}{n-k} \right]$$

[$|a| < 1$; $n=1, 2, 3, \dots$]

$$75. \int_0^{2\pi} x^{-1} \frac{\sin x}{a \pm b \cos 2x} dx = \frac{\pi}{2\sqrt{a^2 - b^2}}$$

[$|a| > |b|$]

$$76. = 0$$

[$|a| < |b|$]

$$77. \int_0^{2\pi} x^{-1} \frac{\sin nx}{1-2a \cos x + a^2} dx = \frac{\pi}{2} \frac{1-a^{2n}}{(1-a^2)^2}$$

[$0 < a < 1$; $n=1, 2, 3, \dots$]

$$78. \int_0^{2\pi} x^{-1} \frac{\sin bx}{1-2a \cos x + a^2} dx = \frac{\pi}{2} \frac{1+a-2a^{[b]+1}}{(1-a^2)(1-a)}$$

[$0 < a < 1$; $b \neq 0, 1, 2, \dots$]

$$79. = \frac{\pi}{2} \frac{a^{[b]} + a^{[b]+1} - 2}{(1-a^2)(1-a)} a^{-[b]}$$

[$a > 1$; $b \neq 0, 1, 2, \dots$]

$$80. \int_0^{4\pi} \frac{\sin(mx/2) dx}{(1-2a \cos x + a^2)^n} = 0.$$

$$81. \int_0^{2\pi} \left[\operatorname{tg}^{\pm \mu} x \left(\frac{\sin 2x}{\cos 2x} \right) + (\mp a)^{\pm \delta} \right] \frac{dx}{1 \mp 2a \cos 2x + a^2} =$$

$$= (\mp 1)^{1-\delta} \frac{\pi}{4a} \left(\frac{\sin(\mu\pi/2)}{\cos(\mu\pi/2)} \right) \left(1 + \operatorname{sgn}(a^2-1) \left| \frac{1-a}{1+a} \right|^\mu \right)$$

[$|a| \neq 1$; $-1-\delta < \operatorname{Re} \mu < 1$; $\delta = \begin{cases} 1 \\ 0 \end{cases}$].

$$82. \int_0^{2\pi} \frac{x^{-1}}{(1-2a \cos 2x + a^2)^{n+1}} \left(\frac{\operatorname{tg} x}{\sin x} \right) dx = \frac{\pi}{2|1-a^2|^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 a^{2k}$$

[$|a| \neq 1$]

$$83. \int_0^{2\pi} x^{-1} \frac{\operatorname{tg} x}{a + b \cos kx} dx = \frac{\pi}{2|b| \sqrt{a^2 - b^2}}$$

[$k=2, 4$; $|a| > |b|$]

$$84. = 0$$

[$k=2, 4$; $|a| < |b|$]

$$85. \int_0^{2\pi} \frac{x^{-1} \sin^{2m+1} x}{1-2a \cos x + a^2} dx =$$

$$= \frac{(-1)^m \pi (1+a)^{2m}}{2^{2m+1} a^{2m+1}} \left(\left| \frac{1-a}{1+a} \right|^{2m-1} - \sum_{k=0}^{2m} \frac{(-1)^k}{k!} \Gamma \left[\frac{m+1/2}{1/2+m-k} \right] \frac{4^k a^k}{(1+a)^{2k}} \right)$$

[$|a| \neq 1$]

$$86. \int_0^{2\pi} \frac{x^{-1} \sin^{2m+1} x \cos^n x}{(1-2a \cos x + a^2)^v} dx = \frac{\pi n!}{2^{2m+1} (2m+n+1)! (1+a)^{2v}} \times$$

$$\times \sum_{k=0}^n \frac{(-1)^k (2m+2n-2k+1)!}{k! (n-k)!} \times$$

$$\times (2m+2k-1)!! {}_2F_1 \left(m+n-k+3/2, v; 2m+n+2; \frac{4a}{(1+a)^2} \right)$$

[$|a| \neq 1$]

$$87. \int_0^{2\pi} \frac{1}{1-2a \cos cx + a^2} \frac{dx}{x^2 + z^2} = \frac{\pi}{2z(1-a^2)} \frac{1 + ae^{\mp cz}}{1 - ae^{\mp cz}}$$

[$c, \operatorname{Re} z > 0$; $\begin{cases} |a| < 1 \\ |a| > 1 \end{cases}$].

$$88. \int_0^{2\pi} \frac{1}{1-2a \cos cx + a^2} \frac{dx}{x^2 - y^2} = -\frac{\pi a \sin cy}{y|1-a^2| (1-2a \cos cy + a^2)}$$

[$c, y > 0$; $|a| \neq 1$]

$$69. \int_0^{\infty} \frac{\cos bx}{1-2a \cos cx + a^2 x^2 + z^2} dx = \frac{\pi}{2z(1-a^2)} \frac{e^{-bz} + ae^{\pm z(b-c)}}{1-a^2 e^{\pm 2iz}} \quad [b, c, \operatorname{Re} z > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$90. \int_0^{\infty} \frac{\cos bx}{1-2a \cos cx + a^2 x^2 - y^2} dx = -\frac{\pi}{2y|1-a^2|} \frac{|1-a^2| \sin by + 2a^{\pm(b+c)/c} \sin cy}{1-2a \cos cy + a^2} \quad [b, c, y > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$91. \int_0^{\infty} \frac{x \sin bx}{1-2a \cos cx + a^2 x^2 + z^2} dx = \frac{\pi}{2} \frac{e^{-bz} - ae^{\pm b/c}}{(1-ae^{-cz})(1-ae^{\pm z})} \quad [a, b, \operatorname{Re} z > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$92. \int_0^{\infty} \frac{x \sin bx}{1-2a \cos cx + a^2 x^2 - y^2} dx = -\frac{\pi}{2|1-a^2|} \frac{a^{\pm b/c} - \cos by}{1-2a \cos cy + a^2} \left\{ \begin{array}{l} 1 \\ a^2 \end{array} \right\} \quad [b, c, y > 0; \left\{ \begin{array}{l} |a| < 1 \\ |a| > 1 \end{array} \right\}].$$

$$93. \int_{-\infty}^{\infty} \frac{1 - \cos ax}{x(x-y)} dx = \frac{\sin ay}{y} \quad [a > 0; \operatorname{Im} y = 0].$$

2.5.17. Интегралы, содержащие $(a \cos x + b \sin x + c)^2$.

$$1. \int_0^{\pi/2} \frac{x}{\cos x + \operatorname{tg} a \sin x} \frac{dx}{\cos x} = -\frac{x}{\operatorname{tg} a} \ln |\cos a| \quad [a > 0].$$

$$2. \int_0^{\pi/2} \frac{x}{\cos x + \operatorname{tg} a \sin x} \frac{dx}{\sin x} = \left(\frac{\pi}{2} + a \right) \ln |\sin a| \quad [a > 0].$$

$$3. \int_0^{\pi/4} \frac{A \cos x + B \sin x}{a \cos x + b \sin x} dx = \frac{Ab - Ba}{2(a^2 + b^2)} \ln \frac{(a+b)^2}{2a^2} + \frac{Aa + Bb}{a^2 + b^2} \frac{\pi}{4} \quad [b \neq -a].$$

$$4. \int_0^{\pi/4} x \frac{\cos x - \sin x}{\cos x + \sin x} dx = \frac{\pi}{4} \ln 2 - \frac{G}{2}.$$

$$5. \int_0^{\pi/4} \frac{x}{\cos x + \sin x} \left(\frac{\sin x}{\cos x} \right)^{-1} dx = -\frac{\pi}{8} \ln 2 + G \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right\}.$$

$$6. \int_0^{\pi/4} \frac{(\pi/4 - x)}{\cos x - \sin x} \frac{dx}{\cos x} = G - \frac{\pi}{8} \ln 2.$$

$$7. \int_0^{\pi/4} \frac{x \operatorname{tg} x}{\cos x + \sin x} \frac{dx}{\cos x} = \frac{\pi}{4} - \frac{\pi}{8} \ln 2 - \frac{\ln 2}{2}.$$

$$8. \int_0^{\pi/4} \frac{x dx}{(a \cos x + b \sin x)^2} = \frac{1}{a^2 + b^2} \ln \frac{a+b}{a} + \frac{\pi}{4} \frac{a-b}{(a^2 + b^2)(a+b)} \quad [a, b > 0].$$

$$9. \int_0^{\pi/2} x \frac{\cos x \mp \sin x}{\cos x \pm \sin x} dx = \pm \frac{\pi}{4} \ln 2 - G.$$

$$10. \int_0^{\pi/2} \frac{x dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 \pm G.$$

$$11. \int_0^{\pi/2} \frac{x dx}{(a \cos x + b \sin x) \sqrt{\sin 2x}} = -\frac{\pi}{\sqrt{2ab}} \left[\operatorname{arctg} \left(\sqrt{\frac{2a}{b}} + 1 \right) + \operatorname{arctg} \left(\sqrt{\frac{2a}{b}} - 1 \right) - \operatorname{arctg} \frac{a}{b} \right] \quad [a, b > 0].$$

$$12. \int_0^{\pi/2} \frac{x dx}{(\sin x + \cos x) \sqrt{\sin 2x}} = \frac{\pi^2}{4\sqrt{2}}.$$

$$13. \int_0^{\pi/2} \frac{x dx}{(a \cos x + b \sin x)^2} = \frac{1}{a^2 + b^2} \left(\ln \frac{b}{a} + \frac{\pi a}{2b} \right) \quad [a, b > 0].$$

$$14. \int_0^{\pi/2} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(a \cos x + b \sin x)^{\mu+\nu}} dx = \frac{B(\mu, \nu)}{a^{\nu} b^{\mu}} \quad [a, b, \operatorname{Re} \mu, \operatorname{Re} \nu > 0].$$

$$15. \int_0^{\pi/4} (a \cos x + b \sin x)^n dx = \frac{1 + (-1)^n}{2} \frac{(n-1)!}{(n)!} 2\pi (a^2 + b^2)^{n/2}.$$

$$16. \int_0^{2\pi} \frac{dx}{a + ib \cos x + ic \sin x} = \frac{2\pi \operatorname{sgn} a}{a^2 + b^2 + c^2} \quad [a \neq 0].$$

$$17. = 0$$

$$18. \int_0^{\pi} x^{-n-1/2} \sin^n x (\cos ax + \sin ax) dx = -\frac{\pi \sqrt{2}}{(2n-1)!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k+a)^{n-1/2} \quad [a > \pi].$$

2.5.18. Интегралы, содержащие $(a \cos^2 x + b \sin^2 x + c)^{-n}$.

1. $\int_0^{\pi/4} \frac{x \sin x dx}{(a^2 \cos^2 x + b^2 \sin^2 x) \sqrt{\cos 2x}} =$
 $= \frac{\pi}{b} \ln \cos \left(\frac{\pi}{8} - \frac{\arccos b}{2} \right) - \frac{\pi}{b} \ln \sin \left(\frac{\pi}{8} + \frac{\arccos b}{2} \right) \quad [a^2 + b^2 = 1].$
2. $\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = I_n \quad [a, b > 0].$
 $I_1 = \frac{\pi}{2ab}, \quad I_2 = \frac{\pi}{4ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \quad I_3 = \frac{\pi}{16ab} \left(\frac{3}{a^4} + \frac{2}{a^2 b^2} + \frac{3}{b^4} \right),$
 $I_4 = \frac{\pi}{32ab} \left(\frac{5}{a^6} + \frac{3}{a^4 b^2} + \frac{3}{a^2 b^4} + \frac{5}{b^6} \right).$
3. $\int_0^{\pi/2} \frac{\cos 2nx dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \left(\frac{2n}{n} \right) \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi \quad [a, b > 0].$
4. $\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^{\mu+\nu}} dx = \frac{a^{-2\nu} b^{-2\mu}}{2} B(\mu, \nu) \quad [a, b, \operatorname{Re} \mu, \operatorname{Re} \nu > 0].$
5. $\int_0^{\pi/2} \frac{\cos^{\rho+2n} x \cos^{\rho} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{\rho+k-1}{k} \frac{b^{\rho-1}}{(2a)^{2n-k+1} (a+b)^{\rho+1-k}} \quad [a, b > 0; \rho > -2n-1].$
6. $\int_0^{\pi/2} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b} \quad [a, b > 0].$
7. $\int_0^{\pi/2} \frac{\cos \mu x \cos^{\mu-1} x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2a(a+b)^{\mu}} \quad [a, b > 0; \mu > -1].$
8. $\int_0^{\pi/2} \frac{\sin \mu x \cos^{\mu-1} x \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2b(a+b)^{\mu}} \quad [a, b, \mu > 0].$
9. $\int_0^{\pi/2} \frac{x \operatorname{ctg} x}{a^2 \cos^2 x - (-1)^k b^2 \sin^2 x} dx = \frac{\pi}{2^k a^2} \ln \frac{a^k + b^k}{b^k} \quad [a, b > 0; k=1 \text{ или } 2].$
10. $\int_0^{\pi/2} \frac{x \sin x dx}{a^2 \cos^2 x - b^2 \sin^2 x} =$
 $= -\frac{2}{b \sqrt{a^2 + b^2}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin \left[(2k+1) \arcsin \frac{b}{\sqrt{a^2 + b^2}} \right] \quad [a, b > 0].$

11. $\int_0^{\pi/2} \frac{\cos 2nx}{1 - b^2 \sin^2 x} dx = \frac{(-1)^n \pi}{2 \sqrt{1-b^2}} \left(\frac{1 - \sqrt{1-b^2}}{b} \right)^{2n} \quad [|b| < 1].$
12. $\int_0^{\pi/2} \frac{A \cos^2 x + B \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{4ab} \left(\frac{A}{a^2} + \frac{B}{b^2} \right) \quad [a, b > 0].$
13. $\int_0^{\pi/2} \frac{x \sin 2x}{(1 + b \sin^2 x)^2} dx = \frac{\pi}{2b} \frac{\sqrt{1+b} - 1}{1+b} \quad [b > -1].$
14. $\int_0^{\pi/2} \frac{x \sin 2x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{2ab^2(a+b)} \quad [a, b > 0].$
15. $\int_0^{\pi/2} \frac{x \cos^2 x dx}{\sin x (a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{2a^4} \ln \left(1 + \frac{a}{b} \right) - \frac{\pi}{4a^3(a+b)} \quad [a, b > 0].$
16. $\int_0^{\pi/2} \frac{x^2 \cos x (1 - a^2 + \sin^2 x)}{(a^2 - \cos^2 x)^2} dx =$
 $= -\frac{\pi^2}{4a^2} + \frac{4}{a} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arcsin a]}{(2k+1)^2} \quad [|a| < 1].$
17. $\int_0^{\pi} \frac{x \sin x dx}{a + b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{b}{a}} \quad [a, b > 0].$
18. $= \frac{\pi}{2 \sqrt{|ab|}} \ln \frac{\sqrt{a} + \sqrt{|b|}}{\sqrt{a} - \sqrt{|b|}} \quad [a > -b > 0].$
19. $\int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a, b > 0].$
20. $\int_0^{\pi} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a} \quad [a, b > 0].$
21. $\int_0^{\pi} \frac{x dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a \sqrt{a^2 - 1}} \quad [|a| > 1].$
22. $= 0 \quad [|a| < 1].$
23. $\int_0^{\pi} \frac{x \cos x dx}{a^2 - \cos^2 x} = \frac{4}{\sqrt{1-a^2}} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arccos a]}{(2k+1)^2} \quad [|a| < 1].$
24. $\int_0^{\pi} \frac{x \sin 2x}{a^2 - \cos^2 x} dx = \pi \ln [4(1-a^2)] \quad [|a| < 1].$

2.5.18. Интегралы, содержащие $(a \cos^2 x + b \sin^2 x + c)^{-n}$.

$$1. \int_0^{\pi/4} \frac{x \sin x dx}{(a^2 \cos^2 x + b^2 \sin^2 x) \sqrt{\cos 2x}} = \frac{\pi}{b} \ln \cos \left(\frac{\pi}{8} - \frac{\arccos b}{2} \right) - \frac{\pi}{b} \ln \sin \left(\frac{\pi}{8} + \frac{\arccos b}{2} \right) \quad [a^2 + b^2 = 1].$$

$$2. \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = I_n \quad [a, b > 0].$$

$$I_1 = \frac{\pi}{2ab}, \quad I_2 = \frac{\pi}{4ab} \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \quad I_3 = \frac{\pi}{16ab} \left(\frac{3}{a^4} + \frac{2}{a^2 b^2} + \frac{3}{b^4} \right),$$

$$I_4 = \frac{\pi}{32ab} \left(\frac{5}{a^6} + \frac{3}{a^4 b^2} + \frac{3}{a^2 b^4} + \frac{5}{b^6} \right).$$

$$3. \int_0^{\pi/2} \frac{\cos 2nx dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \left(\frac{2n}{n} \right) \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi \quad [a, b > 0].$$

$$4. \int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^{\mu+\nu}} dx = \frac{\pi^{-2\mu-2\nu}}{2} B(\mu, \nu) \quad [a, b, \operatorname{Re} \mu, \operatorname{Re} \nu > 0].$$

$$5. \int_0^{\pi/2} \frac{\cos^{p+2n} x \cos^{p-k} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} dx = \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{p+k-1}{k} \frac{b^{p-1}}{(2a)^{2n-k+1} (a+b)^{p-k}} \quad [a, b > 0; p > -2n-1].$$

$$6. \int_0^{\pi/2} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b} \quad [a, b > 0].$$

$$7. \int_0^{\pi/2} \frac{\cos \mu x \cos^{\mu-1} x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2a(a+b)^{\mu}} \quad [a, b > 0; \mu > -1].$$

$$8. \int_0^{\pi/2} \frac{\sin \mu x \cos^{\mu-1} x \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2b(a+b)^{\mu}} \quad [a, b, \mu > 0].$$

$$9. \int_0^{\pi/2} \frac{x \operatorname{ctg} x}{a^2 \cos^2 x - (-1)^k b^2 \sin^2 x} dx = \frac{\pi}{2^k a^2} \ln \frac{a^k + b^k}{b^k} \quad [a, b > 0; k=1 \text{ или } 2].$$

$$10. \int_0^{\pi/2} \frac{x \sin x dx}{a^2 \cos^2 x - b^2 \sin^2 x} = -\frac{2}{b \sqrt{a^2 + b^2}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin \left[(2k+1) \arcsin \frac{b}{\sqrt{a^2 + b^2}} \right] \quad [a, b > 0].$$

$$11. \int_0^{\pi/2} \frac{\cos 2nx}{1 - b^2 \sin^2 x} dx = \frac{(-1)^n \pi}{2 \sqrt{1-b^2}} \left(\frac{1 - \sqrt{1-b^2}}{b} \right)^{2n} \quad [|b| < 1].$$

$$12. \int_0^{\pi/2} \frac{A \cos^2 x + B \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{4ab} \left(\frac{A}{a^2} + \frac{B}{b^2} \right) \quad [a, b > 0].$$

$$13. \int_0^{\pi/2} x \frac{\sin 2x}{(1 + b \sin^2 x)^2} dx = \frac{\pi}{2b} \frac{\sqrt{1+b} - 1}{1+b} \quad [b > -1].$$

$$14. \int_0^{\pi/2} \frac{x \sin 2x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx = \frac{\pi}{2ab^2(a+b)} \quad [a, b > 0].$$

$$15. \int_0^{\pi/2} \frac{x \cos^2 x dx}{\sin x (a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{2a^4} \ln \left(1 + \frac{a}{b} \right) - \frac{\pi}{4a^2(a+b)} \quad [a, b > 0].$$

$$16. \int_0^{\pi/2} x^2 \frac{\cos x (1 - a^2 + \sin^2 x)}{(a^2 - \cos^2 x)^2} dx = -\frac{\pi^2}{4a^4} + \frac{4}{a} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arcsin a]}{(2k+1)^2} \quad [|a| < 1].$$

$$17. \int_0^{\pi} \frac{x \sin x dx}{a + b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{b}{a}} \quad [a, b > 0].$$

$$18. \int_0^{\pi} \frac{x \sin x dx}{a + b \cos^2 x} = \frac{\pi}{2 \sqrt{|ab|}} \ln \frac{\sqrt{a} + \sqrt{|b|}}{\sqrt{a} - \sqrt{|b|}} \quad [a > -b > 0].$$

$$19. \int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a, b > 0].$$

$$20. \int_0^{\pi} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a} \quad [a, b > 0].$$

$$21. \int_0^{\pi} \frac{x dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a \sqrt{a^2 - 1}} \quad [|a| > 1].$$

$$22. \int_0^{\pi} \frac{x dx}{a^2 - \cos^2 x} = 0 \quad [|a| < 1].$$

$$23. \int_0^{\pi} \frac{x \cos x dx}{a^2 - \cos^2 x} = \frac{4}{\sqrt{1-a^2}} \sum_{k=0}^{\infty} \frac{\sin [(2k+1) \arccos a]}{(2k+1)^2} \quad [|a| < 1].$$

$$24. \int_0^{\pi} \frac{x \sin 2x}{a^2 - \cos^2 x} dx = \pi \ln [1 - a^2] \quad [|a| < 1].$$

$$= 2\pi \ln [2(1 - a^2 + a\sqrt{a^2 - 1})]$$

$$[|a| > 1]$$

$$26. \int_0^\pi \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} dx = \frac{\pi^2}{2} \frac{\sqrt{a^2 - 1} - a}{a(a^2 - 1)}$$

$$[|a| > 1]$$

$$27. \int_0^\pi x^2 \frac{1 - a^2 + (2a^2 - 1) \sin^2 x}{(a^2 - \cos^2 x)^3} dx = \pi \ln [4(1 - a^2)]$$

$$[|a| < 1]$$

$$28. = 2\pi \ln [2(1 - a^2 + |a|\sqrt{a^2 - 1})]$$

$$[|a| > 1]$$

$$29. \int_0^\pi x^2 \frac{\cos x (a^2 - 1 - \sin^2 x)}{(a^2 - \cos^2 x)^3} dx = \frac{\pi}{a} \ln \frac{1 - a}{1 + a}$$

$$[|a| \neq 0 \neq 1]$$

$$30. \int_0^\pi \frac{1}{x} \frac{\sin x \cos 2mx \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2a} \frac{b^{2m-1}}{(a+b)^{2m}} \quad [a, b > 0; n = 2m \text{ или } 2m-1]$$

$$31. \int_0^\pi \frac{1}{x} \frac{\sin^k x \cos^l x}{a^2 \cos^2 x + b^2 \sin^2 x} dx = I_{k,l}^a$$

$$[a, b > 0]$$

$$I_{1,0}^a = \frac{\pi}{2ab} \frac{b^2 - a^2}{b^2 + a^2}, \quad I_{1,1}^a = I_{1,0}^a = \frac{\pi}{2a} \frac{b}{a^2 + b^2}, \quad I_{2,0}^a = -\frac{\pi}{2b} \frac{a}{a^2 + b^2}$$

$$32. \int_0^\infty \frac{1}{x^2 + z^2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi \operatorname{sh} 2z}{4z (b^2 \operatorname{sh}^2 z - a^2 \operatorname{ch}^2 z)} \left(\frac{b}{a} - \frac{a}{b} - \frac{2}{\operatorname{sh} 2z} \right)$$

$$[a, b, \operatorname{Re} z > 0; \operatorname{th} z \neq a/b]$$

$$33. \int_0^\infty \frac{x \sin 2x dx}{(x^2 + z^2) (a^2 \cos^2 x + b^2 \sin^2 x)} = \frac{\pi}{2 (b^2 \operatorname{sh}^2 z - a^2 \operatorname{ch}^2 z)} \left(\frac{b-a}{b+a} e^{-2z} \right)$$

$$[a, b, \operatorname{Re} z > 0]$$

$$34. \int_0^\infty \frac{\sin^k x \cos^l x}{(a^2 \cos^2 x + b^2 \sin^2 x)^m} \frac{dx}{x} = I_{k,l}^m$$

$$[a, b > 0]$$

$$I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{2ab}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{2b(a+b)}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{2a(a+b)},$$

$$I_{1,1}^{m-1} = \frac{\pi}{2b(a+b)}, \quad I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{4} \frac{a^2 + b^2}{a^2 b^2}, \quad I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{4ab^2},$$

$$I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{4ab^2}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{16} \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^2 b^4},$$

$$I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{16} \frac{3a^2 + b^2}{a^2 b^2}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{16} \frac{a^2 + 3b^2}{a^2 b^2},$$

$$I_{1,0}^m = \frac{\pi}{32} \frac{5a^4 + 3a^2 b^2 + 3a^2 b^4 + 5b^4}{a^2 b^4}, \quad I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{32} \frac{5a^4 + 2a^2 b^2 + b^4}{a^2 b^4},$$

$$I_{1,0}^m = I_{1,0}^{m-1} = \frac{\pi}{32} \frac{5a^4 + b^4}{a^2 b^4}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{32} \frac{a^4 + 2a^2 b^2 + 5b^4}{a^2 b^4},$$

$$I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{32} \frac{a^4 + 5b^4}{a^2 b^4}, \quad I_{1,1}^m = I_{1,1}^{m-1} = \frac{\pi}{32} \frac{a^2 + b^2}{a^2 b^4}.$$

$$35. \int_0^\pi \frac{1}{x} \frac{(1 - \cos x) \cos^l x}{(a^2 \cos^2 x + b^2 \sin^2 x)^m} \frac{dx}{\sin x} = I_m^l$$

$$[a, b > 0]$$

$$I_1^0 = \frac{\pi}{2ab}, \quad I_1^1 = \frac{\pi}{2a(a+b)}, \quad I_1^2 = \frac{\pi}{4} \frac{a^2 + b^2}{a^2 b^2},$$

$$I_2^0 = \frac{\pi}{4a^2 b}, \quad I_2^1 = \frac{\pi}{16} \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^2 b^2}, \quad I_2^2 = \frac{\pi}{16} \frac{a^2 + 3b^2}{a^2 b^2},$$

$$I_2^3 = \frac{\pi}{32} \frac{a^4 + 2a^2 b^2 + 5b^4}{a^2 b^2}, \quad I_2^4 = \frac{\pi}{32} \frac{a^2 + 5b^2}{a^2 b^2}.$$

2.5.19. Интегралы, содержащие $(a \cos^2 x + b \sin^2 x + c)^{1/2}$.

$$1. \int_0^a \frac{x}{\sin x \sqrt{tg^2 a \cos^2 x - \sin^2 x}} dx = -\frac{\pi}{2tg a} \ln \left| tg \frac{\pi - 2a}{4} \right|$$

$$[a > 0]$$

$$2. \int_0^a \frac{x \sin x dx}{\cos^2 x \sqrt{tg^2 a \cos^2 x - \sin^2 x}} = \frac{\pi}{4 \cos a} (1 - \cos a)$$

$$[a > 0]$$

$$3. \int_0^a \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 a - \sin^2 x}} = \frac{\pi}{\cos^2 a} \frac{\sin^2 a}{2}$$

$$[a > 0]$$

$$4. \int_0^a \frac{x \sin x dx}{(1 - \sin^2 b \sin^2 x) \sqrt{\sin^2 a - \sin^2 x}} =$$

$$= (2 \cos b \sqrt{1 - \sin^2 b \sin^2 a})^{-1} \pi \ln \frac{\cos b + \sqrt{1 - \sin^2 b \sin^2 a}}{2 \cos a \cos^2 (b/2)}$$

$$[a > 0]$$

$$5. \int_0^{\pi/2} \frac{x \cos x dx}{\sqrt{\sin^2 x - \sin^2 a}} = \frac{\pi}{2} \ln (1 + \cos a)$$

$$[|a| < \pi/2]$$

$$6. \int_0^{\pi/4} \frac{x \sin x dx}{\cos x \sqrt{\cos^2 x + k^2 \sin^2 x}} = \frac{1}{1 - k^2} \left[F \left(\frac{\pi}{4}, \sqrt{1 - k^2} \right) - \frac{\pi}{2 \sqrt{2(1 + k^2)}} \right]$$

$$[0 < k < 1]$$

$$7. \int_0^{\pi/2} \left(1 - k^2 \left(\frac{\sin x}{\cos x} \right)^2 \right)^{-1/2} dx = (ik)^{b-1} K \left(\sqrt{1 + k^{2/3}} \right)$$

$$[0 < k < 1; b = \frac{1}{6}]$$

$$8. \int_0^{\pi/2} \frac{\left(\frac{\sin x}{\cos x} \right)^2 dx}{\sqrt{1 + \sin^2 x}} = \pm \sqrt{2} E \left(\frac{1}{\sqrt{2}} \right) \mp 2^{1/2} K \left(\frac{1}{\sqrt{2}} \right)$$

9. $\int_0^{\pi/2} \operatorname{tg}^2 \frac{x}{2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = 2\sqrt{1-k^2} - 2E(k) + K(k)$ $[0 < k < 1]$
10. $\int_0^{\pi/2} x \sin 2x \sqrt{1-k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2} dx =$
 $= -\frac{1}{9k^2} \left[\frac{3\pi}{2} \left\{ \frac{1}{k^2} \right\} + k'^2 K(k) - 2(1+k'^2) E(k) \right]$ $[0 < k < 1; k' = \sqrt{1-k^2}]$
11. $\int_0^{\pi/2} \cos 2nx \sqrt{1-k^2 \sin^2 x} dx =$
 $= (-1)^{n+1} \frac{\pi}{2} \sum_{j=0}^{\infty} \frac{(2n+2j-3)!! (2n+2j-1)!!}{j! (2n+j)!} \left(\frac{k}{2} \right)^{2n+2j}$ $[0 < k < 1]$
12. $\int_0^{\pi/2} \frac{x \sin 2x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{1}{k^2} [2E(k) - \pi \sqrt{1-k^2}]$ $[0 < k < 1]$
13. $\int_0^{\pi/2} \frac{x \sin 2x}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{k^2} [\pi - 2E(k)]$ $[0 < k < 1]$
14. $\int_0^{\pi/2} \frac{\cos 2nx}{\sqrt{1-k^2 \sin^2 x}} dx = (-1)^n \frac{\pi}{2} \sum_{j=0}^{\infty} \frac{[(2n+2j-1)!!]^2}{j! (2n+j)!} \left(\frac{k}{2} \right)^{2n+2j}$ $[0 < k < 1]$
15. $\int_0^{\pi/2} \frac{\left\{ \frac{\sin x}{\cos x} \right\}^2 \left(1 - a^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1} dx}{\sqrt{1-k^2 \sin^2 x}} =$
 $= 2(k')^{-2} \left(\sin 2\beta \sqrt{1-k'^2 \sin^2 \beta} \right)^{-1/2} \times$
 $\times \left[\frac{\pi}{2} - K(k) E(\beta, k') - E(k) F(\beta, k') + K(k) F(\beta, k') \right]$
 $\left[\left\{ \frac{a^2 - 1 - k'^2 \sin^2 \beta}{a^2 - \cos^2 \beta} \right\}; 0 < k < 1; k' = \sqrt{1-k^2}; \beta = \left\{ \frac{1}{2} \right\} \right]$
16. $\int_0^{\pi} \sin nx \left(\sqrt{1-k^2 \cos^2 x} \right)^{\pm 1} dx = 0$ $[0 < k < 1; n=2, 4, 6, \dots]$
17. $= \frac{2}{n} (1-k^2)^{\pm 1/2} {}_2F_2 \left(-\frac{1}{2}, \frac{1}{2}, 1; 1+\frac{n}{2}, 1-\frac{n}{2}; \frac{k^2}{k^2-1} \right)$ $[0 < k < 1; n=1, 3, 5, \dots]$
18. $\int_0^{2\pi} \cos nx \left(\sqrt{1-k^2 \cos^2 x} \right)^{\pm 1} dx = 0$ $[0 < k < 1; n=1, 3, 5, \dots]$
19. $= \frac{\pi(n-3)!! k^n}{2^{n/2} (n/2)!} \left\{ \frac{-1}{n-1} \right\} {}_2F_1 \left(\frac{n-1}{2}, \frac{n+1}{2}; n+1; k^2 \right)$ $[0 < k < 1; n=2, 4, 6, \dots]$

20. $\int_0^{\pi/2} \frac{x dx}{\sqrt{1-k^2 \sin^2 x}} = \pi n^2 K(k)$ $[0 < k < 1]$
21. $\int_0^{\pi/2} \frac{x \sin x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{(-1)^n \pi n}{2k} \ln \frac{1-k}{1+k}$ $[0 < k < 1]$
22. $\int_0^{\pi/2} \frac{\sin x}{x} \sqrt{1-k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2} dx = E(k)$ $[0 < k < 1]$
23. $\int_0^{\pi/2} \frac{\operatorname{tg} x}{x} \sqrt{1-k^2 \left\{ \frac{\sin rx}{\cos rx} \right\}^2} dx = E(k)$ $[0 < k < 1; r=1, 2]$
24. $\int_0^{\pi/2} \frac{\sin x}{x} \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = K(k)$ $[0 < k < 1]$
25. $\int_0^{\pi/2} \frac{\sin x}{x} \left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \frac{1}{\sqrt{2}} K \left(\frac{1}{\sqrt{2}} \right)$
26. $\int_0^{\pi/2} \frac{\operatorname{tg} x}{x} \left(1 - k^2 \left\{ \frac{\sin rx}{\cos rx} \right\}^2 \right)^{-1/2} dx = K(k)$ $[r=1, 2; 0 < k < 1]$
27. $\int_0^{\pi/2} \frac{1}{x} \operatorname{tg} x \left(1 + \left\{ \frac{\sin rx}{\cos rx} \right\}^2 \right)^{-1/2} dx = \frac{1}{\sqrt{2}} K \left(\frac{1}{\sqrt{2}} \right)$ $[r=1, 2]$
28. $\int_0^{\pi/2} \frac{\sin x \cos^n x}{x \sqrt{1-k^2 \sin^2 x}} dx = I_n$ $[0 < k < 1; k' = \sqrt{1-k^2}]$
 $I_1 = I_2 = k^{-2} [E(k) - k'^2 K(k)],$
 $I_3 = I_4 = 3^{-1} k^{-6} [(2-3k^2) k'^2 K(k) - 2(k'^2 - k^2) E(k)].$
29. $\int_0^{\pi/2} \frac{\sin x \cos^n x}{x \sqrt{1-k^2 \cos^2 x}} dx = J_n$ $[0 < k < 1]$
 $J_1 = J_2 = k^{-2} [K(k) - E(k)],$
 $J_3 = J_4 = 3^{-1} k^{-6} [(2+k^2) K(k) - 2(1+k^2) E(k)].$
30. $\int_0^{\pi/2} \frac{1}{x} \sin x \cos^n x \left(1 + \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \mp \sqrt{2} \left[E \left(\frac{1}{\sqrt{2}} \right) - \frac{1}{2} \left\{ \frac{2}{1} \right\} K \left(\frac{1}{\sqrt{2}} \right) \right]$ $[r=1, 2]$
31. $\int_0^{\pi/2} \frac{1}{x} \sin^2 x \cos^n x \left(1 - k^2 \left\{ \frac{\sin x}{\cos x} \right\}^2 \right)^{-1/2} dx = \frac{1}{2k^2} [(1+k^2) E(k) - 2k'^2 K(k)]$ $[r=1, 2; 0 < k < 1; k' = \sqrt{1-k^2}]$

$$32. \int_0^{\infty} \frac{\sin^3 x \cos x}{x \sqrt{1-k^2 \sin^2 2x}} dx = \frac{1}{4k^2} [K(k) - E(k)] \quad [0 < k < 1]$$

$$33. \int_0^{\infty} \frac{\sin^3 x \cos x}{x \sqrt{1-k^2 \cos^2 2x}} dx = \frac{1}{4k^2} [E(k) - (1-k^2) K(k)] \quad [0 < k < 1]$$

$$34. \int_0^{\infty} \frac{1}{x} \sin^3 x \cos x \left(1 + \frac{\sin 2x}{\cos 2x}\right)^{-1/2} dx = \frac{1}{4} \left[\frac{1}{2} \left(\frac{1}{2} \right) K\left(\frac{1}{\sqrt{2}}\right) - E\left(\frac{1}{\sqrt{2}}\right) \right]$$

$$35. \int_0^{\infty} \frac{\sin^2 x \lg x}{x \sqrt{1-k^2 \sin^2 x}} dx = \frac{1}{k^2} [K(k) - E(k)] \quad [0 < k < 1]$$

$$36. \int_0^{\infty} \frac{\sin^2 x \lg x}{x \sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{k^2} [E(k) - (1-k^2) K(k)] \quad [0 < k < 1]$$

$$37. \int_0^{\infty} \frac{1}{x} \cos^2 2x \lg x \left(1 - k^2 \frac{\sin 2x}{\cos 2x}\right)^{-1/2} dx = \pm \frac{1}{k^2} \left[E(k) - (1-k^2) K(k) \right] \quad [0 < k < 1]$$

$$38. \int_0^{\infty} \frac{\cos^2 2x \lg x}{x \sqrt{1+\sin^2 2x}} dx = \sqrt{2} \left[K\left(\frac{1}{\sqrt{2}}\right) - E\left(\frac{1}{\sqrt{2}}\right) \right]$$

$$39. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\sin^4 x \lg x}{\cos^4 x \lg(x/2)} \right\} \left(1 - k^2 \frac{\cos x}{\sin x}\right)^{-1/2} dx = \frac{1}{3k^4} [(2 \pm 3k^2) k'^2 K(k) - 2(k'^2 - k^2) E(k)] \quad [0 < k < 1; k' = \sqrt{1-k^2}]$$

$$40. \int_0^{\infty} \frac{1}{x} \left\{ \frac{\sin^4 x \lg x}{\cos^4 x \lg(x/2)} \right\} \left(1 - k^2 \frac{\sin x}{\cos x}\right)^{-1/2} dx = \frac{1}{3k^4} [(2 + k^2) K(k) - 2(1 + k^2) E(k)] \quad [0 < k < 1]$$

$$41. \int_0^{\infty} \frac{\sin^2 4x \lg x}{x \sqrt{1-k^2 \sin^2 2x}} dx = \frac{4}{3k^4} [(2-k^2) E(k) - 2(1-k^2) K(k)] \quad [0 < k < 1]$$

$$42. \int_0^{\infty} \frac{\sqrt{1-k^2 \sin^2 x}}{x^2 + z^2} dx = \frac{1}{2z} \operatorname{sh} 2x \left[\left(\frac{1}{\operatorname{sh}^2 z} + k^2 \right) \Pi\left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k\right) - k^2 K(k) \right] \quad [\operatorname{Re} z > 0; 0 < k < 1]$$

$$43. \int_0^{\infty} \frac{dx}{(x^2 + z^2) \sqrt{1-k^2 \sin^2 x}} = \frac{1}{z} \operatorname{cth} z \Pi\left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k\right) \quad [0 < k < 1; \operatorname{Re} z > 0]$$

$$44. \int_0^{\infty} \frac{x \sin x dx}{(x^2 + z^2) \sqrt{1-k^2 \sin^2 x}} = \operatorname{ch} z \left[K(k) - \Pi\left(\frac{\pi}{2}, \frac{1}{\operatorname{sh}^2 z}, k\right) \right] \quad [0 < k < 1; \operatorname{Re} z > 0]$$

$$2.5.20. \text{ Интегралы от } A(x) \left\{ \frac{\sin ax}{\cos ax} \right\}^u \left\{ \frac{\sin(bx+c)}{\cos(bx+c)} \right\}^v$$

$$A(x) \left\{ \frac{\sin ax}{\cos ax} \right\}^u \left\{ \frac{\cos(bx+c)}{\sin(bx+c)} \right\}^v$$

$$1. \int_0^a \frac{\sin [c(a-x)]}{(x+z)^2} dx = c \{ \sin t [\Phi_1(t-ac) - \Phi_1(t)] - \cos t [\Phi_2(t-ac) - \Phi_2(t)] \} \\ \Phi_1(t) = t^{-1} \cos t + \operatorname{si}(t), \quad \Phi_2(t) = t^{-1} \sin t - \operatorname{ci}(t) \\ [t = c(x+a); a, c > 0; |\arg x| < \pi]$$

$$2. \int_0^a \frac{x dx}{\cos x \cos(a-x)} = -\frac{a}{\sin a} \ln \cos a \quad [0 < a < \pi/2]$$

$$3. \int_0^{\pi} \sin^{\mu-1} x \cos \left[a \left(\frac{\pi}{2} - x \right) \right] dx = 2^{\mu-1} \Gamma \left[\frac{(\mu-a)/2}{\mu-a, \mu+a} \right] \quad [\operatorname{Re} \mu > 0]$$

$$4. \int_0^{\pi} \sin^{n-1} x \cos [m(a-x)] dx = \frac{\pi \cos [m(a-\pi/2)]}{2^{n-1} n B \left(\frac{n+m+1}{2}, \frac{n-m+1}{2} \right)} \quad [n \geq m; m, n = 1, 2, 3, \dots]$$

$$5. \int_0^{2\pi} [a + b \cos(a-x)]^n \left\{ \frac{\sin mx}{\cos mx} \right\} dx = \frac{2\pi i^m}{(1+i)^m} (a^2 + b^2)^{n/2} \left\{ \frac{\sin cm}{\cos cm} \right\} P_{\frac{n}{2}}^m(c) \\ [c = a/\sqrt{a^2 + b^2}]$$

$$6. \int_0^{2\pi} [a + b \cos(c-x)]^n (d + h \cos x)^{n-1} dx = 2\pi \frac{g^n}{\sqrt{d^2 - h^2}} P_{\frac{n}{2}} \left(\frac{ad - bh \cos c}{g} \right) \\ [g = \sqrt{(a^2 - b^2)(d^2 - h^2)}; |b| < a; |h| < d]$$

$$7. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\sin(bx+c)}{\cos(bx+c)} \right\} dx = b^{-\alpha} \Gamma(\alpha) \left\{ \frac{\sin(c+\alpha\pi/2)}{\cos(c+\alpha\pi/2)} \right\} \quad [b > 0; 0 < \operatorname{Re} \alpha < 1]$$

$$8. \int_0^{\infty} x^{2\alpha-1} \frac{\sin(\pi\alpha/2 - bx)}{x^2 - y^2} dx = -\frac{\pi}{2} y^{\alpha-2} \cos\left(\frac{\pi\alpha}{2} - by\right) \quad [b > 0; 0 < \alpha < 2]$$

$$2.5.21. \text{ Интегралы от } \left\{ \frac{\sin(ax^p + bx^q + c)}{\cos(ax^p + bx^q + c)} \right\}$$

$$1. \int_0^{1+\sigma} x^{\sigma} \sin(ax^2 - 2ax) dx = \frac{(-1)^{\sigma+1}}{2} \{ \operatorname{Si}(a) - \pi [C^2(a) + S^2(a)] \}$$

$$[\sigma = 0 \text{ или } 1; a > 0]$$

$$2. \int_0^{\infty} \frac{1}{x} \sin(ax^2 - 2ax) dx = \frac{1}{2} \operatorname{Si}(a) + \frac{\pi}{2} [1 - C(a) - S(a)] \quad [a > 0].$$

$$3. \int_0^{\infty} \frac{\sin(ax^2 + 2bx)}{\cos(ax^2 + 2bx)} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left(\cos \frac{b^2}{a} \mp \sin \frac{b^2}{a} \right) \quad [a > 0].$$

$$4. \int_0^{\infty} \frac{1}{x} \sin(ax^2 \mp 2ax) dx = \frac{\pi}{2} \left[\frac{1}{2} - C^2(a) - S^2(a) \mp C(a) \mp S(a) \right] \quad [a > 0].$$

$$5. \int_0^{\infty} \cos(x^3 \mp ax) dx = \frac{\pi}{3\sqrt{3}} \left\{ \frac{J_{-1/3}(c) + J_{1/3}(c)}{J_{-1/3}(c) - J_{1/3}(c)} \right\} \quad [a > 0; c = 2(a/3)^{1/3}].$$

$$6. \int_0^{\infty} \frac{\sin(ax^p + bx^q)}{\cos(ax^p + bx^q)} dx = \frac{1}{p} \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \left(\frac{\sin c}{\cos c}\right) \\ \left[c = \frac{k(q-p)+1}{2p} \pi; a, b, p, q > 0 \right].$$

$$7. \int_{-\infty}^{\infty} \frac{\sin(ax^2 + 2bx + c)}{\cos(ax^2 + 2bx + c)} dx = \sqrt{\frac{\pi}{a}} \left(\frac{\sin d}{\cos d} \right) \quad \left[d = \frac{\pi}{4} + \frac{ac - b^2}{a}; a > 0 \right].$$

$$8. \int_{-\infty}^{\infty} \frac{\sin a \sqrt{|x|} \operatorname{sgn} x}{x-y} dx = \cos a \sqrt{|y|} + e^{-a \sqrt{|y|}} \quad [a > 0; \operatorname{Im} y = 0].$$

2.5.22. Интегралы от $x^\alpha \left\{ \frac{\sin ax^p \sin bx}{\cos ax^p \cos bx} \right\}$, $x^\alpha \left\{ \frac{\sin ax^p \cos bx}{\cos ax^p \sin bx} \right\}$.

Обозначение: $\delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$1. \int_0^{\infty} \frac{\sin ax^2}{\cos ax^2} \sin bx dx = \\ = \sqrt{\frac{\pi}{2a}} \left[\left\{ \frac{\cos [b^2/(4a)]}{\sin [b^2/(4a)]} \right\} C\left(\frac{b^2}{4a}\right) \pm \left\{ \frac{\sin [b^2/(4a)]}{\cos [b^2/(4a)]} \right\} S\left(\frac{b^2}{4a}\right) \right] \quad [a, b > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} \frac{\sin ax^2}{\cos ax^2} \sin bx dx = \\ \frac{b}{2} a^{-(\alpha+1)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \left\{ \frac{\cos [(1-\alpha)\pi/4]}{\sin [(1-\alpha)\pi/4]} \right\} {}_2F_3\left(\frac{\alpha+3}{4}, \frac{\alpha+1}{4}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^2}{64a^2}\right) \mp \\ \frac{b^2 a^{-(\alpha+3)/2}}{12} \Gamma\left(\frac{\alpha+3}{2}\right) \left\{ \frac{\cos [(1+\alpha)\pi/4]}{\sin [(1+\alpha)\pi/4]} \right\} {}_2F_3\left(\frac{5+\alpha}{4}, \frac{3+\alpha}{4}; \frac{3}{2}, \frac{5}{4}, \frac{7}{4}; -\frac{b^2}{64a^2}\right) \\ [a, b > 0; -1-2\delta < \operatorname{Re} \alpha < 2].$$

$$3. \int_0^{\infty} x \frac{\sin ax^2}{\cos ax^2} \sin bx dx = \frac{b}{4a} \sqrt{\frac{\pi}{2a}} \left(\sin \frac{b^2}{4a} \pm \cos \frac{b^2}{4a} \right) \quad [a, b > 0].$$

2.5.22]

$$4. \int_0^{\infty} x^{-1/2} \frac{\sin ax^2}{\cos ax^2} \sin bx dx = \mp \frac{\pi}{4} \sqrt{\frac{b}{a}} \left(\sin \frac{b^2}{8a} \pm \cos \frac{b^2}{8a} \right) J_{1/4}\left(\frac{b^2}{8a}\right) \quad [a, b > 0].$$

$$5. \int_0^{\infty} \frac{\sin ax^2}{\cos ax^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left(\cos \frac{b^2}{4a} \mp \sin \frac{b^2}{4a} \right) \quad [a, b > 0].$$

$$6. \int_0^{\infty} x^{\alpha-1} \frac{\sin ax^2}{\cos ax^2} \cos bx dx = \\ = \frac{a^{-\alpha/2}}{2} \Gamma\left(\frac{\alpha}{2}\right) \left\{ \frac{\sin (\alpha\pi/4)}{\cos (\alpha\pi/4)} \right\} {}_2F_3\left(\frac{2+\alpha}{4}, \frac{\alpha}{4}; \frac{1}{2}, \frac{1}{4}, \frac{3}{4}; -\frac{b^2}{64a^2}\right) \mp \\ \mp \frac{b^2}{4} a^{-\alpha/2-1} \Gamma\left(\frac{\alpha}{2}+1\right) \left\{ \frac{\cos (\alpha\pi/4)}{\sin (\alpha\pi/4)} \right\} {}_2F_3\left(1+\frac{\alpha}{4}, \frac{2+\alpha}{4}; \frac{3}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{b^2}{64a^2}\right) \\ [a, b > 0; -2\delta < \operatorname{Re} \alpha < 2].$$

$$7. \int_0^{\infty} x \frac{\sin ax^2}{\cos ax^2} \cos bx dx = \\ = \frac{\delta}{2a} \mp \frac{b}{2a} \sqrt{\frac{\pi}{2a}} \left[\left\{ \frac{\sin [b^2/(4a)]}{\cos [b^2/(4a)]} \right\} C\left(\frac{b^2}{4a}\right) \mp \left\{ \frac{\cos [b^2/(4a)]}{\sin [b^2/(4a)]} \right\} S\left(\frac{b^2}{4a}\right) \right] \quad [a, b > 0].$$

$$8. \int_0^{\infty} x^{-1/2} \frac{\sin ax^2}{\cos ax^2} \cos bx dx = \mp \frac{\pi}{2} \sqrt{\frac{b}{2a}} \left\{ \frac{\sin c}{\cos c} \right\} J_{-1/4}\left(\frac{b^2}{8a}\right) \\ [c = (b^2 - \pi a)/(8a); a, b > 0].$$

$$9. \int_0^{\infty} x^{1/2} \cos ax^2 \frac{\sin bx}{\cos bx} dx = -\frac{\pi}{8\sqrt{2}} \left(\frac{b}{a}\right)^{3/2} \left[\left\{ \frac{\sin [(b^2 - \pi a)/(8a)]}{\cos [(b^2 + \pi a)/(8a)]} \right\} J_{\pm 1/4}\left(\frac{b^2}{8a}\right) + \right. \\ \left. + \left\{ \frac{\cos [(b^2 - \pi a)/(8a)]}{\sin [(b^2 + \pi a)/(8a)]} \right\} J_{\pm 3/4}\left(\frac{b^2}{8a}\right) \right] \quad [a, b > 0].$$

$$10. \int_0^{\infty} \frac{1}{x^2} \sin ax^2 \cos bx dx = \frac{b\pi}{2} \left[S\left(\frac{b^2}{4a}\right) - C\left(\frac{b^2}{4a}\right) + \sqrt{\pi a} \sin \frac{b^2 + \pi a}{4a} \right] \quad [a, b > 0].$$

$$11. \int_0^{\infty} \frac{\sin ax^2 \sin bx}{\cos ax^2 \cos bx} dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left[J_{1/3}(c) + J_{-1/3}(c) \mp \frac{\sqrt{3}}{\pi} K_{1/3}(c) \right] \\ [c = 2b^{3/2}/(3a)^{3/2}; a, b > 0].$$

$$12. \int_0^{\infty} x^{-1/2} \sin(2a\sqrt{x}) \sin cx \frac{\sin bx}{\cos bx} dx = \\ = \sqrt{\frac{\pi}{2(c-b)}} \left[\left\{ \frac{\sin d}{\cos d} \right\} C(d) \mp \left\{ \frac{\cos d}{\sin d} \right\} S(d) \right] \mp \\ \mp \sqrt{\frac{\pi}{2(c+b)}} \left[\left\{ \frac{\sin h}{\cos h} \right\} C(h) \mp \left\{ \frac{\cos h}{\sin h} \right\} S(h) \right] \\ [c > b > 0; a > 0; d = \frac{a^2}{c-b}, h = \frac{a^2}{c+b}].$$

$$13. = \mp \frac{1}{2} \sqrt{\frac{\pi}{c}} \left[\cos \frac{a^2}{2c} C \left(\frac{a^2}{2c} \right) \mp \frac{\left\{ \cos \left[\frac{a^2}{2(2c)} \right] \right\}}{\left\{ \sin \left[\frac{a^2}{2(2c)} \right] \right\}} S \left(\frac{a^2}{2c} \right) \right] \\ [c = b > 0; a > 0].$$

$$14. \int_0^\infty x^{-1/2} \sin(2a \sqrt{x}) \cos cx \frac{\sin bx}{\cos bx} dx = \\ = \sqrt{\frac{\pi}{2(c+b)}} \left[\frac{\cos h}{\sin h} C(h) \pm \frac{\sin h}{\cos h} S(h) \right] - \\ - \sqrt{\frac{\pi}{2(c-b)}} \left[\frac{\sin d}{\cos d} S(d) \mp \frac{\cos d}{\sin d} C(d) \right] \\ [c > b > 0; a > 0; d = \frac{a^2}{c-b}, h = \frac{a^2}{c+b}].$$

$$15. = \frac{1}{2} \sqrt{\frac{\pi}{c}} \left[\frac{\left\{ \cos \left[\frac{a^2}{2(2c)} \right] \right\}}{\left\{ \sin \left[\frac{a^2}{2(2c)} \right] \right\}} C \left(\frac{a^2}{2c} \right) \pm \sin \frac{a^2}{2c} S \left(\frac{a^2}{2c} \right) \right] \\ [c = b > 0; a > 0].$$

$$16. \int_0^\infty x^{-1/2} \cos(2a \sqrt{x}) \sin cx \frac{\sin bx}{\cos bx} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2(c-b)}} (\cos d \pm \sin d) \mp \frac{1}{2} \sqrt{\frac{\pi}{2(c+b)}} (\cos h \pm \sin h) \\ [c > b > 0; a > 0; d = \frac{a^2}{c-b}, h = \frac{a^2}{c+b}].$$

$$17. = \mp \frac{1}{4} \sqrt{\frac{\pi}{c}} \left(\cos \frac{a^2}{2c} \pm \sin \frac{a^2}{2c} \right) [b = c > 0; a > 0].$$

$$18. \int_0^\infty x^{-1/2} \cos(2a \sqrt{x}) \cos cx \frac{\sin bx}{\cos bx} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2(c+b)}} (\cos h \mp \sin h) + \frac{1}{2} \sqrt{\frac{\pi}{2(c-b)}} (\sin d \mp \cos d) \\ [c > b > 0; a > 0; d = \frac{a^2}{c-b}, h = \frac{a^2}{c+b}].$$

$$19. = \frac{1}{4} \sqrt{\frac{\pi}{c}} \left(\cos \frac{a^2}{2c} \mp \sin \frac{a^2}{2c} \right) [b = c > 0; a > 0].$$

2.5.23. Интегралы от $A(x) \frac{\sin(ax+b/x)}{\cos(ax+b/x)}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^\infty \frac{x^{\alpha-1}}{\sqrt{x^2+1}} \frac{\sin(ax+a/x)}{\cos(ax+a/x)} dx = \pm \left(\frac{\pi}{2} \right)^{3/2} \sqrt{a} [J_\mu(a) J_\nu(a) - Y_\mu(a) Y_\nu(a)] \\ [4\mu = -1-2\alpha-2\delta, 4\nu = 3+2\alpha-2\delta; a > 0; -1 < \operatorname{Re} \alpha < 2].$$

$$2. \int_0^1 \frac{x^{\alpha-1}}{\sqrt{1-x^2}} \frac{\sin(ax-a/x)}{\cos(ax-a/x)} dx = \mp \sqrt{\frac{\pi a}{2}} J_{(1+\delta-1)/2}(a) K_{(1-\delta)/2}(a) \\ [\operatorname{Re} \alpha > -1; a > 0].$$

2.5.24.]

$$3. \int_0^\infty \frac{\sin(ax+a/x) \sin(ax-a/x)}{\cos(ax+a/x) \cos(ax-a/x)} \frac{dx}{1+x^2} = \frac{\pi}{4} \frac{\sin(-2a)}{\exp(-2a)} [a > 0].$$

$$4. \int_0^\infty x^{\alpha-1} \frac{\sin(ax+b/x)}{\cos(ax+b/x)} dx = \frac{\pi}{2} \left(\frac{b}{a} \right)^{\alpha/2} \left\{ \frac{\cos(\alpha\pi/2)}{\sin(\alpha\pi/2)} \right\} [J_{-\alpha}(2\sqrt{ab}) \pm J_\alpha(2\sqrt{ab})] \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$5. \int_0^\infty x^{-1/2} \frac{\sin(ax+b/x)}{\cos(ax+b/x)} dx = \sqrt{\frac{\pi}{a}} \left\{ \frac{\sin c}{\cos c} \right\} [c = 2\sqrt{ab} + \pi/4; a, b > 0].$$

$$6. \int_0^\infty x^{-3/2} \frac{\sin(ax+b/x)}{\cos(ax+b/x)} dx = \sqrt{\frac{\pi}{b}} \left\{ \frac{\sin c}{\cos c} \right\} [c = 2\sqrt{ab} + \pi/4; a, b > 0].$$

$$7. \int_0^\infty x^{\alpha-1} \frac{\sin(ax-b/x)}{\cos(ax-b/x)} dx = 2 \left(\frac{b}{a} \right)^{\alpha/2} \left\{ \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} \right\} K_\alpha(2\sqrt{ab}) \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$8. \int_0^\infty x^{-1/2} \frac{\sin(ax-b/x)}{\cos(ax-b/x)} dx = \sqrt{\frac{\pi}{2a}} e^{-2\sqrt{ab}} [a, b > 0].$$

$$9. \int_0^\infty x^{-3/2} \frac{\sin(ax-b/x)}{\cos(ax-b/x)} dx = \mp \sqrt{\frac{\pi}{2b}} e^{-2\sqrt{ab}} [a, b > 0].$$

$$10. \int_0^\infty x^{-1} \frac{\sin(ax-b/x)}{\cos(ax-b/x)} dx = \frac{1}{2b} \sqrt{\frac{\pi}{2}} [a, b > 0].$$

$$11. \int_0^\infty \frac{x^\delta}{x^2+z^2} \frac{\sin(ax-b/x)}{\cos(ax-b/x)} dx = \frac{\pi}{2} z^{\delta-1} \exp\left(-az - \frac{b}{z}\right) \\ [a, b, \operatorname{Re} z > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

2.5.24. Интегралы от $x^\alpha \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx}$, $x^\alpha \frac{\sin(a/x) \cos bx}{\cos(a/x) \sin bx}$.

$$1. \int_0^\infty x^{\alpha-1} \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} dx = \\ = \frac{\pi}{4} \left(\frac{a}{b} \right)^{\alpha/2} \frac{1}{\sin(\alpha\pi/2)} [J_{\pm\alpha}(2\sqrt{ab}) - J_{\mp\alpha}(2\sqrt{ab}) + 2\pi^{-1} \sin \alpha\pi K_\alpha(2\sqrt{ab})] \\ [ab > 0; |\operatorname{Re} \alpha| < 1 + \delta; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$2. \int_0^\infty \frac{1}{x} \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} dx = \pm \frac{\pi}{2} Y_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) [a, b > 0].$$

$$3. \int_0^{\infty} x^{-1/2} \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} dx = \\ = \pm \frac{1}{2} \sqrt{\frac{\pi}{2b}} [\sin(2\sqrt{ab}) - \cos(2\sqrt{ab}) \pm e^{-2\sqrt{ab}}] \quad [a, b > 0].$$

$$4. \int_0^{\infty} x^{-3/2} \frac{\sin(a/x) \sin bx}{\cos(a/x) \cos bx} dx = \\ = \pm \frac{1}{2} \sqrt{\frac{\pi}{2a}} [\sin(2\sqrt{ab}) - \cos(2\sqrt{ab}) \pm e^{-2\sqrt{ab}}] \quad [a, b > 0].$$

$$5. \int_0^{\infty} x^{2-1} \frac{\cos(a/x) \sin bx}{\sin(a/x) \cos bx} dx = \\ = \frac{\pi}{4} \left(\frac{a}{b}\right)^{1/2} \frac{1}{\cos(\alpha\pi/2)} [J_{\alpha}(2\sqrt{ab}) + J_{-\alpha}(2\sqrt{ab}) \pm 2\pi^{-1} \sin \alpha\pi K_{\alpha}(2\sqrt{ab})] \\ [a, b > 0; |\operatorname{Re} \alpha| < 1].$$

$$6. \int_0^{\infty} x^{-1/2} \frac{\cos(a/x) \sin bx}{\sin(a/x) \cos bx} dx = \\ = \frac{1}{2} \sqrt{\frac{\pi}{2b}} [\sin(2\sqrt{ab}) + \cos(2\sqrt{ab}) \pm e^{-2\sqrt{ab}}] \quad [a, b > 0].$$

$$7. \int_0^{\infty} x^{-3/2} \frac{\cos(a/x) \sin bx}{\sin(a/x) \cos bx} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} [\sin(2\sqrt{ab}) + \cos(2\sqrt{ab}) \mp e^{-2\sqrt{ab}}] \\ [a, b > 0].$$

2.5.25. Интегралы от $A(x) \frac{\sin(c\sqrt{a^2 \pm x^2}) \sin bx}{\cos(c\sqrt{a^2 \pm x^2}) \cos bx}$,

$$A(x) \left\{ \frac{\sin(c\sqrt{a^2 \pm x^2}) \cos bx}{\cos(c\sqrt{a^2 \pm x^2}) \sin bx} \right\}.$$

$$1. \int_0^a \sin(c\sqrt{a^2 - x^2}) \cos bx dx = \frac{\pi}{2} \frac{ac}{\sqrt{b^2 + c^2}} J_2(a\sqrt{b^2 + c^2}) \quad [a > 0].$$

$$2. \int_0^a \frac{\sin(c\sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx dx = \\ = \frac{\pi}{2} \sqrt{\frac{\pi c}{2}} J_{1/2} \left[\frac{a}{2} (\sqrt{c^2 + b^2} - b) \right] J_{1/2} \left[\frac{a}{2} (\sqrt{c^2 + b^2} + b) \right] \quad [a > 0].$$

$$3. \int_0^a \frac{\cos(c\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} \cos bx dx = \frac{\pi}{2} J_0(a\sqrt{b^2 + c^2}) \quad [a > 0].$$

$$4. \int_0^a \frac{\cos(c\sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx dx = \\ = \left(\frac{\pi}{2}\right)^{3/2} \sqrt{c} J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - b) \right] J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + b) \right] \quad [a > 0].$$

$$5. \int_0^a \sqrt{a^2 - x^2} \cos(c\sqrt{a^2 - x^2}) \cos bx dx = \\ = \frac{\pi}{2} \frac{a^2 c^2}{b^2 + c^2} \left[J_0(a\sqrt{b^2 + c^2}) + \frac{b^2 - c^2}{ac^2 \sqrt{b^2 + c^2}} J_1(a\sqrt{b^2 + c^2}) \right] \quad [a > 0].$$

$$6. \int_0^a \frac{\cos(c\sqrt{a^2 - x^2})}{\sqrt{x(a^2 - x^2)}} \cos bx dx = \\ = \left(\frac{\pi}{2}\right)^{3/2} \sqrt{b} J_{-1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] J_{1/4} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right] \quad [a > 0].$$

$$7. \int_0^{\infty} \frac{\sin(c\sqrt{x^2 + y^2})}{x \sqrt{x^2 + y^2}} \sin cx dx = \frac{\pi}{2y} \sin cy \quad [c, y > 0].$$

$$8. \int_0^{\infty} \frac{\sin(c\sqrt{x^2 + y^2})}{(x^2 + y^2)^{3/2}} \cos bx dx = \frac{\pi c}{2y} e^{-by} \quad [b > y > 0].$$

$$9. \int_0^{\infty} \frac{\sin(c\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \cos bx dx = \frac{\pi}{2} J_0(y\sqrt{c^2 - b^2}) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \left[\begin{Bmatrix} c > b > 0 \\ b > c > 0 \end{Bmatrix}; y > 0 \right].$$

$$10. \int_0^{\infty} \frac{\sin(c\sqrt{x^2 + y^2})}{\sqrt{x(x^2 + y^2)}} \cos bx dx = \\ = \left(\frac{\pi}{2}\right)^{3/2} \sqrt{b} J_{-1/4} \left[\frac{y}{2} (c - \sqrt{c^2 - b^2}) \right] J_{1/4} \left[\frac{y}{2} (c + \sqrt{c^2 - b^2}) \right] \quad [c > b > 0; y > 0].$$

$$11. \int_0^{\infty} \frac{\sin(c\sqrt{x^2 + y^2}) \cos bx}{(x^2 + a^2) \sqrt{x^2 + y^2}} dx = \frac{\pi e^{-ab} \sin(c\sqrt{y^2 - a^2})}{2a \sqrt{y^2 - a^2}} \\ [b > c > 0; a, y > 0; a \neq y].$$

$$12. = \pi c (2a)^{-1} e^{-ab} \quad [b > c > 0; a = y > 0].$$

$$13. \int_0^{\infty} x^{\delta} \frac{\cos(c\sqrt{x^2 + y^2})}{x^2 + a^2} \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} dx = \frac{\pi}{2a^{1-\delta}} e^{-ab} \cos(c\sqrt{y^2 - a^2}) \\ [b > c > 0; a, y > 0; \delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}].$$

$$14. \int_0^{\infty} x^{-1/2} \frac{\cos(c\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} \left\{ \frac{\sin bx}{\cos bx} \right\} dx =$$

$$= -\frac{\pi}{2} \sqrt{\frac{nb}{2}} J_{\pm 1/4} \left[\frac{y}{2} (c - \sqrt{c^2 - b^2}) \right] Y_{1/4} \left[\frac{y}{2} (c + \sqrt{c^2 - b^2}) \right]$$

$[b > b > 0; \nu > 0]$

$$15. \int_0^{\infty} \frac{\cos(c\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} \cos bx dx = K_0(y\sqrt{b^2-c^2})$$

$[b > c > 0; \nu > 0]$

$$= -\frac{\pi}{2} Y_0(y\sqrt{c^2-b^2})$$

$[c > b > 0; \nu > 0]$

2.5.26. Интегралы, содержащие $\left\{ \frac{\lg ax}{\operatorname{ctg} ax} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/4} \lg^{\mu} x dx = \frac{1}{2} \beta \left(\frac{\mu+1}{2} \right)$$

$[Re \mu > -1]$

$$2. \int_0^{\pi/4} \lg^m x dx = \frac{(-1)^{m(m+1)/2}}{4} \left\{ \frac{\pi}{\ln 4} \right\} + \sum_{k=0}^{[m/2]-1} \frac{(-1)^k}{m-2k-1}$$

$[m = \{0, 2, 4, \dots\}]$

$$3. \int_0^{\pi/4} x \lg^m x dx = I_m,$$

$$I_1 = \frac{G}{2} - \frac{\pi}{8} \ln 2, \quad I_2 = -\left(\frac{\ln 2}{2} + \frac{\pi^2}{32} - \frac{\pi}{4} \right), \quad I_3 = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} \ln 2 - \frac{G}{2}.$$

$$4. \int_0^{\pi/4} x^n \lg x dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^n \sum_{h=1}^{\infty} \frac{(4^h-1) \zeta(2h)}{4^{2h-1} (n+2h)}$$

$[n = 1, 2, 3, \dots]$

$$5. \int_0^{\pi/4} x^n \operatorname{ctg} x dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^n \left[\frac{2}{n} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (n+2k)} \right]$$

$[n = 1, 2, 3, \dots]$

$$6. \int_0^{\pi/4} x \operatorname{ctg} x dx = \frac{\pi}{8} \ln 2 + \frac{G}{2}.$$

$$7. \int_0^{\pi/2} \lg^{\mu} x dx = \frac{\pi}{2 \cos(\mu\pi/2)}$$

$[|Re \mu| < 1]$

$$8. \int_0^{\pi/2} x^n \operatorname{ctg} x dx = \left(\frac{\pi}{2} \right)^n \left[\frac{1}{n} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k (n+2k)} \right]$$

$[n = 1, 2, 3, \dots]$

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$$9. \int_0^{\pi/2} x \operatorname{ctg} x dx = \frac{\pi}{2} \ln 2. \quad 10. \int_0^{\pi} x \operatorname{tg} x dx = -\pi \ln 2.$$

$$11. \int_0^{\pi} \frac{\lg(x \pm 1/2) dx}{x} = 2 - (1 \pm 1/2) \pi \operatorname{sgn} a.$$

$$12. \int_0^{\pi/4} \lg^{\mu} x \left\{ \frac{\sin x}{\cos x} \right\}^2 dx = \frac{1 \pm \mu}{2} \beta \left(\frac{\mu+1}{2} \right)$$

$[Re \mu > -1]$

$$13. \int_0^{\pi/4} \frac{\lg(x \pm 1/2) \lg x}{\cos 2x} dx = \frac{G}{2} - \frac{\pi}{8} \ln 2.$$

$$14. \int_0^{\pi/2} \lg^{\mu} x \left\{ \frac{\sin 2x}{\cos 2x} \right\} dx = \pm \frac{\pi \mu}{2} \left\{ \frac{\sin(\mu\pi/2)}{\cos(\mu\pi/2)} \right\}$$

$[|Re \mu| < 1 + \delta]$

$$15. \int_0^{\pi/2} \lg^{\mu} x \sin^{\nu-1} x \left\{ \frac{\sin vx}{\cos vx} \right\} dx = \frac{\cos v}{\sin v} B(\mu + \nu - 1, 1 - \mu)$$

$[2c = (\mu + \nu)\pi; Re \mu < 1; Re(\mu + \nu) > 1 - \delta]$

$$16. \int_0^{\pi/2} \frac{\lg^{\mu}(x/2)}{2 \pm \sin x} dx = \frac{1}{3} \left\{ \psi \left[\frac{(\mu+2)/3}{2} \right] - \psi \left[\frac{(\mu+1)/3}{2} \right] \right\}$$

$[Re \mu > -1]$

$$17. \int_0^{\pi/2} \frac{\lg^m(x/2)}{1 + \cos(l\pi/n) \sin x} dx =$$

$$= \frac{1}{2n \sin(l\pi/n)} \sum_{k=0}^{n-1} (-1)^{k-1} \sin \frac{kl}{n} \pi \left[\psi \left(\frac{n+m+k}{2n} \right) - \psi \left(\frac{m+k}{2n} \right) \right]$$

$[l+n=1, 3, 5, \dots]$

$$18. = \frac{1}{n \sin(l\pi/n)} \sum_{k=0}^{[n-1]/2} (-1)^{k-1} \sin \frac{kl}{n} \pi \left[\psi \left(\frac{n+m-k}{n} \right) - \psi \left(\frac{m+k}{n} \right) \right]$$

$[l+n=0, 2, 4, \dots; m=1, 2, 3, \dots]$

$$19. \int_0^{\pi/2} \operatorname{ctg}^{\mu} x \cos^{\nu-1} x \left\{ \frac{\sin vx}{\cos vx} \right\} dx = \frac{\cos(\mu\pi/2)}{\sin(\mu\pi/2)} B(\mu + \nu - 1, 1 - \mu)$$

$[Re \mu < 1 + \delta; Re(\mu + \nu) > 1]$

$$20. \int_0^{\pi/2} \operatorname{ctg} x \cos^{\nu-2} x \sin vx dx = \pi/2$$

$[Re \nu > 0]$

$$21. \int_0^{\pi/2} \frac{x \operatorname{ctg} x}{\cos 2x} dx = \frac{\pi}{4} \ln 2.$$

$$22. \int_0^{\pi/2} x \frac{\operatorname{ctg}^{\mu-1} x}{\sin^2 x} dx = \frac{\pi}{2\mu \cos(\mu\pi/2)}$$

$$[0 < \operatorname{Re} \mu < 1]$$

$$23. \int_0^{\pi/2} \frac{x dx}{(\operatorname{tg}^{2/3} x + 1)(\sin x \cos^3 x)^{2/3}} = \frac{\pi 2^{1/3} \pi^2}{4}$$

$$24. \int_0^{\pi/2} \frac{x (\operatorname{tg} x)^{2pq-1}}{(\operatorname{tg}^{2p} x + 1)^{2q} \cos^2 x} dx = 2^{-2q-1} \frac{\pi^{3/2}}{p} \Gamma\left[\frac{q}{q+1/2}\right]$$

$$[q > 0]$$

$$25. \int_0^a \frac{x \operatorname{tg} x}{\sqrt{\operatorname{tg}^2 a - \operatorname{tg}^2 x} (\operatorname{tg}^2 b - \operatorname{tg}^2 x) \cos^2 x} dx =$$

$$= \frac{\pi}{2\sqrt{\operatorname{tg}^2 b - \operatorname{tg}^2 a}} \left[a + \operatorname{arctg} \frac{\cos b}{\sin(b-a) \sin(b+a)} - \frac{\pi}{2} \right]$$

$$[0 < a < b < \pi/2]$$

$$26. \int_0^{\infty} \frac{\operatorname{tg}^{\mu} ax}{x^2 + z^2} dx = \frac{\pi}{2z \cos(\mu\pi/2)} \operatorname{th}^{\mu} az$$

$$[a, \operatorname{Re} z > 0; |\operatorname{Re} \mu| < 1]$$

$$27. \int_0^{\infty} \frac{x}{x^2 + z^2} \left\{ \frac{\operatorname{tg} ax}{\operatorname{ctg} ax} \right\} dx = \frac{\pi}{2} \left\{ \frac{\operatorname{th} az}{\operatorname{cth} az} \right\}$$

$$[a, \operatorname{Re} z > 0]$$

$$28. \int_0^{\infty} x^{-2n} \sin^{2n-1} x \cos(2n-1)x \operatorname{tg} x dx = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} 2^{2n-1} \pi |B_{2n}|$$

$$[n = 1, 2, 3, \dots]$$

$$29. \int_0^{\infty} x^{-n} \sin^{2m} x \cos 2mx \operatorname{tg} x dx = 0$$

$$[n = 1, 2, 3, \dots, 2m+1]$$

$$30. \int_0^{\infty} x^{-n} \sin^{2m-1} x \cos 2mx \operatorname{tg} x dx = 0$$

$$[n = 1, 2, 3, \dots, 2m]$$

2.5.27. Интегралы, содержащие тригонометрические функции от тригонометрических функций.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} \left\{ \frac{\sin(a \sin x) \sin(2n+1)x}{\cos(a \sin x) \cos 2nx} \right\} dx = \frac{\pi}{2} J_{2n+\delta}(a)$$

$$[|\arg a| < \pi]$$

$$2. \int_0^{\pi/2} \left\{ \frac{\sin(a \cos x)}{\cos(a \cos x)} \right\} \cos bx dx =$$

$$= \frac{1}{b} \left(\frac{ab}{1-b^2} \right)^{\delta} \left\{ \frac{\cos(b\pi/2)}{\sin(b\pi/2)} \right\} {}_2F_2 \left(\frac{\delta+1}{2}; 1, \frac{b+\delta}{2}, 1, \frac{\delta-b}{2}; -\frac{a^2}{4} \right) =$$

$$= (-b)^{1-\delta} \left\{ \frac{\cos(b\pi/2)}{\sin(b\pi/2)} \right\} {}_2J_{\delta+1, \delta}(a) \quad [b > 0]$$

$$3. \int_0^{\pi/2} \left\{ \frac{\sin(a \cos x)}{\cos(a \cos x)} \right\} \cos nx dx = \frac{\pi}{2} \left[\frac{\sin(n\pi/2)}{\cos(n\pi/2)} J_n(a) - \frac{\cos(n\pi/2)}{\sin(n\pi/2)} E_n(a) \right]$$

$$4. \int_0^{\pi/2} x \left\{ \frac{\sin(n\pi \sin x) \cos x}{\sin(n\pi \cos x) \sin x} \right\} dx = \pm \frac{1}{2n} J_e(n\pi) + \frac{(\mp 1)^{n+1}}{2n} \quad [n = 1, 2, 3, \dots]$$

$$5. \int_0^{\pi/2} \sin(x \operatorname{tg} x - vx) \sin^{v-2} x dx = 0 \quad [\operatorname{Re} v > 0; |\arg z| < \pi]$$

$$6. \int_0^{\pi/2} \cos(x \operatorname{tg} x - vx) \cos^v x dx = \pi z^{v/2-1-v/2} \frac{1}{\Gamma(1+(v+\gamma)/2)} W_{\gamma/2, -(v+1)/2}(2z)$$

$$[\operatorname{Re} v > -1; |\arg z| < \pi; \gamma + v \neq -2, -4, -6, \dots]$$

$$7. \int_0^{\pi/2} \cos(x \operatorname{tg} x + vx) \cos^v x dx = 2^{-v-1} \pi e^{-z}$$

$$[\operatorname{Re} v > -1; |\arg z| < \pi]$$

$$8. \int_0^{\pi/2} \cos(x \operatorname{tg} x - vx) \cos^{v-2} x dx = \frac{\pi}{\Gamma(v)} e^{-z} z^{v-2}$$

$$[\operatorname{Re} v > 1; |\arg z| < \pi]$$

$$9. \int_0^{\pi/2} \sin(n \operatorname{tg} x + vx) \frac{\cos^{v+1} x}{\sin x} dx = \frac{\pi}{2}$$

$$[\operatorname{Re} v > 0; n = 1, 2, 3, \dots]$$

$$10. \int_0^{\pi/2} \frac{\sin(x \operatorname{tg} x - nx) + \sin nx}{\sin x} \cos^{n-1} x dx = \pi \quad [|\arg z| < \pi; n = 1, 2, 3, \dots]$$

$$11. \int_0^{\pi} \sin(a \sin x) \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 2 \cos \frac{b\pi}{2} \left\{ \frac{\sin(b\pi/2)}{\cos(b\pi/2)} \right\} {}_2J_{\delta, \delta}(a) \quad [b > 0]$$

$$12. \int_0^{\pi} \cos(a \sin x) \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -2b \sin \frac{b\pi}{2} \left\{ \frac{\sin(b\pi/2)}{\cos(b\pi/2)} \right\} {}_2J_{-\delta, \delta}(a) \quad [b > 0]$$

$$13. \int_0^{\pi} \left\{ \frac{\sin(a \sin x) \sin nx}{\cos(a \sin x) \cos nx} \right\} dx = [1 \mp (-1)^n] \frac{\pi}{2} J_n(a)$$

$$14. \int_0^{\pi} \left\{ \frac{\sin(a \sin x) \cos nx}{\cos(a \sin x) \sin nx} \right\} dx = [(-1)^{n+1} \mp 1] \frac{\pi}{2} E_n(a)$$

$$15. \int_0^{\pi} \left\{ \frac{\sin(a \cos x)}{\cos(a \cos x)} \right\} \cos nx dx = \pi \left\{ \frac{\sin(n\pi/2)}{\cos(n\pi/2)} \right\} J_n(a)$$

$$16. \int_0^{\pi} \left\{ \frac{\sin(vx - a \sin x)}{\cos(vx - a \sin x)} \right\} dx = \pi \left\{ \frac{E_v(a)}{J_v(a)} \right\}$$

$$17. \int_0^{\pi} \cos (n x-a \sin x) d x=\pi J_n(a) .$$

$$18. \int_{-\pi}^{\pi} \frac{\sin (a \cos x)}{\cos (a \cos x)} \sin n x d x=0 .$$

$$19. \int_{-\pi}^{\pi} \frac{\sin (a \cos x)}{\cos (a \cos x)} \cos n x d x=2 \pi\left\{\frac{\sin (n \pi / 2)}{\cos (n \pi / 2)}\right\} J_n(a) .$$

2.5.28. Интегралы, содержащие x^a и тригонометрические функции от тригонометрических функций.

$$1. \int_0^{\pi / 2} x\left\{\frac{\sin (a \operatorname{tg} x)}{\cos (a \operatorname{tg} x) \operatorname{tg} x}\right\} d x=\pm \frac{\pi}{4} e^{-a}\left[C+\ln (2 a)-\operatorname{Ei}(-2 a)\right] \quad[a>0] .$$

$$2. \int_0^{\pi / 2} x \cos (a \operatorname{tg} x) \frac{d x}{\sin 2 x}=-\frac{\pi}{4} \operatorname{Ei}(-a) \quad[a>0] .$$

$$3. \int_0^{\pi / 2} x \sin (a \operatorname{ctg} x) \frac{d x}{\sin ^2 x}=\frac{1-e^{-a}}{2 a} \pi \quad[a>0] .$$

$$4. \int_0^{\pi} x \cos (n \pi \cos x) \sin x d x=\delta_0 n \pi .$$

$$5. \int_0^{\pi} x \sin (n \pi \cos x) \sin x d x=\frac{1}{n}\left[(-1)^n-J_0(n \pi)\right] \quad[n=1,2,3, \ldots] .$$

$$6. \int_0^{\infty} \frac{1}{x} \sin (a \operatorname{tg} x)\left\{\frac{1}{\cos x}\right\} d x=\frac{\pi}{2}\left(1-e^{-a}\right) \quad[a>0] .$$

$$7. \int_0^{\infty} \frac{1}{x} \sin (a \operatorname{tg} x)\left\{\frac{\sin 2 x}{\cos ^2 x \operatorname{tg}(x / 2)}\right\} d x=\frac{\pi}{4}(1+a) e^{-a}\left\{\begin{matrix} 2 \\ 1 \end{matrix}\right\} \quad[a>0] .$$

$$8. \int_0^{\infty} \frac{1}{x} \sin (a \operatorname{tg} x) \operatorname{tg} x\left\{\frac{\operatorname{tg} x}{\operatorname{tg}(x / 2)}\right\} d x=\frac{\pi}{2} e^{-a} \quad[a>0] .$$

$$9. \int_0^{\infty} \frac{1}{x} \sin (a \operatorname{tg} x) \operatorname{ctg} x\left\{\frac{\sin x}{\operatorname{tg}(x / 2)}\right\} d x=\frac{\pi}{2}\left(1-e^{-a}\right) \quad[a>0] .$$

$$10. \int_0^{\infty} \frac{1}{x} \cos (a \operatorname{tg} x)\left\{\frac{\operatorname{tg} x}{\operatorname{tg}(x / 2)}\right\} d x=\frac{\pi}{2} e^{-a} \quad[a>0] .$$

$$11. \int_0^{\infty} \frac{1}{x} \cos (a \operatorname{tg} x)\left\{\frac{\sin ^2 x}{\sin ^2 x \operatorname{tg} x}\right\} d x=\frac{\pi}{16}(1-a) e^{-a}\left\{\begin{matrix} 4 \\ 1 \end{matrix}\right\} \quad[a>0] .$$

$$12. \int_0^{\infty} x \sin (a \operatorname{tg}^2 x) \frac{d x}{x^2+z^2}=\frac{\pi}{2}\left[\exp (-a \operatorname{th} z)-e^{-a}\right] \quad[a, \operatorname{Re} z>0] .$$

$$13. \int_0^{\infty} \cos (a \operatorname{tg}^2 x) \cos x \frac{d x}{x^2+z^2}=\frac{\pi}{2 z}\left[\operatorname{ch} z \exp (-a \operatorname{th} z)-e^{-a} \operatorname{sh} z\right] \quad[a, \operatorname{Re} z>0] .$$

$$14. \int_0^{\infty} x \frac{\cos (a \operatorname{tg}^2 x)}{\sin 2 x} \frac{d x}{x^2+z^2}=\frac{\pi}{2 \operatorname{sh} 2 z} \exp (-a \operatorname{th} z) \quad[a, \operatorname{Re} z>0] .$$

$$15. \int_0^{\infty} x \cos (a \operatorname{tg}^2 x) \operatorname{tg} x \frac{d x}{x^2+z^2}=\frac{\pi}{2 \operatorname{ch} z}\left[e^{-a} \operatorname{ch} z-\exp (-a \operatorname{th} z) \operatorname{sh} z\right] \quad[a, \operatorname{Re} z>0] .$$

$$16. \int_0^{\infty} x \cos (a \operatorname{tg}^2 x) \operatorname{ctg} 2 k x \frac{d x}{x^2+z^2}=\frac{\pi}{2}\left[\operatorname{cth} 2 k z \exp (-a \operatorname{th} z)-e^{-a}\right] \quad[k=1 \text { или } 2 ; a, \operatorname{Re} z>0] .$$

2.5.29. Интегралы, содержащие разности тригонометрических и алгебраических функций.

$$1. \int_0^{\infty} \frac{1-\cos b x}{x} d x=C+\ln (a b)-\operatorname{ci}(a b) \quad[a, b>0] .$$

$$2. \int_0^a\left(\frac{1}{x}-\operatorname{ctg} x\right) d x=\ln \frac{a}{\sin a} \quad[a>0] .$$

$$3. \int_0^{\infty} \frac{1-\cos ^n x}{x^2} d x=\frac{\pi n}{4}\left(\frac{2 n}{m}\right) \quad\left[m=\left[\frac{n+1}{2}\right] ; n=1,2,3, \ldots\right] .$$

$$4. \int_0^{\infty} x^{\alpha-1}(\cos ^{2 n} b x-1) d x=2^{-2 n} \sqrt{\pi} b^{-\alpha}(2 \cdot) \Gamma\left[\frac{\alpha / 2}{(1-\alpha) / 2}\right] \sum_{k=0}^{n-1} \frac{(n-k)^{-\alpha}}{k!(2 n-k)!} \quad[b>0 ; -2<\operatorname{Re} \alpha<0 ; n=1,2,3, \ldots] .$$

$$5. \int_0^{\infty} x^{\alpha-1}\left(\cos ^{2 n} b x-\frac{(2 n) !}{2^{2 n}(n) !^2}\right) d x=2^{-2 n} \sqrt{\pi} b^{-\alpha}(2 n) \Gamma\left[\frac{\alpha / 2}{(1-\alpha) / 2}\right] \sum_{k=0}^{n-1} \frac{(n-k)^{-\alpha}}{k!(2 n-k)!} \quad[b>0 ; 0<\operatorname{Re} \alpha<1 ; n=1,2,3, \ldots] .$$

$$6. \int_0^{\infty} \frac{x^n-\sin ^n x}{x^{n+2}} d x=J_n \quad[n=1,2,3, \ldots] .$$

$$J_n = \frac{\pi}{2^n (n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n-k}.$$

$$J_1 = \frac{\pi}{4}, \quad J_2 = \frac{\pi}{3}, \quad J_3 = \frac{13}{32}\pi, \quad J_4 = \frac{7}{15}\pi.$$

$$7. \int_0^{\infty} \left(\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)} \right) \frac{dx}{x} = \frac{C}{2} - \frac{3}{4}.$$

$$8. \int_0^{\infty} \left(\frac{\sin x}{x} - \frac{1}{1+x} \right) \frac{dx}{x} = 1 - C.$$

$$9. \int_0^{\infty} \left(\cos x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -C$$

{k=1, 2}.

$$10. \int_0^{\infty} \left(\frac{1}{x} - \operatorname{ctg} x \right) \frac{dx}{x} = \frac{\pi}{2}.$$

$$11. \int_0^1 \left(\cos ax \pm \cos \frac{a}{x} \right) \frac{dx}{1 \pm x^2} = \frac{\pi}{2} \left\{ \frac{\exp(-a)}{\sin a} \right\}$$

{a>0}.

$$12. \int_0^{\pi/4} \frac{\pi/4 - x \operatorname{tg} x}{\cos 2x} dx = \frac{\pi}{8} \ln 2 + \frac{G}{2}.$$

$$13. \int_0^{\pi/2} \frac{(1-x \operatorname{ctg} x)}{\sin^2 x} dx = \frac{\pi}{4}, \quad 14. \int_0^{\pi/2} \frac{4x^2 \cos x + (\pi-x)x}{\sin x} dx = \pi^2 \ln 2.$$

$$15. \int_0^a \frac{1 - \cos bx}{x} dx - \int_a^{\infty} \frac{\cos bx}{x} dx = C + \ln(ab)$$

{a, b>0}.

$$16. \int_0^{\infty} \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a}$$

{a, b>0}.

$$17. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} |b-a|$$

{a, b>0}.

$$18. \int_0^{\infty} \frac{\cos^n ax - \cos^n bx}{x} dx = \left[1 - \frac{1+(-1)^n}{2} \frac{(n-1)!}{n!} \right] \ln \frac{b}{a}$$

{a, b>0; n=1, 2, 3, ...}.

$$19. \int_0^{\infty} \frac{\sin^{2n} ax - \sin^{2n} bx}{x} dx = -\frac{(2n-1)!}{(2n)!} \ln \frac{b}{a}$$

{a, b>0; n=1, 2, 3, ...}.

$$20. \int_0^{\infty} \frac{a \sin bx - b \sin ax}{x^2} dx = ab \ln \frac{a}{b}$$

{a, b>0}.

$$21. \int_0^{\infty} \frac{\cos ax - \cos bx}{x(x+z)} dx = -\frac{1}{z} \left[\operatorname{ci}(az) \cos az + \operatorname{si}(az) \sin az - \operatorname{ci}(bz) \cos bz - \operatorname{si}(bz) \sin bz + \ln \frac{b}{a} \right]$$

{a, b>0; |arg z|<\pi}.

$$22. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2(x^2+z^2)} dx = \frac{\pi}{2z^2} [(b-a)z + e^{-bz} - e^{-az}]$$

{a, b, Re z>0}.

$$23. \int_0^{\infty} \frac{\cos ax + \sin ax}{x^{1/2}(x^2+z^2)} dx = \frac{\pi e^{-az}}{z \sqrt{2}} z^{\pm 1/2}$$

{Re z>0; a>0}.

$$24. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^m} dx = J_{m,n}^n$$

$$J_{1,2}^0 = \frac{\sqrt{2\pi}}{3}, \quad J_1^1 = \frac{\pi}{6}, \quad J_1^2 = \frac{1}{4}, \quad J_2^2 = \frac{\pi}{15},$$

$$J_2^3 = \frac{1}{4} \ln 3 + \frac{1}{3}, \quad J_3^3 = \frac{3\pi}{32}, \quad J_3^4 = \frac{9}{40} \ln 3 - \frac{1}{10},$$

$$J_3^5 = \frac{5}{192} \pi, \quad J_4^5 = \frac{23}{280} - \frac{27}{1120} \ln 3, \quad J_4^6 = \frac{47}{2560} \pi.$$

$$25. \int_0^{\infty} \frac{3-4 \sin^2 ax}{x} \sin^2 ax dx = \frac{\ln 2}{2}$$

{a>0}.

$$26. \int_0^{\infty} \left(\cos ax \pm \cos \frac{a}{x} \right) \frac{dx}{1 \pm x^2} = \pi \left\{ \frac{\exp(-a)}{\sin a} \right\}$$

{a>0}.

$$27. \int_0^{\pi/4} \frac{\sin x - x \cos x}{(\sin 2x)^{3/2}} dx = \frac{x}{2\sqrt{2}} (\sqrt{2}-1).$$

$$28. \int_0^{\pi/4} \left[\left(\frac{\pi}{2} - x \right) \operatorname{tg} x - \operatorname{ctg} x \right] \frac{dx}{\cos 2x} = -\frac{\pi}{4} \ln 2.$$

$$29. \int_0^{\infty} \frac{1}{x} \sum_{k=1}^n a_k \cos b_k x dx = -\sum_{k=1}^n a_k \ln b_k$$

{b_k>0; \sum_{k=1}^n a_k=0}.

$$30. \int_0^{\infty} \frac{\cos^n ax \cos(anx) - \cos^n bx \cos(bnx)}{x} dx = (1-2^{-n}) \ln \frac{b}{a}$$

{a, b>0; n=1, 2, 3, ...}.

$$31. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^{n+1}} \sin bx \, dx = I_n^n \quad (b > 0)$$

$$I_1^n = \frac{|1-b^2|}{4} \ln \frac{1+b}{|1-b|} + \frac{b}{2} \quad (b \neq 1), \quad I_1^n = \frac{1}{2} \quad (b=1);$$

$$I_1^n = \frac{\pi}{4} b \left(1 - \frac{b^2}{3}\right) \quad (b \leq 1), \quad I_1^n = \frac{\pi}{6} \quad (b \geq 1);$$

$$I_2^n = \frac{b}{4} \ln \frac{|4-b^2|}{b^2} + \frac{2-b^2}{b(4-b^2)} \quad (b \neq 2);$$

$$I_2^n = \frac{\pi}{8} (2-b^2) \quad (b < 2), \quad I_2^n = -\frac{\pi}{8} \quad (b=2), \quad I_2^n = 0 \quad (b > 2);$$

$$I_3^n = \frac{b(b^2-6)}{24} \ln \frac{b^2}{|4-b^2|} + \frac{1}{6} \ln \frac{b+2}{|b-2|} - \frac{b}{6} \quad (b \neq 2);$$

$$I_3^n = \frac{1}{3} (\ln 2 - 1) \quad (b=2); \quad I_3^n = \frac{\pi}{96} b (b^2 - 12b + 16) \quad (b \leq 2);$$

$$I_3^n = 0 \quad (b > 2);$$

$$I_4^n = \frac{b^3}{480} (20-b^2) \ln \frac{b^2}{|4-b^2|} + \frac{4-5b^2}{60} \ln \frac{2+b}{|2-b|} + b \frac{22+b^2}{120} \quad (b \neq 2);$$

$$I_4^n = \frac{13}{30} - \frac{8}{15} \ln 2 \quad (b=2); \quad I_4^n = \frac{\pi}{2880} (192b - 80b^3 + 30b^4 - b^6) \quad (b \leq 2);$$

$$I_4^n = \frac{\pi}{18} \quad (b > 2).$$

$$32. \int_0^{\infty} \frac{(\sin x - x \cos x)^n}{x^{n+1}} \cos bx \, dx = J_n^n \quad (b > 0)$$

$$J_1^n = \frac{b}{2} \ln \frac{|1-b|}{1+b} + 1 \quad (b \neq 1); \quad J_1^n = \frac{\pi}{4} (1-b^2) \quad (b \leq 1);$$

$$J_1^n = 0 \quad (b \geq 1); \quad J_2^n = -\frac{\pi}{4} b \quad (b < 2); \quad J_2^n = 0 \quad (b \geq 2);$$

$$J_2^n = \frac{b^2-2}{8} \ln \frac{b^2}{|4-b^2|} - \frac{1}{2} \quad (b \neq 2); \quad J_2^n = \frac{\pi}{24} (b^2 - 6b + 4) \quad (b \leq 2);$$

$$J_3^n = 0 \quad (b \geq 2); \quad J_3^n = \frac{b^3}{96} (12-b^2) \ln \frac{b^2}{|4-b^2|} - \frac{b}{6} \ln \frac{2+b}{|2-b|} + \frac{b^2+6}{24} \quad (b \neq 2);$$

$$J_3^n = \frac{5}{12} - \frac{2}{3} \ln 2 \quad (b=2).$$

2.5.30. Интегралы от $e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}^n$.

$$1. \int_0^{\pi/2} e^{-px} \sin^n x \, dx = p^{1-2\lambda} n! \prod_{k=0}^{[n/2]-1} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ 1 - p^{2\lambda} e^{\pi/2} \left[1 + \sum_{k=1}^{[n/2]} \frac{(p^2 + 4\lambda^2)(p^2 + 4(\lambda+1)^2) \dots (p^2 + 4(\lambda+k-1)^2)}{(2k+2\lambda)!} \right] \right\}$$

$$\left[\begin{matrix} \lambda=1/2, & n=1, 3, 5, \dots \\ \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad |\arg p| < \pi$$

$$2. \int_0^{\pi/2} e^{-px} \cos^n x \, dx = p^{1-2\lambda} n! \prod_{k=0}^{[n/2]-1} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ (-1)^{n-1} e^{-p\pi/2} + p^{2\lambda} \left[1 + \sum_{k=1}^{[n/2]} \frac{(p^2 + 4\lambda^2)(p^2 + 4(\lambda+1)^2) \dots (p^2 + 4(\lambda+k-1)^2)}{(2k+2\lambda)!} \right] \right\}$$

$$\left[\begin{matrix} \lambda=1/2, & n=1, 3, 5, \dots \\ \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad |\arg p| < \pi$$

$$3. \int_0^{\pi} e^{-px} \sin^2 x \, dx = \frac{\pi e^{-p\pi/2}}{2^v (v+1) B\left(\frac{v+ip}{2} + 1, \frac{v-ip}{2} + 1\right)} \quad [\operatorname{Re} v > -1]$$

$$4. \int_0^{\pi} e^{imx} \left\{ \frac{\sin nx}{\cos nx} \right\} dx = \pi \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} \delta_{m,n}$$

$$5. \int_0^{\pi} e^{-px} \sin^n x \, dx = n! p^{1-2\lambda} [1 - (-1)^{2\lambda} e^{-m\pi}] \prod_{k=0}^{[n/2]-1} [p^2 + (2k+2\lambda)^2]^{-1}$$

$$\left[\begin{matrix} \lambda=1/2, & n=1, 3, 5, \dots \\ \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad |\arg p| < \pi$$

$$6. \int_0^{\pi(m+1/2)} e^{-px} \cos^n x \, dx =$$

$$= n! e^{-p\pi/2} (-p)^{1-2\lambda} \{ \exp[-m\pi(p+2i\lambda)] - 1 \} \prod_{k=0}^{[n/2]-1} [p^2 + (2k+2\lambda)^2]^{-1}$$

$$\left[\begin{matrix} \lambda=1/2, & n=1, 3, 5, \dots \\ \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad |\arg p| < \pi$$

$$7. \int_{\pi/2}^{\infty} e^{-px} \left\{ \frac{\sin x}{\cos x} \right\}^n dx = An! e^{-p\pi/2} \prod_{k=0}^{[n/2]-1} [p^2 + (2k+2\lambda)^2]^{-1} \times$$

$$\times \left\{ 1 + \delta \left[\frac{p^2 + 4\lambda^2}{(2+2\lambda)!} + \frac{[p^2 + 4\lambda^2][p^2 + (2\lambda+2)^2]}{(4+2\lambda)!} + \dots \right. \right.$$

$$\left. \dots + \frac{[p^2 + 4\lambda^2][p^2 + (2\lambda+2)^2] \dots [p^2 + (n-2)^2]}{n!} \right\}$$

$$\left[\begin{matrix} A = -(-p)^{\delta}, & \lambda=1/2, & n=1, 3, 5, \dots \\ A = p, & \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad \operatorname{Re} p > 0; \quad \delta = \begin{cases} 1 \\ 0 \end{cases}$$

$$8. \int_0^{\infty} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{1}{b^2 + p^2} \left\{ \begin{matrix} b \\ p \end{matrix} \right\} \quad [\operatorname{Re} p > |b|]$$

$$9. \int_0^{\infty} e^{-px} \sin^2 bx \, dx = 2b^2 p^{-1} (p^2 + 4b^2)^{-2} \quad [\operatorname{Re} p > 2|b|]$$

$$10. \int_0^{\infty} e^{-px} \sin^n bx \, dx = b^n p^{1-2\lambda} n! \prod_{k=0}^{[n/2]-1} [p^2 + b^2 (2k+2\lambda)^2]^{-1}$$

$$\left[\begin{matrix} \lambda=1/2, & n=1, 3, 5, \dots \\ \lambda=0, & n=2, 4, 6, \dots \end{matrix} \right]; \quad \operatorname{Re} p > n|b|$$

$$11. \int_0^{\infty} e^{-px} \cos^n bx \, dx = 2^{1-n} \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \frac{p}{p^2 + (n-2k)^2 b^2} + \frac{1+(-1)^n}{2} \cdot \frac{1}{2^n p} \left(\frac{n+1}{2} \right) \quad [\operatorname{Re} p > n |\operatorname{Im} b|; n = 1, 2, 3, \dots]$$

2.5.31. Интегралы от $x^\alpha e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\pi/2} x^n e^{-px} \left\{ \frac{\sin x}{\cos x} \right\} dx = (-1)^n \frac{d^n}{dp^n} \left[\frac{p^{1-\delta} + (-p)^\delta e^{-p\pi/2}}{1+p^2} \right]$$

$$2. \int_0^{\pi} x^n e^{-px} \left\{ \frac{\sin x}{\cos x} \right\} dx = (-1)^n \frac{d^n}{dp^n} \left(p^{1-\delta} \frac{1+e^{-\pi p}}{1+p^2} \right)$$

$$3. \int_{\pi/2}^{\infty} x^n e^{-px} \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{e^{-p\pi/2}}{(p^2+1)^2} \left[\frac{\pi p^\delta}{2} (p^2+1) + \left\{ \frac{p^2-1}{2p} \right\} \right] \quad [\operatorname{Re} p > 0]$$

$$4. \int_0^{\infty} x^n e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{b^\delta}{p^{\alpha+\delta}} \Gamma(\alpha+\delta) {}_2F_1 \left(\frac{\alpha+1}{2}, \frac{\alpha}{2} + \delta; \delta + \frac{1}{2}; -\frac{b^2}{p^2} \right) = \frac{\Gamma(\alpha)}{(b^2+p^2)^{\alpha/2}} \left\{ \frac{\sin c}{\cos c} \right\} \quad [c = \alpha \arctg(b/p); \operatorname{Re} \alpha > -\delta; \operatorname{Re} p > |\operatorname{Im} b|]$$

$$5. \int_0^{\infty} x^n e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^n \frac{\partial^n}{\partial p^n} \frac{b^\delta p^{1-\delta}}{p^2+b^2} = n! \left(\frac{p}{p^2+b^2} \right)^{n+1} \sum_{k=0}^{[(n+1-\delta)/2]} (-1)^k \binom{n+1}{2k+\delta} \left(\frac{b}{p} \right)^{2k+\delta} \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$6. \int_0^{\infty} x^n e^{-px} \sin bx \, dx = I_n \quad [\operatorname{Re} p > |\operatorname{Im} b|; n = -1, 0, 1, 2, 3, 4]$$

$$I_{-1} = \arctg \frac{b}{p}, \quad I_0 = \frac{b}{b^2+p^2}, \quad I_1 = \frac{2pb}{(p^2+b^2)^2}, \quad I_2 = \frac{2b(3p^2-b^2)}{(b^2+p^2)^3}, \\ I_3 = \frac{24bp(p^2-b^2)}{(b^2+p^2)^4}, \quad I_4 = \frac{24b}{(b^2+p^2)^5} (5p^4-10p^2b^2+b^4).$$

$$7. \int_0^{\infty} x^n e^{-px} \cos bx \, dx = J_n \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$J_0 = \frac{p}{p^2+b^2}, \quad J_1 = \frac{p^2-b^2}{(b^2+p^2)^2}, \quad J_2 = \frac{2p(p^2-3b^2)}{(b^2+p^2)^3},$$

$$J_3 = \frac{6}{(b^2+p^2)^4} (p^4-6p^2b^2+b^4), \quad J_4 = \frac{24p}{(b^2+p^2)^5} (p^4-10p^2b^2+5b^4).$$

$$8. \int_0^{\infty} x^{-3/2} e^{-px} \sin bx \, dx = \sqrt{2\pi} (\sqrt{p^2+b^2}-p)^{1/2} \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$9. \int_0^{\infty} x^{n-1/2} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{\partial^n}{\partial p^n} \left(\frac{\sqrt{b^2+p^2}-p}{b^2+p^2} \right)^{1/2} \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$10. \int_0^{\infty} x^{2n-1} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}^n dx = 2^{1-n} \Gamma(\alpha) \sum_{k=0}^{[(n-1)/2]} (\mp 1)^{[n/2]-k} \binom{n}{k} [b^2(n-2k)^2+p^2]^{-\alpha/2} \left\{ \frac{\sin c}{\cos c} \right\} + \frac{1+(-1)^n}{2} 2^{-n} p^{-\alpha} \Gamma(\alpha) \left(\frac{n}{2} \right) \quad [c = \alpha \arctg(b(p-n-2k)/p); \operatorname{Re} \alpha > -\delta; \operatorname{Re} p > n |\operatorname{Im} b|]$$

$$11. \int_0^{\infty} \frac{e^{-px}}{x^m} \sin^n bx \, dx = J_m^n \quad [\operatorname{Re} p > n |\operatorname{Im} b|; m \leq n; m, n = 1, 2, 3, \dots]$$

$$J_1^{2n} = \frac{(-1)^{n+1}}{4^n} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \ln [p^2 + 4(n-k)^2 b^2] - 2^{-2n} \binom{2n}{n} \ln p,$$

$$J_1^{2n+1} = \frac{(-1)^n}{4^n} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \arctg \left[\frac{b}{p} (2n-2k+1) \right],$$

$$J_1 = \arctg \frac{b}{p}, \quad J_1^2 = \frac{1}{4} \ln \left(1 + \frac{4b^2}{p^2} \right),$$

$$J_1^3 = b \arctg \frac{2b}{p} - \frac{p}{4} \ln \left(1 + \frac{4b^2}{p^2} \right).$$

2.5.32. Интегралы от $A(x) e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_{-a}^a \sqrt{a^2-x^2} e^{-px} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = -\frac{i\delta a^2}{2} \{ I_0[a(p-ib)] \mp I_0[a(p+ib)] \} \quad [a, \operatorname{Re} p > 0; |\arg b| < \pi]$$

2. $\int_{-\infty}^{\infty} x \sqrt{a^2 - x^2} e^{-px} \frac{\sin bx}{\cos bx} dx = \frac{i\delta}{2} a^3 \{I_1[a(p - ib)] - I_1[a(p + ib)]\}$
 $[a, \operatorname{Re} p > 0; |\arg b| < \pi].$
3. $\int_0^{\infty} \frac{(x+z)^n}{x} e^{-px} \sin bx dx = (-1)^n b p^{-n-1} \sum_{k=0}^{\infty} (-1)^k \left(\frac{b}{2p}\right)^{2k} L_{n-2k-1}^{(-n-2k-1)}(p^2)$
 $[Re p > |\operatorname{Im} b|; |\arg z| < \pi].$
4. $\int_0^{\infty} \frac{x e^{-px}}{x^4 + z^4} \cos px dx = \frac{\pi}{4z^3} e^{-pz} \sqrt[4]{z^2}$
 $[p > 0; |\arg z| < \pi/4].$
5. $\int_0^{\infty} \frac{x e^{-px}}{x^4 - y^4} \cos px dx = -\frac{\pi}{4y^3} e^{-py} \sin py$
 $[p, y > 0].$
6. $\int_0^{\pi/2} e^{i p x} \sin^{\mu} x \cos^{\nu} x dx = 2^{-\mu-\nu-1} \times$
 $\times \left\{ e^{i(\mu+\nu)/2} B\left(\frac{p-\mu-\nu}{2}, \nu+1\right) {}_2F_1\left(-\mu, \frac{p-\mu-\nu}{2}; 1+\frac{p-\mu+\nu}{2}; -1\right) + \right.$
 $\left. + e^{i(\mu+1)/2} B\left(\frac{p-\mu-\nu}{2}, \mu+1\right) {}_2F_1\left(-\nu, \frac{p-\mu-\nu}{2}; 1+\frac{p+\mu-\nu}{2}; -1\right) \right\}$
 $[Re \mu, Re \nu > -1].$

2.5.33. Интегралы от $x^{\alpha} e^{px} \frac{\sin ax}{\cos ax} \frac{\sin bx}{\cos bx}$, $x^{\alpha} e^{px} \sin^{\mu} ax \cos^{\nu} bx$.

1. $\int_0^{\pi/2} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x dx = e^{i\pi/2} B(\mu, \nu) =$
 $= 2^{1-\mu-\nu} e^{i\pi/2} \left\{ \frac{1}{\mu} {}_2F_1(1-\nu, 1; 1+\mu; -1) + \frac{1}{\nu} {}_2F_1(1-\mu, 1; 1+\nu; -1) \right\}$
 $[Re \mu, Re \nu > 0].$
2. $\int_0^{\pi} e^{i p x} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{\pi i e^{i\mu(p-\nu)/2}}{2^{1+\nu-\mu} \Gamma(\mu) \Gamma(\nu)} \times$
 $\times {}_2F_1\left(1-\nu, \frac{p-\mu-\nu}{2}; 1+\frac{\mu-\nu+p}{2}; -1\right)$
 $[Re \mu, Re \nu > 0].$
3. $\int_0^{\pi} e^{-px} \sin^{\mu-1} x \cos x dx = \frac{\pi p^{2-\mu} e^{-p\pi/2}}{(\mu^2 + p^2) \Gamma(\frac{\mu+ip}{2}, \frac{\mu-ip}{2})}$
 $[Re \mu > 0].$
4. $\int_0^{\infty} e^{-px} \sin^{\alpha} ax \frac{\sin bx}{\cos bx} dx =$
 $= \frac{2^{-\alpha-1}}{\alpha(n+1)} \exp\left(\frac{1-\delta+n}{2} \pi i\right) \left[\left(\frac{b+ip+na}{2}\right)^{-1} \pm (-1)^n \left(\frac{b-ip+na}{2}\right)^{-1} \right]$
 $[a, b, Re p > 0; \delta = \{0\}].$

5. $\int_0^{\infty} e^{-px} \frac{\sin bx}{\sin cx} dx = \frac{1}{2c} \left[\Psi\left(\frac{c+b-ip}{2c}\right) - \Psi\left(\frac{c-b-ip}{2c}\right) \right]$
 $[b, Re p > 0; c \neq 0].$
6. $\int_0^{\infty} e^{-px} \frac{\sin nx}{\sin x} dx = 2p \sum_{k=1}^{[n/2]} \left[p^2 + \left(2k - \frac{1+(-1)^n}{2}\right)^2 \right]^{-1} + \frac{1-(-1)^n}{2} \frac{1}{p}$
 $[Re p > 0].$
7. $\int_0^{\infty} e^{-px} \cos(2n+1)x \operatorname{tg} x dx = \frac{2n+1}{p^2 + (2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k (2k+1)}{p^2 + (2k+1)^2}$
 $[Re p > 0].$
8. $\int_0^{\infty} \frac{1}{x} e^{-px} \sin ax \sin bx dx = \frac{1}{4} \ln \frac{p^2 + (a+b)^2}{p^2 + (a-b)^2}$
 $[Re p > |\operatorname{Im} a| + |\operatorname{Im} b|].$
9. $\int_0^{\infty} \frac{1}{x^2} e^{-px} \sin ax \sin bx dx =$
 $= \frac{a}{2} \operatorname{arctg} \frac{2pb}{p^2 + a^2 - b^2} + \frac{b}{2} \operatorname{arctg} \frac{2pa}{p^2 + b^2 - a^2} + \frac{p}{4} \ln \frac{p^2 + (a-b)^2}{p^2 + (a+b)^2}$
 $[Re p \geq |\operatorname{Im} a| + |\operatorname{Im} b|].$
10. $\int_0^{\infty} \frac{1}{x} e^{-px} \sin ax \cos bx dx = \frac{1}{2} \operatorname{arctg} \frac{2ap}{p^2 + b^2 - a^2} + \left\{ \begin{matrix} 0 \\ \pi/2 \end{matrix} \right\}$
 $[Re p > |\operatorname{Im} a| + |\operatorname{Im} b|; \frac{p^2 + b^2 - a^2}{p^2 + b^2 + a^2}].$

2.5.34. Интегралы от $x^{\alpha} R(e^{ax}) \frac{\sin bx}{\cos bx}$.

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

1. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} + 1} \frac{\sin bx}{\cos bx} dx = \Gamma(\alpha) \left\{ b^{-\alpha} \frac{\sin(\alpha\pi/2)}{\cos(\alpha\pi/2)} + \right.$
 $\left. + \frac{i\delta}{2} (2i)^{-\alpha} \left[\zeta\left(\alpha, \frac{a+ib}{2a}\right) + \zeta\left(\alpha, \frac{a-ib}{2a}\right) - \zeta\left(\alpha, \frac{ib}{2a}\right) - \zeta\left(\alpha, -\frac{ib}{2a}\right) \right] \right\}$
 $[Re \alpha > -\delta; Re a > |\operatorname{Im} b|].$
2. $\int_0^{\infty} \frac{\sin bx}{e^{ax} + 1} dx = \frac{1}{2b} - \frac{\pi}{2a \operatorname{sh}(\delta\pi/a)}$
 $[Re a > |\operatorname{Im} b|].$
3. $\int_0^{\infty} \frac{x^{\alpha-1}}{e^{ax} - 1} \frac{\sin bx}{\cos bx} dx = \frac{i\delta}{2} e^{-\alpha} \Gamma(\alpha) \left[\zeta\left(\alpha, 1+i\frac{b}{a}\right) - \zeta\left(\alpha, 1-i\frac{b}{a}\right) \right]$
 $[Re \alpha > 1-\delta; Re a > |\operatorname{Im} b|].$

$$4. \int_0^{\infty} \frac{\sin bx}{e^{ax}-1} dx = \frac{\pi}{2a} \operatorname{cth} \frac{b\pi}{a} - \frac{1}{2b} \quad [\operatorname{Re} a > |\operatorname{Im} b|]$$

$$5. \int_0^{\infty} \frac{x \cos bx}{e^{ax}-1} dx = \frac{1}{2b^2} - \frac{\pi^2}{2a^2 \operatorname{sh}^2(b\pi/a)} \quad [\operatorname{Re} a > |\operatorname{Im} b|]$$

$$6. \int_0^{\infty} \frac{\sin^2 bx}{e^{ax}-1} dx = \frac{1}{2a} \left[C + \operatorname{Re} \psi \left(2i \frac{b}{a} \right) \right] \quad [\operatorname{Re} a > 2|\operatorname{Im} b|]$$

$$7. \int_0^{\infty} \frac{e^{-ax}}{(e^{-ax}+1)^2} \cos bx dx = \frac{\pi b}{2a^2 \operatorname{sh}(b\pi/2)} \quad [\operatorname{Re} a > |\operatorname{Im} b|]$$

$$8. \int_0^{\infty} \frac{e^{-px} (1-e^{-ax})^{p-1} \left(\frac{\sin bx}{\cos bx} \right) dx}{\cos bx} = \frac{(-i)^p}{2a} \left[B\left(\beta, \frac{p-ib}{a}\right) \mp B\left(\beta, \frac{p+ib}{a}\right) \right] \\ [\operatorname{Re} a > 0; \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} \beta > -\delta]$$

$$9. \int_0^{\infty} \frac{\sin bx}{e^{ax}-e^{cx}} dx = \frac{1}{2i(a-c)} \left[\psi \left(\frac{a+ib}{a-c} \right) - \psi \left(\frac{a-ib}{a-c} \right) \right] \\ [\max(\operatorname{Re} a, \operatorname{Re} c) > |\operatorname{Im} b|]$$

$$10. \int_0^{\infty} \frac{x^m \left(\frac{\sin bx}{\cos bx} \right)}{e^{mcx}-e^{icx}} dx = \\ = (-1)^n \frac{\partial^{2n+1-\delta}}{\partial b^{2n+1-\delta}} \left[\frac{\pi}{4c} \left\{ \operatorname{th} \left[\frac{b\pi}{2c} \right] \right\} - \frac{1}{2b} \left\{ 0 \right\} - \sum_{k=1}^{[(m-1)/2]} \frac{b}{b^2+c^2(2k-2\lambda)^2} \right] \\ \left[b, c > 0; \begin{cases} \lambda=1/2, m=1, 3, 5, \dots \\ \lambda=0, m=2, 4, 6, \dots \end{cases} \right]$$

$$11. \int_{-\infty}^{\infty} \frac{e^{ix}}{(e^x+1)^{\delta}} \left(\frac{\sin bx}{\cos bx} \right) dx = I_{\delta}^{\pm} \quad [|\operatorname{Im} b|=0; 0 < \operatorname{Re} a+|\operatorname{Im} b| < \pi k]$$

$$I_{\delta}^{\pm} = \frac{1}{\Gamma(\delta)} \left\{ \operatorname{Im} z \right\}, \quad z = \Gamma(a+ib) \Gamma(p-a-ib),$$

$$I_1^{\pm} = \frac{\pi}{\sin^2 a\pi \operatorname{ch}^2 b\pi + \cos^2 a\pi \operatorname{sh}^2 b\pi} \left\{ \frac{\cos a\pi \operatorname{sh} b\pi}{\sin a\pi \operatorname{ch} b\pi} \right\},$$

$$I_4^{\pm} = -\frac{\pi}{\sin^2 a\pi \operatorname{ch}^2 b\pi + \cos^2 a\pi \operatorname{sh}^2 b\pi} \left[(1-\mu) \left\{ \frac{\cos a\pi \operatorname{sh} b\pi}{\sin(-a\pi) \operatorname{ch} b\pi} \right\} + \right. \\ \left. + b \left\{ \frac{\sin a\pi \operatorname{ch} b\pi}{\cos a\pi \operatorname{sh} b\pi} \right\} \right],$$

$$I_n^{\pm} = \frac{1}{n-1} \left[(n-1-a) I_{n-1}^{\pm} \mp b I_{n-1}^{\mp} \right] \quad [n=2, 3, 4, \dots]$$

2.5.35. Интегралы от $f(x, e^{ax}, \sin bx, \cos bx)$.

$$1. \int_0^{\infty} e^{-x} \operatorname{sgn}(\sin x) dx = -\operatorname{th} \frac{\pi}{2}$$

$$2. \int_0^{\infty} e^{-px} |\sin bx| dx = \frac{b}{p^2+b^2} \operatorname{cth} \frac{p\pi}{2b} \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$3. \int_0^{\infty} e^{-px} |\cos bx| dx = \frac{1}{p^2+b^2} \left[p + \frac{b}{\operatorname{sh} [p\pi/(2b)]} \right] \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$4. \int_0^{\infty} e^{-px} |\sin bx|^v dx = \frac{\pi}{(v+1) 2^{v+1} b} \left[\operatorname{sh} \frac{p\pi}{2b} B\left(1+\frac{v}{2}, \frac{ip}{2b}, 1+\frac{v}{2}, -\frac{ip}{2b}\right) \right]^{\frac{1}{2}} \\ [2 \operatorname{Re} p > \operatorname{Re} v, |\operatorname{Im} b|]$$

$$5. \int_0^{\infty} \frac{e^{-px}}{\operatorname{ch} a \pm \cos x} dx = \frac{p}{\operatorname{sh} a} \left[\frac{1}{p^2+1} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k e^{-ka}}{p^2+k^2} \right] \quad [a, \operatorname{Re} p > 0]$$

$$6. \int_0^{\pi/2} \frac{(e^{ix} \cos x)^n + (e^{-ix} \sin x)^n}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \frac{\pi}{ab} \left(\frac{b}{a+b} \right)^n \quad [a, b > 0]$$

$$7. \int_{-\pi/2}^{\pi/2} e^{ipx} \cos^{k-1} x (a^2 e^{ix} + b^2 e^{-ix})^v dx = \\ = \frac{\pi c^{2v} F(-v, (\mu+v \mp p-1)/2; (\mu-v \pm p+1)/2; (a^2/b^2)^{\pm 1})}{2^{k-1} \mu B((\pm p+\mu-v+1)/2, (v+\mu \mp p+1)/2)} \\ [c = \max(a, b); \operatorname{Re} \mu > 0; \left[\frac{a}{b} \leq \frac{b}{a} \right]]$$

$$8. \int_{-\pi}^{\pi} (a + \cos x)^v e^{ipx} dx = \frac{2\pi}{(v+1)_n} (a^2-1)^{v/2} p^n \left(\frac{a}{\sqrt{a^2-1}} \right) \quad [a > 1]$$

2.5.36. Интегралы, содержащие $e^{-ax^2-cx} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\infty} e^{-ax^2} \left(\frac{\sin bx}{\cos bx} \right) dx = \sqrt{\frac{\pi}{4a}} e^{-b^2/(4a)} \left\{ -i \operatorname{erf} \left[\frac{ib/(2\sqrt{a})}{1} \right] \right\} \quad [\operatorname{Re} a > 0]$$

$$2. \int_0^{\infty} e^{-ax^2} \left(\frac{\sin bx}{\cos bx} \right) dx = \frac{b\delta}{2} a^{-(\delta+1)/2} \Gamma\left(\frac{\alpha+\delta}{2}\right) {}_1F_1\left(\frac{\alpha+\delta}{2}; \delta+\frac{1}{2}; -\frac{b^2}{4a}\right) =$$

$$= 2^{-(\alpha+3)/2} \sqrt{\pi} \left\{ \sec(\alpha\pi/2) \right\} a^{-\alpha/2} \exp\left(-\frac{b^2}{8a}\right) \left[D_{\alpha-1}\left(-\frac{b}{\sqrt{2a}}\right) \mp D_{\alpha-1}\left(\frac{b}{\sqrt{2a}}\right) \right] \\ [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\delta]$$

$$3. \int_0^{\infty} x^{-1/2} e^{-ax^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{\pi}{2} \sqrt{\frac{b}{2a}} \exp\left(-\frac{b^2}{8a}\right) I_{\pm 1/4}\left(\frac{b^2}{8a}\right) \quad [\operatorname{Re} a > 0].$$

$$4. \int_0^{\infty} x^{1/2} e^{-ax^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \pm \frac{\pi}{4} \left(\frac{b}{2a}\right)^{3/2} \exp\left(-\frac{b^2}{8a}\right) \left[I_{\pm 1/4}\left(\frac{b^2}{8a}\right) - I_{\pm 3/4}\left(\frac{b^2}{8a}\right) \right] \quad [\operatorname{Re} a > 0].$$

$$5. \int_0^{\infty} x^{2n+1/2} e^{-ax^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = (-1)^n \sqrt{\frac{\pi}{a}} 2^{-2n-1} e^{-b^2/(4a)} H_{2n, \delta} \left(\frac{b}{2\sqrt{a}} \right) \quad \left[\operatorname{Re} a > 0; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$6. \int_0^{\infty} \frac{1}{x} e^{-ax^2} \sin bx \, dx = \frac{\pi}{2} \operatorname{erf} \left(\frac{b}{2\sqrt{a}} \right) \quad [\operatorname{Re} a > 0].$$

$$7. \int_0^{\infty} x e^{-ax^2} \cos bx \, dx = \frac{1}{2a} + \frac{ib}{2} \sqrt{\frac{\pi}{a^3}} \exp\left(-\frac{b^2}{4a}\right) \operatorname{erf} \left(\frac{ib}{2\sqrt{a}} \right) \quad [\operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} e^{-2ax^2 - cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{i\delta}{4} \sqrt{\frac{\pi}{a}} \left\{ \exp\left[\frac{(c+ib)^2}{4a}\right] \operatorname{erfc} \left(\frac{c+ib}{2\sqrt{a}} \right) \mp \exp\left[\frac{(c-ib)^2}{4a}\right] \operatorname{erfc} \left(\frac{c-ib}{2\sqrt{a}} \right) \right\} \quad [\operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} x^{\alpha-1} e^{-ax^2 - cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{i\delta \Gamma(\alpha)}{2(a^2)^{\alpha/2}} \exp\left(\frac{c^2 - b^2}{8a}\right) \left[\exp\left(\frac{ibc}{4a}\right) D_{-\alpha} \left(\frac{c+ib}{\sqrt{2a}} \right) \mp \exp\left(-\frac{ibc}{4a}\right) D_{-\alpha} \left(\frac{c-ib}{\sqrt{2a}} \right) \right] \quad [\operatorname{Re} a > 0; \operatorname{Re} \alpha > -\delta].$$

$$10. \int_0^{\infty} x e^{-ax^2 - cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = -\frac{i\delta}{8a} \sqrt{\frac{\pi}{a}} \left\{ (c+ib) \exp\left[\mp \frac{(c+ib)^2}{4a}\right] \operatorname{erfc} \left(\frac{c+ib}{2\sqrt{a}} \right) \mp \right. \\ \left. \mp (c-ib) \exp\left[\mp \frac{(c-ib)^2}{4a}\right] \operatorname{erfc} \left(\frac{c-ib}{2\sqrt{a}} \right) \right\} + \frac{1-\delta}{2a} \quad [\operatorname{Re} a > 0].$$

$$11. \int_0^{\infty} x^{(1 \pm 1)/2} \frac{e^{-ax^2}}{x^2 + z^2} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \frac{\pi}{4} z^{\delta-1} e^{az^2} \left[\pm e^{-bz} \operatorname{erf} \left(z \sqrt{a} - \frac{b}{2\sqrt{a}} \right) - e^{bz} \operatorname{erf} \left(z \sqrt{a} + \frac{b}{2\sqrt{a}} \right) + 2 \left\{ \frac{\sin bz}{\operatorname{ch} bz} \right\} \right] \quad [\operatorname{Re} a, \operatorname{Re} z > 0].$$

$$12. \int_0^{\infty} e^{-ax^2} \sin(2n+1)x \left\{ \frac{\sin x}{\cos x} \right\}^{-1} dx = \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^n (\pm 1)^k \exp\left(-\frac{k^2}{a}\right) \right] \quad [\operatorname{Re} a > 0].$$

$$13. \int_0^{\infty} x e^{-ax^2} \operatorname{tg} cx \, dx = \frac{c}{a} \sqrt{\frac{\pi}{a}} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{c^2 k^2}{a}\right) \quad [\operatorname{Re}(c^2/a) > 0].$$

$$14. \int_0^{\infty} \frac{e^{-ax^2} dx}{1 - 2c \cos x + c^2} = \frac{1}{|1 - c^2|} \sqrt{\frac{\pi}{a}} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \exp\left(-\frac{k^2}{a}\right) c^{\pm k} \right] \quad \left[\operatorname{Re} a > 0; \begin{cases} c < 1 \\ c > 1 \end{cases} \right].$$

2.5.37. Интегралы от $x^{\alpha-1} e^{-px-q/x} \left\{ \frac{\sin bx}{\cos bx} \right\}$.Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-q/x} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = i\delta \left(\frac{q}{b}\right)^{\alpha/2} \left[\exp\left(-\frac{\alpha\pi i}{4}\right) K_{\alpha}(2\sqrt{q/b}) \mp \exp\left(\frac{\alpha\pi i}{4}\right) K_{\alpha}(2\sqrt{q/b}) \right] \quad [\delta, \operatorname{Re} q > 0; \operatorname{Re} \alpha < 1].$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = i\delta q^{\alpha/2} \left[(p+ib)^{-\alpha/2} K_{\alpha}(2\sqrt{pq/p+ib}) \mp (p-ib)^{-\alpha/2} K_{\alpha}(2\sqrt{pq/p-ib}) \right] \quad [\operatorname{Re} p > \operatorname{Im} b; \operatorname{Re} q > 0].$$

$$3. \int_0^{\infty} x^{-3/2} e^{-px-q/x} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{\pi}{q}} e^{-2\sqrt{pq}} \left\{ \frac{\sin(2z_+ \sqrt{q})}{\cos(2z_+ \sqrt{q})} \right\} \quad \left[2z_{\pm} = \sqrt{p^2 + b^2} \pm p; \operatorname{Re} p > \operatorname{Im} b; \operatorname{Re} q > 0 \right].$$

$$4. \int_0^{\infty} x^{-1/2} e^{-px-q/x} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \\ = \sqrt{\frac{\pi}{p^2 + b^2}} e^{-2\sqrt{pq}} \left[\frac{\sin(2z_+ \sqrt{q})}{\cos(2z_+ \sqrt{q})} \right] \pm \frac{\sin(2z_- \sqrt{q})}{\sin(2z_- \sqrt{q})} \quad \left[2z_{\pm} = \sqrt{p^2 + b^2} \pm p; \operatorname{Re} p > \operatorname{Im} b; \operatorname{Re} q > 0 \right].$$

$$3. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \sin bx \, dx = J_{\alpha} \quad [b, \operatorname{Re} p, \operatorname{Re} z > 0; \operatorname{Re} \alpha > -1]$$

$$J_{2n} = (-1)^n z \frac{\partial^{2n-2}}{\partial b^{2n-2}} \left[\frac{b}{\sqrt{p^2+b^2}} K_1(z\sqrt{p^2+b^2}) \right].$$

$$4. \int_0^{\infty} x^{\alpha-1} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \cos bx \, dx = I_{\alpha} \quad [b, \operatorname{Re} \alpha, \operatorname{Re} p, \operatorname{Re} z > 0]$$

$$I_{2n+1} = (-1)^n \frac{\partial^{2n}}{\partial b^{2n}} [K_0(z\sqrt{p^2+b^2})].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left(\sqrt{x^2 + \frac{z^2}{2}} \right) (x^2+z^2)^{\alpha/2-1} \exp(-p\sqrt{x^2+z^2}) \left(\frac{\sin bx}{\cos bx} \right) dx = \\ = \left(\frac{2}{b} \right)^{\alpha-1/2} z^{\alpha+1/2} \Gamma \left[\frac{\alpha+\delta/2}{(\delta+1)/2-\alpha} \right] K_{\alpha+1/2}(bz) \quad [\operatorname{Re} \alpha > -\delta; b, \operatorname{Re} z > 0]$$

$$6. \int_0^{\infty} x^{1-\delta} \left(\sqrt{x^2+z^2} - z \right)^{\pm 1/2} \frac{\exp(-p\sqrt{x^2+z^2})}{\sqrt{x^2+z^2}} \left(\frac{\sin bx}{\cos bx} \right) dx = \\ = b^{\delta} \sqrt{\frac{\pi}{2}} \frac{(p^2+b^2+p)^{\mp 1/2}}{\sqrt{p^2+b^2}} \exp(-z\sqrt{p^2+b^2}) \quad [b, \operatorname{Re} p, \operatorname{Re} z > 0; \delta = \frac{1}{2}, 1]$$

2.5.40. Интегралы, содержащие показательную функцию от показательной и тригонометрических функций $\left(\frac{\sin bx}{\cos bx} \right)$.

$$1. \int_{-\infty}^{\infty} \exp(ax - e^x) \left(\frac{\sin bx}{\cos bx} \right) dx = |\Gamma(a+ib)| \left\{ \frac{\sin \varphi}{\cos \varphi} \right\} \\ [\varphi = \arg \Gamma(a+ib), \arg \Gamma(z) = -i \ln \frac{\Gamma'(z)}{\Gamma(z)}; \operatorname{Re} a > |\operatorname{Im} b|]$$

$$2. \int_{-\pi/2}^{\pi/2} x e^{z \sin x} \cos x \, dx = \frac{\pi}{2} [\operatorname{ch} z - I_0(z)] \quad [| \arg z | < \pi]$$

$$3. \int_0^{\pi} e^{z \cos x} \cos nx \, dx = \pi I_n(z) \quad [| \arg z | < \pi]$$

$$4. \int_0^{\pi} x e^{z \cos x} \sin x \, dx = \frac{\pi}{z} [I_0(z) - e^{-z}] \quad [| \arg z | < \pi]$$

$$5. \int_0^{\pi} e^{iz \cos x} \cos nx \, dx = i^n \pi J_n(z) \quad [| \arg z | < \pi]$$

$$6. \int_0^{\pi} x e^{iz \cos x} \sin^{v-1} x \, dx = \sqrt{\pi} \left(\frac{2}{z} \right)^{(v-1)/2} \Gamma \left(\frac{v}{2} \right) J_{(v-1)/2}(z) \quad [\operatorname{Re} v > 0; | \arg z | < \pi]$$

$$7. \int_0^{\pi} e^{\pm i(vx - z \sin x)} dx = \pi [J_v(z) \pm iE_v(z)] \quad [| \arg z | < \pi]$$

$$8. \int_0^{\pi} [e^{i[(v-1)x - z \sin x]} - e^{i[(v+1)x - z \sin x]}] dx = \\ = 2\pi [J_v^*(z) + iE_v^*(z)] \quad [| \arg z | < \pi]$$

$$9. \int_{-\pi}^{\pi} \frac{dx}{1 - \exp(-a - ib \cos x - ic \sin x)} = 2\pi \sum_{k=0}^{\infty} e^{-ka} J_0(k\sqrt{b^2+c^2}) \quad [a, b, c > 0]$$

$$10. \int_a^{a+2\pi} e^{i(nx - z \sin x)} dx = 2\pi J_n(z) \quad [| \arg z | < \pi; n = 0, \pm 1, \pm 2, \dots]$$

$$11. \int_0^{\infty} e^{-(p^2 - z \sin x)} dx = \frac{1}{p} + 2 \sum_{k=1}^{\infty} \frac{p}{p^2 + k^2} J_k \left(\frac{kz}{p} \right) \quad [\operatorname{Re} p > 0; | \arg z | < \pi]$$

$$12. \int_0^{\pi/2} \frac{\cos^{\mu-1} x}{\sin^{2\mu} x} e^{\pm i(1-\mu)x - y \cot x} dx = \\ = \pm \frac{i}{2} e^{\mp i\pi/2} \sqrt{\frac{\pi}{y}} \gamma^{1+\mu} \Gamma(\mu) H_{\mu-1/2}^{(1/2)} \left(\frac{y}{2} \right) \quad [\operatorname{Re} y, \operatorname{Re} \mu > 0]$$

$$13. \int_0^{\pi/2} \frac{\sin 2n x}{\sin^{2n+2} x} \frac{dx}{e^{k\pi \cot x} - 1} = \frac{(-1)^{n-1}}{2n+1} \left\{ \frac{n}{2^{2n-1}} \right\} \\ \left[k = \frac{1}{2}; n = 1, 2, 3, \dots \right]$$

$$14. \int_0^{\infty} e^{-t^2 x} \frac{\sin x \, dx}{x \cos^2 x} = \frac{\sqrt{\pi}}{2}$$

2.5.41. Интегралы, содержащие $A(x)$, $e^{f(x)}$, $\left(\frac{\sin(ax^2+bx^{\pm 1}+c)}{\cos(ax^2+bx^{\pm 1}+c)} \right)$.

Обозначение: $\delta = \frac{1}{2}$.

$$1. \int_0^{\infty} e^{-px} \left(\frac{\sin(bx+c)}{\cos(bx+c)} \right) dx = \frac{1}{p^2+b^2} \left\{ p \left(\frac{\sin c}{\cos c} \right) \pm \delta \left(\frac{\cos c}{\sin c} \right) \right\}$$

$$2. \int_0^{\infty} e^{-px} \left(\frac{\sin a\{x\}}{\cos a\{x\}} \right) dx = \frac{e^p - 1}{p(e^{2p} - 2e^p \cos a + 1)} \left\{ \frac{\sin a}{e^p - \cos a} \right\} \quad [a, \operatorname{Re} p > 0]$$

3. $\int_0^{\infty} e^{-px} b^x \left\{ \frac{\sin a [x]}{\cos a [x]} \right\} dx = \frac{e^p - 1}{p(e^{2p} - 2be^p \cos a + b^2)} \left\{ \frac{b \sin a}{e^p - b \cos a} \right\}$
 $[a > 0; b \geq 1; \operatorname{Re} p > \ln b]$
4. $\int_0^{\infty} e^{-px} \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx = \sqrt{\frac{\pi}{2b}} \left\{ \frac{\sin z}{\cos z} \right\} \left[\frac{1}{2} - S(z) \right] \pm \sqrt{\frac{\pi}{2b}} \left\{ \frac{\cos z}{\sin z} \right\} \left[\frac{1}{2} - C(z) \right]$
 $[z = p^2/(4b); b, \operatorname{Re} p > 0]$
5. $\int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx =$
 $= \frac{1}{2} \Gamma(\alpha) (2b)^{-\alpha/2} \left\{ \exp \left[-i \left(\frac{\alpha\pi}{4} + \frac{p^2}{8b} \right) \right] D_{-\alpha} \left(\frac{p}{\sqrt{2b}} e^{-\pi i/4} \right) + \right.$
 $\left. + \exp \left[i \left(\frac{\alpha\pi}{4} + \frac{p^2}{8b} \right) \right] D_{-\alpha} \left(\frac{p}{\sqrt{2b}} e^{\pi i/4} \right) \right\} \quad [b, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -2\theta]$
6. $\int_0^{\infty} x^{-1/2} e^{-px} \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx = \frac{\pi}{4} \sqrt{\frac{p}{b}} \left[J_{1/4} \left(\frac{p^2}{8b} \right) \left\{ \frac{\cos c}{\sin c} \right\} \pm Y_{1/4} \left(\frac{p^2}{8b} \right) \left\{ \frac{\sin c}{\cos c} \right\} \right]$
 $[c = p^2/(8b) + \pi/8; b, \operatorname{Re} p > 0]$
7. $\int_0^{\infty} x e^{-px} \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx = \frac{\delta}{2b} + \frac{p}{2} \sqrt{\frac{\pi}{2}} b^{-3/2} \left\{ \frac{\sin z}{\cos z} \right\} \left[\frac{1}{2} - C(z) \right] \mp$
 $\mp \left\{ \frac{\cos z}{\sin z} \right\} \left[\frac{1}{2} - S(z) \right] \quad [z = p^2/(4b); b, \operatorname{Re} p > 0]$
8. $\int_0^{\infty} \frac{e^{-px}}{x} \sin bx^2 dx = \frac{\pi}{2} \left[\frac{1}{2} - C(z) \right]^2 + \frac{\pi}{2} \left[\frac{1}{2} - S(z) \right]^2$
 $[z = p^2/(4b); b, \operatorname{Re} p > 0]$
9. $\int_0^{\infty} x^{\alpha-1} e^{-px} \left\{ \frac{\sin (bx^2 + px)}{\cos (bx^2 + px)} \right\} dx = \frac{\Gamma(\alpha)}{(2b)^{\alpha/2}} \exp \left(-\frac{2p^2}{b} \right) \left\{ \frac{\sin (\alpha\pi/2)}{\cos (\alpha\pi/2)} \right\} D_{-\alpha} \left(\frac{p}{\sqrt{b}} \right)$
 $[b > 0; \operatorname{Re} \alpha > -\delta; |\arg p| < \pi/4]$
10. $\int_0^{\infty} x e^{-px} \left\{ \frac{\sin bx^2 \sin px}{\cos bx^2 \cos px} \right\} dx = \frac{\pi}{4} \sqrt{\frac{\pi}{2b^3}} \exp \left(-\frac{p^2}{2b} \right) \quad [b > 0; |\arg p| < \pi/4]$
11. $\int_0^{\infty} x^2 e^{-px} [\sin (bx^2 \pm px) \pm \cos (bx^2 \pm px)] dx =$
 $= 2^{-3/2} b^{-5/2} \sqrt{\pi} (b - p^2) \exp \left(-\frac{p^2}{2b} \right) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad [b > 0; |\arg p| < \pi/4]$
12. $\int_0^{\infty} e^{-ax^2} \left\{ \frac{\sin cx^2}{\cos cx^2} \right\} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + c^2}} e^{-aA} \left[B_+ \left\{ \frac{\sin cA}{\cos cA} \right\} \mp \right.$
 $\left. \mp B_- \left\{ \frac{\cos cA}{\sin cA} \right\} \right] = \frac{\sqrt{\pi}}{2 \sqrt{a^2 + c^2}} e^{-aA} \left\{ \frac{\sin D}{\cos D} \right\}$
 $\left[A = \frac{b^2}{4(a^2 + c^2)}, B_{\pm} = \frac{1}{\sqrt{2}} (\sqrt{a^2 + c^2} \pm a)^{1/2}, 2D = \operatorname{arctg} (c/a) - Ac; b > 0; \operatorname{Re} a > |\operatorname{Im} c| \right]$

13. $\int_0^{\infty} x^{-1/2} e^{-px - q/x} \left\{ \frac{\sin (bx + c/x)}{\cos (bx + c/x)} \right\} dx = \frac{\sqrt{\pi}}{r} e^{-2rs \cos (A+B)} \left\{ \frac{\sin d}{\cos d} \right\}$
 $[d = A + 2rs \sin (A+B), r = \frac{1}{2} \sqrt{p^2 + b^2}, s = \frac{1}{2} \sqrt{q^2 + c^2},$
 $2A = \operatorname{arctg} (b/p), 2B = \operatorname{arctg} (c/q); \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} q > |\operatorname{Im} c|]$
14. $\int_0^{\infty} x^{-3/2} e^{-px - q/x} \left\{ \frac{\sin (bx + c/x)}{\cos (bx + c/x)} \right\} dx = \frac{\sqrt{\pi}}{s} e^{-2rs \cos (A+B)} \left\{ \frac{\sin f}{\cos f} \right\}$
 $[f = B + 2rs \sin (A+B), r, s, A, B \text{ см. 2.5.41.13}; \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} q > |\operatorname{Im} c|]$
15. $\int_{-\infty}^{\infty} x^{\alpha} e^{-(ax^2 + bx + c)} \left\{ \frac{\sin (qx + r)}{\cos (qx + r)} \right\} dx =$
 $= \left(-\frac{1}{2a} \right)^{\alpha} \sqrt{\frac{\pi}{a}} \exp \left(\frac{b^2 - 4ac - q^2}{4a} \right) \sum_{k=0}^{[n/2]} \frac{n! a^k}{(n-2k)! k!} \times$
 $\times \sum_{l=0}^{n-2k} \binom{n-2k}{l} \frac{b^{n-2k-l} q^l}{i^{n-2k-l}} \left\{ \frac{\sin h}{\cos h} \right\} \quad [h = r - bq/(2a) - \pi i/2; \operatorname{Re} a > 0]$
16. $\int_{-\infty}^{\infty} e^{-(ax^2 + bx + c)} \left\{ \frac{\sin (px^2 + qx + r)}{\cos (px^2 + qx + r)} \right\} dx =$
 $= \frac{\sqrt{\pi}}{\sqrt{a^2 + p^2}} \exp \left[\frac{a(b^2 - 4ac) - (aq^2 - 2bpq + 4cr^2)}{4(a^2 + p^2)} \right] \times$
 $\times \left\{ \frac{\sin \left[\frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - 4pr) - (b^2 p - 2abq + 4a^2 r)}{4(a^2 + p^2)} \right]}{\cos \left[\frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - 4pr) - (b^2 p - 2abq + 4a^2 r)}{4(a^2 + p^2)} \right]} \right\} \quad [\operatorname{Re} a > 0]$
17. $\int_0^{\infty} e^{-ax^2 + bx^4} \left[2x \sqrt{a} \left\{ \frac{\cos cx^3}{\sin cx^3} \right\} \mp \sqrt{b} \left\{ \frac{\sin cx^3}{\cos cx^3} \right\} \right] dx = \frac{\pi}{2} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$
 $[c = 2\sqrt{ab}; a, b > 0]$
- 2.5.42. Интегралы от $A(x) e^{-px} \left\{ \frac{\sin \sqrt{ax^2 + bx + c}}{\cos \sqrt{ax^2 + bx + c}} \right\}$.
- Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.
1. $\int_0^a \frac{e^{-px}}{\sqrt{x} \sqrt{a^2 - x^2}} \cos (c \sqrt{a^2 - x^2}) dx =$
 $= -\frac{\pi}{2} \sqrt{\pi p} [J_{1/4}(z_+) J_{1/4}(z_-) - J_{-1/4}(z_+) J_{-1/4}(z_-)]$
 $[2z_{\pm} = c \pm \sqrt{c^2 - p^2}; a, c > 0]$
2. $\int_0^a \frac{e^{-px}}{\sqrt{ax - x^2}} \cos (c \sqrt{ax - x^2}) dx = \pi e^{-ap/2} J_0 \left(\frac{a}{2} \sqrt{c^2 - p^2} \right) \quad [a, c > 0]$

3. $\int_a^\infty x^k e^{-px} \sin(c \sqrt{x^2 - a^2}) dx = \frac{a^{1+k} c p^k}{(\rho^2 + c^2)^{(1+k)/2}} K_{1+k}(a \sqrt{\rho^2 + c^2})$
 $[k = 0 \text{ или } 1; a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
4. $\int_a^\infty \frac{e^{-px}}{\sqrt{x+a}} \left\{ \frac{\sin(c \sqrt{x^2 - a^2})}{\cos(c \sqrt{x^2 - a^2})} \right\} dx =$
 $= \sqrt{\frac{\pi}{2}} \frac{c^\delta}{\sqrt{\rho^2 + c^2}} (\sqrt{\rho^2 + c^2} + \rho)^{1/2 - \delta} \exp(-a \sqrt{\rho^2 + c^2})$
 $[a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
5. $\int_a^\infty \frac{e^{-px}}{\sqrt{x-a}} \cos(c \sqrt{x^2 - a^2}) dx = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\rho + \sqrt{\rho^2 + c^2}}{\rho^2 + c^2}} \exp(-a \sqrt{\rho^2 + c^2})$
 $[a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
6. $\int_a^\infty (x-a)^{\alpha-1} (x+a)^{-\alpha-1/2} e^{-px} \left\{ \frac{\sin(c \sqrt{x^2 - a^2})}{\cos(c \sqrt{x^2 - a^2})} \right\} dx =$
 $= \frac{2^{1/2}}{a \sqrt{c}} \Gamma\left(\alpha + \frac{\delta}{2}\right) W_{1/4 - \alpha, 1/4}(z_+) M_{\alpha - 1/4, \pm 1/4}(z_-)$
 $[z_\pm = a(\sqrt{\rho^2 + c^2} \pm \rho); a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
7. $\int_a^\infty \frac{e^{-px}}{\sqrt{x-a}(x+a)} \left\{ \frac{\sin(c \sqrt{x^2 - a^2})}{\cos(c \sqrt{x^2 - a^2})} \right\} dx = \frac{(-i)^\delta \pi}{\sqrt{2a}} e^{ap} \operatorname{erfc}(z_+) \left\{ \frac{\operatorname{erf}(iz_-)}{1} \right\}$
 $[z_\pm = a(\sqrt{\rho^2 + c^2} \pm \rho); a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
8. $\int_a^\infty \frac{e^{-px}}{(x^2 - a^2)^{3/4}} \left\{ \frac{\sin(c \sqrt{x^2 - a^2})}{\cos(c \sqrt{x^2 - a^2})} \right\} dx = \sqrt{\frac{\pi c}{2}} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[z_\pm = \frac{a}{2}(\sqrt{\rho^2 + c^2} \pm \rho); a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
9. $\int_a^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2 - a^2}} \cos(c \sqrt{x^2 - a^2}) dx = \frac{1}{\sqrt{2\pi\rho}} K_{1/4}(z_+) K_{1/4}(z_-)$
 $[z_\pm = \frac{a}{2}(\sqrt{\rho^2 + c^2} \pm \rho); a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
10. $\int_a^\infty \frac{x^m}{\sqrt{x^2 - a^2}} e^{-px} \cos(c \sqrt{x^2 - a^2}) dx = \frac{a^m \rho^m}{(\rho^2 + c^2)^{m/2}} K_m(a \sqrt{\rho^2 + c^2})$
 $[m = 0 \text{ или } 1; a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$
11. $\int_a^\infty \frac{(x + \sqrt{x^2 - a^2})^v + (x - \sqrt{x^2 - a^2})^v}{\sqrt{x^2 - a^2}} e^{-px} \left\{ \frac{\sin(c \sqrt{x^2 - a^2})}{\cos(c \sqrt{x^2 - a^2})} \right\} dx =$
 $= 2a^v \frac{(\sin d)}{(\cos d)} K_v(a \sqrt{\rho^2 + c^2}) [d = \operatorname{arctg}(c/\rho); a > 0; \operatorname{Re} p > |\operatorname{Im} c|]$

12. $\int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x^2 + z^2}} \left\{ \frac{\sin(b \sqrt{x^2 + z^2})}{\cos(b \sqrt{x^2 + z^2})} \right\} dx =$
 $= \frac{\pi}{4} \sqrt{\frac{1}{\pi\rho}} \left[\frac{Y_{1/4}(z_+)}{J_{1/4}(z_+)} J_{1/4}(z_-) + \frac{Y_{1/4}(z_-)}{Y_{1/4}(z_+)} Y_{1/4}(z_-) \right]$
 $[2z_\pm = z(\sqrt{\rho^2 + b^2} \pm b); \operatorname{Re} z > 0; \operatorname{Re} p > |\operatorname{Im} b|]$
13. $\int_0^\infty e^{-px} \sin(b \sqrt{x^2 + xz}) dx = \frac{bz}{2 \sqrt{\rho^2 + b^2}} e^{\rho z/2} K_1\left(\frac{z}{2} \sqrt{\rho^2 + b^2}\right)$
 $[\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
14. $\int_0^\infty \frac{1}{\sqrt{x}} e^{-px} \cos(b \sqrt{x^2 + xz}) dx =$
 $= \sqrt{\frac{\pi}{2}} \sqrt{\frac{\rho + \sqrt{\rho^2 + b^2}}{\rho^2 + b^2}} \exp\left[-\frac{z}{2}(\sqrt{\rho^2 + b^2} - \rho)\right]$
 $[\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
15. $\int_0^\infty \frac{e^{-px}}{\sqrt{x+z}} \left\{ \frac{\sin(b \sqrt{x^2 + xz})}{\cos(b \sqrt{x^2 + xz})} \right\} dx =$
 $= \sqrt{\frac{\pi}{2}} b^\delta (\sqrt{\rho^2 + b^2} + \rho)^{1/2 - \delta} (\rho^2 + b^2)^{-1/2} \exp\left[-\frac{z}{2}(\sqrt{\rho^2 + b^2} - \rho)\right]$
 $[\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
16. $\int_0^\infty x^{\alpha-1} (x+z)^{-\alpha-1/2} e^{-px} \left\{ \frac{\sin(b \sqrt{x^2 + xz})}{\cos(b \sqrt{x^2 + xz})} \right\} dx =$
 $= \frac{2^{\delta+1/2}}{z \sqrt{b}} \Gamma\left(\alpha + \frac{\delta}{2}\right) e^{\rho z/2} W_{1/4 - \alpha, 1/4}(z_+) M_{\alpha - 1/4, \pm 1/4}(z_-)$
 $[z_\pm = \frac{z}{2}(\sqrt{\rho^2 + b^2} \pm \rho); 2 \operatorname{Re} \alpha > -\delta; \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
17. $\int_0^\infty \frac{e^{-px}}{\sqrt{x(x+z)}} \left\{ \frac{\sin(b \sqrt{x^2 + xz})}{\cos(b \sqrt{x^2 + xz})} \right\} dx = \frac{\pi}{\sqrt{z}} e^{\rho z} \operatorname{erfc}(z_+) \left\{ \frac{-i \operatorname{erf}(iz_-)}{1} \right\}$
 $[z_\pm = \frac{z}{2}(\sqrt{\rho^2 + b^2} \pm \rho); \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
18. $\int_0^\infty \frac{e^{-px}}{\sqrt{x} \sqrt{x+z}} \cos(b \sqrt{x^2 + xz}) dx = e^{\rho z/2} K_0\left(\frac{z}{2} \sqrt{\rho^2 + b^2}\right)$
 $[\operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$
19. $\int_0^\infty (x^2 + xz)^{-3/4} e^{-px} \left\{ \frac{\sin(b \sqrt{x^2 + xz})}{\cos(b \sqrt{x^2 + xz})} \right\} dx = \sqrt{\frac{\pi b}{2}} e^{\rho z/2} K_{1/4}(z_+) I_{\pm 1/4}(z_-)$
 $[z_\pm = \frac{z}{4}(\sqrt{\rho^2 + b^2} \pm \rho); \operatorname{Re} p > |\operatorname{Im} b|; |\arg z| < \pi]$

$$20. \int_0^{\infty} \frac{1}{\sqrt{x^2+xz}} \left[\left(\frac{z}{2} + x + \sqrt{x^2+xz} \right)^{\alpha} \left(\frac{z}{2} + x - \sqrt{x^2+xz} \right)^{\beta} \right] \times \\ \times e^{-\rho x} \frac{\left(\sin(b\sqrt{x^2+xz}) \right)}{\cos(b\sqrt{x^2+xz})} dx = 2 \left(\frac{z}{2} \right)^{\alpha} \frac{\left(\sin d \right)}{\cos d} e^{\rho z/2} K_{\nu} \left(\frac{z}{2} \sqrt{p^2+b^2} \right) \\ [d = \nu \arctg(b/p); \operatorname{Re} \rho > |\operatorname{Im} b|; |\arg z| < \pi].$$

2.5.43. Интегралы от $A(x) e^{f(x)} \frac{\left(\sin [cx/(x^2 \pm a^2)] \right)}{\cos [cx/(x^2 \pm a^2)]}$.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^a \frac{x^{\alpha-1}}{\sqrt{a^2-x^2}} \exp \left(-b \frac{a^2+x^2}{a^2-x^2} \right) \frac{\left(\sin [cx/(a^2-x^2)] \right)}{\cos [cx/(a^2-x^2)]} dx = \\ = 2^{\delta} + (2\alpha-3)/4 \sqrt{\frac{a}{c}} \Gamma \left(\frac{\alpha+\delta}{2} \right) (V \sqrt{4a^2b^2+c^2} + 2ab)^{1/4} \times \\ \times D_{\alpha-1} \left(\sqrt{\frac{V \sqrt{4a^2b^2+c^2} + 2ab}{a}} \right) M_{\alpha/2-1/4, \pm 1/4} \left(\frac{V \sqrt{4a^2b^2+c^2} - 2ab}{2a} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha > -\delta].$$

$$2. \int_a^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp \left(-\frac{b}{x^2-a^2} \right) \frac{\left(\sin [cx/(x^2-a^2)] \right)}{\cos [cx/(x^2-a^2)]} dx = \\ = \frac{a^{\alpha-1}}{\sqrt{c} 2^{\alpha/2+1/4-\delta}} \exp \left(\frac{b}{2a^2} \right) \Gamma \left(\frac{1+\delta-\alpha}{2} \right) (V \sqrt{b^2+a^2c^2} + b)^{1/4} \times \\ \times D_{\alpha-1} \left(\sqrt{\frac{V \sqrt{b^2+a^2c^2} + b}{a}} \right) M_{1/4-\alpha/2, \pm 1/4} \left(\frac{V \sqrt{b^2+a^2c^2} - b}{2a^2} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha < 1+\delta].$$

$$3. \int_a^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2-a^2}} \exp \left(-b \frac{x^2+a^2}{x^2-a^2} \right) \frac{\left(\sin [cx/(x^2-a^2)] \right)}{\cos [cx/(x^2-a^2)]} dx = \\ = 2^{\delta} - (2\alpha+1)/4 a^{\alpha} - 3/4 c^{-1/2} \Gamma \left(\frac{1+\delta-\alpha}{2} \right) (V \sqrt{4a^2b^2+c^2} + 2ab)^{1/4} \times \\ \times D_{\alpha-1} \left(\sqrt{\frac{V \sqrt{4a^2b^2+c^2} + 2ab}{a}} \right) M_{1/4-\alpha/2, \pm 1/4} \left(\frac{V \sqrt{4a^2b^2+c^2} - 2ab}{2a} \right) \\ [a, b, c > 0; \operatorname{Re} \alpha < 1+\delta].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp \left(-\frac{b}{x^2+z^2} \right) \frac{\left(\sin [cx/(x^2+z^2)] \right)}{\cos [cx/(x^2+z^2)]} dx = \\ = \frac{2^{\delta}}{\sqrt{2\pi c}} \Gamma \left(\frac{1-\alpha+\delta}{2} \right) \Gamma \left(\frac{\delta+\alpha}{2} \right) e^{\alpha-1/2} \exp \left(-\frac{b}{2z^2} \right) \times \\ \times M_{1/4-\alpha/2, \pm 1/4} \left(\frac{b+V \sqrt{b^2-c^2z^2}}{2z^2} \right) M_{1/4-\alpha/2, \pm 1/4} \left(\frac{b-V \sqrt{b^2-c^2z^2}}{2z^2} \right) \\ [b, c, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1+\delta].$$

$$5. \int_0^{\infty} \frac{x^{\alpha-1}}{\sqrt{x^2+z^2}} \exp \left(-b \frac{z^2-x^2}{z^2+x^2} \right) \frac{\left(\sin [cx/(x^2+z^2)] \right)}{\cos [cx/(x^2+z^2)]} dx = \\ = \frac{2^{1/2} \pm 1/2 \alpha - 1/2}{\sqrt{\pi c}} \Gamma \left(\frac{\alpha+\delta}{2} \right) \Gamma \left(\frac{1-\alpha+\delta}{2} \right) \times \\ \times M_{1/4-\alpha/2, \pm 1/4} \left(\frac{2zb+V \sqrt{4b^2z^2-c^2}}{2z} \right) M_{1/4-\alpha/2, \pm 1/4} \left(\frac{2zb-V \sqrt{4b^2z^2-c^2}}{2z} \right) \\ [b, c, \operatorname{Re} z > 0; -\delta < \operatorname{Re} \alpha < 1+\delta].$$

2.5.44. Интегралы, содержащие показательную и тригонометрические функции от тригонометрических функций.

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/2} e^{z \cos x} \frac{\left(\sin (z \sin x) \right)}{\cos (z \sin x)} dx = \left\{ \begin{array}{l} -E_1(z) + C_1(-z) \\ -S_1(-z) \end{array} \right\} \quad [|\arg(-z)| < \pi].$$

$$2. \int_0^{\pi/2} e^{z \cos x} \frac{\left(\sin (z \sin x) \right)}{\cos (z \sin x)} dx = \left\{ \begin{array}{l} 2 \operatorname{sh} (z) \\ \pi \end{array} \right\} \quad [|\arg z| < \pi].$$

$$3. \int_0^{\pi} e^{z \cos x} \frac{\left(\sin (z \sin x) \sin nx \right)}{\cos (z \sin x) \cos nx} dx = \frac{\pi 2^n}{2n!} \quad [|\arg z| < \pi; n=1, 2, 3, \dots].$$

$$4. \int_0^{\pi} e^{z \cos x} \cos (z \sin x + bx) dx = z^{-b} \sin (bz) \gamma(b, z) \quad [|\arg z|, |\arg b| < \pi].$$

$$5. \int_0^{\pi} e^{z \cos x} \cos (z \sin x + x/2) dx = \sqrt{\frac{\pi}{z}} \operatorname{erf}(\sqrt{z}) \quad [|\arg z| < \pi].$$

$$6. \int_0^{\pi} e^{z(\cos bx + \cos cx)} \frac{\left(\sin (z \sin bx) \sin (z \sin cx) \right)}{\cos (z \sin bx) \cos (z \sin cx)} dx = \\ = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{z^{(b+c)k}}{\Gamma(1+bk) \Gamma(1+ck)} + \{0\}.$$

$$7. \int_0^{\pi} e^{z \cos x} \frac{\sin (z \sin x)}{\sin x} dx = \pi \operatorname{sh} z \quad [|\arg z| < \pi].$$

$$8. \int_0^{\pi} e^{z \cos x} \cos (z \sin x) \frac{\sin 2nx}{\sin x} dx = \pi \sum_{k=0}^{n-1} \frac{z^{2k+1}}{(2k+1)!} \quad [|\arg z| < \pi].$$

$$9. \int_0^{\pi} e^{z \cos x} \sin (z \sin x) \operatorname{tg} \frac{x}{2} dx = \pm \pi (1 - e^z) \quad [|\arg z| < \pi].$$

$$10. \int_0^{\pi} e^{z \cos x} \frac{\cos(z \sin x)}{\cos x + c} dx = \frac{\pi}{\sqrt{c^2 - 1}} e^{z(\sqrt{c^2 - 1} - c)} \quad [|\arg z| < \pi; |c| > 1]$$

$$11. \int_0^{\pi} e^{z \cos x} \sin(z \sin x) \frac{\sin bx}{1 - 2a \cos bx + a^2} dx = \frac{\pi}{2a} \sum_{k=1}^{\infty} \frac{a^k z^{kb}}{\Gamma(kb + 1)} \quad [|\arg z| < \pi; |a| < 1]$$

$$12. \int_0^{\pi} e^{z \cos x} \cos(z \sin x) \frac{1 - a \cos bx}{1 - 2a \cos bx + a^2} dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{a^k z^{kb}}{\Gamma(kb + 1) + \pi} \quad [|\arg z| < \pi; |a| < 1]$$

$$13. \int_0^{2\pi} e^{a \cos x + b \sin x} \left\{ \frac{\sin(p \cos x + q \sin x + nx)}{\cos(p \cos x + q \sin x + nx)} \right\} dx =$$

$$= (-i)^0 \pi [(n-b)^2 + (q+a)^2]^{-n/2} [(A+iB)^{n/2} I_n(1) \overline{C-iD} + (A-iB)^{n/2} I_n(1) \overline{C+iD}]$$

$$[(n-b)^2 + (q+a)^2 > 0; A=a^2-b^2+p^2-q^2, B=2(ab+pq), C=a^2+b^2-p^2-q^2, D=2(ap-bq)]$$

$$14. \int_0^{2\pi} e^{a \cos x + b \sin x} \left\{ \frac{\sin(b \cos x - a \sin x + nx)}{\cos(b \cos x - a \sin x + nx)} \right\} dx = \frac{2\pi}{n!} (\mu^2 + b^2)^{n/2} \frac{(\sin \varphi)}{(\cos \varphi)} \quad [q = n \arctg(b/a)]$$

$$15. \int_0^{\pi} \frac{1}{x} e^{a \cos x} \sin(a \sin x) \cos nx dx = \frac{\pi}{2} \left[e^a - \sum_{k=0}^{n-1} \frac{a^k}{k!} - \frac{a^n}{2n!} \right] \quad [a > 0]$$

$$16. \int_0^{\pi} \frac{1}{x} e^{a \cos x} \cos(a \sin x) \sin nx dx = \frac{\pi}{2} \left[\sum_{k=0}^{n-1} \frac{a^k}{k!} + \frac{a^n}{2n!} \right] \quad [a > 0; n=1, 2, 3, \dots]$$

$$17. \int_0^{\pi} \frac{1}{x} e^{a \cos x} \sin(a \sin x + nx) dx = \frac{\pi}{2} e^a \quad [a > 0]$$

$$18. \int_0^{\pi} \frac{x^{\delta} e^{a \cos x}}{x^2 - y^2} \left\{ \frac{\sin(a \sin x)}{\cos(a \sin x)} \right\} dx = \pm \frac{\pi}{2} y^{\delta-1} a \cos ac \left\{ \frac{\cos(a \sin ac)}{\sin(a \sin ac)} \right\} - \frac{\pi \delta}{2}$$

$$[a, c, y > 0; \delta = \{1\}]$$

$$19. \int_0^{\pi} \frac{x^{\delta} e^{a \cos x}}{x^2 + z^2} \left\{ \frac{\sin(a \sin x + bx)}{\cos(a \sin x + bx)} \right\} dx = \frac{\pi}{2} z^{\delta-1} \exp(-bx + ac) \left\{ \frac{\sin(a \sin cx)}{\cos(a \sin cx)} \right\}$$

$$[a, b, c, \operatorname{Re} z > 0; \delta = \{1\}]$$

$$20. \int_0^{\pi} \frac{x^{\delta} e^{a \cos x}}{(x^2 + z^2) \sin cx} \left\{ \frac{\sin(a \sin cx + cx)}{\cos(a \sin cx + cx)} \right\} dx = \frac{\pi z^{-\delta}}{2 \sinh cz} [e^a - \exp(ae^{-cz} - cz)]$$

$$[a, c, \operatorname{Re} z > 0; \delta = \{1\}]$$

$$21. \int_0^{\infty} \frac{e^{a \cos cx}}{x^2 + z^2} \frac{\sin(a \sin cx)}{\sin cx} dx = \frac{\pi}{2z \sinh cz} [e^a - \exp(ae^{-cz})] \quad [a, c, \operatorname{Re} z > 0]$$

$$22. \int_0^{\infty} \frac{e^{a \cos cy}}{x^2 + y^2} \frac{\sin(a \sin cx)}{\sin cx} dx = \frac{\pi}{2y \sin cy} [-e^a + e^{a \cos cy} \cos(a \sin cy)]$$

$$[a, c, y > 0]$$

$$23. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{1 - e^{a \cos cx} \cos(a \sin cx)}{\sin cx} dx = \frac{\pi}{2 \sinh cz} [e^a - \exp(ae^{-cz})]$$

$$[a, c, \operatorname{Re} z > 0]$$

$$24. \int_0^{\infty} \frac{x}{x^2 + y^2} \frac{1 - e^{a \cos cy} \cos(a \sin cy)}{\sin cy} dx = \frac{\pi}{2 \sin cy} e^{a \cos cy} \sin(a \sin cy)$$

$$[a, c, y > 0]$$

$$25. \int_0^{\infty} \frac{e^{a \cos cx}}{x^2 + z^2} \sin(a \sin cx) \operatorname{th}^{\pm 1} cx dx = \pm \frac{\pi}{2z} \operatorname{th}^{\pm 1} cz [\exp(ae^{-cz}) - e^a]$$

$$[a, c, \operatorname{Re} z > 0]$$

$$26. \int_0^{\pi/2} e^{-\rho \operatorname{th} x} \sin^{\nu-1} x \cos^{\mu-1} x \left\{ \frac{\sin[p/2 - (\mu-1)x]}{\cos[p/2 - (\mu-1)x]} \right\} dx =$$

$$= \frac{\sqrt{\pi}}{2} \rho^{1/2-\mu} \Gamma(\mu) \left\{ Y_{\mu-1/2}^{(p/2)} \right\} \quad [\nu=1/2-\mu \text{ или } \nu=1-2\mu; \rho, \operatorname{Re} \mu > 0]$$

$$27. \int_0^{\pi/2} e^{-\rho \operatorname{th} x} \cos^{\nu-1} x \cos[2\rho \operatorname{th} x - (\nu+1)x] dx = -\frac{1}{2} \Gamma\left(-\frac{\nu}{2}\right) \rho^{\nu/2} e^{\rho} \sin \frac{\nu\pi}{2}$$

$$[\operatorname{Re} \nu < 0; \rho > 0]$$

2.5.45. Интегралы, содержащие разности функций $A(x)$, e^{px} , $\left\{ \frac{\sin bx}{\cos bx} \right\}$.

$$1. \int_0^{\pi} \frac{1 - e^{-ax}}{x^n} \sin bx dx = I_n \quad [\operatorname{Re} a, b > 0]$$

$$I_1 = \arctg \frac{a}{b}, \quad I_2 = a \arctg \frac{\theta}{a} + \frac{b}{2} \ln \left(1 + \frac{a^2}{b^2} \right).$$

$$2. \int_0^{\infty} \frac{1 - e^{-ax}}{x} \cos bx dx = \frac{1}{2} \ln \left(1 + \frac{a^2}{b^2} \right) \quad [\operatorname{Re} a, b > 0]$$

$$3. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx dx = \ln b - \frac{1}{2} [\psi(ib) + \psi(-ib)] \quad [b > 0]$$

$$4. \int_0^{\infty} e^{-px} \frac{1 - \cos ax}{x} dx = \frac{1}{2} \ln \left(1 + \frac{a^2}{p^2} \right) \quad [\operatorname{Re} p > |\operatorname{Im} a|].$$

$$5. \int_0^{\infty} e^{-px} \frac{1 - \cos ax}{x^2} dx = a \operatorname{arctg} \frac{a}{p} - \frac{p}{2} \ln \left(1 + \frac{a^2}{p^2} \right) \quad [\operatorname{Re} p > |\operatorname{Im} a|].$$

$$6. \int_0^{\infty} e^{-px} \frac{x - \sin x}{x^3} dx = I_3 \quad [\operatorname{Re} p > 0].$$

$$I_1 = p^{-1} - \operatorname{arctg} p, \quad I_2 = 2^{-1} \ln(1 + p^{-2}) + p \operatorname{arctg} p - 1, \\ I_3 = 2^{-1} [p \ln(1 + p^{-2}) + (p^2 + 1) \operatorname{arctg} p - p].$$

$$7. \int_0^{\infty} \frac{1 - \cos ax}{x(e^{2\pi x} - 1)} dx = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a} \quad [a > 0].$$

$$8. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C.$$

$$9. \int_0^{\infty} \frac{\cos ax - e^{-ax}}{x(x^4 + z^4)} dx = \frac{\pi}{2z^4} \exp \left(-\frac{az}{\sqrt{2}} \right) \sin \left(\frac{az}{\sqrt{2}} \right) \quad [a > 0; |\arg z| < \pi/4].$$

$$10. \int_0^{\infty} \frac{e^{-ax} - e^{-cx} \cos bx}{x} dx = \frac{1}{2} \ln \frac{c^2 + b^2}{a^2} \quad [\operatorname{Re} a > 0; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$11. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x} \cos bx dx = \frac{1}{2} \ln \frac{b^2 + c^2}{b^2 + a^2} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Im} b|].$$

$$12. \int_0^{\infty} \frac{e^{-ax} - e^{-cx}}{x^2} \sin bx dx = \frac{b}{2} \ln \frac{b^2 + c^2}{b^2 + a^2} + c \operatorname{arctg} \frac{b}{c} - a \operatorname{arctg} \frac{b}{a} \quad [\operatorname{Re} a, \operatorname{Re} c > |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{1}{x} (ae^{-bx} - e^{-cx} \frac{\sin ax}{x}) dx = \frac{a}{2} \ln \frac{a^2 + c^2}{b^2} + c \operatorname{arctg} \frac{a}{c} - a \operatorname{arctg} \frac{a}{b} \quad [\operatorname{Re} b > 0; \operatorname{Re} c > |\operatorname{Im} a|].$$

$$14. \int_0^{\infty} \frac{1 - \cos ax}{x [e^{2\pi cx} - e^{2\pi(n-1)c}]} dx = \frac{1}{2} \ln \frac{2c \operatorname{sh} [a\pi/(2c)]}{a\pi} - \frac{1}{2} \sum_{k=1}^{n-1} \ln \left(1 + \frac{a^2}{4k^2 c^2} \right) \quad [a, c > 0; n=1, 2, 3, \dots].$$

$$15. \int_0^{\infty} e^{-px} \frac{\cos ax - \cos bx}{x^n} dx = I_n \quad [\operatorname{Re} p > |\operatorname{Im} a|, |\operatorname{Im} b|]. \\ I_1 = \frac{1}{2} \ln \frac{p^2 + b^2}{p^2 + a^2}, \quad I_2 = \frac{p}{2} \ln \frac{p^2 + a^2}{p^2 + b^2} + b \operatorname{arctg} \frac{b}{p} - a \operatorname{arctg} \frac{a}{p}.$$

$$16. \int_0^{\infty} e^{-px} \frac{\sin x - x \cos x}{x^3} dx = J_3 \quad [\operatorname{Re} p > 0],$$

$$J_1 = \operatorname{arctg} p - \frac{p}{p^2 + 1}, \quad J_2 = 1 - p \operatorname{arctg} p, \quad J_3 = \frac{1}{2} [(p^2 + 1) \operatorname{arctg} p - p].$$

$$17. \int_0^{\infty} \frac{ae^{-bx} \sin cx - ce^{-hx} \sin ax}{x^2} dx = \frac{ac}{2} \ln \frac{a^2 + h^2}{c^2 + b^2} + c \operatorname{arctg} \frac{h}{a} - ab \operatorname{arctg} \frac{b}{c} \quad [\operatorname{Re} b > |\operatorname{Im} c|; \operatorname{Re} h > |\operatorname{Im} a|].$$

$$18. \int_0^{\infty} \frac{\cos ax - \cos bx}{x [e^{m\pi x} - e^{(m-2)\pi x}]} dx = \\ = \frac{1}{2} \left[\left\{ \ln \operatorname{ch} [b\pi/(2c)] - \ln \operatorname{ch} [a\pi/(2c)] \right\} - \sum_{k=1}^{(m-1)/2} \ln \frac{b^2 + c^2 (2k - 2\lambda)^2}{a^2 + c^2 (2k - 2\lambda)^2} + \right. \\ \left. + \left\{ \ln (a/b) \right\} \right] \quad [a, b, \operatorname{Re} c > 0; \left\{ \begin{smallmatrix} \lambda=1/2, m=1, 3, 5, \dots \\ \lambda=0, m=2, 4, 6, \dots \end{smallmatrix} \right\}].$$

$$19. \int_0^{\infty} e^{-px} \frac{2x \cos x - \sin x}{x^2} \sin x dx = \frac{p}{4} \ln \left(1 + \frac{4}{p^2} \right) \quad [\operatorname{Re} p > 0].$$

$$20. \int_0^{\infty} e^{-px} \frac{2x \cos x - \sin x}{x^2} \sin x dx = \frac{1}{2} \pi (1 - e^{-1}).$$

2.5.46. Интегралы от $\prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^{\nu_k}$,

$$\prod_k \left\{ \frac{\operatorname{sh} a_k x}{\operatorname{ch} a_k x} \right\}^{\mu_k} \left\{ \frac{\cos b_k x}{\sin b_k x} \right\}^{\nu_k}.$$

Обозначение: $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$.

$$1. \int_0^{\pi/2} \operatorname{ch} ax \cos^2 bx dx = n! b^{n+2\lambda} a^{1-2\lambda} \frac{\operatorname{ch} [a\pi/(2b)]}{\operatorname{sh} [a\pi/(2b)]} \prod_{k=0}^{(n+1)/2} [a^2 + (2k + 2\lambda)^2 b^2]^{-1/2} \\ \left[|\arg b|, |\arg a| < \pi; \left\{ \begin{smallmatrix} \lambda=1/2, n=1, 3, 5, \dots \\ \lambda=0, n=2, 4, 6, \dots \end{smallmatrix} \right\} \right].$$

$$2. \int_0^{\infty} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$3. \int_0^{\infty} \frac{\sin^2 bx}{\operatorname{sh}^2 cx} dx = \frac{\pi b \operatorname{cth} (b\pi/c) - c}{2c^2} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$4. \int_0^{\infty} \frac{\sin bx}{\operatorname{ch} cx} dx = \frac{i}{2c} \left[\Psi \left(\frac{c - ib}{4c} \right) - \Psi \left(\frac{c + ib}{4c} \right) \right] - \frac{\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch} cx} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

$$6. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch}^v cx} dx = \frac{2^{v-2}}{c \Gamma(v)} \Gamma\left(\frac{v}{2} + \frac{ib}{2c}\right) \Gamma\left(\frac{v}{2} - \frac{ib}{2c}\right) \quad [\operatorname{Re}(vc) > |\operatorname{Im} b|].$$

$$7. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch}^n cx} dx = \frac{2^{n-1} \pi}{(n-1)!} b^{1-2n} \operatorname{sech} \left\{ \frac{b\pi}{(2c)} \right\} \prod_{k=1}^{[(n-1)/2]} \left[\frac{\lambda^2}{4c^2} + (k-\lambda)^2 \right] \\ \left[n \operatorname{Re} c > |\operatorname{Im} b|; \begin{cases} \lambda = 1/2, & n = 1, 3, 5, \dots \\ \lambda = 0, & n = 2, 4, 6, \dots \end{cases} \right].$$

$$8. \int_0^{\infty} \frac{\operatorname{sh} ax \{\sin bx\}}{\operatorname{sh} cx \{\cos bx\}} dx = \\ = \frac{\pi}{2c} [\operatorname{ch}(b\pi/c) + \cos(a\pi/c)]^{-1} \left\{ \frac{\operatorname{sh}(b\pi/c)}{\sin(a\pi/c)} \right\} + \frac{\delta i}{2c} \left[\Psi\left(\frac{a+c+ib}{2c}\right) - \Psi\left(\frac{a+c-ib}{2c}\right) \right] \\ [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$9. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{sh} cx} \sin bx dx = \frac{\pi}{2c} \frac{\operatorname{sh}(b\pi/c)}{\operatorname{ch}(b\pi/c) + \cos(a\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$10. \int_0^{\infty} \frac{\{\operatorname{sh} ax \sin bx\}}{\{\operatorname{ch} ax \cos bx\} \operatorname{ch} cx} dx = \\ = \frac{\pi}{c} \frac{\sin[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)]}{\{\cos[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)]\}} \left(\operatorname{th} \frac{b\pi}{c} + \cos \frac{a\pi}{c} \right)^{-1} \\ [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$11. \int_0^{\infty} \frac{\{\operatorname{sh} ax \cos bx\}}{\{\operatorname{ch} ax \sin bx\} \operatorname{ch} cx} dx = \\ = \frac{1}{4c} \left(\frac{1}{-i} \right) \left[\Psi\left(\frac{3c-a-ib}{4c}\right) - \Psi\left(\frac{3c+a+ib}{4c}\right) \pm \Psi\left(\frac{3c-a+ib}{4c}\right) \mp \Psi\left(\frac{3c+a-ib}{4c}\right) \right] + \\ + 2\pi [\cos(a\pi/c) + \operatorname{ch}(b\pi/c)]^{-1} \left\{ \frac{\operatorname{sh}(a\pi/c)}{i \operatorname{sh}(b\pi/c)} \right\} \quad [\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|].$$

$$12. \int_0^{\infty} \frac{\{\sin ax \sin bx\}}{\{\cos ax \cos bx\} \operatorname{ch} cx} dx = \\ = \frac{\pi}{c} \frac{\{\operatorname{sh}[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)]\}}{\{\operatorname{ch}[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)]\}} \left(\operatorname{th} \frac{a\pi}{c} + \operatorname{ch} \frac{b\pi}{c} \right)^{-1} \\ [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$13. \int_0^{\infty} \frac{\sin ax \cos bx}{\operatorname{sh} cx} dx = \frac{\pi \operatorname{sh}(a\pi/c)}{2c [\operatorname{ch}(a\pi/c) + \operatorname{ch}(b\pi/c)]} \quad [\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

$$14. \int_0^{\infty} \frac{\operatorname{sh} cx \sin bx}{\operatorname{ch}^2 cx} dx = \frac{\pi b}{2c^2 \operatorname{ch}[b\pi/(2c)]} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

2.5.47.]

$$15. \int_0^{\infty} \operatorname{sh}^{v-1} ax \frac{\{\sin bx\}}{\{\cos bx\}} dx = \\ = \frac{1}{2a} \frac{1}{\pi} \frac{\operatorname{sh}[b\pi/(2a)]}{\operatorname{ch}[b\pi/(2a)]} \Gamma\left[\frac{(v+\delta)/2}{1-v/2}\right] \Gamma\left(\frac{1-v}{2} + \frac{ib}{2a}\right) \Gamma\left(\frac{1-v}{2} - \frac{ib}{2a}\right) \\ [\operatorname{Re} v > -\delta; \operatorname{Re}\{(v-1)a\} < -|\operatorname{Im} b|].$$

$$16. \int_0^{\infty} \operatorname{sh}^{v-1} ax \operatorname{ch} ax \frac{\{\sin bx\}}{\{\cos bx\}} dx = \\ = \frac{b}{4a^v \Gamma(v)} \frac{\{\operatorname{ch}[b\pi/(2a)]\}}{\{\operatorname{sh}[b\pi/(2a)]\}} \Gamma\left[\frac{(v+\delta)/2}{(1-v)/2}\right] \Gamma\left(\frac{ib-va}{2a}\right) \Gamma\left(-\frac{ib+va}{2a}\right) \\ [\operatorname{Re} v > -\delta; \operatorname{Re}(va) < -|\operatorname{Im} b|].$$

$$17. \int_0^{\infty} \left[1 - \left\{ \frac{\operatorname{th} ax}{\operatorname{cth} ax} \right\} \right] \sin bx dx = \frac{1}{b} - \frac{\pi}{2a} \left\{ \frac{1}{\operatorname{cth}[b\pi/(2a)]} \right\} \quad [b, \operatorname{Re} a > 0].$$

2.5.47. Интегралы от $x^\alpha \prod_k \frac{\{\operatorname{sh} a_k x\}^{\mu_k}}{\{\operatorname{ch} a_k x\}^{\nu_k}} \frac{\{\sin b_k x\}^{\nu_k}}{\{\cos b_k x\}^{\mu_k}}$,

$$x^\alpha \prod_k \frac{\{\operatorname{sh} a_k x\}^{\mu_k}}{\{\operatorname{ch} a_k x\}^{\nu_k}} \frac{\{\cos b_k x\}^{\nu_k}}{\{\sin b_k x\}^{\mu_k}}.$$

Обозначение: $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

$$1. \int_0^{\infty} \frac{x^{\alpha-1} \{\sin bx\}}{\operatorname{sh} cx \{\cos bx\}} dx = \frac{\Gamma(\alpha) i^\delta}{(2c)^\alpha} \left\{ \zeta\left[\alpha, \frac{1}{2} \left(1 + \frac{ib}{c}\right)\right] \mp \zeta\left[\alpha, \frac{1}{2} \left(1 - \frac{ib}{c}\right)\right] \right\} \\ [\operatorname{Re} \alpha > 1 - \delta; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$2. \int_0^{\infty} \frac{x^{2n+1-\delta} \{\sin bx\}}{\operatorname{sh} cx \{\cos bx\}} dx = (-1)^n \frac{\pi}{2c} \frac{\partial^{2n+1-\delta}}{\partial b^{2n+1-\delta}} \operatorname{th} \frac{b\pi}{2c} \quad \left[\operatorname{Re} c > |\operatorname{Im} b|; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$3. \int_0^{\infty} \frac{\sin^2 bx}{x \operatorname{sh} cx} dx = \frac{1}{2} \ln \operatorname{ch} \frac{b\pi}{c} \quad [\operatorname{Re} c > 2|\operatorname{Im} b|].$$

$$4. \int_0^{\infty} \frac{x^{\alpha-1} \{\sin bx\}}{\operatorname{ch} cx \{\cos bx\}} dx = \\ = \frac{i^\delta \Gamma(\alpha)}{(4c)^\alpha} \left[\zeta\left(\alpha, \frac{c+ib}{4c}\right) \mp \zeta\left(\alpha, \frac{c-ib}{4c}\right) \pm \zeta\left(\alpha, \frac{3c-ib}{4c}\right) \mp \zeta\left(\alpha, \frac{3c+ib}{4c}\right) \right] \\ [\operatorname{Re} \alpha > -\delta; \operatorname{Re} c > |\operatorname{Im} b|].$$

$$5. \int_0^{\infty} \frac{x^{2n+\delta} \{\sin bx\}}{\operatorname{ch} cx \{\cos bx\}} dx = (-1)^{n+\delta} \frac{\pi}{2c} \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} \operatorname{sech} \frac{b\pi}{2c} \quad \left[\operatorname{Re} c > |\operatorname{Im} b|; \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$6. \int_0^{\infty} \frac{\sin bx}{x \operatorname{ch} cx} dx = 2 \operatorname{arctg} \left[\exp \left(\frac{b\pi}{2c} \right) \right] - \frac{\pi}{2} \quad [\operatorname{Re} c > |\operatorname{Im} b|].$$

7. $\int_0^{\infty} x \frac{\sin bx}{\cosh^2 cx} dx = \frac{\pi}{4c^3} \operatorname{cosech}^2 \frac{b\pi}{2c} \left(b\pi \operatorname{ch} \frac{b\pi}{2c} - 2c \operatorname{sh} \frac{b\pi}{2c} \right)$ $[2 \operatorname{Re} c > |\operatorname{Im} b|]$
8. $\int_0^{\infty} \frac{\cos bx}{x} \left\{ \operatorname{th} ax \right\} \operatorname{cth} ax dx = \pm \ln \left\{ \frac{\operatorname{cth} [b\pi/(4a)]}{2 \operatorname{sh} [b\pi/(2a)]} \right\}$ $[b, \operatorname{Re} a > 0]$
9. $\int_0^{\infty} \frac{\operatorname{sh} ax \sin bx}{x \operatorname{sh} cx} dx = \operatorname{arctg} \left(\operatorname{tg} \frac{a\pi}{2c} \operatorname{th} \frac{b\pi}{2c} \right)$ $[\operatorname{Re} c > \operatorname{Re} a; +|\operatorname{Im} b|]$
10. $\int_0^{\infty} \frac{\sin ax \sin bx}{x \operatorname{sh} cx} dx = \frac{1}{2} \left[\ln \operatorname{ch} \left(\frac{a+b}{2c} \pi \right) - \ln \operatorname{ch} \left(\frac{a-b}{2c} \pi \right) \right]$ $[\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|]$
11. $\int_0^{\infty} \frac{\operatorname{ch} bx \sin^2 ax}{x \operatorname{sh} cx} dx = \frac{1}{4} \ln \frac{\operatorname{ch} (2a\pi/c) + \cos (b\pi/c)}{1 + \cos (b\pi/c)}$ $[\operatorname{Re} c > \operatorname{Re} b; +|\operatorname{Im} a|]$
12. $\int_0^{\infty} \frac{\operatorname{sh} ax \cos bx}{x \operatorname{ch} cx} dx = \frac{1}{2} \ln \frac{\operatorname{ch} [b\pi/(2c)] + \sin [a\pi/(2c)]}{\operatorname{ch} [b\pi/(2c)] - \sin [a\pi/(2c)]}$ $[\operatorname{Re} c > \operatorname{Re} a; +|\operatorname{Im} b|]$
13. $\int_0^{\infty} \frac{\cos ax \sin bx}{x \operatorname{ch} cx} dx = \operatorname{arctg} \frac{\operatorname{sh} [b\pi/(2c)]}{\operatorname{ch} [a\pi/(2c)]}$ $[\operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|]$

2.5.49. Интегралы, содержащие $A(x) (a + b \operatorname{ch} x)^v \left(\frac{\sin cx}{\cos cx} \right)^t$

1. $\int_0^a \frac{\cos bx}{(\operatorname{ch} a - \operatorname{ch} x)^v} dx = \sqrt{\frac{\pi}{2}} \frac{\Gamma(1-v)}{\operatorname{sh}^{v-1/2} a} P_{v-1/2}^{v-1/2}(\operatorname{ch} a)$ $[a > 0; \operatorname{Re} v < 1]$
2. $\int_0^{\infty} \frac{\cos bx}{a + \operatorname{ch} cx} dx = \frac{\pi}{c \sqrt{a^2 - 1}} \operatorname{cosech} \frac{b\pi}{c} \sin \left(\frac{b}{c} \operatorname{Arch} a \right)$ $[a > 1; \operatorname{Re} c > |\operatorname{Im} b|]$
3. $= -\frac{\pi}{c \sqrt{a^2 - 1}} \operatorname{sech} \frac{b\pi}{c} \sin \left[\frac{b}{c} \operatorname{Arch} (-a) \right]$ $[a < -1; \operatorname{Re} c > |\operatorname{Im} b|]$
4. $= \frac{\pi}{c \sqrt{1 - a^2}} \operatorname{cosech} \frac{b\pi}{c} \operatorname{sh} \left(\frac{b}{c} \operatorname{arccos} a \right)$ $[-1 < a < 1; \operatorname{Re} c > |\operatorname{Im} b|]$
5. $\int_0^{\infty} \frac{\cos bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \operatorname{sh} (b\pi/c)}$ $[\operatorname{Re} c > |\operatorname{Im} b|]$
6. $\int_0^{\infty} \frac{\cos bx}{(a + \operatorname{ch} cx)^v} dx = \frac{e^{b\pi/c}}{c} (a^2 - 1)^{-v/2} \Gamma \left[\frac{v - ib/c}{v} \right] Q_{v-1}^{ib/c} \left(\frac{a}{\sqrt{a^2 - 1}} \right)$ $[b, \operatorname{Re} (cv) > 0; a \notin [-1, 1]]$

7. $\int_0^{\infty} \frac{\cos bx}{\sqrt{a + \operatorname{ch} cx}} dx = \frac{\pi}{c \sqrt{2}} \operatorname{sech} \frac{b\pi}{c} P_{ib/c-1/2}(a)$ $[-1 < a < 1; \operatorname{Re} c > 2|\operatorname{Im} b|]$
8. $\int_0^{\infty} \frac{\cos bx}{(1 + \operatorname{ch} cx)^v} \operatorname{ch} cx dx = \frac{2^{v-1}}{c} B \left(v + \frac{ib}{c}, v - \frac{ib}{c} \right)$ $[\operatorname{Re} v > 0; \operatorname{Re} (vc) > \operatorname{Re} c + |\operatorname{Im} b|]$
9. $\int_0^{\infty} \frac{x \sin bx}{1 + \operatorname{ch} cx} dx = \frac{\pi}{c^3} \operatorname{cosech}^2 \frac{b\pi}{c} \left(b\pi \operatorname{ch} \frac{b\pi}{c} - c \operatorname{sh} \frac{b\pi}{c} \right)$ $[\operatorname{Re} c > |\operatorname{Im} b|]$
10. $\int_0^{\infty} \frac{x^{2n+\delta}}{(1 + \operatorname{ch} cx)^v} \left(\frac{\sin bx}{\cos bx} \right) dx = \frac{(-1)^{n+\delta-1} \pi}{c^2} \Gamma \left[\frac{2-v}{v-2}, 2v \right] \frac{\partial^{2n+\delta}}{\partial b^{2n+\delta}} \frac{b}{\operatorname{sh} (b\pi/c)}$ $[\operatorname{Re} (vc) > |\operatorname{Im} b|; \delta = \{ \frac{1}{2} \}]$
11. $\int_0^{\infty} \frac{(\operatorname{sh} ax \sin bx)}{(\operatorname{ch} ax \cos bx)} \frac{dx}{\cos t + \operatorname{ch} cx} = \frac{\pi}{c \sin t} \left(\operatorname{ch} \frac{2b\pi}{c} - \cos \frac{2a\pi}{c} \right)^{-1} \times$
 $\times \left\{ \frac{\sin [a(\pi-t)/c] \operatorname{sh} [b(\pi+t)/c] - \sin [a(\pi+t)/c] \operatorname{sh} [b(\pi-t)/c]}{\cos [a(\pi-t)/c] \operatorname{ch} [b(\pi+t)/c] - \cos [a(\pi+t)/c] \operatorname{ch} [b(\pi-t)/c]} \right\}$
 $[0 < t < \pi; \operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|; \pi \operatorname{Re} c > |\operatorname{Re} (\bar{c}t)|]$
12. $\int_0^{\infty} \frac{(\operatorname{sh} ax \sin bx)}{(\operatorname{ch} ax \cos bx)} \frac{dx}{1 + \operatorname{ch} cx} = \frac{2\pi}{c^2 [\operatorname{ch} (2b\pi/c) - \cos (2a\pi/c)]} \times$
 $\times \left[b \left\{ \frac{\sin (a\pi/c) \operatorname{ch} (b\pi/c)}{\cos (a\pi/c) \operatorname{sh} (b\pi/c)} \right\} + a \left\{ \frac{\cos (a\pi/c) \operatorname{sh} (b\pi/c)}{\sin (a\pi/c) \operatorname{ch} (b\pi/c)} \right\} \right]$ $[\operatorname{Re} c > |\operatorname{Re} a| + |\operatorname{Im} b|]$
13. $\int_0^{\pi} \frac{\operatorname{sh} cx \sin bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi}{c} \operatorname{ch} \frac{bt}{c} \operatorname{cosech} \frac{b\pi}{c}$ $[b, c > 0; 0 < t < \pi]$
14. $\int_0^{\pi} \frac{\operatorname{ch} cx \cos bx}{\cos t + \operatorname{ch} cx} dx = -\frac{\pi}{c} \operatorname{sh} \frac{bt}{c} \operatorname{ctg} t \operatorname{cosech} \frac{b\pi}{c}$ $[b, c > 0; 0 < t < \pi]$
15. $\int_0^{\infty} \frac{\operatorname{ch} cx \cos bx}{1 + \operatorname{ch} cx} dx = -\frac{\pi b}{c^2} \operatorname{cosech} \frac{b\pi}{c}$ $[b, c > 0]$
16. $\int_0^{\infty} \frac{\operatorname{sh} (cx/2) \sin bx}{\cos t + \operatorname{ch} cx} dx = \frac{\pi \operatorname{sh} (bt/c)}{2c \operatorname{ch} (b\pi/c) \sin (t/2)}$ $[\operatorname{Re} c > 2|\operatorname{Im} b|; 0 < t < \pi]$
17. $\int_0^{\infty} \frac{\operatorname{sh} (cx/2) \sin bx}{1 + \operatorname{ch} cx} dx = \frac{\pi b}{c^2 \operatorname{ch} (b\pi/c)}$ $[\operatorname{Re} c > 2|\operatorname{Im} b|]$

$$18. \int_0^{\infty} \frac{\operatorname{sh} cx \sin bx}{(\cos t + \operatorname{ch} cx)^2} dx = \frac{b\pi \operatorname{sh}(bt/c)}{c^2 \sin t \operatorname{sh}(b\pi/c)} \quad [\operatorname{Re} c > |\operatorname{Im} b|; |t| < \pi]$$

$$19. \int_0^{\infty} \frac{\cos bx}{(x^2 + z^2)(\cos t + \operatorname{ch} cx)} dx = \frac{\pi e^{-bz}}{2z(\cos cz + \cos t)} + \\ + \frac{c\pi}{\operatorname{sh} t} \sum_{k=0}^{\infty} \left\{ \frac{\exp[-b\pi[(2k+1)\pi - t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) - t]^2} - \frac{\exp[-b\pi[(2k+1)\pi + t]/(\pi c)]}{c^2 z^2 - [\pi(2k+1) + t]^2} \right\} \\ [\operatorname{Re} c > |\operatorname{Im} b|; |t| < \pi; \operatorname{Re} z > 0]$$

2.5.49. Интегралы, содержащие $A(x)(a + b \operatorname{ch}^2 x)^{-1} \left\{ \frac{\sin cx}{\cos cx} \right\}$.

$$1. \int_0^{\infty} \frac{\cos bx}{1 + \operatorname{ch}^2 cx} dx = \frac{\pi}{2c} \operatorname{cosec} \frac{b\pi}{2c} \sin \frac{b \ln(1 + \sqrt{2})}{2c} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|]$$

$$2. \int_0^{\infty} \frac{\operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \cos bx dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} \cos \frac{b \ln(1 + \sqrt{2})}{2c} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|]$$

$$3. \int_0^{\infty} \frac{\cos bx}{\operatorname{sh}^2 t + \operatorname{ch}^2 cx} dx = \frac{\pi}{c} \left(\operatorname{sh} 2t \operatorname{sh} \frac{b\pi}{2c} \right)^{-1} \sin \frac{bt}{c} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|]$$

$$4. \int_0^{\infty} \frac{\cos bx}{\sqrt{\operatorname{sh}^2 t + \operatorname{ch}^2 cx}} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} P_{1/2}(2a)^{-1/2} (\operatorname{ch} 2t) \quad [2 \operatorname{Re} c > |\operatorname{Im} b|]$$

$$5. \int_0^{\infty} \frac{x^{-1/2}}{1 + 2 \operatorname{ch}^2 cx} \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \sqrt{\frac{2\pi}{3}} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\sqrt{c^2 k^2 + b^2} + kc}{c^2 k^2 + b^2} \right)^{1/2} \sin \frac{k\pi}{2} \quad [2 \operatorname{Re} c > |\operatorname{Im} b|]$$

$$6. \int_0^{\infty} \frac{x^{2n+1} \operatorname{ch} cx}{1 + \operatorname{ch}^2 cx} \sin bx dx = \\ = (-1)^{n+1} \frac{\pi}{2c} \frac{d^{n+1}}{db^{2n+1}} \left[\operatorname{sech} \frac{b\pi}{2a} \cos \frac{b \ln(1 + \sqrt{2})}{2a} \right] \quad [\operatorname{Re} c > |\operatorname{Im} b|]$$

2.5.50. Интегралы, содержащие $A(x)(\operatorname{ch} ax \pm \cos bx)^{-1} \left\{ \frac{\sin cx}{\cos cx} \right\}$.

$$1. \int_0^{\infty} \frac{(\operatorname{sh} ax \sin bx)}{(\operatorname{ch} ax \cos bx)} \frac{dx}{\operatorname{ch} 2ax + \cos 2bx} = \frac{\pi}{4(a^2 + b^2)} \left\{ \frac{b}{a} \right\} \quad [b, \operatorname{Re} a > 0]$$

$$2. \int_0^{\infty} \frac{x^{2k-1} \operatorname{sh} ax \sin bx}{\operatorname{ch} 2ax + \cos 2bx} dx = \frac{\Gamma(\alpha)}{(a^2 + b^2)^{\alpha/2}} \sin \left(\alpha \operatorname{arctg} \frac{b}{a} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\alpha}} \\ [b, \operatorname{Re} a > 0; \operatorname{Re} \alpha > -2]$$

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$$3. \int_0^{\infty} \frac{\sin nx}{x(\operatorname{ch} x + \cos x)} dx = \frac{\pi}{4} \quad [n = 1, 2, 3, \dots]$$

$$4. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\sin bx}{\operatorname{ch} cx - \cos bx} dx = \frac{b}{2c} \frac{b^2 + 2c^2}{b^2 + c^2} - \operatorname{arctg} \frac{b}{c} \quad [by = 2\pi; b, \operatorname{Re} c > 0]$$

$$5. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\sin bx}{\operatorname{ch} cx \pm \cos bx} dx = \pm \operatorname{arctg} \frac{b}{c} + \left\{ \frac{-b^{-1}}{bc(c^2 + b^2)^{-1}} \right\} \\ [by = \pi; y, \operatorname{Re} c > 0]$$

$$6. \int_0^{\infty} \frac{x}{x^2 - y^2} \frac{\operatorname{ch} ax \sin bx}{\operatorname{ch} 2ax - \cos 2bx} dx = -\frac{b^3}{2a(a^2 + b^2)} \quad [by = \pi; b, \operatorname{Re} a > 0]$$

2.5.51. Интегралы, содержащие $(x^2 + z^2)^{-1} \left\{ \frac{\operatorname{sh} ax}{\operatorname{ch} ax} \right\}^{-1} \left\{ \frac{\sin bx}{\cos bx} \right\}$.

$$1. \int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\sin bx}{\operatorname{sh} cx} dx = \frac{\pi}{2z^2} - \frac{\pi e^{-bz}}{2z \sin cz} - c\pi \sum_{k=1}^{\infty} \frac{(-1)^k e^{-bk\pi/c}}{\pi^2 k^2 - c^2 z^2} \\ [\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi n]$$

$$2. = (-1)^n b \frac{e^{-b\pi/c}}{2n} + \frac{c}{2\pi n} \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} e^{-bk\pi/c} + \\ + (-1)^n \frac{c}{2\pi n} e^{-b\pi/c} \ln(1 + e^{-b\pi/c}) + \frac{c}{2\pi n} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{(1+z)^{n-1}}{z} \ln(1+z) \right) \Big|_{z=e^{-b\pi/c}} \\ [cz = \pi n; \operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; n = 1, 2, 3, \dots]$$

$$3. = -\frac{b}{2} \operatorname{ch} \frac{b\pi}{c} + \frac{c}{\pi} \operatorname{sh} \frac{b\pi}{c} \ln \left(2 \operatorname{ch} \frac{b\pi}{2c} \right) \quad [cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|]$$

$$4. = c \operatorname{sh} \frac{b\pi}{2c} - \frac{2c}{\pi} \operatorname{ch} \frac{b\pi}{2c} \operatorname{arctg} \left(\operatorname{sh} \frac{b\pi}{2c} \right) = \\ = \frac{2c}{\pi} [e^{b\pi/(2c)} \operatorname{arctg} e^{-b\pi/(2c)} - e^{-b\pi/(2c)} \operatorname{arctg} e^{b\pi/(2c)}] \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|]$$

$$5. = -2\sqrt{2} c e^{-b\pi/(4c)} + \frac{2\sqrt{2} c}{\pi} \operatorname{sh} \frac{b\pi}{4c} \ln \frac{\operatorname{ch}[b\pi/(4c)] + 1/\sqrt{2}}{\operatorname{ch}[b\pi/(4c)] - 1/\sqrt{2}} + \\ + \frac{4\sqrt{2}}{\pi} c \operatorname{ch} \frac{b\pi}{4c} \operatorname{arctg} \frac{1}{\sqrt{2} \operatorname{sh}[b\pi/(4c)]} \quad [4cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|]$$

$$6. \int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\sin bx}{\operatorname{ch} cx} dx = \pi^2 \sum_{k=0}^{\infty} (-1)^k \frac{(k+1/2)^3 e^{-bz} - (k+1/2) e^{-b\pi(k+1/2)/c}}{\pi^2 (k+1/2)^2 - c^2 z^2} \\ [\operatorname{Re} c > |\operatorname{Im} b|; \operatorname{Re} z > 0; cz \neq \pi(n+1/2)]$$

$$7. = -2 \operatorname{sh} \frac{b\pi}{2c} + \frac{\pi}{2} \operatorname{sh} \frac{b\pi}{c} - 2 \operatorname{ch} \frac{b\pi}{c} \operatorname{arctg} \operatorname{th} \frac{b\pi}{4c} \quad [2cz = \pi; \operatorname{Re} c > |\operatorname{Im} b|]$$

8.
$$= \frac{\pi}{\sqrt{2}} e^{-b\pi/(4c)} + \frac{\text{sh}[b\pi/(4c)]}{\sqrt{2}} \ln \frac{\text{ch}[b\pi/(4c)] + 1/\sqrt{2}}{\text{ch}[b\pi/(4c)] - 1/\sqrt{2}} -$$

$$- \sqrt{2} \text{ch} \frac{b\pi}{4c} \text{arctg} \frac{1}{\sqrt{2} \text{sh}[b\pi/(4c)]} \quad [4cz = \pi; \text{Re } c > |\text{Im } b|].$$
9.
$$\int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\cos bx}{\text{sh } cx} dx = \frac{\pi e^{-bz}}{2 \sin cz} + \pi^2 \sum_{k=1}^{\infty} \frac{(-1)^k k e^{-k\pi/c}}{\pi^2 k^2 - c^2 z^2}$$

$$[\text{Re } c > |\text{Im } b|; \text{Re } z > 0; cz \neq \pi n].$$
10.
$$= -\frac{1}{2} + \frac{\pi b}{2c} e^{-b\pi/c} + \text{ch} \frac{b\pi}{c} \ln(1 + e^{-b\pi/c}) =$$

$$= -\frac{b\pi}{2c} \text{sh} \frac{b\pi}{c} - \frac{1}{2} + \frac{1}{2} \text{ch} \frac{b\pi}{c} \ln \left[2 \left(1 + \text{ch} \frac{b\pi}{c} \right) \right] \quad [cz = \pi; \text{Re } c > |\text{Im } b|].$$
11.
$$= 2 \text{sh} \frac{b\pi}{2c} \text{arctg}(e^{-b\pi/(4c)}) + \frac{\pi}{2} e^{-b\pi/(2c)} - 1 =$$

$$= \frac{\pi}{2} \text{ch} \frac{b\pi}{2c} - 1 + \text{sh} \frac{b\pi}{2c} \text{arctg} \text{sh} \frac{b\pi}{2c} \quad [2cz = \pi; \text{Re } c > |\text{Im } b|].$$
12.
$$\int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\cos bx}{\text{ch } cx} dx = \frac{\pi}{z} \sum_{k=0}^{\infty} (-1)^k \frac{\pi (k+1/2)^2 e^{-bx} - c^2 e^{-b\pi(2k+1)/(2c)}}{\pi^2 (k+1/2)^2 - c^2 z^2}$$

$$[\text{Re } c > |\text{Im } b|; \text{Re } z > 0; cz \neq \pi(n+1/2)].$$
13.
$$= \frac{(-1)^n}{2n+1} \exp \left[-\frac{b}{c} (n+1/2) \pi \right] \left[b + \frac{c}{\pi} \ln(1 + e^{-b\pi/c}) \right] +$$

$$+ \frac{ce^{-b\pi/(2c)}}{\pi(2n+1)} \sum_{k=0}^{n-1} \frac{(-1)^k}{k-n} e^{-bk\pi/c} + \frac{ce^{-b\pi/(2c)}}{\pi(2n+1)(n+1)} F(n+1, 1; n+2, -e^{-b\pi/c})$$

$$[cz = \pi(n+1/2); \text{Re } c > |\text{Im } b|].$$
14.
$$= \frac{2c}{\pi} \text{ch} \frac{b\pi}{2c} - e^{b\pi/c} \text{arctg}(e^{-b\pi/(2c)}) - e^{-b\pi/c} \text{arctg}(e^{b\pi/(2c)})$$

$$[cz = \pi; \text{Re } c > |\text{Im } b|].$$
15.
$$= be^{-b\pi/(2c)} + \frac{2c}{\pi} \text{ch} \frac{b\pi}{c} \ln(1 + e^{-2b\pi/c}) \quad [2cz = \pi; \text{Re } c > |\text{Im } b|].$$
16.
$$= 2\sqrt{2} ce^{-b\pi/(4c)} + \frac{4\sqrt{2}}{\pi} \text{sh} \frac{b\pi}{4c} \text{arctg} \frac{1}{\sqrt{2} \text{sh}[b\pi/(4c)]} -$$

$$- \frac{2\sqrt{2}c}{\pi} \text{ch} \frac{b\pi}{4c} \ln \frac{\text{ch}[b\pi/(4c)] + 1/\sqrt{2}}{\text{ch}[b\pi/(4c)] - 1/\sqrt{2}} \quad [4cz = \pi; \text{Re } c > |\text{Im } b|].$$
17.
$$\int_0^{\infty} \frac{\text{th } ax \sin bx}{x^2 + z^2} dx = b \text{ch} \frac{b\pi}{2a} - \frac{2a}{\pi} \text{sh} \frac{b\pi}{2a} \ln \left(2 \text{sh} \frac{b\pi}{2a} \right) \quad [2az = \pi; b, \text{Re } z > 0]$$
18.
$$= -\frac{\pi}{2z} e^{bz} + \frac{\text{sh } bz}{z} \ln \text{cth} \frac{bz}{2} + \frac{2}{z} \text{ch } bz \text{arctg } e^{bz} \quad [4az = \pi; b, \text{Re } z > 0]$$
19.
$$\int_0^{\infty} \frac{x \text{th } ax \cos bx}{x^2 + z^2} dx = -bz e^{-bz} + \text{sh } bz \ln(1 - e^{-2bz}) \quad [2az = \pi; b, \text{Re } z > 0]$$

20.
$$= -\frac{\pi}{2} e^{bz} \text{ch } bz \ln \text{cth} \frac{bz}{2} + 2 \text{sh } bz \text{arctg } e^{bz} \quad [4az = \pi; b, \text{Re } z > 0].$$
21.
$$\int_0^{\infty} \frac{\text{ch } ax \sin bx}{x^2 + z^2} dx = \frac{2a}{\pi} \text{sh} \frac{b\pi}{2a} \ln \text{cth} \frac{b\pi}{4a} \quad [2az = \pi; b, \text{Re } z > 0].$$
22.
$$\int_0^{\infty} \frac{x \text{th } ax \cos bx}{x^2 + z^2} dx = -2 + \frac{\pi}{2} e^{-bz} + \text{ch } bz \ln \text{cth} \frac{bz}{2} + 2 \text{sh } bz \text{arctg } e^{-bz}$$

$$[4az = \pi; b, \text{Re } z > 0].$$
23.
$$\int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\text{sh } ax \cos bx}{\text{ch } ax \sin bx} \frac{dx}{\text{sh } cx} = \pm \frac{\pi}{2z \sin cz} \frac{\text{sh } az}{\cos az} \pm$$

$$\pm c\pi \sum_{k=1}^{\infty} (-1)^k \frac{\exp(-kb\pi/c)}{\pi^2 k^2 - c^2 z^2} \frac{\text{sh}(ak\pi/c)}{\cos(ak\pi/c)} + \left\{ \begin{matrix} 0 \\ \pi(2cz^2 - 1) \end{matrix} \right\}$$

$$[\text{Re } c > |\text{Re } a| + |\text{Im } b|; \text{Re } z > 0; cz \neq \pi n].$$
24.
$$= \frac{1}{2} \exp \left(-\frac{b\pi}{c} \right) \left[\pm b \frac{\text{sh}(a\pi/c)}{\cos(a\pi/c)} - a \frac{\text{ch}(a\pi/c)}{\sin(a\pi/c)} \right] +$$

$$+ \frac{c}{2\pi} \left\{ \text{ch}(b\pi/c) \sin(a\pi/c) \ln \left[1 + 2 \exp \left(-\frac{b\pi}{c} \right) \cos \frac{a\pi}{c} + \exp \left(-\frac{2b\pi}{c} \right) \right] \mp \right.$$

$$\left. - \frac{c}{\pi} \frac{\text{sh}(b\pi/c) \cos(a\pi/c)}{\text{ch}(b\pi/c) \sin(a\pi/c)} \right\} \text{arctg} \frac{\sin(a\pi/c)}{\exp(b\pi/c) + \cos(a\pi/c)}$$

$$[cz = \pi; \text{Re } c > |\text{Re } a| + |\text{Im } b|].$$
25.
$$= \pm c \exp \left(-\frac{b\pi}{2c} \right) \frac{\text{sh}[a\pi/(2c)]}{\cos[a\pi/(2c)]} \mp$$

$$\mp \frac{c}{\pi} \frac{\text{ch}[b\pi/(2c)] \cos[a\pi/(2c)]}{\text{sh}[b\pi/(2c)] \sin[a\pi/(2c)]} \ln \frac{\text{ch}[b\pi/(2c)] + \sin[a\pi/(2c)]}{\text{ch}[b\pi/(2c)] - \sin[a\pi/(2c)]} +$$

$$+ \frac{2c}{\pi} \frac{\text{sh}[b\pi/(2c)] \sin[a\pi/(2c)]}{\text{ch}[b\pi/(2c)] \cos[a\pi/(2c)]} \text{arctg} \frac{\cos[a\pi/(2c)]}{\text{sh}[b\pi/(2c)]}$$

$$[2cz = \pi; \text{Re } c > |\text{Re } a| + |\text{Im } b|].$$
26.
$$\int_0^{\infty} \frac{1}{x^2 + z^2} \frac{\text{sh } ax \sin bx}{\text{ch } ax \cos bx} \frac{dx}{\text{ch } cx} =$$

$$= \exp \left(-\frac{b\pi}{2c} \right) \left[b \frac{\text{sh}[a\pi/(2c)]}{\cos[a\pi/(2c)]} \mp a \frac{\text{ch}[a\pi/(2c)]}{\sin[a\pi/(2c)]} \right] \mp$$

$$\mp \frac{c}{\pi} \frac{\text{sh}[b\pi/(2c)] \sin[a\pi/(2c)]}{\text{ch}[b\pi/(2c)] \cos[a\pi/(2c)]} \ln \left[1 + 2 \exp \left(-\frac{b\pi}{c} \right) \cos \frac{a\pi}{c} + \exp \left(-\frac{2b\pi}{c} \right) \right] +$$

$$+ \frac{c}{\pi} \frac{\text{ch}[b\pi/(2c)] \cos[a\pi/(2c)]}{\text{sh}[b\pi/(2c)] \sin[a\pi/(2c)]} \text{arctg} \left[\frac{\sin(a\pi/c)}{1 + \exp(-b\pi/c) \cos(a\pi/c)} \right] \left\{ \frac{1}{\exp(-b\pi/c)} \right\}$$

$$[2cz = \pi; \text{Re } c > |\text{Re } a| + |\text{Im } b|].$$
27.
$$\int_0^{\infty} \frac{x}{x^2 + z^2} \frac{\text{sh } ax \sin bx}{\text{ch } ax \cos bx} \frac{dx}{\text{sh } cx} =$$

$$= \frac{\pi e^{-bz}}{2 \sin cz} \frac{\text{sh } az}{\cos az} + \pi^2 \sum_{k=1}^{\infty} (-1)^k \frac{k \exp(-kb\pi/c)}{\pi^2 k^2 - c^2 z^2} \frac{\text{sh}(ak\pi/c)}{\cos(ak\pi/c)}$$

$$[\text{Re } c > |\text{Re } a| + |\text{Im } b|; \text{Re } z > 0; cz \neq \pi n].$$

$$28. = \frac{\pi}{2c} \exp\left(-\frac{b\pi}{c}\right) \left[\frac{1}{2} \left\{ \frac{\sin(a\pi/c)}{\cos(a\pi/c)} \right\} + a \left\{ \frac{\cos(a\pi/c)}{\sin(a\pi/c)} \right\} - \frac{1}{2} \{0\} \right] +$$

$$+ \frac{1}{2} \left\{ \frac{\operatorname{sh}(b\pi/c) \sin(a\pi/c)}{\operatorname{ch}(b\pi/c) \cos(a\pi/c)} \right\} \ln \left[1 + 2 \exp\left(-\frac{b\pi}{c}\right) \cos \frac{a\pi}{c} + \exp\left(-\frac{2b\pi}{c}\right) \right] +$$

$$+ \frac{\operatorname{ch}(b\pi/c) \cos(a\pi/c)}{\operatorname{sh}(b\pi/c) \sin(a\pi/c)} \operatorname{arctg} \frac{\sin(a\pi/c)}{\exp(b\pi/c) + \cos(a\pi/c)}$$

[$c = \pi$; $\operatorname{Re} c > 0$; $\operatorname{Re} a + \operatorname{Im} b > 0$]

$$29. = \left\{ \begin{array}{l} 0 \\ -1 \end{array} \right\} + \frac{\pi}{2} \exp\left(-\frac{b\pi}{2c}\right) \left\{ \frac{\sin[a\pi/(2c)]}{\cos[a\pi/(2c)]} \right\} +$$

$$+ \frac{1}{2} \left\{ \cos[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)] \right\} \frac{\operatorname{ch}[b\pi/(2c)] + \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] - \sin[a\pi/(2c)]} +$$

$$+ \frac{1}{2} \left\{ \sin[a\pi/(2c)] \operatorname{ch}[b\pi/(2c)] \right\} \frac{\operatorname{ch}[b\pi/(2c)] - \sin[a\pi/(2c)]}{\operatorname{ch}[b\pi/(2c)] + \sin[a\pi/(2c)]} +$$

$$+ \frac{\sin[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)]}{\cos[a\pi/(2c)] \operatorname{sh}[b\pi/(2c)]} \operatorname{arctg} \frac{\cos[a\pi/(2c)]}{\operatorname{sh}[b\pi/(2c)]}$$

[$2c = \pi$; $\operatorname{Re} c > 0$; $\operatorname{Re} a + \operatorname{Im} b > 0$]

$$30. \int_0^{\infty} \frac{\operatorname{ch} x}{\operatorname{ch}^2 x - 1} \left[(\operatorname{ch} x + \sqrt{\operatorname{ch}^2 x - 1})^{\nu} - (\operatorname{ch} x - \sqrt{\operatorname{ch}^2 x - 1})^{\nu} \right] \cos bx \, dx =$$

$$= \frac{\pi \sin \nu \pi}{\operatorname{ch} b\pi + \cos \nu \pi} \quad [b, \operatorname{Re} \nu > 0]$$

2.5.52. Интегралы, содержащие $A(x)$, $\operatorname{sh} bx$, $\operatorname{ch} bx$, $\sin ax^2$, $\cos bx^2$.

$$1. \int_0^{\infty} \frac{x^{\nu}}{\operatorname{sh} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \frac{1}{\pi} +$$

$$I_1^+ = \frac{1}{4\pi}, \quad I_2^+ = \frac{1}{16\pi}, \quad I_1^- = \frac{1}{8}, \quad I_2^- = \frac{1}{16} \left(\frac{1}{4} - \frac{3}{\pi^2} \right).$$

$$2. \int_0^{\infty} \frac{x^{\nu}}{\operatorname{ch} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \pm \frac{1}{8} \left\{ \frac{1}{2\pi} \right\} - \frac{1}{8} \frac{1}{2}.$$

$$3. \int_0^{\infty} \frac{\sin bx}{\operatorname{sh} cx} \left\{ \frac{\sin ax^2}{\cos ax^2} \right\} dx = \pm \frac{\pi}{2c} \operatorname{cosech} \frac{b\pi}{2c} \left[\left\{ \frac{\sin[b^2\pi/(4c^2)]}{\cos[b^2\pi/(4c^2)]} \right\} - \left\{ \frac{0}{\operatorname{ch}(b\pi/c)} \right\} \right]$$

[$a\pi = c^2$; $a, b, \operatorname{Re} c > 0$]

$$4. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch} cx} \left\{ \frac{\sin ax^2}{\cos ax^2} \right\} dx = -\frac{\pi}{c} \sum_{k=0}^{\infty} (\pm 1)^k \exp \left[-\left(k + \frac{1}{2}\right) \frac{b\pi}{c} \right] \left\{ \frac{\sin d_k}{\cos d_k} \right\} +$$

$$+ \sqrt{\frac{\pi}{a}} \sum_{k=0}^{\infty} \exp \left[-\frac{bc}{a} \left(k + \frac{1}{2}\right) \right] \left\{ \frac{\sin h_k}{\cos h_k} \right\}$$

[$d_k = \frac{a\pi^2}{c^2} \left(k + \frac{1}{2}\right)^2$, $h_k = \frac{\pi}{4} - \frac{b^2}{4a} + \frac{c^2}{a} \left(k + \frac{1}{2}\right)^2$; $a, b, \operatorname{Re} c > 0$]

$$5. \int_0^{\infty} \frac{\cos bx}{\operatorname{ch} cx} \left\{ \frac{\sin ax^2}{\cos ax^2} \right\} dx = \frac{\pi}{2c} \operatorname{sech} \frac{b\pi}{2c} \left[\left\{ \frac{\cos d}{\sin d} \right\} + \frac{1}{\sqrt{2}} \right]$$

[$d = b^2\pi/(4c^2)$, $a\pi = c^2$; $a, b, \operatorname{Re} c > 0$]

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$$6. \int_0^{\infty} \frac{\sin bx}{\operatorname{th} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \pm \frac{1}{2} \operatorname{th} \frac{b}{2} \left\{ \frac{\sin[\pi/4 + b^2/(4\pi)]}{\cos[\pi/4 + b^2/(4\pi)]} - 1 \right\} \quad [b > 0]$$

$$7. \int_0^{\infty} \frac{\operatorname{sh} bx}{\operatorname{sh} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \frac{1}{2} \operatorname{cosec} \frac{b}{2} \left\{ \frac{\sin[b^2/(4\pi)]}{\cos(b/2) - \cos[b^2/(4\pi)]} \right\} \quad [| \operatorname{Re} b | < \pi]$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} bx}{\operatorname{ch} \pi x} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx = \pm 2^{-3/2} \operatorname{sec} \frac{b}{2} \left[\sqrt{2} \left\{ \frac{\cos[b^2/(4\pi)]}{\sin[b^2/(4\pi)]} \right\} - 1 \right] \quad [| \operatorname{Re} b | < \pi]$$

$$9. \int_0^{\infty} \frac{\cos bx}{1 + 2 \operatorname{ch}(2\pi x/3)} \left\{ \frac{\sin \pi x^2}{\cos \pi x^2} \right\} dx =$$

$$= \pm \left(4 \operatorname{ch} \frac{b}{3} - 2 \right)^{-1} \left\{ \frac{\cos d}{\sin d} \right\} + \left\{ \frac{-\sqrt{3}}{1} \right\} \quad [d = \frac{\pi}{12} - \frac{b^2}{4\pi}, b > 0]$$

$$10. \int_0^{\infty} \frac{x \operatorname{sh}(\pi x/2) \sin(\pi x/2) \sin ax^2}{\operatorname{ch} \pi x + \cos \pi x} dx = \frac{1}{4} \frac{\partial}{\partial z} \left(z, e^{-2\pi z} \right) \Big|_{z=0} \quad [a > 0]$$

2.5.53. Интегралы от $A(x) \left\{ \frac{\operatorname{sh}(\sqrt{a^2 - x^2}) \sin bx}{\operatorname{ch}(\sqrt{a^2 - x^2}) \cos bx} \right\}$,

$$A(x) \left\{ \frac{\operatorname{sh}(\sqrt{a^2 - x^2}) \cos bx}{\operatorname{ch}(\sqrt{a^2 - x^2}) \sin bx} \right\}.$$

$$1. \int_0^a \operatorname{sh}(c \sqrt{a^2 - x^2}) \cos bx \, dx = \frac{\pi ac}{2 \sqrt{c^2 - b^2}} I_2(a \sqrt{c^2 - b^2}) \quad [0 < b < c; a > 0]$$

$$2. = \frac{\pi a^2 c}{4} \quad [b = c > 0; a > 0]$$

$$3. = \frac{\pi ac}{2 \sqrt{b^2 - c^2}} J_1(a \sqrt{b^2 - c^2}) \quad [0 < c < b; a > 0]$$

$$4. \int_0^a \frac{\operatorname{sh}(c \sqrt{a^2 - x^2})}{(a^2 - x^2)^{3/4}} \cos bx \, dx =$$

$$= \frac{\pi}{2} \sqrt{\frac{a}{c}} J_4 \left[\frac{a}{2} (b - \sqrt{b^2 - c^2}) \right] J_{1/4} \left[\frac{a}{2} (b + \sqrt{b^2 - c^2}) \right] \quad [a > 0]$$

$$5. \int_0^a \frac{\operatorname{ch}(c \sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} \cos bx \, dx = \frac{\pi}{2} I_0(a \sqrt{c^2 - b^2}) \quad [0 < b < c; a > 0]$$

$$6. = \frac{\pi}{2} \quad [b = c > 0; a > 0]$$

$$7. = \frac{\pi}{2} J_0(a \sqrt{b^2 - c^2}) \quad [0 < c < b; a > 0]$$

$$9. \int_0^{\infty} \frac{\sin t a \sin x}{x} \sin x \sin 2n x \cos (a \cos x) dx =$$

$$= (-1)^{n-1} \frac{\pi a^{2n-1}}{8(2n-1)!} \left[1 + \frac{a^2}{2n(2n+1)} \right] \quad [a > 0].$$

$$10. \int_0^{\infty} \frac{\operatorname{ch} (a \sin x)}{x} \sin x \cos (2n-1)x \cos (a \cos x) dx =$$

$$= (-1)^{n-1} \frac{\pi a^{2n-2}}{8(2n-2)!} \left[1 - \frac{a^2}{2n(2n-1)} \right] \quad [a > 0].$$

$$11. \int_0^{\infty} \frac{\operatorname{sh} (a \sin x)}{x} \cos x \cos 2n x \cos (a \cos x) dx =$$

$$= \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n \pi a^{2n-1}}{8(2n-1)!} \left[1 - \frac{3a^2}{2n(n+1)} \right] \quad [a > 0].$$

$$12. \int_0^{\infty} \frac{\operatorname{ch} (a \sin bx)}{x^2 - y^2} \frac{\sin (a \cos bx)}{\cos (a \cos bx)} dx =$$

$$= \frac{\pi}{2y} \frac{\cos (a \cos by)}{\sin (a \cos by)} \operatorname{sh} (a \sin by) \quad [a, b, y > 0].$$

$$13. \int_0^{\infty} \frac{x \operatorname{sh} (a \sin bx)}{x^2 - y^2} \frac{\sin (a \cos bx)}{\cos (a \cos bx)} dx =$$

$$= \frac{\pi}{2} \left[\frac{\cos (a \cos by)}{\sin (a \cos by)} \operatorname{ch} (a \sin by) - \frac{1}{a \cos by} \right] \quad [a, b, y > 0].$$

$$14. \int_0^{\infty} \frac{\operatorname{ch} (a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} \frac{\sin (2a \cos^2 x)}{\cos (2a \cos^2 x)} dx = \frac{\pi}{2bc} \frac{\sin d}{\cos d}$$

$$[d = 2ac(b+c)^{-1}; a, b, c > 0].$$

$$15. \int_0^{\infty} \frac{\sin (a \sin x)}{\sin x} \sin (a \cos x) dx = \frac{\pi}{2} \sin a \quad [a > 0].$$

2.5.56. Интегралы, содержащие e^{-px} , $\operatorname{sh} ax$, $\operatorname{ch} ax$, $\sin bx$, $\cos bx$.

$$1. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} cx \sin cx}{\operatorname{ch} cx \cos cx} dx = \frac{p}{p^4 + 4c^4} \left(\frac{2c^2}{p^2} \right)$$

$$[Re p > |Re c| + |Im c|]$$

$$2. \int_0^{\infty} e^{-px} \frac{\operatorname{sh} cx \cos cx}{\operatorname{ch} cx \sin cx} dx = \frac{c(p^2 + 2c^2)}{p^4 + 4c^4}$$

$$[Re p > |Re c| + |Im c|]$$

$$3. \int_0^{\infty} e^{-px} \operatorname{sh}^{\mu} cx \frac{\sin bx}{\cos bx} dx =$$

$$= \frac{(-1)^{\delta} \Gamma(\mu+1)}{2^{\mu+1} \Gamma(c)} \left\{ \Gamma \left[\frac{(p-\mu c-ib)/(2c)}{(p+\mu c-ib)/(2c)+1} \right] + \Gamma \left[\frac{(p-\mu c+ib)/(2c)}{(p+\mu c+ib)/(2c)+1} \right] \right\}$$

$$\left[Re p > |Re(\mu c)| + |Im b|; Re \mu > -1-\delta, \delta = \begin{cases} 1 \\ 0 \end{cases} \right].$$

$$4. \int_0^{\infty} e^{-px} \frac{\sin bx}{\sin cx} dx = \frac{1}{2ci} \left[\Psi \left(\frac{p+ic+ib}{2c} \right) - \Psi \left(\frac{p+ic-ib}{2c} \right) \right]$$

$$[Re p > |Re c| + |Im b|].$$

$$5. \int_0^{\infty} e^{-cx} \frac{\sin bx}{\sin cx} dx = \frac{\pi}{2c} \operatorname{ch} \frac{b\pi}{2c} - \frac{1}{b}$$

$$[b, Re c > 0].$$

$$6. \int_0^{\infty} \frac{\operatorname{sh} ax \sin bx}{e^{cx} - 1} dx = -\frac{b}{2(a^2 + b^2)} + \frac{\pi}{2c} \frac{\operatorname{sh}(2b\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} +$$

$$+ \frac{i}{2c} \left[\Psi \left(\frac{a+ib}{c} + i \right) - \Psi \left(\frac{a-ib}{c} + i \right) \right] \quad [b > 0; Re c > |Re a|].$$

$$7. \int_0^{\infty} \frac{\operatorname{sh} ax \cos bx}{e^{cx} \pm 1} dx = \mp \frac{a}{2(a^2 + b^2)} \pm \frac{\pi}{\sigma c} \frac{\sin(a\sigma\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} \left\{ \frac{\operatorname{ch}(b\pi/c)}{1} \right\}$$

$$[\sigma = \begin{cases} 1 \\ 2 \end{cases}; b > 0; Re c > |Re a|].$$

$$8. \int_0^{\infty} \frac{\operatorname{ch} ax \sin bx}{e^{cx} \pm 1} dx = \pm \frac{b}{2(a^2 + b^2)} \mp \frac{\pi}{\sigma c} \frac{\operatorname{sh}(b\sigma\pi/c)}{\operatorname{ch}(2b\pi/c) - \cos(2a\pi/c)} \left\{ \frac{\cos(a\pi/c)}{1} \right\}$$

$$[\sigma = \begin{cases} 1 \\ 2 \end{cases}; b > 0; Re c > |Re a|].$$

2.5.57. Интегралы, содержащие e^{ix} , $\operatorname{sh} ax$, $\operatorname{ch} ax$, $\sin bx$, $\cos bx$.

$$1. \int_0^{\infty} e^{-px^2} \frac{\operatorname{sh} cx \sin bx}{\operatorname{ch} cx \cos bx} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp \left(-\frac{c^2 - b^2}{4p} \right) \left\{ \frac{\sin [bc/(2p)]}{\cos [bc/(2p)]} \right\}$$

$$[Re p > 0].$$

$$2. \int_0^{\infty} x^{2p-1} \frac{\operatorname{sh} cx \sin cx}{\operatorname{ch} cx \cos cx} dx = \frac{1}{4} \sqrt{\frac{\pi}{p^3}} \left[\frac{\sin d}{\cos d} \pm \frac{\cos d}{\sin d} \right]$$

$$[d = c^2/(2p); Re p > 0].$$

$$3. \int_0^{\infty} x^{-1/2} e^{-px^2} \frac{\operatorname{sh} cx \sin cx}{\operatorname{ch} cx \cos cx} dx = \frac{\pi}{2^{5/4} \sqrt{p}} \sqrt{\frac{c}{p}} J_{-1/4} \left(\frac{c^2}{4p} \right) \left\{ \frac{\cos [c^2/(4p) + \pi/4]}{\sin [c^2/(4p) + \pi/4]} \right\}$$

$$[Re p > 0].$$

$$4. \int_0^{\infty} x e^{-px^2} \left\{ \frac{\operatorname{ch} cx \sin bx}{\operatorname{sh} cx \cos bx} \right\} dx = \\ = \frac{\sqrt{\pi}}{8p^2} \sqrt{\frac{b^2+c^2}{p}} \exp \left(-\frac{b^2+c^2}{4p^2} \right) \left[c \left\{ \frac{\sin d}{\cos d} \right\} \pm b \left\{ \frac{\cos d}{\sin d} \right\} \right] \\ [d = bc(3p)^{-1/2} (b^2+c^2); \operatorname{Re} p > 0]$$

$$5. \int_0^{\infty} x e^{-px^2} \left\{ \frac{\operatorname{ch} cx \sin cx}{\operatorname{sh} cx \cos cx} \right\} dx = \frac{c}{4p} \sqrt{\frac{\pi}{p}} \left(\cos \frac{c^2}{2p} \pm \sin \frac{c^2}{2p} \right) \quad [p > 0]$$

$$6. \int_0^{\infty} x^{-1/2} e^{-px^2} \left\{ \frac{\operatorname{ch} cx \sin cx}{\operatorname{sh} cx \cos cx} \right\} dx = \pm \frac{\pi}{2^{5/4}} \sqrt{\frac{c}{p}} J_{1/4} \left(\frac{c^2}{4p} \right) \left\{ \cos [c^2/(4p) - \pi/4] \right\} \\ [\operatorname{Re} p > 0]$$

$$7. \int_0^{\infty} x^{2e} e^{-px^2} \operatorname{sh} px^2 \cos bx dx = \sqrt{\frac{\pi}{2}} \exp \left(-\frac{b^2}{8c} \right) - \frac{\pi b}{4} \operatorname{erfc} \left(\frac{b}{\sqrt{8c}} \right) \\ [b, \operatorname{Re} p > 0]$$

$$8. \int_0^{\infty} e^{-px^2} \operatorname{ch} cx \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx = \frac{\sqrt{\pi}}{2(p^2+b^2)^{1/4}} \exp \left[-\frac{p^2}{4(p^2+b^2)} \right] \left\{ \frac{\sin d}{\cos d} \right\} \\ \left[d = \frac{bc^2}{4(p^2+b^2)} + \frac{1}{2} \operatorname{arctg} \frac{b}{p}; b, \operatorname{Re} p > 0 \right]$$

$$9. \int_0^{\infty} x^{-1/2} e^{-px^2} (\operatorname{sh} cx \pm \sin cx) dx = \frac{\pi}{4} \sqrt{\frac{c}{p}} J_{1/4} \left(\frac{c^2}{8p} \right) \left\{ \frac{\operatorname{ch} [c^2/(8p)]}{\operatorname{sh} [c^2/(8p)]} \right\} \\ [\operatorname{Re} p > 0]$$

$$10. \int_0^{\infty} x^{-1/2} e^{-px^2} (\operatorname{ch} cx \pm \cos cx) dx = \frac{\pi}{4} \sqrt{\frac{c}{2p}} J_{1/4} \left(\frac{c^2}{8p} \right) \left\{ \frac{\operatorname{ch} [c^2/(8p)]}{\operatorname{sh} [c^2/(8p)]} \right\} \\ [\operatorname{Re} p > 0]$$

$$11. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} (x/2) \sin bx dx = \operatorname{Im} K_{1/2+ib}(a) \quad [a, b > 0]$$

$$12. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{ch} (x/2) \cos bx dx = \operatorname{Re} K_{1/2+ib}(a) \quad [a, b > 0]$$

$$13. \int_0^{\infty} e^{-px^2} \operatorname{sh} \left(\frac{x}{\operatorname{ch} c - \cos x} \right) dx = \pi e^{c^2 p} \left[\theta_3 \left(2cp, \frac{4p}{i\pi} \right) + \theta_3 \left(2cp, \frac{4p}{i\pi} \right) \right] - \frac{1}{2} \sqrt{\frac{\pi}{p}} \\ [\operatorname{Re} c, \operatorname{Re} p > 0]$$

$$\theta_3(u, x) = \frac{1}{\sqrt{\pi x}} \left[\sum_{k=0}^{\infty} e^{-(u+h)^2/x} - \sum_{h=-1}^{\infty} e^{-(v+h)^2/x} \right]$$

2.6. ЛОГАРИФИЧЕСКАЯ ФУНКЦИЯ

2.6.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входит логарифмическая функция. С помощью замены переменной они могут быть преобразованы к интегралам от показательной и степенной функций. Некоторые интегралы от логарифмов можно получить путем дифференцирования по параметру. Например,

$$\int_0^{\infty} x^{\alpha-1} \ln^{\alpha} (1+x) dx = \int_0^{\infty} x^{\alpha} e^x (e^x-1)^{\alpha-1} dx = \\ = \frac{\partial^{\alpha}}{\partial p^{\alpha}} \int_0^{\infty} x^{\alpha-1} (1+x)^p dx \Big|_{p=0} = \frac{\partial^{\alpha}}{\partial p^{\alpha}} \Gamma(\alpha, -p-\alpha) \Big|_{p=0} \quad [-n < \operatorname{Re} \alpha < 0]$$

2.6.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие логарифмическую, а также степенную, алгебраические, показательную и тригонометрические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Значения параметров, входящих в интегралы общего вида, приводятся в фигурных скобках после каждого интеграла; $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия.

$$1. \int_0^{\infty} x^{h-1} (y^x - x^y)^{\beta-1} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\ = \frac{c}{yd} y^{h+\beta-r-1} \sum_{n=0}^{p-1} \frac{1}{h+1} B \left(\frac{\alpha+h+1}{r}, \beta \right) \left(\pm \frac{cy}{d} \right)^h \chi_1 + \frac{y^{\alpha+r\beta-r}}{r} B \left(\frac{\alpha}{r}, \beta \right) \ln d \\ [|\operatorname{Im} h| \leq |d|]$$

$$2. = \frac{dy^{\alpha+\beta-r-1}}{xr} \sum_{h=0}^{p-1} \frac{1}{h+1} B \left(\frac{\alpha-h-1}{r}, \beta \right) \left(\pm \frac{d}{cy} \right)^h \chi_2 + \\ + y^{\alpha} (\beta-1) \left(\frac{d}{c} \right)^{\alpha} \pi \sum_{h=0}^{q-1} \frac{(1+\beta)_h}{h! (\alpha+hr)} \left\{ \operatorname{cosec} [(\alpha+hr)\pi] \right\} \left(\frac{d}{cy} \right)^{hr} \chi_3 + \\ + y^{\alpha+\beta-r-2\beta} B \left(\frac{\alpha}{r}, \beta \right) \left[\psi \ln(cy) + \psi \left(\frac{\alpha}{r} \right) - \psi \left(\frac{\alpha}{r} + \beta \right) \right] \quad [|\operatorname{Im} h| > |d|]$$

(Если $r > 0$ — произвольное, то $p=q=\infty$ и $\chi_j=1$, $j=1, 2, 3$; если $r > 0$ — рациональное ($r=p/q$) то

$$\chi_1 = {}_2F_1 \left(1, (h+1)/p, \Delta(q, (\alpha+h+1)r^{-1}); z \right), \\ \chi_2 = {}_2F_1 \left(1, (h+1)/p, \Delta(q, \beta + (\alpha+h+1)r^{-1}); z \right), \\ \chi_3 = {}_2F_1 \left(1, (h+1)/p, \Delta(q, 1 - \beta - r^{-1}(\alpha-h-1)); z \right), \\ \chi_2 = {}_2F_1 \left(1, \Delta(q, 1+h-\beta), (\alpha+hr)/p; z^{-1} \right), \quad z = \left(\pm \frac{cy}{d} \right)^p \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; \{ |\arg(cx+d)| < \pi \text{ при } 0 < x < y \}]$$

$$4. \int_0^{\infty} x e^{-p x^2} \left\{ \frac{\operatorname{ch} cx \sin bx}{\operatorname{sh} cx \cos bx} \right\} dx = \\ = \frac{\sqrt{\pi}}{2 p^{3/2}} \sqrt{\frac{b^2 + c^2}{p}} \exp \left(-\frac{b^2 + c^2}{4 p^2} \right) \left[c \left\{ \frac{\sin d}{\cos d} \right\} \pm b \left\{ \frac{\cos d}{\sin d} \right\} \right] \\ [d = bc(3p)^{-1/2} (b^2 + c^2); \operatorname{Re} p > 0].$$

$$5. \int_0^{\infty} x e^{-p x^2} \left\{ \frac{\operatorname{ch} cx \sin cx}{\operatorname{sh} cx \cos cx} \right\} dx = \frac{c}{4 p} \sqrt{\frac{\pi}{p}} \left(\cos \frac{c^2}{2p} \pm \sin \frac{c^2}{2p} \right) \quad [\operatorname{Re} p > 0].$$

$$6. \int_0^{\infty} x^{-1/2} e^{-p x^2} \left\{ \frac{\operatorname{ch} cx \sin cx}{\operatorname{sh} cx \cos cx} \right\} dx = \pm \frac{\pi}{2^{5/4}} \sqrt{\frac{c}{p}} J_{1/4} \left(\frac{c^2}{4p} \right) \left\{ \cos [c^2/(4p) - \pi/4] \right\} \\ [\operatorname{Re} p > 0].$$

$$7. \int_0^{\infty} x e^{-p x^2} \operatorname{sh} p x^2 \cos b x dx = \sqrt{\frac{\pi c}{2}} \exp \left(-\frac{b^2}{8c} \right) - \frac{\pi b}{4} \operatorname{erfc} \left(\frac{b}{\sqrt{8c}} \right) \\ [b, \operatorname{Re} c > 0].$$

$$8. \int_0^{\infty} e^{-p x^2} \operatorname{ch} cx \left\{ \frac{\sin bx^2}{\cos bx^2} \right\} dx = \frac{\sqrt{\pi}}{2(p^2 + b^2)^{1/4}} \exp \left[-\frac{p^2}{4(p^2 + b^2)} \right] \left\{ \frac{\sin d}{\cos d} \right\} \\ \left[d = \frac{bc^2}{4(p^2 + b^2)} + \frac{1}{2} \operatorname{arctg} \frac{b}{p}; b, \operatorname{Re} p > 0 \right].$$

$$9. \int_0^{\infty} x^{-1/2} e^{-p x^2} (\operatorname{sh} cx \pm \sin cx) dx = \frac{\pi}{4} \sqrt{\frac{c}{2p}} I_{1/4} \left(\frac{c^2}{8p} \right) \left\{ \frac{\operatorname{ch} [c^2/(8p)]}{\operatorname{sh} [c^2/(8p)]} \right\} \\ [\operatorname{Re} p > 0].$$

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$$11. \int_0^{\infty} e^{-a \operatorname{ch} x} \operatorname{sh} (x/2) \sin bx dx = \operatorname{Im} K_{1/2+ib}(a) \quad [a, b > 0].$$

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$$13. \int_0^{\infty} e^{-p x^2} \operatorname{sh} \left(\frac{c}{\operatorname{ch} c - \cos x} \right) dx = \pi e^{c^2 p} \left[\theta_3 \left(2cp, \frac{4p}{i\pi} \right) + \theta_3 \left(2cp, \frac{4p}{i\pi} \right) \right] - \frac{1}{2} \sqrt{\frac{\pi}{p}} \\ [\operatorname{Re} c, \operatorname{Re} p > 0].$$

$$\theta_3(u, x) = \frac{1}{\sqrt{x}} \left[\sum_{k=0}^{\infty} e^{-u^2 + k^2/x} - \sum_{k=-1}^{\infty} e^{-(u+k)^2/x} \right].$$

2.6. ЛОГАРИФИЧЕСКАЯ ФУНКЦИЯ

2.6.1. Введение.

В этом разделе содержатся интегралы, в подынтегральные выражения которых входит логарифмическая функция. С помощью замены переменной они могут быть преобразованы к интегралам от показательной и степенной функций. Некоторые интегралы от логарифмов можно получить путем дифференцирования по параметру. Например,

$$\int_0^{\infty} x^{\alpha-1} \ln^2(1+x) dx = \int_0^{\infty} x^{\alpha} e^x (e^x - 1)^{\alpha-1} dx = \\ = \frac{\partial^2}{\partial p^2} \int_0^{\infty} x^{\alpha-1} (1+x)^p dx \Big|_{p=0} = \frac{\partial^2}{\partial p^2} \Gamma(\alpha, -p-\alpha) \Big|_{p=0}. \quad [-n < \operatorname{Re} \alpha < 0].$$

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$$1. \int_0^{\infty} x^{\alpha-1} (y^x - x^y)^{\beta-1} \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = \\ = \frac{c}{rd} y^{\alpha+\beta-r-1} \sum_{h=0}^{p-1} \frac{1}{h+1} B \left(\frac{\alpha+h+1}{r}, \beta \right) \left(\mp \frac{cy}{d} \right)^h x_1 + \frac{(p+r-1-r)}{r} B \left(\frac{\alpha}{r}, \beta \right) \ln d \\ [|\operatorname{Im} \alpha| \leq |d|].$$

$$2. = \frac{dy^{\alpha+\beta-r-1}}{dx} \sum_{h=0}^{p-1} \frac{1}{h+1} B \left(\frac{\alpha-h-1}{r}, \beta \right) \left(\mp \frac{d}{cy} \right)^h x_2 + \\ + y^{\alpha+\beta-1} \left(\frac{d}{c} \right)^{\alpha} \sum_{h=0}^{q-1} \frac{(1+h)\beta}{h! (\alpha+hr)} \left\{ \operatorname{cosec} [(\alpha+hr)\pi] \left(\frac{d}{cy} \right)^{hr} x_3 + \right. \\ \left. + y^{\alpha+\beta-r-r-2} B \left(\frac{\alpha}{r}, \beta \right) \left[r \ln(cy) + \psi \left(\frac{\alpha}{r} \right) - \psi \left(\frac{\alpha}{r} + \beta \right) \right] \right\} \quad [|\operatorname{Im} \alpha| > |d|].$$

(Если $r > 1$ — произвольное, то $p = q = \infty$ и $y_1 = 1$, $j = 1, 2, 3$; если $r > 0$ — рациональное ($r = p/q$) то

$$x_1 = q + \frac{r}{q} q + 1 \left(\frac{1}{1+(h+1)/p}, \Delta(q, (\alpha+h+1)r^{-1}); z \right), \\ x_2 = q + \frac{r}{q} q + 1 \left(\frac{1}{1+(h+1)/p}, \Delta(q, \beta + (\alpha+h+1)r^{-1}); z^{-1} \right), \\ x_3 = q + \frac{r}{q} q + 1 \left(\frac{1}{1+(h+1)/p}, \Delta(q, 1 - \beta - r^{-1}(\alpha-h-1)); z^{-1} \right), \\ x_4 = q + \frac{r}{q} q + 1 \left(\frac{1}{\Delta(q, 1+h-\beta), (\alpha+hr)/p}; z^{-1} \right), \quad z = \left(\mp \frac{cy}{d} \right)^p. \\ \left\{ \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; \left\{ \begin{array}{l} |\arg(cx+d)| < \pi \text{ при } 0 < x < y \\ c, d > 0 \end{array} \right\} \right\}.$$

$$\begin{aligned}
 3. \int_0^{\infty} x^{\alpha-1} (x^r - y^r)^{\beta-1} \frac{\{\ln(cx+d)\}}{\{\ln cx-d\}} dx = \\
 = y^{r(\beta-1)} \left\{ \int_0^{y^{-1}} x^{r(1-\beta)-1} (y^r - x^r)^{\beta-1} \ln(dx+c) dx + \right. \\
 \left. + \frac{y^{\alpha}}{r^2} B\left(1-\beta-\frac{\alpha}{r}, \beta\right) \left[r \ln y - \psi\left(1-\beta-\frac{\alpha}{r}\right) + \psi\left(1-\frac{\alpha}{r}\right) \right] \right\} \\
 \left[0 < \operatorname{Re} \beta < 1 - \operatorname{Re} \alpha/r; \begin{cases} |\arg(cx+d)| < \pi \text{ при } x > y \\ \alpha, d > 0 \end{cases} \right]
 \end{aligned}$$

$$\begin{aligned}
 4. \int_0^{\infty} x^{\alpha-1} (x^r + y^r)^{\beta-1} \frac{\{\ln(cx+d)\}}{\{\ln cx-d\}} dx = \\
 = \frac{c}{rd} y^{r(\beta-1)} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha+h+1}{r}, 1-\rho-\frac{\alpha+h+1}{r}\right) \left(\pm \frac{cy}{d}\right)^h \chi_1 + \\
 + \left(\frac{c}{d}\right)^{r-\rho-\alpha} \pi \sum_{h=0}^{q-1} \frac{(-1)^h (1-\rho)_h}{h! (r-r\rho-\alpha+rh)} \frac{\{\operatorname{cosec}[(r-r\rho-\alpha+rh)\pi]\}}{\{\operatorname{ctg}[(r-r\rho-\alpha+rh)\pi]\}} \left(\frac{cy}{d}\right)^{rh} \chi_3 + \\
 + \frac{y^{r(\beta-1)}}{r} B\left(\frac{\alpha}{r}, 1-\rho-\frac{\alpha}{r}\right) \ln d \quad [cy \leq |d|]
 \end{aligned}$$

$$\begin{aligned}
 5. = y^{r(\beta-1)} \left(\frac{d}{c}\right)^{\alpha} \pi \sum_{h=0}^{q-1} \frac{(1-\rho)_h (-1)^h}{h! (\alpha+rh)} \frac{\{\operatorname{cosec}[(\alpha+rh)\pi]\}}{\{\operatorname{ctg}[(\alpha+rh)\pi]\}} \left(\frac{d}{cy}\right)^{rh} \chi_3 + \\
 + y^{r(\beta-1)+\alpha-1} \frac{d}{cr} \sum_{h=0}^{p-1} \frac{1}{h+1} B\left(\frac{\alpha-h-1}{r}, 1-\rho+\frac{h+1-\alpha}{r}\right) \left(\pm \frac{d}{cy}\right)^h \chi_1 + \\
 + r^{-2} y^{r(\beta-1)+\alpha} B\left(1-\rho-\frac{\alpha}{r}, \frac{\alpha}{r}\right) \left[r \ln(cy) + \psi\left(\frac{\alpha}{r}\right) - \psi\left(1-\rho-\frac{\alpha}{r}\right) \right] \\
 [cy > |d|]
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $\rho = q = \infty$ и $\chi_j = 1$, $j = 1, 2, 3, 4$; если $r > 0$ — рациональное ($r = p/q$), то

$$\begin{aligned}
 \chi_1 &= {}_{q+2}F_{q+1} \left(1, (h+1)/p, \Delta(q, (\alpha+h+1)r^{-1}); z \right), \\
 \chi_2 &= {}_{q+2}F_{q+1} \left(1, \Delta(q, 1-\rho+h), (1-\rho+h)q^{-1}-\alpha\rho^{-1}; z \right), \\
 \chi_3 &= {}_{q+2}F_{q+1} \left(1, \Delta(q, 1-\rho+h), (\alpha+rh)\rho^{-1}; z^{-1} \right), \\
 \chi_4 &= {}_{q+2}F_{q+1} \left(1, \Delta(q, 1-\rho+(1+h-\alpha)r^{-1}), (h+1)/p; z^{-1} \right), \\
 z &= (-1)^q \left(\pm \frac{cy}{d} \right)^{\alpha} \\
 [\operatorname{Re} z > 0, \operatorname{Re}(\alpha+rp) < r; r|\arg y| < \pi; \begin{cases} |\arg(cx+d)| < \pi \text{ при } 0 < x < \infty \\ \alpha, d > 0 \end{cases}]
 \end{aligned}$$

$$\begin{aligned}
 6. \int_0^{\infty} \frac{x^{\alpha-1}}{x^r - y^r} \frac{\{\ln(cx+d)\}}{\{\ln cx-d\}} dx = -\frac{\pi y^{\alpha-r+1}}{rd} \sum_{h=0}^{p-1} \frac{1}{h+1} \operatorname{ctg} \left(\frac{\alpha+h+1}{r} \pi \right) \times \\
 \times \left(\pm \frac{cy}{d} \right)^h \chi_1 - \pi^2 \left(\frac{c}{d} \right)^{r-\alpha} \sum_{h=0}^{q-1} \frac{1}{\alpha-r-rh} \frac{\{\operatorname{cosec}[(r+rh-\alpha)\pi]\}}{\{\operatorname{ctg}[(r+rh-\alpha)\pi]\}} \left(\frac{cy}{d} \right)^{rh} \chi_3 - \\
 - \frac{\pi}{r} y^{\alpha-r} \operatorname{ctg} \frac{\alpha\pi}{r} \ln d \quad [cy \leq |d|]
 \end{aligned}$$

$$\begin{aligned}
 7. = -\pi^2 y^{-r} \left(\frac{d}{c} \right)^{\alpha} \sum_{h=0}^{q-1} \frac{1}{\alpha+rh} \frac{\{\operatorname{cosec}[(\alpha+rh)\pi]\}}{\{\operatorname{ctg}[(\alpha+rh)\pi]\}} \left(\frac{d}{cy} \right)^{rh} \chi_3 + \\
 + \frac{\pi d y^{\alpha-r-1}}{rc} \sum_{h=0}^{p-1} \frac{1}{h+1} \operatorname{ctg} \left(\frac{h-\alpha}{r} \pi \right) \left(\pm \frac{d}{cy} \right)^h \chi_1 - \\
 - \pi r^{-2} y^{\alpha-r} \operatorname{ctg} \frac{\alpha\pi}{r} \left[r \ln(cy) - 2\pi \operatorname{cosec} \frac{2\alpha}{r} \pi \right] \quad [cy > |d|]
 \end{aligned}$$

{Если $r > 0$ — произвольное, то $\rho = q = \infty$ и $\chi_j = 1$, $j = 1, 2, 3, 4$; если $r > 0$ — рациональное ($r = p/q$), то

$$\begin{aligned}
 \chi_1 &= {}_2F_1(1, (h+1)p^{-1}; 1+(h+1)p^{-1}; z), \\
 \chi_2 &= {}_2F_1(1, (r+rh-\alpha)p^{-1}; 1+(r+rh-\alpha)p^{-1}; z), \\
 \chi_3 &= {}_2F_1(1, (\alpha+rh)p^{-1}; 1+(\alpha+rh)p^{-1}; z^{-1}), \\
 \chi_4 &= {}_2F_1(1, (h+1)p^{-1}; 1+(h+1)p^{-1}; z^{-1}), \quad z = (\pm cy/d)^{\alpha} \\
 [0 < \operatorname{Re} \alpha < r; y > 0; \begin{cases} |\arg(cx+d)| < \pi \text{ при } x > 0 \\ \alpha, d > 0 \end{cases}]
 \end{aligned}$$

$$\begin{aligned}
 8. \int_0^{\infty} x^{\alpha-1} \frac{(y^{r/2} + \sqrt{x^2 + y^2})^{\nu}}{(x^r + y^r)^{\rho}} \frac{\{\ln(cx+d)\}}{\{\ln cx-d\}} dx = U_{\rho} \\
 U_{\rho} = \frac{cy^{2(1+\nu/2-\rho)+1}}{dr} \sum_{h=0}^{p-1} \frac{2^{\nu+2(h+\alpha+1)/r}}{h+1} \left(\frac{\nu r}{\nu r + \alpha + h + 1} \right)^{1-2\rho} \times \\
 \times B\left(\frac{\alpha+h+1}{r}, 2\rho-\nu-\frac{2}{r}(\alpha+h+1)\right) \left(\pm \frac{cy}{d}\right)^h \chi_1 + \\
 + \nu^{1-2\rho} \pi \left(\frac{c}{d}\right)^{r(1-\nu)/2-\alpha} y^{r/2-\rho} \sum_{h=0}^{q-1} \frac{(-1)^h ((1-\nu)/2)_h ((1+\nu)/2)_h}{h! (3/2-2\rho)_h [r(1-\nu)/2+hr-\alpha]} \times \\
 \times \frac{\{\operatorname{cosec}[r(1-\nu)/2+hr-\alpha]\pi\}}{\{\operatorname{ctg}[r(1-\nu)/2+hr-\alpha]\pi\}} \left(\frac{cy}{d}\right)^{rh} \chi_3 + \\
 + \nu^{2\rho} \left(\frac{c}{d}\right)^{2\rho-\alpha-r\nu/2} y^{\rho} \sum_{h=0}^{q-1} \frac{(-1)^h (2\rho-\nu/2)_h (2\rho+\nu/2)_h}{h! (1/2+2\rho)_h [(2\rho-\nu/2+h)r-\alpha]} \times \\
 \times \frac{\{\operatorname{cosec}[(2\rho-\nu/2+h)r-\alpha]\pi\}}{\{\operatorname{ctg}[(2\rho-\nu/2+h)r-\alpha]\pi\}} \left(\frac{cy}{d}\right)^{rh} \chi_3 + \\
 + \frac{\ln d}{r} 2^{\nu+2\alpha/r} \left(\frac{\nu r}{\nu r + \alpha} \right)^{1-2\rho} y^{\alpha-2\rho+\nu/2} B\left(\frac{\alpha}{r}, 2\rho-\nu-\frac{2\alpha}{r}\right) \\
 [cy \leq |d|]
 \end{aligned}$$

$$\chi_2 = 2\tilde{F}_{2,2+1} \left(\begin{matrix} 1, (\alpha + h + r\delta) p^{-1}; \\ 1 + (\alpha + h + r\delta) p^{-1}, \Delta(q, h+1), \Delta(q, h+\delta+1/2) \end{matrix} \middle| z \right),$$

$$z = (-1)^r \left(-\frac{d}{c} \right)^p \left(\frac{b}{2q^2} \right)^q,$$

$$[-r\delta < \operatorname{Re} \alpha < r; b > 0; \left(\begin{matrix} \operatorname{arg}(cs+d) < \pi \text{ при } x > 0 \\ c, d > 0 \end{matrix} \right)].$$

$$12. \int_0^\infty x^{\alpha-1} \cos bx^r \left\{ \frac{\ln(cx+d)}{\ln|cx-d|} \right\} dx = U(\delta) \Big|_{\delta=0} \quad [U(\delta) \text{ см. в 2.6.2.11}].$$

2.6.3. Интегралы от $x^\alpha \ln^\sigma x$.

$$1. \int_0^a x^{\alpha-1} \ln^\sigma x \frac{a}{x} dx = a^\alpha \alpha^{-\sigma-1} \Gamma(\sigma+1) \quad [a, \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1].$$

$$2. \int_0^1 x^{\alpha-1} \ln^\sigma x [x - (-1)^\sigma \alpha^{-\sigma-1} n!] \quad [\operatorname{Re} \alpha > 0].$$

$$3. \int_0^{\infty} \frac{a \ln(1+bx) - b \ln(1+ax)}{x^2} dx = ab \ln \frac{a}{b} \quad [a, b > 0].$$

2.6.4. Интегралы от $\frac{x^\alpha}{(x^\mu + x^\nu)^\mu} \ln^\sigma x$.

$$1. \int_0^a \frac{x^{\alpha-1}}{(x^\mu + x^\nu)^\mu} \ln^\sigma x dx = a^{\alpha-\mu\sigma} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(1-\rho-k)_k}{k! (\alpha+\mu k)^{\sigma+1}} \quad [a, \mu, \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1].$$

$$2. \int_0^1 \frac{x^{\mu\rho/2-1} \ln x}{(x^\mu + x^\nu)^\mu} dx = \frac{\ln z}{\mu z^{1/\mu}} B\left(\frac{\rho}{2}, \frac{\rho}{2}\right) \quad [\mu, \operatorname{Re} \rho > 0; \mu |\arg z| < \pi].$$

$$3. \int_0^1 \frac{x^{\alpha-1} \ln x}{1+x^\mu} dx = J_\mu^\alpha \quad [\mu, \operatorname{Re} \alpha > 0].$$

$$J_\mu^\alpha = \frac{1}{\mu^\alpha} \beta'\left(\frac{\alpha}{\mu}\right), \quad J_\mu^\mu = \frac{(-1)^\alpha}{\mu^\alpha} \left[\frac{\pi^2}{12} + \sum_{k=1}^{n-1} \frac{(-1)^k}{k^3} \right],$$

$$J_\mu^{1/2} = -\frac{4}{\mu^2} G, \quad J_\mu^{1/2-1} - J_\mu^{1/2+1} = -\frac{\pi^2 \sin(\pi/\mu)}{\mu^2 \cos^2(\pi/\mu)}.$$

$$4. \int_0^1 \frac{x^{\alpha-1} \ln^\sigma x}{1+x^\mu} dx = J_\mu^{\alpha, \sigma} \quad [\mu, \operatorname{Re} \alpha > 0; \sigma = 1, 2, 3, \dots]$$

$$J_\mu^{\alpha, \sigma} = \frac{1}{\mu^{\sigma+1}} \beta^{(\sigma)}\left(\frac{\alpha}{\mu}\right), \quad J_\mu^{2l} = \frac{2^{2l}-1}{2^{2l}\mu^{2l+1}} (2l)! \zeta(2l+1),$$

$$J_\mu^{1/2-1} = \frac{1-2^{1/2-1}}{2^{1/2}\mu^{3/2}} \pi^{1/2} B_{2/2}, \quad J_\mu^{1/2, \sigma} = (-1)^\sigma \left(\frac{2}{\mu}\right)^{\sigma+1} n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\sigma+1}},$$

$$J_\mu^{1/2, 2l} = \left(\frac{\pi}{\mu}\right)^{2l+1} \frac{1}{2} |E_{2l}|, \quad J_\mu^{1/2, 2} = \frac{2}{\mu^3} \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3}, \quad J_\mu^{1/2, 2} = \frac{\pi^3}{2\mu^3},$$

$$J_\mu^{1/2, 3} + J_\mu^{3/2, 3} = \frac{\pi^3}{\mu^3} \pi^2, \quad J_\mu^{1/2, 3} = (-1)^n \mu^{-4} \left[\frac{7\pi^4}{120} + 6 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^4} \right],$$

$$J_\mu^{1/2, 4} = \frac{5}{24} \pi^4, \quad J_\mu^{1/2, 5} = -\frac{31}{252\mu^5} \pi^4, \quad J_\mu^{1/2, 6} = \frac{61}{2} \left(\frac{\pi}{\mu}\right)^7, \quad J_\mu^{1/2, 7} = -\frac{127}{240} \left(\frac{\pi}{\mu}\right)^8.$$

$$5. \int_0^1 \frac{x^{\alpha-1} \ln^\sigma x}{(1+x^\mu)^\mu} dx = J_\mu^{\alpha, \sigma} \quad [\mu, \operatorname{Re} \alpha > 0; \sigma = 1, 2, 3, \dots]$$

$$J_\mu^{1, 1} = -\frac{\ln 2}{\mu^2}, \quad J_\mu^{3/2, 1} = -\frac{4 \ln 2}{\mu^2},$$

$$J_\mu^{1, 2} = \frac{(-1)^n}{\mu^2} \left[\ln 2 - \frac{(n-1)\pi^2}{12} - \sum_{k=1}^{n-2} \frac{(-1)^k}{k^2} (n-k-1) \right], \quad J_\mu^{1, 2} = \frac{\pi^2}{6\mu^2},$$

$$J_\mu^{1/2, 2} = \frac{\pi^3}{2\mu^3}, \quad J_\mu^{1/2, 2} = \frac{(-1)^n}{\mu^3} \left[\frac{3}{2} (n-1) \zeta(3) + 2 \sum_{k=1}^{n-2} \frac{(-1)^k}{k^3} (n-k-1) - \frac{\pi^2}{6} \right],$$

$$J_\mu^{1/2, 3} + (-1)^n J_\mu^{1/2, 3} = \mu^{-4} \left[6 \sum_{k=1}^{n-2} (-1)^{k-1} \frac{n-k-1}{k^4} - \frac{7\pi^4}{120} (n-1) \right], \quad J_\mu^{1/2, 4} = \frac{7\pi^4}{30\mu^4}.$$

$$6. \int_0^\infty \frac{x^{\alpha-1} \ln^\sigma x}{(x^\mu + x^\nu)^\mu} dx = \frac{1}{\mu} \left(\frac{\partial}{\partial \alpha} \right)^\sigma \left[z^{\alpha-\mu\sigma} B\left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu}\right) \right] \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$7. \int_0^\infty \frac{x^{\alpha-1} \ln x}{(x^\mu + x^\nu)^\mu} dx = \frac{z^{\alpha-\mu}}{\mu^2} B\left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu}\right) \left[\mu \ln z + \psi\left(\frac{\alpha}{\mu}\right) - \psi\left(\rho - \frac{\alpha}{\mu}\right) \right] \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$8. \int_0^\infty \frac{x^{\alpha-1} \ln x}{(x^\mu + x^\nu)^\mu} dx = \frac{z^{\alpha-\mu} (1-\alpha/\mu)_{m-2} \pi}{(m-1)! \mu^2 \sin(\alpha\pi/\mu)} \left(\mu \ln z - \sum_{k=1}^{m-1} \frac{\mu}{k-\alpha} - \pi \cot \frac{\alpha\pi}{\mu} \right) \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu m; m = 1, 2, 3, \dots].$$

$$9. \int_0^\infty \frac{\ln x}{(x^\mu + x^\nu)^{m+1/2}} dx = \frac{2z^{1/2-m}}{2m-1} \left(\ln z + 2 \ln 2 - \sum_{k=1}^{m-2} \frac{1}{k} - 2 \sum_{k=m-1}^{2m-3} \frac{1}{k} \right) \quad [\arg z < \pi; m = 1, 2, 3, \dots].$$

$$10. \int_0^\infty \frac{x^{\alpha-1} \ln^2 x}{(x^\mu + x^\nu)^\mu} dx = \frac{z^{\alpha-\mu}}{\mu^3} B\left(\frac{\alpha}{\mu}, \rho - \frac{\alpha}{\mu}\right) \{ [\mu \ln z + \psi(\alpha/\mu) - \psi(\rho - \alpha/\mu)]^2 + \psi'(\alpha/\mu) + \psi'(\rho - \alpha/\mu) \} \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu \operatorname{Re} \rho].$$

$$11. \int_0^{\infty} \frac{x^{\mu/2-1} \ln^{2n} x}{(x^4+1)^{\mu}} dx = 2^{n-1} 2^{\mu-1} (2n)! \sum_{k=0}^{\infty} \frac{(1-\rho-k)_n}{k! (1+2k)^{n+1}} \quad [\mu, \operatorname{Re} \rho > 0].$$

$$12. \int_0^{\infty} \frac{x^{\mu/2-1} \ln^n x}{x^4+1} dx = 2^{n-1} \frac{1+(-1)^n}{2} n! \mu^{-n-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} \quad [\mu > 0].$$

$$13. \int_0^{\infty} \frac{x^{\mu-1} \ln^{2n+1} x}{(x^4+1)^2} dx = 0 \quad [\mu > 0].$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{x^4+x^4} dx = \frac{\pi}{\mu} \frac{\partial^n}{\partial \alpha^n} \frac{x^{\alpha} \mu}{\sin(\alpha\pi/\mu)} \quad [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu].$$

$$15. \int_0^{\infty} \frac{x^{\alpha-1} \ln^2 x}{x^4+2^{\mu}} dx = \frac{\pi 2^{2-\mu}}{\mu^2 \sin(\alpha\pi/\mu)} \left[\mu^2 \ln^2 z - 2\pi \mu \ln z \operatorname{ctg} \frac{\alpha\pi}{\mu} + \frac{\pi^2}{\sin^2(\alpha\pi/\mu)} \left(1 + \cos^2 \frac{\alpha\pi}{\mu} \right) \right] \\ [\mu > 0; \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu].$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^4+x^4)(x^2+z^2)} \ln \frac{x}{a} dx = 0 \quad [z > 0; \alpha - \mu < 2].$$

2.6.5. Интегралы от $\frac{x^{\alpha}}{(a^4-x^4)^{\mu}} \ln^{\sigma} x$.

$$1. \int_0^a \frac{x^{\alpha-1} \ln^n x}{(a^4-x^4)^{\mu-\beta}} dx = \frac{a^{4(\beta-1)}}{\mu} \frac{\partial^n}{\partial \alpha^n} \left[a^{\alpha} B \left(\beta, \frac{\alpha}{\mu} \right) \right] \\ [\alpha, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -n \text{ при } a=1)].$$

$$2. \int_0^a x^{\alpha-1} (a^4-x^4)^m \ln^n x dx = (-1)^n n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(\alpha+\mu k)^{n+1}} \quad [\alpha, \mu > 0].$$

$$3. \int_0^a x^{\alpha-1} (a^4-x^4)^m \ln^{\sigma} \frac{a}{x} dx = a^{2+\mu\sigma} \Gamma(\sigma+1) \sum_{k=0}^m \frac{(-1)^k (1+m-k)_n}{k! (\alpha+\mu k)^{\sigma+1}} \\ [\alpha, \mu, \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -m-1].$$

$$4. \int_0^a \frac{x^{\alpha-1} \ln x}{(a^4-x^4)^{1-\beta}} dx = \frac{a^{4(\beta-1)}}{\mu^2} B \left(\beta, \frac{\alpha}{\mu} \right) \left[\mu \ln a + \psi \left(\frac{\alpha}{\mu} \right) - \psi \left(\beta + \frac{\alpha}{\mu} \right) \right] \\ [\alpha, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -1 \text{ при } a=1)].$$

$$5. \int_0^a \frac{x^{\alpha-1} \ln^2 x}{(a^4-x^4)^{1-\beta}} dx = \frac{a^{4(\beta-1)}}{\mu^3} B \left(\beta, \frac{\alpha}{\mu} \right) \left\{ \left[\mu \ln a + \psi \left(\frac{\alpha}{\mu} \right) - \psi \left(\beta + \frac{\alpha}{\mu} \right) \right]^2 + \psi' \left(\frac{\alpha}{\mu} \right) - \psi' \left(\beta + \frac{\alpha}{\mu} \right) \right\} \\ [\alpha, \mu, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0 \text{ (Re } \beta > -2 \text{ при } a=1)].$$

$$6. \int_0^1 \frac{x^{\mu-1}}{z^{\mu}-x^{\mu}} \ln^{\sigma} \frac{1}{x} dx = \frac{\Gamma(\sigma+1)}{\mu^{\sigma+1}} \sum_{k=1}^{\infty} \frac{z^{\mu k}}{k^{\sigma+1}} \quad [|z| > 1; \mu > 0; \operatorname{Re} \sigma > -1].$$

$$7. \int_0^1 \frac{x^{\sigma-1}}{1-x^4} \ln^{\sigma} \frac{1}{x} dx = \Gamma(\sigma+1) \mu^{-\sigma-1} \zeta \left(\sigma+1, \frac{\sigma}{\mu} \right) \quad [\operatorname{Re} \alpha, \operatorname{Re} \sigma > 0].$$

$$8. \int_0^1 \frac{x^{\mu-1}}{1-x^4} \ln^{\sigma} \frac{1}{x} dx = \Gamma(\sigma+1) \mu^{-\sigma-1} \zeta(\sigma+1) \quad [\mu, \operatorname{Re} \sigma > 0].$$

$$9. \int_0^1 \frac{x^{\alpha-1}}{1-x^4} \ln^n x dx = I_{\mu}^{\alpha, n}, \quad I_{\mu}^{\alpha, n} = -\mu^{-n-1} \psi^{(n)} \left(\frac{\alpha}{\mu} \right) \\ [\mu, \operatorname{Re} \alpha > 0; n=1, 2, 3, \dots].$$

$$I_{\mu}^{\alpha, 1} = \frac{1}{\mu^2} \left(\sum_{k=1}^{n-1} \frac{1}{k^2} - \frac{\pi^2}{6} \right) \quad [n=1, 2, 3, \dots].$$

$$I_{\mu}^{\alpha, 1} + I_{\mu}^{\alpha-m, 1} = -\frac{\pi^2}{k^2 \sin^2(m\pi/k)} \quad [k > m].$$

$$I_{\mu}^{\mu/2, 1} = -\frac{\pi^2}{2\mu^3}, \quad I_{\mu}^{\mu/2-1, 1} + I_{\mu}^{\mu/2+1, 1} = -\frac{\pi^2}{\mu^2 \cos^2(\pi/\mu)}.$$

$$I_{\mu}^{\mu/2, 1} = \frac{4}{\mu^2} \left(1 - \frac{\pi^2}{8} \right), \quad I_{\mu}^{\mu/6, 2} - I_{\mu}^{\mu/3, 2} = \frac{8\sqrt{3}}{\mu^3} \pi^2.$$

$$I_{\mu}^{\mu(n+p), 2} = \frac{2}{\mu^3} \sum_{k=n}^{\infty} \frac{1}{(k+p)^2} \quad [p=1 \text{ или } 1/2].$$

$$I_{\mu}^{(n+1)\mu, 3} = \frac{1}{\mu^4} \left[6 \sum_{k=0}^{n-1} \frac{1}{(k+1)^4} - \frac{\pi^4}{15} \right], \quad I_{\mu}^{(n+1/2)\mu, 3} = \frac{1}{\mu^4} \left[96 \sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} - \pi^4 \right].$$

$$I_{\mu}^{\mu, 2} = -\frac{8}{63} \left(\frac{\pi}{\mu} \right)^6, \quad I_{\mu}^{\mu, 2} = -\frac{8}{15} \left(\frac{\pi}{\mu} \right)^8.$$

$$10. \int_a^1 \frac{x^{\mu-1}}{1-x^4} \ln^{2n-1} x dx = (-1)^n \frac{(2n)!}{4n\mu^{2n}} B_{2n} \quad [\mu > 0; n=1, 2, 3, \dots].$$

$$11. \int_0^1 \frac{x^{\mu/2-1}}{1-x^4} \ln^{2n-1} x dx = \left(\frac{2n}{\mu} \right)^{2n} \frac{1-2^{2n}}{4n} B_{2n} \quad [\mu > 0; n=1, 2, 3, \dots].$$

$$12. \int_0^1 \frac{x^{\mu/2-1}}{1-x^4} \ln^{2n} x dx = (2n)! \mu^{-2n-1} (2^{2n}-1) \zeta(2n+1) \quad [\mu > 0; n=1, 2, 3, \dots].$$

$$13. \int_0^1 \frac{x^{\alpha-1}}{(1-x)^{\alpha}} \ln^n x \, dx = J_{\mu}^{\alpha, n} \quad [\mu, \operatorname{Re} \alpha > 0; n = 1, 2, 3, \dots]$$

$$J_{\mu}^{\alpha, 1} - J_{\mu}^{\alpha+1, 1} = \mu^{-2} \left[\sum_{k=1}^{n-2} \frac{n-k-1}{k^2} - \frac{\pi^2}{6} (n-1) \right],$$

$$J_{\mu}^{\alpha, 2} - J_{\mu}^{\alpha+1/2, 2} = \mu^{-2} \left[4 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^2} - \frac{\pi^2}{2} \right],$$

$$J_{\mu}^{\alpha, 3} = \frac{2\pi^2}{3\mu^3}, \quad J_{\mu}^{\alpha+1, 3} = \frac{2}{\mu^3} \left(\frac{\pi^2}{3} - \pi^2(3) + \sum_{k=1}^{n-1} \frac{n-k}{k^3} \right),$$

$$J_{\mu}^{\alpha, 3} - J_{\mu}^{\alpha+1, 3} = \frac{1}{\mu^4} \left(6 \sum_{k=1}^{n-1} \frac{n-k}{k^3} - \frac{n\pi^4}{15} \right),$$

$$J_{\mu}^{\alpha, 2, 3} - J_{\mu}^{\alpha+1/2, 3} = \frac{1}{\mu^4} \left(96 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^4} - n\pi^4 \right),$$

$$J_{\mu}^{\alpha, 4} = \frac{4\pi^4}{15\mu^4}, \quad J_{\mu}^{\alpha+2, 2m} + J_{\mu}^{\alpha+2, 2m} = \left(\frac{2\pi}{\mu} \right)^{2m} \frac{2^{2m}-1}{\mu} |B_{2m}|,$$

$$14. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{(x^2-1)^m} dx = \frac{(-1)^m \pi}{\mu (m-1)!} \frac{\partial^m}{\partial x^m} \left[\left(1 - \frac{\alpha}{\mu} \right)_{m-1} \operatorname{ctg} \frac{\alpha\pi}{\mu} \right] \\ [0 < \operatorname{Re} \alpha < \mu m; n \geq m; m, n = 1, 2, 3, \dots]$$

$$15. \int_0^{\infty} \frac{x^{1/2-1} \ln^n x}{x^2-1} dx = \frac{2^{n+1}-1}{(n+1)\mu^{n+1}} \pi^{n+1} |B_{n+1}| (1 - (-1)^n) \quad [\mu > 0]$$

$$16. \int_0^{\infty} \frac{x^{\alpha-1} \ln^n x}{x^2-y^2} dx = -\frac{\pi}{\mu} \frac{\partial^n}{\partial x^n} \left(y^{\alpha-\mu} \operatorname{ctg} \frac{\alpha\pi}{\mu} \right) \quad [\mu, y > 0; 0 < \operatorname{Re} \alpha < \mu]$$

$$17. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{x^2-y^2} dx = \frac{\pi y^{\alpha-\mu}}{\mu^2} \left[\frac{\pi}{\sin^2(\alpha\pi/\mu)} - \mu \ln y \operatorname{ctg} \frac{\alpha\pi}{\mu} \right] \\ [\mu, y > 0; 0 < \operatorname{Re} \alpha < \mu]$$

$$18. \int_0^{\infty} \frac{x^{\alpha-1} \ln^2 x}{x^2-y^2} dx = \frac{\pi y^{\alpha-\mu}}{\mu^3} \left[\frac{2\pi}{\sin^2(\alpha\pi/\mu)} \left(\mu \ln y - \pi \operatorname{ctg} \frac{\alpha\pi}{\mu} \right) - \mu^2 \ln^2 y \operatorname{ctg} \frac{\alpha\pi}{\mu} \right] \\ [\mu, y > 0; 0 < \operatorname{Re} \alpha < \mu]$$

2.6.6. Интегралы от $\frac{x^{\alpha} \ln^{\sigma} x}{ax^2+bx+c}$.

$$1. \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} = \frac{\ln(ab)}{2(b-a)} \ln \frac{(a+b)^2}{4ab} \quad [0 < a < b]$$

$$2. \int_a^b \frac{\ln^2 x}{(x+1)^2} dx = \frac{1}{1-2\pi} \sum_{k=0}^{n-1} \binom{2n+1}{2k} (2i)^{2n-2k} \int_a^b \frac{\ln^{2n-2k} x}{(x+1)^2} dx \\ [a=0, b=\infty, \text{ или } a=0, b=1, \text{ или } a=1, b=\infty].$$

$$3. \int_0^{\infty} \frac{\ln x}{x^2+a^2} dx = \frac{\pi}{2a} \ln a - \frac{\pi}{a} \quad [a > 0].$$

$$4. \int_0^1 \frac{x^{\alpha-1}}{x^2-2x \cos \gamma + a^2} \ln^{\sigma} \frac{1}{x} dx = \frac{\Gamma(\sigma+1)}{\sin \gamma} \sum_{k=1}^{\infty} \frac{\sin(k\gamma) a^{-k-1}}{(\alpha+k-1)^{\sigma+1}} \\ [a > 1; 0 < \gamma < 2\pi; \operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1].$$

$$5. \int_0^1 \frac{x^{\alpha-1} \ln^{\sigma} x}{x^2-2x \cos \gamma + a^2} dx = \pi \cos \gamma \frac{\partial^{\sigma}}{\partial \alpha^{\sigma}} \left\{ a^{\alpha} \frac{\operatorname{sh}[(\alpha-1)(\gamma-\pi)]}{\sin \alpha\pi} \right\} \\ [a > 0; 0 < \gamma < 2\pi; \operatorname{Re} \alpha > 0].$$

$$6. \int_0^1 \frac{\ln^2 x}{x^2-2x \cos \gamma + 1} dx = \frac{\gamma(\gamma-2\pi)(\gamma-\pi)}{6 \sin \gamma} \quad [0 < \gamma < 2\pi].$$

$$7. \int_0^1 \frac{\ln^4 x}{x^2-2x \cos \gamma + 1} dx = \frac{\gamma(\gamma-2\pi)}{30 \sin \gamma} (4\pi^2 - 3\gamma^2 + 6\gamma\pi) \quad [0 < \gamma < 2\pi].$$

$$8. \int_0^1 \frac{x^k \ln^{\sigma} x}{x^2 \pm x - 1} dx = J_{\pm}^{k, \sigma}, \\ J_{-}^{0,1} = -0,7813024129 \dots, \quad J_{-}^{0,1} = -1,17195361935 \dots, \\ J_{-}^{1,1} = -0,15766014915 \dots, \quad J_{-}^{1,1} = -0,3118211319 \dots, \\ J_{+}^{0,2} = \frac{8\pi^2}{81\sqrt{3}}, \quad J_{-}^{0,2} = \frac{10\pi^2}{81\sqrt{3}}.$$

$$9. \int_0^1 \frac{(y^{\alpha}-x^{\alpha}) \ln x}{x^2-2x \cos \gamma + 1} dx = \frac{\pi \sin[(\gamma-\pi)\alpha]}{\sin \gamma \sin \alpha\pi} \{ \pi \operatorname{ctg} \alpha\pi - (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] \} \\ [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1].$$

$$10. \int_0^1 \frac{(x^{\alpha}+x^{-\alpha}) \ln^2 x}{x^2-2x \cos \gamma + 1} dx = \\ = \frac{\pi \sin[(\gamma-\pi)\alpha]}{\sin \gamma \sin \alpha\pi} \{ \gamma^2 - 2\gamma\pi + 2\pi \operatorname{ctg} \alpha\pi \{ (\gamma-\pi) \operatorname{ctg} [\alpha(\gamma-\pi)] - \pi \operatorname{ctg} \alpha\pi \} \} \\ [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1].$$

$$11. \int_0^1 x^{\alpha-1} \left(1 + \frac{x \ln x}{1-x} \right) \frac{dx}{1-x} = \alpha \psi'(\alpha) - 1 \quad [\operatorname{Re} \alpha > 0].$$

$$12. \int_0^1 \left(1 + \frac{x \ln x}{1-x} \right) \frac{dx}{1-x} = \frac{\pi^2}{6} - 1.$$

$$13. \int_0^1 x^{\alpha-1} \left[\frac{1+(x-1) \ln x}{1-x} + \frac{x \ln x}{(1-x)^2} \right] dx = \Psi'(\alpha) - 1 \quad [\operatorname{Re} \alpha > 0]$$

$$14. \int_0^{\infty} \frac{\ln^p x}{x(x-z)} dx = \frac{\Gamma(\sigma+1)}{z} \operatorname{Li}_{\sigma+1}(z) \quad [|\arg(1-z)| < \pi, \operatorname{Re} \sigma > -1 \text{ или } |z|=1, \operatorname{Re} \sigma > 0]$$

$$15. \int_0^{\infty} \frac{\ln x}{x(x-y)} dx = \frac{1}{y} \left[\frac{\pi^2}{3} - \operatorname{Li}_3\left(\frac{1}{y}\right) - \frac{1}{2} \ln^2 y \right] \quad [y > 1]$$

$$16. \int_0^{\infty} \frac{\ln^2 x}{x(x-y)} dx = \frac{1}{y} \left[\operatorname{Li}_3\left(\frac{1}{y}\right) - \frac{1}{6} \ln^3 y + \frac{\pi^2}{3} \ln y \right] \quad [y > 1]$$

$$17. \int_0^{\infty} \frac{\ln^a x dx}{(x+a)(x+b)} = I_n \quad [I_m a = I_m b = 0]$$

$$I_0 = \frac{1}{a-b} \ln \left| \frac{a}{b} \right|,$$

$$I_1 = \frac{1}{2(a-b)} (\ln^2 |a| - \ln^2 |b| + A \operatorname{sgn} a) \quad [A=0 \text{ при } ab > 0; A=\pi^2 \text{ при } ab < 0]$$

$$I_2 = \frac{1}{3(a-b)} (\ln^3 |a| - \ln^3 |b| + B \pi^2) \quad [B = \ln |a/b| \text{ при } a, b > 0; B = \ln(ab^2) \text{ при } a > 0, b < 0; B = -\ln(a^2 b) \text{ при } a < 0, b > 0; B = 2 \ln |b/a| \text{ при } a, b < 0]$$

$$18. \int_0^{\infty} \frac{x^{\alpha-1} \ln^a x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{\pi}{\sin \gamma} \frac{\partial^a}{\partial \alpha^a} \left\{ a^{\alpha-1} \frac{\sin[(\alpha-1)(\gamma-\pi)]}{\sin \alpha \pi} \right\} \quad [a > 0; 0 < \gamma < 2\pi, 0 < \operatorname{Re} \alpha < 2]$$

$$19. \int_0^{\infty} \frac{x^{\alpha-1} \ln x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{\pi a^{\alpha-1}}{\sin \gamma \sin^2 \alpha \pi} \{ \ln a \sin \alpha \pi \sin[(\alpha-1)(\gamma-\pi)] + \gamma \sin \alpha \pi \cos[(\alpha-1)(\gamma-\pi)] + \pi \sin[\gamma(\alpha-1)] \} \quad [a > 0; 0 < \gamma < 2\pi; 0 < \operatorname{Re} \alpha < 2, \alpha \neq 1]$$

$$20. \int_0^{\infty} \frac{\ln x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{(\pi-\gamma) \ln a}{a \sin \gamma} \quad [a > 0; 0 < \gamma < 2\pi]$$

$$21. \int_0^{\infty} \frac{(x^a - x^{-a}) \ln x}{x^2 - 2x \cos \gamma + 1} dx = \frac{2\pi \sin[\alpha(\gamma-\pi)]}{\sin \gamma \sin \alpha \pi} \{ \pi \operatorname{ctg} \alpha \pi - (\gamma-\pi) \operatorname{ctg}[\alpha(\gamma-\pi)] \} \quad [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1/2]$$

$$22. \int_0^{\infty} \frac{(x^2 + x^{-2}) \ln x}{x^2 - 2x \cos \gamma + 1} dx = \frac{2\pi \sin[\alpha(\gamma-\pi)]}{\sin \gamma \sin \alpha \pi} \{ \gamma^2 - 2\gamma\pi + 2\pi \operatorname{ctg} \alpha \pi \{ (\gamma-\pi) \operatorname{ctg}[\alpha(\gamma-\pi)] - \pi \operatorname{ctg} \alpha \pi \} \} \quad [0 < \gamma < 2\pi; |\operatorname{Re} \alpha| < 1/2]$$

$$23. \int_0^{\infty} \frac{\ln^2 x}{x^2 - 2ax \cos \gamma + a^2} dx = \frac{\pi-\gamma}{a \sin \gamma} \left(\ln^2 a + \frac{4}{3} \pi^2 \right) \quad [a > 0; 0 < \gamma < 2\pi]$$

$$24. \int_0^{\infty} \frac{\ln^{2n-1} x}{x^2 + bx + 1} dx = 0 \quad [|b| < 2; n=1, 2, 3, \dots]$$

2.6.7. Интегралы от $x^\alpha (a^2 - x^2)^p \ln^n x$.

$$1. \int_0^x x^{1/n-1} (a^2 - x^2)^{-1/n} \ln x dx = \frac{2 \ln a - \pi \operatorname{ctg}[\pi/(2n)]}{8a^{1/n} \cos[\pi/(2n)]} B\left(\frac{1}{2n}, \frac{1}{2n}\right) \quad [a > 0; n=2, 3, 4, \dots]$$

$$2. \int_0^x x^2 (a^2 - x^2)^{n/2-1} \ln x dx = Ba^{n+1} \frac{(n-2)!!}{(n+1)!!} \left[\ln a + \frac{n}{n+1} - \sum_{k=1}^{[n/2]} \frac{1}{2k-r} - (1-r) \ln 2 \right] \quad [a > 0; B=\pi/2, r=0 \text{ при } n=1, 3, 5, \dots; B=1, r=1 \text{ при } n=2, 4, 6, \dots]$$

$$3. \int_0^x \frac{x^m \ln x}{\sqrt{a^2 - x^2}} dx = Aa^m \frac{(m-1)!!}{m!!} \left[(-1)^{m+1} \ln 2 + \ln a + (-1)^{m+1} \sum_{k=1}^m \frac{(-1)^k}{k} \right] \quad [a > 0; A=\pi/2 \text{ при } m=0, 2, 4, \dots; A=1 \text{ при } m=1, 3, 5, \dots]$$

$$4. \int_0^a x^m \sqrt{a^2 - x^2} \ln x dx = Aa^{m+2} \frac{(m-1)!!}{(m+2)!!} \left[\ln a + (-1)^{m+1} \ln 2 + (-1)^{m+1} \sum_{k=1}^m \frac{(-1)^k}{k} - \frac{1}{m+2} \right] \quad [a > 0; A=\pi/2 \text{ при } m=0, 2, 4, \dots; A=1 \text{ при } m=1, 3, 5, \dots]$$

$$5. \int_0^a (a^2 - x^2)^{n/2-1} \ln x dx = Ba^{n-1} \frac{(n-2)!!}{(n-1)!!} \left[\ln a - \sum_{k=1}^{[n/2]} \frac{1}{2k-r} - (1-r) \ln 2 \right] \quad [a > 0; B=\pi/2, r=0 \text{ при } n=1, 3, 5, \dots; B=1, r=1 \text{ при } n=2, 4, 6, \dots]$$

$$6. \int_0^a \frac{x^m \ln^2 x}{\sqrt{a^2 - x^2}} dx = Aa^m \frac{m!!}{(m+1)!!} \left\{ \left[\ln(2a) + \sum_{k=1}^m \frac{(-1)^k}{k} \right]^2 + (-1)^m \left[\frac{\pi^2}{12} + \sum_{k=1}^m \frac{(-1)^k}{k^2} \right] \right\} \quad [a > 0; A=\pi/2 \text{ при } m=0, 2, 4, \dots; A=1 \text{ при } m=1, 3, 5, \dots]$$

$$7. \int_0^1 \frac{\ln x}{\sqrt{2-x^2}} dx = -\frac{\pi}{8} \ln 2 - \frac{G}{2}$$

2.6.8. Интегралы от $A(x) \ln^n x$.

1. $\int_0^1 \frac{\ln x dx}{(x^2 - a^2)(b^2 - x^2)} = \frac{1}{2b} K\left(\frac{b^2 - a^2}{b}\right) \ln(ab)$ [0 < a < b].
2. $\int_0^a \frac{\ln x dx}{(a^2 - x^2)(x^2 + z^2)} = \frac{1}{2b} \left[K\left(\frac{a}{b}\right) \ln(ab) - \frac{\pi}{2} K\left(\frac{z}{b}\right) \right]$ [a, Re z > 0].
3. $\int_0^x \frac{\ln x dx}{(a^2 - x^2)(b^2 - x^2)} = \frac{1}{2b} \left[K\left(\frac{a}{b}\right) \ln(ab) - \frac{\pi}{2} K\left(\frac{\sqrt{b^2 - a^2}}{b}\right) \right]$ [0 < a < b].
4. $\int_0^1 \frac{1+x}{1-x} \ln x dx = 1 - \frac{\pi^2}{3}$. 5. $\int_0^1 \frac{1+x}{1-x^2} \ln x dx = -\frac{3+\pi^2}{27}$.
6. $\int_0^1 \frac{\ln x}{(x^2 + z^2)\sqrt{1-x^2}} dx = \frac{\pi}{2z\sqrt{z^2+1}} \ln \frac{z}{z+\sqrt{z^2+1}}$ [Re z > 0].
7. $\int_0^1 \frac{x^2 \ln x dx}{(1-x^2)(1+x^2)} = -\frac{\pi^2}{16(2+\sqrt{2})}$.
8. $\int_0^1 \left(\frac{1+x}{1-x}\right)^p \frac{\ln x}{1-x^2} dx = \frac{1}{2p} \left[\psi(-p) - \psi\left(-\frac{p}{2}\right) \right]$ [0 < Re p < 1].
9. $\int_0^\infty x^{1/2} \frac{(x^{1/2} \pm \sqrt{x^2-1})^v}{\sqrt{x^2-1}} \ln x dx = \pm \frac{2}{v\mu^2} \left[\psi\left(\frac{v}{4}\right) - \psi\left(\frac{2+v}{4}\right) \right] - \frac{4}{\mu^2 v^2}$ [± Re v < 0; μ > 0].
10. $\int_0^\infty \frac{\ln x dx}{(x^2 + z^2)(x^2 + y^2)} = \frac{\ln(z/y)}{2z} K\left(\frac{\sqrt{z^2 - y^2}}{z}\right)$ [μ > y > 0].
11. $\int_0^\infty x^{2r-1} \frac{(\sqrt{x^2+1} \pm 1)^v}{(x^2+1)^r} \ln^n x dx =$
 $= \frac{(v-1)^{2r}}{\mu} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[2^{2r/2} \Gamma\left(\frac{v}{2} + \frac{v}{2} + \alpha/\mu, 2r - v - 2\alpha/\mu\right) \right]$
[r=0 или 1/2; μ > 0; μ Re(±v-v)/2 < Re α < μr - μ Re v/2].
12. $\int_0^\infty x^{2r-1} \frac{(\sqrt{x^2+1} \pm 1)^v}{(x^2+1)^r} \ln x dx =$
 $= \frac{(v-1)^{2r}}{\mu^2} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[\frac{\alpha/\mu + (v \mp v)/2, 2r - v - 2\alpha/\mu}{1 - \alpha/\mu - (v \pm v)/2} \right] \times$
 $\times \left[2 \ln 2 + \psi\left(\frac{\alpha}{\mu} + \frac{v \mp v}{2}\right) + \psi\left(1 - \frac{\alpha}{\mu} - \frac{v \pm v}{2}\right) - 2\psi\left(2r - v - \frac{2\alpha}{\mu}\right) \right]$
[r=0 или 1/2; μ > 0; μ Re(±v-v)/2 < Re α < μr - μ Re v/2].

13. $\int_0^\infty x^{2r-1} \frac{(\sqrt{x^2+1} \pm x^{1/2})^v}{(x^2+1)^r} \ln^n x dx =$
 $= \frac{(v-1)^{2r}}{\mu} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[\frac{2\alpha/\mu, r \mp v/2 - \alpha/\mu}{1 - r + \alpha/\mu \mp v/2} \right]$
[r=0 или 1/2; μ > 0; 0 < Re α < μr - μ Re v/2 + μr].
14. $\int_0^\infty x^{2r-1} \frac{(\sqrt{x^2+1} \pm x^{1/2})^v}{(x^2+1)^r} \ln x dx =$
 $= \frac{(v-1)^{2r}}{\mu^2} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[\frac{2\alpha/\mu, r \mp v/2 - \alpha/\mu}{1 - r + \alpha/\mu \mp v/2} \right] - 2 \ln 2 + 2\psi\left(\frac{2\alpha}{\mu}\right) -$
 $- \psi\left(1 - r + \frac{\alpha}{\mu} \mp \frac{v}{2}\right) + \psi\left(r + \frac{v}{2} - \frac{\alpha}{\mu}\right)$
[r=0 или 1/2; μ > 0; 0 < Re α < μr - μ Re v/2].
15. $\int_0^1 x^{2r-1} \frac{(1 + \sqrt{1-x^2})^v + (-1)^{2r-1} (1 - \sqrt{1-x^2})^v}{(1-x^2)^r} \ln^n x dx =$
 $= \frac{2^v}{\mu} v^{1-2r} \frac{\partial^r}{\partial x^r} \left[2^{2r/2} \Gamma\left(\frac{\alpha/\mu, v + \alpha/\mu}{2\alpha/\mu + v + 1 - 2r}\right) \right]$
[r=0 или 1/2; μ, Re α, Re(α+μv) > 0].
16. $\int_0^1 x^{2r-1} \frac{(1 + \sqrt{1-x^2})^v + (-1)^{2r-1} (1 - \sqrt{1-x^2})^v}{(1-x^2)^r} \ln x dx =$
 $= \frac{2^{v+\alpha/\mu}}{\mu^2} v^{1-2r} \Gamma\left(\frac{\alpha/\mu, v + \alpha/\mu}{2\alpha/\mu + v + 1 - 2r}\right) \left\{ \ln 2 + \psi\left(\frac{\alpha}{\mu}\right) + \right.$
 $\left. + \psi\left(v + \frac{\alpha}{\mu}\right) - 2\psi\left(\frac{2\alpha}{\mu} + v + 1 - 2r\right) \right\}$
[r=0 или 1/2; μ, Re α, Re(α+μv) > 0].
17. $\int_0^1 x^{2r-1} \frac{(\sqrt{1+x^2} + \sqrt{1-x^2})^v + (-1)^{2r-1} (\sqrt{1+x^2} - \sqrt{1-x^2})^v}{(1-x^2)^r} \ln^n x dx =$
 $= \frac{v^{1-2r}}{\mu} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[\frac{\alpha/(2\mu), \alpha/(2\mu) + v/2}{\alpha/\mu + v/2 + 1 - 2r} \right]$
[r=0 или 1/2; μ, Re α, Re(α+μv) > 0].
18. $\int_0^1 x^{2r-1} \frac{(\sqrt{1+x^2} + \sqrt{1-x^2})^v + (-1)^{2r-1} (\sqrt{1+x^2} - \sqrt{1-x^2})^v}{(1-x^2)^r} \ln x dx =$
 $= \frac{v^{1-2r}}{\mu^2} 2^{v+2r-2} \frac{\partial^r}{\partial x^r} \left[\frac{\alpha/(2\mu), \alpha/(2\mu) + v/2}{\alpha/\mu + v/2 + 1 - 2r} \right] \left\{ 2 \ln 2 + \right.$
 $\left. + \psi\left(\frac{\alpha}{2\mu}\right) + \psi\left(\frac{\alpha}{2\mu} + \frac{v}{2}\right) - 2\psi\left(\frac{\alpha}{\mu} + \frac{v}{2} + 1 - 2r\right) \right\}$
[r=0 или 1/2; μ, Re α, Re(α+μv) > 0].

2.6.9. Интегралы от $x^\alpha \ln^n(ax+b)$.

$$1. \int_0^a x^{\alpha-1} \ln^n(ax) dx = a^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k)!} \frac{\partial^k}{\partial \rho^k} \left(a^\rho \Gamma \left[\frac{k-\rho}{\rho} \right] \right)_{\rho=0} \\ [a > 0; \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -n \text{ при } a=1)]$$

$$2. \int_0^a x^{\alpha-1} \ln(ax) dx = a^\alpha \alpha^{-1} [\ln(2a) - \beta(\alpha+1)] \\ [a > 0; \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -1 \text{ при } a=1)]$$

$$3. \int_0^a x^{\alpha-1} \ln^n(ax) dx = \frac{a^\alpha}{n} \left[(-1)^{n-1} \sum_{k=1}^n \frac{(-1)^k}{k} + \ln a + \frac{1-(-1)^n}{2} \ln 4 \right] \\ [a > 0]$$

$$4. \int_0^a x^{\alpha-1/2} \ln(ax) dx = \frac{a^{\alpha+1/2}}{2\alpha+1} \left[2 \ln(2a) + (-1)^n \left(\pi - 4 \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right) \right] \\ [a > 0]$$

$$5. \int_0^a x^{\alpha-1} \ln^n(a-x) dx = a^\alpha \frac{\partial^n}{\partial \beta^n} [a^\beta B(\alpha, \beta+1)]_{\beta=1} \\ [a, \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -n \text{ при } a=1)]$$

$$6. \int_0^1 \frac{1}{x} \ln^n(1-x) dx = (-1)^n n! \zeta(n+1).$$

$$7. \int_0^a x^{\alpha-1} \ln(a-x) dx = \alpha^{-1} a^\alpha [\ln a - C - \psi(\alpha+1)] \\ [a, \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -1 \text{ при } a=1)]$$

$$8. \int_0^1 x^n \ln(1 \pm x) dx = \frac{(-1)^{n+1}}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} + \frac{1-(-1)^{n+1}}{n+1} \ln 2.$$

$$9. \int_0^1 \frac{\ln(1 \pm x)}{x} dx = \frac{\pi^2}{24} (-1 \pm 3), \quad 10. \int_0^{1/2} \frac{\ln(1-x)}{x} dx = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.$$

$$11. \int_0^1 x^{-3/2} \ln(1 \pm x) dx = \pi \left\{ \frac{1}{0} \right\} - \left\{ \frac{2}{4} \right\} \ln 2.$$

$$12. \int_0^a x^{\alpha-1} \ln^2(ax) dx = \\ = a^\alpha \alpha^{-1} \left\{ 2 (\ln a - C) [\ln 2 - \beta(\alpha+1)] + \ln^2 a + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k(\alpha+k)} \right\} \\ [a > 0; \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -2 \text{ при } a=1)]$$

$$13. \int_0^a x^{\alpha-1} \ln^2(a-x) dx = a^\alpha \alpha^{-1} \{ [\ln a - C - \psi(1+\alpha)]^2 + \pi^2/6 - \psi'(1+\alpha) \} \\ [a > 0; \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -2 \text{ при } a=1)]$$

$$14. \int_0^1 \frac{\ln^2(1+x)}{x} dx = 2 \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} - \frac{\pi^2 C}{6}.$$

$$15. \int_0^1 \frac{\ln^2(1-x)}{x} dx = 2\pi^2/3. \quad 16. \int_0^1 \frac{\ln^2(1-x)}{x^2} dx = \frac{\pi^2}{3}.$$

$$17. \int_1^{n+1} \ln[x] dx = \ln(n!).$$

$$18. \int_0^a x^{\alpha-1} \ln^n(x+a) dx = a^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \rho^k} \left(\Gamma \left[\frac{k-\rho}{\rho} \right] \frac{a^\rho}{k-\alpha-\rho} \right)_{\rho=0} \\ [a > 0; \operatorname{Re} \alpha < 0]$$

$$19. \int_0^a x^{\alpha-1} \ln(x \pm a) dx = \mp a^\alpha \alpha^{-1} \left\{ \frac{\beta(-\alpha) + \ln(2a)}{\psi(-\alpha) - \ln a + C} \right\} \\ [a > 0; \operatorname{Re} \alpha < 0]$$

$$20. \int_a^\infty x^{-n-1} \ln(x+a) dx = \frac{a^{-n}}{n} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k} + \ln a + \frac{1-(-1)^n}{2} \ln 4 \right] \\ [a > 0]$$

$$21. \int_0^\infty x^{\alpha-1} \ln|1 \pm x| dx = \frac{\pi}{\alpha \sin \alpha \pi} \left\{ \frac{1}{\cos \alpha \pi} \right\} \\ [-1 < \operatorname{Re} \alpha < 0]$$

$$22. \int_a^\infty x^{-(n+3/2)} \ln(x+a) dx = \frac{a^{-n-1/2}}{2n+1} \left[2 \ln(2a) + (-1)^n \left(\pi - 4 \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \right) \right] \\ [a > 0]$$

$$23. \int_a^\infty x^{\alpha-1} \ln^2(x+a) dx = \alpha^{-1} a^\alpha \left\{ 2 \left(C + \frac{1}{\alpha} - \ln a \right) [\ln 2 + \beta(-\alpha)] + \right. \\ \left. + \frac{2}{\alpha} C - \ln^2 a - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-\alpha)^2} + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k(k-\alpha)} \right\} \\ [a > 0; \operatorname{Re} \alpha < 0]$$

$$24. \int_a^\infty x^{\alpha-1} \ln^2(x-a) dx = -\alpha^{-1} a^\alpha \left\{ [C + \psi(-\alpha) - \ln a]^2 + \frac{\pi^2}{6} + \psi'(-\alpha) \right\} \\ [a > 0; \operatorname{Re} \alpha < 0]$$

2.6.10. Интегралы от $x^\alpha(ax+b)^\beta \ln^n(cx+d)$.

$$1. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\alpha} \ln^n(ax) dx = (-1)^n a^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k)!} \frac{\partial^k}{\partial \rho^k} \left(a^\rho \Gamma \left[\frac{\rho+k}{\rho} \right] \right) \\ [a > 0; \operatorname{Re} \alpha > 0 \text{ (Re } \alpha > -n \text{ при } a=1)]$$

$$2. \int_0^a \frac{1}{(a+x)^\rho} \ln^n(a+x) dx = (-1)^n \frac{\partial^n}{\partial \rho^n} \left(a^{1-\rho} \frac{2^{1-\rho}-1}{1-\rho} \right) \quad [a > 0].$$

$$3. \int_0^1 (1 \pm x)^{\beta-1} \ln^n(1 \pm x) dx = \mp \frac{(-1)^n n!}{\beta+1} + (1 \pm 1) n! 2^{\beta-1} \sum_{k=0}^n \frac{(-1)^k (\ln 2)^{n-k}}{(n-k)! \beta^{k+1}} \quad [Re \beta > 0 \text{ для } \ln^n(1 \pm x)].$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln(a+x) dx = a^{\alpha-\rho} \beta(\alpha) [\ln a + \psi(\rho)] - a^{2-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k \beta(\rho)_k}{(\alpha+k) k!} \psi(\rho+k) \quad [a > 0; Re \alpha > 0 (Re \alpha > -1 \text{ при } a=1)].$$

$$5. \int_0^a \frac{\ln(a+x)}{(a+x)^\rho} dx = -\frac{a^{1-\rho}}{(1-\rho)^2} \{ (2^{1-\rho}-1) [1 + (\rho-1) \ln a] + (\rho-1) 2^{1-\rho} \ln 2 \} \quad [a > 0; \rho \neq 1].$$

$$6. \int_0^a \frac{\ln(a+x)}{a+x} dx = \ln 2 \ln(2a) \quad [a > 0].$$

$$7. \int_0^a x^{-1/2} \frac{\ln(a+x)}{a+x} dx = 2a^{-1/2} \left(\frac{\pi}{2} \ln 2 + \frac{\pi}{4} \ln a - G \right) \quad [a > 0].$$

$$8. \int_0^1 \frac{\ln(1+x)}{1+x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}.$$

$$9. \int_0^1 \frac{\ln(1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi(\rho) - \psi(\rho+k)].$$

$$10. \int_0^1 \frac{\ln(1+x)}{(1+x)^4} dx = -\frac{1}{24} \left(\ln 2 + \frac{2}{9} \right).$$

$$11. \int_0^1 x^2 \frac{\ln(1+x)}{(1+x)^4} dx = \frac{1}{24} \left(\frac{25}{9} - 7 \ln 2 \right).$$

$$12. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln^2(a+x) dx = a^{2-\rho} \beta(\alpha) [(\ln a + \psi(\rho))^2 - \psi'(\rho)] + a^{\alpha-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k) k!} [\psi^2(\rho+k) + \psi'(\rho+k) - 2\psi(\rho+k) (\ln a + \psi(\rho))] \quad [a > 0; Re \alpha > 0 (Re \alpha > -2 \text{ при } a=1)].$$

$$13. \int_0^1 \frac{\ln^2(1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi^2(\rho+k) - \psi'(\rho+k) - 2\psi(\rho+k) \psi(\rho)].$$

$$14. \int_0^1 \frac{\ln^3(1+x)}{x^2(1+x)^\rho} dx = \left\{ \psi^3(\rho) - \psi'(\rho) + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)!} [\psi^3(\rho+k+2) + \psi'(\rho+k+2) - 2\psi(\rho+k+2) \psi(\rho)] \right\}.$$

$$15. \int_0^1 \frac{\ln(1-x)}{x(1+x)^\rho} dx = \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} (C + \psi(2+k)) - \frac{\pi^2}{6}.$$

$$16. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln(1+x) dx = \frac{1}{\beta} {}_3F_2(1, 1, 1; 2, \beta+1; -1) \quad [Re \beta > 0].$$

$$17. \int_0^1 \frac{\ln^2(1-x)}{x(1+x)^\rho} dx = \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} [\psi'(k+2) - \psi(k+2) (\psi(k+2) + 2C) - C^2 - \pi^2/6].$$

$$18. \int_0^1 \frac{\ln^3(1-x)}{x^2(1+x)^\rho} dx = a^{1-\rho} \left\{ \rho \psi'(1) + \frac{\pi^2}{3} + \rho(\rho+1) \sum_{k=0}^{\infty} \frac{(\rho+2)_k (-1)^k}{(k+2)(k+1)^2 k!} \left[\frac{\pi^2}{6} - \psi'(2+k) + (C + \psi(2+k))^2 \right] \right\}.$$

$$19. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln^2(1+x) dx = -2 \left[\sum_{k=0}^{\infty} \frac{(-1)^{k+1} k! \psi(k+1)}{(1+\beta)_k (k+1)} - C {}_3F_2(1, 1, 1; 2, 1+\beta; -1) \right] \quad [Re \beta > 0].$$

$$20. \int_0^1 \frac{(1-x)^{\beta-1}}{x^2} \ln^2(1+x) dx = \frac{2}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(\beta+1)_k (k+2)} (C + \psi(k+2)) \quad [Re \beta > 0].$$

$$21. \int_0^1 \frac{\ln(1 \pm x)}{\sqrt{x}} \frac{dx}{1+x} = \frac{3 \pm 1}{4} \pi \ln 2 - 2G.$$

$$2. \int_0^a \frac{1}{(a+x)^\rho} \ln^a(a+x) dx = (-1)^a \frac{\partial^a}{\partial \rho^a} \left(\frac{a^{1-\rho} - 1}{1-\rho} \right) \quad [a > 0]$$

$$3. \int_0^1 (1 \pm x)^{\beta-1} \ln^a(1 \pm x) dx = \mp \frac{(-1)^a a!}{\beta^{a+1}} + (1 \pm 1) n! 2^{\beta-1} \sum_{k=0}^n \frac{(-1)^k (1 \pm 2)^{a-k}}{(n-k)! \beta^{k+2}} \quad [\operatorname{Re} \beta > 0 \text{ для } \ln^a(1 \pm x)]$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln(a+x) dx =$$

$$= a^{\alpha-\rho} \beta(\alpha) [\ln a + \psi(\rho)] - a^{\alpha-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k (\rho)_k}{(\alpha+k)! k!} \psi(\rho+k) \quad [a > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -1 \text{ при } a=1)]$$

$$5. \int_0^a \frac{\ln(a+x)}{(a+x)^\rho} dx =$$

$$= -\frac{a^{1-\rho}}{(1-\rho)^2} \{ (2^{1-\rho} - 1) [1 + (\rho-1) \ln a] + (\rho-1) 2^{1-\rho} \ln 2 \} \quad [a > 0; \rho \neq 1]$$

$$6. \int_0^a \frac{\ln(a+x)}{a+x} dx = \ln 2 \ln(2a) \quad [a > 0]$$

$$7. \int_0^a x^{-1/2} \frac{\ln(a+x)}{a+x} dx = 2a^{-1/2} \left(\frac{\pi}{2} \ln 2 + \frac{\pi}{4} \ln a - G \right) \quad [a > 0]$$

$$8. \int_0^1 \frac{\ln(1+x)}{1+x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}$$

$$9. \int_0^1 \frac{\ln(1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi(\rho) - \psi(\rho+k)]$$

$$10. \int_0^1 \frac{\ln(1+x)}{(1+x)^4} dx = -\frac{1}{24} \left(\ln 2 + \frac{2}{9} \right)$$

$$11. \int_0^1 x^2 \frac{\ln(1+x)}{(1+x)^4} dx = \frac{1}{24} \left(\frac{25}{9} - 7 \ln 2 \right)$$

$$12. \int_0^a \frac{x^{\alpha-1}}{(a+x)^\rho} \ln^2(a+x) dx = a^{\alpha-\rho} \beta(\alpha) [(\ln a + \psi(\rho))^2 - \psi'(\rho)] +$$

$$+ a^{\alpha-\rho} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha+k)! k!} [\psi^2(\rho+k) + \psi'(\rho+k) - 2\psi(\rho+k) (\ln a + \psi(\rho))] \quad [a > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -2 \text{ при } a=1)]$$

$$13. \int_0^1 \frac{\ln^2(1+x)}{(1+x)^\rho} \frac{dx}{x} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} [\psi^2(\rho+k) - \psi'(\rho+k) - 2\psi(\rho+k) \psi(\rho)]$$

$$14. \int_0^1 \frac{\ln^2(1+x)}{x^2(1+x)^\rho} dx = \left\{ \psi^2(\rho) - \psi'(\rho) + \right.$$

$$\left. + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)!} [\psi^2(\rho+k+2) + \psi'(\rho+k+2) - 2\psi(\rho+k+2) \psi(\rho)] \right\}$$

$$15. \int_0^1 \frac{\ln(1-x)}{x(1+x)^\rho} dx = \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} (C + \psi(2+k)) - \frac{\pi^2}{6}$$

$$16. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln(1+x) dx = \frac{1}{\beta} {}_2F_2(1, 1, 1; 2, \beta+1; -1) \quad [\operatorname{Re} \beta > 0]$$

$$17. \int_0^1 \frac{\ln^2(1-x)}{x(1+x)^\rho} dx =$$

$$= \rho \sum_{k=0}^{\infty} \frac{(\rho+1)_k (-1)^k}{(k+1)^2 k!} [\psi'(k+2) - \psi(k+2) (\psi(k+2) + 2C) - C^2 - \pi^2/6]$$

$$18. \int_0^1 \frac{\ln^2(1-x)}{x^2(1+x)^\rho} dx = a^{-1-\rho} \left\{ \rho \psi'(1) + \frac{\pi^2}{3} + \right.$$

$$\left. + \rho(\rho+1) \sum_{k=0}^{\infty} \frac{(\rho+2)_k (-1)^k}{(k+2)(k+1)^2 k!} \left[\frac{\pi^2}{6} - \psi'(2+k) + (C + \psi(2+k))^2 \right] \right\}$$

$$19. \int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln^2(1+x) dx =$$

$$= \frac{2}{\beta} \left[\sum_{k=0}^{\infty} \frac{(-1)^{k+1} k! \psi(k+1)}{(1+\beta)_k (k+1)} - C {}_2F_2(1, 1, 1; 2, 1+\beta; -1) \right] \quad [\operatorname{Re} \beta > 0]$$

$$20. \int_0^1 \frac{(1-x)^{\beta-1}}{x^2} \ln^2(1+x) dx = \frac{2}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(\beta+1)_k (k+2)} [C + \psi(k+2)] \quad [\operatorname{Re} \beta > 0]$$

$$21. \int_0^1 \frac{\ln(1 \pm x)}{\sqrt{x}} \frac{dx}{1+x} = \frac{3 \pm 1}{4} \pi \ln 2 - 2G$$

22. $\int_0^a x^{\alpha-1} \frac{\ln^a (a-x)}{(cx+d)^\beta} dx =$
 $= d^{-\beta} a^\alpha \frac{\partial^\alpha}{\partial \beta^\alpha} \left[a^\beta B(\alpha, \beta+1) {}_2F_1 \left(\rho, \alpha; \alpha+\beta+1; -\frac{ac}{d} \right) \right]_{\beta=\alpha}$
 $[a, \operatorname{Re} \alpha > 0, |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a]$
23. $\int_0^a x^{\alpha-1} \frac{\ln(a-x)}{(cx+d)^\beta} dx = \frac{d^{-\beta} a^\alpha}{\alpha} {}_2F_1 \left(\rho, \alpha; \alpha+1; -\frac{ac}{d} \right) (\ln a - C) -$
 $- d^{-\beta} a^\alpha \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\alpha+k) k!} \left(-\frac{ac}{d} \right)^k \psi(\alpha+1+k)$
 $[a, \operatorname{Re} \alpha > 0, |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a, |ac| < |d|]$
24. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln^a(a-x) dx = a^{\alpha-1} \frac{\partial^\alpha}{\partial \beta^\alpha} [a^\beta B(\alpha, \beta)]$
 $[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -n \text{ при } a=1)]$
25. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln(a-x) dx = a^{\alpha+\beta-1} B(\alpha, \beta) [\ln a + \psi(\beta) - \psi(\alpha+\beta)]$
 $[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -1 \text{ при } a=1)]$
26. $\int_0^1 \frac{(1-x)^{\beta-1}}{x} \ln(1-x) dx = -\psi'(\beta)$ $[\operatorname{Re} \beta > 0]$
27. $\int_0^a \frac{x^{\alpha-1}}{(a-x)^{1-\beta}} \ln^2(a-x) dx =$
 $= \frac{a^{\alpha+\beta-1}}{\beta} B(\beta, \alpha) \{ [\psi(\beta) - \psi(\beta+\alpha) + \ln a]^2 + \psi'(\beta) - \psi'(\beta+\alpha) \}$
 $[a, \operatorname{Re} \beta > 0; \operatorname{Re} \alpha > 0 (\operatorname{Re} \alpha > -2 \text{ при } a=1)]$
28. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln^a(cx+d) dx =$
 $= a^{\alpha+\beta-1} B(\alpha, \beta) \frac{\partial^\alpha}{\partial \beta^\alpha} \left[{}_2F_1 \left(-\rho, \alpha; \alpha+\beta; -\frac{ac}{d} \right) \right]_{\beta=\alpha}$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a]$
29. $\int_0^a (a-x)^{\beta-1} \ln(cx+d) dx = \int_0^a x^{\beta-1} \ln(a+d-cx) dx$
30. $\int_0^1 \ln \frac{1+x}{2} \frac{dx}{1-x} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$
31. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \ln(cx+d) dx = a^{\alpha+\beta-1} B(\alpha, \beta) \ln J +$
 $+ d^{-1} a^{\alpha+\beta} B(\alpha+1, \beta) {}_3F_2(\alpha+1, 1, 1; 2, \alpha+\beta+1; -ac/d)$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a]$

32. $\int_0^a \frac{\ln |(1-b)x/a+b|}{\sqrt{x}} \frac{dx}{a-x} = -\frac{2}{\sqrt{a}} \arccos^2 \sqrt{\frac{b}{a}}$ $[a > 0; 0 < b \leq a]$
33. $\int_0^a \frac{\ln(a-cx)}{\sqrt{x} \sqrt{a-x}} dx = 2\pi \ln \frac{1+\sqrt{1-c}}{2} + \pi \ln a$ $[0 < c \leq 1]$
34. $\int_0^a \frac{\ln(d+cx)}{\sqrt{a-x} \sqrt{x}} dx = \pi a \ln \frac{d+\sqrt{d(d+ac)}}{2} + \frac{\pi a}{2} \frac{\sqrt{d}-\sqrt{d+ac}}{\sqrt{d}+\sqrt{d+ac}}$
 $[a, b, c > 0]$
35. $\int_0^1 \frac{\ln(1-cx)}{\sqrt{x} \sqrt{1-x}} dx = 2\pi (\sqrt{1-c}-1)$ $[0 < c < 1]$
36. $\int_0^1 \frac{\ln(1-cx)}{x \sqrt{1-x}} dx = \frac{\pi^2}{2} - 2 \arccos^2 \sqrt{c}$ $[0 < c \leq 1]$
37. $= \pi^2/2 + 2 \ln(\sqrt{c} + \sqrt{c-1})$ $[c \geq 1]$
38. $\int_0^a x^{\alpha-1} \frac{\ln^a(cx+d)}{(cx+d)^\beta} dx = (-1)^a \frac{\partial^\alpha}{\partial \beta^\alpha} \left[\frac{a^{\alpha-\beta} c^{-\beta}}{\rho-\alpha} {}_2F_1 \left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac} \right) \right]$
 $[\operatorname{Re}(\alpha-\rho) < 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x < \infty]$
39. $\int_0^a x^{\alpha-1} \frac{\ln(cx+d)}{(cx+d)^\beta} dx =$
 $= \frac{a^{\alpha-\beta} c^{-\beta}}{\rho-\alpha} \left\{ [\ln(ac) + \psi(\rho)] {}_2F_1 \left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac} \right) - \right.$
 $\left. - (\rho-\alpha) \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\rho-\alpha+k) k!} \left(-\frac{d}{ac} \right)^k \left[\psi(\rho+k) - \frac{1}{\rho-\alpha+k} \right] \right\}$
 $[\operatorname{Re}(\alpha-\rho) < 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x < \infty, |d| < |ac|]$
40. $\int_1^{\infty} \frac{\ln(x+1)}{\sqrt{x(x+1)}} dx = \frac{3+\pi}{4} \pi \ln 2 + 2G$
41. $\int_0^{\infty} x^{\alpha-1} \frac{\ln^a(x-a)}{(cx+d)^\beta} dx =$
 $= a^{\alpha-\beta} c^{-\beta} \frac{\partial^\alpha}{\partial \beta^\alpha} \left[a^\beta B(\beta+1, \rho-\alpha-\beta) {}_2F_1 \left(\rho, \rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac} \right) \right]_{\beta=\alpha}$
 $[\operatorname{Re}(\alpha-\rho) < 0; a > 0, |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty]$
42. $\int_a^{\infty} x^{\alpha-1} \frac{\ln(x-a)}{(cx+d)^\beta} dx = \frac{a^{\alpha-\beta} c^{-\beta}}{\rho-\alpha} \left\{ (\ln a - C) {}_2F_1 \left(\rho, \rho-\alpha; 1+\rho-\alpha; -\frac{d}{ac} \right) - \right.$
 $\left. - (\rho-\alpha) \sum_{k=0}^{\infty} \frac{(\rho)_k}{(\rho-\alpha+k) k!} \left(-\frac{d}{ac} \right)^k \psi(\rho-\alpha+k) \right\}$
 $[\operatorname{Re}(\alpha-\rho) < 0; a > 0, |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty, |d| < |ac|]$

$$43. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln^n (x-a) dx = a^{\alpha-1} \frac{\partial^n}{\partial \beta^n} [a^{\beta} B(\beta, 1-\alpha-\beta)]$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1$].

$$44. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln (x-a) dx =$$

$$= a^{\alpha+\beta-1} B(\beta, 1-\alpha-\beta) [\ln a + \psi(\beta) - \psi(1-\alpha-\beta)]$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1$].

$$45. \int_a^{\infty} x^{\alpha-1} \frac{\ln^2 (x-a)}{(x-a)^{1-\beta}} dx =$$

$$= a^{\alpha+\beta-1} B(1-\alpha-\beta, \beta) \{ [\psi(\beta) - \psi(1-\alpha-\beta) + \ln a]^2 + \psi'(\beta) + \psi'(1-\alpha-\beta) \}$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1$].

$$46. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln^n (cx+d) dx = a^{\alpha+\beta-1} \frac{\partial^n}{\partial \rho^n} \times$$

$$\times [(ac)^{\rho} B(\beta, 1-\alpha-\beta-\rho) {}_3F_1(-\rho, 1-\alpha-\beta-\rho; 1-\alpha-\rho; -d/(ac))]_{\rho=0}$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1; |\arg(cx+d)| < \pi$ при $a \leq x < \infty$].

$$47. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln (cx+d) dx =$$

$$= a^{\alpha+\beta-1} B(\beta, 1-\alpha-\beta) [\ln(ac) + \psi(1-\alpha) - \psi(1-\alpha-\beta)] +$$

$$+ d c^{-1} a^{\alpha+\beta-2} B(\beta, 2-\alpha-\beta) {}_3F_2(2-\alpha-\beta, 1, 1; 2, 2-\alpha; -d/(ac))$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1; |\arg(cx+d)| < \pi$ при $a \leq x < \infty$].

$$48. \int_0^{\infty} \frac{x^{\alpha-1}}{(cx+d)^{\rho}} \ln^n (cx+d) dx = (-1)^n \left(\frac{d}{c}\right)^{\alpha} \frac{\partial^n}{\partial \rho^n} [d^{-\rho} B(\alpha, \rho-\alpha)]$$

[$\operatorname{Re} \alpha, \operatorname{Re}(\rho-\alpha) > 0; |\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$49. \int_0^{\infty} \frac{x^{\alpha-1}}{(cx+d)^{\rho}} \ln (cx+d) dx = \frac{d^{\alpha-\rho}}{c^{\alpha}} B(\alpha, \rho-\alpha) [\ln d + \psi(\rho) - \psi(\rho-\alpha)]$$

[$\operatorname{Re} \alpha, \operatorname{Re}(\rho-\alpha) > 0; |\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$50. \int_0^{\infty} \frac{x^{\alpha-1}}{cx+d} \ln (cx+d) dx = \frac{d^{\alpha-1} \pi}{c^{\alpha} \sin \alpha \pi} [\ln d - C - \psi(1-\alpha)]$$

[$0 < \operatorname{Re} \alpha < 1; |\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$51. \int_0^{\infty} \frac{x^{\alpha-1}}{cx+d} \ln^2 (cx+d) dx = \frac{d^{\alpha-1} \pi}{c^{\alpha} \sin \alpha \pi} \{ [\ln d - C - \psi(1-\alpha)]^2 - \pi^2/6 + \psi'(1-\alpha) \}$$

[$0 < \operatorname{Re} \alpha < 1; |\arg(cx+d)| < \pi$ при $0 \leq x < \infty$].

$$52. \int_0^{\infty} \frac{\ln(x+1)}{(x+1)^{\rho}} \frac{dx}{x} = \psi'(\rho)$$

[$\operatorname{Re} \rho > 0$].

$$53. \int_0^{\infty} x^{\alpha-1} \frac{\ln^2 (x+a)}{(x+a)^{\rho}} dx =$$

$$= a^{\alpha-\rho} B(\alpha, \rho-\alpha) \{ [\psi(\rho) - \psi(\rho-\alpha) + \ln a]^2 - \psi'(\rho) + \psi'(\rho-\alpha) \}$$

[$|\arg a| < \pi; \operatorname{Re}(\alpha-\rho) < 0; \operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -2$ при $a=1$)].

$$54. \int_0^{\infty} \frac{\ln^2 (1+x)}{x(1+x)^{\rho}} dx = -\psi'(\rho)$$

[$\operatorname{Re} \rho > 0$].

$$55. \int_0^{\infty} \frac{\ln^2 (1+x)}{x^2 (1+x)^{\rho}} dx = 2\psi'(\rho+1) + \rho\psi''(\rho+1)$$

[$\operatorname{Re} \rho > 1$].

$$56. \int_0^{\infty} x^{\alpha-1} \frac{\ln (x+b)}{(x+a)^{\rho}} dx = a^{\alpha-\rho} B(\alpha, \rho-\alpha) [\ln b + \psi(\rho) - \psi(\rho-\alpha) -$$

$$- \frac{\alpha}{\rho} (1-ab^{-1}) {}_3F_2(\alpha+1, 1, 1; 2, \rho+1; 1-ab^{-1})]$$

[$\operatorname{Re}(\alpha-\rho) < 0; |\arg a|, |\arg b| < \pi; \operatorname{Re} \alpha > 0$ ($\operatorname{Re} \alpha > -1$ при $b=1$)].

$$57. \int_0^{\infty} \frac{\ln (x+1)}{x(x+a)^{\rho}} dx = \frac{a^{-\rho}}{\rho} [\rho\psi'(\rho) - (1-a) {}_3F_2(1, 1, 1; 2, \rho+1; 1-a)]$$

[$\operatorname{Re} \rho > 0; |\arg a| < \pi$].

$$58. \int_0^{\infty} \frac{x^{\alpha-1}}{(ax+b)^{\rho}} \ln^n (cx+d) dx =$$

$$= (-1)^n \left(\frac{d}{c}\right)^{\alpha} b^{-\sigma} \frac{\partial^n}{\partial \rho^n} [d^{-\rho} B(\alpha, \rho+\sigma-\alpha) {}_2F_1(\sigma, \alpha; \sigma+\rho; 1-ad/(bc))]_{\rho=0}$$

[$\operatorname{Re}(\alpha-\sigma) < 0, \operatorname{Re} \alpha > 0; |\arg(cx+d)|, |\arg(ax+b)| < \pi$ при $0 \leq x < \infty$].

$$59. \int_a^{\infty} x^{\alpha-1} (x-a)^{\beta-1} \ln (x+a) dx =$$

$$= a^{\alpha+\beta-1} \{ B(\beta, 1-\alpha-\beta) [\ln a + \psi(1-\alpha) - \psi(1-\alpha-\beta)] +$$

$$+ B(\beta, 2-\alpha-\beta) {}_3F_2(2-\alpha-\beta, 1, 1; 2-\alpha, 2; -1) \}$$

[$a, \operatorname{Re} \beta > 0; \operatorname{Re}(\alpha+\beta) < 1$].

$$60. \int_0^{\infty} \frac{x^{\alpha-1}}{(ax+b)^{\sigma}} \ln (cx+d) dx = \left(\frac{b}{a}\right)^{\alpha} b^{-\sigma} B(\alpha, \sigma-\alpha) \{ [\ln d - \psi(\sigma-\alpha)] +$$

$$+ \left(\frac{ad}{bc}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \left(1 - \frac{ad}{bc}\right)^k \psi(\sigma+k) \}$$

[$\operatorname{Re}(\alpha-\sigma) < 0, \operatorname{Re} \alpha > 0; |1-ad/(bc)| < 1; |\arg(cx+d)|, |\arg(ax+b)| < \pi$ при $0 \leq x < \infty$].

$$61. \int_0^{\infty} \frac{\ln (cx+d)}{x(ax+b)} dx = \frac{2\pi}{\sqrt{ab}} \ln \frac{\sqrt{ad} + \sqrt{bc}}{\sqrt{ad} - \sqrt{bc}}$$

[$|\arg(cx+d)|, |\arg(ax+b)| < \pi$ при $0 \leq x < \infty$].

$$62. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{x(ax+b)}} dx = \frac{2\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{bc}{ad}}$$

$[a, b > 0; |\arg(ax+d)| < \pi \text{ при } 0 \leq x < \infty]$

$$63. \int_0^{\infty} x^{-1/2} \frac{\ln(cx+d)}{(ax+b)^2} dx = \frac{\pi}{2} a^{-1/2} b^{-1/2} \left(\ln \frac{\sqrt{bc} + \sqrt{ad}}{\sqrt{a}} + \frac{\sqrt{bc}}{\sqrt{bc} + \sqrt{ad}} \right)$$

$[|\arg(cx+d)|, |\arg(ax+b)| < \pi \text{ при } 0 \leq x < \infty]$

$$64. \int_0^{\infty} \frac{\ln|a-x|}{\sqrt{x(b+x)}} dx = \frac{\pi}{\sqrt{b}} \ln(a+b)$$

$[a, b > 0]$

2.6.11. Интегралы от $x^\alpha (ax^2+b)^\beta \ln(cx+d)$.

$$1. \int_0^{\infty} \frac{\ln(c^2+ax)}{x^2+x^2} dx = \frac{1}{c} \operatorname{arctg} \frac{a}{c} \left[\ln c + \frac{1}{2} \ln(a^2+c^2) \right]$$

$[a, c > 0]$

$$2. \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - \frac{1}{16}$$

$$3. \int_0^1 x^{1/2} \frac{\ln(1+x)}{(1+x^2)^2} dx = \frac{\pi}{16} (\ln 2 + 1) \pm \frac{3}{8} \ln 2$$

$$4. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{a^2-x^2}} dx = \frac{\pi}{2} \ln \frac{d+\sqrt{d^2-a^2c^2}}{2}$$

$[a > 0; d > a/c]$

$$5. = (\pi/2) \ln(2ac/d^2)$$

$[0 < d < a/c]$

$$6. \int_0^{\infty} \frac{\ln(a+x)}{\sqrt{a^2-x^2}} dx = \frac{\pi}{2} \ln \frac{a}{2} \pm 2G$$

$[a > 0]$

$$7. \int_0^a \frac{\ln(cx+1)}{x\sqrt{a^2-x^2}} dx = \frac{\pi^2}{8a} - \frac{1}{2a} \arccos^2(ac)$$

$[a > 0; |ac| \leq 1]$

$$8. \int_0^{\infty} x \frac{\ln(cx+d)}{\sqrt{a^2-x^2}} dx =$$

$$= \frac{1}{2} \left[\pi (1 - \sqrt{d^2-a^2c^2}) + 2 \sqrt{d^2-a^2c^2} \arcsin(a/d) + 2ac (\ln d - 1) \right]$$

$[d \geq a/c; a > 0]$

$$9. = \frac{1}{2} \left[\pi + 2 \sqrt{a^2c^2-d^2} \ln \frac{a+\sqrt{a^2c^2-d^2}}{d} + 2ac (\ln d - 1) \right]$$

$[0 < d \leq a/c; a > 0]$

$$10. \int_1^{\infty} \frac{\ln(x \pm 1)}{x^2+1} dx = \frac{\pi}{8} \ln 2 + \{G\}$$

$$11. \int_1^{\infty} \frac{\ln(x-1)}{x^2+x+1} dx = \frac{\pi \ln 3}{6 \sqrt{3}}$$

$$12. \int_0^{\infty} \frac{\ln(cx+d)}{(ax+b)^2} dx = \frac{1}{b^2-ad} \left(\frac{c}{a} \ln \frac{bc}{a} - \frac{d}{b} \ln d \right)$$

$[bc \neq ad; |\arg(cx+d)| < \pi \text{ при } 0 \leq x < \infty]$

$$13. = \frac{1}{a^2d} (\ln d + 1)$$

$[bc = ad; |\arg(cx+d)| < \pi \text{ при } 0 \leq x < \infty]$

$$14. \int_0^{\infty} \frac{\ln|1 \pm x|}{1+x^2} dx = \frac{\pi}{4} \ln 2 \pm G$$

$$15. \int_0^{\infty} \frac{\ln(ax+b)}{x^2+c^2} dx = \frac{a}{b} {}_2F_2 \left(1, \frac{1}{2}; \frac{1}{2}, \frac{3}{2}; \frac{3}{2}, \frac{3}{2}; \frac{a^2c^2}{b^2} \right) +$$

$$+ \frac{1}{2a} \left[\ln \frac{b}{ac} \cdot \ln \frac{b+ac}{b-ac} - \frac{\pi}{2} \ln \left(1 - \frac{a^2c^2}{b^2} \right) \right] + \pi \ln b$$

$[Re a > 0; |\arg(ax+b)| < \pi \text{ при } x > 0]$

$$16. \int_0^{\infty} x^{1/2} \frac{\ln|a-x|}{(b+x)^2} dx = \frac{\pi b^{1/2}}{2b} \ln(a+b) \pm \frac{\pi b^{1/2}}{a+b}$$

$[a, b > 0]$

$$17. \int_0^{\infty} \frac{x \ln|cx+d|}{(x^2+x^2)^2} dx = \frac{c^2}{2(d^2+c^2z^2)} \left(\ln|cx+d| + \frac{\pi d}{2cz} + \frac{d^2}{c^2z^2} \ln d \right)$$

$[d, Re z > 0; Im z = 0]$

$$18. \int_0^{\infty} \frac{x^2-x^2}{(x^2+x^2)^2} \ln(cx+d) dx = \frac{c}{d^2+c^2z^2} \left(d \ln \left| \frac{cx}{d} \right| - \frac{cz\pi}{2} \right)$$

$[d, Re z > 0; Im z = 0]$

2.6.12. Интегралы от $x^2 (a_1x+b_1)^\beta (a_2x+b_2)^\gamma \ln^a(cx+d)$.

$$1. \int_a^b \frac{\ln|x-y|}{(x-a)^{\alpha-1}(b-x)^{\alpha}} dx = \pi \frac{\ln(b-a) + \psi(\alpha) + C}{\sin \alpha \pi} +$$

$$+ \operatorname{ctg} \alpha \pi \left(\frac{y-a}{b-y} \right)^{\alpha} {}_2F_1 \left(1, \alpha; 1+\alpha; -\frac{y-a}{b-y} \right)$$

$[a < y < b; 0 < Re \alpha < 1]$

$$2. \int_a^b \left(\frac{x-a}{b-x} \right)^{\alpha} \ln|x-y| dx = \pi \alpha (b-a) \frac{\ln(b-a) + \psi(\alpha+1) + C - 1}{\sin \alpha \pi} -$$

$$- \pi \frac{x-a}{\sin \alpha \pi} + \pi \frac{\operatorname{ctg} \alpha \pi}{\alpha+1} (y-a)^{\alpha-1} (b-y)^{-\alpha} {}_2F_1 \left(1, \alpha; 2+\alpha; -\frac{y-a}{b-y} \right)$$

$[Re \alpha = 0, \alpha \neq 0; a < y < b]$

$$3. \int_a^b \frac{\ln|x-y|}{\sqrt{(x-a)(b-x)}} dx = \pi \ln \frac{b-a}{4}$$

$[a < y < b]$

$$4. \int_0^1 \ln(1-x) \frac{1-(\pm x)^{n+1}}{1-x} dx = - \sum_{j=1}^{n+1} \frac{(\pm 1)^j}{j} \sum_{k=1}^j \frac{1}{k}$$

5. $\int_0^1 \ln(1+x) \frac{1-(\pm x)^{n+1}}{1 \mp x} dx = 2 \ln 2 \sum_{k=0}^{[n/2]} \frac{(\mp 1)^k}{2k+1} \pm \sum_{j=1}^{n+1} \frac{1}{j} \sum_{k=1}^j \frac{(\mp 1)^{k-1}}{k}.$
6. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln^s (a-x)}{(cx+d)^\rho} dx =$
 $= d^{-\rho} a^{\alpha+\beta-1} \frac{\partial^s}{\partial \beta^s} [a^\beta B(\alpha, \beta) {}_2F_1(\rho, \alpha; \alpha+\beta; -ac/d)]$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a].$
7. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln(a-x)}{(cx+d)^\rho} dx =$
 $= d^{-\rho} a^{\alpha+\beta-1} B(\alpha, \beta) \left\{ {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) [\ln a + \psi(\beta)] - \right.$
 $\left. - \sum_{k=0}^{\infty} \frac{(\rho)_k (\alpha)_k}{(\alpha+\beta)_k k!} \left(-\frac{ac}{d}\right)^k \psi(\alpha+\beta+k) \right\}$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |ac| < d; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a].$
8. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln^s (cx+d)}{(cx+d)^\rho} dx =$
 $= (-1)^s a^{\alpha+\beta-1} B(\alpha, \beta) \left(\frac{\partial}{\partial \rho} \right)^s \left[d^{-\rho} {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) \right]$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a].$
9. $\int_0^a x^{\alpha-1} (a-x)^{\beta-1} \frac{\ln(cx+d)}{(cx+d)^\rho} dx =$
 $= a^{\alpha+\beta-1} B(\alpha, \beta) d^{-\rho} \left\{ [\ln d + \psi(\rho)] {}_2F_1\left(\rho, \alpha; \alpha+\beta; -\frac{ac}{d}\right) - \right.$
 $\left. - \sum_{k=0}^{\infty} \frac{(\rho)_k (\alpha)_k}{(\alpha+\beta)_k k!} \left(-\frac{ac}{d}\right)^k \psi(\rho+k) \right\}$
 $[a, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |ac| < d; |\arg(cx+d)| < \pi \text{ при } 0 \leq x \leq a].$
10. $\int_0^1 \frac{(1-x^2) \ln(1+x)}{(ax+b)^2 (bx+a)^2} dx =$
 $= \frac{1}{a^2-b^2} \left\{ \frac{1}{a-b} \left[\frac{a+b}{ab} \ln(a+b) - \frac{\ln b}{a} - \frac{\ln a}{b} \right] + \frac{4 \ln 2}{b^2-a^2} \right\}$
 $[a, b > 0, a \neq b].$
11. $\int_0^1 \frac{(1+x^2) \ln(1+x)}{(a^2+x^2)(a^2x^2+1)} dx = \frac{1}{2a(a^2+1)} \left[\frac{\pi}{2} \ln(a^2+1) - 2 \operatorname{arctg} a \ln a \right]$
 $[a > 0].$

12. $\int_0^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln^s (x-a)}{(cx+d)^\rho} dx =$
 $= \frac{d^{-\rho}}{c^{\alpha+\beta-1}} \left[a^{\alpha+\beta-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \right]$
 $[\operatorname{Re}(\rho-\alpha-\beta) > -1; \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty].$
13. $\int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln(x-a)}{(cx+d)^\rho} dx =$
 $= a^{\alpha+\beta-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \left\{ {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \times \right.$
 $\times [\ln a + \psi(\beta)] - \sum_{k=0}^{\infty} \frac{(\rho)_k (1+\rho-\alpha-\beta)_k}{(1+\rho-\alpha)_k k!} \left(-\frac{ac}{d}\right)^k \psi(1+\rho-\alpha-\beta+k) \Big\}$
 $[\operatorname{Re}(\alpha+\beta-\rho) < 1; a, \operatorname{Re} \beta > 0; |ac| > d; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty].$
14. $\int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln^s (cx+d)}{(cx+d)^\rho} dx =$
 $= (-1)^s \frac{\partial^s}{\partial \rho^s} \left[a^{\alpha+\beta-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \times \right.$
 $\times {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) \Big]$
 $[\operatorname{Re}(\alpha+\beta-\rho) < 1; \operatorname{Re} \beta > 0; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty].$
15. $\int_a^\infty x^{\alpha-1} (x-a)^{\beta-1} \frac{\ln(cx+d)}{(cx+d)^\rho} dx = a^{\alpha+\beta-1} c^{-\rho} B(\beta, 1+\rho-\alpha-\beta) \times$
 $\times \left\{ {}_2F_1\left(\rho, 1+\rho-\alpha-\beta; 1+\rho-\alpha; -\frac{d}{ac}\right) [\ln(ac) + \psi(\rho)] - \right.$
 $\left. - \sum_{k=0}^{\infty} \frac{(\rho)_k (1+\rho-\alpha-\beta)_k}{(1+\rho-\alpha)_k k!} \left(-\frac{d}{ac}\right)^k [\psi(\rho+k) + \psi(1+\rho-\alpha-\beta+k)] - \right.$
 $\left. - \psi(1+\rho-\alpha+k) \right\}$
 $[\operatorname{Re}(\alpha+\beta-\rho) < 1; a, \operatorname{Re} \beta > 0; |d| < |ac|; |\arg(cx+d)| < \pi \text{ при } a \leq x < \infty].$
16. $\int_0^\infty \frac{x^{\alpha-1} \ln^s (cx+d)}{(ax+b)^\sigma (cx+d)^\rho} dx =$
 $= (-1)^s \left(\frac{d}{c} \right)^{\alpha-\sigma} b^{-\sigma} \frac{\partial^s}{\partial \rho^s} \left[d^{-\rho} B(\alpha, \rho+\sigma-\alpha) {}_2F_1\left(\sigma, \alpha; \sigma+\rho; 1-\frac{ad}{bc}\right) \right]$
 $[\operatorname{Re} \alpha > 0; \operatorname{Re}(\alpha-\sigma-\rho) < 0; |\arg(ax+b)|, |\arg(cx+d)| < \pi \text{ при } 0 \leq x < \infty].$
17. $\int_0^\infty \frac{x^{\alpha-1} \ln(cx+d)}{(ax+b)^\sigma (cx+d)^\rho} dx = \frac{d^{\alpha-\rho}}{c^{\sigma+\rho}} B(\alpha, \rho+\sigma-\alpha) \left\{ {}_2F_1\left(\sigma, \alpha; \sigma+\rho; 1-\frac{ad}{bc}\right) \times \right.$
 $\times [\ln d - \psi(\rho+\sigma-\alpha)] + \sum_{k=0}^{\infty} \frac{(\sigma)_k (\alpha)_k}{(\sigma+\rho)_k k!} \left(1-\frac{ad}{bc}\right)^k \psi(\sigma+\rho+k) \Big\}$
 $[\operatorname{Re}(\alpha-\sigma-\rho) < 0; \operatorname{Re} \alpha > 0; \left|1-\frac{ad}{bc}\right| < 1; |\arg(cx+d)|, |\arg(ax+b)| < \pi \text{ при } 0 \leq x < \infty].$

$$18. \int_0^{\infty} x^{\alpha-1} (x+a)^k (x+2a)^{\rho} \ln(x+2a) dx =$$

$$= a^{k+\rho+1} \Gamma(\alpha+\rho+1) \Gamma\left[\frac{\alpha+\rho-k}{\rho}\right] \left\{ \ln(2a) + \psi(\rho) - \right.$$

$$\left. - \psi(\rho-\alpha-k) - k(\alpha+\rho-1)^{-1} \right\}$$

$$[k=0 \text{ или } 1; \operatorname{Re} \alpha > 0, \operatorname{Re}(\alpha-\rho) < -k; |\arg a| < \pi]$$

$$19. \int_0^{\infty} \frac{\ln(cx+d)}{\sqrt{x(ax+b)(gx+h)}} dx =$$

$$= \frac{2\pi}{\sqrt{d(hg-ah)}} \left[\sqrt{\frac{dg}{h}} \ln \left(1 + \sqrt{\frac{ch}{dg}} \right) - \sqrt{\frac{dg}{h}} \ln \left(1 + \sqrt{\frac{hc}{da}} \right) \right] +$$

$$+ \frac{\pi \ln d}{b \ln a (\sqrt{ah} + \sqrt{bg})} \quad [bs \neq ah; a, b, c, d, g, h > 0].$$

$$20. \int_0^{\infty} \frac{\sqrt{x} \ln(cx+d)}{(ax+b)(gx+h)} dx =$$

$$= \frac{2\pi d}{bg-ah} \left[\sqrt{\frac{b}{ad}} \ln \left(1 + \sqrt{\frac{bc}{ad}} \right) - \sqrt{\frac{h}{gd}} \ln \left(1 + \sqrt{\frac{hc}{gd}} \right) \right] + \frac{\pi \ln d}{\sqrt{ag}(\sqrt{bg} + \sqrt{ah})}$$

$$[bg \neq ah; a, b, c, d, g, h > 0].$$

$$21. \int_0^{\infty} \frac{1-x^2}{(ax+b)^2 (bx+a)^2} \ln(1+x) dx = \frac{1}{ab(a^2-b^2)} \ln \frac{b}{a}$$

$$[a \neq b; a, b > 0].$$

2.6.13. Интегралы от $x^{\alpha} \ln^n \frac{ax+b}{cx+d}$.

$$1. \int_a^b \ln \frac{x+b}{x+a} \frac{dx}{x} = \frac{1}{2} \ln^2 \frac{b}{a} \quad [0 < a < b].$$

$$2. \int_a^b \ln \frac{c+x}{c-x} \frac{dx}{(x^2-a^2)(b^2-x^2)} = \frac{\pi}{b} F\left(\arcsin \frac{b}{c}, \frac{a}{b}\right) \quad [|b| < |c|, a < b].$$

$$3. \int_a^b \ln^n \frac{b-x}{x-a} \frac{dx}{x-y} = \sum_{k=0}^{[(n+1)/2]} (\pi i)^{2k} a_{2k}^n \ln n+1 - 2k \frac{b-y}{y-a}$$

$[a < y < b; a_{2k}^n \text{ находятся по рекуррентным формулам}]$

$$\sum_{j=0}^m \binom{m}{2j+1} a_{2k-2j-1}^n = \binom{m}{2k}, \quad k=0, 1, 2, \dots, \left[\frac{n+1}{2}\right], \quad \binom{k}{k+1} \equiv 0.$$

$$4. \int_0^{\infty} \ln^n \frac{a-x}{x} dx = -\frac{1+(-1)^n}{2(n+1)} \sum_{k=0}^{n/2-1} \binom{n+1}{2k} (\pi i)^{n-2k} \int_0^a \ln^{2k} \frac{a-x}{x} dx \quad [a > 0]$$

2.6.13.]

$$5. \int_0^a x^n \ln \frac{a+x}{a-x} dx = \frac{2}{n+1} a^{n+1} \left\{ \sum_{k=0}^{[(n-1)/2]} \frac{1}{n-2k} + \frac{1+(-1)^n}{2} \ln 2 \right\} \quad [a > 0].$$

$$6. \int_0^{\pi} \ln \frac{a+x}{a-x} \frac{dx}{x} = \frac{\pi^2}{4} \quad [a \neq 0].$$

$$7. \int_0^{\sqrt{2}-1} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(\sqrt{2}-1).$$

$$8. \int_0^{\sqrt{5}-2} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{12} - \frac{1}{6} \ln^2(\sqrt{5}-2).$$

$$9. \int_0^{(\sqrt{5}-1)/2} \ln \frac{1+x}{1-x} \frac{dx}{x} = \frac{\pi^2}{6} - \frac{3}{2} \ln^2 \frac{\sqrt{5}-1}{2}.$$

$$10. \int_0^{\pi} x^{-1/2} \ln \frac{a+bx}{a-bx} \frac{dx}{\sqrt{a-x}} = 2\pi \ln \operatorname{clg} \frac{\pi - \arcsin b}{4} \quad [-1 \leq b \leq 1; a > 0].$$

$$11. \int_0^{\pi} \ln \frac{b+x}{b-x} \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{a} \arcsin \frac{a}{b} \quad [0 < a \leq b].$$

$$12. \int_0^{\pi} \ln^{\sigma} \frac{a+x}{a-x} dx = 2a\Gamma(\sigma+1) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\sigma}} \quad [\operatorname{Re} \sigma > 0].$$

$$13. \int_0^{\pi} \ln^{\sigma} \frac{a+x}{a-x} \frac{dx}{(a^2-x^2)^{\rho}} = (2a)^{1-\rho} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(2\rho-k-1)_k (k+1-\rho)^{-1-\sigma}}{k!} \quad [\operatorname{Re} \sigma > -1; \operatorname{Re} \rho < 1, a > 0].$$

$$14. \int_0^{\pi} \ln \frac{a+x}{a-x} \frac{dx}{\sqrt{a^2-x^2}} = 4G.$$

$$15. \int_0^{\infty} x^{\alpha-1} \ln \frac{x+a}{x-a} dx = -\frac{a^{\alpha}}{\alpha} \left[\psi\left(\frac{1-\alpha}{2}\right) + C + \ln 4 \right] \quad [\operatorname{Re} \alpha < 1, \alpha \neq 0; a > 0].$$

$$16. \int_0^{\infty} \ln \frac{x+a}{x-a} \frac{dx}{x} = \frac{\pi^2}{4} \quad [a > 0].$$

$$17. \int_0^{\infty} x^n \ln \frac{x+a}{x-a} dx = \frac{a^{1-n}}{n-1} \left[\frac{1+(-1)^n}{2} \ln 4 + \sum_{k=1}^{[(n-1)/2]} \frac{1}{k} \right] +$$

$$+ [1-(-1)^n] \sum_{k=1}^{[(n-1)/2]} \frac{1}{2k-1} \quad [a > 0; n=2, 3, 4, \dots].$$

$$18. \int_a^\infty \ln^\sigma \frac{x+a}{x-a} dx = 2a\Gamma(\sigma+1)\zeta(\sigma)$$

[Re $\sigma > 1$; $a > 0$].

$$19. \int_a^\infty \ln^\sigma \frac{x+a}{x-a} \frac{dx}{(x^2-a^2)^\rho} = (2a)^{1-2\rho} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(2\rho-k-1)_\sigma (-1)^k}{k! (k-\rho+1)^{\sigma+1}}$$

[$a > 0$; Re $\rho < 1$; Re $(\sigma+2\rho) > 1$].

$$20. \int_a^\infty \ln \frac{x+a}{x-a} \frac{dx}{\sqrt{x^2-a^2}} = \frac{\pi^2}{2}$$

[$a > 0$].

$$21. \int_0^\infty x^{\alpha-1} \ln \frac{cx+d}{cx-b} dx = \frac{\pi x^{-\alpha}}{\alpha \sin \alpha\pi} (d^\alpha - b^\alpha)$$

[$b, c, d > 0$; $0 < \operatorname{Re} \alpha < 1$].

$$22. \int_0^\infty x^{\alpha-1} \ln \left| \frac{cx+d}{cx-b} \right| dx = \frac{\pi (d^2 - b^2 \cos \alpha\pi)}{\alpha c^2 \sin \alpha\pi}$$

[$b, c, d > 0$; $0 < \operatorname{Re} \alpha < 1$].

$$23. \int_0^\infty x^{\alpha-1} \ln \left| \frac{x+a}{x-a} \right| dx = \frac{\pi a^\alpha}{\alpha} \operatorname{tg} \frac{\alpha\pi}{2}$$

[$a > 0$; Re $\alpha < 1$, $\alpha \neq 0$].

$$24. \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x} = \frac{\pi^2}{2}$$

[$a > 0$].

$$25. \int_0^\infty \ln \frac{x+a}{x} \frac{dx}{x+a} = \frac{\pi^2}{4}$$

[$a > 0$].

$$26. \int_a^\infty \ln \left| \frac{x-a}{x-b} \right| \frac{dx}{x-y} = 0$$

[$y < a < b$ или $y > b > a$].

$$27. = \pi$$

[$a < y < b$].

$$28. \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x(x^2+z^2)} = \frac{\pi}{z^2} \operatorname{arctg} \frac{z}{a}$$

[$a, \operatorname{Re} z > 0$].2.6.14. Интегралы от $A(x) \ln(ax^2+bx+c)$.Обозначение: $k' = \sqrt{1-k^2}$, $0 < k < 1$.

$$1. \int_0^1 x^n \ln(1+x^2) dx = \frac{1}{n+1} \left\{ \frac{\pi}{2} (-1)^{[n/2]} + \ln 2 - \frac{2}{n+1} + 2 \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{n-2k-1} \right\}$$

$$2. \int_0^1 \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \frac{\ln a}{c(c+d)} + \frac{b}{ad^2+bc^2} \left[2\sqrt{\frac{a}{b}} \operatorname{arctg} \sqrt{\frac{b}{a}} + \frac{bc-ad}{b(c+d)} \ln \frac{a+b}{a} - 2\frac{c}{d} \ln \frac{c+d}{c} \right]$$

[$a, b, c, d > 0$].

$$3. \int_0^1 \frac{\ln(1-x^2)}{(a+bx)\sqrt{1-x^2}} dx = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{a+\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$$

[$a, b > 0$; $a \neq b$].

$$4. \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = (2-k^2) K(k) - (2-\ln k') E(k).$$

$$5. \int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}} \ln(1-k^2x^2) dx = k'^2 (1+k'^2 - k'^2 \ln k') K(k) - (2-\ln k') E(k).$$

$$6. \int_0^1 \frac{\ln(1-x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} K(k) - \frac{\pi}{2} K(k').$$

$$7. \int_0^1 \frac{\ln(1+kx^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2+2k}{1-k} K(k) - \frac{\pi}{8} K(k').$$

$$8. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln k' K(k).$$

$$9. \int_0^1 x^2 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = \frac{1}{8k^2} \{ (-2+11k^2-6k^4+3k'^2 \ln k') K(k) + [2-10k^2-3(1-2k^2) \ln k'] E(k) \}.$$

$$10. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{k'^2} [(k^2-2) K(k) + (2+\ln k') E(k)].$$

$$11. \int_0^1 \frac{x^2 \ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{(k'h)^2} [(2+\ln k') E(k) - (1+k'^2+k'^2 \ln k') K(k)].$$

$$12. \int_0^1 x^2 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{k^2} [(k^2-2+\ln k') K(k) + (2-\ln k') E(k)].$$

$$13. \int_0^\infty \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \frac{\ln b}{cd} + \frac{b}{ad^2+bc^2} \left(\pi \sqrt{\frac{a}{b}} + 2\frac{c}{d} \ln \frac{c}{d} + \frac{ad}{bc} \ln \frac{a}{b} \right)$$

[$a, b, c, d > 0$].

$$14. \int_0^\infty \frac{\ln(a+bx^2)}{c+dx^2} dx = \frac{\pi}{bd} \ln \left(\sqrt{a} + \sqrt{\frac{bc}{a}} \right)$$

[$a, b, c, d > 0$].

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$$18. \int_a^\infty \ln \frac{x+a}{x-a} dx = 2a\Gamma(\sigma+1)\zeta(\sigma) \quad [\operatorname{Re} \sigma > 1; a > 0].$$

$$19. \int_a^\infty \ln \frac{x+a}{x-a} \frac{dx}{(x^2-a^2)^\rho} = (2a)^{1-2\rho} \Gamma(\sigma+1) \sum_{k=0}^{\infty} \frac{(2\rho-k-1)_k (-1)^k}{k! (k-\rho+1)^{\rho+1}} \quad [a > 0; \operatorname{Re} \rho < 1; \operatorname{Re}(\sigma+2\rho) > 1].$$

$$20. \int_a^\infty \ln \frac{x+a}{x-a} \frac{dx}{\sqrt{x^2-a^2}} = \frac{\pi^2}{2} \quad [a > 0].$$

$$21. \int_0^\infty x^{2-1} \ln \frac{cx+d}{cx+b} dx = \frac{\pi c^{-\alpha}}{\alpha \sin \alpha \pi} (d^\alpha - b^\alpha) \quad [b, c, d > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$22. \int_0^\infty x^{\alpha-1} \ln \left| \frac{cx+d}{cx-b} \right| dx = \frac{\pi (d^2-b^2 \cos \alpha \pi)}{\alpha c^\alpha \sin \alpha \pi} \quad [b, c, d > 0; 0 < \operatorname{Re} \alpha < 1].$$

$$23. \int_0^\infty x^{2-1} \ln \left| \frac{x+a}{x-a} \right| dx = \frac{\pi a^\alpha}{\alpha} \lg \frac{\alpha \pi}{2} \quad [a > 0; |\operatorname{Re} \alpha| < 1, \alpha \neq 0].$$

$$24. \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x} = \frac{\pi^2}{2} \quad [a > 0].$$

$$25. \int_0^\infty \ln \frac{x+a}{x} \frac{dx}{x+a} = \frac{\pi^2}{4} \quad [a > 0].$$

$$26. \int_0^\infty \ln \left| \frac{x-a}{x-b} \right| \frac{dx}{x-y} = 0 \quad [y < a < b \text{ или } y > b > a].$$

$$27. = \pi \quad [a < y < b].$$

$$28. \int_0^\infty \ln \left| \frac{x+a}{x-a} \right| \frac{dx}{x(x^2+z^2)} = \frac{\pi}{z^2} \operatorname{arctg} \frac{z}{a} \quad [a, \operatorname{Re} z > 0].$$

2.6.14. Интегралы от $A(x) \ln(ax^2+bx+c)$.

Обозначение: $k' = \sqrt{1-k^2}$, $0 < k < 1$.

$$1. \int_0^1 x^n \ln(1+x^2) dx = \frac{1}{n+1} \left\{ \frac{\pi}{2} (-1)^{[n/2]} + \ln 2 - \frac{2}{n+1} + 2 \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{n-2k-1} \right\}.$$

$$2. \int_0^1 \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \frac{\ln a}{c(c+d)} + \frac{b}{ad^2+bc^2} \left[2\sqrt{\frac{a}{b}} \operatorname{arctg} \sqrt{\frac{b}{a}} + \frac{bc-ad}{b(c+d)} \ln \frac{a+b}{a} - 2\frac{c}{d} \ln \frac{c+d}{c} \right] \quad [a, b, c, d > 0].$$

$$3. \int_0^1 \frac{\ln(1-x^2)}{(a+bx) \sqrt{1-x^2}} dx = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} \quad [a, b > 0; a \neq b].$$

$$4. \int_0^1 \frac{\sqrt{1-k^2x^2}}{1-x^2} \ln(1-k^2x^2) dx = (2-k^2) K(k) - (2-\ln k') E(k).$$

$$5. \int_0^1 \frac{\sqrt{1-x^2}}{1-k^2x^2} \ln(1-k^2x^2) dx = k'^2 (1+k'^2 - k'^2 \ln k') K(k) - (2-\ln k') E(k).$$

$$6. \int_0^1 \frac{\ln(1-x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} K(k) - \frac{\pi}{2} K(k').$$

$$7. \int_0^1 \frac{\ln(1 \pm kx^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2+2k}{1-k} K(k) - \frac{\pi}{8} K(k').$$

$$8. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln k' K(k).$$

$$9. \int_0^1 x^2 \sqrt{\frac{1-k^2x^2}{1-x^2}} \ln(1-k^2x^2) dx = -9k^2 \{ (-2+11k^2-6k^4+3k'^2 \ln k') K(k) + [2-10k^2-3(1-2k^2) \ln k'] E(k) \}.$$

$$10. \int_0^1 \frac{\ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{k'^2} [(k^2-2) K(k) + (2+\ln k') E(k)].$$

$$11. \int_0^1 \frac{x^2 \ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)^3}} = \frac{1}{(k'k)^2} [(2+\ln k') E(k) - (1+k'^2+k'^2 \ln k') K(k)].$$

$$12. \int_0^1 \frac{x^2 \ln(1-k^2x^2) dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{k^2} [(k^2-2+\ln k') K(k) + (2-\ln k') E(k)].$$

$$13. \int_0^\infty \frac{\ln(a+bx^2)}{(c+dx)^2} dx = \frac{\ln a}{cd} + \frac{b}{ad^2+bc^2} \left(\pi \sqrt{\frac{a}{b}} + 2\frac{c}{d} \ln \frac{c}{d} + \frac{ad}{bc} \ln \frac{a}{b} \right) \quad [a, b, c, d > 0].$$

$$14. \int_0^\infty \frac{\ln(a+bx^2)}{c+dx^2} dx = \frac{\pi}{bd} \ln \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{dc}{b}} \right) \quad [a, b, c, d > 0].$$

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$$15. \int_0^{\infty} \frac{x^{\alpha-1}}{x+z} \ln(x^2+z^2) dx =$$

$$= \frac{\pi z^{\alpha-1}}{\sin \alpha \pi} \left[\ln(2z^2) - (1-\alpha) \sin \frac{\alpha \pi}{2} \beta\left(\frac{1-\alpha}{2}\right) - (2-\alpha) \cos \frac{\alpha \pi}{2} \beta\left(1-\frac{\alpha}{2}\right) \right]$$

$[0 < \operatorname{Re} \alpha < 1 \text{ при } z \neq 1; -2 < \operatorname{Re} \alpha < 1 \text{ при } z = 1; \operatorname{Re} z > 0].$

$$16. \int_0^1 \ln(1+2bx+x^2) \frac{dx}{x} = \frac{\pi^2}{6} - \frac{\arccos^2 b}{2} \quad [b \leq 1].$$

$$17. \int_0^1 \frac{\ln(1+2bx-ax^2)}{\sqrt{x}\sqrt{1-x}} dx = 2\pi \ln \frac{\sqrt{c}+1+\sqrt{(c+1)^2+a}}{4}$$

$[0 < 2b < a \leq 1; c = 1+2b-a].$

$$18. \int_0^{\infty} x^{\alpha-1} \ln(1+2bx+x^2) dx = \frac{2\pi \cos(\alpha \arccos b)}{\alpha \sin \alpha \pi} \quad [-1 < b \leq 1; -1 < \operatorname{Re} \alpha < 0].$$

$$19. \int_{-\infty}^{\infty} \ln(a^2-2abx+x^2) \frac{dx}{x^2+z^2} = \frac{\pi}{z} \ln(z^2+2az\sqrt{1-b^2+a^2})$$

$[a, \operatorname{Re} z > 0; -1 < b \leq 1].$

$$20. \int_0^{1/2} \ln^n[x(1-x)] \frac{dx}{1-x} = I_n \quad [n = 1, 2, 3, \dots].$$

$$I_n = -\frac{2^n}{n+1} \ln^{n+1} \frac{1}{2} + \frac{1}{2} \frac{\partial^n}{\partial v^n} \left[B(v, v+1) - \frac{1}{v} \right]_{v=1/2},$$

$$I_4 = \frac{16}{5} \ln^5 2 + 8 [18\zeta_5(5) - \pi^2 \zeta_5(3)].$$

2.6.15. Интегралы от $A(x) \ln \frac{P(x)}{Q(x)}$.

$$1. \int_0^1 x^{\alpha-1} \ln(1+x+x^2+\dots+x^{n+1}) dx = \alpha^{-1} \left[\psi(\alpha+1) - \psi\left(\frac{\alpha}{n+2}+1\right) \right]$$

$[\operatorname{Re} \alpha > -1, \alpha \neq 0].$

$$2. = \pi^2(n+1)/(6n+12) \quad [n=0].$$

$$3. \int_0^{\infty} x^{\alpha-1} \ln(1+x+x^2+\dots+x^{n+1}) dx = \pi \alpha^{-1} \left(\operatorname{ctg} \frac{\alpha \pi}{n+2} - \operatorname{ctg} \alpha \pi \right)$$

$[-1 < \operatorname{Re} \alpha < 0].$

$$4. \int_0^1 \frac{\ln(x^{-1} \pm x)}{1+x^2} dx = \frac{\pi}{8} (3 \pm 1) \ln 2.$$

$$5. \int_0^1 \frac{x \ln(x^{-1} \pm x)}{1+x^4} dx = \frac{\pi}{16} \ln 2 \pm \frac{G}{4}.$$

$$6. \int_1^{\infty} \frac{\ln(x+x^{-1})}{1+x^2} dx = \frac{\pi}{8} (3 \pm 1) \ln 2.$$

$$7. \int_1^{\infty} \frac{x \ln(x+x^{-1})}{1+x^4} dx = \frac{\pi}{16} \ln 2 \pm \frac{G}{4}.$$

$$8. \int_0^1 x^{\alpha-1} \ln \frac{(1+x)(a^2+x)}{(a+x)^2} dx = \frac{\pi(a^2-1)^{\alpha}}{\alpha \sin \alpha \pi}$$

$[\operatorname{Re} \alpha > -1; a > 0].$

$$= \ln^2 a \quad [\alpha=0; a>0].$$

$$9. \int_0^1 \ln \frac{(1-ax)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{2} \frac{1}{\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln(1+a)$$

$[a>0].$

$$10. \int_0^{\infty} \ln \frac{x^2+a^2}{x^2+b^2} dx = (a-b)\pi$$

$[a, b \geq 0].$

$$11. \int_0^{\infty} \ln \left| \frac{x^2-a^2}{x^2-b^2} \right| \frac{dx}{x-y} = \pm \pi$$

$[0 < a < |y| < b].$

$$12. = 0 \quad [\pm y \notin [a, b]; 0 < a < b].$$

$$13. \int_0^{\infty} x^{\alpha-1} \ln \frac{(x+a)^2+c^2}{(x+b)^2+c^2} dx =$$

$$= -\frac{2}{\alpha} \left[(b^2+c^2)^{\alpha/2} \cos\left(\alpha \operatorname{arctg} \frac{c}{b}\right) - (a^2+c^2)^{\alpha/2} \cos\left(\alpha \operatorname{arctg} \frac{c}{a}\right) \right]$$

$[0 < \operatorname{Re} \alpha < 1; a, b, c > 0].$

$$14. \int_0^{\infty} x^{\alpha-1} \ln \frac{a^2+2abx+x^2}{(a+x)^2} dx = \frac{2\pi a^{\alpha} [\cos(\pi \arccos b) - 1]}{\alpha \sin \alpha \pi}$$

$[|\operatorname{Re} \alpha| < 1; -1 < b \leq 1; a > 0].$

$$15. \int_0^{\infty} \ln \frac{(1+x)(x+a^2)}{(x+a)^2} \frac{dx}{x} = \ln^2 a$$

$[a > 0].$

2.6.16. Интегралы от $A(x) \ln^{\sigma}(\sqrt{ax^4+bx^2+cx^0})$.

$$1. \int_0^1 \frac{\ln(\sqrt{x^2-c^2}+x)}{\sqrt{1-x^2}} dx =$$

$$= \left(\arccos c + \varphi - \frac{\pi}{2} \right) \ln c + \frac{1}{2} L(\arccos a + \varphi) - \frac{1}{2} L(\arccos a - \varphi) - L(\varphi),$$

$$\cos \varphi = \sqrt{\frac{1-a^2}{1-c^2}} \quad [c \leq a < 1].$$

$$2. = (\arccos c - \pi/2) \ln c \quad [a = c < 1].$$

$$3. \int_0^a x^{\alpha-1} \ln \left(\sqrt{a^2 - x^2} + a \right) dx = \frac{a^\alpha}{\alpha} \left\{ \ln a + \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{\alpha+1}{2} \right] - \frac{1}{\alpha} \right\} \quad [a, \operatorname{Re} \alpha > 0]$$

$$4. \int_0^a \frac{x^{\alpha-1}}{(x^2 + z^2)^{\mu}} \ln \left(\sqrt{x^2 + z^2} + z^{1/2} \right) dx = \frac{a^{\alpha-1} z^{-1/2}}{\mu} \frac{\partial}{\partial \nu} \left[2^{1-\nu} \frac{\Gamma(\mu+\nu/2)}{\Gamma(\mu)} \left(\frac{\alpha + \frac{\nu-1}{2}}{\mu + \frac{\nu-1}{2}} \right)^{-1} \times \right. \\ \left. \times {}_2F_2 \left(\frac{\alpha + \frac{\nu-1}{2}}{\mu + \frac{\nu-1}{2}}, \frac{1+\nu}{2}, \frac{2r+1}{2}, 1 + \frac{\alpha + \frac{\nu-1}{2}}{\mu + \frac{\nu-1}{2}}, \frac{1+\nu}{2}, -\frac{a^2}{z^2} \right) \right]_{\nu=0} \\ [r=0 \text{ или } 1/2; a, \mu, \operatorname{Re} \alpha > 0; \mu, \arg z < \pi]$$

$$5. \int_0^a \frac{x^{\alpha-1}}{\sqrt{x^2 + z^2}} \ln \left(\sqrt{x^2 + z^2} + z^{1/2} \right) dx = \\ = \pm a^\alpha (az)^{-1/2} \sum_{k=0}^{\infty} \frac{(1/2)_k}{\alpha + \mu k} \left(-\frac{a^2}{z^2} \right)^k \left[\ln(2z^{1/2}) + \right. \\ \left. + \frac{1}{2} (1 \mp 1) \left(\frac{1}{\alpha + \mu k} - \ln a \right) - \psi(2k+1) + \psi(k+1) \right] \\ [a, \mu, \operatorname{Re} \alpha > 0; \mu, \arg z < \pi]$$

$$6. \int_0^a x^{\alpha-1} \ln \left(\sqrt{x^2 + z^2} + z^{1/2} \right) dx = \\ = -\frac{a^\alpha}{2} \sum_{k=1}^{\infty} \frac{(1/2)_k}{(\alpha + \mu k) k!} \left(-\frac{a^2}{z^2} \right)^k \pm \frac{a^\alpha}{\alpha} \left[\ln(2z^{1/2}) - \mu \frac{1 \mp 1}{2} (\alpha \ln a - 1) \right] \\ [a, \mu, \operatorname{Re} \alpha > 0; \mu, \arg z < \pi]$$

$$7. \int_0^a \ln \left(\sqrt{x^2 + a^2} + x \right) dx = a \left[1 - \sqrt{2} + \ln(1 + \sqrt{2}) + \ln a \right] \quad [a > 0]$$

$$8. \int_0^1 \ln \left(\sqrt{x^2 + 1} + x \right) \frac{dx}{x} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k! (2k+1)^2}$$

$$9. \int_0^a \frac{\ln \left(\sqrt{x^2 + a^2} + x \right)}{\sqrt{x^2 + a^2}} dx = \frac{1}{2} \ln(1 + \sqrt{2}) \left[\ln(1 + \sqrt{2}) + 2 \ln a \right] \quad [a > 0]$$

$$10. \int_0^a x \frac{\ln \left(\sqrt{x^2 + a^2} + x \right)}{\sqrt{x^2 + a^2}} dx = a \left[(\sqrt{2} - 1) \ln a + \sqrt{2} \ln(\sqrt{2} + 1) - 1 \right] \quad [a > 0]$$

$$11. \int_0^1 \frac{\ln(ax + \sqrt{1-b^2x^2})}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln 2 - 2L \left(\frac{\arcsin a}{2} \right) - 2L \left(\frac{\pi - \arcsin a}{2} \right) \\ [a^2 + b^2 = 1]$$

$$12. \int_0^1 \frac{\ln \left(1 + \sqrt{a^2 - c^2 x^2} \right)}{1 - a^2 + c^2 x^2} dx = \frac{1}{2c \sqrt{1-a^2}} \left\{ \left(\arcsin b \pm \gamma \mp \frac{\pi}{2} \right) \ln(1-a^2) + \right. \\ \left. + 2L(\arcsin b) \mp [2L(\gamma) - L(\arcsin b + \gamma) + L(\arcsin b - \gamma)] \right\} \\ \left[b = \frac{c}{\sqrt{1+c^2-a^2}}; c^2 < a^2; \cos \gamma = \frac{b}{a}; 0 < b < a < 1 \right]$$

$$13. \int_0^1 \frac{\ln \left(1 + \sqrt{1-k^2 x^2} \right)}{\sqrt{(1-x^2)(1-k^2 x^2)}} dx = \frac{\ln k}{2} K(k) + \pi \left(-\frac{1}{4} + \frac{1}{2} \right) K(k') \\ [0 < k < 1; k' = \sqrt{1-k^2}]$$

$$14. \int_0^1 \frac{\ln \left(\sqrt{1+kx} + \sqrt{1-kx} \right)}{\sqrt{(1-x^2)(1-k^2 x^2)}} dx = \frac{\ln 4k}{4} K(k) + \frac{\pi}{8} (2 \mp 1) K(k') \\ [0 < k < 1; k' = \sqrt{1-k^2}]$$

$$15. \int_0^1 \ln \frac{1+a\sqrt{1-x^2}}{1-a\sqrt{1-x^2}} \frac{dx}{1-x^2} = \pi \arcsin a \quad [|a| \leq 1]$$

$$16. \int_0^1 \ln \frac{1+a\sqrt{1-x^2}}{1-a\sqrt{1-x^2}} \frac{dx}{1-x^2} = \frac{\pi a}{\sqrt{1-a^2}} \left(\frac{\sqrt{1+z^2} + a + z\sqrt{1-a^2}}{\sqrt{1+z^2} - a - z\sqrt{1-a^2}} \right)^{1/2} \\ [|a| < 1; \operatorname{Re} z > 0]$$

$$17. \int_0^1 x \ln \left| \frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right| \frac{dx}{1-x^2} = \frac{\pi^2}{4}$$

$$18. \int_0^1 \ln \frac{1+a\sqrt{1-k^2 x^2}}{1-a\sqrt{1-k^2 x^2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \pi F(\arcsin a, \sqrt{1-k^2}) \\ [|a| < 1; 0 < k < 1]$$

$$19. \int_0^a \ln \frac{a+b\sqrt{(1-x^2)(a^2-x^2)}}{a-b\sqrt{(1-x^2)(a^2-x^2)}} \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{ab + \sqrt{a^2 b^2 + (1 + \sqrt{1-b^2})^2}}{1 + \sqrt{1-b^2}} \\ [a > 0; |b| \leq 1]$$

$$20. \int_0^a \ln \frac{\sqrt{a^2 - b^2 x^2} - x \sqrt{1-b^2}}{a-x} \frac{dx}{x} = \frac{1}{2} \arcsin^2 b \quad [|b| \leq 1]$$

$$21. \int_a^\infty x^{\alpha-1} \ln(x + \sqrt{x^2 - a^2}) dx = -\frac{a^\alpha}{\alpha} \left\{ \ln a + \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{\alpha+1}{2} \right] \right\} \\ [a > 0; \operatorname{Re} \alpha < 0]$$

$$22. \int_a^\infty x^{\alpha-1} \ln(x + \sqrt{x^2 - a^2}) dx = \frac{a^\alpha}{n} \left(\ln a + B \frac{(n-2)!!}{(n-1)!} \right) \\ [a > 0; B = \pi/2 \text{ при } n=1, 3, 5, \dots; B=1 \text{ при } n=2, 4, 6, \dots]$$

$$23. \int_a^{\infty} \ln(a + \sqrt{x^2 + a^2}) \frac{dx}{x^2} = \frac{1}{a} [2 - \sqrt{2} + \ln(1 + \sqrt{2}) + \ln a] \quad (a > 0).$$

$$24. \int_1^{\infty} \ln^{\sigma}(x + \sqrt{x^2 - 1}) \frac{dx}{(x^2 - 1)^{\beta}} = 2^{1-2\beta} \Gamma(\sigma + 1) \sum_{k=0}^{\infty} \frac{(-1)^k (2\beta - k)_k}{k! (2k + 1 - 2\beta)^{\sigma+1}} \quad \{ \operatorname{Re} \beta < 1/2; \operatorname{Re}(\sigma + \beta) > 0 \}.$$

$$25. \int_1^{\infty} \ln(x + \sqrt{x^2 - 1}) \frac{dx}{x^2 - 1} = \frac{\pi^2}{4}.$$

$$26. \int_a^{\infty} \frac{\ln(a + \sqrt{x^2 + a^2})}{x \sqrt{x^2 + a^2}} dx = \frac{1}{2a} \ln(1 + \sqrt{2}) [\ln(1 + \sqrt{2}) - 2 \ln a] + a^{-1} \sum_{k=0}^{\infty} \frac{(1/2 - k)_k}{k! (2k + 1)^2} \quad (a > 0).$$

$$27. \int_a^{\infty} x^k (x^2 - a^2)^{\alpha-1} (x \pm \sqrt{x^2 - a^2})^{\nu} \ln(x \pm \sqrt{x^2 - a^2}) dx = \\ = \left(\frac{\pi}{2} \right)^{k+2} k^{-2\alpha-1} a^{\nu+2\alpha+k-1} \Gamma \left[\frac{2\alpha}{(1 \mp \nu + k)/2 + \alpha} \right] \times \\ \times \left[2 \ln a \mp \psi \left(\frac{1 - k \mp \nu}{2} - \alpha \right) \pm \psi \left(\frac{1 \mp \nu + k}{2} + \alpha \right) + \frac{2k}{\nu} \right] \\ \{ k=0 \text{ или } 1; a, \operatorname{Re} \alpha > 0; \operatorname{Re}(2 \pm \nu) < 1 - k \}.$$

$$28. \int_0^{\infty} x^{\alpha-1} \frac{\ln(\sqrt{x^{\mu} + z^{\mu}} + z^{\mu/2})}{\sqrt{x^{\mu} + z^{\mu}}} dx = \\ = \frac{z^{\alpha-\mu/2}}{\mu} 2^{2\alpha/\mu-1} B \left(\frac{\alpha}{\mu}, 1 - \frac{2\alpha}{\mu} \right) \left[\mu \ln z + 2\beta \left(1 - 2 \frac{\alpha}{\mu} \right) - (1 \mp 1) \pi \operatorname{ctg} \left(\frac{\alpha \pi}{\mu} \right) \right] \\ \{ \mu > 0, 0 < \operatorname{Re} \alpha < \mu/2, \mu, \arg z < \pi \}.$$

$$29. \int_0^{\infty} x^{\alpha-1} \frac{\ln(\sqrt{x^{\mu} + z^{\mu}} \pm z^{\mu/2})}{\sqrt{x^{\mu} + z^{\mu}}} dx = \\ = \frac{z^{\alpha-\mu/2}}{\mu} 2^{-2\alpha/\mu} B \left(\frac{2\alpha}{\mu}, 1 - \frac{\alpha}{\mu} \right) \left[\mu \ln z \pm \pi \operatorname{tg} \left(\frac{\alpha \pi}{\mu} \right) \right] \\ \{ \mu > 0, \mu |\arg z| < \pi; 0 < \operatorname{Re} \alpha < \mu/2, z \neq 1; (2 \operatorname{Re} \alpha < \mu \text{ при } z = 1) \}.$$

$$30. \int_0^{\infty} \frac{\ln(\sqrt{x^{\mu} + 1} + x^{\mu/2})}{\sqrt{x^{\mu} + 1}} \frac{dx}{x} = \frac{\pi^2}{2\mu} \quad (\mu > 0).$$

$$31. \int_0^{\infty} x^{\alpha-1} \ln(\sqrt{x^{\mu} + 1} \pm x^{\mu/2}) dx = \mp 2^{-2\alpha/\mu} \alpha^{-1} B(2\alpha/\mu, -\alpha/\mu) \\ \{ \mu > 0; -\mu < 2 \operatorname{Re} \alpha < 0 \}.$$

$$32. \int_0^{\infty} \frac{\ln(\sqrt{x^2 + z^2} + x)}{x^2 + z^2} dx = \frac{\pi}{2z} \ln z + \frac{2G}{z} \quad (\operatorname{Re} z > 0).$$

$$33. \int_0^{\infty} x \frac{\ln(\sqrt{x^2 + z^2} + x)}{(x^2 + z^2)^{\rho}} dx = -\frac{z^{2-2\rho}}{2(1-\rho)} \left\{ \ln z + \frac{\sqrt{\pi}}{2} \Gamma \left[\frac{\rho-1}{\rho-1/2} \right] \right\} \\ (\operatorname{Re} \rho > 1; \operatorname{Re} z > 0).$$

$$34. \int_0^{\infty} x^{\alpha-1} \frac{\ln(\sqrt{x^2 + 1} + x)}{\sqrt{x^2 + 1}} dx = 2^{1-\alpha} \Gamma(\sigma + 1) \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha - k)_k}{k! (2k + 1 - \alpha)^{\sigma+1}} \\ (\operatorname{Re} \alpha < 1; \operatorname{Re}(\alpha + 2\sigma) > 0).$$

$$35. \int_0^{\infty} \frac{x}{(x^2 + z^2)^{\rho}} \ln \frac{\sqrt{x^2 + z^2} + x}{\sqrt{x^2 + z^2} - x} dx = \frac{z^{2-2\rho-1}}{\rho-1} B(\rho-1, \rho-1) \\ (\operatorname{Re} z > 0; \operatorname{Re} \rho > 1).$$

$$36. \int_0^{\infty} \frac{x}{(x^2 + z^2)^{\rho}} \ln \frac{\sqrt{x^2 + 2z^2} + x}{\sqrt{x^2 + 2z^2} - x} dx = \frac{2^{\rho-1} z^{2-2\rho}}{\rho-1} B \left(\frac{\rho-1}{2}, \frac{\rho-1}{2} \right) \\ (\operatorname{Re} z > 0; \operatorname{Re} \rho > 1).$$

$$37. \int_0^{\infty} \ln \frac{\sqrt{x^2 + z^2} + a^2}{\sqrt{x^2 + z^2} - a^2} \frac{dx}{x^2 + z^2} = \pi \arcsin \frac{a}{z} \quad (0 < a < |z|; \operatorname{Re} z > 0).$$

$$38. \int_0^{\infty} x^{\alpha-1} \left\{ \ln \frac{a^2 + b^2 + x^2 + \sqrt{[x^2 + (a+b)^2][x^2 + (a-b)^2]}}{V[x^2 + (a+b)^2][x^2 + (a-b)^2]} - \ln(2ab) \right\} dx = \\ = \frac{ab}{b^2 - a^2} \frac{\alpha/2}{\alpha/2} \Gamma \left[\frac{\alpha/2}{1 - \alpha/2}, \frac{(3-\alpha)/2}{(3-\alpha)/2} \right] Q_{-\alpha/2-1} \left(\frac{b^2 + a^2}{b^2 - a^2} \right) \quad (0 < \operatorname{Re} \alpha < 1).$$

$$39. \int_0^{\infty} x^{\alpha-1} \frac{(\sqrt{x^{\mu} + z^{\mu}} + z^{\mu/2})^{\nu}}{(x^{\mu} + z^{\mu})^{\rho}} \ln(\sqrt{x^{\mu} + z^{\mu}} \pm z^{\mu/2}) dx = \\ = \frac{z^{\alpha+(1-2\rho)\mu/2}}{\mu} (\pm \nu)^{1-2r} \left(\frac{\alpha}{\mu} + \frac{\nu \mp \nu}{2} \right)^{2r-1} 2^{\nu+2\alpha/\mu-1} \times \\ \times B \left(\frac{\nu \mp \nu}{2} + \frac{\alpha}{\mu}, 2r - \nu - \frac{2\alpha}{\mu} \right) \times \\ \times \left[\ln(4z^{\mu}) - 2\psi \left(\nu - 2 \frac{\alpha}{\mu} + 2r \right) + 2\psi \left(\frac{1 \pm 1}{2} \mp \nu \mp \frac{\alpha}{\mu} \right) - 2\nu^{2r-1} (1 - 2r) \right] \\ \{ r=0 \text{ или } 1/2; \mu > 0; 2 \operatorname{Re} \alpha > \mu \operatorname{Re}(\pm \nu - \nu); \operatorname{Re}(\mu \nu + 2\alpha) < 2\mu r; \mu, \arg z < \pi \}.$$

$$40. \int_0^{\infty} x^{\alpha-1} \frac{(\sqrt{x^{\mu} + z^{\mu}} + z^{\mu/2})^{\nu}}{(x^{\mu} + z^{\mu})^{\rho}} \ln(\sqrt{x^{\mu} + z^{\mu}} \pm z^{\mu/2}) dx = \\ = \frac{z^{\alpha+(1-2\rho)\mu/2}}{\mu} (\mp \nu)^{1-2r} 2^{2r-2\alpha/\mu-1} \left(\frac{\alpha}{\mu} \mp \frac{\nu}{2} \right)^{2r-1} B \left(2 \frac{\alpha}{\mu}, r \mp \frac{\nu}{2} - \frac{\alpha}{\mu} \right) \times \\ \times \left[\mu \ln z \mp \psi \left(r \mp \frac{\nu}{2} - \frac{\alpha}{\mu} \right) \pm \psi \left(1 \mp \frac{\nu}{2} + \frac{\alpha}{\mu} - r \right) + 2\nu^{2r-1} (1 - 2r) \right] \\ \{ r=0 \text{ или } 1/2; \mu > 0; \operatorname{Re}(2 \pm \mu \nu) < 2r\mu; \mu |\arg z| < \pi; \operatorname{Re} \alpha > 0 (2 \operatorname{Re} \alpha > -\mu \text{ при } z=1) \}.$$

$$\begin{aligned}
 41. \int_0^{\infty} \frac{x^k}{(x^2+z^2)^\rho} & [(1-x^2+z^2-x)^v \ln(1-x^2+z^2-x) + \\
 & + (-1)^k (1-x^2+z^2-x)^v \ln(1-x^2+z^2-x)] dx = \\
 & = 2^{1/2} \pi^{1/2} k^{-2\rho+2v-1} \left(\frac{v}{\rho-1}\right)^k B\left(\frac{v-k-1}{2}, \rho, \rho-\frac{v+k+1}{2}\right) \times \\
 & \times \left[2 \ln z + \psi\left(\frac{v-k-1}{2} + \rho\right) - \psi\left(\rho-\frac{v+k+1}{2}\right) + 2kv\right] \\
 & \quad [k=0 \text{ или } 1; \operatorname{Re} z > 0; 2 \operatorname{Re} \rho > k+1; \operatorname{Re} v \geq 1] \\
 42. \int_0^{\infty} x^k \frac{(2z^2+x^2)^{k-1/2}}{(x^2+z^2)^\rho} & [(1-x^2+2z^2-x)^v \ln(1-x^2+2z^2-x) + \\
 & + (-1)^k (1-x^2+2z^2-x)^v \ln(1-x^2+2z^2-x)] dx = \\
 & = 2^{v/2+\rho-2k-1/2} z^{2k-v-2\rho} \left(\frac{v}{\rho-1}\right)^k B\left(\frac{v-2k+2\rho}{4}, \frac{2\rho-v-2k}{4}\right) \times \\
 & \times \left[2 \ln(2z^2) + \psi\left(\frac{v-2k+2\rho}{4}\right) - \psi\left(\frac{2\rho-v-2k}{4}\right) + 4kv\right] \\
 & \quad [k=0 \text{ или } 1; \operatorname{Re} z > 0; 2 \operatorname{Re} \rho > 2k+1; \operatorname{Re} v \geq 1]
 \end{aligned}$$

2.6.17. Интегралы, содержащие $A(x)$ и $\frac{1}{\ln^a x}$.

$$\begin{aligned}
 1. \int_0^1 \frac{(x^\beta - x^\gamma) dx}{\ln x} &= \ln \frac{\beta+1}{\gamma+1} \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > -1] \\
 2. \int_0^1 (x^\alpha - 1)^n \frac{dx}{\ln x} &= \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} \ln(1+\alpha k) \\
 & \quad [n \operatorname{Re} \alpha > -1; n=1, 2, 3, \dots] \\
 3. \int_0^1 \frac{(x-1)^2 dx}{x^3+1 \ln x} &= \ln \frac{\pi}{4} \\
 4. \int_0^1 \frac{(x^\beta - x^\gamma)^2 dx}{x^2-1 \ln x} &= -\ln(\cos \beta\pi) \quad [|\operatorname{Re} \beta| < 1/2] \\
 5. \int_0^1 \frac{x^\beta - x^\gamma}{(1-cx)^\alpha \ln x} dx &= \ln \frac{\beta+1}{\gamma+1} + \sum_{k=1}^{\infty} \binom{n+k-1}{k} c^k \ln \frac{\beta+k+1}{\gamma+k+1} \\
 & \quad [c < 1; \operatorname{Re} \beta, \operatorname{Re} \gamma > -1] \\
 6. \int_0^1 \frac{x^\beta - x^\gamma}{x^\mu+1 \ln x} dx &= \ln \Gamma \left[\frac{(\beta+\mu+1)/(2\mu), (\gamma+1)/(2\mu)}{(\gamma+\mu+1)/(2\mu), (\beta+1)/(2\mu)} \right] \\
 & \quad [\mu > 0; \operatorname{Re} \beta, \operatorname{Re} \gamma > -1] \\
 7. \int_0^1 \frac{x^{\alpha-1} - x^{-\alpha}}{x^\mu+1 \ln x} dx &= \ln \left(\frac{\alpha\pi}{2\mu} \right) \quad [\mu > 0; 0 < \operatorname{Re} \alpha < 1]
 \end{aligned}$$

2.6.17.]

$$\begin{aligned}
 8. \int_0^1 x^\mu \frac{x^\nu-1}{x^\mu+1} \frac{dx}{\ln x} &= \ln \frac{\pi}{2} \quad [\mu > 0] \\
 9. \int_0^1 x^{\alpha-1} (x^\beta-1)^n (x^\gamma-1)^m \frac{dx}{\ln x} &= \\
 &= \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{j=0}^m (-1)^j \binom{m}{j} \ln [\alpha + \gamma(m-j) + \beta(n-k)] \\
 & \quad [\operatorname{Re} \alpha > \max(0, -n \operatorname{Re} \beta, -m \operatorname{Re} \gamma, -n \operatorname{Re} \beta - m \operatorname{Re} \gamma)] \\
 10. \int_0^1 \frac{x^{\alpha-1}}{x^\beta+1} (x^\beta-1)^n \frac{dx}{\ln x} &= \sum_{k=0}^n (-1)^k \binom{n}{k} \ln \Gamma \left(\frac{\beta(n-k)+\alpha}{\gamma} \right) \\
 & \quad [\operatorname{Re} \alpha > \max(0, \operatorname{Re} \gamma, -n \operatorname{Re} \beta, \operatorname{Re} \gamma - n \operatorname{Re} \beta)] \\
 11. \int_0^1 x^{\alpha-1} (x^\beta-1) (x^\gamma-1) \frac{dx}{\ln x} &= \ln \frac{\alpha(\beta+\gamma+\alpha)}{(\beta+\alpha)(\gamma+\alpha)} \\
 & \quad [\operatorname{Re} \alpha > \max(0, -\operatorname{Re} \beta, -\operatorname{Re} \gamma, -\operatorname{Re} \beta - \operatorname{Re} \gamma)] \\
 12. \int_0^1 \frac{x-1}{x+1} \frac{dx}{(x^2+1) \ln x} &= \ln 2 \\
 13. \int_0^1 x^2 \frac{x-1}{x+1} \frac{dx}{(x^2+1) \ln x} &= \ln \frac{\pi}{2\sqrt{2}} \\
 14. \int_0^1 x^{\alpha-1} \frac{(x^\beta-1)(x^\gamma-1)}{x^\mu-1} \frac{dx}{\ln x} &= \ln \Gamma \left[\frac{\alpha/\mu, (\alpha+\beta+\gamma)/\mu}{(\alpha+\beta)/\mu, (\alpha+\gamma)/\mu} \right] \\
 & \quad [\mu > 0; \operatorname{Re} \alpha > \max(0, -n \operatorname{Re} \beta, -\operatorname{Re} \gamma, -\operatorname{Re} \beta - \operatorname{Re} \gamma)] \\
 15. \int_0^1 \frac{(x-1)^2}{1+2x \cos(m\pi/n) + x^2} \frac{dx}{\ln x} &= \frac{1}{\sin(m\pi/n)} \sum_{k=1}^{(n-1)/r} (-1)^k \sin \frac{k m \pi}{n} \times \\
 & \times \ln \Gamma \left[\frac{r \frac{n-(-1)^r k+1}{2n}, r \frac{n-(-1)^r k+1}{2n}, r \frac{k+2}{2n}, r \frac{k}{2n}}{r \frac{k+1}{2n}, r \frac{k+1}{2n}, r \frac{n-(-1)^r k}{2n}, r \frac{n-(-1)^r k+2}{2n}} \right] \\
 & \quad [m < n; r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l, l=1, 2, 3, \dots] \\
 16. \int_0^1 \frac{(x^\beta - x^\gamma)}{(x-1)^\alpha} \frac{dx}{\ln x} &= \ln \Gamma \left[\frac{\beta+1}{\gamma+1} \right] \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > -1] \\
 17. \int_0^1 \left[1 - \frac{(x^\beta-1)(x^\gamma-1)}{x-1} \right] \frac{dx}{x \ln x} &= \ln B(\beta, \gamma) \quad [\operatorname{Re} \beta, \operatorname{Re} \gamma > 0] \\
 18. \int_0^1 \left(\frac{x^{\beta\mu}-1}{x^\mu-1} - \frac{x^\beta-1}{x-1} \right) \frac{dx}{x \ln x} &= \mu \ln \beta \quad [\mu, \operatorname{Re} \beta > 0]
 \end{aligned}$$

$$19. \int_0^1 \left(\frac{1}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = -\psi(q)$$

[Re q > 0]

$$20. \int_1^\infty \left[\frac{(\mu-1)(x-1)}{x^2} - \frac{1}{x} + \frac{1}{x^{\mu+1}} \right] \frac{dx}{(x-1) \ln x} = \ln \Gamma(\mu)$$

[μ > 0]

$$21. \int_0^\infty \frac{x^\beta - x^\gamma}{x^{\alpha+1} \ln x} dx =$$

$$= \frac{1}{2} \ln \left(\sin \frac{\beta+1}{2\mu} \pi \right) - \ln \left(\cos \frac{\beta+1}{2\mu} \pi \right) + \ln \left(\sin \frac{\gamma+1}{2\mu} \pi \right) + \ln \left(\cos \frac{\gamma+1}{2\mu} \pi \right)$$

[μ > 0; Re β, Re γ > -1/2]

$$22. \int_0^\infty \frac{(x^\beta - x^{-\beta})^2}{x^2 - 1} \frac{dx}{\ln x} = -2 \ln(\cos \beta \pi)$$

[Re β < 1/2]

$$23. \int_0^1 \ln[a + (1-a)x] \frac{dx}{\ln x} = - \sum_{k=1}^\infty \frac{\ln(k+1)}{k} \left(\frac{a-1}{a} \right)^k$$

[a > 1/2]

$$24. \int_0^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\ln x} \ln(1+cx) dx = \sum_{k=1}^\infty \frac{c^k}{k} \ln \frac{\alpha+k}{\beta+k} + \ln \frac{\alpha}{\beta}$$

[Re α, Re β > 0; |c| < 1]

$$25. \int_0^1 x^{a-1} \frac{(x^\beta - 1)^n (x^\gamma - 1)^m}{\ln^2 x} dx = (-1)^{m+n} \sum_{k=0}^n (-1)^k \binom{n}{k} \times$$

$$\times \sum_{j=0}^m (-1)^j \binom{m}{j} \{ \alpha + (n-k)\beta + (m-j)\gamma \} \ln [\alpha + (n-k)\beta + (m-j)\gamma]$$

[Re α > max(0, -n Re β, -m Re γ, -n Re β - m Re γ)]

$$26. \int_0^1 x^{\alpha-1} \frac{(x^\beta - 1)(x^\gamma - 1)}{\ln^2 x} dx =$$

$$= \alpha \ln \alpha - (\alpha + \beta) \ln(\alpha + \beta) - (\alpha + \gamma) \ln(\alpha + \gamma) + (\alpha + \beta + \gamma) \ln(\alpha + \beta + \gamma)$$

[Re α, Re(α + β), Re(α + γ), Re(α + β + γ) > 0]

$$27. \int_0^1 \frac{(x^\beta - x^\gamma)^2}{\ln^2 x} dx =$$

$$= (2\gamma + 1) \ln(2\gamma + 1) - 2(\gamma + \beta + 1) \ln(\gamma + \beta + 1) + (2\beta + 1) \ln(2\beta + 1)$$

[Re γ, Re β > -1/2]

$$28. \int_0^1 \left[\frac{1}{\ln^2 x} - \frac{x}{(1-x)^2} \right] dx = C - \frac{1}{2}$$

$$29. \int_0^1 \left[\frac{1}{\ln^2 x} + \frac{(\rho-2)x - \rho + 1}{(x-1)^2} x^{\rho-1} \right] dx = \rho - \psi(\rho) - \frac{3}{2}$$

[Re ρ > 3/2]

$$30. \int_0^1 [x^{\rho-1}(q-r) + x^{q-1}(r-p) + x^{r-1}(p-q)] \frac{dx}{\ln^2 x} =$$

$$= p(q-r) \ln p + q(r-p) \ln q + r(p-q) \ln r \quad [p, q, r > 0]$$

$$31. \int_0^1 \left[\frac{x^{\rho-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right.$$

$$+ \frac{x^{s-1}}{(s-p)(s-q)(s-r)} \left. \right] \frac{dx}{\ln^2 x} = \frac{1}{2} \left[\frac{\rho^2 \ln p}{(p-q)(p-r)(p-s)} + \right.$$

$$+ \frac{q^2 \ln q}{(q-p)(q-r)(q-s)} + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)} \left. \right]$$

[p, q, r, s > 0]

$$32. \int_0^1 \left(\frac{x-1}{\ln x} - x \right) \frac{dx}{\ln x} = \ln 2 - 1$$

$$33. \int_0^1 \left(\frac{x^q - 1}{x \ln x} - q \right) \frac{dx}{\ln x} = q(\ln q - 1)$$

[q > 0]

$$34. \int_0^1 x^{\alpha-1} \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)} \right] \frac{dx}{\ln x} = -\ln \Gamma(\alpha) + \left(\alpha - \frac{1}{2} \right) \ln \alpha - \alpha + \ln \sqrt{2\pi}$$

[Re α > 0]

$$35. \int_0^1 \left(\frac{1}{x \ln x} + \frac{2}{1-x^2} \right) \frac{dx}{\ln x} = \ln 2$$

$$36. \int_0^1 \left[\frac{1}{\ln x} + \frac{x(3-x)}{2(1-x)} \right] \frac{dx}{x \ln x} = \frac{\ln(2\pi)}{2}$$

$$37. \int_0^1 \left(\frac{1}{\ln x} + \frac{1+x^2}{1-x^2} \right) \frac{dx}{\ln x} = \ln 2 - 1$$

$$38. \int_0^1 \left[p-1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x} \right) x^{p-1} \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) \ln p + p - \frac{\ln(2\pi)}{2}$$

[p > 0]

$$39. \int_0^1 \ln(1+x) \frac{x \ln x + 1 - x}{x \ln^2 x} dx = \ln \frac{4}{\pi}$$

$$40. \int_0^\infty x^{\alpha-1} \ln(1+x^{-\alpha}) \left[(p+\alpha)x^\rho - (q+\alpha)x^q + \frac{x^q - x^\rho}{\ln x} \right] \frac{dx}{\ln x} =$$

$$= -\alpha \ln \left(\operatorname{tg} \frac{q\pi}{2\alpha} \operatorname{ctg} \frac{p\pi}{2\alpha} \right) \quad [p+\alpha, q+\alpha < 0; p, q > 0]$$

$$41. \int_0^1 \left(\frac{x^q - 1}{x \ln^2 x} - \frac{q}{x \ln x} - \frac{q^2}{2} \right) \frac{dx}{\ln x} = \frac{q^2}{2} \ln q - \frac{3}{4} q^2 \quad [2.6.18, \quad |q > 0|]$$

$$42. \int_0^1 \left[\left(p - \frac{1}{2} \right) x^p + \frac{1}{2} (x^{2p-1} - 1) \left(1 - \frac{1}{\ln x} \right) \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) (\ln p - 1) \quad [p > 0]$$

$$43. \int_0^1 \left[\left(q - \frac{1}{2} \right) \frac{x^{q-1} - x^{q-1}}{\ln x} + p \frac{x^{p-1}}{1-x^p} - r \frac{x^{r-1}}{1-x^r} \right] \frac{dx}{\ln x} = (p-r) [1/2 - q - \ln \Gamma(q) + \ln \sqrt{2\pi}] \quad [q > 0]$$

$$44. \int_0^1 \left(\frac{x^q - 1}{x \ln^2 x} - \frac{q}{x \ln^2 x} - \frac{q^2}{2x \ln x} - \frac{q^3}{6} \right) \frac{dx}{\ln x} = \frac{q^3}{6} \ln q - \frac{11}{36} q^3 \quad [q > 0]$$

$$45. \int_0^1 \frac{x - (1 - \ln x)^q}{x \ln x} dx = -\psi(-q) \quad [\operatorname{Re} q < 0]$$

2.6.18. Интегралы от $\frac{A(x) \ln^m x}{\ln^m x + a}$

$$1. \int_0^1 \frac{dx}{\ln x + a} = e^{-a} \operatorname{Ei}(a) \quad [a \neq 0]$$

$$2. \int_0^1 \frac{x^{a-1} dx}{(\ln x + a)^n} = \frac{1}{(n-1)!} \left[\alpha^{n-1} e^{-\alpha a} \operatorname{Ei}(\pm \alpha a) - \sum_{k=1}^{n-1} (n-k-1)! \alpha^{k-1} (\pm a)^{k-n} \right] \quad \left[\begin{array}{l} [a, \operatorname{Re} \alpha > 0, n=1, 3, 5, \dots] \\ [a, \operatorname{Re} \alpha > 0, n=1, 2, 3, \dots] \end{array} \right]$$

$$3. \int_0^{\infty} \frac{dx}{x^2 (\ln x + a)} = -e^a \operatorname{Ei}(-a) \quad [a \neq 0]$$

$$4. \int_0^1 \frac{x^{a-1} \ln^m x}{\ln^2 x + a^2} dx = \frac{1}{a} \frac{\partial^m}{\partial a^m} [\sin \alpha a \operatorname{ci}(\alpha a) - \cos \alpha a \operatorname{si}(\alpha a)] \quad [a, \operatorname{Re} \alpha > 0]$$

$$5. \int_0^1 \frac{x^{a-1} \ln x}{\ln^2 x + a^2} dx = \sin \alpha a \operatorname{si}(\alpha a) + \cos \alpha a \operatorname{ci}(\alpha a) \quad [a, \operatorname{Re} \alpha > 0]$$

$$6. \int_0^1 \frac{x^{a-1/2-1}}{\ln^2 x + a^2} \frac{dx}{1+x^2} = \frac{1}{2a} \beta \left(\frac{a+1}{4\pi} \right) \quad [a \pm \pi; a > 0]$$

$$7. \quad = \frac{\mu}{8\pi} (4 - \pi) \quad [\mu = 2\pi]$$

$$8. \quad = \frac{\mu \ln 2}{2\pi} \quad [\mu = \pi]$$

$$9. \quad = \frac{\mu \sqrt{2}}{2\pi} [\pi + 2 \ln(\sqrt{2} - 1)] \quad [2\mu = \pi]$$

$$10. \int_0^1 \frac{x^a - x^{-a}}{\ln^2 x + a^2} \frac{dx}{1-x^2} = \frac{2\pi}{a} \sum_{k=1}^{\infty} (-1)^k \frac{\sin k\alpha\pi}{2a + k\pi} \quad [a > 0; \operatorname{Re} \alpha < 1]$$

$$11. \int_0^1 \frac{\ln x}{\ln^2 x + a^2} \frac{dx}{1-x^2} = \frac{1}{4} (1 - 2 \ln 2) \quad [a = \pi]$$

$$12. \quad = \frac{1}{4} (2 - \pi) \quad [2a = \pi]$$

$$13. \quad = \frac{1}{2\sqrt{2}} [2\sqrt{2} - \pi + 2 \ln(\sqrt{2} - 1)] \quad [4a = \pi]$$

$$14. \int_0^1 \frac{x \ln x}{\ln^2 x + a^2} \frac{dx}{1-x^2} = \frac{1}{2} \left[\frac{\pi}{2a} + \ln \frac{a}{a} + \psi \left(\frac{a}{\pi} \right) \right] \quad [\operatorname{Re} a > 0]$$

$$15. \int_0^1 \frac{\ln x}{(\ln^2 x + a^2)^2} \frac{dx}{1-x} = -\frac{\pi^2}{a^4} \sum_{k=1}^{\infty} \left(\frac{2\pi}{a} \right)^{2k-2} |B_{2k}| \quad [\operatorname{Re} a > 0]$$

$$16. \int_0^1 \frac{\ln(1-x^2) dx}{x (\ln^2 x + a^2)} = -\frac{\pi}{a} \ln \Gamma \left(\frac{a+2\pi}{2\pi} \right) + \frac{\pi}{2a} \ln(a\pi) - \frac{\mu}{2} \left(\ln \frac{2\pi}{a\mu} + 1 \right) \quad [a, \mu > 0]$$

2.6.19. Интегралы от произведений логарифмов.

$$1. \int_0^1 \ln x \ln(1 \pm x) dx = 2 - \frac{3}{24} \pi^2 - (1 \pm 1) \ln 2$$

$$2. \int_0^1 x^n \ln x \ln(1 \pm x) dx = \frac{(1 \pm 1)^n}{n+1} \left[\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n+1} + \frac{1}{k} \right) - \frac{3}{24} \pi^2 \right] + \frac{(1 \pm 1) [(-1)^{n+1} - 1]}{2(n+1)^2} \ln 2$$

$$3. \int_0^1 \ln^m x \ln^n(1-x) \frac{dx}{x} = I_{m,n} \quad [m, n=1, 2, 3, \dots]$$

$$I_{m,n} = \frac{\partial^{m+n}}{\partial v^m \partial \mu^n} B(v, \mu+1) \Big|_{v=\mu=1}, \quad I_{m,1} = (-1)^{m+1} m! \zeta(m+2),$$

$$I_{1,2} = -\frac{\pi^4}{180}, \quad I_{2,2} = 8\zeta_5(5) - \frac{2}{3} \pi^2 \zeta_3(3), \quad I_{3,2} = 6\zeta_7(3) - \frac{\pi^6}{105}$$

$$4. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1 \pm x) \frac{dx}{x} = -\Gamma(\sigma+1) \zeta(\sigma+2) \left\{ \frac{2^{-\sigma-1}}{1} - 1 \right\} \quad [\operatorname{Re} \sigma > -1].$$

$$5. \int_0^1 \ln^{2n} x \ln(1 \pm x) \frac{dx}{x} = -\frac{\pi^{2n+2} 2^{2n+1}}{(2n+1)(2n+2)} B_{2n+2} \left\{ \frac{2^{-2n-1}}{1} - 1 \right\}.$$

$$6. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1 - bx^u) \frac{dx}{x} = -\frac{\Gamma(\sigma+1)}{\mu^{\sigma+1}} \operatorname{Li}_{\sigma+1}(b) \quad [\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1].$$

$$7. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1 - 2bx^u \cos \gamma + b^2 x^{2u}) \frac{dx}{x} = -2\Gamma(\sigma+1) \mu^{-\sigma-1} \sum_{k=1}^{\infty} \frac{b^k \cos(k\gamma)}{k^{2\sigma+2}} \\ [\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1].$$

$$8. \int_0^{\infty} x^{\alpha-1} \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \\ = \frac{\pi^2}{(a-1) \sin^2 \alpha \pi} [(\alpha^{\alpha-1} + 1) \ln a - 2\pi(\alpha^{\alpha-1} - 1) \operatorname{ctg} \alpha \pi] \\ [\alpha > 0; a \neq 1; 0 < \operatorname{Re} \alpha < 2; \alpha \neq 1].$$

$$9. \int_0^{\infty} \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \frac{\ln a}{6(a-1)} (4\pi^2 + \ln^2 a) \quad [a > 0; a \neq 1].$$

$$10. \int_0^{\infty} \frac{\ln |1 \pm x|}{x} \frac{x \ln x - x - a}{(x+a)^2} dx = \frac{1}{2(a-1)} \left(\ln^2 a + \frac{0}{\pi^2} \right) \quad [a > 0].$$

$$11. \int_0^{\infty} \frac{\ln^2 x dx}{(x-1)^2} = \frac{2}{3} \pi^2.$$

$$12. \int_0^{\infty} \frac{\ln x}{x^2} \ln(1 + ax^2) dx = \pi \sqrt{a} \left(1 - \frac{\ln a}{2} \right) \quad [\operatorname{Re} a > 0].$$

$$13. \int_0^{\infty} \ln x \ln \frac{x^2 + a^2}{x^2 + b^2} dx = \pi \left(b - a + \ln \frac{a}{b} \right) \quad [a, b > 0].$$

$$14. \int_0^{\infty} \ln x \ln \frac{a^2 + 2bx + x^2}{a^2 - 2bx + x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad [b \leq a].$$

$$15. \int_0^{\infty} \ln(1 + a^2 x^2) \ln(b^2 + c^2 x^2) \frac{dx}{x^3} = \\ = \frac{2\pi}{b} [(ab+c) \ln(ab+c) - ab \ln a - c \ln c] \quad [a, b, c > 0].$$

$$16. \int_0^{\infty} \ln(1 + a^2 x^2) \ln(1 + c^2 x^2) \frac{dx}{x^3} = \\ = \frac{2\pi}{3} [ac(a+c) + a^2 \ln a + c^2 \ln c - (a^3 + c^3) \ln(a+c)] \quad [a, c > 0].$$

$$17. \int_0^{\infty} \ln(1 + a^2 x^{2\lambda}) \ln(b^2 + c^2 x^{2\lambda}) \frac{dx}{x^{1+\lambda}} = \frac{2\pi}{c} [(b+ac) \ln(b+ac) - b \ln b - ac] \\ [a, b, c > 0].$$

2.6.20. Интегралы, содержащие $\ln \ln x$.

$$1. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{x + e^{2\lambda}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-k\lambda} (C + \ln k) \quad [\operatorname{Im} \lambda = 0].$$

$$2. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k}.$$

$$3. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x)^2} = \frac{1}{2} \left(\ln \frac{\pi}{2} - C \right).$$

$$4. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1 + 2x \cos \gamma + x^2} = \frac{\pi}{2 \sin \gamma} \ln \left\{ (2\pi)^{\gamma/\pi} \Gamma \left[\frac{(\pi+\gamma)/(2\pi)}{(\pi-\gamma)/(2\pi)} \right] \right\} \quad [|\gamma| < \pi].$$

$$5. \int_0^1 \ln \ln \frac{1}{x} \frac{x^{n-1}}{1+x^2+x^4+\dots+x^{2n}} dx = \frac{\pi}{2n+2} \operatorname{tg} \frac{\pi}{2n+2} \ln \frac{2\pi}{r} + \\ + \frac{\pi}{n+1} \sum_{k=1}^{n/r} (-1)^{k-1} \sin \frac{k\pi}{n+1} \ln \Gamma \left[\frac{r(n+1 - (-1)^k k)/(2n+2)}{rk/(2n+2)} \right] \\ [r=2 \text{ при } n=2l, r=1 \text{ при } n=2l-1, l=1, 2, 3, \dots].$$

$$6. \int_0^1 x^{2\sigma-1} \ln \ln \frac{1}{x} \ln^\sigma \frac{1}{x} dx = \alpha^{-\sigma-1} \Gamma(\sigma+1) \{ \psi(\sigma+1) - \ln \alpha \} \\ [\operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1].$$

$$7. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x^2) \sqrt{\ln(1/x)}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + 2 \ln 2 + C].$$

2.6.21. Интегралы от $x^\alpha e^{-\rho x^\mu} \ln^\pi x$.

$$1. \int_0^{\infty} x^{2\alpha-1} e^{-\rho x^\mu} \ln^\pi x dx = \frac{1}{\mu} \left(\frac{\partial}{\partial \alpha} \right)^\pi \left[\rho^{-\alpha/\mu} \Gamma \left(\frac{\alpha}{\mu} \right) \right] \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0].$$

$$2. \int_0^{\infty} x^{2\alpha-1} e^{-\rho x^\mu} \ln x dx = \mu^{-2} \rho^{-\alpha/\mu} \Gamma(\alpha/\mu) \{ \psi(\alpha/\mu) - \ln \rho \} \quad [\mu, \operatorname{Re} \alpha, \operatorname{Re} \rho > 0].$$

$$4. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1 \pm x) \frac{dx}{x} = -\Gamma(\sigma+1) \zeta(\sigma+2) \left\{ \frac{2^{-\sigma-1} - 1}{1} \right\} \quad [\operatorname{Re} \sigma > -1]$$

$$5. \int_0^1 \ln^{2n} x \ln(1 \pm x) \frac{dx}{x} = -\frac{\pi^{2n+2} 2^{2n+1}}{(2n+1)(2n+2)} B_{2n+2} \left\{ \frac{2^{-2n-1} - 1}{1} \right\}.$$

$$6. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1-bx^\mu) \frac{dx}{x} = -\frac{\Gamma(\sigma+1)}{\mu^{\sigma+1}} \operatorname{Li}_{\sigma+2}(b) \quad (\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1).$$

$$7. \int_0^1 \ln^\sigma \frac{1}{x} \ln(1-2bx^i \cos \gamma + b^2 x^{2i}) \frac{dx}{x} = -2\Gamma(\sigma+1) \mu^{-\sigma-1} \sum_{k=1}^{\infty} \frac{b^k \cos(k\gamma)}{k^{\sigma+2}} \quad (\mu > 0; \operatorname{Re} \sigma > -1; |b| < 1).$$

$$8. \int_0^{\infty} x^{\alpha-1} \ln x \ln \frac{x}{a(x-1)(x-a)} dx = \frac{\pi^2}{(a-1) \sin^2 \alpha \pi} [(\alpha^{\alpha-1} + 1) \ln a - 2\pi(\alpha^{\alpha-1} - 1) \operatorname{ctg} \alpha \pi] \quad (a > 0; a \neq 1; 0 < \operatorname{Re} \alpha < 2, \alpha \neq 1).$$

$$9. \int_0^{\infty} \ln x \ln \frac{x}{a(x-1)(x-a)} \frac{dx}{x} = \frac{\ln a}{6(a-1)} (4\pi^2 + \ln^2 a) \quad (a > 0, a \neq 1).$$

$$10. \int_0^{\infty} \frac{\ln |1 \pm x|}{x} \frac{x \ln x - x - a}{(x+a)^2} dx = \frac{1}{2(a+1)} \left(\ln^2 a + \frac{0}{a^2} \right) \quad (a > 0).$$

$$11. \int_0^{\infty} \frac{\ln^2 x dx}{(x-1)^2} = \frac{2}{3} \pi^2.$$

$$12. \int_0^{\infty} \frac{\ln x}{x^2} \ln(1+ax^2) dx = \pi \sqrt{a} \left(1 - \frac{\ln a}{2} \right) \quad [\operatorname{Re} a > 0].$$

$$13. \int_0^{\infty} \ln x \ln \frac{x^2+a^2}{x^2+b^2} dx = \pi \left(b-a + \ln \frac{a^a}{b^b} \right) \quad (a, b > 0).$$

$$14. \int_0^{\infty} \ln x \ln \frac{a^2+2bx+x^2}{a^2-2bx+x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad (|b| \leq a).$$

$$15. \int_0^{\infty} \ln(1+a^2x^2) \ln(b^2+c^2x^2) \frac{dx}{x^2} = \frac{\pi^2}{b} [(ab+c) \ln(ab+c) - ab \ln a - c \ln c] \quad (a, b, c > 0).$$

$$16. \int_0^{\infty} \ln(1+a^2x^2) \ln(1+c^2x^2) \frac{dx}{x^2} = \frac{2\pi}{3} [ac(a+c) + a^2 \ln a + c^2 \ln c - (a^2+c^2) \ln(a+c)] \quad (a, c > 0).$$

$$17. \int_0^{\infty} \ln(1+a^2x^2) \ln(b^2+c^2x^2) \frac{dx}{x^{1+\lambda}} = \frac{2\pi}{c} [(b+ac) \ln(b+ac) - b \ln b - ac] \quad (a, b, c > 0).$$

2.6.20. Интегралы, содержащие $\ln \ln x$.

$$1. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{x+e^{i\lambda}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-i\lambda k} (C + \ln k) \quad (\operatorname{Im} \lambda = 0).$$

$$2. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k}.$$

$$3. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x)^2} = \frac{1}{2} \left(\ln \frac{\pi}{2} - C \right).$$

$$4. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{1+2x \cos \gamma + x^2} = \frac{\pi}{2 \sin \gamma} \ln \left\{ (2\pi)^{\gamma/\pi} \Gamma \left[\frac{(\pi+\gamma)/(2\pi)}{(\pi-\gamma)/(2\pi)} \right] \right\} \quad (|\gamma| < \pi).$$

$$5. \int_0^1 \ln \ln \frac{1}{x} \frac{x^{n-1}}{1+x^2+x^4+\dots+x^{2n}} dx = \frac{\pi}{2n+2} \operatorname{tg} \frac{\pi}{2n+2} \ln \frac{2\pi}{r} + \frac{\pi}{n+1} \sum_{k=1}^{n/r} (-1)^{k-1} \sin \frac{k\pi}{n+1} \ln \Gamma \left[\frac{r(n+1-(-1)^k k)/(2n+2)}{rk/(2n+2)} \right] \quad (r=2 \text{ при } n=2l, r=1 \text{ при } n=2l-1, l=1, 2, 3, \dots).$$

$$6. \int_0^1 x^{2-1} \ln \ln \frac{1}{x} \ln^\sigma \frac{1}{x} dx = \alpha^{-\sigma-1} \Gamma(\sigma+1) [\psi(\sigma+1) - \ln \alpha] \quad (\operatorname{Re} \alpha > 0; \operatorname{Re} \sigma > -1).$$

$$7. \int_0^1 \ln \ln \frac{1}{x} \frac{dx}{(1+x^2) \sqrt{\ln(1/x)}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} (\ln(2k+1) + 2 \ln 2 + C).$$

2.6.21. Интегралы от $x^\alpha e^{-px^\mu} \ln^\alpha x$.

$$1. \int_0^{\infty} x^{2-1} e^{-px^\mu} \ln^\alpha x dx = \frac{1}{\mu} \left(\frac{\partial}{\partial \alpha} \right)^\alpha \left[p^{-\alpha/\mu} \Gamma \left(\frac{\alpha}{\mu} \right) \right] \quad (\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0).$$

$$2. \int_0^{\infty} x^{2-1} e^{-px^\mu} \ln x dx = \mu^{-2} p^{-\alpha/\mu} \Gamma(\alpha/\mu) [\psi(\alpha/\mu) - \ln p] \quad (\mu, \operatorname{Re} \alpha, \operatorname{Re} p > 0).$$

$$3. \int_0^{\infty} x^{\alpha-1} e^{-px} \ln x \, dx = \frac{n!}{p^{\alpha+1}} \left(\sum_{k=1}^n \frac{1}{k} - C - \ln p \right)$$

[Re p > 0].

$$4. \int_0^{\infty} e^{-px} \ln x \, dx = -\frac{1}{p} (C + \ln p)$$

[Re p > 0].

$$5. \int_0^{\infty} x^{\alpha-1/2} e^{-px} \ln x \, dx = \frac{\sqrt{\pi} (2\alpha-1)!!}{2^{\alpha} p^{\alpha+1/2}} \left[2 \sum_{k=1}^{\alpha} \frac{1}{2k-1} - C - \ln(4p) \right]$$

[Re p > 0].

$$6. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x}} \ln x \, dx = -\sqrt{\frac{\pi}{p}} [C + \ln(4p)]$$

[Re p > 0].

$$7. \int_0^{\infty} x^{\alpha-1} e^{-px} \ln^2 x \, dx = \mu^{-3} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \left\{ \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right]^2 + \Psi'\left(\frac{\alpha}{\mu}\right) \right\}$$

[μ, Re α, Re p > 0].

$$8. \int_0^{\infty} x^{\alpha-1} e^{-px} \ln^2 x \, dx = \mu^{-3} p^{-\alpha/\mu} \Gamma\left(\frac{\alpha}{\mu}\right) \left\{ \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right]^2 + \right. \\ \left. + 3 \left[\Psi\left(\frac{\alpha}{\mu}\right) - \ln p \right] \Psi'\left(\frac{\alpha}{\mu}\right) + \Psi''\left(\frac{\alpha}{\mu}\right) \right\}$$

[μ, Re α, Re p > 0].

2.6.22. Интегралы от $x^{\alpha} e^{ax^2+bx} \ln^{\alpha} x$.

$$1. \int_0^{\infty} x^{\alpha-1} e^{-px^2-2qx} \ln^{\alpha} x \, dx = \left(\frac{\partial}{\partial \alpha} \right)^{\alpha} [(4p)^{-\alpha/2} \Gamma(\alpha) \Psi(\alpha/2, 1/2; q^2/p)]$$

[Re α, Re p > 0] или [Re α > 0, Re p = 0, Re q > 0] или [0 < Re α < 2, Re p = Re q = 0, Im p ≠ 0].

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px^2-2qx} \ln x \, dx = -2^{-\alpha-1} p^{-\alpha/2} \Gamma(\alpha) \ln p \Psi(\alpha/2, 1/2; q^2/p) +$$

$$+ \frac{p^{-\alpha/2}}{4} \Gamma\left(\frac{\alpha}{2}\right) \sum_{k=0}^{\infty} \frac{(\alpha/2)_k \Psi(\alpha/2+k)}{(1/2)_k k!} \left(\frac{q^2}{p}\right)^k -$$

$$- \frac{q}{2} p^{-(\alpha+1)/2} \Gamma\left(\frac{\alpha+1}{2}\right) \sum_{k=0}^{\infty} \frac{((\alpha+1)/2)_k \Psi((\alpha+1)/2+k)}{(3/2)_k k!} \left(\frac{q^2}{p}\right)^k$$

[Re α, Re p > 0] или [Re α > 0, Re p = 0, Re q > 0] или [0 < Re α < 2, Re p = Re q = 0, Im p ≠ 0].

$$3. \int_0^{\infty} x (px^2 - qx - 1) e^{-px^2+2qx} \ln x \, dx =$$

$$= \frac{1}{4p} + \frac{q}{4p} \sqrt{\frac{\pi}{p}} \exp(q^2/p) [1 + \operatorname{erf}(q/\sqrt{p})]$$

[Re p > 0].

$$4. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \ln^{\alpha} x \, dx = 2 \left(\frac{\partial}{\partial \alpha} \right)^{\alpha} \left[\left(\frac{q}{p} \right)^{\alpha/2} K_{\alpha}(2\sqrt{pq}) \right]$$

[Re p, Re q > 0].

$$5. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \ln x \, dx = -2\pi \operatorname{sig} \alpha \Gamma(\alpha/2) K_{\alpha}(2\sqrt{pq}) -$$

$$- \frac{\pi}{\sin \alpha \pi} [\ln p l_{-\alpha}(2\sqrt{pq}) + \ln q l_{\alpha}(2\sqrt{pq}) -$$

$$- p^{-\alpha} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! \Gamma(k-\alpha+1)} \Psi(k-\alpha+1) - q^{\alpha} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! \Gamma(k+\alpha+1)} \Psi(k+\alpha+1)]$$

[Re p, Re q > 0; α ≠ 0, ±1, ±2, ...].

$$6. \int_0^{\infty} x^{\alpha-1} e^{-px-q/x} \ln x \, dx = \frac{(-1)^{\alpha} q^{\alpha+1}}{2} \sum_{k=0}^{\infty} \frac{(pq)^k}{k! (k+n+1)!} \times$$

$$\times [\ln^2 q - \ln^2 p - 2 \ln q \Psi(k+n+2) + \Psi^2(k+n+2) - \Psi^2(k+1) +$$

$$+ \Psi'(k+1) - \Psi'(k+n+2)] + p^{-n-1} \sum_{k=0}^n \frac{(-pq)^k}{k!} (n-k)! [\Psi(n-k+1) - \ln p]$$

[Re p, Re q > 0; n = -1, 0, 1, 2, ...].

$$7. \int_0^{\infty} e^{-px-q/x} \ln x \frac{dx}{x} = \ln \frac{q}{p} K_0(2\sqrt{pq})$$

[Re p, Re q > 0].

$$8. \int_0^{\infty} x^{-1} \pm 1/2 e^{-px-q/x} \ln x \, dx =$$

$$= \sqrt{\frac{\pi}{p}} \left(\frac{p}{q} \right)^{(1 \mp 1)/4} \left[\frac{1}{2} e^{-2\sqrt{pq}} \ln \frac{q}{p} \mp e^{2\sqrt{pq}} \operatorname{Ei}(-4\sqrt{pq}) \right]$$

[Re p, Re q > 0].

$$9. \int_0^{\infty} \left(p - \frac{q}{x^2} \right) e^{-px-q/x} \ln x \, dx = 2K_0(2\sqrt{pq})$$

[Re p, Re q > 0].

2.6.23. Интегралы от $x^{\alpha} e^{px} \ln(a+bx)$.

$$1. \int_0^1 x e^x \ln(1-x) \, dx = 1 - e.$$

$$2. \int_a^{\infty} \frac{e^{-px}}{x} \ln(bx-ab) \, dx = \frac{1}{2} \operatorname{Ei}^2(-ap) - \ln(ab) \operatorname{Ei}(-2ap)$$

[a, b, Re p > 0].

$$3. \int_0^{\infty} \frac{x^{\alpha-1} e^{-px}}{(a+bx)^{\mu}} \ln^{\alpha}(a+bx) \, dx =$$

$$= (-1)^{\alpha} \left(\frac{a}{b} \right)^{\alpha} \Gamma(\alpha) \left(\frac{\partial}{\partial \lambda} \right)^{\alpha} \left[a^{-\alpha} \Psi\left(\alpha, 1-\rho+\alpha, \frac{ap}{b}\right) \right]$$

[Re p > 0; at z(a+bx) < π при z ≥ 0; Re α > 0 (Re α > -n при α = -1)].

$$4. \int_0^{\infty} x^{\alpha-1} e^{-px} \ln(a+bx) dx = \\ = -\left(\frac{a}{b}\right)^{\alpha} \frac{\pi}{\alpha \sin \alpha \pi} {}_1F_1\left(\alpha; \alpha+1; \frac{ap}{b}\right) + \Gamma(\alpha) p^{-\alpha} \left\{ \ln \frac{ap}{b} - \psi(\alpha) \right\} + \\ + \frac{ap}{b(1-\alpha)} {}_2F_2\left(1, 1; 2, 2-\alpha; \frac{ap}{b}\right) \\ \text{[Re } \rho > 0; |\arg(a+bx)| < \pi \text{ при } x \geq 0; \text{ Re } \alpha > 0 \text{ (Re } \alpha > -1 \text{ при } \alpha = -1)]$$

$$5. \int_0^{\infty} e^{-px} \ln \left| \frac{(a+bx)}{(a-bx)} \right| dx = \frac{1}{p} \left[\ln a - e^{\frac{ap}{b}} \text{Ei} \left(-\frac{ap}{b} \right) \right] \\ \text{[Re } \rho > 0; \left\{ \begin{array}{l} |\arg(a/b)| < \pi \\ a, b > 0 \end{array} \right\}]$$

$$6. \int_0^{\infty} \frac{e^{-px}}{x} \ln(1+bx) dx = \frac{1}{2} \left(\ln^2 \frac{p}{b} + 2\text{C} \ln \frac{p}{b} + \text{C}^2 + \frac{\pi^2}{2} \right) - \\ - \sum_{k=1}^{\infty} \frac{2k^{-1} + \psi(k) - \ln p + \ln b}{k!k} \left(\frac{p}{b} \right)^k \quad \text{[Re } \rho, \text{ Re } b > 0]$$

$$7. \int_0^{\infty} e^{-px} \ln(x+a+b) \left(\frac{1}{x+a} + \frac{1}{x+b} \right) dx = (1 + \ln a \ln b) \ln(a+b) + \\ + e^{-i\pi a/b} \text{Ei}(-ap) \text{Ei}(-bp) + [1 - \ln(ab)] \text{Ei}(-ap - bp) \quad [a, b, \text{Re } \rho > 0]$$

$$8. \int_0^{\infty} e^{-px} \ln \left| \frac{x+a}{x-a} \right| dx = \frac{2}{p} [\text{shi}(ap) \text{chi}(ap) - \text{chi}(ap) \text{sh}(ap)] \\ [a, \text{Re } \rho > 0]$$

2.6.24. Интегралы от $A(x) e^{-px} \ln(ax^2+bx+c)$.

$$1. \int_0^a \frac{e^{-px}}{\sqrt{x(a-x)}} \ln[x(a-x)] dx = \\ = \pi e^{-ap/2} \left[\frac{\pi}{2} Y_0\left(\frac{ap}{2}\right) + \left(\ln \frac{a^2}{8} - \text{C} \right) J_0\left(\frac{ap}{2}\right) \right] \quad [a, \text{Re } \rho > 0]$$

$$2. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2-a^2}} \ln(x^2-a^2) dx = -\left(\text{C} + \ln \frac{2p}{a} \right) K_0(ap) \quad [a, \text{Re } \rho > 0]$$

$$3. \int_{a+b}^{\infty} \frac{e^{-px}}{x} \ln[(x-a)(x-b)] dx = \\ = \text{Ei}(-ap) \text{Ei}(-bp) - \ln(ab) \text{Ei}(-ap-bp) \quad [a, b, \text{Re } \rho > 0]$$

$$4. \int_0^{\infty} e^{-px} \ln[(a+x)(b+x)] \frac{dx}{x+a+b} = \\ = e^{p(a+b)} \{ \text{Ei}(-ap) \text{Ei}(-bp) - \ln(ab) \text{Ei}(-ap-bp) \} \\ \text{[Re } \rho > 0; |\arg(a+b)| < \pi]$$

$$5. \int_0^{\infty} e^{-px} \ln(x^2+a^2) dx = \frac{2}{p} [\ln a - \text{ci}(ap) \cos ap - \text{si}(ap) \sin ap] \\ \text{[Re } a, \text{Re } \rho > 0]$$

$$8. \int_0^{\infty} e^{-px} \ln \left| \frac{x^2-a^2}{x^2+a^2} \right| dx = \frac{1}{p} [2 \ln a - e^{ap} \text{Ei}(-ap) - e^{-ap} \text{Ei}(ap)] \\ \text{[Re } \rho > 0; \left\{ \begin{array}{l} a > 0 \\ |\text{Im } a| > 0 \end{array} \right\}]$$

$$7. \int_0^{\infty} \frac{e^{-px}}{x} \ln(1+a^2x^2) dx = \text{ci}^2\left(\frac{p}{a}\right) + \text{si}^2\left(\frac{p}{a}\right) \quad \text{[Re } a, \text{Re } \rho > 0]$$

$$8. \int_0^{\infty} \frac{e^{-px}}{x} \ln|1-a^2x^2| dx = \text{Ei}\left(\frac{p}{a}\right) \text{Ei}\left(-\frac{p}{a}\right) \quad [a, \text{Re } \rho > 0]$$

2.6.25. Интегралы от $e^{-px} \ln A(x)$.

$$1. \int_a^{\infty} e^{-px} \ln(x + \sqrt{x^2-a^2}) dx = \frac{1}{p} [K_0(ap) + \ln ae^{-ap}] \quad [a, \text{Re } \rho > 0]$$

$$2. \int_a^{\infty} e^{-px} \ln(\sqrt{x+a} + \sqrt{x-a}) dx = \frac{1}{2p} [K_0(ap) + \ln(2a) e^{-ap}] \quad [a, \text{Re } \rho > 0]$$

$$3. \int_0^{\infty} e^{-px} \ln(\sqrt{x^2+z} + \sqrt{x}) dx = \frac{1}{2p} [e^{pz/2} K_0(pz/2) + \ln z] \\ \text{[Re } \rho > 0; |\arg z| < \pi]$$

$$4. \int_0^{\infty} e^{-px} \ln(\sqrt{x+ib} + \sqrt{x-id}) dx = \frac{\pi}{4p} [H_0(bp) - Y_0(bp) + \ln \sqrt{2b}] \\ [b, \text{Re } \rho > 0]$$

$$5. \int_0^{\infty} e^{-px} \ln(\sqrt{x^2+z^2} + x) dx = \frac{\pi}{2p} [H_0(pz) - Y_0(pz)] + \frac{\ln z}{p}$$

$$8. \int_0^{\infty} x^n e^{-px} \ln(\sqrt{x^2+z^2} + x) dx = \\ = \frac{(-1)^n \pi}{2} \frac{\partial^n}{\partial p^n} \left\{ \frac{1}{p} [H_0(pz) - Y_0(pz) + \frac{2}{\pi} \ln z] \right\} \quad \text{[Re } \rho, \text{Re } z > 0]$$

$$7. \int_0^{\infty} x e^{-px} \ln(\sqrt{x^2+z^2} + x) dx = \\ = \frac{\pi}{2p^2} [H_0(pz) - Y_0(pz)] + \frac{\pi z}{2p} [H_1(pz) - Y_1(pz)] - \frac{z}{p} + \frac{1}{p^2} \ln z \\ \text{[Re } \rho, \text{Re } z > 0]$$

$$8. \int_0^{\infty} e^{-\rho x} \ln(Vx^2 + 2zx + z) dx = \frac{1}{\rho} [e^{\rho z} K_0(\rho z) + \ln z]$$

$$[\operatorname{Re} \rho > 0; |\arg z| < \pi]$$

$$9. \int_0^{\infty} \frac{e^{-\rho x}}{x^2 + z^2} \ln(Vx^2 + z^2 + x) dx =$$

$$= \frac{\pi}{2} \ln z [H_0(\rho z) - Y_0(\rho z)] - S_{-1,0}(\rho z) \quad [\operatorname{Re} \rho, \operatorname{Re} z > 0]$$

2.6.26. Интегралы от $R(e^x) \ln x$.

$$1. \int_0^{\infty} \frac{\ln x}{e^x + a} dx = a^{-1} \left[\sum_{k=1}^{\infty} \frac{(-a)^k}{k} \ln k - C \ln(1+a) \right] \quad [-1 < a \leq 1]$$

$$2. \int_0^{\infty} \frac{x^{n-1} \ln x}{(e^x + 1)^n} dx = \frac{\Gamma(n)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k (k+n-1)!}{k! (k+n)^n} [\psi(n) - \ln(k+n)]$$

$$[\operatorname{Re} n > 0; n = 1, 2, 3, \dots]$$

$$3. \int_0^{\infty} \frac{e^x \ln x}{e^{2x} + 2e^x \cos \gamma + 1} dx =$$

$$= \frac{1}{2 \sin \frac{\gamma}{2}} \left\{ \gamma \ln(2\pi) - \pi \ln \pi + \pi \ln \cos \frac{\gamma}{2\pi} + 2\pi \ln \Gamma\left(\frac{n+\gamma}{2\pi}\right) \right\} \quad [0 < \gamma < \pi]$$

$$4. \int_0^{\infty} \frac{e^x \ln x}{(e^x + 1)^3} dx = -\frac{C}{2} + \frac{1}{2} \ln \frac{\pi}{2}$$

2.6.27. Интегралы, содержащие x^{α} , $e^{-\rho x}$ и $\frac{1}{\ln^{\alpha}(x+a)}$.

$$1. \int_0^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\ln x} e^{-\rho x} dx = \sum_{k=0}^{\infty} \frac{(-\rho)^k}{k!} \ln \frac{\alpha+k}{\beta+k} \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg \rho| < \pi]$$

$$2. \int_0^{\infty} e^{-x} \frac{dx}{\ln x} = 0$$

$$3. \int_0^{\infty} (x^{\alpha-1} - x^{\beta-1}) e^{-\rho x} \frac{dx}{\ln x} = \frac{1}{\rho} [\lambda(\rho, \alpha-1) - \lambda(\rho, \beta-1)]$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} \rho > 0]$$

$$4. \int_0^{\infty} \left[e^{-x} - \frac{x(1+x)^{-q-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln q \quad [\operatorname{Re} q > 0]$$

$$5. \int_0^{\infty} \left[(q-1)e^{-x} + \frac{(1+x)^{-q} - (1+x)^{-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln \Gamma(q) \quad [\operatorname{Re} q > 0]$$

$$6. \int_0^{\infty} \frac{x^{n-1} e^{-\rho x}}{\ln^2 x + \pi^2} dx = (-1)^{n-1} [\psi^{(n-1)}(\rho) - e^{\rho}] \quad [\operatorname{Re} \rho > 0]$$

2.6.28. Интегралы от $x^{\alpha} \ln(a + be^{-\rho x})$

$$1. \int_0^{\infty} x^n \ln(1 - e^{-ax}) dx = -\left(\frac{2}{a}\right)^{n+1} \frac{\pi^n \beta_{n,2}}{(n+2)(n+1)} \quad [a > 0]$$

$$2. \int_0^{\infty} \ln(1 \pm e^{-ax}) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln \Gamma\left(\frac{az}{2\pi} + \frac{3 \mp 1}{4} \pi\right) -$$

$$- \frac{\pi}{4z} \left[\ln(2\pi az) \pm \ln\left(\frac{2\pi}{az}\right) \right] + \frac{a}{2} \left(1 + \ln \frac{2\pi}{az} \right) \quad [a, \operatorname{Re} z > 0]$$

$$3. \int_0^{\infty} \ln \frac{a + be^{-\rho x}}{a + be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q} \quad [a + b, \rho q > 0]$$

2.6.29. Интегралы от $x^{\alpha} R(\operatorname{sh} x, \operatorname{ch} x) \ln x$.

$$1. \int_1^{\infty} \frac{\operatorname{sh} mx}{\operatorname{sh} nx} \ln x dx = \frac{\pi}{2n} \operatorname{tg} \frac{m\pi}{2n} \ln \frac{2\pi}{r} +$$

$$+ \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma\left[\frac{(n-1)^r k}{kr/(2n)}\right]$$

$$[r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l; l=1, 2, 3, \dots]$$

$$2. \int_1^{\infty} \frac{\operatorname{ch} mx}{\operatorname{ch} nx} \ln x dx = \frac{\pi}{2n} \ln\left(\frac{2\pi}{r}\right) \sec \frac{m\pi}{2n} +$$

$$+ \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \Gamma\left[\frac{r[2n - (-1)^r(2k-1)]/(4n)}{r(2k-1)/(4n)}\right]$$

$$[r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l; l=1, 2, 3, \dots]$$

$$3. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} ax} dx = \frac{\pi}{2a} \left[4 \ln \Gamma\left(\frac{3}{4}\right) - \ln(\pi a) \right] \quad [\operatorname{Re} a > 0]$$

$$4. \int_0^{\infty} \frac{\ln x}{\sqrt{x} \operatorname{ch} x} dx = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + \ln 4 + C]$$

$$5. \int_0^{\infty} \frac{\ln x}{\operatorname{ch}^2 x} dx = \ln \frac{\pi}{4} - C$$

$$6. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} x + c} dx = \frac{\pi}{\sqrt{1-c^2}} \left[2 \ln \Gamma\left(\frac{1+l}{l}\right) + \frac{1}{2} \ln(1+c) + \right.$$

$$\left. + (l-1) \ln \pi + (l-1/2) \ln 2 \right] \quad [-1 < c \leq 1; \pi l = \arccos c]$$

$$8. \int_0^{\infty} e^{-px} \ln(\sqrt{x^2+2x}+x+z) dx = \frac{1}{p} [e^{pz} K_0(pz) + \ln z]$$

$$[\operatorname{Re} p > 0; |\arg z| < \pi]$$

$$9. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2+z^2}} \ln(\sqrt{x^2+z^2}+x) dx =$$

$$= \frac{\pi}{2} \ln z [H_0(pz) - Y_0(pz)] - S_{-1,0}(pz) \quad [\operatorname{Re} p, \operatorname{Re} z > 0]$$

2.6.26. Интегралы от $R(e^x) \ln x$.

$$1. \int_0^{\infty} \frac{\ln x}{e^x + a} dx = a^{-1} \left[\sum_{k=1}^{\infty} \frac{(-a)^k}{k} \ln k - \zeta \ln(1+a) \right] \quad [-1 < a \leq 1]$$

$$2. \int_0^{\infty} \frac{x^{2n-1} \ln x}{(e^x + 1)^n} dx = \frac{\Gamma(n)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k (k+n-1)!}{k! (k+n)^n} [\psi(n) - \ln(k+n)]$$

$$[\operatorname{Re} a > 0; n=1, 2, 3, \dots]$$

$$3. \int_0^{\infty} \frac{e^x \ln x}{e^x + 2e^x \cos \gamma + 1} dx =$$

$$= \frac{1}{2 \sin \gamma} \left\{ \gamma \ln(2\pi) - \pi \ln \pi + \pi \ln \cos \frac{\gamma}{2\pi} + 2\pi \ln \Gamma\left(\frac{\pi+\gamma}{2\pi}\right) \right\} \quad [0 < \gamma < \pi]$$

$$4. \int_0^{\infty} \frac{e^x \ln x}{(e^x + 1)^2} dx = -\frac{\zeta}{2} + \frac{1}{2} \ln \frac{\pi}{2}$$

2.6.27. Интегралы, содержащие x^α , e^{-px} и $\frac{1}{\ln^a(x+a)}$.

$$1. \int_0^1 \frac{x^{2\alpha-1} - x^{\beta-1}}{\ln x} e^{-px} dx = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \ln \frac{\alpha+k}{\beta+k} \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; |\arg p| < \pi]$$

$$2. \int_0^{\infty} e^{-x} \frac{dx}{\ln x} = 0,$$

$$3. \int_0^{\infty} (x^{\alpha-1} - x^{\beta-1}) e^{-px} \frac{dx}{\ln x} = \frac{1}{p} [\lambda(p, \alpha-1) - \lambda(p, \beta-1)]$$

$$[\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} p > 0]$$

$$4. \int_0^{\infty} \left[e^{-x} - \frac{x(1+x)^{-q-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln q$$

$$[\operatorname{Re} q > 0]$$

$$5. \int_0^{\infty} \left[(q-1)e^{-x} + \frac{(1+x)^{-q} - (1+x)^{-1}}{\ln(1+x)} \right] \frac{dx}{x} = \ln \Gamma(q)$$

$$[\operatorname{Re} q > 0]$$

2.6.27

$$6. \int_0^{\infty} \frac{x^{n-1} e^{-px}}{\ln^2 x + \pi^2} dx = (-1)^{n-1} [\psi^{(n)}(p) - e^p] \quad [\operatorname{Re} p > 0]$$

2.6.28. Интегралы от $x^\alpha \ln(a+be^{-px})$

$$1. \int_0^{\infty} x^n \ln(1-e^{-ax}) dx = -\left(\frac{2}{a}\right)^{n+1} \frac{\pi^{n+2} B_{n+2}}{(n+2)(n+1)} \quad [a > 0]$$

$$2. \int_0^{\infty} \ln(1 \pm e^{-ax}) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln \Gamma\left(\frac{az}{2\pi} + \frac{3 \mp 1}{4}\right) - \frac{\pi}{4z} \left[\ln(2\pi az) \pm \ln\left(\frac{2\pi}{az}\right) \right] + \frac{a}{2} \left(1 + \ln \frac{2\pi}{az} \right) \quad [a, \operatorname{Re} z > 0]$$

$$3. \int_0^{\infty} \ln \frac{a+be^{-px}}{a+be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q} \quad [a+b, pq > 0]$$

2.6.29. Интегралы от $x^\alpha R(\sinh x, \cosh x) \ln x$.

$$1. \int_0^{\infty} \frac{\sinh mx}{\sinh nx} \ln x dx = \frac{\pi}{2n} \ln \frac{m\pi}{2n} \ln \frac{2\pi}{r} + \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \Gamma\left[\frac{(n-(-1)^k k)r/(2n)}{kr/(2n)}\right]$$

$$[r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l; l=1, 2, 3, \dots]$$

$$2. \int_0^{\infty} \frac{\cosh mx}{\cosh nx} \ln x dx = \frac{\pi}{2n} \ln \left(\frac{2\pi}{r}\right) \sec \frac{m\pi}{2n} + \frac{\pi}{n} \sum_{k=1}^{(n-1)/r} (-1)^{k-1} \cos \frac{(2k-1)\pi}{2n} m\pi \ln \Gamma\left[\frac{r[2n-(-1)^k(2k-1)]/(4n)}{r(2k-1)/(4n)}\right]$$

$$[r=1 \text{ при } m+n=2l-1, r=2 \text{ при } m+n=2l; l=1, 2, 3, \dots]$$

$$3. \int_0^{\infty} \frac{\ln x}{\cosh ax} dx = \frac{\pi}{2a} \left[4 \ln \Gamma\left(\frac{3}{4}\right) - \ln(\pi a) \right] \quad [\operatorname{Re} a > 0]$$

$$4. \int_0^{\infty} \frac{\ln x}{\sqrt{x} \cosh x} dx = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + \ln 4 + \zeta]$$

$$5. \int_0^{\infty} \frac{\ln x}{e^{1/x^2} x} dx = \ln \frac{\pi}{4} - \zeta,$$

$$6. \int_0^{\infty} \frac{\ln x}{\cosh x + c} dx = \frac{\pi}{1-c^2} \left[2 \ln \Gamma\left(\frac{1+f}{t}\right) + \frac{1}{2} \ln(1+c) + (t-1) \ln \pi + (t-1/2) \ln 2 \right] \quad [-1 < c \leq 1; \pi t = \arccos c]$$

$$7. \int_0^{\infty} \frac{\ln x}{2 \operatorname{ch}^2 x + 1} dx = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^{\infty} (-1)^k \frac{C + \ln(4k)}{\sqrt{k}} \sin \frac{\pi k}{3}.$$

$$8. \int_0^{\infty} x^{2\alpha-1} \frac{\operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = 2\Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^2} \quad [\operatorname{Re} \alpha > 0].$$

$$9. \int_0^{\infty} x^n \frac{(n+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = \frac{(-1)^n}{2^n} \beta^{(n)} \left(\frac{1}{2} \right).$$

$$10. \int_0^{\infty} x^{2n} \frac{(2n+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} \ln x dx = - \left(\frac{2}{\pi} \right)^{2n+1} |E_{2n}|.$$

$$11. \int_0^{\infty} x^{2n-1} \frac{n \operatorname{sh} 2ax - ax}{\operatorname{sh}^2 ax} \ln x dx = - \frac{1}{n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0; n = 1, 2, 3, \dots].$$

$$12. \int_0^{\infty} x^{2n-1} \frac{2ne^{-ax} \operatorname{sh} ax - ax}{\operatorname{sh}^2 ax} \ln x dx = - \frac{1}{2n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0; n = 1, 2, 3, \dots].$$

$$13. \int_0^{\infty} x^{2n} \frac{2ax \operatorname{ch} ax - (2n+1) \operatorname{sh} ax}{\operatorname{sh}^2 ax} \ln x dx = \frac{1}{a} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [a > 0; n = 1, 2, 3, \dots].$$

$$14. \int_0^{\infty} x^{2n} \frac{2ax \operatorname{sh} ax - (2n+1) \operatorname{ch} ax}{\operatorname{ch}^2 ax} \ln x dx = \frac{2}{a} (2^{2n-1} - 1) \left(\frac{\pi}{2a} \right)^{2n} |B_{2n}| \quad [a > 0; n = 1, 2, 3, \dots].$$

$$15. \int_0^{\infty} x^{n-1} \frac{ax \operatorname{ch} ax - n \operatorname{sh} ax}{\operatorname{sh}^2 ax} \ln x dx = \frac{2^{n-1} - 1}{n} \left(\frac{\pi}{a} \right)^n |B_n| \quad [a > 0; n = 2, 4, 6, \dots].$$

$$16. = 2(n-1)! \frac{2^n - 1}{(2a)^n} \zeta(n) \quad [a > 0; n = 1, 3, 5, \dots].$$

2.6.30. Интегралы от $A(x)R(\operatorname{sh} x, \operatorname{ch} x) \ln A(x)$.

$$1. \int_0^{\infty} \frac{\ln(a+bx^2)}{\operatorname{ch} cx} dx = \frac{2\pi}{c} \ln \Gamma \left[\frac{(2\sqrt{a}c + 3\pi\sqrt{b})/(4\pi\sqrt{b})}{(2\sqrt{a}c + \pi\sqrt{b})/(4\pi\sqrt{b})} \right] - \frac{\pi}{c} \ln \frac{c}{2\pi\sqrt{b}} \quad [a, b, c > 0].$$

$$2. \int_0^{\infty} \frac{\ln(1+bx^2)}{\operatorname{sh}^2 cx} dx = \frac{2}{c} \left[\ln \frac{c}{\pi\sqrt{b}} + \frac{\pi\sqrt{b}}{2c} - \psi \left(1 + \frac{c}{\pi\sqrt{b}} \right) \right] \quad [a, b, c > 0].$$

$$3. \int_0^{\infty} \frac{\operatorname{ch} cx}{\operatorname{sh}^2 cx} \ln \left(1 + \frac{4c^2}{\pi^2} x^2 \right) dx = \frac{\pi-2}{c} \quad [c > 0].$$

$$4. \int_0^{\infty} \frac{\operatorname{ch} cx}{\operatorname{sh}^2 cx} \ln \left(1 + \frac{16c^2}{\pi^2} x^2 \right) dx = \frac{1}{c} [\pi\sqrt{2} - 4 + 2\sqrt{2} \ln(1+\sqrt{2})] \quad [c > 0].$$

$$5. \int_0^{\infty} \frac{\operatorname{sh}(2\pi x \sqrt{b/3})}{\operatorname{sh}(\pi x \sqrt{b})} \ln(a+bx^2) dx = \sqrt{\frac{3}{b}} \ln \Gamma \left[\frac{(\sqrt{a}+4)/6, (\sqrt{a}+5)/6}{(\sqrt{a}+1)/6, (\sqrt{a}+2)/6} \right] + \sqrt{\frac{3}{b}} \ln 6 \quad [a, b > 0].$$

$$6. \int_0^{\infty} \frac{\operatorname{ch} cx + cx \operatorname{sh} cx}{x^2 \operatorname{ch}^2 cx} \ln \left(1 + \frac{4c^2}{\pi^2} x^2 \right) dx = \frac{4c}{\pi} \ln 2 \quad [c > 0].$$

$$7. \int_0^{\infty} \frac{\operatorname{ch} cx + cx \operatorname{sh} cx}{x^2 \operatorname{ch}^2 cx} \ln \left(1 + \frac{c^2}{\pi^2} x^2 \right) dx = \frac{c}{\pi} (4-\pi) \quad [c > 0].$$

$$8. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2-a^2}} \operatorname{sh}(b\sqrt{x^2-a^2}) \ln \frac{x+\sqrt{x^2-a^2}}{a} dx = \frac{1}{2} \ln \frac{p+b}{p-b} K_0(a\sqrt{p^2-b^2}) \quad [a > 0; \operatorname{Re} p > |\operatorname{Re} b|].$$

$$9. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2+2zx}} \operatorname{sh}(b\sqrt{x^2+2zx}) \ln \frac{x+z+\sqrt{x^2+2zx}}{z} dx = e^{z^2/p} \ln \frac{p+b}{p-b} K_0(z\sqrt{p^2-b^2}) \quad [\operatorname{Re} p > |\operatorname{Re} b|; |\arg z| < \pi].$$

2.6.31. Интегралы, содержащие логарифмическую функцию от гиперболических функций.

$$1. \int_0^{\infty} e^{-px} \ln(\operatorname{sh} ax) dx = -\frac{1}{p} \left[\ln 2 + \frac{a}{2p} + \psi \left(\frac{p}{2a} \right) - C \right] \quad [a, \operatorname{Re} p > 0].$$

$$2. \int_0^{\infty} e^{-nax} \ln(\operatorname{sh} ax) dx = \frac{1}{2an} \left[\frac{1}{n} + \ln 2 - 2\beta(2n+1) \right] \quad [a > 0; n = 1, 2, 3, \dots].$$

$$3. \int_0^{\infty} e^{-px} \ln(\operatorname{ch} ax) dx = \frac{1}{p} \left[\beta \left(\frac{p}{2a} \right) - \frac{a}{p} \right] \quad [a, \operatorname{Re} p > 0].$$

$$4. \int_{-\infty}^{\infty} x \ln \operatorname{ch} x dx = 0. \quad 5. \int_0^{\infty} \ln \operatorname{ch} x \frac{dx}{1-x^2} = 0.$$

$$6. \int_0^{\infty} \ln \operatorname{ch} x \frac{dx}{\operatorname{ch} 2x} = \frac{G}{2} - \frac{\pi}{8} \ln 2. \quad 7. \int_0^{\infty} \ln \operatorname{th} x dx = -\pi^2/6.$$

8. $\int_0^{\infty} x^{\alpha-1} \ln \operatorname{th} x \, dx = -\frac{\pi^{(\alpha+2)/2}}{\alpha \sin(\alpha\pi/2)} (2^{\alpha-1}-1) \zeta(-\alpha)$ [$\operatorname{Re} \alpha > 1$]
9. $\int_0^{\infty} \ln \operatorname{th} x \frac{dx}{\operatorname{ch} x} = -\frac{\pi}{2} \ln 2$
10. $\int_0^{\infty} x^{\alpha-1} \ln \frac{\operatorname{ch} ax + \operatorname{sh} b}{\operatorname{ch} ax - \operatorname{sh} b} \, dx = -\left(\frac{2}{\alpha}\right)^{\alpha} \frac{\pi^{\alpha+1}}{\alpha \sin(\alpha\pi/2)} \times$
 $\times \left[\zeta\left(-\alpha, \frac{\pi-2b}{4i}\right) - \zeta\left(-\alpha, \frac{\pi+2b}{4i}\right) + \zeta\left(-\alpha, \frac{3\pi+2b}{4i}\right) - \zeta\left(-\alpha, \frac{3\pi-2b}{4i}\right) \right]$
[$\operatorname{Re} \alpha > 0$; $2b > \pi$]

2.6.32. Интегралы от $A(x) \frac{(\sin bx)}{(\cos bx)} \ln^a x$.

1. $\int_0^{\infty} x^{\alpha-1} \frac{(\sin bx)}{(\cos bx)} \ln^a x \, dx = \frac{\partial^a}{\partial a^a} \left[b^{-a} \Gamma(\alpha) \frac{(\sin(\alpha\pi/2))}{(\cos(\alpha\pi/2))} \right]$
[$b > 0$; $\left\{ \frac{-1}{0} \right\} < \operatorname{Re} \alpha < 1$]
2. $\int_0^{\infty} x^{\alpha-1} \frac{(\sin bx)}{(\cos bx)} \ln x \, dx = \frac{\Gamma(\alpha)}{b^{\alpha}} \frac{(\sin(\alpha\pi/2))}{(\cos(\alpha\pi/2))} \left[\Psi(\alpha) - \ln b \pm \frac{\pi}{2} \lg \mp \frac{\alpha\pi}{2} \right]$
[$b > 0$; $\left\{ \frac{-1}{0} \right\} < \operatorname{Re} \alpha < 1$]
3. $\int_0^{\infty} \frac{\sin bx}{x} \ln x \, dx = -\frac{\pi}{2} (C + \ln b)$ [$b > 0$]
4. $\int_0^{\infty} \frac{1}{\sqrt{x}} \frac{(\sin bx)}{(\cos bx)} \ln x \, dx = -\sqrt{\frac{\pi}{2b}} \left[\ln(4b) + C \mp \frac{\pi}{2} \right]$ [$b > 0$]
5. $\int_0^{\infty} x^{\alpha-1} \frac{(\sin bx)}{(\cos bx)} \ln^2 x \, dx =$
 $= \frac{\Gamma(\alpha)}{b^{\alpha}} \frac{(\sin(\alpha\pi/2))}{(\cos(\alpha\pi/2))} \left\{ \left[\Psi(\alpha) - \ln b \pm \frac{\pi}{2} \lg \mp \frac{\alpha\pi}{2} \right]^2 + \Psi'(\alpha) - \frac{\pi^2}{4} \frac{(\sin(\alpha\pi/2))^{-2}}{(\cos(\alpha\pi/2))} \right\}$
[$b > 0$; $\left\{ \frac{-1}{0} \right\} < \operatorname{Re} \alpha < 1$]
6. $\int_0^{\infty} \frac{\sin bx}{x} \ln^2 x \, dx = \frac{\pi}{2} \left(C^2 + \frac{\pi^2}{12} + 2C \ln b + \ln^2 b \right)$ [$b > 0$]
7. $\int_0^{\infty} \frac{\sin^2 bx}{x^2} \ln x \, dx = -\frac{b\pi}{2} [C + \ln(2b) - 1]$ [$b > 0$]

8. $\int_0^{\infty} \frac{1}{x^2+z^2} \frac{(x \sin bx)}{(\cos bx)} \ln(ax) \, dx =$
 $= \frac{\pi}{2} x^{(1-1)/2} [e^{-bz} \ln(az) + e^{bz} \operatorname{Ei}(-bz) - e^{-bz} \operatorname{Ei}(bz)]$ [$b, \operatorname{Re} z > 0$; $|\arg a| < \pi$]
9. $\int_0^{\infty} \frac{1}{x^2+y^2} \frac{(x \sin bx)}{(\cos bx)} \ln(ax) \, dx =$
 $= \frac{\pi}{2} y^{(1-1)/2} \left[\frac{(\cos by)}{(\sin by)} [\ln(iy) - \operatorname{ci}(by)] - \operatorname{si}(by) \frac{(\sin by)}{(\cos by)} \right]$
[$b, y > 0$; $|\arg a| < \pi$]
10. $\int_0^{\infty} \frac{\cos ax - \cos bx}{x} \ln x \, dx = \ln \frac{a}{b} \left[C + \frac{1}{2} \ln(ab) \right]$ [$a, b > 0$]
11. $\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} \ln x \, dx = \frac{\pi}{2} [(a-b)(C-1) + a \ln a - b \ln b]$ [$a, b > 0$]

2.6.33. Интегралы от $A(x) \frac{(\sin bx)}{(\cos bx)} \ln A(x)$.

1. $\int_0^a \sin bx \ln(a-x) \, dx = \frac{1}{b} \{ \ln a - \sin ab \operatorname{Si}(ab) - \cos ab [\operatorname{ci}(ab) - C - \ln b] \}$
[$a, b > 0$]
2. $\int_0^{\infty} \frac{\cos bx}{x} \ln(1+ax) \, dx = \frac{1}{2} \left[\operatorname{Si}^2\left(\frac{b}{a}\right) + \operatorname{Ci}^2\left(\frac{b}{a}\right) \right]$ [$a, b > 0$]
3. $\int_0^{\infty} \sin bx \ln \left| \frac{x+a}{x-a} \right| \, dx = \frac{\pi}{b} \sin ab$ [$a, b > 0$]
4. $\int_0^{\infty} \cos bx \ln \left| \frac{x+a}{x-a} \right| \, dx = \frac{1}{b} \left[\frac{\pi}{2} (\cos bc - \cos ab) + \right.$
 $\left. + \cos bc \operatorname{Si}(bc) + \cos ab \operatorname{Si}(ab) - \sin bc \operatorname{ci}(bc) - \sin ab \operatorname{ci}(ab) \right]$ [$a, b, c > 0$]
5. $\int_0^{\infty} \frac{\cos bx}{x} \ln \left| \frac{x-1-a}{x-1-a} \right| \, dx = -\pi \operatorname{Si}(ab)$ [$a > 0, b \geq 0$]
6. $\int_0^a \cos bx \ln(a^2-x^2) \, dx = \frac{\sin ab}{b} \left[\operatorname{ci}(2ab) - C - \ln \frac{b}{a} \right] - \frac{\cos ab}{b} \operatorname{Si}(2ab)$
[$a, b > 0$]
7. $\int_0^a \frac{\cos bx}{\sqrt{a^2-x^2}} \ln(a^2-x^2) \, dx = \frac{\pi^2}{4} Y_0(ab) - \frac{\pi}{2} J_0(ab) \left(C + \ln \frac{2b}{a} \right)$ [$a, b > 0$]

[a, b > 0].

[a > 0, b > 0].

[b, Re z > 0].

[b, Re z > 0].

[b, y > 0].

[b, Re a, Re c > 0].

[b, Re a, Re c > 0].

[b, g, h, Re a > 0].

[b, Re a > 0; |Im c| ≤ Re a].

[b, Re z > 0].

[b, Re z > 0].

[b > 0].

8. $\int_0^{\infty} \frac{\sin bx}{x} \ln(1+ax^2) dx = -\pi \operatorname{Ei}\left(-\frac{b}{\sqrt{a}}\right)$
9. $\int_0^{\infty} \frac{\cos bx}{x^2} \ln(1+ax^2) dx = \pi \left[b \operatorname{Ei}\left(-\frac{b}{\sqrt{a}}\right) + \sqrt{b} \exp\left(-\frac{b}{\sqrt{a}}\right) \right]$
10. $\int_0^{\infty} \frac{\cos bx}{x^2+z^2} \ln(x^2+z^2) dx = -\left(C + \ln \frac{2b}{z}\right) K_0(bz)$
11. $\int_0^{\infty} \cos bx \ln\left(1+\frac{z^2}{x^2}\right) dx = \frac{\pi}{b} (1-e^{-bz})$
12. $\int_0^{\infty} \cos bx \ln\left|1-\frac{y^2}{x^2}\right| dx = \frac{\pi}{b} |1-\cos(by)|$
13. $\int_0^{\infty} \cos bx \ln \frac{a^2+x^2}{c^2+x^2} dx = \frac{\pi}{b} (e^{-bc} - e^{-ac})$
14. $\int_0^{\infty} x \sin bx \ln \frac{a^2+x^2}{c^2+x^2} dx = \frac{\pi}{b^2} [(1+bc)e^{-bc} - (1+ab)e^{-ab}]$
15. $\int_0^{\infty} \frac{\sin bx}{x} \ln \frac{a^2+x^2}{g^2+x^2} dx = \pi \left[\operatorname{Ei}\left(-\frac{ab}{h}\right) - \operatorname{Ei}\left(-\frac{ab}{g}\right) \right]$
16. $\int_0^{\infty} \sin bx \ln \frac{a^2+(x+z)^2}{a^2+(x-c)^2} dx = \frac{2\pi}{b} e^{-ab} \sin(bc)$
17. $\int_0^{\infty} \frac{\sin bx}{\sqrt{x^2+z^2}} \ln(x+\sqrt{x^2+z^2}) dx = \frac{\pi}{2} \{K_0(bz) + \ln z [I_0(bz) - L_0(bz)]\}$
18. $\int_0^{\infty} \frac{\cos bx}{\sqrt{x^2+z^2}} \ln(x+\sqrt{x^2+z^2}) dx = \frac{1}{2} [S_{1,0}(ibz) + S_{-1,0}(-ibz)] + \ln z K_0(bz)$
19. $\int_0^{\infty} \frac{\cos bx}{x} \ln(x+\sqrt{x^2-1}) dx =$

$$= -\frac{1}{4} \sum_{k=1}^{\infty} \frac{2\psi(k) + 3k^{-1} - 2 \ln(b/2)}{k(k!)^2} \left(-\frac{b^2}{4}\right)^k - \frac{1}{4} \left(C + \frac{\pi^2}{3}\right)$$

2.6.34. Интегралы, содержащие $\ln \left\{ \frac{\sin bx}{\cos bx} \right\}$.

1. $\int_0^{\pi/4} x^{n-1} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx =$
 $= -\frac{1}{2\alpha} \left(\frac{\pi}{4}\right)^{\alpha} \left[\ln 2 + \frac{2}{\alpha} \zeta(1) - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(\alpha+2k)} \left\{ \frac{1}{2^{2k-1}} - 1 \right\} \right]$ [Re α > -1/2]
2. $\int_0^{\pi/4} x \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{7+28}{128} \zeta(3) - \frac{\pi}{8} G - \frac{\pi^2}{32} \ln 2$
3. $\int_0^{\pi/4} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\pi}{4} \ln 2 - \frac{G}{2}$
4. $\int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+2} x} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx =$
 $= -\frac{1}{n+1} \left[\frac{\ln 2}{2} + \frac{(-1)^{[n/2]}}{4} \left\{ \frac{\pi}{\ln 4} \right\} + \sum_{k=0}^{[n/2]-1} \frac{(-1)^k}{n-2k-1} \right] + \frac{1+1+2+(-1)^n}{2(n+1)^2}$
5. $\int_0^{\pi/4} \cos^{v-1} 2x \sin 2x \ln \sin x dx = -(4v)^{-1} [C + \psi(v+1) + \ln 2]$ [Re v > 0].
6. $\int_0^{\pi/4} \cos^{v-1} 2x \lg 2x \ln \cos x dx = \frac{1}{4(1-v)} \beta(v)$ [Re v > 0].
7. $\int_0^{\pi/2} x^{a-1} \ln \sin x dx = -\frac{1}{\alpha} \left(\frac{\pi}{2}\right)^{\alpha} \left[\frac{1}{\alpha} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(\alpha+2k)} \right]$ [Re α > 0].
8. $\int_0^{\pi/2} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\pi}{2} \ln 2$
9. $\int_0^{\pi/2} x \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \pm \frac{7}{16} \zeta(3) - \frac{\pi^2}{8} \ln 2$
10. $\int_0^{\pi/2} x^2 \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = \frac{+7-1}{32} \pi^2 \zeta(3) - \frac{\pi^3}{24} \ln 2$
11. $\int_0^{\pi/2} \left\{ \frac{\sin x}{\cos x} \right\}^{\mu-1} \ln \left\{ \frac{\sin x}{\cos x} \right\} dx = -\frac{\sqrt{\pi}}{2} \Gamma \left[\frac{\mu/2}{(\mu+1)/2} \right] \beta(\mu)$ [Re μ > 0].

$$12. \int_0^{\pi/2} \left(\frac{\sin x}{\cos x} \right)^n \ln \left(\frac{\sin x}{\cos x} \right) dx = A \frac{(-1)^{n+1} (n-1)!}{n!!} \left[\ln 2 + \sum_{k=1}^n \frac{(-1)^k}{k} \right]$$

[A=1, если n=1, 3, 5, ...; A=π/2, если n=0, 2, 4, ...]

$$13. \int_0^{\pi/2} \left(\frac{\sin x}{\cos x} \right) \ln \left(\frac{\sin x}{\cos x} \right) dx = \ln 2 - 1.$$

$$14. \int_0^{\pi/2} \left(\frac{\sin x}{\cos x} \right)^{\mu-1} \ln \left(\frac{\cos x}{\sin x} \right) dx =$$

$$= -\frac{\sqrt{\pi}}{4} \Gamma \left[\frac{\mu/2}{(\mu+1)/2} \right] \left[C + 2 \ln 2 + \psi \left(\frac{\mu+1}{2} \right) \right] \quad [\operatorname{Re} \mu > -1]$$

$$15. \int_0^{\pi/2} \left(\frac{\sin x}{\cos x} \right)^n \ln \left(\frac{\cos x}{\sin x} \right) dx = I_n$$

[n=-2, -1, 0, 1, 2, ...]

$$I_{-2} = -\frac{\pi}{2}, \quad I_{-1} = -\frac{\pi^2}{8}, \quad I_0 = -\frac{\pi}{2} \ln 2, \quad I_1 = -1,$$

$$I_2 = -\frac{\pi}{8} (1 + 2 \ln 2), \quad I_n = -\frac{A (n-1)!}{2 n!!} \left[\sum_{k=1}^{[(n+1)/2]} \frac{1}{k-r} + B \right]$$

[A=π/2, r=0, B=2 ln 2, если n=2, 4, 6, ...; A=1, r=1/2, B=0, если n=1, 3, 5, ...]

$$16. \int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \ln \left(\frac{\sin x}{\cos x} \right) dx = \frac{1}{4} B \left(\frac{\mu}{2}, \frac{\nu}{2} \right) \left[\frac{\Gamma(\mu/2)}{\Gamma(\nu/2)} - \psi \left(\frac{\mu+\nu}{2} \right) \right]$$

[{Re μ > 0, Re ν > -2}].

$$17. \int_0^{\pi/2} \frac{\cos^{\nu-1} x}{\sin x} \ln \cos x dx = -\frac{1}{4} \psi' \left(\frac{\nu}{2} \right)$$

[Re ν > 0]

$$18. \int_0^{\pi/2} \cos 2nx \ln \left(\frac{\sin x}{\cos x} \right) dx = -\frac{\pi}{4n} \left\{ \frac{1}{(-1)^n} \right\}$$

[n=1, 2, 3, ...]

$$19. \int_0^{\pi/2} x \left(\frac{\sin x}{\cos x} \right) \ln \left(\frac{\cos x}{\sin x} \right) dx = \pm (\ln 2 - 2) - \frac{\pi}{4} (1 \mp 1).$$

$$20. \int_0^{\pi/2} \cos^{\nu-1} x \sin^{\nu} x \ln \cos x dx = 2^{-\nu-2} \pi [C + \psi(\nu) - 1/\nu - \ln 4] \quad [\nu > 0]$$

$$21. \int_0^{\pi/2} \frac{1}{(a \sin x + b \cos x)^2} \ln \left(\frac{\sin x}{\cos x} \right) dx = \frac{1}{a^2 + b^2} \left[\mp \left| \frac{a}{b} \right|^{1/2} \ln \left| \frac{a}{b} \right| - \frac{\pi}{2} \right]$$

[a > 0; b ≠ 0; Im b = 0]

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$$22. \int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \ln \left(\frac{\sin x}{\cos x} \right) dx = \frac{\pi}{2ab} \ln \left[\frac{1}{a+b} \left| \frac{b}{a} \right| \right]$$

[a, b > 0]

$$23. \int_0^{\pi/2} \frac{\sin 2x}{(a \sin^2 x + b \cos^2 x)^2} \ln \left(\frac{\sin x}{\cos x} \right) dx = \frac{1}{2(b-a)} \left(\frac{b^{-1}}{a} \right) \ln \frac{a}{b}$$

[a, b > 0]

$$24. \int_0^{\pi/2} \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^3} \ln \left(\frac{\sin x}{\cos x} \right) dx = \frac{\pi}{2(a+b)} \left(\frac{b^{-1}}{a} \right)$$

[a, b > 0]

$$25. \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2 x}} \ln \left(\frac{\sin x}{\cos x} \right) dx = -\frac{1}{2} K(k) \ln k -$$

$$-\frac{\pi}{4} K(1-k^2) + \frac{1}{8} (1+k) \ln(1-k^2) K(k) \quad [0 < k < 1]$$

$$26. \int_0^{\pi} \sin^{\mu-1} x \ln \sin x dx = -\sqrt{\pi} \Gamma \left[\frac{\mu/2}{(\mu+1)/2} \right] \beta(\mu)$$

[Re μ > 0]

$$27. \int_0^{\pi} x \sin^{\mu-1} x \ln \sin x dx = -\frac{\pi}{2} B \left(\frac{1}{2}, \frac{\mu}{2} \right) \beta(\mu)$$

[Re μ > 0]

$$28. \int_0^{\pi} x \ln \left(\frac{\sin x}{\cos x} \right) dx = -\frac{\pi^2}{2} \ln 2.$$

$$29. \int_0^{\pi} x^2 \ln \sin x dx = -\frac{\pi^3}{3} \ln 2 - \frac{\pi}{2} \zeta(3).$$

$$30. \int_0^{\pi} x \ln |\sin x| dx = -\frac{\pi^2 n^2}{2} \ln 2. \quad 31. \int_0^{\pi} \sin 2nx \ln \sin x dx = 0.$$

$$32. \int_0^{\pi} \sin [(2n+1)x] \ln \sin x dx = \frac{2}{2n+1} \left(\ln 2 - 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} - \frac{1}{2n+1} \right).$$

$$33. \int_0^{\pi} \cos [(2n+1)x] \ln \sin x dx = 0.$$

$$34. \int_0^{\pi} \cos 2nx \ln \sin x dx = -\frac{\pi}{2n} \quad [n=1, 2, 3, \dots]$$

$$35. \int_0^{\pi} \sin^{\mu} x \cos \left[\mu \left(\frac{\pi}{2} - x \right) \right] \ln \sin x dx = -2^{-\mu} \pi \ln 2 \quad [\mu > -1]$$

$$36. \int_0^{\pi} \sin^{2n} 2x \cos 2x \ln \sin x dx = -\frac{\pi}{4n+2} \frac{(2n-1)!}{(2n)!}.$$

$$37. \int_0^{\pi} \frac{\ln \sin x dx}{a+b \cos x} = \frac{\pi}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \quad [a>0, a>b].$$

$$38. \int_0^{\pi n} \frac{1}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin bx}{\cos bx} \right\} dx = \frac{n\pi}{|1-a^2|} \ln \frac{1+a^{2n\varphi}}{2} \quad [n=1, 2; \varphi = \operatorname{sgn}(1-|a|), |a| \neq 1].$$

$$39. \int_0^{\pi} \frac{\cos x \ln \sin x dx}{1-2a \cos x + a^2} = \frac{\pi}{2a} \left| \frac{1+a^2}{1-a^2} \right| \ln(1-a^{\varphi}) - \frac{\pi a^{\varphi} \ln 2}{|1-a^2|} \quad [|a| \neq 1; \varphi = \operatorname{sgn}(1-|a|)].$$

$$40. \int_0^{\pi} \frac{\cos x}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} dx = \frac{\pi}{2a|1-a^2|} [(1+a^2) \ln(1+a^{\varphi}) - 2a^{1+\varphi} \ln 2] \quad [|a| \neq 1; \varphi = \operatorname{sgn}(1-|a|)].$$

$$41. \int_0^{2\pi} \frac{\cos(x/2)}{1-2a \cos x + a^2} \ln \left\{ \frac{\sin bx}{\cos bx} \right\} dx = 0 \quad [|a| \neq 1].$$

$$42. \int_0^{2\pi} \frac{\cos^2(x/2) \ln \sin(x/2)}{1-2a \cos x + a^2} dx = \frac{\pi}{2a} \left| \frac{a+1}{a-1} \right| \ln(1-a^{\varphi}) - \frac{\pi a^{(1-\varphi)/2}}{|a-1|} \ln 2 \quad [|a| \neq 1; \varphi = \operatorname{sgn}(1-|a|)].$$

$$43. \int_0^{\pi} \frac{\sin x}{\sqrt{1+\sin^2 x}} \ln \sin x dx = -\frac{\pi}{8} \ln 2.$$

$$44. \int_0^{\pi} \frac{\sin^3 x}{\sqrt{1+\sin^2 x}} \ln \sin x dx = \frac{\ln 2 - 1}{4}.$$

$$45. \int_0^{\pi} \frac{\ln |\cos x|}{x^2} dx = -\frac{\pi}{2}.$$

$$46. \int_0^{\infty} \ln \left\{ \frac{|\sin ax|}{|\cos ax|} \right\} \frac{dx}{x^2+z^2} = \frac{\pi}{2z} \ln \frac{1+e^{-2az}}{2} \quad [a, \operatorname{Re} z > 0].$$

$$47. \int_0^{\infty} \ln \left\{ \frac{|\sin ax|}{|\cos ax|} \right\} \frac{dx}{x^2-y^2} = \frac{(1 \pm 1)\pi^2}{8y} - \frac{\pi a}{2} \quad [a, y > 0].$$

$$48. \int_0^{\infty} \frac{\cos bx}{x^2} \ln |\cos ax| dx = \frac{\pi}{2} (b \ln 2 - a) \quad [a, b > 0].$$

$$49. \int_0^{\infty} \frac{\cos bx}{x^2+z^2} \ln |\cos ax| dx = \frac{\pi}{2z} [\operatorname{ch} bz \ln(1+e^{-2az}) - e^{-bz} \ln 2] \quad [0 < b < 2a; \operatorname{Re} z > 0].$$

$$50. \int_0^{\infty} \frac{\cos bx}{x^n(x^2+z^2)} \ln |\cos ax| dx = I_n \quad [a, b, \operatorname{Re} z > 0],$$

$$I_1 = \frac{\pi}{2} z^{-2} [\operatorname{sh} bz \ln(1+e^{-2az}) - (1-e^{-bz}) \ln 2],$$

$$I_2 = \frac{\pi}{2} z^{-3} [(bz+e^{-bz}) \ln 2 - az - \operatorname{ch} bz \ln(1+e^{-2az})].$$

$$51. \int_0^{\infty} \frac{1}{x^2+z^2} x^{-1} \sin bx \ln \cos^2 ax dx = \pi \ln(1+e^{-2az}) z^{-4} \operatorname{sh} bz - z^{-3} + \pi z^{-2} e^{-bz} \ln 2 \quad [0 \leq b < 2a; \operatorname{Re} z > 0].$$

$$52. \int_0^{\infty} \frac{\lg x}{x} \ln |\cos x| dx = -\frac{\pi}{2} \ln 2.$$

$$53. \int_0^{\infty} \frac{1}{x^2+z^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln(2+2 \cos x) dx = -\pi \left\{ \frac{\operatorname{sh} bz}{z^{-1} \operatorname{ch} bz} \right\} \ln(1+e^{-z}) \quad [\operatorname{Re} z > 0; 0 < b < 1].$$

$$54. \int_0^{\infty} \frac{1}{x^2+z^2} \left\{ \frac{x \sin bx}{\cos bx} \right\} \ln(2-2 \cos x) dx = -\pi \left\{ \frac{\operatorname{sh} bz}{z^{-1} \operatorname{ch} bz} \right\} \ln(1-e^{-z}) \quad [\operatorname{Re} z > 0; 0 < b < 1].$$

2.6.35. Интегралы, содержащие $\ln^n \left\{ \frac{\sin bx}{\cos bx} \right\}$.

$$1. \int_0^{\pi/4} \lg x \ln^{2n} \cos 2x dx = \frac{2^{2n}-1}{2^{2n+1}} (2n)! \zeta(2n+1) \quad [n=1, 2, 3, \dots].$$

$$2. \int_0^{\pi/4} \lg x \ln^{2n+1} \cos 2x dx = \frac{1-2^{2n+1}}{4(n+1)} \pi^{2n+2} |B_{2n+2}|.$$

$$3. \int_0^{\pi/4} \cos^{v-1} 2x \lg x \ln^n \cos 2x dx = \frac{1}{2} \beta^{(n)}(v) \quad [\operatorname{Re} v > 0].$$

$$4. \int_0^{\pi/4} \sin^{u-1} 2x \lg \left(\frac{\pi}{4} \pm x \right) \ln^n \sin 2x dx = \frac{((-1)^n 2^{-n} n! \zeta(n+1, \mu))}{2^{-1} \beta^{(n)}(\mu)} \quad [\operatorname{Re} \mu > 0].$$

$$5. \int_0^{\pi/6} \ln^2(2 \sin x) dx = \frac{7\pi^3}{216}, \quad 6. \int_0^{\pi/6} x \ln^3(2 \sin x) dx = \frac{17\pi^4}{25 \cdot 920}.$$

$$7. \int_0^{\pi/2} \ln^2 \left(\frac{\sin x}{\cos x} \right) dx = \frac{\pi}{2} \left(\ln^2 2 + \frac{\pi^2}{12} \right).$$

$$8. \int_0^{\pi/2} \ln^n (2 \sin x) dx = I_n.$$

$$I_1 = 0, \quad I_2 = \frac{\pi^2}{24}, \quad I_3 = -\frac{3\pi}{4} \zeta(3), \quad I_4 = \frac{19}{480} \pi^4.$$

$$9. \int_0^{\pi/2} x \ln^2 (2 \sin x) dx = \text{Li}_3 \left(\frac{1}{2} \right) + \frac{1}{24} \ln^4 2 - \frac{\pi^2}{24} \ln^2 2 + \frac{7}{8} \ln 2 \zeta(3) - \frac{19}{2880} \pi^4.$$

$$10. \int_0^{\pi/2} x^2 \ln^2 (2 \sin x) dx = \pi \text{Li}_4 \left(\frac{1}{2} \right) + \frac{\pi}{24} \ln^4 2 + \frac{7}{8} \pi \ln 2 \zeta(3) - \frac{37\pi^5}{2880}.$$

$$11. \int_0^{\pi/2} \ln^2 (2 \sin x) \ln (2 \cos x) dx = \frac{1}{8} \pi \zeta(3).$$

$$12. \int_0^{\pi} x^2 \ln^2 (2 \sin x) dx = -\frac{7\pi^5}{240}.$$

2.6.36. Интегралы от $A(x, \sin x, \cos x) \ln(a \cos x + b \sin x + c)$.

$$1. \int_0^{\pi/2} \ln(1 \pm \sin x) dx = -\frac{\pi}{2} \ln 2 \pm 2G.$$

$$2. \int_0^{\pi/2} \ln(1 \pm \cos x) dx = -\frac{\pi}{2} \ln 2 \pm 2G.$$

$$3. \int_0^{\pi/2} \ln \left(1 + a \frac{(\sin x)}{(\cos x)} \right) \frac{(\sin x)}{(\cos x)} dx = \frac{\pi^2}{8} - \frac{1}{2} \arccos^2 a \quad (|a| < 1).$$

$$4. \int_0^{\pi/2} \frac{\cos x \ln(1 \pm a \cos x)}{1 - a^2 \cos^2 x} dx = \frac{1}{a \sqrt{1-a^2}} \left[L \left(\frac{\pi}{2} - \arccos a \right) - \frac{\arccos a}{2} \ln(1-a^2) + \frac{1 \pm a}{4} \pi \ln(1-a^2) \right] \quad (0 < a < 1).$$

$$5. \int_0^{\pi} \ln(a \pm b \cos x) dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \quad (a, b > 0).$$

$$6. \int_0^{\pi} \ln(1 + b \cos x) \frac{dx}{\cos x} = \pi \arcsin b \quad (|b| < 1).$$

$$7. \int_0^{\pi} \frac{x \sin bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = -\pi \operatorname{sh} bz \ln(1 \pm e^{-z}) - \frac{\pi}{2} e^{-bz} \ln 2 \quad (b, \operatorname{Re} z > 0).$$

$$8. \int_0^{\pi} \frac{\cos bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = \frac{\pi}{2} \left[\operatorname{ch} bz \ln(1 \pm e^{-z}) - \frac{1}{2} e^{-bz} \ln 2 \right] \quad (b, \operatorname{Re} z > 0).$$

$$9. \int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi \ln |a| \quad (|b| \leq |a|).$$

$$10. \int_0^{\pi} \frac{(\sin(x/2))^n}{(\cos(x/2))^n} \ln(1 - 2a \cos x + a^2) dx = \pm \frac{\pi n}{2} \quad (|a| < 1).$$

$$11. = \pi \ln |a| \pm \pi/(2a) \quad (|a| > 1).$$

$$12. \int_0^{\pi} \frac{\ln(1 - 2a \cos x + a^2) dx}{1 - 2b \cos x + b^2} = \frac{2\pi}{|1-b^2|} \ln(1 - ab^{\pm 1}) \quad \left[\begin{matrix} (|b| < 1), & |a| \leq 1 \\ (|b| > 1), & |a| > 1 \end{matrix} \right].$$

$$13. = \frac{2\pi}{|1-b^2|} \ln |a - b^{\pm 1}| \quad \left[\begin{matrix} (|b| < 1), & |a| \leq 1 \\ (|b| > 1), & |a| > 1 \end{matrix} \right].$$

$$14. \int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi n \ln |a| \quad (|b| \leq |a|).$$

$$15. \int_0^{\pi} \cos mx \ln(1 - 2a \cos x + a^2) dx = -\frac{\pi n}{m} a^{\pm m} \quad \left[\begin{matrix} (|a| < 1), & n=1, 2; m=1, 2, 3, \dots \\ (|a| > 1), & \end{matrix} \right].$$

$$16. \int_0^{\pi} \frac{(\sin x \sin mx)}{(\cos x \cos mx)} \ln(1 - 2a \cos x + a^2) dx = \frac{\pi n}{2} \left(\frac{b^{m+1}}{m+1} - \frac{b^{m-1}}{m-1} \right) \quad \left[\begin{matrix} n=1, 2; b = \left\{ \begin{matrix} a, & |a| < 1 \\ a^{-1}, & |a| > 1 \end{matrix} \right\} \end{matrix} \right].$$

$$17. \int_0^{2\pi} \cos((m-1/2)x) \ln(1 - 2a \cos x + a^2) dx = 0 \quad (|a| \leq 1).$$

$$18. \int_0^{2\pi} \frac{(\sin(x/2) \sin nx)}{(\cos(x/2) \cos nx)} \ln(1 - 2a \cos x + a^2) dx = 0 \quad (|a| \neq 1).$$

$$19. \int_0^{2\pi} \frac{(\sin(x/2) \sin((n-1/2)x))}{(\cos(x/2) \cos((n-1/2)x))} \ln(1 - 2a \cos x + a^2) dx = \pi \left[\pm \frac{b^n}{n} - \frac{b^{n-1}}{n-1} \right] \quad \left[b = \begin{cases} a, & |a| < 1 \\ a^{-1}, & |a| > 1 \end{cases}; n=2, 3, 4, \dots \right].$$

$$20. \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln(1 - a e^{-z}) \quad (|a| \leq 1; \operatorname{Re} z > 0).$$

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$$3. \int_0^{\pi/2} \ln \left(1 + a \frac{\sin x}{\cos x} \right) \frac{\sin x}{\cos x} dx = \frac{\pi^2}{8} - \frac{1}{2} \arccos^2 a \quad (|a| < 1).$$

$$4. \int_0^{\pi/2} \frac{\cos x \ln(1 + a \cos x)}{1 - a^2 \cos^2 x} dx = \frac{1}{a \sqrt{1-a^2}} \left[L \left(\frac{\pi}{2} - \arccos a \right) - \frac{\arccos a}{2} \ln(1-a^2) + \frac{1}{4} \pi \ln(1-a^2) \right] \quad (0 < a < 1).$$

$$5. \int_0^{\pi} \ln(a \pm b \cos x) dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \quad (a, b > 0).$$

$$6. \int_0^{\pi} \ln(1 + b \cos x) \frac{dx}{\cos x} = \pi \arcsin b \quad (|b| < 1).$$

$$7. \int_0^{\infty} \frac{x \sin bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = -\pi \operatorname{sh} bz \ln(1 \pm e^{-z}) - \frac{\pi}{2} e^{-bz} \ln 2 \quad (b, \operatorname{Re} z > 0).$$

$$8. \int_0^{\infty} \frac{\cos bx}{x^2 + z^2} \ln(1 \pm \cos x) dx = \frac{\pi}{z} \left[\operatorname{ch} bz \ln(1 \pm e^{-z}) - \frac{1}{2} e^{-bz} \ln 2 \right] \quad (b, \operatorname{Re} z > 0).$$

$$9. \int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi \ln |a| \quad (|b| \leq |a|).$$

$$10. \int_0^{\pi} \frac{(\sin(x/2))^2}{(\cos(x/2))^2} \ln(1 - 2a \cos x + a^2) dx = \pm \frac{\pi a}{2} \quad (|a| < 1).$$

$$11. = \pi \ln |a| \pm \pi/(2a) \quad (|a| > 1).$$

$$12. \int_0^{\pi} \frac{\ln(1 - 2a \cos x + a^2) dx}{1 - 2b \cos x + b^2} = \frac{2\pi}{|1-b^2|} \ln(1 - ab \pm 1) \quad \left[\begin{array}{l} |b| < 1, \\ |a| > 1 \end{array} \right]$$

$$13. = \frac{2\pi}{|1-b^2|} \ln |a - b \pm 1| \quad \left[\begin{array}{l} |b| < 1, \\ |a| > 1 \end{array} \right].$$

$$14. \int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2) dx = 2\pi \ln |a| \quad (|b| \leq |a|).$$

$$15. \int_0^{\pi} \cos mx \ln(1 - 2a \cos x + a^2) dx = -\frac{\pi n}{m} a^{\pm m} \quad \left[\begin{array}{l} |a| < 1, \\ |a| > 1 \end{array} \right]; \quad n=1, 2; \quad m=1, 2, 3, \dots$$

$$16. \int_0^{\pi} \frac{(\sin x \sin mx)}{(\cos x \cos mx)} \ln(1 - 2a \cos x + a^2) dx = \frac{\pi n}{2} \left(\pm \frac{b^{m+1}}{m+1} - \frac{b^{m-1}}{m-1} \right) \quad \left[\begin{array}{l} n=1, 2; \quad b = \begin{cases} a, & |a| < 1 \\ a^{-1}, & |a| > 1 \end{cases} \end{array} \right].$$

$$17. \int_0^{2\pi} \cos \{(m-1/2)x\} \ln(1 - 2a \cos x + a^2) dx = 0 \quad (|a| \neq 1).$$

$$18. \int_0^{2\pi} \frac{(\sin(x/2) \sin nx)}{(\cos(x/2) \cos nx)} \ln(1 - 2a \cos x + a^2) dx = 0 \quad (|a| \neq 1).$$

$$19. \int_0^{2\pi} \frac{(\sin(x/2) \sin \{(n-1/2)x\})}{(\cos(x/2) \cos \{(n-1/2)x\})} \ln(1 - 2a \cos x + a^2) dx = \pi \left[\pm \frac{b^n}{n} - \frac{b^{n-1}}{n-1} \right] \quad \left[\begin{array}{l} n = \begin{cases} a, & |a| < 1 \\ a^{-1}, & |a| > 1 \end{cases} \\ a=2, 3, 4, \dots \end{array} \right].$$

$$20. \int_0^{\infty} \ln(1 - 2a \cos x + a^2) \frac{dx}{x^2 + z^2} = \frac{\pi}{z} \ln(1 - a e^{-z}) \quad (|a| \leq 1; \operatorname{Re} z > 0).$$

$$21. \quad = \frac{\pi}{2} \ln(|a| - \operatorname{sgn} a e^{-z}) \quad [|a| \geq 1; \operatorname{Re} z > 0]$$

$$22. \quad \int_0^{\infty} \ln(1 - 2a \cos x + a^2) \frac{dx}{(x^2 + z^2)^2} = \frac{\pi}{2z^3} \ln(1 - ae^{-z}) - \frac{\pi a}{2z^3(e^z - a)} \quad [|a| \leq 1; \operatorname{Re} z > 0]$$

$$23. \quad = \frac{\pi}{2z^3} \ln(|a| - \operatorname{sgn} a e^{-z}) - \frac{\pi a e^{-z}}{2z^3 a - e^{-z}} \quad [|a| \geq 1; \operatorname{Re} z > 0]$$

$$24. \quad \int_0^{\infty} \frac{\sin bx}{x} \ln(1 - 2a \cos x + a^2) dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{a^k}{k} [1 + \operatorname{sgn}(b-k)] \quad [b > 0; |a| < 1]$$

$$25. \quad = -\frac{\pi}{2} \left\{ \sum_{k=1}^{[b]} \frac{a^{-k}}{k} [1 + \operatorname{sgn}(b-k)] - 2 \ln |a| \right\} \quad [b > 0; |a| > 1]$$

$$26. \quad \int_0^{\infty} \frac{\cos bx}{x^2 + z^2} \ln(1 - 2a \cos x + a^2) dx = \frac{\pi}{2} \{ \operatorname{ch} bz \ln(1 - ae^{-z}) + \sum_{k=1}^{[b]} \frac{a^k}{k} \operatorname{sh}[z(b-k)] \} \quad [|a| < 1; b, \operatorname{Re} z > 0]$$

$$27. \quad = \frac{\pi}{2} \left\{ \operatorname{ch} bz \ln(|a| - \operatorname{sgn} a e^{-z}) + \sum_{k=1}^{[b]} \frac{a^{-k}}{k} \operatorname{sh}[z(b-k)] - \operatorname{sh} bz \ln |a| \right\} \quad [|a| > 1; b, \operatorname{Re} z > 0]$$

$$28. \quad \int_0^{\pi/4} \ln(\cos x \pm \sin x) dx = -\frac{\pi}{8} \ln 2 \pm \frac{1}{2} G.$$

$$29. \quad \int_0^{2\pi} \ln(a + b \cos x + c \sin x) dx = 2\pi \ln \left[(a + \sqrt{a^2 - b^2 - c^2})/2 \right] \quad [a^2 > b^2 + c^2]$$

2.6.37. Интегралы, содержащие $\ln \frac{a_1 \cos x + b_1 \sin x + c_1}{a_2 \cos x + b_2 \sin x + c_2}$.

$$1. \quad \int_0^{\pi/2} \ln \frac{a + b \sin x}{a - b \sin x} \frac{dx}{\sin x} = \pi \arcsin \frac{b}{a} \quad [|b| \leq |a|]$$

$$2. \quad \int_0^{\pi/2} \ln \frac{a + b \cos x}{a - b \cos x} \frac{dx}{\cos x} = \pi \arcsin \frac{b}{a} \quad [|b| \leq |a|]$$

$$3. \quad \int_0^{\pi/2} \frac{\cos x}{1 - a^2 \cos^2 x} \ln \frac{1 + b \cos x}{1 - b \cos x} dx = \frac{\pi}{2a \sqrt{1 - a^2}} \ln \frac{1 + ab + \sqrt{(1 - a^2)(1 - b^2)}}{1 - ab + \sqrt{(1 - a^2)(1 - b^2)}} \quad [0 < b < a < 1]$$

$$4. \quad \int_0^{\pi} \ln \frac{1 + \sin x}{1 + a \sin x} \frac{dx}{\sin x} = \arccos a \quad [|a| < 1]$$

$$5. \quad \int_0^{\pi} \sin(2n+1)x \ln \frac{1 + 2a \cos x + a^2}{1 - 2a \cos x + a^2} dx = (-1)^n \frac{2\pi a^{2n+1}}{2n+1} \quad \left[\begin{matrix} |a| < 1 \\ |a| > 1 \end{matrix} \right]$$

$$6. \quad \int_0^{2\pi} \cos mx \ln \frac{1 - 2a \cos x + a^2}{1 - 2a \cos nx + a^2} dx = \frac{2\pi}{m} (na^{\pm m/n} - a^{\pm m}) \quad \left[\begin{matrix} |a| < 1 \\ |a| > 1 \end{matrix} \right]$$

$$7. \quad \int_0^{2\pi} \operatorname{ctg} \frac{x}{2} \ln \frac{1 + 2a \cos x + a^2}{1 + 2a \cos nx + a^2} dx = 0 \quad [\ln a = 0]$$

$$8. \quad \int_0^{\infty} \ln \frac{1 + \sin x}{1 - \sin x} \frac{dx}{x} = \frac{\pi^2}{2}, \quad 9. \quad \int_0^{\infty} \ln \left| \frac{\cos x + \sin x}{\cos x - \sin x} \right| \frac{dx}{x} = \frac{\pi^2}{4}.$$

$$10. \quad \int_0^{\infty} \ln \frac{1 - 2a \cos bx + a^2}{1 - 2a \cos cx + a^2} \frac{dx}{x} = 2 \ln(1 - a^{\pm 1}) \ln \left| \frac{c}{b} \right| \quad \left[\begin{matrix} -1 \leq a < 1 \\ |a| > 1 \text{ или } a = -1 \end{matrix} \right]$$

2.6.38. Интегралы, содержащие $\ln(a \cos^2 x + b \sin^2 x + c)$.

Обозначение: $k' = \sqrt{1 - k^2}$, $0 < k < 1$.

$$1. \quad \int_0^{\pi/2} \ln(1 + b \sin^2 x) dx = \pi \ln \frac{1 + \sqrt{1+b}}{2} \quad [b \geq -1]$$

$$2. \quad \int_0^{\pi/2} \frac{(\sin x)^2}{(\cos x)^2} \ln(1 + b \sin^2 x) dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+b}}{2} + \frac{1}{2} \cdot \frac{1 - \sqrt{1+b}}{1 + \sqrt{1+b}} \right) \quad [b \geq -1]$$

$$3. \quad \int_0^{\pi/2} \frac{\ln(1 - a^2 \cos^2 x)}{1 - b^2 \cos^2 x} dx = -\frac{\pi}{\sqrt{1-b^2}} \ln \frac{1 + \sqrt{1-b^2}}{\sqrt{1-b^2} + \sqrt{1-a^2}} \quad [a^2, b^2 < 1]$$

$$4. \quad \int_0^{\pi/2} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \frac{|a| + |b|}{2} \quad [ab \neq 0]$$

$$5. \quad \int_0^{\pi/2} x^n \sin x \cos x \left(1 - k^2 \left(\frac{\sin x}{\cos x} \right)^2 \right)^{-n/2} \ln \left(1 - k^2 \left(\frac{\sin x}{\cos x} \right)^2 \right) dx = J_{n,2}^1$$

$$J_{-1}^1 = \frac{1}{27k^2} \left[3\pi \left\{ k'^3 (1 - 3 \ln k') \right\} \pm (22k'^3 + 6k'^4 - 3k'^2 \ln k') K(k) \mp (2 - k^2) (14 - 6 \ln k') E(k) \right],$$

$$J_1^{\pm} = \pm k^{-2} \left\{ (2-k^2) K(k) - (4-\ln k') E(k) \right\} + \pi k^{-2} \left\{ \frac{k'}{1-k'} (1-\ln k') \right\},$$

$$J_1^{-1} = \pm k^{-2} \left\{ \left(2-k^2 - \left\{ \frac{k'}{1-k'} \right\} \ln k' \right) K(k) - (2-\ln k') E(k) \right\},$$

$$J_1^{\pm 1} = \pm k^{-2} (2+\ln k') K(k) + \pi k^{-2} \left\{ \frac{(1+\ln k')/k'}{-1} \right\},$$

$$6. \int_0^{\pi/2} \ln \frac{1+\sin \varphi \cos^2 x}{1-\sin \varphi \cos^2 x} dx = \pi \ln \frac{1+\sin(\varphi/2)}{\cos(\varphi/2)} \quad [|\varphi| < \pi/2],$$

$$7. \int_0^{\pi/2} \frac{\sin x}{x} \left(1-k^2 \frac{(\sin x)^2}{(\cos x)^2} \right)^{-1/2} \ln \left(1 \pm k^2 \frac{(\sin x)^2}{(\cos x)^2} \right) dx = \\ = \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} K(k) - \frac{\pi}{8} K(\sqrt{1-k^2}).$$

$$8. \int_0^{\infty} \frac{\sin^m x \cos^l x}{x} \left(1-k^2 \frac{(\sin x)^2}{(\cos x)^2} \right)^{-m/2} \ln \left(1-k^2 \frac{(\sin x)^2}{(\cos x)^2} \right) dx = I_m^l, \quad l \neq 0,$$

$$I_{1,1}^{-1} = I_{1,1}^0 = (2-k^2) K(k) - (2-\ln k') E(k), \quad I_{1,1}^0 = I_{1,1}^{-1} = \ln k' K(k),$$

$$I_{1,1}^{\pm} = \pm k^{-2} \left\{ \left(2-k^2 - \left\{ \frac{k'}{1-k'} \right\} \ln k' \right) K(k) - (2-\ln k') E(k) \right\},$$

$$I_{1,1}^{-1} = I_{1,1}^0 = \pm k^{-2} \left\{ \left(k^2-2 + \left\{ \frac{1}{k'} \right\} \ln k' \right) K(k) + (2-\ln k') E(k) \right\},$$

$$I_{1,1}^0 = I_{1,1}^{-1} = k'^{-2} [(k^2-2) K(k) + (2+\ln k') E(k)],$$

$$I_{1,1}^{\pm} = I_{1,1}^0 = \pm k^{-2} \left\{ \frac{1}{k'^2} \right\} \left[\left(2-k^2 + \left\{ \frac{1}{k'^2} \right\} \ln k' \right) K(k) - (2+\ln k') E(k) \right],$$

$$I_{2,0}^0 = I_{2,0}^{-1} = \pm k^{-2} \left\{ \frac{k'}{1} \right\} \left[\left(2-k^2 + \left\{ \frac{k'}{1} \right\} \ln k' \right) K(k) - (2+\ln k') E(k) \right].$$

$$9. \int_0^{\infty} \ln (a^2 \cos^2 cx + b^2 \sin^2 cx) \frac{dx}{x^2+z^2} = \frac{\pi}{z} [\ln (a \operatorname{ch} cz + b \operatorname{sh} cz) - cz] \\ [a, b, c, \operatorname{Re} z > 0].$$

$$10. \int_0^{\pi/2} \frac{\lg(x/2) \ln(\sin^2 x + k' \cos^2 x)}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{2} \ln \frac{2(k')^3}{1+k'} K(k),$$

$$11. \int_0^{\pi/2} \frac{1}{x} \left(\frac{\lg x}{\sin x} \right) \frac{\ln(\sin^2 x + k' \cos^2 x)}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{2} \ln \frac{2(\sqrt{k'})^3}{1+k'} K(k),$$

$$12. \int_0^{\pi/2} \frac{\ln(\sin x + \sqrt{\sin^2 x - \sin^2 \varphi})}{1-a^2 \cos^2 x} dx = \\ = -\frac{1}{\sqrt{1-a^2}} \left[\operatorname{arctg} \left(\frac{\lg \varphi}{\sqrt{1-a^2}} \right) \ln \sin \varphi + \frac{\pi}{2} \ln \frac{1+\sqrt{1-a^2}}{\sqrt{1-a^2} + \sqrt{1-a^2 \cos^2 \varphi}} \right] \\ [1, a| < 1; 0 < \varphi < \pi/2].$$

2.6.39. Интегралы, содержащие $\ln A(\lg x)$.

$$1. \int_0^{\pi/4} \ln \lg x dx = -G, \quad 2. \int_0^{\pi/4} \sin 2x \ln \lg x dx = -\frac{1}{2} \ln 2.$$

$$3. \int_0^{\pi/4} \frac{\ln \lg x}{\cos 2x} dx = -\frac{\pi^2}{8},$$

$$4. \int_0^{\pi/4} \frac{\cos 2x}{1 \pm a \sin 2x} \ln \lg x dx = -\frac{\operatorname{arcsin} a}{4a} (\pi \mp \operatorname{arcsin} a) \quad [|a| < 1].$$

$$5. \int_0^{\pi/4} \frac{\cos 2x \ln \lg x}{1 \pm a \sin^2 2x} dx = -\frac{\pi}{4\sqrt{a}} \left\{ \operatorname{Arsh} \sqrt{\frac{1}{a}} \right\} \operatorname{arcsin} \sqrt{\frac{1}{a}} \quad [|a| < 1].$$

$$6. \int_0^{\pi/4} \ln(1 \pm \lg x) dx = \frac{\pi}{8} \ln 2 - \left\{ \frac{0}{G} \right\},$$

$$7. \int_0^{\pi/4} \ln \lg \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8},$$

$$8. \int_0^{\pi/4} \ln \lg \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\lg 2x} = \pm \frac{\pi^2}{16},$$

$$9. \int_0^{\pi/4} \ln \lg \left(\frac{\pi}{4} \pm x \right) (\ln \lg x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2}-1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|,$$

$$10. \int_0^{\pi/4} \ln \lg \left(\frac{\pi}{4} \pm x \right) (\ln \lg x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1-2^{2n+1}}{2^{2n+1}n} (2n)! \zeta(2n+1) \\ (n=1, 2, 3, \dots).$$

$$11. \int_0^{\pi/4} \ln \lg \left(\frac{\pi}{4} \pm x \right) (\ln \sin 2x)^n \frac{dx}{\lg 2x} = \frac{(-1)^n}{2} n! \zeta(n+2),$$

$$12. \int_0^{\pi/4} \frac{\sin 2x \ln \lg x}{1-a^2 \sin^2 2x} dx = \frac{1}{2a\sqrt{1-a^2}} [L(\operatorname{arcsin} a) - \operatorname{arcsin} a \ln 2] \quad [|a| < 1].$$

$$13. \int_0^{\pi/4} \frac{\sin^4 x \ln \lg x dx}{(\sin^2 \varphi + \lg^2 v \sin^2 2x) \sqrt{\sin^2 2x - \sin^2 \varphi}} = \\ = -\frac{\pi}{2} \frac{\cos^2 v}{\sin \varphi \sin v} \ln \frac{\sin v + \sqrt{1-\cos^2 \varphi \cos^2 v}}{\sin \varphi (1+\sin v)} \quad [0 < \varphi < \pi/2, 0 < v < \pi/2].$$

$$14. \int_0^{\pi/2} \ln \lg x dx = 0.$$

$$15. \int_0^{\pi/2} \cos^{a-1} x \cos(a+1)x \ln \lg x dx = \frac{\pi}{2a} \quad [a > 0].$$

$$16. \int_0^{\pi/2} \cos^{a-1} x \operatorname{ctg} x \sin(a+1)x \ln \lg x dx = \frac{\pi}{2} [\zeta + \psi(a+1)] \quad [a > -1].$$

$$17. \int_0^{\pi/2} \lg x \ln \lg(x/2) (\ln \cos x)^n dx = (-1)^n 2^{-n-2} n! \zeta(n+2, 1/2).$$

$$18. \int_0^{\pi/2} \frac{\ln \lg x}{\sqrt{1-k^2 \sin^2 x}} dx = -\ln \sqrt{1-k^2} K(k) \quad [0 < k < 1].$$

$$19. \int_0^{\pi/2} \ln(1+\lg x) dx = \frac{\pi}{4} \ln 2 + G. \quad 20. \int_0^{\pi} x \ln |\lg x| dx = 0.$$

$$21. \int_0^{\infty} \frac{\ln |\lg x|}{x^2+x^2} dx = \frac{\pi}{2z} \ln \operatorname{th} z \quad [\operatorname{Re} z > 0].$$

$$22. \int_0^{\infty} \ln \left| \frac{1+\lg x}{1-\lg x} \right| \frac{dx}{x} = \frac{\pi^2}{4}.$$

$$23. \int_0^{\infty} \ln \left(1+b^2 \left(\frac{\lg ax}{\operatorname{ctg} ax} \right)^2 \right) \frac{dx}{x^2+x^2} = \frac{\pi}{2} \ln \left(1+b \left(\frac{\operatorname{th} az}{\operatorname{cth} az} \right) \right) \quad [a, b, \operatorname{Re} z > 0].$$

2.6.40. Интегралы, содержащие показательную, логарифмическую, гиперболические и тригонометрические функции.

$$1. \int_0^{\infty} e^{-px} \frac{\sin bx}{\cos bx} \ln x dx = \frac{1}{b^2+p^2} \left[\left\{ -\frac{p}{b} \right\} \operatorname{arctg} \frac{b}{p} - \zeta \left\{ \frac{b}{p} \right\} - \frac{1}{2} \left\{ \frac{b}{p} \right\} \ln(b^2+p^2) \right] \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$2. \int_0^{\infty} x^{\alpha-1} e^{-px} \frac{\sin bx}{\cos bx} \ln x dx = \Gamma(\alpha) (p^2+b^2)^{-\alpha/2} \left\{ \frac{\sin c}{\cos c} \right\} \left[\psi(\alpha) - \frac{1}{2} \ln(p^2+b^2) \pm \operatorname{arctg} \frac{b}{p} \left\{ \frac{\operatorname{ctg} c}{\operatorname{tg} c} \right\} \right] \\ [c = \alpha \operatorname{arctg} \frac{b}{p}, \operatorname{Re} p > |\operatorname{Im} b|; \operatorname{Re} \alpha > \left\{ \frac{-1}{0} \right\}].$$

$$3. \int_0^{\infty} e^{-px} \frac{\sin bx}{x} \ln x dx = -\operatorname{arctg} \frac{b}{p} \left[\zeta + \frac{1}{2} \ln(p^2+b^2) \right] \quad [\operatorname{Re} p > |\operatorname{Im} b|]$$

$$4. \int_0^{\infty} e^{-qx} \frac{\sin bx}{x} \ln x dx = -\frac{i}{2} \ln \left(\frac{b}{q} \right) [K_0(2e^{-\pi i/4} \sqrt{bq}) - K_0(2e^{\pi i/4} \sqrt{bq})] + \\ + \frac{\pi}{4} [K_0(2e^{\pi i/4} \sqrt{bq}) + K_0(2e^{-\pi i/4} \sqrt{bq})] \quad [b, \operatorname{Re} q > 0].$$

$$5. \int_0^{\infty} \sin bx \ln(1+e^{-ax}) dx = -\frac{1}{4b} \left[4 \ln 2 - \psi \left(1 + \frac{ib}{2a} \right) - \psi \left(1 - \frac{ib}{2a} \right) + \psi \left(\frac{a+ib}{2a} \right) + \psi \left(\frac{a-ib}{2a} \right) \right] \quad [b, \operatorname{Re} a > 0].$$

$$6. \int_0^{\infty} \sin bx \ln(1-e^{-ax}) dx = -\frac{1}{2b} \left[2\zeta + \psi \left(1 + \frac{ib}{a} \right) + \psi \left(1 - \frac{ib}{a} \right) \right] \quad [b, \operatorname{Re} a > 0].$$

$$7. \int_0^{\infty} \cos bx \ln(1 \pm e^{-ax}) dx = \frac{a}{2b^2} - \frac{\pi}{2b \operatorname{sh}(b\pi/a)} \left\{ \operatorname{ch}(b\pi/a) \right\} \quad [b, \operatorname{Re} a > 0].$$

$$8. \int_0^{\infty} \operatorname{sh} ax \sin bx \ln(1-e^{-2ax}) dx = \frac{a}{a^2+b^2} \left(\frac{2ab}{a^2+b^2} - \frac{\pi}{2} \operatorname{th} \frac{b\pi}{2a} \right) \quad [b, \operatorname{Re} a > 0].$$

$$9. \int_0^{\infty} \operatorname{ch} ax \cos bx \ln(1-e^{-2ax}) dx = \frac{a(b^2-a^2)}{(a^2+b^2)^2} - \frac{\pi b}{2(a^2+b^2)} \operatorname{th} \frac{b\pi}{2a} \quad [b, \operatorname{Re} a > 0].$$

$$10. \int_0^{\infty} \cos bx \ln \left(1 + \frac{\cos a}{\operatorname{ch} x} \right) dx = \frac{\pi}{b \operatorname{sh} b\pi} \left(\operatorname{ch} \frac{b\pi}{2} - \operatorname{ch} ab \right) \quad [b > 0; |\operatorname{Re} a| < \pi].$$

$$11. \int_0^{\infty} \cos bx \ln \frac{\operatorname{ch} x + \sin a}{\operatorname{ch} x - \sin a} dx = \frac{\pi \operatorname{sh} ab}{b \operatorname{ch}(b\pi/2)} \quad [b > 0; 2|\operatorname{Re} a| < \pi].$$

$$12. \int_0^{\infty} \cos bx \ln \frac{\operatorname{ch} x + \cos a}{\operatorname{ch} x + \cos c} dx = \frac{\pi}{b \operatorname{sh} ab} (\operatorname{ch} cb - \operatorname{ch} ab) \quad [b > 0; |\operatorname{Re} a|, |\operatorname{Re} c| < \pi].$$

$$13. \int_0^{\infty} e^{-px} \ln \left\{ \frac{2|\sin ax|}{2|\cos ax|} \right\} dx = -p \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(p^2+4a^2k^2)} \quad [\operatorname{Re} p > 0].$$

$$14. \int_0^{\infty} e^{-px} \ln |\operatorname{ctg} ax| dx = 2p \sum_{k=1}^{\infty} \frac{1}{(2k-1)(p^2+4(2k-1)^2 a^2)} \quad [\operatorname{Re} p > 0].$$

$$15. \int_0^{\pi} e^{a \cos x} \{\ln(2a \sin^2 x) + C\} dx = -\pi K_0(a) \quad [a > 0]$$

$$16. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2-1}} \sin(b\sqrt{x^2-1}) \ln(x+\sqrt{x^2-1}) dx = \operatorname{arctg}\left(\frac{b}{p}\right) K_0(\sqrt{p^2+b^2}) \quad [Re p > |Im b|]$$

$$17. \int_0^{\infty} \frac{e^{-px}}{\sqrt{x^2+2ax}} \sin(b\sqrt{x^2+2ax}) \ln \frac{a+x+\sqrt{x^2+2ax}}{a} dx = \\ = e^{ap} \operatorname{arctg}\left(\frac{b}{p}\right) K_0(a\sqrt{b^2+p^2}) \quad [Re p > |Im b|]$$

$$18. \int_0^{\infty} x^{z-1} e^{-x} \frac{\{\sin(b \ln x)\}}{\{\cos(b \ln x)\}} dx = |\Gamma(z+ib)| \frac{\{\sin \arg \Gamma(\alpha+ib)\}}{\{\cos \arg \Gamma(\alpha+ib)\}} \\ [Re z > |Im b|; \arg \Gamma(x) = -i \ln \frac{\Gamma(x)}{|\Gamma(x)|}]$$

2.7. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

2.7.1. Введение.

В этом разделе содержатся интегралы, в подынтегральных выражениях которых входят обратные тригонометрические функции. Они могут быть образованы к интегралам от логарифмической функции и от степенных функций, если воспользоваться соответствующими формулами, приведенными в приложении I. В ряде случаев интегралы от обратных тригонометрических функций с помощью замены переменной сводятся к интегралам от тригонометрических функций. Например,

$$\int_0^1 \arcsin x dx = -i \int_0^1 \ln(ix + \sqrt{1-x^2}) dx = \int_0^{\pi/2} x \cos x dx.$$

2.7.2. Интегралы общего вида.

В этом пункте помещены формулы, которые позволяют находить интегралы, содержащие обратные тригонометрические, а также степенную, алгебраические и тригонометрические функции. Некоторые частные случаи этих интегралов, соответствующие конкретным значениям входящих в них параметров, приведены в последующих пунктах. Интегралы общего вида представляются с помощью функций V_j , $j=1, 2, 3, \dots$, помещенных в приложении III. Конкретные значения параметров, входящих в эти функции, приводятся в фигурных скобках после каждого интеграла. В п. 2.7.2 приняты следующие обозначения: $r=p/q$, где p и q — взаимно простые натуральные числа, если не указаны другие условия;

$$\delta = \begin{cases} 1, & \text{если взят } \sin bx, \\ 0, & \text{если взят } \cos bx; \end{cases} \quad \epsilon = \begin{cases} (-1)^{[n/2]+k}, & \text{если взят } \sin bx, \\ 1, & \text{если взят } \cos bx; \end{cases}$$

$$\gamma = \begin{cases} 1/2, & \text{если взят } \sin^n bx, \quad n=1, 3, 5, \dots, \\ 0, & \text{если взят } \sin^n bx, \quad n=2, 4, 6, \dots, \\ 0, & \text{если взят } \cos^n bx, \quad n=1, 2, 3, \dots; \end{cases}$$

$$\xi = r^{\frac{\alpha}{2}} \left(\frac{\alpha}{2} + \gamma + h \right), \quad \tilde{b} = \frac{by}{2} (n-2k).$$

$$1. \int_0^y x^{z-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} \arcsin\left(\frac{x}{y}\right)^r dx = I(\sigma) \Big|_{\sigma=1},$$

$$I(\sigma) = U + \frac{1+(-1)^n}{2} 2^{-n-1} \frac{n! y^{\frac{\alpha}{2}} \sqrt{\pi}}{\alpha [(n/2)!]^2} \left\{ (-1)^{\sigma} \Gamma\left[\frac{1/2+\alpha/(2r)}{1+\alpha/(2r)}\right] + \sigma \sqrt{\pi} \right\}, \\ U = \frac{(-1)^{\sigma} 2^{-n-1} \pi}{r y^{\frac{\alpha}{2}}} \sum_{k=0}^{[(n-1)/2]} e^{\binom{n}{k}} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \tilde{b}^{2\gamma+2h} \frac{1}{\xi \Gamma(2\gamma+1/2+h)} \left\{ \Gamma\left[\frac{1/2+\xi}{1+\xi}\right] \times \right. \\ \times {}_2F_{2p+1}\left(1, \xi/q, \Delta(q, 1/2+\xi); 1+\xi/q, \Delta(q, 1+\xi), \Delta(p, 2\gamma+1/2+h), \Delta(p, 1+h)\right) \Big|_{\sigma=1} + \\ \left. + (-1)^{\sigma} \sigma \sqrt{\pi} {}_2F_{2p+1}\left(1, \xi/q; \Delta(p, 2\gamma+1/2+h), \Delta(p, 1+h), 1+\xi/q\right) \right\}, \\ z = (-1)^{\sigma} \tilde{b}^{2\gamma+2p} \quad [b > 0; Re \alpha > -r\sigma - \delta n].$$

$$2. \int_0^y x^{z-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} \arccos\left(\frac{x}{y}\right)^r dx = J(\sigma) \Big|_{\sigma=0} \quad [\text{см. 2.7.2.1}].$$

$$3. \int_0^y x^{z-1} \frac{\{\sin bx\}^n}{\{\cos bx\}} \arcsin\left(\frac{y}{x}\right)^r dx = J(\sigma) \Big|_{\sigma=1}.$$

$$J(\sigma) = U - \frac{1+(-1)^n}{2} 2^{-n-1} \frac{n! y^{\frac{\alpha}{2}} \sqrt{\pi}}{\alpha [(n/2)!]^2} \left\{ (-1)^{\sigma} \Gamma\left[\frac{1/2-\alpha/(2r)}{1-\alpha/(2r)}\right] + \sigma \sqrt{\pi} \right\}, \\ U = \frac{(-1)^{\sigma} 2^{-n-1} \pi}{r y^{\frac{\alpha}{2}}} \sum_{k=0}^{[(n-1)/2]} e^{\binom{n}{k}} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \tilde{b}^{2\gamma+2h} \left\{ \Gamma\left[\frac{1/2-\xi}{1-\xi}, \frac{1/2+2\gamma+h}{1-\xi}\right] \times \right. \\ \times {}_2F_{2p+1}\left(1, \frac{\xi}{q}, \Delta(q, \xi); 1+\frac{\xi}{q}, \Delta(p, 1+h), \Delta\left(q, \frac{1}{2}+\xi\right), \Delta\left(p, \frac{1}{2}+2\gamma+h\right)\right) - \\ - \frac{(-1)^{\sigma} \sqrt{\pi}}{2^n} y^z \sum_{j=0}^{q-1} e^{\binom{n}{j}} \sum_{l=0}^{q-1} \frac{(1/2)_j}{j! (1+2j)} \times \\ \times \tilde{b}^{r(1+2j)-2} \Gamma\left[\frac{\gamma+(\alpha-r)/2-jr}{\gamma+(1+r-\alpha)/2+jr}\right] \times \\ \times {}_2F_{2p+1}\left(\frac{1+2j}{2q}, 1, \Delta\left(q, \frac{1}{2}+j\right); 1+\frac{1+2j}{2q}, \Delta(q, 1+j), \Delta\left(p, \gamma+\frac{1+r-\alpha}{2}+jr\right), \Delta(p, \beta)\right) \Big|_{\sigma=1} + \\ \left. + (-1)^{\sigma} \pi^{3/2} 2^{-n-1} y^{\frac{\alpha}{2}} \Gamma\left[\frac{\gamma+\alpha}{1/2+\gamma-\alpha}\right] \sum_{k=0}^{[(n-1)/2]} e^{\binom{n}{k}} \tilde{b}^{-\alpha} - \right. \\ \left. - \sigma \pi^{3/2} 2^{-n-1} y^{\frac{\alpha}{2}} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} \tilde{b}^{2\gamma+2h} \frac{1}{\Gamma(1/2+2\gamma+h)} \times \right. \\ \left. \times {}_2F_{2p+1}\left(1, \xi/q; \Delta(p, 1/2+2\gamma+h), \Delta(p, 1+h)\right) \right\}, \\ z = (-1)^{\sigma} \tilde{b}^{2\gamma+2p} \quad [\beta = 1-\gamma+ir+(r-\alpha)/2; y, b > 0; Re \alpha < 0]$$

$$4. \int_0^{\infty} x^{\alpha-1} \left(\frac{\sin bx}{\cos bx} \right)^n \arccos \left(\frac{y}{x} \right)^r dx = J(\alpha) |_{\sigma=\alpha} \quad [\text{см. 2.7.2.3}].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left(\frac{\sin bx}{\cos bx} \right)^n \sin \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = K(\sigma) |_{\sigma=\alpha},$$

$$K(\sigma) = V_{35} + \frac{1+(-1)^n}{2} 2^{-n-\alpha/r} \frac{\pi n!}{r [(n/2)!]^2} \left(\frac{v}{2} \right)^{\sigma} y^{\alpha+r(\sigma-1)} \times$$

$$\times \Gamma \left[\frac{\alpha/r}{\alpha/(2r) + (v+1+\sigma)/2}, \frac{\alpha/(2r) + (1+\sigma-v)/2} \right]$$

$$\left\{ E_{35} = \left(\frac{v}{2} \right)^{\sigma} \frac{\pi^{3/2}}{2^{n-1} r} 2^{-n} \Gamma \left[\frac{2\xi}{\xi + (1+\sigma+v)/2}, \frac{\xi + (1+\sigma-v)/2}{2\gamma + 1/2 + h} \right], \right.$$

$$u = \alpha + r(\sigma-1), \quad a_1 = \xi, \quad b_1 = \xi + 1/2, \quad c_1 = \xi + (1+\sigma+v)/2,$$

$$d_1 = \xi + (1+\sigma-v)/2, \quad e_1 = 2\gamma + 1/2 + h, \quad z = (-b^2/p^2)^p \}$$

$$[v, b > 0; \operatorname{Re} \alpha > -\delta n].$$

$$6. \int_0^{\infty} x^{\alpha-1} (y^{2r} - x^{2r})^{-1/2} \left(\frac{\sin bx}{\cos bx} \right)^n \cos \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = K(\sigma) |_{\sigma=\alpha}$$

$$[\text{см. 2.7.2.5}].$$

$$7. \int_0^{\infty} x^{\alpha-1} \left(\frac{\sin bx}{\cos bx} \right)^n \sin \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = L(\sigma) |_{\sigma=\alpha},$$

$$L(\sigma) = V_{36} + V_{38} + V_{37} + \frac{1+(-1)^n}{2} 2^{-n+\alpha/r-1} \frac{\pi n! v^{\sigma}}{r [(n/2)!]^2} \times$$

$$\times y^{\alpha+r(\sigma-1)} \Gamma \left[\frac{1-\sigma-\alpha/r}{1+v/2-\alpha/(2r)}, \frac{1-v/2-\alpha/(2r)} \right]$$

$$\left\{ E_{36} = \pi^{3/2} v^{\sigma} \frac{2^{2\xi-n}}{r} \Gamma \left[\frac{1-\sigma-2\xi}{1+v/2-\xi}, \frac{1-v/2-\xi}{2\gamma + 1/2 + h} \right], \right.$$

$$E_{38} = \pi^{3/2} v^{\sigma} 2^{-n-\sigma} \frac{(-1)^j}{(1/2)^j} \Gamma \left[\frac{1-d_1}{1-a_1}, \frac{1-b_1}{1-b_2}, \frac{e_1}{e_2} \right],$$

$$E_{37} = -\pi^{3/2} v^{\sigma} 2^{1-n-\sigma} \frac{(-1)^j}{(3/2)^j} \Gamma \left[\frac{1-d_3}{1-a_3}, \frac{1-b_3}{1-b_3}, \frac{e_3}{e_3} \right],$$

$$u = \alpha + r(\sigma-1), \quad v_1 = r(1-\sigma+2j)-\alpha, \quad v_2 = r(2-\sigma+2j)-\alpha,$$

$$a_1 = \xi - v/2, \quad b_1 = \xi + v/2, \quad c_1 = (1+\sigma)/2 + \xi, \quad d_1 = \sigma/2 + \xi,$$

$$e_1 = 2\gamma + 1/2 + h, \quad a_2 = (1-v-\sigma)/2 + j, \quad b_2 = (1+v-\sigma)/2 + j,$$

$$c_2 = 1/2 + j, \quad d_2 = 1-\gamma-\alpha/2 + r[(1-\sigma)/2 + j],$$

$$e_2 = \gamma + (1-\alpha)/2 + r[(1-\sigma)/2 + j], \quad a_3 = 1-(\sigma+v)/2 + j,$$

$$b_3 = 1-(\sigma-v)/2 + j, \quad c_3 = 3/2 + j,$$

$$d_3 = 1-\gamma-\alpha/2 + r(1-\sigma/2 + j), \quad e_3 = \gamma + (1-\alpha)/2 + r(1-\sigma/2 + j),$$

$$z = (-b^2/p^2)^p \} \quad [v, b > 0; \operatorname{Re} \alpha < r(1-\sigma)].$$

$$8. \int_0^{\infty} x^{\alpha-1} (x^{2r} - y^{2r})^{-1/2} \left(\frac{\sin bx}{\cos bx} \right)^n \cos \left[v \arccos \left(\frac{y}{x} \right)^r \right] dx = L(\sigma) |_{\sigma=\alpha}$$

$$[\text{см. 2.7.2.7}].$$

$$9. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left(\frac{\sin bx}{\cos bx} \right)^n \sin \left[v \operatorname{arctg} \left(\frac{x}{y} \right)^r \right] dx = M(\sigma) |_{\sigma=\alpha},$$

$$M(\sigma) = V_{39} + V_{38} + V_{37} +$$

$$+ \frac{1+(-1)^n}{2} \frac{(-1)^{\sigma} \sqrt{\pi} 2^{2/r-n} n!}{\Gamma(-v) [(n/2)!]^2} y^{\alpha+r v} \Gamma \left[\frac{\sigma/2 + \alpha/(2r)}{(1+\sigma)/2 - \alpha/(2r)}, \frac{-v - \alpha/r}{(1+\sigma)/2 - \alpha/(2r)} \right]$$

$$\left\{ E_{39} = (-1)^{\sigma} \frac{2^{2\xi-n} \pi}{r \Gamma(-v)} \Gamma \left[\frac{-v-2\xi}{(1+\sigma)/2 - \xi}, \frac{\sigma/2 + \xi}{2\gamma + 1/2 + h} \right], \right.$$

$$E_{38} = (-1)^{\sigma+j} \frac{2^{-v-n-1} \pi}{\Gamma(-v) (1/2)^j} \Gamma \left[\frac{a_1}{1-b_1}, \frac{1-d_1}{e_1} \right],$$

$$E_{37} = (-1)^{\sigma+j-1} \frac{2^{-v-n} \pi}{\Gamma(-v) (3/2)^j} \Gamma \left[\frac{a_3}{1-b_3}, \frac{1-d_3}{e_3} \right],$$

$$u = \alpha + r v, \quad v_1 = r(2j-v)-\alpha, \quad v_2 = r(2j+1-v)-\alpha,$$

$$a_1 = \sigma/2 + \xi, \quad b_1 = (1-\sigma)/2 + \xi, \quad c_1 = 1 + v/2 + \xi,$$

$$d_1 = (1+v)/2 + \xi, \quad e_1 = 2\gamma + 1/2 + h, \quad a_2 = (\sigma-v)/2 + j,$$

$$b_2 = (1-\sigma-v)/2 + j, \quad c_2 = 1/2 + j, \quad d_2 = 1-\gamma-\alpha/2 + r(j-v/2),$$

$$c_3 = (1-\alpha)/2 + \gamma + r(j-v/2), \quad a_3 = (1-v+\sigma)/2 + j,$$

$$b_3 = 1-(\sigma+v)/2 + j, \quad c_3 = 3/2 + j, \quad d_3 = 1-\gamma-\alpha/2 + r[j + (1-v)/2],$$

$$e_3 = (1-\alpha)/2 + \gamma + r[j + (1-v)/2], \quad z = (-1)^{\sigma} v^{\sigma} y^{\alpha} p^{-2p}$$

$$\left\{ b, y > 0; -n\delta - r\sigma < \operatorname{Re} \alpha < \frac{1-(-1)^n}{2} - r \operatorname{Re} v \right\}.$$

$$10. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left(\frac{\sin bx}{\cos bx} \right)^n \cos \left[v \operatorname{arctg} \left(\frac{x}{y} \right)^r \right] dx = M(\sigma) |_{\sigma=\alpha}$$

$$[\text{см. 2.7.2.9}].$$

$$11. \int_0^{\infty} x^{\alpha-1} (x^{2r} + y^{2r})^{v/2} \left(\frac{\sin bx}{\cos bx} \right)^n \sin \left[v \operatorname{arctg} \left(\frac{y}{x} \right)^r \right] dx = N(\sigma) |_{\sigma=\alpha},$$

$$N(\sigma) = V_{39} + V_{38} +$$

$$+ \frac{1+(-1)^n}{2} \frac{(-1)^{\sigma} \sqrt{\pi} 2^{-v-n-\alpha/r} n!}{\Gamma(-v) [(n/2)!]^2} y^{\alpha+r v} \Gamma \left[\frac{\alpha/r, (\sigma-v)/2 - \alpha/(2r)}{(1+\sigma+v)/2 + \alpha/(2r)} \right]$$

$$\left\{ E_{39} = \frac{(-1)^{\sigma} \pi}{r \Gamma(-v)} 2^{-n-v-2\xi} \Gamma \left[\frac{2\xi, (\sigma-v)/2 - \xi}{(1+\sigma+v)/2 + \xi}, \frac{2\gamma + 1/2 + h}{(1+\sigma+v)/2 + \xi} \right], \right.$$

$$E_{38} = \frac{(-1)^{\sigma} \pi}{r \Gamma(-v)} 2^{-n-v-j-\sigma} \Gamma \left[\frac{\sigma-v+2j, 1-d_1}{c_3, e_3} \right],$$

$$u = \alpha + v r, \quad v_1 = r(2j+\sigma-v)-\alpha, \quad a_1 = \xi, \quad b_1 = 1/2 + \xi,$$

$$c_1 = (1+\sigma+v)/2 + \xi, \quad d_1 = 1+(v-\sigma)/2 + \xi, \quad e_1 = 1/2 + 2\gamma + h,$$

$$a_2 = (\sigma-v)/2 + j, \quad b_2 = (1+\sigma-v)/2 + j, \quad c_2 = 1/2 + \sigma + j,$$

$$d_2 = 1-\gamma-\alpha + r[j + (\sigma-v)/2], \quad e_2 = 1/2 + \gamma - \alpha/2 + r[j + (\sigma-v)/2],$$

$$z = (-1)^{\sigma} y^{\sigma} b^{\alpha} p^{-2p} \quad [b, y > 0; -n\delta < \operatorname{Re} \alpha < (1-(-1)^n)/2 + r\sigma - r \operatorname{Re} v].$$

$$12. \int_0^{\infty} x^{2r-1} (x^{2r} + y^{2r})^{1/2r} \left\{ \frac{\sin bx}{\cos bx} \right\}^n \cos \left[v \operatorname{arctg} \left(\frac{y}{x} \right)^r \right] dx = N(\sigma) \ln \sigma$$

{см. 2.7.2.11}.

2.7.3. Интегралы, содержащие $\left\{ \frac{\arcsin bx}{\arccos bx} \right\}$.

$$1. \int_0^1 \arcsin(e^{i\varphi} x) dx = \operatorname{sgn}(\cos \varphi) \arcsin(\sqrt{1 - \sin \varphi}) - \\ - \cos \varphi + \sqrt{2 \sin \varphi} \cos[(2\varphi + \pi)/4] + i \left\{ \ln(\sqrt{1 + \sin \varphi} + \sqrt{\sin \varphi}) + \right. \\ \left. + \sin \varphi - \sqrt{2 \sin \varphi} \sin[(2\varphi + \pi)/4] \right\} \quad [0 \leq \varphi \leq \pi].$$

$$2. \int_0^1 x^{\alpha-1} \left\{ \frac{\arcsin x}{\arccos x} \right\} dx = \frac{\pi}{2\alpha} \left[\frac{1}{0} \right] - \frac{1}{\pi} \Gamma \left[\frac{(\alpha+1)/2}{1+\alpha/2} \right] \quad \left[\operatorname{Re} \alpha > -\frac{1}{0} \right].$$

$$3. \int_0^1 \frac{x}{\sqrt{(1-x^2)(1-k^2x^2)}} \left\{ \frac{\arcsin kx}{\arccos kx} \right\} dx = \frac{\pi}{2k} \left\{ \ln \sqrt{1-k^2} \right\} \quad [0 < k < 1].$$

$$4. \int_0^1 e^{-px} \arcsin x dx = \frac{\pi}{2p} [I_0(p) - L_0(p) - e^{-p}] \quad [|\operatorname{arg} p| < \pi].$$

$$5. \int_0^1 x e^{-px} \arcsin x dx = \frac{\pi}{2p^2} [L_0(p) - I_0(p) + p L_1(p) - p I_1(p)] + \frac{1}{p} \quad [|\operatorname{arg} p| < \pi].$$

$$6. \int_0^1 \sin(n\pi x) \left\{ \frac{\arcsin x}{\arccos x} \right\} dx = \frac{1}{2n} [(-1)^n - J_0(n\pi)] \quad [n = 1, 2, 3, \dots].$$

$$7. \int_0^1 \ln x \left\{ \frac{\arcsin x}{\arccos x} \right\} dx = \pm (2 - \ln 2) - \left\{ \frac{\pi}{2} \right\}.$$

$$8. \int_0^1 \frac{\arccos x}{\ln x} dx = - \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1}.$$

$$9. \int_{-1}^1 \cos ax \left\{ \frac{\arcsin x}{\arccos x} \right\} dx = \frac{\pi}{a} \left[\frac{\operatorname{ch} a}{e^{-a}} - I_0(a) \right] \quad [a \neq 0].$$

$$10. \int_{-1}^1 \operatorname{ch} ax \arccos x dx = \frac{\pi}{a} \operatorname{sh} a \quad [a \neq 0].$$

$$11. \int_{-1}^1 \left\{ \frac{\sin(n\pi x)}{\cos(n\pi x)} \right\} \arccos x dx = \frac{1}{n} \left\{ \frac{(-1)^n - J_0(n\pi)}{0} \right\} \quad [n = 1, 2, 3, \dots].$$

2.7.4. Интегралы от $A(x)$, $\left\{ \frac{\operatorname{arctg} x}{\operatorname{arccotg} x} \right\}^n$.

$$1. \int_0^1 \left\{ \frac{\operatorname{arctg}(e^{i\varphi} x)}{\operatorname{arccotg}(e^{i\varphi} x)} \right\} dx = \frac{\pi}{4} - \frac{\cos \varphi}{2} \ln(2 \cos \varphi) - \frac{\varphi}{2} \sin \varphi \pm \\ \pm \frac{1}{2} \left[\frac{1}{2} \ln \frac{1 + \sin \varphi}{1 - \sin \varphi} + \sin \varphi \ln(2 \cos \varphi) - \varphi \cos \varphi \right] \quad [|\varphi| < \pi/2].$$

$$2. \int_0^1 \operatorname{arctg}^2 x dx = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 - G.$$

$$3. \int_0^a x^{\alpha-1} \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccotg}(x/a)} \right\} dx = \frac{a^\alpha}{4\alpha} \left[\pi + 2\beta \left(\frac{\alpha+1}{2} \right) \right] \quad \left[a > 0; \operatorname{Re} \alpha > -\frac{1}{0} \right].$$

$$4. \int_0^a \frac{1}{x} \operatorname{arctg}(x/a) dx = G.$$

$$5. \int_0^a x^n \left\{ \frac{\operatorname{arctg}(x/a)}{\operatorname{arccotg}(x/a)} \right\} dx = \\ = \frac{a^{n+1}}{n+1} \left\{ \frac{\pi}{4} - (-1)^{[n+1]/2} \left[\frac{\pi(1-(-1)^n)}{8} + \frac{1+(-1)^n}{4} \ln 2 \right] + \sum_{k=0}^{[n-1]/2} \frac{(-1)^k}{n-2k} \right\} \quad [a > 0].$$

$$6. \int_0^{\infty} \operatorname{arccotg}^n x dx = I_n \quad [n = 2, 3, 4, \dots].$$

$$I_n = n \left(\frac{\pi}{2} \right)^{n-1} \left[\frac{1}{n-1} + \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(n+2k-1)(2k)!} \right], \quad I_2 = \pi \ln 2.$$

$$7. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\operatorname{arctg} x}{\operatorname{arccotg} x} \right\} dx = \frac{\pi}{2\alpha \cos(\alpha\pi/2)} \quad [0 < \operatorname{Re} \alpha < 1].$$

$$8. \int_0^a \frac{\operatorname{arctg} bx}{x \sqrt{a^2 - x^2}} dx = \frac{\pi}{2a} \ln(ab + \sqrt{1 + a^2 b^2}) \quad [a, \operatorname{Re} b > 0].$$

$$9. \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \left\{ \frac{\operatorname{arctg} bx}{\operatorname{arccotg} bx} \right\} dx = \frac{\pi}{4b} (ab \pm 2\sqrt{1 + a^2 b^2} \mp ab \mp 2) \quad [a, \operatorname{Re} b > 0].$$

$$10. \int_0^a \frac{x \operatorname{arctg} bx}{\sqrt{a^2 - x^2} (x^2 - y^2)} dx = \frac{\pi}{2 \sqrt{y^2 - a^2}} \left[\operatorname{arccotg} by - \operatorname{arctg} \sqrt{\frac{1 + a^2 b^2}{b^2 (y^2 - a^2)}} \right] \quad [0 < a < y; \operatorname{Re} b > 0].$$

$$11. \int_0^a \frac{x \operatorname{arctg} x}{\sqrt{c^2 - x^2} (x^2 - y^2)} dx = I(a, c, y) \quad [0 < a \leq c < y]$$

$$I\left(\frac{1}{\sqrt{3}}, 1, \sqrt{3}\right) = -\frac{\sqrt{3}}{576} \pi^2, \quad I\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 1\right) = -\frac{\sqrt{3}}{96} \pi^2,$$

$$I\left(\sqrt{\frac{5}{3}}, \sqrt{2}, \sqrt{3}\right) = -\frac{\pi^2}{30}, \quad I\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right) = \sqrt{10} \left(\frac{2\pi}{15} \operatorname{arctg} \sqrt{\frac{5}{3}} - \frac{\pi^2}{25} \right),$$

$$I\left(\frac{1}{\sqrt{3}}, \sqrt{2}, \sqrt{3}\right) = \frac{\pi}{6} \operatorname{arctg} \sqrt{\frac{5}{3}} - \frac{\pi^2}{20}, \quad I(\sqrt{2}, \sqrt{2}, \sqrt{3}) = -\frac{\pi^2}{12},$$

$$I(1, \sqrt{2}, \sqrt{3}) = -\frac{\pi^2}{96}.$$

$$12. \int_0^\infty \frac{\operatorname{arctg}^n bx}{x^2 + z^2} dx = I_n \quad [b, \operatorname{Re} z > 0],$$

$$I_1 = \frac{1}{2} \left[\frac{\pi^2}{6} - \frac{1}{4} \ln^2(1+bz) - \frac{1}{2} \operatorname{Li}_2\left(\frac{1}{1+bz}\right) - \frac{1}{2} \operatorname{Li}_2(1-bz) \right],$$

$$I_2 = \frac{\pi}{4z} \left[\frac{\pi^2}{6} - \ln^2 \frac{1+bz}{2} - 2 \operatorname{Li}_2\left(\frac{1-bz}{2}\right) \right].$$

$$13. \int_0^\infty \frac{\operatorname{arctg} bx}{x(x+z)} dx = \frac{\pi}{2} [\operatorname{arctg}(\sqrt{2bz+1}) + \operatorname{arctg}(\sqrt{2bz-1}) - \operatorname{arctg} bz]$$

$$[b > 0; |\arg z| < \pi].$$

$$14. \int_0^\infty \frac{\operatorname{arctg} bx}{x(x^2+z^2)} dx = \frac{\pi}{2z^2} \ln(1+bz) \quad [b, \operatorname{Re} z > 0].$$

$$15. \int_0^\infty \frac{\operatorname{arctg} ax - \operatorname{arctg} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b} \quad [a, b > 0].$$

$$16. \int_0^\infty \operatorname{arctg} ax \operatorname{arctg} bx dx = \frac{\pi}{2} \ln \frac{(a+b)^{1/a+1/b}}{a^{1/b} b^{1/a}} \quad [a, b > 0].$$

$$17. \int_{-\infty}^\infty \frac{\operatorname{arctg} ax - \operatorname{arctg} bx}{x} \operatorname{arctg} cx dx = \frac{1}{2} \pi^2 \ln \frac{a}{b} \quad [a, b, c > 0].$$

$$18. \int_0^\infty \frac{\arcsin ax \operatorname{arctg} bx}{x^2} dx = \frac{b\pi}{2} \ln \frac{a + \sqrt{a^2 + b^2}}{a \sqrt{a^2 + b^2}} +$$

$$+ \frac{\pi a}{2} \ln \frac{b + \sqrt{a^2 + b^2}}{a} - \frac{\pi a}{2} \operatorname{arctg} \frac{b}{a} \quad [a, b > 0].$$

$$19. \int_0^\infty \frac{\operatorname{arctg} ax \operatorname{arctg} bx}{x^2} dx = \frac{\pi}{2} \ln \frac{(a+b)e^{ab}}{a^2 b^2} \quad [a, b > 0].$$

$$20. \int_0^\infty \frac{1}{x^2} \left(1 - \frac{1}{x} \operatorname{arctg} x \right) dx = \frac{\pi}{4}.$$

2.7.5. Интегралы от $A(x)$, e^{-px} , $\left\{ \frac{\operatorname{arctg} bx}{\operatorname{arctg} bx} \right\}$.

$$1. \int_0^\infty e^{-px} \left\{ \frac{\operatorname{arctg} bx}{\operatorname{arctg} bx} \right\} dx = \frac{\pi}{2p} \left\{ \frac{0}{1} \right\} \pm \frac{1}{p} \left[\operatorname{ci}\left(\frac{p}{b}\right) \sin \frac{p}{b} - \operatorname{si}\left(\frac{p}{b}\right) \cos \frac{p}{b} \right]$$

$$[b, \operatorname{Re} p > 0].$$

$$2. \int_0^\infty x^{2-\alpha} e^{-px} \left\{ \frac{\operatorname{arctg} bx}{\operatorname{arctg} bx} \right\} dx =$$

$$= \frac{\pi}{2p^\alpha} \Gamma(\alpha) \left\{ \frac{1}{0} \right\} \mp \frac{p^{1-\alpha}}{b} \Gamma(\alpha-1) {}_2F_2\left(\frac{1}{2}, 1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{p^2}{4b^2}\right) \mp$$

$$\mp \frac{p^{1-\alpha}}{2(\alpha+1) \sin(\alpha\pi/2)} {}_1F_2\left(\frac{1+\alpha}{2}; \frac{3}{2}, \frac{3+\alpha}{2}; -\frac{p^2}{4b^2}\right) \mp$$

$$\mp \frac{p^{1-\alpha}}{2\alpha \cos(\alpha\pi/2)} {}_1F_2\left(\frac{\alpha}{2}; \frac{1}{2}, 1+\frac{\alpha}{2}; -\frac{p^2}{4b^2}\right) \quad [b, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -1].$$

$$3. \int_0^\infty x^{-p} e^{-px} \left\{ \frac{\operatorname{arctg} bx}{\operatorname{arctg} bx} \right\} dx = \frac{1}{p^2} \left[\left\{ \frac{0}{\pi/2} \right\} + \operatorname{ci}\left(\frac{p}{b}\right) \sin \frac{p}{b} \mp \operatorname{si}\left(\frac{p}{b}\right) \cos \frac{p}{b} \right] \mp$$

$$\mp \frac{1}{pb} \left[\operatorname{si}\left(\frac{p}{b}\right) \sin \frac{p}{b} - \operatorname{ci}\left(\frac{p}{b}\right) \cos \frac{p}{b} \right] \quad [b, \operatorname{Re} p > 0].$$

$$4. \int_0^\infty x^{2-1} e^{-px^2} \operatorname{arctg} bx dx =$$

$$= \frac{\pi p^{-\alpha/2}}{4} \Gamma\left(\frac{\alpha}{2}\right) - \frac{\pi b^{-\alpha}}{2\alpha \cos(\alpha\pi/2)} {}_1F_1\left(\frac{\alpha}{2}; 1+\frac{\alpha}{2}; -\frac{p}{b^2}\right) -$$

$$- \frac{p^{1-\alpha/2}}{2b} \Gamma\left(\frac{\alpha-1}{2}\right) {}_2F_2\left(\frac{1}{2}, 1; \frac{3}{2}, \frac{3-\alpha}{2}; -\frac{p}{b^2}\right) \quad [a, \operatorname{Re} p > 0; \operatorname{Re} \alpha > -1].$$

$$5. \int_0^\infty \frac{1}{(x^2+1)^2} \left\{ \frac{x^2}{x^2-x} \right\} e^{-x^2} \operatorname{arctg} x dx = \frac{\sqrt{\pi}}{8} \left\{ \frac{4}{3} \right\}.$$

$$6. \int_0^\infty \frac{\operatorname{arctg} bx}{e^x - 1} dx = \frac{\pi}{c} \left[\ln \Gamma\left(\frac{c}{2\pi b}\right) - \frac{e-\pi b}{2\pi b} \ln \frac{c}{2\pi b} + \frac{c}{2\pi b} - \frac{1}{2} \ln(2\pi) \right]$$

$$[b, \operatorname{Re} c > 0].$$

$$7. \int_0^\infty \frac{\operatorname{arctg} x^3}{e^{2\pi x} - 1} dx = \frac{1}{4} \ln(2\pi) - \frac{\pi}{4\sqrt{3}} - \frac{1}{2} \ln(1+e^{-2\pi\sqrt{3}}).$$

$$8. \int_0^\infty \frac{\operatorname{arctg} x^3}{e^{2\pi x} - 1} dx = \frac{1}{8} \ln(12\pi) - \frac{\pi}{4\sqrt{3}} + \frac{1}{4} \ln(1+e^{-2\pi\sqrt{3}}).$$

$$9. \int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arctg} \frac{1}{x} - e^{-bx} \right) \frac{dx}{x} = C + \ln b$$

[b > 0].

$$10. \int_0^{\infty} \frac{\operatorname{arctg} bx}{\operatorname{sh} cx} dx = \frac{\pi}{c} \left[\ln \Gamma \left(\frac{c}{2\pi b} \right) - \ln \Gamma \left(\frac{c + \pi b}{2\pi b} \right) - \ln \left(2 \sqrt{\frac{b\pi}{c}} \right) \right]$$

[b, Re c > 0].

2.7.6. Интегралы от $A(x)$, $\left\{ \frac{\sin ax}{\cos ax} \right\}$, $\left\{ \frac{\operatorname{arctg} bx}{\operatorname{arctg} bx} \right\}$.

Обозначение: $\delta = \frac{1}{10}$.

$$1. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\} \operatorname{arctg} ax dx =$$

$$= \frac{\pi b^{\delta} a^{-\alpha-\delta} \{\sin(\alpha\pi/2)\}^{-1}}{2(\alpha+\delta) \{\cos(\alpha\pi/2)\}} {}_2F_2 \left(\frac{\alpha+\delta}{2}; 1 + \frac{\alpha+\delta}{2}, \delta + \frac{1}{2}; \frac{b^2}{4a^2} \right) \pm$$

$$\pm \frac{b^{1-\alpha}}{a \{\sin(\alpha\pi/2)\}} \Gamma(\alpha-1) {}_2F_2 \left(\frac{1}{2}, 1; \frac{3}{2}, \frac{3-\alpha}{2}; 1 - \frac{\alpha}{2}; \frac{b^2}{4a^2} \right) +$$

$$+ \frac{b^{\alpha\pi}}{2 \Gamma(\alpha)} \frac{\{\sin(\alpha\pi/2)\}}{\{\cos(\alpha\pi/2)\}} \quad [-\delta < \operatorname{Re} \alpha < 1; a, b > 0].$$

$$2. \int_0^{\infty} \frac{\cos bx}{x} \operatorname{arctg} ax dx = -\frac{\pi}{2} \operatorname{Ei} \left(-\frac{b}{a} \right) \quad [a, b > 0].$$

$$3. \int_0^{\infty} \sin bx \operatorname{arctg} ax dx = \frac{\pi}{2b} (1 - e^{-b/a}) \quad [a, b > 0].$$

$$4. \int_0^{\infty} \cos bx \operatorname{arctg} ax dx = \frac{1}{2b} \left[e^{-b/a} \operatorname{Ei} \left(\frac{b}{a} \right) - e^{b/a} \operatorname{Ei} \left(-\frac{b}{a} \right) \right] \quad [a, b > 0].$$

$$5. \int_0^{\infty} x^{\alpha-1} \left\{ \frac{\sin bx}{\cos bx} \right\} \operatorname{arctg} \frac{a}{x} dx =$$

$$= \frac{b^{\delta} a^{\alpha-\delta} \pi \{\sin(\alpha\pi/2)\}^{-1}}{2(\alpha+\delta) \{\cos(\alpha\pi/2)\}} {}_2F_2 \left(\frac{\alpha+\delta}{2}; \delta + \frac{1}{2}, 1 + \frac{\alpha+\delta}{2}; \frac{a^2 b^2}{4} \right) \mp$$

$$\mp a b^{1-\alpha} \Gamma(\alpha-1) \frac{\{\cos(\alpha\pi/2)\}}{\{\sin(\alpha\pi/2)\}} {}_2F_2 \left(1, \frac{1}{2}; \frac{3}{2}, 1 - \frac{\alpha}{2}; \frac{3-\alpha}{2}; \frac{a^2 b^2}{4} \right)$$

$$[-\delta < \operatorname{Re} \alpha < 2; a, b > 0].$$

$$6. \int_0^{\infty} \cos bx \operatorname{arctg} \frac{a}{x} dx = \frac{\pi}{b} e^{-b\sqrt{a/2}} \sin \left(b \sqrt{\frac{a}{2}} \right) \quad [a, b > 0].$$

$$7. \int_0^{\infty} \frac{\sin cx}{\cos cx + d} \operatorname{arctg} \frac{a}{x} dx = \frac{\pi}{c} \ln \frac{d + 1 + \sqrt{d^2 - 1}}{e^{-ac} + d + 1} \frac{d^2 - 1}{d^2 - 1} \quad [a, c > 0; d > 1].$$

$$8. \int_0^{\infty} \frac{\operatorname{tg} x}{b^2 \sin^2 x + c^2 \cos^2 x} \operatorname{arctg} \frac{a}{x} dx = \frac{\pi}{2b^2} \ln \left(1 + \frac{b}{c} \operatorname{th} a \right) \quad [a, b, c > 0].$$

$$9. \int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arctg} \frac{1}{x} - \cos bx \right) \frac{dx}{x} = C + \ln b \quad [b > 0].$$

2.7.7. Интегралы, содержащие $\operatorname{arctg} ax$ и логарифмическую функцию.

$$1. \int_0^1 \ln x \operatorname{arctg} x dx = \frac{\ln 2}{2} + \frac{\pi^2}{48} - \frac{\pi}{4}.$$

$$2. \int_0^1 \frac{1}{x} \ln^{v-1} x \operatorname{arctg} bx dx = \Gamma(v) b \sum_{k=0}^{\infty} \frac{(-b^2)^k}{(2k+1)^{v+1}} \quad [b| < 1; \operatorname{Re} v > 0].$$

$$3. \int_0^1 \ln^{n-1} x (\ln x + n) \operatorname{arctg} x dx = (-2)^{n-1} n! (2^n - 1) \zeta(n+1).$$

$$4. \int_0^{\infty} \frac{1}{x^2} \ln(1 + a^2 x^2) \operatorname{arctg} bx dx = \frac{\pi}{2} [ab + (a^2 - b^2) \ln(a+b) + b^2 \ln b - a^2 \ln a] \quad [a, b > 0].$$

$$5. \int_0^{\infty} \frac{\ln^{v-1} (x + \sqrt{x^2 + 1})}{\sqrt{x^2 + 1}} \operatorname{arctg} x dx =$$

$$= -\frac{\pi (2\pi)^v}{v \cos(v\pi/2)} \{ \zeta(-v, 3/4) - \zeta(-v, 1/4) \} \quad [-1 < \operatorname{Re} v < 0].$$

2.7.8. Интегралы, содержащие $\left\{ \frac{\operatorname{arctg} f(x)}{\operatorname{arctg} g(x)} \right\}$.

$$1. \int_0^{\infty} \sin bx \operatorname{arctg} \frac{2ax}{x^2 + c^2} dx = \frac{\pi}{b} e^{-b\sqrt{a^2 + c^2}} \operatorname{sh} ab \quad [a, b, c > 0].$$

$$2. \int_0^{\infty} x^{\alpha-1} \operatorname{arctg} (ae^{-x}) dx = a^{-\alpha-1} \Gamma(\alpha) \Phi(-a^2, \alpha+1, 1/2) \quad [a, \operatorname{Re} \alpha > 0].$$

$$3. \int_0^{\infty} \frac{x}{x^2 + z^2} \operatorname{arctg} \frac{a \sin cx}{1 + a \cos cx} dx = \frac{\pi}{2} \ln(1 + ae^{-cz}) \quad [a > -e^{cz}].$$

$$4. \int_0^1 \left(\frac{1-x^2}{1-x^2-x^2} \right)^{1/2} \operatorname{arctg} (\operatorname{tg} \lambda \sqrt{1-k^2 x^2}) dx =$$

$$= \frac{\pi}{2k} [E(\lambda, k) - \operatorname{ctg} \lambda (1 - \sqrt{1-k^2 \sin^2 \lambda})] - \frac{\pi(1-k^2)}{2k} F(\lambda, k) \frac{\{1\}}{\{0\}} \quad [0 < k < 1].$$

$$5. \int_0^{\pi} \left\{ \frac{\sin nx \cos x}{\cos nx \sin x} \right\} \operatorname{arctg} \frac{a \sin x}{1 - a \cos x} dx = \frac{\pi}{4} \left(\frac{a^{n+1}}{n+1} \pm \frac{a^{n-1}}{n-1} \right) \quad [0 < a < 1]$$

$$6. \int_0^{\pi} \frac{\cos^n x}{\sin x} \operatorname{arctg} \frac{a \sin x}{1 - a \cos x} dx = -\frac{\pi}{2} \ln [(1-a)(1+a)^{2n-1}] \quad [n=0, 1; a > 0]$$

$$7. \int_0^{\pi/2} \left\{ \frac{\operatorname{arctg}(a \sqrt{1-k^2 \sin^2 x})}{\operatorname{arctg}(a \sqrt{1-k^2 \sin^2 x})} \right\} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} = \frac{\pi}{2} \left\{ F(\operatorname{arctg} a, k) \right. \\ \left. - F(\operatorname{arctg}(a \sqrt{1-k^2}), k) \right\} \quad [0 < k < 1]$$

2.7.9. Интегралы, содержащие $\left\{ \frac{\operatorname{arctg} x}{\operatorname{arctg} x} \right\}$.

$$1. \int_0^{\infty} x^{\nu} (x^2+1)^{\nu/2} e^{-\rho x} \left\{ \frac{\sin(\nu \operatorname{arctg} x + c)}{\cos(\nu \operatorname{arctg} x + c)} \right\} dx = \\ = -\frac{\sqrt{\pi}}{2} \rho^{-\nu-1/2} \Gamma(\nu+1) \left[\left\{ \frac{\sin(\rho/2+c)}{\cos(\rho/2+c)} \right\} V_{\nu+1/2} \left(\frac{\rho}{2} \right) \pm \left\{ \frac{\cos(\rho/2+c)}{\sin(\rho/2+c)} \right\} J_{\nu+1/2} \left(\frac{\rho}{2} \right) \right] \\ [Re \rho > 0; Re \nu > -1]$$

$$2. \int_0^{\infty} \frac{\sin(\nu \operatorname{arctg} x)}{(x^2+1)^{\nu/2} (e^{2\pi x}+1)} dx = \frac{1}{2(\nu-1)} - \frac{\zeta(\nu)}{2^{\nu}} \quad [Re \nu > 1]$$

$$3. \int_0^{\infty} \frac{\sin(\nu \operatorname{arctg} ax)}{(a^2 x^2+1)^{\nu/2} (e^{2\pi x}-1)} dx = \frac{\zeta(\nu, a^{-1})}{2a^{\nu}} - \frac{1}{2a(\nu-1)} - \frac{1}{4} \\ [Re a > 0; Re \nu > 1]$$

$$4. \int_0^{\infty} \frac{(x+1)^{\nu/2}}{\sqrt{x}} e^{-\rho x} \cos bx \cos(\nu \operatorname{arctg} \sqrt{x}) dx = \\ = \sqrt{\frac{\pi}{2}} (2\rho)^{-(\nu+1)/2} \exp\left(-\frac{4\rho^2-b^2}{8\rho}\right) \left[e^{b/2} D_{\nu}\left(\frac{2\rho+a}{\sqrt{2\rho}}\right) + e^{-b/2} D_{\nu}\left(\frac{2\rho-a}{\sqrt{2\rho}}\right) \right] \\ [b, Re \rho > 0]$$

$$5. \int_0^{\infty} (\lambda^2+y^2)^{-1/2} (x^2+z^2)^{-\nu/2} \left\{ \frac{\sin[\mu \operatorname{arctg}(x/y)] \sin[\nu \operatorname{arctg}(x/z)]}{\cos[\mu \operatorname{arctg}(x/y)] \cos[\nu \operatorname{arctg}(x/z)]} \right\} dx = \\ = \frac{\pi}{2(y+z)^{\mu+\nu-1}} \Gamma\left[\frac{\mu+\nu-1}{2}\right] \quad [\mu, z > 0; Re(\mu+\nu) > 1]$$

$$6. \int_0^{\infty} \frac{x^{\alpha-1}}{(x^2+2ax \cos \lambda + a^2)^{\nu}} \left\{ \frac{\sin(2\rho \operatorname{arctg}[a \sin \lambda/(x+a \cos \lambda)])}{\cos(2\rho \operatorname{arctg}[a \sin \lambda/(x+a \cos \lambda)])} \right\} dx = \\ = a^{\alpha-2\nu} B(\alpha, 2\nu-\alpha) \left\{ \frac{\sin[(2\rho-\alpha)\lambda]}{\cos[(2\rho-\alpha)\lambda]} \right\} \quad [0 < Re \alpha < 2 Re \nu; a > 0; 0 \leq \lambda < \pi]$$

2.8. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

Интегралы, содержащие обратные гиперболические функции, сводятся к интегралам, содержащим гиперболические функции, с помощью подстановок вида $y = \operatorname{Arsh} x$, $y = \operatorname{Arch} x$, $y = \operatorname{Arth} x$, $y = \operatorname{Arcth} x$.

Кроме того, они могут быть сведены к интегралам от логарифмической, степенной и других функций, если воспользоваться формулами, приведенными в приложении I. Полученные таким образом интегралы отнесены в соответствующие разделы.

Например, с помощью формулы

$$\operatorname{ch}(\nu \operatorname{Arsh} x) = \frac{1}{2} [(\sqrt{x^2+1} + x)^{\nu} + (\sqrt{x^2+1} - x)^{-\nu}]$$

получаем

$$\int_0^{\infty} x^{\alpha-1} e^{-\rho x} \operatorname{ch}(\nu \operatorname{Arsh} x) dx = \\ = \frac{1}{2} \int_0^{\infty} x^{\alpha-1} e^{-\rho x} (\sqrt{x^2+1} + x)^{\nu} dx + \frac{1}{2} \int_0^{\infty} x^{\alpha-1} e^{-\rho x} (\sqrt{x^2+1} - x)^{-\nu} dx \\ [Re \alpha, Re \rho > 0]$$

Глава 3. КРАТНЫЕ ИНТЕГРАЛЫ

3.1. ДВОЙНЫЕ ИНТЕГРАЛЫ

3.1.1. Введение.

В этом разделе помещены двойные интегралы и указаны некоторые способы их приведения к одномерным интегралам. Кроме того, указаны различные системы координат на плоскости, которые могут быть использованы при вычислении двойных интегралов.

Отметим, что основным способом вычисления двойных интегралов является сведение их к повторным [20]. Например, для вычисления двойного интеграла

$$\iint_G f(x, y) dx dy \text{ его можно свести к одному или нескольким повторным}$$

$$\int_a^b \int_{x_1(y)}^{x_2(y)} f(x, y) dx dy \text{ или } \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y) dy dx$$

$$\text{или}$$

$$\int_a^b \int_{x_1(y)}^{x_2(y)} f(x, y) dx dy \text{ или } \int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y) dy dx$$

при этом сначала вычисляются внутренние интегралы.

В частности, если G — треугольник, ограниченный прямыми $y=0$, $y=x$ и $x=a$ ($a>0$), то

$$\iint_G f(x, y) dx dy = \int_0^a \int_0^x f(x, y) dy dx = \int_0^a \int_y^a f(x, y) dx dy.$$

Кроме того, часто используется метод замены переменных интегрирования [20].

Пусть задано взаимно однозначное отображение $x=x(u, v)$, $y=y(u, v)$ области Γ плоскости переменных u, v на область G плоскости переменных x, y , причем

- x, y как функции u, v непрерывны в Γ вместе с первыми частными производными;
- якобиан

$$\Delta = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v \\ \partial y / \partial u & \partial y / \partial v \end{vmatrix} \neq 0 \text{ в } \Gamma.$$

Тогда справедлива следующая формула:

$$\iint_G f(x, y) dx dy = \iint_{\Gamma} f[x(u, v), y(u, v)] |\Delta| du dv.$$

3.1. ДВОЙНЫЕ ИНТЕГРАЛЫ

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3.1.2.1

Приведем некоторые системы координат на плоскости (dl — элемент длины, dS — элемент площади) [16].

1) Полярные координаты:

$$x = u \cos v, \quad y = u \sin v,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi \quad (\text{обычно используются обозначения } u=r, v=\varphi),$$

$$dl^2 = du^2 + u^2 dv^2, \quad dS = u du dv.$$

2) Обобщенные полярные координаты:

$$x = au \cos v, \quad y = bu \sin v,$$

$$a, b > 0, \quad 0 \leq u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = (a^2 \cos^2 v + b^2 \sin^2 v) du^2 + (b^2 - a^2) u \sin 2v du dv + (a^2 \sin^2 v + b^2 \cos^2 v) u^2 dv^2,$$

$$dS = abu du dv.$$

3) Эллиптические координаты ($x, y > 0$):

$$x = \left[\frac{(u+a^2)(v+a^2)}{a^2-b^2} \right]^{1/2}, \quad y = \left[\frac{(u+b^2)(v+b^2)}{b^2-a^2} \right]^{1/2},$$

$$0 \leq b^2 < a^2, \quad -b^2 < u < \infty, \quad -a^2 < v < -b^2,$$

$$dl^2 = \frac{1}{4} \frac{u-v}{(u+a^2)(u+b^2)} du^2 - \frac{u-v}{(v+a^2)(v+b^2)} dv^2,$$

$$dS = \frac{u-v}{4(u+a^2)(u+b^2)(v+a^2)(v+b^2)} du dv.$$

4) Вырожденные эллиптические координаты:

$$x = \operatorname{ch} u \cos v, \quad y = \operatorname{sh} u \sin v,$$

$$-\infty < u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = (\operatorname{ch}^2 u - \cos^2 v) (du^2 + dv^2), \quad dS = (\operatorname{ch}^2 u - \cos^2 v) du dv.$$

5) Параболические координаты:

$$x = u^2 - v^2, \quad y = 2uv,$$

$$-\infty < u < \infty, \quad 0 \leq v < \infty,$$

$$dl^2 = 4(u^2 + v^2) (du^2 + dv^2), \quad dS = 4(u^2 + v^2) du dv.$$

6) Биполярные координаты:

$$x = \frac{\operatorname{ch} u}{\operatorname{ch} u + \cos v}, \quad y = \frac{\sin v}{\operatorname{ch} u + \cos v},$$

$$-\infty < u < \infty, \quad 0 \leq v < 2\pi,$$

$$dl^2 = \frac{du^2 + dv^2}{(\operatorname{ch} u + \cos v)^2}, \quad dS = \frac{1}{(\operatorname{ch} u + \cos v)^2} du dv.$$

3.1.2. Общие формулы; интегралы от алгебраических функций.

В этом пункте предполагается, что все рассматриваемые интегралы существуют и $a, b > 0$.

$$1. \quad \iint_{\Gamma} x^{\alpha-1} y^{\beta-1} \left[\left(\frac{x}{a} \right)^p + \left(\frac{y}{b} \right)^q \right] dx dy =$$

$$\left(\frac{a}{\alpha} \right)^p \left(\frac{b}{\beta} \right)^q \leq 1$$

$$= \frac{a^{\alpha} b^{\beta}}{pq} \Gamma \left[\frac{\alpha}{p} + \frac{\beta}{q} \right] \int_0^1 f(t) t^{\alpha/p + \beta/q - 1} dt \quad [\alpha, \beta, p, q > 0].$$

$$2. \iint_{x^2+y^2 \leq 1} f(\alpha x + \beta y + \gamma) dx dy = 2 \int_{-1}^1 \sqrt{1-t^2} f(t \sqrt{\alpha^2 + \beta^2} + \gamma) dt$$

[$\alpha^2 + \beta^2 \neq 0$; α, β, γ — действительные числа]

$$3. \iint_{|x|+|y| \leq 1} f(x+y) dx dy = \int_{-1}^1 f(t) dt.$$

$$4. \int_0^1 \int_0^{1-x} f(xy) (1-x)^{\alpha-1} y^{\alpha} (1-y)^{\beta-1} dx dy =$$

$$= B(\alpha, \beta) \int_0^1 f(t) (1-t)^{\alpha+\beta-1} dt \quad [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0].$$

$$5. \int_0^1 \int_0^{1-x} x^{\alpha-1} y^{\beta-1} (1-x)^{\gamma-\alpha-1} (1-y)^{\gamma-\beta-1} (1-xy)^{-\gamma} dx dy =$$

$$= \Gamma \left[\begin{matrix} \alpha, \beta, \gamma-\alpha, \gamma-\beta \\ \gamma, \gamma \end{matrix} \right] {}_2F_1(\alpha, \beta; \gamma; z)$$

[$\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$; $\operatorname{Re} \gamma > \operatorname{Re} \beta > 0$; $|\arg(1-z)| < \pi$].

$$6. \iint_{\substack{x \geq 0, y \geq 0 \\ x+y \leq 1}} x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta-\beta' \\ \gamma \end{matrix} \right] F_3(\alpha, \beta, \beta', \gamma; u, v) \quad [\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta-\beta') > 0].$$

$$7. \iint_{\substack{x \geq 0, y \geq 0 \\ x+y \leq 1}} x^{\beta-1} y^{\beta'-1} (1-x-y)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta-\beta' \\ \gamma \end{matrix} \right] F_3(\alpha, \alpha', \beta, \beta', \gamma; u, v)$$

[$\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta-\beta') > 0$].

$$8. \int_0^1 \int_0^{1-x} x^{\beta-1} y^{\beta'-1} (1-x)^{\gamma-\beta-1} (1-y)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} dx dy =$$

$$= \Gamma \left[\begin{matrix} \beta, \beta', \gamma-\beta, \gamma'-\beta' \\ \gamma, \gamma' \end{matrix} \right] F_3(\alpha, \beta, \beta', \gamma, \gamma'; u, v)$$

[$\operatorname{Re} \beta, \operatorname{Re} \beta', \operatorname{Re}(\gamma-\beta), \operatorname{Re}(\gamma'-\beta') > 0$].

$$9. \int_0^1 \int_0^{1-x} \frac{x^{\alpha-1} y^{\beta-1} (1-x)^{\gamma-\alpha-1} (1-y)^{\gamma'-\beta-1} dx dy}{(1-ux)^{\gamma+\gamma'-\alpha-1} (1-vy)^{\gamma+\gamma'-\beta-1} (1-ux-vy)^{\alpha+\beta-\gamma-\gamma'+1}} =$$

$$= \Gamma \left[\begin{matrix} \alpha, \beta, \gamma-\alpha, \gamma'-\beta \\ \gamma, \gamma' \end{matrix} \right] F_3(\alpha, \beta, \gamma, \gamma'; u(1-v), v(1-u))$$

[$\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re}(\gamma-\alpha), \operatorname{Re}(\gamma'-\beta) > 0$].

$$10. \iint_{\substack{x \geq 0, y \geq 0 \\ \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q \leq 1}} x^{\alpha-1} y^{\beta-1} dx dy = \frac{a^{\alpha} b^{\beta}}{\alpha q + \beta p} \Gamma \left[\begin{matrix} \alpha/p, \beta/q \\ \alpha/p + \beta/q \end{matrix} \right]$$

[$p, q, \operatorname{Re} \alpha, \operatorname{Re} \beta > 0$].

$$11. \iint_{\substack{x \geq 0, y \geq 0 \\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1}} \frac{xy}{\sqrt{x^2+y^2}} dx dy = \frac{ab}{3(a+b)}.$$

$$12. \iint_{x^2+y^2 \leq r^2} \frac{dx dy}{(a^2+x^2+y^2)^a} = \begin{cases} \frac{\pi}{1-a} [(a^2+r^2)^{1-a} - a^{2(1-a)}] & [a \neq 1], \\ \pi \ln(1+r^2) & [a = 1] \end{cases}$$

3.1.3. Интегралы, содержащие показательную функцию.
Условие: $a, b, p, q > 0$.

$$1. \int_0^{\infty} \int_0^{\infty} f(ax+by) e^{-px-qy} dx dy =$$

$$= -\frac{1}{pq(b\rho-aq)} \left[aq \int_0^{\infty} f(x) e^{-px/a} dx - bp \int_0^{\infty} f(y) e^{-qy/b} dy \right].$$

$$2. \int_0^{\infty} \int_0^{\infty} f(|x-y|) e^{-px-qy} dx dy = \frac{1}{pq(p+q)} \left[p \int_0^{\infty} e^{-qy} f(y) dy + q \int_0^{\infty} e^{-px} f(x) dx \right].$$

$$3. \int_0^{\infty} \int_0^{\infty} f(xy) e^{-px-qy} dx dy = 2 \int_0^{\infty} K_0(2\sqrt{pq}t) f(t) dt.$$

$$4. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{x+y}} f\left(\frac{xy}{x+y}\right) e^{-px-qy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq(p+q)}} \int_0^{\infty} e^{-(\sqrt{p}+\sqrt{q})^2 t} f(t) dt.$$

$$5. \int_0^{\infty} \int_0^{\infty} x^{\mu} y^{\nu} (x+y)^{\lambda} e^{-px-qy} dx dy =$$

$$= \frac{\Gamma(\mu+1) \Gamma(\nu+1) \Gamma(\mu+\nu+\lambda+2)}{\Gamma(\mu+\nu+2) p^{\mu+1} q^{\nu+1}} {}_2F_1\left(-\lambda, \mu+1; \mu+\nu+2; \frac{p-q}{p}\right)$$

[$\operatorname{Re} \mu, \operatorname{Re} \nu, \operatorname{Re}(\mu+\nu+\lambda+2) > 0$].

$$6. \int_0^{\infty} \int_0^{\infty} x^{\mu-1} y^{\nu-1} e^{-px-qy} dx dy = \frac{\Gamma(\mu) \Gamma(\nu)}{p^{\mu} q^{\nu}} \quad [\operatorname{Re} \mu, \operatorname{Re} \nu > 0].$$

$$7. \int_0^{\infty} \int_0^{\infty} (x+y)^{\nu-1} e^{-px-qy} dx dy = \Gamma(\nu) \frac{p^{\nu}-q^{\nu}}{(pq)^{\nu}(p-q)} \quad [\operatorname{Re} \nu > 0].$$

$$8. \int_0^{\infty} \int_0^{\infty} \frac{1}{x+y} e^{-px-qy} dx dy = \frac{1}{p-q} \ln \frac{q}{p} \quad [p \neq q].$$

$$9. \int_0^{\infty} \int_0^{\infty} (xy)^{\nu/2} (x+y)^{-(\nu+1)/2} e^{-px-qy} dx dy = \Gamma\left(\frac{\nu}{2}+1\right) \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})^{\nu+1}}.$$

$$10. \int_0^{\infty} \int_0^{\infty} \sqrt{x+y} e^{-px-qy} dx dy = \frac{\sqrt{\pi}}{2(pq)^{3/4}} (\sqrt{p}+\sqrt{q}).$$

$$11. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})}.$$

$$12. \int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{x+y} e^{-px-xy} dx dy = \frac{\pi}{2\sqrt{pq}(\sqrt{p}+\sqrt{q})^3}.$$

$$13. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy}(x+y)} e^{-px-xy} dx dy = 2\sqrt{\frac{\pi}{p}} K\left(\sqrt{1-\frac{q}{p}}\right).$$

$$14. \int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y(x+y)}} e^{-px-xy} dx dy = \\ = \frac{\sqrt{\pi}}{\sqrt{p(p-q)}} \left[K\left(\sqrt{1-\frac{q}{p}}\right) - E\left(\sqrt{1-\frac{q}{p}}\right) \right].$$

$$15. \int_0^\infty \int_0^\infty \frac{\sqrt{x+y}}{\sqrt{xy}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{q\sqrt{p}} E\left(\sqrt{1-\frac{q}{p}}\right).$$

$$16. \int_0^\infty \int_0^\infty \frac{\sqrt{y}}{\sqrt{x(x+y)^3}} e^{-px-xy} dx dy = \frac{2\sqrt{\pi p}}{p-q} \left[K\left(\sqrt{1-\frac{q}{p}}\right) - E\left(\sqrt{1-\frac{q}{p}}\right) \right].$$

$$17. \int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{\sqrt{(x+y)^3}} e^{-px-xy} dx dy = \\ = \frac{\sqrt{\pi}(p+q)}{\sqrt{p(p-q)^3}} K\left(\sqrt{1-\frac{q}{p}}\right) - \frac{2\sqrt{\pi p}}{(p-q)^2} E\left(\sqrt{1-\frac{q}{p}}\right).$$

$$18. \int_0^\infty \int_0^\infty \frac{1}{[x^2y(x+y)]^{1/4}} e^{-px-xy} dx dy = \frac{2\sqrt{2}\Gamma(3/4)}{p^{1/4}q^{1/4}} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$$

$$19. \int_0^\infty \int_0^\infty \frac{1}{[xy^3(x+y)^2]^{1/4}} e^{-px-xy} dx dy = \frac{2\sqrt{2}\Gamma(1/4)}{p^{1/4}} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$$

$$20. \int_0^\infty \int_0^\infty \left[\frac{y}{x(x+y)^3}\right]^{1/4} e^{-px-xy} dx dy = \\ = \frac{4\sqrt{2}\Gamma(5/4)p^{1/4}}{\sqrt{q}(p-q)} E\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right] - \\ - \frac{2\sqrt{2}\Gamma(5/4)p^{1/4}}{\sqrt{q}(\sqrt{p}-\sqrt{q})} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$$

$$21. \int_0^\infty \int_0^\infty \left[\frac{x}{y(x+y)^3}\right]^{1/4} e^{-px-xy} dx dy = \\ = \frac{4\sqrt{2}\Gamma(3/4)p^{1/4}}{p-q} E\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right] - \\ - \frac{2\sqrt{2}\Gamma(3/4)p^{1/4}}{\sqrt{q}(\sqrt{p}-\sqrt{q})} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$$

$$22. \int_0^\infty \int_0^\infty \frac{(xy)^{(v-3)/2}}{(x+y)^{v/2} [xy/(x+y)+2a]^{(v+1)/2}} e^{-px-xy} dx dy = \\ = \frac{2^{v/2}\sqrt{\pi}\Gamma\left(\frac{v+1}{2}\right)}{a(v-1)(v-2)} e^{a(\sqrt{p}+\sqrt{q})^2} D_{2-v}\left[2\sqrt{a}(\sqrt{p}+\sqrt{q})\right] \quad (\operatorname{Re} v > -2).$$

$$24. \int_0^\infty \int_0^\infty \frac{xy}{(x+y)^{3/2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})^3}.$$

$$25. \int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y(x+y)}} e^{-px-xy} dx dy = \frac{\pi}{\sqrt{p}(\sqrt{p}+\sqrt{q})}.$$

$$26. \int_0^\infty \int_0^\infty (xy+1)^{-3/2} e^{-px-xy} dx dy = \\ = \frac{8}{\pi} [\cos(2\sqrt{pq}) \operatorname{ci}(2\sqrt{pq}) - \sin(2\sqrt{pq}) \operatorname{si}(2\sqrt{pq})].$$

$$27. \int_0^\infty \int_0^\infty \frac{(xy)^{-1/2}}{4xy+1} e^{-px-xy} dx dy = \frac{\pi^2}{2} [H_0(\sqrt{pq}) - Y_0(\sqrt{pq})].$$

$$28. \int_0^\infty \int_0^\infty \frac{1}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{1}{\sqrt{p^2+q^2}} \ln \frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}.$$

$$29. \int_0^\infty \int_0^\infty (Vx^2+y^2+x)^{1/2} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{2(pq)^{3/2}} \left(\sqrt{2q} + \frac{p^{3/2}}{p+\sqrt{2pq+q}} \right).$$

$$30. \int_0^\infty \int_0^\infty (Vx^2+y^2-x)^{1/2} e^{-px-xy} dx dy = \frac{1}{2} \sqrt{\frac{\pi}{pq}} \left(\frac{\sqrt{p}+\sqrt{2q}}{p+\sqrt{2pq+q}} \right).$$

$$31. \int_0^\infty \int_0^\infty \frac{(Vx^2+y^2+x)^{1/2}}{Vx^2+y^2} e^{-px-xy} dx dy = \frac{\sqrt{\pi}(\sqrt{p}+\sqrt{2q})}{\sqrt{pq}(p+\sqrt{2pq+q})}.$$

$$32. \int_0^\infty \int_0^\infty \frac{(Vx^2+y^2-x)^{1/2}}{Vx^2+y^2} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{q}(p+\sqrt{2pq+q})}.$$

11. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{x+y}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})}.$
12. $\int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{x+y} e^{-px-xy} dx dy = \frac{\pi}{2\sqrt{pq}(\sqrt{p}+\sqrt{q})^2}.$
13. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy(x+y)}} e^{-px-xy} dx dy = 2\sqrt{\frac{\pi}{p}} K\left(\sqrt{1-\frac{q}{p}}\right).$
14. $\int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y(x+y)}} e^{-px-xy} dx dy =$
 $= \frac{\sqrt{\pi}}{\sqrt{p(p-q)}} \left[K\left(\sqrt{1-\frac{q}{p}}\right) - E\left(\sqrt{1-\frac{q}{p}}\right) \right].$
15. $\int_0^\infty \int_0^\infty \frac{\sqrt{x+y}}{\sqrt{xy}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{q\sqrt{p}} E\left(\sqrt{1-\frac{q}{p}}\right).$
16. $\int_0^\infty \int_0^\infty \frac{\sqrt{y}}{\sqrt{x(x+y)^3}} e^{-px-xy} dx dy = \frac{2\sqrt{\pi p}}{p-q} \left[K\left(\sqrt{1-\frac{q}{p}}\right) - E\left(\sqrt{1-\frac{q}{p}}\right) \right].$
17. $\int_0^\infty \int_0^\infty \frac{\sqrt{xy}}{\sqrt{(x+y)^3}} e^{-px-xy} dx dy =$
 $= \frac{\sqrt{\pi}(p+q)}{\sqrt{p(p-q)^3}} K\left(\sqrt{1-\frac{q}{p}}\right) - \frac{2\sqrt{\pi p}}{(p-q)^3} E\left(\sqrt{1-\frac{q}{p}}\right).$
18. $\int_0^\infty \int_0^\infty \frac{1}{[x^2 y (x+y)]^{1/4}} e^{-px-xy} dx dy = \frac{2\sqrt{2}\Gamma(3/4)}{\sqrt{p}^{1/4} q^{1/4}} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$
19. $\int_0^\infty \int_0^\infty \frac{1}{[xy^3 (x+y)^3]^{1/4}} e^{-px-xy} dx dy = \frac{2\sqrt{2}\Gamma(1/4)}{p^{1/4}} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$
20. $\int_0^\infty \int_0^\infty \left[\frac{y}{x(x+y)^3}\right]^{1/4} e^{-px-xy} dx dy =$
 $= \frac{4\sqrt{2}\Gamma(5/4)p^{1/4}}{\sqrt{q}(p-q)} E\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right] -$
 $- \frac{2\sqrt{2}\Gamma(5/4)p^{1/4}}{\sqrt{q}(\sqrt{p}-\sqrt{q})} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$

21. $\int_0^\infty \int_0^\infty \left[\frac{x}{y(x+y)^2}\right]^{1/4} e^{-px-xy} dx dy =$
 $= \frac{4\sqrt{2}\Gamma(3/4)p^{1/4}}{p-q} E\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right] -$
 $- \frac{2\sqrt{2}\Gamma(3/4)p^{1/4}}{\sqrt{q}(\sqrt{p}-\sqrt{q})} K\left[\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{q}{p}}\right)}\right].$
22. $\int_0^\infty \int_0^\infty \frac{(xy)^{(v-1)/2}}{(x+y)^{v/2} [xy/(x+y)+2a]^{(v+1)/2}} e^{-px-xy} dx dy =$
 $= \frac{2^{v/2} \sqrt{\pi} \Gamma\left(\frac{v+1}{2}\right)}{a(v-1)(v-2)} e^{a(\sqrt{p}+\sqrt{q})^2} D_{2-v}\left[2\sqrt{a}(\sqrt{p}+\sqrt{q})\right] \quad [\operatorname{Re} v > -2].$
24. $\int_0^\infty \int_0^\infty \frac{xy}{(x+y)^{3/2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{pq}(\sqrt{p}+\sqrt{q})^2}.$
25. $\int_0^\infty \int_0^\infty \frac{\sqrt{x}}{\sqrt{y(x+y)}} e^{-px-xy} dx dy = \frac{\pi}{\sqrt{p}(\sqrt{p}+\sqrt{q})}.$
26. $\int_0^\infty \int_0^\infty (xy+1)^{-3/2} e^{-px-xy} dx dy =$
 $= \frac{8}{\pi} [\cos(2\sqrt{pq}) \operatorname{ci}(2\sqrt{pq}) - \sin(2\sqrt{pq}) \operatorname{si}(2\sqrt{pq})].$
27. $\int_0^\infty \int_0^\infty \frac{(xy)^{-1/2}}{4xy+1} e^{-px-xy} dx dy = \frac{\pi^2}{2} [\operatorname{H}_0(\sqrt{pq}) - \gamma_0(\sqrt{pq})].$
28. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{1}{\sqrt{p^2+q^2}} \ln \frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}.$
29. $\int_0^\infty \int_0^\infty (Vx^2+y^2+x)^{1/2} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{2(pq)^{3/2}} \left(\sqrt{2q} + \frac{p^{3/2}}{p+\sqrt{2pq+q}} \right).$
30. $\int_0^\infty \int_0^\infty (Vx^2+y^2-x)^{1/2} e^{-px-xy} dx dy = \frac{1}{2} \sqrt{\frac{\pi}{pq^3}} \left(\frac{\sqrt{p}+\sqrt{2q}}{p+\sqrt{2pq+q}} \right).$
31. $\int_0^\infty \int_0^\infty \frac{(Vx^2+y^2+x)^{1/2}}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}(\sqrt{p}+\sqrt{2q})}{\sqrt{pq}(p+\sqrt{2pq+q})}.$
32. $\int_0^\infty \int_0^\infty \frac{(Vx^2+y^2-x)^{1/2}}{\sqrt{x^2+y^2}} e^{-px-xy} dx dy = \frac{\sqrt{\pi}}{\sqrt{q}(p+\sqrt{2pq+q})}.$

$$33. \int_0^{\infty} \int_0^{\infty} \frac{(Vx^2 + 2axy + y^2 - ax - y)^{1/2}}{Vx^2 + 2axy + y^2} e^{-px - qy} dx dy = \frac{V(1-a)\pi}{Vp(p+V)2(a+1)pq+q} \quad [a \geq -1].$$

$$34. \int_0^{\infty} \int_0^{\infty} e^{-px - qy - xy} dx dy = e^{-pq} \operatorname{Ei}(pq).$$

$$35. \int_0^{\infty} \int_0^{\infty} e^{-px - qy - xy} dx dy = -e^{-pq} \operatorname{Ei}(-pq).$$

$$36. \int_0^{\infty} \int_0^{\infty} x^{\nu-1} e^{-px - qy - xy} dx dy = \Gamma(\nu) p^{-\nu/2} q^{\nu/2-1} e^{pq/2} W_{-\nu/2, (1-\nu)/2}(pq).$$

$$37. \int_0^{\infty} \int_0^{\infty} y^{\nu} e^{-px - qy - x^2 y} dx dy = \frac{\pi^{\nu/2}}{\sin \pi \nu} 2^{-\nu-3/2} p^{(\nu+1)/2} q^{-(\nu+1)/4} [H_{-\nu-1/2}(p\sqrt{q}) - Y_{-\nu-1/2}(p\sqrt{q})] \quad [-1/2 < \nu < 0].$$

$$38. \int_0^{\infty} \int_0^{\infty} e^{-px - qy - x^2 y} dx dy = \frac{1}{Vq} [\sin(p\sqrt{q}) \operatorname{ci}(p\sqrt{q}) - \cos(p\sqrt{q}) \operatorname{si}(p\sqrt{q})].$$

$$39. \int_0^{\infty} \int_0^{\infty} ye^{-px - qy - x^2 y} dx dy = -\cos(p\sqrt{q}) \operatorname{ci}(p\sqrt{q}) - \sin(p\sqrt{q}) \operatorname{si}(p\sqrt{q}).$$

$$40. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha x^2 + 2\beta xy + \gamma y^2) e^{-(\alpha x^2 + 2\beta xy + \gamma y^2)} dx dy = \frac{\pi}{2} (c\alpha - 2b\beta + a\gamma) (ac - b^2)^{-3/2} \quad [a > 0; ac > b^2].$$

$$41. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^n e^{-(x^2 - 2axy + y^2)/[2(1-a^2)]} dx dy = 2\pi n! (1-a^2)^{(n+1)/2} p_n \left(\frac{-ia}{V1-a^2} \right) \quad [|a| < 1].$$

$$42. \int_0^{\infty} \int_0^{\infty} e^{-px - qy - (x/y)} dx dy = \frac{1}{p^2} e^{q/p} \operatorname{Ei} \left(-\frac{q}{p} \right) + \frac{1}{pq}.$$

$$43. \int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-px - qy - (x/y)} dx dy = -\frac{1}{p} e^{q/p} \operatorname{Ei} \left(-\frac{q}{p} \right).$$

$$44. \int_0^{\infty} \int_0^{\infty} \frac{1}{Vxy} e^{-px - qy - (x/y)} dx dy = \frac{\pi}{Vpq} e^{q/p} \operatorname{erfc} \left(\sqrt{\frac{q}{p}} \right).$$

$$45. \int_0^{\infty} \int_0^{\infty} \frac{1}{V y} e^{-px - qy - \alpha^2 x^2/(4y)} dx dy = \frac{V\pi}{Vq(p+\alpha Vq)} \quad [|\arg \alpha| < \pi/4].$$

$$46. \int_0^{\infty} \int_0^{\infty} \frac{1}{V xy} e^{-px - qy - x^2/(4y)} dx dy = \frac{\pi}{Vq(p+Vq)^{1/2}}.$$

$$47. \int_0^{\infty} \int_0^{\infty} xy^{-3/2} e^{-px - qy - \alpha^2 x^2/(4y)} dx dy = \frac{2V\pi}{\alpha} \frac{1}{p+\alpha Vq} \quad [|\arg \alpha| \leq \pi/4; \alpha \neq 0].$$

$$48. \int_0^{\infty} \int_0^{\infty} x^{\nu} y^{-\nu-1} e^{-px - qy - x^2/(4y)} dx dy = \frac{2^{\nu+1} \pi q^{\nu/2}}{\sin \pi \nu V p^2 - q} \operatorname{sh} \left(\nu \operatorname{Arch} \frac{p}{Vq} \right) \quad [|\operatorname{Re} \nu| < 1].$$

$$49. \int_0^{\infty} \int_0^{\infty} \frac{1}{x V y} (1 - e^{nx}) e^{-px - qy - x^2/(4y)} dx dy = \sqrt{\frac{\pi}{q}} \ln \frac{p+Vq-a}{p+Vq} \quad [p+Vq > 0].$$

$$50. \int_0^{\infty} \int_0^{\infty} \frac{1}{V xy} e^{-px - qy - xy/(x+y)} dx dy = \frac{\pi (Vp+Vq)}{pq [1 + (Vp+Vq)^2]^{1/2}}.$$

$$51. \int_0^{\infty} \int_0^{\infty} \frac{V xy}{x+y} e^{-px - qy - xy/(x+y)} dx dy = \frac{\pi}{2pq(p+q+2Vpq+1)^{3/2}}.$$

$$52. \int_0^{\infty} \int_0^{\infty} \frac{V xy}{x+y} e^{-px - qy - xy/(x+y)} dx dy = \frac{\pi (Vp+Vq)}{2Vpq [1 + (Vp+Vq)^2]^{3/2}}.$$

$$53. \int_0^{\infty} \int_0^{\infty} \frac{(xy)^{2\nu-1/2}}{(x+y)^{\nu+1}} e^{-px - qy - \alpha^2 xy/(x+y)} dx dy = \frac{\pi \Gamma(2\nu)}{2^{\nu+1} \alpha^{\nu} pq [(Vp+Vq)^2 + \alpha^2]^{-\nu/2}} p^{-\nu-1} \left[\frac{Vp+Vq}{(Vp+Vq)^2 + \alpha^2} \right] \quad [\operatorname{Re} \nu > -1/2].$$

$$54. \int_0^{\infty} \int_0^{\infty} (xy)^{\nu-1} e^{-px - qy - 1/(x^2 y)} dx dy = 4q^{-\nu/2} K_{\nu} [(2pVq)^{1/2}] K_{\nu} [(2pVq)^{1/2} e^{-\pi i/4}].$$

$$55. \int_0^{\infty} \int_0^{\infty} \frac{1}{V xy} e^{-px - qy - 1/(x^2 y)} dx dy = \pi \sqrt{\frac{2}{pq}} e^{-2p^{1/2} q^{1/4}}.$$

$$56. \int_0^{\infty} \int_0^{\infty} (xy)^{-1/4} e^{-px-xy} \pm 2a \sqrt{xy} \, dx \, dy = \\ = \frac{2\sqrt{\pi} (pq)^{1/4}}{pq-a^2} E \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] + \\ + \frac{2\sqrt{\pi} (pq)^{-1/4} (-\sqrt{pq} \pm a)}{pq-a^2} K \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] \quad [pq > a^2].$$

$$57. \int_0^{\infty} \int_0^{\infty} (xy)^{-3/4} e^{-px-xy} \pm 2a \sqrt{xy} \, dx \, dy = \\ = 4\sqrt{\pi} (pq)^{-1/4} K \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq}} \right)} \right] \quad [pq > a^2].$$

$$58. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{x^2+y^2}} e^{-px-xy-a\sqrt{x^2+y^2}} \, dx \, dy = \\ = \frac{1}{\sqrt{p^2+q^2-a^2}} \ln \frac{p+q+a+\sqrt{p^2+q^2-a^2}}{p+q-a-\sqrt{p^2+q^2-a^2}}.$$

3.1.4. Интегралы, содержащие гиперболические функции.

Условие: $a, b, c, p, q > 0$.

$$1. \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f'(a \operatorname{sh} x + b \operatorname{ch} x \operatorname{sh} y + c \operatorname{ch} x \operatorname{ch} y) \operatorname{ch} x \, dx \, dy = \\ = -\frac{2\pi \operatorname{sgn} c}{\sqrt{c^2-a^2-b^2}} f(\operatorname{sgn} c \sqrt{c^2-a^2-b^2}) \quad [c^2 > a^2+b^2; \lim_{x \rightarrow +\infty} f(x) = 0].$$

$$2. \int_0^{\ln(1+\sqrt{2})} \int_0^{\ln(1+\sqrt{2})} \frac{\operatorname{ch} x \operatorname{ch} y \, dx \, dy}{(1-k^2 \operatorname{sh} x \operatorname{sh} y) \sqrt{\operatorname{sh} x \operatorname{sh} y (1-\operatorname{sh} x) (1-\operatorname{sh} y)}} = 2\pi K(k).$$

$$3. \int_0^{\ln(1+\sqrt{2})} \int_0^{\ln(1+\sqrt{2})} \frac{\operatorname{ch} x \operatorname{ch} y \, dx \, dy}{\sqrt{(1-\operatorname{sh}^2 x \operatorname{sh}^2 y) (1-\operatorname{sh}^2 x) (1-\operatorname{sh}^2 y)}} = K^2 \left(\frac{\sqrt{2}}{2} \right).$$

$$4. \int_0^k \int_0^{\infty} \frac{y \, dx \, dy}{\sqrt{\operatorname{ch}^2 x - y^2}} = E(k) - (1-k^2) K(k).$$

$$5. \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{y^3 \, dx \, dy}{\operatorname{ch}^2 x \sqrt{\operatorname{ch}^2 x - y^2}} = \frac{1}{3} [(2-k^2) E(k) - 2(1-k^2) K(k)].$$

$$6. \int_0^{\infty} \int_0^{\infty} \operatorname{sh}(x+y) e^{-px-xy} \, dx \, dy = \frac{p+q}{(p^2-1)(q^2-1)}.$$

$$7. \int_0^{\infty} \int_0^{\infty} \operatorname{ch}(x+y) e^{-px-xy} \, dx \, dy = \frac{pq+1}{(p^2-1)(q^2-1)}.$$

$$8. \int_0^{\infty} \int_0^{\infty} \operatorname{ch} \sqrt{xy} e^{-px-xy} \, dx \, dy = \frac{4}{4pq-1} + 4(4pq-1)^{-3/2} \arcsin \frac{1}{2\sqrt{pq}} \quad [pq > 1/4].$$

$$9. \int_0^{\infty} \int_0^{\infty} \frac{1}{xy} (\operatorname{ch} \sqrt{xy} - 1) e^{-px-xy} \, dx \, dy = -2 \arcsin^2 \frac{1}{2\sqrt{pq}} \quad [pq \geq 1/4].$$

$$10. \int_0^{\infty} \int_0^{\infty} \frac{1}{\sqrt{xy}} \operatorname{sh} \sqrt{xy} e^{-px-xy} \, dx \, dy = \frac{4}{(4pq-1)^{1/2}} \arcsin \frac{1}{2\sqrt{pq}} \quad [pq > 1/4].$$

$$11. \int_0^{\infty} \int_0^{\infty} \frac{1}{(xy^2)^{1/4}} (\operatorname{ch} \sqrt{xy} - \sqrt{xy}) e^{-px-xy} \, dx \, dy = \\ = \frac{2}{\sqrt{pq}} \left[1 - (4pq-1)^{1/2} \arcsin \frac{1}{2\sqrt{pq}} \right] \quad [pq \geq 1/4].$$

$$12. \int_0^{\infty} \int_0^{\infty} \frac{\operatorname{sh}(a\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} e^{-px-xy} \, dx \, dy = \frac{1}{2\sqrt{p^2+q^2-a^2}} \ln \frac{pq-a\sqrt{p^2+q^2-a^2}}{pq+a\sqrt{p^2+q^2-a^2}}.$$

$$13. \int_0^{\infty} \int_0^{\infty} \frac{\operatorname{ch}(a\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} e^{-px-xy} \, dx \, dy = \\ = \frac{1}{2\sqrt{p^2+q^2-a^2}} \ln \frac{p^2+pq+q^2+(p+q)\sqrt{p^2+q^2-a^2}}{p^2+pq+q^2-(p+q)\sqrt{p^2+q^2-a^2}}.$$

3.1.5. Интегралы, содержащие тригонометрические функции.

Условие: $a, b, c, p, q > 0; k^2 < 1$.

$$1. \int_0^{\pi} \int_0^{2\pi} f(a \sin x \sin y + b \cos x \sin y + c \cos y) \sin y \, dx \, dy = \\ = 2\pi \int_{-1}^1 f(t \sqrt{a^2+b^2+c^2}) \, dt \quad [\text{Функция } f(t) \text{ непрерывна при } |t| < \sqrt{a^2+b^2+c^2}].$$

$$2. \int_0^{\pi} \int_0^{\pi} f' \left(a \operatorname{ctg} x - \frac{b \cos y - c}{\sin x \sin y} \right) \frac{dx \, dy}{\sin^2 x \sin y} = \\ = -\frac{2\pi \operatorname{sgn} c}{\sqrt{c^2-a^2-b^2}} f(\operatorname{sgn} c \sqrt{c^2-a^2-b^2}) \quad [c^2 > a^2+b^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$3. \int_0^{\infty} \int_0^{\pi} f'(a \cos x \operatorname{sh} y + b \operatorname{ch} y) \operatorname{sh} y \, dx \, dy = -\frac{\pi \operatorname{sgn} b}{\sqrt{b^2-a^2}} f(\operatorname{sgn} b \sqrt{b^2-a^2}) \quad [b^2 > a^2; \lim_{t \rightarrow +\infty} f(t) = 0].$$

4. $\int_0^\infty \int_0^{2\pi} f' [(a \sin x + b \cos x) \operatorname{sh} y + c \operatorname{ch} y] \operatorname{sh} y \, dx \, dy =$
 $= -\frac{2\pi \operatorname{sgn} c}{\sqrt{c^2 - a^2 - b^2}} f(\operatorname{sgn} c \sqrt{c^2 - a^2 - b^2}) \quad [c^2 > a^2 + b^2; \lim_{t \rightarrow +\infty} f(t) = 0].$
5. $\int_0^\infty \int_0^\pi f(a \cos x \operatorname{sh} y + b \operatorname{ch} y) \sin x \operatorname{sh}^2 y \, dx \, dy =$
 $= 2 \int_0^\infty f(\operatorname{sgn} b \sqrt{b^2 - a^2} \operatorname{ch} y) \operatorname{sh}^2 y \, dy \quad [b^2 > a^2; \lim_{t \rightarrow +\infty} f(t) = 0].$
6. $\int_0^{\pi/2} \int_0^{\pi/2} \cos(2a \sin x \sin y) \, dx \, dy = \frac{\pi^2}{4} J_0^2(a).$
7. $\int_0^\pi \int_0^\pi \frac{dx \, dy}{4 - \alpha^2(1 - \cos x)(1 - \cos y)} = \frac{\pi}{2\alpha} K\left(\frac{1}{\alpha}\right) \quad [\alpha > 1].$
8. $\int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{3} K^2\left(\frac{\sqrt{2}}{2}\right).$
9. $\int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \cos^2 y \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{72} \left[20 K^2\left(\frac{\sqrt{2}}{2}\right) - 9\pi^2 K^2\left(\frac{\sqrt{2}}{2}\right) \right].$
10. $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \sin^2 x \sin^2 y} \, dx \, dy = \frac{1}{8} \left[4 K^2\left(\frac{\sqrt{2}}{2}\right) + \pi^2 K^{-2}\left(\frac{\sqrt{2}}{2}\right) \right].$
11. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = \frac{\pi}{2 \sqrt{1 - k^2}}.$
12. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} \, dx \, dy = K(k).$
13. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{dx \, dy}{\sqrt{1 - \sin^2 x \cos^2 y}} = K^2\left(\frac{\sqrt{2}}{2}\right).$
14. $\int_0^{\pi/2} \int_0^y \frac{\operatorname{ctg} y \, dx \, dy}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{4} K(\sqrt{1 - k^2}) + \frac{1}{2} \ln k K(k).$
15. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin y \, dx \, dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2} \operatorname{cosec} \alpha.$

16. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{dx \, dy}{\sqrt{1 - k_1^2 \sin^2 x - k_2^2 \sin^2 y}} =$
 $= \frac{2}{1 + \sqrt{1 - k_2^2}} K\left(\frac{\sqrt{1 - k_1^2} - \sqrt{1 - k_1^2 - k_2^2}}{1 + \sqrt{1 - k_2^2}}\right) K'\left(\frac{\sqrt{1 - k_1^2} + \sqrt{1 - k_1^2 - k_2^2}}{1 + \sqrt{1 - k_2^2}}\right) \quad [k_1^2 + k_2^2 < 1].$
17. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos^2 x \, dx \, dy}{\sqrt{1 - \sin^2 x \sin^2 y}} = \frac{1}{8} \left[4 K^2\left(\frac{\sqrt{2}}{2}\right) - \pi^2 K^{-2}\left(\frac{\sqrt{2}}{2}\right) \right].$
18. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{\cos^2 x \cos^2 y \, dx \, dy}{\sqrt{1 - \sin^2 x \cos^2 y}} = \frac{1}{8} \left[4 K^2\left(\frac{\sqrt{2}}{2}\right) - 3\pi^2 K^{-2}\left(\frac{\sqrt{2}}{2}\right) \right].$
19. $\int_0^{\pi/2} \int_0^y \frac{\sin 2y \, dx \, dy}{(1 - k^2 \sin^2 y) \sqrt{1 - k^2 \sin^2 x}} = -\frac{K(k)}{4k^2} \ln(1 - k^2).$
20. $\int_0^y \int_0^y \frac{dx \, dy}{\sqrt{(1 - k^2 \sin^2 x)(1 - k^2 \sin^2 y)}} = \frac{F^2(q, k)}{2}.$
21. $\int_0^{\pi/2} \int_0^y \frac{dx \, dy}{\sqrt{(1 - k^2 \sin^2 x)(1 - k^2 \sin^2 y)}} = \frac{K^2(k)}{2}.$
22. $\int_0^{\pi/2} \int_0^{\pi/2} \frac{[k^2 \cos^2 x + (1 - k^2) \cos^2 y] \, dx \, dy}{\sqrt{(1 - k^2 \sin^2 x)[1 - (1 - k^2) \sin^2 y]}} = \frac{\pi}{2}.$
23. $\int_0^{\pi/2} \int_0^y \frac{\sqrt{1 - k^2 \sin^2 x}}{1 - k^2 \sin^2 y} \, dx \, dy = \frac{E(k) K(k)}{2} - \frac{\ln(1 - k^2)}{4}.$
24. $\int_0^{\pi/2} \int_0^y \sqrt{\frac{1 - k^2 \sin^2 x}{1 - k^2 \sin^2 y}} \sin 2y \, dx \, dy =$
 $= \frac{1}{k^2} \left\{ (2 - k^2) K(k) - \left[2 + \frac{1}{4} \ln(1 - k^2) \right] E(k) \right\}.$
25. $\int_0^{\pi/2} \int_0^y \sqrt{\frac{1 - k^2 \sin^2 x}{1 - k^2 \sin^2 y}} \sin y \, dx \, dy = \frac{\pi}{2 \sqrt{1 - k^2}}.$
26. $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{(1 + \sin x)(1 + \sin y)}{\sin x \sin y}} \frac{dx \, dy}{1 - k^2 \sin x \sin y} = 2\pi K(k).$
27. $\int_0^k \int_0^{\pi/2} y \sqrt{1 - y^2 \sin^2 x} \, dx \, dy = \frac{1}{3} [(1 + k^2) E(k) - (1 - k^2) K(k)].$

28. $\int_0^1 \int_0^{\pi/2} \sqrt{1-y^2 \sin^2 x} dx dy = G + \frac{1}{2}.$
29. $\int_0^1 \int_0^{\pi/2} \sqrt{\cos^2 x + y^2 \sin^2 x} dx dy = \frac{\pi^2}{8}.$
30. $\int_0^k \int_0^{\pi/2} y \sqrt{1-y^2 \sin^2 x} dx dy =$
 $= \frac{1}{3} [(1+k^2) E(\varphi, k) - (1-k^2) F(\varphi, k) - (1-\sqrt{1-k^2 \sin^2 \varphi}) \operatorname{ctg} \varphi].$
31. $\int_0^1 \int_0^{\pi/2} y \sqrt{1-y^2 \sin^2 x} dx dy = \frac{\sin^2 \varphi - \cos \varphi + 1}{3 \sin \varphi}.$
32. $\int_0^k \int_0^{\pi/2} \frac{\sqrt{1-y^2 \sin^2 x}}{1-y^2} dx dy = k K(k).$
33. $\int_0^1 \int_0^{\pi/2} \frac{dx dy}{\sqrt{\cos^2 x + y^2 \sin^2 x}} = \frac{\pi^2}{4}.$ 34. $\int_0^1 \int_0^{\pi/2} \frac{dx dy}{\sqrt{1-y^2 \sin^2 x}} = 2G.$
35. $\int_0^k \int_0^{\pi/2} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = E(k) - (1-k^2) K(k).$
36. $\int_0^1 \int_0^{\pi/2} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = \frac{1 - \cos \varphi}{\sin \varphi}.$
37. $\int_0^k \int_0^{\pi/2} \frac{y dx dy}{\sqrt{1-y^2 \sin^2 x}} = E(\varphi, k) - (1-k^2) F(\varphi, k) + (\sqrt{1-k^2 \sin^2 \varphi} - 1) \operatorname{ctg} \varphi.$
38. $\int_0^k \int_0^{\pi/2} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} =$
 $= (k^2 - \alpha^2) \Pi(\varphi, \alpha^2, k) + E(\varphi, k) - F(\varphi, k) + (\sqrt{1-k^2 \sin^2 \varphi} - 1) \operatorname{ctg} \varphi.$
39. $\int_0^k \int_0^{\pi/2} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} = (k^2 - \alpha^2) \Pi(\pi/2, \alpha^2, k) - K(k) + E(k).$
40. $\int_0^1 \int_0^{\pi/2} \frac{y dx dy}{(1-\alpha^2 \sin^2 x) \sqrt{1-y^2 \sin^2 x}} =$
 $= (1-\alpha^2) \Pi(\varphi, \alpha^2, 1) - \alpha^2 \Pi(\varphi, \alpha^2, 0) + (1 - \cos \varphi) / \sin \varphi - \ln(\operatorname{tg} \varphi + \sec \varphi).$

41. $\int_0^1 \int_0^{\pi/2} \frac{dx dy}{(1+y) \sqrt{1-y^2 \sin^2 x}} = \frac{\pi^2}{8}.$
42. $\int_0^{\infty} \int_0^{\infty} e^{-p \sqrt{x^2+y^2}} \cos ax \cos by dx dy = \frac{\pi}{2} \frac{p}{(a^2+b^2+p^2)^{3/2}}.$
43. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \sin(x+y) dx dy = \frac{p+q}{(p^2+1)(q^2+1)}.$
44. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \cos(x+y) dx dy = \frac{pq-1}{(p^2+1)(q^2+1)}.$
45. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \cos(xy) dx dy = \sin(pq) \operatorname{ci}(pq) - \cos(pq) \operatorname{si}(pq).$
46. $\int_0^{\infty} \int_0^{\infty} \frac{1}{x+y} e^{-px-xy} \sin \frac{xy}{x+y} dx dy = \sqrt{\frac{\pi}{pq}} \frac{\sqrt{p} + \sqrt{q}}{(\sqrt{p} + \sqrt{q})^3 + 1}.$
47. $\int_0^{\infty} \int_0^{\infty} \frac{1}{x+y} e^{-px-xy} \cos \frac{xy}{x+y} dx dy = \sqrt{\frac{\pi}{pq}} \frac{(\sqrt{p} + \sqrt{q})^3}{(\sqrt{p} + \sqrt{q})^3 + 1}.$
48. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \sin(x \sqrt{y}) dx dy = \sqrt{\frac{\pi}{q}} - \pi p e^{p^2/q} \operatorname{erfc}(p \sqrt{q}).$
49. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \cos(x \sqrt{y}) dx dy = -p e^{p^2/q} \operatorname{Ei}(-p^2/q).$
50. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \sin(y \sqrt{x}) dx dy = \sqrt{\frac{\pi}{2p}} e^{pq^2/2} D_{-3/2}(q \sqrt{2p}).$
51. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \sin(y \sqrt{2x}) dx dy = \sqrt{\frac{\pi}{2}} n! p^{(n-2)/2} e^{pq^2/2} D_{-(n+1/2)}(q \sqrt{2p}).$
52. $\int_0^{\infty} \int_0^{\infty} e^{-px-xy} \sin(2 \sqrt{xy}) dx dy = \frac{\pi \sqrt{a}}{2} (pq+a)^{-3/2}.$
53. $\int_0^{\infty} \int_0^{\infty} y^{n-2-px-xy} \sin \sqrt{xy} dx dy = \frac{\pi (2n+1)! p^n}{n! (4pq+1)^{n+3/2}}.$
54. $\int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-px-xy} \sin(x \sqrt{y}) dx dy = \frac{\pi}{p} e^{p^2/q} \operatorname{erfc}(p \sqrt{q}).$

55. $\int_0^\infty \int_0^\infty \frac{1}{y} e^{-px-xy} \sin(2\sqrt{axy}) dx dy = \frac{\pi}{p} \sqrt{\frac{a}{pq+a}}.$
56. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{y}} e^{-px-xy} \sin(x\sqrt{y}) dx dy = -e^{p^2/4} \text{Ei}(-p^2/4).$
57. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{y}} e^{-px-xy} \cos(2\sqrt{axy}) dx dy = \frac{\sqrt{\pi q}}{pq+a}.$
58. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{y}} e^{-px-xy} \sin(2\sqrt{axy}) dx dy = \frac{1}{pq+a} \sqrt{\frac{\pi a}{p}}.$
59. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{y}} e^{-px-xy} \cos(x\sqrt{y}) dx dy = e^{-p^2/4} \text{erfc}(p\sqrt{y}).$
60. $\int_0^\infty \int_0^\infty \frac{x^{n-1/2}}{\sqrt{y}} e^{-px-xy} \cos\sqrt{xy} dx dy = \frac{2 \dots (2n)! q^n}{n! (4pq+1)^{n+1/2}}.$
61. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy}} e^{-px-xy} \cos(2\sqrt{axy}) dx dy = \frac{\pi}{\sqrt{pq+a}}.$
62. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy}} e^{-px-xy} \cos 2\sqrt{xy} \cos 2\sqrt{2ay} dx dy = \pi (pq+1)^{-1/2} e^{-n p/(pq+1)}.$
63. $\int_0^\infty \int_0^\infty (xy)^{-1/2} e^{-px-xy} [\cos(2a\sqrt{xy}) \pm \sin(2a\sqrt{xy})] dx dy =$
 $= \sqrt{\pi} (pq+a^2)^{-3/4} \left\{ 2\text{E} \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq+a^2}} \right)} \right] - \right.$
 $\left. - \text{K} \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq+a^2}} \right)} \right] \right\}.$
64. $\int_0^\infty \int_0^\infty (xy)^{-3/4} e^{-px-xy} [\cos(2a\sqrt{xy}) \pm \sin(2a\sqrt{xy})] dx dy =$
 $= \sqrt{\pi} (pq+a^2)^{-1/4} \text{K} \left[\sqrt{\frac{1}{2} \left(1 \pm \frac{a}{\sqrt{pq+a^2}} \right)} \right].$
65. $\int_0^\infty \int_0^\infty \frac{1}{xy} e^{-px-xy} \sin\sqrt{xy} dx dy = 2\pi \ln \left(\sqrt{1 + \frac{1}{4pq}} + \frac{1}{2\sqrt{pq}} \right).$

66. $\int_0^\infty \int_{-\pi}^\pi e^{-y(a+ib\cos x+ic\sin x)} dx dy = \frac{2\pi}{\sqrt{a^2+b^2+c^2}}.$
67. $\int_0^\infty \int_0^\infty \frac{1}{xy} (\cos\sqrt{xy}-1) e^{-px-xy} dx dy = -2 \text{Arsh}^2 \frac{1}{2\sqrt{pq}}.$
68. $\int_0^\infty \int_0^\infty \cos\sqrt{xy} e^{-px-xy} dx dy = \frac{4}{4pq+1} - 4(4pq+1)^{-3/2} \text{Arsh} \frac{1}{2\sqrt{pq}}.$
69. $\int_0^\infty \int_0^\infty \frac{1}{\sqrt{xy}} e^{-px-xy} \sin\sqrt{xy} dx dy = \frac{4}{(4pq+1)} \text{Arsh} \frac{1}{2\sqrt{pq}}.$
70. $\int_0^\infty \int_0^\infty (xy)^{-1/2} e^{-px-xy} (\sqrt{xy} - \sin\sqrt{xy}) dx dy =$
 $= -\frac{2}{pq} \left[1 - (1+4pq)^{1/2} \text{Arsh} \frac{1}{2\sqrt{pq}} \right].$

3.1.6. Интегралы, содержащие логарифмическую функцию.

Условие: $a, p, q > 0.$

1. $\int_0^\infty \int_0^\infty (\ln xy + 2C) e^{-px-xy} dx dy = -\frac{1}{pq} \ln pq$
2. $\int_0^\infty \int_0^\infty [\ln(x+y) + C] e^{-px-xy} dx dy = \frac{q \ln p - p \ln q}{pq(p+q)}.$
3. $\int_0^\infty \int_0^\infty \ln(x^2+y^2) e^{-px-xy} dx dy = -\frac{2}{pq} \left[C + \ln p - \frac{2p^2(\ln p - \ln q) + \pi pq}{2(p^2+q^2)} \right].$
4. $\int_0^\infty \int_0^\infty \ln \left(1 + \frac{y^2}{x^2} \right) e^{-px-xy} dx dy = \frac{2p(\ln p - \ln q) + \pi q}{q(p^2+q^2)}.$
5. $\int_0^\infty \int_0^\infty \frac{y^x - y^y}{\ln y} e^{-px-xy} dx dy = \lambda(qe^p, a).$
6. $\int_0^1 \int_0^1 e^{xy \ln(xy)} dx dy = \int_0^1 x^x dx.$

3.1.7. Интегралы, содержащие обратные тригонометрические и обратные гиперболические функции.
Условие: $p, q > 0$.

$$1. \int_0^{\infty} \int_0^{\infty} \operatorname{arctg} \frac{y}{x} e^{-px-xy} dx dy = \frac{2q(\ln q - \ln p) + \pi p}{2q(p^2 + q^2)}.$$

$$2. \int_0^{\infty} \int_0^{\infty} \operatorname{arctg} \sqrt{\frac{y}{x}} e^{-px-xy} dx dy = \frac{\pi}{2q(p + \sqrt{pq})}.$$

$$3. \int_0^{\infty} \int_0^{\infty} \operatorname{Arsh} \frac{x}{y} e^{-px-xy} dx dy = \frac{1}{p\sqrt{p^2 + q^2}} \ln \frac{p+q + \sqrt{p^2 + q^2}}{p+q - \sqrt{p^2 + q^2}}.$$

3.2. ТРОЙНЫЕ ИНТЕГРАЛЫ

3.2.1. Введение.

В этом разделе помещены тройные интегралы и указаны некоторые способы их приведения к одномерным интегралам. Кроме того, указаны различные системы координат в пространстве, которые могут быть использованы при вычислении тройных интегралов.

Отметим, что основным способом вычисления тройных интегралов является сведение их к повторным [20], которые находят путем последовательного интегрирования по каждой из переменных в отдельности.

Кроме того, часто используется метод замены переменных интегрирования. Пусть задано взаимно однозначное отображение $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ области Ω пространства переменных x, y, z на область V пространства переменных u, v, w , причем

а) x, y, z как функции u, v, w непрерывны в Ω вместе со всеми первыми частными производными;
б) якобиан

$$\Delta = \begin{vmatrix} \partial x / \partial u & \partial x / \partial v & \partial x / \partial w \\ \partial y / \partial u & \partial y / \partial v & \partial y / \partial w \\ \partial z / \partial u & \partial z / \partial v & \partial z / \partial w \end{vmatrix} \neq 0.$$

Тогда справедлива следующая формула:

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_V f(x(u, v, w), y(u, v, w), z(u, v, w)) |\Delta| du dv dw.$$

3. Некоторые системы координат в пространстве (dl — элемент длины, dS — элемент площади, dV — элемент объема) [16].

1) Цилиндрические координаты:

$$\begin{aligned} x &= u \cos v, & y &= u \sin v, & z &= w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & -\infty &< w < \infty, \\ dl^2 &= du^2 + u^2 dv^2 + dw^2, \\ dS^2 &= (u du dv)^2 + (u dw)^2 + (u dv dw)^2, \\ dV &= u du dv dw, \end{aligned}$$

(обычно используются обозначения $u = \rho$, $v = \varphi$, $w = z$).

3.2. ТРОЙНЫЕ ИНТЕГРАЛЫ

3.2.1.]

2) Сферические координаты:

$$\begin{aligned} x &= u \cos v \sin w, & y &= u \sin v \sin w, & z &= u \cos w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & 0 &\leq w \leq \pi, \\ dl^2 &= du^2 + u^2 \sin^2 w dv^2 + u^2 dw^2, \\ dS^2 &= (u \sin w du dv)^2 + (u du dw)^2 + (u^2 \sin w dv dw)^2, \\ dV &= u^2 \sin w du dv dw \end{aligned}$$

(обычно используются обозначения $u = \rho$, $v = \varphi$, $w = \psi$).

3) Обобщенные сферические координаты:

$$\begin{aligned} x &= au \cos v \sin w, & y &= bu \sin v \sin w, & z &= cu \cos w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & 0 &\leq w \leq \pi, \\ dV &= abc u^2 \sin w du dv dw. \end{aligned}$$

3') $x = au \cos^m v \sin^n w$, $y = bu \sin^m v \sin^n w$, $z = cu \cos^m w$.

$$dV = abc m n u^2 \cos^{m-1} v \sin^{m-1} w \cos^{n-1} w \sin^{2n-1} w du dv dw.$$

4) Эллипсоидальные координаты:

$$\begin{aligned} x &= \frac{(u+a^2)(v+a^2)(w+a^2)}{(b^2-a^2)(c^2-a^2)}, & y &= \frac{(u+b^2)(v+b^2)(w+b^2)}{(a^2-b^2)(c^2-b^2)}, \\ z &= \frac{(u+c^2)(v+c^2)(w+c^2)}{(a^2-c^2)(b^2-c^2)}, \end{aligned}$$

$$0 \leq c^2 < b^2 < a^2, \quad -a^2 < w < -b^2 < v < -c^2 < u < \infty,$$

$$dl^2 = \frac{1}{4} [\mu(u)(u-v)(u-w) du^2 - \mu(v)(u-v)(v-w) dv^2 + \mu(w)(u-w)(v-w) dw^2],$$

$$dS^2 = \frac{1}{16} \{ -\mu(u)\mu(v)(u-v)^2(u-w)(v-w)(du dv)^2 + \mu(u)\mu(w)(u-v)(u-w)^2(v-w)(du dw)^2 - \mu(v)\mu(w)(u-v)(u-w)(v-w)^2(dv dw)^2 \},$$

$$dV = \frac{1}{8} \sqrt{-\mu(u)\mu(v)\mu(w)(u-v)(u-w)(v-w)} du dv dw,$$

$$\mu(t) = [(t+a^2)(t+b^2)(t+c^2)]^{-1}.$$

5) Вырожденные эллипсоидальные «вытянутые» координаты:

$$\begin{aligned} x &= sh u \cos v \sin w, & y &= sh u \sin v \sin w, & z &= ch u \cos w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & 0 &\leq w \leq \pi, \\ dl^2 &= (sh^2 u + \sin^2 w) (du^2 + dw^2) + sh^2 u \sin^2 w dv^2, \\ dS^2 &= (sh^2 u + \sin^2 w) sh^2 u \sin^2 w (du^2 + dw^2) dv^2 + (sh^2 u + \sin^2 w)^2 (du dw)^2, \\ dV &= (sh^2 u + \sin^2 w) sh u \sin w du dv dw. \end{aligned}$$

6) Вырожденные эллипсоидальные «сплюснутые» координаты:

$$\begin{aligned} x &= ch u \cos v \sin w, & y &= ch u \sin v \sin w, & z &= sh u \cos w, \\ 0 &\leq u < \infty, & 0 &\leq v < 2\pi, & 0 &\leq w \leq \pi, \\ dl^2 &= (sh^2 u + \cos^2 w) (du^2 + dw^2) + sh^2 u \sin^2 w dv^2, \\ dS^2 &= (sh^2 u + \cos^2 w) ch^2 u \sin^2 w (du^2 + dw^2) dv^2 + (sh^2 u + \cos^2 w) (du dw)^2, \\ dV &= (sh^2 u + \cos^2 w) ch u \sin w du dv dw. \end{aligned}$$

7) Сферо-конические координаты:

$$\lambda^2 = \frac{u(v+a^2)(w+b^2)}{(a^2-b^2)a^2}, \quad \mu^2 = \frac{u(v+b^2)(w+a^2)}{(b^2-a^2)b^2}, \quad z^2 = \frac{uvw}{a^2b^2},$$

$$-a^2 < w < -b^2 < v < 0 < u < \infty,$$

$$dV = \frac{1}{4} \left[\frac{du^3}{u} - \lambda(v)u(v-w)dv^2 + \lambda(w)u(v-w)dw^2 \right],$$

$$dS^2 = \frac{1}{16} [-\lambda(v)(v-w)(du dv)^2 + \lambda(w)(v-w)(du dw)^2 -$$

$$- \lambda(v)\lambda(w)u^2(v-w)^2(du dw)^2],$$

$$dV = \frac{1}{6} \sqrt{-u\lambda(v)\lambda(w)(v-w)} du dv dw,$$

$$\lambda(t) = [t(t+a^2)(t+b^2)]^{-1/2}.$$

8) Параболоидальные координаты:

$$x = 2uw \cos v, \quad y = 2uw \sin v, \quad z = u^2 - w^2,$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad 0 \leq w < \infty,$$

$$dt^2 = 4[(u^2 + w^2)(du^2 + dw^2) + u^2w^2 dv^2],$$

$$dS^2 = 16[(u^2 + w^2)u^2w^2(du^2 + dw^2) dv^2 + (u^2 + w^2)^2 du^2 dw^2],$$

$$dV = 8(u^2 + w^2)uw du dv dw.$$

9) Торевидальные координаты:

$$x = \frac{\operatorname{sh} u \cos v}{\operatorname{ch} u - \cos w}, \quad y = \frac{\operatorname{sh} u \sin v}{\operatorname{ch} u - \cos w}, \quad z = \frac{\sin w}{\operatorname{ch} u - \cos w},$$

$$0 \leq u < \infty, \quad 0 \leq v < 2\pi, \quad -\pi \leq w \leq \pi,$$

$$dV = \frac{1}{(\operatorname{ch} u - \cos w)^3} (du^2 + \operatorname{sh}^2 u dv^2 + dw^2),$$

$$dS^2 = \frac{1}{(\operatorname{ch} u - \cos w)^4} [\operatorname{sh}^2 u (du^2 + dw^2) dv^2 + du^2 dw^2],$$

$$dV = \frac{\operatorname{sh} u}{(\operatorname{ch} u - \cos w)^3} du dv dw.$$

10) Биполярные координаты:

$$x = \frac{\sin u \cos v}{\operatorname{ch} w - \cos u}, \quad y = \frac{\sin u \sin v}{\operatorname{ch} w - \cos u}, \quad z = \frac{\operatorname{sh} w}{\operatorname{ch} w - \cos u},$$

$$0 \leq u < \pi, \quad 0 \leq v < 2\pi, \quad -\infty < w < \infty,$$

$$dV = \frac{1}{(\operatorname{ch} w - \cos u)^3} (du^2 + \sin^2 u dv^2 + dw^2),$$

$$dS^2 = \frac{1}{(\operatorname{ch} w - \cos u)^4} [\sin^2 u (du^2 + dw^2) dv^2 + du^2 dv^2],$$

$$dV = \frac{\sin u}{(\operatorname{ch} w - \cos u)^3} du dv dw,$$

3.2.2. Общие формулы; интегралы от алгебраических функций.

Условие: $a, b, c, p, q, r, \alpha, \beta, \gamma > 0$.

$$1. \int_{x \geq 0, y \geq 0, z \geq 0} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} \left[\left(\frac{x}{a} \right)^p + \left(\frac{y}{b} \right)^q + \left(\frac{z}{c} \right)^r \right] dx dy dz =$$

$$\left(\frac{x}{a} \right)^p + \left(\frac{y}{b} \right)^q + \left(\frac{z}{c} \right)^r \leq 1$$

$$= \frac{a^{\alpha} b^{\beta} c^{\gamma}}{pqr} \Gamma \left[\frac{\alpha/p + \beta/q + \gamma/r}{\alpha/p + \beta/q + \gamma/r + 1} \right] \int_0^1 f(t) t^{\alpha/p + \beta/q + \gamma/r - 1} dt$$

[при условии, что интеграл справа сходится абсолютно].

$$2. \int_{x \geq 0, y \geq 0, z \geq 0} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx dy dz = \frac{a^{\alpha} b^{\beta} c^{\gamma}}{pqr} \times$$

$$\left(\frac{x}{a} \right)^p + \left(\frac{y}{b} \right)^q + \left(\frac{z}{c} \right)^r \leq 1$$

$$\times \Gamma \left[\frac{\alpha/p + \beta/q + \gamma/r}{\alpha/p + \beta/q + \gamma/r + 1} \right].$$

$$3. \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^{\alpha-1} y^{\beta-1} z^{\gamma-1} f(x+y+z) dx dy dz =$$

$$= \Gamma \left[\frac{\alpha + \beta + \gamma}{\alpha + \beta + \gamma} \right] \int_0^a f(t) t^{\alpha + \beta + \gamma - 1} dt$$

[при условии, что интеграл справа сходится абсолютно].

$$4. \int_{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1} \frac{xyz dx dy dz}{x^2 + y^2 + z^2} =$$

$$= \frac{a^2 b^2 c^2}{8(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)} \left[b^2 c^2 \ln \frac{c}{b} + c^2 a^2 \ln \frac{a}{c} + a^2 b^2 \ln \frac{b}{a} \right].$$

$$5. \int_{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1} dx dy dz = \frac{a^3 \Gamma^4(1/4)}{48\pi \sqrt{2}}.$$

$$6. \int_{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1} \frac{dx dy dz}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} = \frac{4}{3} \pi r^3 (a^2 + b^2 + c^2)^{-1/2}$$

[$a^2 + b^2 + c^2 > r^2$].

$$7. \int_{\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 + \left(\frac{z}{c} \right)^2 \leq 1} \frac{dx dy dz}{\sqrt{x^2 + y^2 + z^2}} =$$

$$= \frac{2\pi abc}{\sqrt{a^2 - c^2}} F \left(\arcsin \frac{\sqrt{a^2 - c^2}}{a}, \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \right) \quad [a > b > c].$$

$$8. \iiint_{\substack{x \geq 0, y \geq 0, z \geq 0, \\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1}} \frac{x y z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \frac{1}{15} a^2 b^2 c^2 \frac{ab + bc + ca}{(a+b)(b+c)(c+a)}.$$

3.2.3. Интегралы, содержащие показательную, гиперболические и тригонометрические функции.
Условие: $a, b, c, p, q, r > 0$.

$$1. \iiint_{x^2 + y^2 + z^2 \leq r^2} e^{-x^2 - y^2 - (z-a)^2} dx dy dz = -\frac{\pi}{a\sqrt{2}} [e^{-(r+a)^2/2} - e^{-(r-a)^2/2}] + \frac{\pi^{3/2}}{2} \left[\operatorname{erf} \left(\frac{r+a}{\sqrt{2}} \right) + \operatorname{erf} \left(\frac{r-a}{\sqrt{2}} \right) \right].$$

$$2. \iiint_0^\infty \frac{(xyz)^{(v-2)/2}}{(xy+yz+xz)^{(v+1)/2}} e^{-px-qy-rz} dx dy dz = \frac{8\pi^{3/2} \left(\frac{v+2}{2} \right)!}{v(v-1)(v-2)} \frac{1}{(\sqrt{p} + \sqrt{q} + \sqrt{r})^{v-2}} \quad [\operatorname{Re} v > -2].$$

$$3. \int_0^\beta \int_0^\beta \int_0^\beta \frac{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z \operatorname{ch} x dy dz}{(1 - \operatorname{sh} x \operatorname{sh} y \operatorname{sh} z) \sqrt{(1 - \operatorname{sh}^2 x)(1 - \operatorname{sh}^2 y)(1 - \operatorname{sh}^2 z)}} dx = 4\pi K^2(k) [\alpha = \ln(\sqrt{2}-1), \beta = \ln(\sqrt{2}+1)].$$

$$4. \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f\{[(a \cos y + b \sin y) \sin x + c \operatorname{ch} x] \operatorname{sh} z + d \operatorname{ch} z\} \operatorname{sh}^2 z \sin x dx dy dz = 4\pi \int_0^\infty f(\operatorname{sgn} d \sqrt{d^2 - a^2 - b^2 - c^2} \operatorname{ch} t) \operatorname{sh}^2 t dt \quad [d^2 > a^2 + b^2 + c^2, \lim_{t \rightarrow +\infty} f(t) = 0].$$

$$5. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} = 4\pi (18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}) K^2[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})].$$

$$6. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x \cos y - \cos x \cos z - \cos y \cos z} = \sqrt{3} \pi K^2\left(\sin \frac{\pi}{12}\right).$$

$$7. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi K^2\left(\frac{\sqrt{2}}{2}\right).$$

3.3. МНОГОМЕРНЫЕ ИНТЕГРАЛЫ

3.3.1. Введение

Обозначения: $x = (x_1, x_2, \dots, x_n)$, $a = (a_1, \dots, a_n)$, $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$, $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, $ax = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$, $dx = dx_1 dx_2 \dots dx_n$.
Замена переменных (сферические координаты):

$$x_1 = \rho \cos \varphi_1,$$

$$x_2 = \rho \sin \varphi_1 \cos \varphi_2,$$

$$x_3 = \rho \sin \varphi_1 \sin \varphi_2 \cos \varphi_3,$$

$$\dots \dots \dots$$

$$x_{n-1} = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \cos \varphi_{n-1},$$

$$x_n = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \sin \varphi_{n-1},$$

$$0 \leq \rho < \infty, \quad 0 \leq \varphi_k \leq \pi, \quad k = 1, 2, \dots, n-2, \quad 0 \leq \varphi_{n-1} < 2\pi,$$

$$dV = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \dots \sin^2 \varphi_{n-2} \sin \varphi_{n-1} d\rho d\varphi_1 d\varphi_2 \dots d\varphi_{n-1} = \text{элемент объема}.$$

При $x^2 \leq 1$ можно использовать формулу

$$\int_{x^2 \leq 1} \dots \int f(x) dx = (-1)^n \int_0^\pi \dots \int f(\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots, \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} \cos \theta_n) \sin^n \theta_1 \sin^{n-1} \theta_2 \dots \sin^{n-2} \theta_{n-1} \sin \theta_n d\theta_1 d\theta_2 \dots d\theta_n.$$

$$1. \int_a^x \varphi(x_1) dx_1 \int_a^{x_1} \varphi(x_2) dx_2 \dots \int_a^{x_{n-1}} \varphi(x_n) dx_n \int_a^{x_n} f(t) dt = \frac{1}{n!} \int_a^x f(t) \left(\int_t^x \varphi(y) dy \right)^n dt.$$

$$2. \int_a^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n \int_0^{x_n} f(t) dt = \frac{1}{n!} \int_a^x (x-t)^n f(t) dt.$$

$$3. \int_0^\infty e^{-ax_1} \int_0^{x_1} e^{-bx_2} \int_0^{x_2} e^{-cx_3} \dots \int_0^{x_{2n-1}} e^{-dx_{2n}} \int_0^{x_{2n}} e^{-ax_{2n+1}} \times \dots \times x_{2n+1}^{n-1} dx_1 dx_2 \dots dx_{2n+1} = \frac{1}{a^{n+1} 2^n n!} \frac{(m+1)!(m+2n)!}{n!(m+n+1)!} \quad [a > 0].$$

3.3.2. Интегралы по области $x_1^2 + x_2^2 + \dots + x_n^2 \leq r^2$

$$1. \int_{x^2 \leq r^2} f(x^2) dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r t^{n/2-1} f(t^2) dt.$$

$$2. \int_{x^2 \leq r^2} f(ax) dx = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-r}^r (r^2 - t^2)^{(n-3)/2} f\left(\frac{t}{a}\right) dt.$$

$$8. \quad \iiint_{\substack{x \geq 0, y \geq 0, z \geq 0, \\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1}} \frac{xyz}{\sqrt{x^2 + y^2 + z^2}} dx dy dz = \frac{1}{15} a^2 b^2 c^3 \frac{ab + bc + ca}{(a+b)(b+c)(c+a)}.$$

3.2.3. Интегралы, содержащие показательную, гиперболические и тригонометрические функции.
Условие: $a, b, c, p, q, r > 0$.

$$1. \quad \iiint_{x^2+y^2+z^2 \leq a^2} e^{-x^2-y^2-(z-a)^2} dx dy dz =$$

$$= \frac{\pi}{4\sqrt{2}} \left[e^{-(a^2/2)^{1/2}} - e^{-(a^2/2)^{1/2}} \right] + \frac{\pi^{3/2}}{2} \left[\operatorname{erf} \left(\frac{a+\sqrt{a^2/2}}{\sqrt{2}} \right) + \operatorname{erf} \left(\frac{a-\sqrt{a^2/2}}{\sqrt{2}} \right) \right].$$

$$2. \int_0^{\sqrt{v-1}} \int_0^{\sqrt{v-1-x^2}} \int_0^{\sqrt{v-1-x^2-y^2}} \frac{(xyz)^2 (v-x)^2}{(xy+yz+zx)^{(v+1)/2}} e^{-x^2-y^2-z^2} dx dy dz =$$

$$= \frac{8\pi\Gamma\left(\frac{v+2}{2}\right)}{v(v-1)(v-2)} \cdot \frac{1}{(\sqrt{v} + \sqrt{v-1})^{v-2}} \quad [Re v > -2].$$

$$3. \int_0^1 \int_0^1 \int_0^1 \frac{ch x ch y ch z dx dy dz}{(1 - sh x sh y sh z) \sqrt{(1 - sh^2 x)(1 - sh^2 y)(1 - sh^2 z)}} = 4\pi K^2(k) [\alpha = \ln(\sqrt{2} - 1), \beta = \ln(\sqrt{2} + 1)].$$

4. $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \{[(a \cos y + b \sin y) \sin x + c \operatorname{ch} x] \operatorname{sh} z + d \operatorname{ch} x\} \operatorname{sh}^2 x \sin x \, dx \, dy \, dz =$
 $= 4\pi \int_0^{\frac{\pi}{2}} \{ \operatorname{sgn} d \sqrt{d^2 - a^2 - b^2 - c^2} \operatorname{ch} t \} \operatorname{sh}^2 t \, dt \quad \left[d^2 > a^2 + b^2 + c^2; \lim_{t \rightarrow +\infty} f(t) = 0 \right]$

$$5. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} =$$

$$= 4\pi (18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}) \text{ К}^2 [(2 - \sqrt{3})(\sqrt{3} - \sqrt{2})].$$

6. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{dx dy dz}{3 - \cos x \cos y - \cos x \cos z - \cos y \cos z} = \sqrt{3} \pi K^2 \left(\sin \frac{\pi}{12} \right).$

7. $\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi K^2 \left(\frac{\sqrt{2}}{2} \right).$

1.3. МНОГОМЕРНЫЕ ИНТЕГРАЛЫ

2.3.1 Введение.

Следующие: $x = (x_1, x_2, \dots, x_n)$, $a = (a_1, \dots, a_n)$ $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$,
 $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, $ax = a_1x_1 + a_2x_2 + \dots + a_nx_n$, $dx = dx_1 dx_2 \dots dx_n$.
 Задача переменных (сферические координаты):

Задана переменных (сферические координаты):

$$x_1 = \rho \cos \varphi_1,$$

$$x_2 = \rho \sin \varphi_1 \cos \varphi_2,$$

$$x_3 = \rho \sin \varphi_1 \sin \varphi_2 \cos \varphi_3,$$

.....

$$x_{n-1} = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-3} \cos \varphi_{n-1},$$

$$x_n = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \sin \varphi_{n-1}$$

$$0 \leq \rho < \infty, \quad 0 \leq \varphi_k \leq \pi, \quad k=1, 2, \dots, n-2, \quad 0 \leq \varphi_{n-1} < 2\pi,$$

$$dV = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \dots \sin^2 \varphi_{n-2} \sin \varphi_{n-1} d\rho d\varphi_1 d\varphi_2 \dots d\varphi_{n-1} -$$

— элемент объема

При $x^2 \leq 1$ можно использовать формулу

$$\int_{x^2 \leq 1} \dots \int f(x) dx = (-1)^n \int_0^\pi \dots \int_0^\pi f(\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots, \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{n-1} \cos \theta_n) \sin^{\alpha-1} \theta_1 \sin^{\alpha-1} \theta_2 \dots \sin^{\alpha-1} \theta_{n-1} \sin \theta_n d\theta_1 d\theta_2 \dots d\theta_n$$

$$\begin{aligned} 1. \quad & \int_a^x \varphi(x_1) dx_1 \int_a^{x_1} \varphi(x_2) dx_2 \dots \int_a^{x_{n-1}} \varphi(x_n) dx_n \int_a^{x_n} f(t) dt = \\ & = \frac{1}{n!} \int_a^x f(t) \left(\int_t^x \varphi(y) dy \right)^n dt. \end{aligned}$$

$$2. \int_a^x dx_1 \int_0^1 dx_2 \dots \int_a^{x_{n-1}} dx_n \int_0^{x_n} f(t) dt = \frac{1}{n!} \int_a^x (x-t)^n f(t) dt.$$

$$3. \int_0^{x_0} e^{-ax_1} \int_0^{x_1} dx_2 \int_{x_2}^{\infty} e^{-ax_{2n-1}} \int_0^{x_{2n-1}} dx_{2n} \int_{x_{2n}}^{\infty} e^{-ax_{2n+1}} \times$$

$$\times x_{2n}^{m+1} \frac{dx_1 dx_2 \dots dx_{2n+1}}{a^{m+1}(m+1)!} = \frac{1}{a^{m+1}(m+1)!} \frac{(m+1)!(m+2n)!}{n! (m+n+1)!} [\sigma > 0].$$

3.3.2. Интегралы по области $x_1^2 + x_2^2 + \dots + x_n^2 \leq r^2$

$$1. \int_{x^2 \leq t^2} \int |f(x^2)| dx = \frac{2n^{n/2}}{\Gamma(n/2)} \int_0^t t^{n-1} |f(t^2)| dt.$$

$$2. \int_{\substack{0 \leq x \leq 1 \\ x \in \mathbb{Q}}} f(ax) dx = \frac{n^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^1 (t^2 - t)^{(n-1)/2} f(t|a|t) dt.$$

$$3. \int_{x^2 \leq r^2} \dots \int f(x^2, ax) dx = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r t^{n-1} dt \int_0^\pi f(t^2, |a|t \cos u) \sin^{n-2} u du$$

$$[|ma_k|=0; k=1, 2, \dots, n].$$

$$4. \int_{x^2 \leq r^2} \dots \int f(ax+b)(r^2-x^2)^m dx = \frac{\Gamma(m+1) \pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-r}^r f(|a|t+b)(r^2-t^2)^{m+n-1/2} dt.$$

$$5. \int_{x^2 \leq r^2} \dots \int f(x^2, ax, bx) dx = \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} \int_{-r}^r dt \int_{-\sqrt{r^2-t^2}}^{\sqrt{r^2-t^2}} dx \int_0^{\sqrt{r^2-t^2-x^2}} v^{n-3} f(t^2+u^2+v^2, |a|t, |b|t+\gamma u) dv$$

$$[|b| > |a|, \gamma = \frac{a^2 b - t(a^2)^{3/2}}{a^2}].$$

$$6. \int_{x^2 \leq r^2} \dots \int f(x^2) g\left(\frac{ax}{x^2}\right) dx = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r f(y^2) y^{n-3} dy \int_{-1}^1 g(|a|t) (1-t^2)^{(n-1)/2} dt.$$

$$7. \int_{x^2 \leq r^2} \dots \int F_1(x_1) F_2(x_2) \dots F_n(x_n) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it(t+iu)} dt}{t+iu} \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-(t+iu)x_k^2} F_k(x_k) dx_k$$

$$[t > 0 - \text{произвольный параметр}].$$

$$8. \int_{x^2 \leq r^2} \dots \int F(x^2) \prod_{k=1}^n \cos a_k x_k dx = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^r F(t^2) t^{n-3} dt \int_{-1}^1 e^{t \cdot a} (1-u^2)^{(n-3)/2} du.$$

$$9. \int_{x^2 \leq r^2} \dots \int dx = \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)}.$$

$$10. \int_{x^2 \leq 1} \dots \int (ax)^{2m} dx = \frac{(2m-1)!!}{2^m} \frac{\pi^{n/2}}{\Gamma(m+1+n/2)} |a|^{2m}.$$

$$11. \int_{x^2 \leq 1} \dots \int (ax)^{2m+1} dx = 0.$$

$$12. \int_{x^2 \leq 1} \dots \int (a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2)^m (bx)^{2l} dx =$$

$$= \frac{\pi^{n/2} (2l)!}{2^{2l} l! (l+m+1+n/2)} D_y^n \left[\left(\sum_{k=1}^n \frac{b_k^2}{1-a_k y} \right)^l \prod_{j=1}^n (1-a_j)^{-1/2} \right] \Big|_{y=0}.$$

$$13. \int_{x^2 \leq 1} \dots \int |x|^{2l} (ax)^{2m} dx = \frac{\pi^{n/2} (2m)!}{2^{2m} m! (n+2l+2n) \Gamma(m+n/2)} |a|^{2m}.$$

$$14. \int_{x^2 \leq 1} \dots \int \frac{(ax)^{2m}}{|x|^{2m}} dx = \frac{\pi^{n/2}}{2^{2m} m!} \Gamma\left[m+1, n/2+m\right] |a|^{2m}.$$

$$15. \int_{x^2 \leq 1} \dots \int \frac{(ax)^{2m} dx}{(b_1 x_1^2 + b_2 x_2^2 + \dots + b_n x_n^2)^v} =$$

$$= \frac{\pi^{n/2}}{2^{2v}} \Gamma\left[v, n/2+v-1, m+1\right] \int_0^\infty t^{v-1} \prod_{j=1}^n (1+b_j t)^{-1/2} \left(\sum_{k=1}^n \frac{a_k}{1+b_k t} \right)^m dt$$

$$[m > 1; 0 < v < m+n/2; b_k > 0, k=1, 2, \dots, n].$$

$$16. \int_{x^2 \leq 1} \dots \int e^{ax} dx = (2\pi)^{n/2} |a|^{-n/2} J_{n/2}(|a|).$$

$$17. \int_{x^2 \leq 1} \dots \int (1-x^2)^p e^{ax} dx = (2\pi)^{n/2} 2^p \Gamma(p+1) |a|^{-p-n/2} J_{p+n/2}(|a|).$$

$$18. \int_{x^2 \leq 1} \dots \int e^{ax/x} dx = \frac{(2\pi)^{n/2}}{n} |a|^{-n/2+1} J_{n/2-1}(|a|).$$

$$19. \int_{x^2 \leq 1} \dots \int e^{-a \cdot (1-x^2)} dx = J(a, n) \quad [a > 0].$$

$$J(a, 2m+1) = \frac{\pi^{(2m+1)/2}}{m! \Gamma\left(\frac{2m+1}{2}\right)} e^{-a} \binom{m+1}{m} \times$$

$$\times \sum_{k=0}^m \binom{m}{k} \binom{m-1}{k} k! a^{m-k} \frac{d^{m-k}}{da^{m-k}} \left[a^{1/2} \left[K_1\left(\frac{a}{2}\right) - K_0\left(\frac{a}{2}\right) \right] \right] =$$

$$= (-1)^m \frac{\pi^{(2m+1)/2}}{\Gamma\left(\frac{2m+1}{2}\right)} e^{-a/2} K_{2m+1}\left(\frac{a}{2}\right).$$

$$J(a, 2m) = \frac{\pi^m}{(m-1)! m!} e^{-a} \frac{d^{m-1}}{da^{m-1}} \{a^{m-1} [1 + a e^{-a} \text{Ei}(-a)]\},$$

$$J(a, 2) = \pi e^{-a} [1 + a e^{-a} \text{Ei}(-a)],$$

$$J(a, 3) = \frac{2\pi a}{3} e^{-a/2} \left[(2a+1) K_1\left(\frac{a}{2}\right) - (2a+3) K_0\left(\frac{a}{2}\right) \right].$$

$$20. \int_{x^2 \leq r^2} \prod_{k=1}^n \cos \alpha_k x_k dx = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{-r}^r (r^2 - t^2)^{(n-1)/2} |a| t dt$$

$$[\operatorname{Im} \alpha_k = 0, k=1, 2, \dots, n].$$

3.3.3. Интегралы по области $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1$.

Условие: $r, a_k, b_k, \alpha_k, \beta_k, \nu_k > 0, k=1, 2, \dots, n$.

$$1. \int_{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1} \frac{F\left(\sum_{k=1}^n a_k^2 x_k^2\right)}{x-b} dx = \frac{\pi^{n/2}}{\Gamma(p+1) \Gamma(n/2-p)} \times$$

$$\times \int_{b_1}^{\infty} y^{p-1} \prod_{i=1}^n \left(\frac{1}{a_i} + \frac{1}{y}\right)^{-1/2} dy \int_{b_2}^{\infty} F(t) \left(t - \sum_{k=1}^n \frac{b_k^2}{y+a_k}\right)^{n/2-p-1} dt$$

$[t_0 = \sum_{k=1}^n \frac{b_k^2}{y+a_k}; b_0=0, \text{ если } b_1, b_2, \dots, b_n \text{ принадлежат области интегрирования}]$

в противном случае: b_0 — положительный корень уравнения $\frac{b_1^2}{\nu_0+a_1^2} + \frac{b_2^2}{\nu_0+a_2^2} + \dots + \frac{b_n^2}{\nu_0+a_n^2} = 1$.

$$2. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1} + \left(\frac{x_2}{a_2}\right)^{\beta_2} + \dots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1}} \prod_{k=1}^n x_k^{\nu_k-1} dx =$$

$$= \Gamma\left[\frac{\nu_1/\beta_1 + \nu_2/\beta_2 + \dots + \nu_n/\beta_n}{\nu_1/\beta_1 + \nu_2/\beta_2 + \dots + \nu_n/\beta_n + 1}\right] \prod_{k=1}^n \frac{a_k^{\nu_k}}{\beta_k}.$$

$$3. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1} + \left(\frac{x_2}{a_2}\right)^{\beta_2} + \dots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1}} \prod_{k=1}^n x_k^{\nu_k-1} dx =$$

$$= \Gamma\left[\frac{\nu_1/\beta_1 + \nu_2/\beta_2 + \dots + \nu_n/\beta_n}{\nu_1/\beta_1 + \nu_2/\beta_2 + \dots + \nu_n/\beta_n + 1}\right] \prod_{k=1}^n \frac{a_k^{\nu_k}}{\beta_k} \int_0^1 f(t) t^{\nu_1/\beta_1 + \nu_2/\beta_2 + \dots + \nu_n/\beta_n - 1} dt$$

$[t_1=1, t_2=+\infty, -\infty < \nu_k < \infty \text{ при } \Omega: (x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \geq 1; t_1=0, t_2=1, \nu_k > 0 \text{ при } \Omega: (x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \leq 1, k=1, 2, \dots, n].$

$$4. \int_{\substack{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1 \\ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1}} \frac{dx}{|x-b|^{2p}} =$$

$$= \frac{\pi^{n/2}}{\Gamma(p+1) \Gamma(n/2-p-1)} \int_0^{\infty} t^{p-1} \left(1 - \sum_{i=1}^n \frac{b_i^2}{t+a_i}\right)^{n/2-p-1} \prod_{i=1}^n \left(\frac{1}{a_i} + \frac{1}{t}\right)^{-1/2} dt$$

$[0 < p < n/2; \nu_0 \text{ см. в 3.3.3.1}].$

$$5. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1}} \frac{1}{(1-x_1^2/a_1^2 - x_2^2/a_2^2 - \dots - x_n^2/a_n^2)^{\mu}} \prod_{k=1}^n x_k^{\nu_k-1} dx =$$

$$= \Gamma\left[\frac{\nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n}{1-\mu + \nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n}\right] \prod_{k=1}^n \frac{a_k^{\nu_k-1}}{\alpha_k} \quad [\mu < 1].$$

$$6. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{a_1}\right)^{\beta_1} + \left(\frac{x_2}{a_2}\right)^{\beta_2} + \dots + \left(\frac{x_n}{a_n}\right)^{\beta_n} \leq 1}} \prod_{k=1}^n x_k^{\nu_k-1} \frac{dx}{(x_1^2 + x_2^2 + \dots + x_n^2)^{\mu}} =$$

$$= \varepsilon \Gamma\left[\frac{\nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n}{\nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n + 1}\right] \frac{1}{\mu - \nu_1/\alpha_1 - \nu_2/\alpha_2 - \dots - \nu_n/\alpha_n} \prod_{k=1}^n \frac{a_k^{\nu_k-1}}{\alpha_k}$$

$[\varepsilon=1, -\infty < \nu_1, \dots, \nu_n < \infty, \mu > \nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n \text{ при } \Omega: (x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \geq 1 \text{ или } [\varepsilon=-1, \mu < \nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n \text{ и } \nu_1, \dots, \nu_n > 0 \text{ при } \Omega: (x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \dots + (x_n/a_n)^{\beta_n} \leq 1].$

$$7. \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} \leq 1}} \prod_{k=1}^n x_k^{\nu_k-1} dx =$$

$$= \frac{\sqrt{\pi}}{2} \Gamma\left[\frac{\nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n}{\beta, (\beta+1)/2}\right] -$$

$$- \Gamma\left[\frac{\nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n}{\beta, (\beta+2)/2}\right] \prod_{k=1}^n \frac{a_k^{\nu_k-1}}{\alpha_k}$$

$[\beta = \nu_1/\alpha_1 + \nu_2/\alpha_2 + \dots + \nu_n/\alpha_n].$

$$8. \int \dots \int_{\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \dots + \left(\frac{x_n}{a_n}\right)^2 \leq 1} \left(1 - \sum_{k=1}^n a_k^2 x_k^2\right)^k f(\theta x) dx =$$

$$= \left(\prod_{k=1}^n a_k\right) \frac{\pi^{(n-1)/2} \Gamma(k+1)}{\Gamma\left(k + \frac{n+1}{2}\right)} \int_0^1 (1-t)^{k+(n-k)/2} f(ct) dt$$

$$\left[c = \sqrt{\sum_{k=1}^n a_k^2 x_k^2} \right]$$

3.3.4. Интегралы по области $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, x_1 + x_2 + \dots + x_n \leq a$.

$$1. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} F_1(x_1) F_2(x_2) \dots F_n(x_n) dx =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it+iy}}{t+iy} \prod_{k=1}^n \int_0^{\infty} e^{-t(x_k+iy)} F_k(x_k) dx_k$$

[$a > 0; t > 0$ — произвольный параметр].

$$2. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} F(x_1 + x_2 + \dots + x_n) e^{ax} \prod_{k=1}^n x_k^{b-1} dx =$$

$$= \frac{1}{2\pi} \prod_{k=1}^n \Gamma(v_k) \int_0^a f(y) dy \int_{-\infty}^{\infty} \frac{e^{y(t+ia)}}{\prod_{k=1}^n (t-a_k+iy)^{v_k}} du$$

[$a > 0; t > 0$ — произвольный параметр].

$$3. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \frac{f(x_1 + x_2 + \dots + x_n)}{(ax+b)^{v_1+v_2+\dots+v_n}} \prod_{k=1}^n x_k^{v_k-1} dx =$$

$$= \Gamma\left[\frac{v_1}{v_1+v_2+\dots+v_n}, \dots, \frac{v_n}{v_1+v_2+\dots+v_n}\right] \int_0^1 f(t) t^{v_1+v_2+\dots+v_n-1} \prod_{k=1}^n (b+a_k t)^{-v_k} dt$$

[$a_k \geq 0, k=1, 2, \dots, n, b > 0$].

$$4. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} (1-x_1)^{\mu_1-1} (1-x_1-x_2)^{\mu_2-1} \dots$$

$$\dots (1-x_1-x_2-\dots-x_{n-1})^{\mu_n-1} f_1(1-x_1) \prod_{k=2}^n x_k^{v_k-1} f_k\left(\frac{1-x_1-x_2-\dots-x_{k-1}}{1-x_1-x_2-\dots-x_{k-1}}\right) dx =$$

$$= \prod_{k=1}^n \int_0^1 t_k^{v_k-1} (1-t_k)^{\mu_k} F_k(1-t_k) dt_k$$

[$\mu_k = \mu_k + \nu_{k+1} + \dots + \nu_n, k=1, 2, \dots, n-1; \mu_n = \mu_n$].

$$5. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq a}} \left(\prod_{k=1}^n x_k^{v_k-1}\right) F(x_1 + x_2 + \dots + x_n) \times$$

$$\times f_1\left(\frac{x_2+x_3+\dots+x_n}{x_1+x_2+\dots+x_n}\right) F_2\left(\frac{x_3+x_4+\dots+x_n}{x_2+x_3+\dots+x_n}\right) \dots F_{n-1}\left(\frac{x_n}{x_{n-1}+x_n}\right) dx =$$

$$= \int_0^a t_1^{v_1-1} F(t_1) dt_1 \prod_{k=2}^n \int_0^1 t_k^{v_k-1} (1-t_k)^{v_{k-1}-1} F_{k-1}(t_k) dt_k$$

[при условии, что интеграл в правой части равенства сходится; $v_k = v_k + v_{k+1} + \dots + v_n$, $1 \leq k \leq n-1$].

$$6. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_n}{a_n} \leq 1}} dx = \frac{a_1 a_2 \dots a_n}{n!} \quad [a_i > 0, i=1, 2, \dots, n].$$

$$7. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} (ax)^m dx = \frac{m!}{(m+n)!} \sum_{k=1}^n \frac{a_k^{m+n-1}}{\varphi'(a_k)}$$

[$\varphi(x) = (x-a_1)(x-a_2)\dots(x-a_n)$].

$$8. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \prod_{k=1}^n x_k^{\alpha_k} dx =$$

$$= \prod_{k=1}^n \Gamma\left[\frac{\sum_{m=1}^k \lambda_m + \sum_{h=k+1}^n \alpha_h + k}{\sum_{m=1}^k \lambda_m + \sum_{m=1}^n \alpha_h + k+1}\right]$$

[$x_k = \frac{b}{m+1}, x_m: \alpha_k, \lambda_k + \alpha_k > -1, k=1, 2, \dots, n$].

$$9. \int \dots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \prod_{k=1}^n (a_1^{(k)} x_1 + a_2^{(k)} x_2 + \dots + a_n^{(k)} x_n)^{m_k} x_k^{v_k-1} dx =$$

$$= \Gamma\left[\frac{v_1, v_2, \dots, v_n}{\sum_{k=1}^n m_k + \sum_{k=1}^n v_k + 1}\right] D_1^{m_1} D_2^{m_2} \dots D_n^{m_n} \prod_{i=1}^n (1-Y_i)^{-v_i} \Big|_{Y_1=Y_2=\dots=Y_n=0}$$

[$Y_i = a_i^{(1)} y_1 + a_i^{(2)} y_2 + \dots + a_i^{(n)} y_n, i=1, 2, \dots, n; v_j > 0, j=1, 2, \dots, n$].

$$10. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{1}{(ax)^{\mu}} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= \Gamma \left[\frac{\nu_1 + \nu_2 + \dots + \nu_n}{\nu_1 + \nu_2 + \dots + \nu_n - \mu + 1; \mu} \right] \int_0^{\infty} t^{\mu-1} \prod_{k=1}^n (1 + at_k)^{-\nu_k} dt$$

$$[a_k, \nu_k, \mu, \nu_1 + \nu_2 + \dots + \nu_n - \mu + 1 > 0].$$

$$11. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq a} \frac{(bx)^m}{(c_0 + cx)^{\nu_1 + \nu_2 + \dots + \nu_n + m + 1}} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= \frac{1}{c_0} \Gamma \left[\frac{\nu_1 + \nu_2 + \dots + \nu_n}{\nu_1 + \nu_2 + \dots + \nu_n + m + 1} \right] \frac{d^m}{dt^m} \prod_{k=1}^n \left(\frac{c_0 + c_k - b_k t}{a} \right)^{-\nu_k} \Big|_{t=0}.$$

$$12. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{ax \leq 1} \frac{dx}{x_1 x_2 \dots x_n (1 - ax)} = \frac{\pi^{(n-1)/2}}{\Gamma \left(\frac{n+1}{2} \right)} \prod_{k=1}^n a_k^{1/2}$$

$$[a_k > 0, k=1, 2, \dots, n].$$

$$13. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{(1 - x_1 - x_2 - \dots - x_n)^{\nu_0 - 1}}{(ax + b)^{\mu}} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= \Gamma \left[\frac{\nu_0 + \nu_1 + \dots + \nu_n}{\nu_0 + \nu_1 + \dots + \nu_n - \mu; \mu} \right] \int_0^{\infty} t^{\mu-1} (1 + bt)^{-\nu_0} \prod_{k=1}^n (1 + (b + a_k)t)^{-\nu_k} dt$$

$$[a_k, b, \nu_0, \nu_k, \mu > 0; \nu_0 + \nu_1 + \dots + \nu_n > \mu; k=1, 2, \dots, n].$$

$$14. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{1}{(a_k + b)^{\nu_k}} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= \Gamma \left[\frac{\nu_0 + \nu_1 + \dots + \nu_n}{\nu_0 + \nu_1 + \dots + \nu_n - \mu; \mu} \right] \prod_{k=1}^n (a_k + b)^{-\nu_k}$$

$$[a_k, b, \nu_0, \nu_k, \mu > 0; \nu_0 + \nu_1 + \dots + \nu_n = \mu; k=1, 2, \dots, n].$$

$$15. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} \frac{1}{(a_0 + ax)^{\mu}} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= (-1)^{\mu} \Gamma \left[\frac{\nu_0 + \nu_1 + \dots + \nu_n}{\nu_0 + \nu_1 + \dots + \nu_n - \mu + m; \mu} \right] \times$$

$$\times \int_0^{\infty} \frac{d^m}{dt^m} \left\{ t^{\mu-m-1} (1 + bt)^{-\nu_0} \prod_{k=1}^n (1 + (a_k + b)t)^{-\nu_k} \right\} dt$$

$$[a_k, b, \nu_0, \nu_k, \mu > 0; \nu_0 + \nu_1 + \dots + \nu_n < \mu; k=1, 2, \dots, n].$$

$$16. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} \left(\prod_{k=1}^n x_k^{\nu_k - 1} \right) e^{-\rho(x_1 + x_2 + \dots + x_n)} dx =$$

$$= \Gamma \left[\frac{\nu_1 + \nu_2 + \dots + \nu_n}{\nu_1 + \nu_2 + \dots + \nu_n - 1; 1} \right] \int_0^1 t^{\nu_1 + \nu_2 + \dots + \nu_n - 1} e^{-\rho t} dt \quad [\nu_k > 0, k=1, 2, \dots, n].$$

$$17. \int_{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0} \int_{x_1 + x_2 + \dots + x_n \leq 1} e^{ax} dx = \sum_{k=1}^n \frac{1}{a_k \Gamma(a_k)} (e^{a_k} - 1)$$

$$[\varphi(x) = (x - a_1)(x - a_2) \dots (x - a_n)].$$

3.3.5. Интегралы вида $\int_a^b \dots \int_a^b f(x) dx$.

$$1. \int_0^{\infty} \dots \int_0^{\infty} f(x_1 + x_2 + \dots + x_n) g(x_1 x_2 \dots x_n) x_1^{1/n-2/n} x_2^{1/n-2/n} \dots x_n^{1/n-2/n} dx =$$

$$= \frac{\Gamma(n) \Gamma(n-1)^{1/2}}{\Gamma \left(\frac{n+1}{2} \right)} \int_0^{\infty} t^{n-1} g(t^n) dt \int_0^{\infty} y^{(n-3)/2} f(y + nt) dy.$$

$$2. \int_0^{\infty} \dots \int_0^{\infty} e^{-(x_1 + x_2 + \dots + x_n)} x_1^{1/n-2/n} \dots x_n^{1/n-2/n} f(x_1 x_2 \dots x_n) dx =$$

$$= \Gamma(n) \Gamma(n-1)^{1/2} \int_0^{\infty} t^{n-1} e^{-t^n} f(t^n) dt.$$

$$3. \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2) e^{ibx} dx =$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma \left(\frac{n+1}{2} \right) \sqrt{a_1 a_2 \dots a_n}} \int_0^{\infty} F(t^2) t^{n-1} dt \int_{-\infty}^{\infty} e^{ibt} (1 - t^2)^{(n-3)/2} dt$$

$$[a_k > 0, k=1, 2, \dots, n; t = b[a_1 + b^2/a_2 + \dots + b^2/a_n]].$$

$$4. \int_0^{\infty} \dots \int_0^{\infty} \frac{1}{(a_0 + ax)^{\mu}} e^{-bx} \prod_{k=1}^n x_k^{\nu_k - 1} dx =$$

$$= \Gamma \left[\frac{\nu_1 + \nu_2 + \dots + \nu_n}{\mu} \right] \int_0^{\infty} e^{-at} t^{\mu-1} \prod_{k=1}^n (b_k + at)^{-\nu_k} dt$$

$$[a_k, b_k, \nu_k, \mu > 0 \text{ и } \mu \geq 0 \text{ и } \mu > 0 \text{ и } a_0 > 0, \text{ либо } a_0 = 0 \text{ и } \nu_1 + \nu_2 + \dots + \nu_n > \mu].$$

$$5. \int_0^{\infty} \dots \int_0^{\infty} \frac{1}{(a_0 + ax)^{\mu}} \prod_{k=1}^n x_k^{\nu_k-1} dx = \\ = \Gamma \left[\begin{matrix} \nu_1, \nu_2, \dots, \nu_n, \mu - \nu_1 - \nu_2 - \dots - \nu_n \\ \mu \end{matrix} \right] a_0^{-\mu} \prod_{k=1}^n \left(\frac{a_k}{a_0} \right)^{\nu_k} \\ [a_k, \mu, \nu_k > 0, k=1, 2, \dots, n].$$

$$6. \int_0^{\infty} \dots \int_0^{\infty} \frac{1}{[1 + (a_1 x_1)^{\mu_1} + (a_2 x_2)^{\mu_2} + \dots + (a_n x_n)^{\mu_n}]^{\nu}} \prod_{k=1}^n x_k^{\nu_k-1} dx = \\ = \Gamma \left[\begin{matrix} \nu_1/\mu_1, \nu_2/\mu_2, \dots, \nu_n/\mu_n, \nu - \nu_1/\mu_1 - \nu_2/\mu_2 - \dots - \nu_n/\mu_n \\ \nu \end{matrix} \right] \prod_{k=1}^n \mu_k^{-1-\mu_k/\nu} a_k^{-\mu_k/\nu} \\ [\nu, a_k, \mu_k, \nu_k > 0, k=1, 2, \dots, n].$$

$$7. \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dx}{[p^2 + (x-a)^2]^{\mu} [q^2 + (x-b)^2]^{\nu}} = \\ = \frac{\pi^{n/2} \Gamma(\mu + \nu - n/2)}{\Gamma(\mu) \Gamma(\nu)} \int_0^1 \frac{t^{\mu-1} (1-t)^{\nu-1} dt}{[p^2 t + q^2 (1-t) + t(1-t)(a-b)^2]^{\mu+\nu-n/2}} \\ [\operatorname{Re} \mu, \operatorname{Re} \nu > 0].$$

$$8. \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dx}{[p^2 + (x-a)^2]^{(n+1)/2} [q^2 + (x-b)^2]^{(n-1)/2}} = \\ = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot \frac{1}{p[(p+q)^2 + (a-b)^2]^{(n-1)/2}}.$$

$$9. \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-Q(x_1, x_2, \dots, x_n) + 2ax + c} dx = \left(\frac{\pi^n}{|A|} \right)^{1/2} e^{-B/A} \\ \left[Q = \sum_{i,j=1}^n a_{ij} x_i x_j - \text{положительно определенная квадратичная форма; } a_{ij} = a_{ji}, A = \\ = \det Q = \det (a_{ij}), B = \det \left(\frac{a_{ij}}{b_{ij}} \right) \right].$$

$$10. \int_0^{\infty} \dots \int_0^{\infty} e^{-\left(x_1 + x_2 + \dots + x_n + \frac{x_1^2 + \dots + x_n^2}{x_1 x_2 \dots x_n} \right)} \frac{1}{x_1} \dots \frac{1}{x_n} = \\ = \frac{1}{\sqrt{n+1}} (2\pi)^{n/2} e^{-(n+1)a} \quad (a > 0).$$

$$11. \int_0^1 \dots \int_0^1 f(x_1, x_2, \dots, x_n) (1-x_1)^{\nu_1-1} \dots (1-x_n)^{\nu_n-1} dx = \\ = \Gamma \left[\begin{matrix} \nu_1, \nu_2, \dots, \nu_n \\ \nu_1 + \nu_2 + \dots + \nu_n \end{matrix} \right] \int_0^1 f(t) (1-t)^{\nu_1 + \nu_2 + \dots + \nu_n - 1} dt \\ [\text{при условии, что интеграл в правой части сходится абсолютно}].$$

$$12. \int_0^1 \dots \int_0^1 (x_1 x_2 \dots x_n)^{z-1} \prod_{k=1}^n (1-x_k)^{\beta_k-1} \prod_{i < j}^n (x_j - x_i)^{2\alpha} dx = \\ = \prod_{k=1}^n \Gamma \left[\begin{matrix} 1 + kz, \alpha + (k-1)z, \beta + (k-1)z \\ 1 + z, \alpha + \beta + (n+k-2)z \end{matrix} \right] \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta > 0; \operatorname{Re} z > \max(-1/n, -\operatorname{Re} \alpha/(n-1), -\operatorname{Re} \beta/(n-1))].$$

$$13. \int_0^1 \dots \int_0^1 (x_1 + x_2 + \dots + x_n)^z dx = \frac{\pi(3n+1)}{12}.$$

$$14. \int_0^1 \dots \int_0^1 \prod_{k=1}^n x_k^{\alpha_k-1} (1-x_k)^{\beta_k-1} (1-x_1 x_2 \dots x_n)^{-\nu} dx = \\ = \Gamma \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1 - \alpha_1, \beta_2 - \alpha_2, \dots, \beta_n - \alpha_n \\ \beta_1, \beta_2, \dots, \beta_n \end{matrix} \right] \times \\ \times {}_{n+1}F_n(\nu, \alpha_1, \alpha_2, \dots, \alpha_n; \beta_1, \beta_2, \dots, \beta_n; z) \\ [\operatorname{Re} \beta_k > \operatorname{Re} \alpha_k > 0; |\arg(1-z)| < \pi].$$

$$15. \int_0^1 \dots \int_0^1 \frac{x_1^{n-1} x_2^{n-2} \dots x_{n-1} dx}{\left(1 + \sum_{k=1}^n x_1^2 x_2^2 \dots x_k^2 \right)^{(n+1)/2}} = \frac{\pi^{(n+1)/2}}{n! 2^{n+1} \Gamma(n/2 + 3/2)}.$$

$$16. \int_0^1 \dots \int_0^1 \frac{x_1^{n-1} x_2^{n-2} \dots x_{n-1} dx}{\left[\frac{1}{(n+1)(n+2)} + \sum_{k=1}^n \frac{x_1^2 x_2^2 \dots x_k^2}{(n-k+1)(n-k+2)} \right]^{(n+1)/2}} = \\ = \frac{(n+1)\pi^{(n+1)/2}}{\sqrt{n+2} \Gamma(n/2 + 3/2)}.$$

Глава 4. КОНЕЧНЫЕ СУММЫ

4.1. ВВЕДЕНИЕ

В этой главе помещены некоторые конечные суммы, содержащие различные элементарные функции. Ответим, что для вычисления конечной суммы $\sum_{k=0}^n a_k$ можно ввести вспомогательную сумму $\sum_{k=0}^n a_k x^k$ (или привести данную сумму к такой форме), записать ее в виде обобщенного гипергеометрического ряда и получить ответ в виде обобщенной гипергеометрической функции. Иногда может оказаться полезной следующая формула замены индекса суммирования:

$$\sum_{k=n_1}^{n_2} a_k = \sum_{k=n-n_2}^{n-n_1} a_{n-k}.$$

в частности,

$$\sum_{k=0}^n a_k = \sum_{k=0}^n a_{n-k}.$$

Всюду в этой главе $l, m, n = 0, 1, 2, 3, \dots$, если не указаны другие условия; кроме того, по определению $\sum_{k=n_1}^{n_2} a_k = 0$ при $n_2 < n_1$.

4.1.1. Суммы вида $\sum_{k=0}^n (\pm 1)^k P(k)$.

1. $\sum_{k=0}^n k^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k (n+1)^{m-k+1}.$
2. $= \sum_{k=1}^m (-1)^k \binom{n+1}{k+1} \sum_{l=1}^k (-1)^l \binom{k}{l} l^m.$
3. $= \frac{1}{m+1} [B_{m+1}(n+1) - B_{m+1}] \quad [m = 1, 2, 3, \dots]$
4. $\sum_{k=1}^n (-1)^{k+1} k^m = \sum_{k=1}^m (-1)^{k+1} \left[\frac{(n+1)}{(k+1)} - 2^{m+1} \frac{(n/2+1)}{(k+1)} \right] \sum_{l=0}^k (-1)^l \binom{k}{l} l^m.$
5. $= \frac{1}{2} (-1)^{n+1} [E_m(n+1) + (-1)^n E_m(0)] \quad [m = 1, 2, 3, \dots]$

4.1.1

6. $\sum_{k=1}^n k = \frac{n(n+1)}{2}.$
7. $\sum_{k=1}^n (-1)^k k = (-1)^n \left[\frac{n+1}{2} \right] \quad \left[\text{здесь } \left[\frac{n+1}{2} \right] - \text{целая часть } (n+1)/2 \right].$
8. $\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1).$
9. $\sum_{k=1}^n (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}.$
10. $\sum_{k=1}^n k^3 = \frac{1}{4} n^2 (n+1)^2.$
11. $\sum_{k=1}^n (-1)^k k^3 = \frac{1}{8} [1 - (-1)^n (1 - 6n^2 + 4n^3)].$
12. $\sum_{k=1}^n k^4 = \frac{1}{30} (n+1)(2n+1)(3n^2 + 2n - 1).$
13. $\sum_{k=1}^n (-1)^k k^4 = (-1)^n \frac{n^4 + 2n^3 - n}{2}.$
14. $\sum_{k=1}^n k^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1).$
15. $\sum_{k=1}^n (-1)^{k+1} k^5 = \frac{1}{4} [1 + (-1)^n (5n^2 - 5n^4 - 2n^6 - 1)].$
16. $\sum_{k=0}^n (k+a)^m = \frac{1}{m+1} [B_{m+1}(n+1+a) - B_{m+1}(a)].$
17. $\sum_{k=0}^n (-1)^k (k+a)^m = \frac{1}{2} \{ (-1)^n E_m(a+n+1) + E_m(a) \} - (-1)^n a^m.$
18. $\sum_{k=0}^n (\pm 1)^k (2k+1)^m =$
 $= \sum_{k=1}^m \frac{(-1)^k}{k!} \sum_{l=1}^k (-1)^l \binom{k}{l} l^m \left[a^k \frac{d^k}{da^k} \sum_{j=0}^n (\pm 1)^j a^{2j+1} \right]_{a=a}.$
19. $\sum_{k=0}^n (2k+1) = (n+1)^2.$
20. $\sum_{k=0}^n (-1)^k (2k+1) = (-1)^n (n+1).$

$$21. \sum_{k=0}^n (2k+1)^2 = \frac{1}{3} (n+1)(2n+1)(2n+3).$$

$$22. \sum_{k=0}^n (-1)^k (2k+1)^2 = (-1)^n 2(n+1)^2 - \frac{1+(-1)^n}{2}.$$

$$23. \sum_{k=0}^n (2k+1)^2 = (n+1)^2 (2n^2 + 4n + 1).$$

$$24. \sum_{k=0}^n (k+a)^l (k+b)^m = \\ = (a-b)^l \sum_{k=0}^n \frac{(-1)^k}{k! (k+m+1)!} (b-a)^{-k} [B_{k+m+1}(n+b+1) - B_{k+m+1}(b)].$$

$$25. \sum_{k=1}^n (k+a)(k+b) = \frac{1}{6} n(n+1)[2n+1+3(a+b)] + nab.$$

$$26. \sum_{k=1}^n k(k+1)\dots(k+m) = \frac{1}{m+2} n(n+1)\dots(n+m+1).$$

$$27. \sum_{k=1}^n k(k+1) = \frac{1}{3} n(n+1)(n+2).$$

$$28. \sum_{k=1}^n k(k+1)(k+2) = \frac{1}{4} n(n+1)(n+2)(n+3).$$

$$29. \sum_{k=1}^n k(k+3)(k+6) = \frac{1}{4} n(n+1)(n+6)(n+7).$$

$$30. \sum_{k=1}^n k(k+4)(k+8) = \frac{1}{4} n(n+1)(n+8)(n+9).$$

$$31. \sum_{k=0}^n (2k+1)(2k+3)(2k+5) = \frac{15}{8} + \frac{1}{8} (2n+1)(2n+3)(2n+5)(2n+7).$$

$$32. \sum_{k=2}^n (3k+1)(3k+4)(3k+7) = \frac{14}{3} + \frac{1}{12} (3n+1)(3n+4)(3n+7)(3n+10).$$

$$33. \sum_{k=1}^n k! = (n+1)! - 1.$$

$$4.1.2. \sum_{k=0}^n k! (n-k)! = S_n, \quad S_1 = 2, \quad S_n = \frac{n+1}{2} S_{n-1} + n!.$$

$$4.1.2. \text{Суммы в виде } \sum \frac{(k+1)^k}{k^2}.$$

$$1. \sum_{k=1}^n \frac{1}{k^2} = \zeta(s) - \zeta(s, n+1).$$

$$2. \sum_{k=1}^n \frac{1}{k^m} = \frac{(-1)^m}{(m-1)!} [\Psi^{(m-1)}(1) - \Psi^{(m-1)}(n+1)] \quad [m=1, 2, 3, \dots]$$

$$3. \sum_{k=1}^n \frac{1}{k} = C + \psi(n+1) = S_n,$$

$$S_1 = 1, \quad S_2 = \frac{3}{2}, \quad S_3 = \frac{11}{6}, \quad S_4 = \frac{25}{12}, \quad S_5 = \frac{137}{60}, \quad S_6 = \frac{147}{60}, \\ S_7 = \frac{1083}{420}, \quad S_8 = \frac{2283}{840}, \quad S_9 = \frac{7129}{2520}, \quad S_{10} = \frac{7331}{2520}.$$

$$4. \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - \Psi'(n+1) = T_n,$$

$$T_1 = 1, \quad T_2 = \frac{5}{4}, \quad T_3 = \frac{49}{36}, \quad T_4 = \frac{205}{144}, \quad T_5 = \frac{5269}{3600}.$$

$$5. \sum_{k=1}^n \frac{1}{k^2} = \zeta(3) + \frac{1}{2} \Psi''(n+1).$$

$$6. \sum_{k=1}^n \frac{(-1)^k}{k^2} = (2^{1-s} - 1) \zeta(s) - 2^{1-s} \zeta\left(s, \frac{n}{2} + 1\right) + \zeta(s, n+1).$$

$$7. \sum_{k=1}^n \frac{(-1)^k}{k^m} = \frac{1}{(m-1)!} (-1)^m [\beta^{(m-1)}(1) - (-1)^n \beta^{(m-1)}(n+1)].$$

$$8. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} = \ln 2 - (-1)^n \beta(n+1) = U_n,$$

$$U_1 = 1, \quad U_2 = \frac{1}{2}, \quad U_3 = \frac{5}{6}, \quad U_4 = \frac{7}{12}, \quad U_5 = \frac{47}{60}, \quad U_6 = \frac{37}{60}, \\ U_7 = \frac{319}{420}, \quad U_8 = \frac{533}{840}, \quad U_9 = \frac{1879}{2520}, \quad U_{10} = \frac{1627}{2520}.$$

$$9. \sum_{k=1}^n \frac{(-1)^{k+1}}{k^3} = \frac{\pi^2}{12} - (-1)^n \beta'(n+1) = V_n,$$

$$V_1 = 1, \quad V_2 = \frac{3}{4}, \quad V_3 = \frac{31}{36}, \quad V_4 = \frac{115}{144}, \quad V_5 = \frac{3019}{3600}.$$

$$16. \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=1}^n \frac{1}{k+n}.$$

4.1.3. Суммы вида $\sum_{k=0}^n \frac{(\pm 1)^k}{(k+a)^2}$.

$$1. \sum_{k=0}^n \frac{1}{(k+a)^2} = \zeta(2, a) - \zeta(2, n+a+1).$$

$$2. \sum_{k=0}^n \frac{(\pm 1)^k}{k+a} = \pm \left\{ \frac{\Psi(a)}{\beta(a)} + (\pm 1)^n \frac{\Psi(n+a+1)}{\beta(n+a+1)} \right\}.$$

$$3. \sum_{k=0}^n \frac{1}{2k+1} = \frac{1}{2} \left[\Psi\left(n+\frac{3}{2}\right) + C \right] + \ln 2.$$

$$4. \sum_{k=0}^n \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + \frac{1}{2} (-1)^n \beta\left(n+\frac{3}{2}\right).$$

$$5. \sum_{k=0}^n \frac{(\pm 1)^k}{(k+a)^2} = \pm \left\{ \frac{\Psi'(a)}{\beta'(a)} - (\pm 1)^n \frac{\Psi'(n+a+1)}{\beta'(n+a+1)} \right\}.$$

$$6. \sum_{k=0}^{n-1} \frac{1}{(2k+1)^2} = \frac{1}{8} \left[\pi^2 - 2\Psi'\left(\frac{1}{2} \pm n\right) \right].$$

4.1.4. Суммы вида $\sum_{k=0}^n \frac{(\pm 1)^k}{k(k+a)(k+b)}$.

$$1. \sum_{k=1}^n \frac{(\pm 1)^k}{k(k+a)} = \frac{1}{a} \left[\left\{ \frac{C}{\ln 2} \right\} + \frac{\Psi(a+1)}{\beta(a+1)} - (\pm 1)^n \frac{\Psi(n+a+1) - \Psi(n+1)}{\beta(n+a+1) - \beta(n+1)} \right].$$

$$2. \sum_{k=1}^n \frac{1}{k(k+m)} = \frac{1}{m} \sum_{k=1}^m \frac{1}{k(k+n)}.$$

$$3. \sum_{k=1}^n \frac{(-1)^k}{k(k+2m)} = \frac{1}{2m} \sum_{k=1}^{2m} (-1)^k \left[\frac{1}{k} - \frac{(-1)^n}{k+n} \right].$$

$$4. \sum_{k=1}^n \frac{1}{(k+m)(k+l)} = \frac{n}{m-l} \sum_{k=1}^{m-l} \frac{1}{(k+l)(k+l+n)}.$$

$$5. \sum_{k=0}^n \frac{1}{(ka+b)(ka+a+b)} = \frac{1}{ab} - \frac{1}{a(na+a+b)}.$$

$$6. \sum_{k=0}^n \frac{1}{(ka+b)(ka+ma+b)} = \frac{1}{ma} \left(\sum_{k=0}^n \frac{1}{ka+b} - \sum_{k=0}^{n+m} \frac{1}{ka+b} \right).$$

$$7. \sum_{k=0}^n \frac{1}{(k+a)(k+b)} = \frac{1}{b-a} [\Psi(n+a+1) - \Psi(a) - \Psi(n+b+1) + \Psi(b)].$$

$$8. \sum_{k=0}^n \frac{(-1)^k}{(k+a)(k+b)} = \frac{1}{b-a} [\beta(a) - \beta(b) + (-1)^n [\beta(n+a+1) - \beta(n+b+1)]].$$

$$9. \sum_{k=0}^n \frac{1}{k^2-a^2} = \frac{1}{2a} [\Psi(n-a+1) - \Psi(n+a+1) + \Psi(a) - \Psi(-a)].$$

$$10. \sum_{k=1}^n \frac{1}{k^2-1} = \frac{n}{2n+1}.$$

$$11. \sum_{k=1}^n \frac{(-1)^k}{k^2-1} = \frac{(-1)^n}{4n} [\beta(n-a+1) - \beta(n+a+1)] - \frac{\pi}{2 \ln 2} - \frac{1}{2n^2}.$$

$$12. \sum_{k=1}^n \frac{(-1)^k}{(k+2m)(k+2l)} = \frac{1}{2(m-l)} \sum_{k=1}^{2(m-l)} (-1)^k \left(\frac{1}{k+2l} - \frac{(-1)^n}{k+2l+n} \right).$$

4.1.5. Суммы вида $\sum_{k=0}^n \frac{(\pm 1)^k}{(k+a)(k+b)(k+c)}$.

$$1. \sum_{k=0}^n \frac{(\pm 1)^{k+1}}{(k+a)(k+b)(k+c)} = \frac{1}{(b-a)(c-a)} \left[(\pm 1)^n \frac{\Psi(n+a+1)}{\beta(n+a+1)} - \frac{\Psi(a)}{\beta(a)} + \frac{1}{(a-b)(c-b)} \left[(\pm 1)^{n+1} \frac{\Psi(n+b+1)}{\beta(n+b+1)} - \frac{\Psi(b)}{\beta(b)} \right] + \frac{1}{(c-a)(c-b)} \left[(\pm 1)^{n+1} \frac{\Psi(n+c)}{\beta(n+c)} - \frac{\Psi(c)}{\beta(c)} \right] \right].$$

$$2. \sum_{k=1}^n \frac{1}{k(k+m)(k+l)} = \frac{n}{m-l} \left[\frac{1}{l} \sum_{k=1}^l \frac{1}{k(k+n)} - \frac{1}{m} \sum_{k=1}^m \frac{1}{k(k+n)} \right].$$

$$3. \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

$$4. \sum_{k=1}^n \frac{(-1)^k}{k(k+2m)(k+2l)} = \frac{1}{4(m-l)} \left\{ \frac{1}{l} \sum_{k=1}^{2l} (-1)^k \left[\frac{1}{k} - \frac{(-1)^n}{k+n} \right] - \right. \\ \left. - \frac{1}{m} \sum_{k=1}^{2m} (-1)^k \left[\frac{1}{k} - \frac{(-1)^n}{k+n} \right] \right\} \quad (m \neq l)$$

$$5. \sum_{k=0}^n \frac{1}{(mk+l)(mk+m+l)(mk+2m+l)} = \\ = \frac{1}{2m} \left[\frac{1}{l(m+l)} - \frac{1}{(mn+m+l)(mn+2m+l)} \right]$$

$$6. \sum_{k=0}^n \frac{1}{(k+a)(k+a+1)(k+a+2)} = \frac{1}{2a(a+1)} - \frac{1}{2(n+a+1)(n+a+2)}$$

$$7. \sum_{k=0}^n \frac{1}{(k+1)(2k+1)(2k+3)} = 2 \sum_{k=1}^n \frac{1}{k+n+1} - \frac{2n+2}{2n+3}$$

$$8. \sum_{k=0}^n \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{12} - \frac{1}{4(2n+3)(2n+5)}$$

$$9. \sum_{k=0}^n \frac{1}{(2k+1)(2k+3)(2k+7)} = \frac{11}{180} - \frac{6n+17}{12(2n+3)(2n+5)(2n+7)}$$

4.1.6. Суммы вида $\sum \frac{P(k)}{Q(k)}$ и другие.

$$1. \sum_{k=1}^n \frac{k}{(k+m)!} m^k = \frac{1}{(m-1)!} - \frac{m^{n+1}}{(m+n)!}$$

$$2. \sum_{k=1}^n \frac{1}{k(k+1) \dots (k+m)} = \frac{1}{m!} \left[\frac{1}{m!} - \frac{1}{(n+1)(n+2) \dots (n+m)} \right]$$

$$3. \sum_{k=0}^n \frac{1}{(k+a)(k+a+1) \dots (k+a+m)} = \\ = \frac{1}{m!} \left[\frac{1}{a(a+1)(a+2) \dots (a+m-1)} - \frac{1}{(n+a+1)(n+a+2) \dots (n+a+m)} \right]$$

$$4. \sum_{k=1}^n \frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{a-b+1} \left[\frac{\Gamma(n+a+1)}{\Gamma(n+b)} - \frac{\Gamma(a)}{\Gamma(b-1)} \right]$$

$$5. \sum_{k=0}^n \frac{\Gamma(a-k)}{\Gamma(b-k)} = \frac{1}{b-a-1} \left[\frac{\Gamma(a-n)}{\Gamma(b-n-1)} - \frac{\Gamma(a+1)}{\Gamma(b)} \right]$$

$$6. \sum_{k=0}^n \frac{1}{k!(k+2)} = 1 - \frac{1}{(n+2)!}$$

$$7. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \sum_{m=1}^k \frac{1}{m} = - \int_0^1 \frac{1-(1-t)^n}{1-t} \ln(1-t) dt$$

$$8. \sum_{k=1}^n \frac{k!}{(2k+1)!} 2^k = \frac{1}{2} - \frac{n!}{(2n+1)!} 2^{n-1}$$

$$9. \sum_{k=1}^n (-1)^{n+k} \frac{(n+k-1)!}{(n-k)!(k!)^2} \sum_{m=1}^{n+k-1} \frac{1}{m} = \frac{1}{n^2}$$

$$10. \sum_{k=0}^n \frac{2^k}{1+a^{2k}} \left\{ \frac{1}{a^{2k}} \right\} = \frac{1}{a-1} \left\{ \frac{1}{-a} \right\} - \frac{2^n}{a^{2n}-1} \left\{ \frac{1}{a^{2n}} \right\}$$

$$11. \sum_{k=0}^{n-1} \frac{1}{1-a^{2k+1}} a^{2k} = \frac{1}{1-a} - \frac{1}{1-a^{2n}}$$

4.1.7. Суммы вида $\sum a_k x^k$.

$$1. \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x} \quad 2. \sum_{k=1}^n kx^k = (1-x)^{-2} [x - (nx-n-1)x^{n+1}]$$

$$3. \sum_{k=1}^n k^2 x^k = (1-x)^{-3} [x(1+x) - x^{n+1} \{(n+1)^2 - (2n^2+2n-1)x + n^2 x^2\}]$$

$$4. \sum_{k=1}^n k^3 x^k = x(1-x)^{-4} [x^2 + 4x + 1 - (n+1)^3 x^n + \\ + (3n^3 + 6n^2 - 4)x^{n+1} - (3n^3 + 3n^2 - 3n + 1)x^{n+2} + n^3 x^{n+3}]$$

$$5. \sum_{k=1}^n k(k+1)x^k = \\ = (1-x)^{-3} [2x - (n+1)(n+2)x^{n+1} + 2n(n+2)x^{n+2} - n(n+1)x^{n+3}]$$

$$6. \sum_{k=0}^n k! x^k = \int_0^\infty \frac{1-(xt)^{n+1}}{1-xt} e^{-t} dt$$

$$7. \sum_{k=0}^n \frac{x^k}{(k+a)!} = \Phi(x, s, a) - x^{n+1} \Phi(x, s, n+a+1)$$

$$8. \sum_{k=1}^n \frac{1}{k} x^k = -\ln(1-x) - x^{n+1} \int_0^1 t^n (1-xt)^{-n-1} dt$$

$$9. \sum_{k=0}^n \frac{(-1)^k}{2k+1} x^{2k+1} = \arctg x + (-1)^n \int_0^x \frac{t^{2n+2} dt}{t^2+1}.$$

$$10. \sum_{k=0}^n \frac{x^k}{k!} = \frac{1}{n!} e^x \Gamma(n+1, x).$$

$$11. \sum_{k=0}^n \frac{(-1)^k}{\Gamma(k+v)} x^k = \frac{n!}{\Gamma(n+v)} L_n^{v-1}(x).$$

$$12. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k! (n-2k)!} x^{2k} = \frac{x^n}{n!} H_n \left(\frac{x}{2x} \right).$$

$$13. \sum_{k=0}^n \frac{(-1)^k}{(2k+\delta)! (n-k)!} x^{2k+\delta} = \frac{(-1)^n}{(2n+\delta)!} H_{2n+\delta} \left(\frac{x}{2} \right) \quad (\delta=0 \text{ или } 1).$$

$$14. \sum_{k=0}^n \frac{\Gamma(n-k+1/2)}{k!} x^{2k} = \pi x^n + 1/2 [H_{n+1/2}(2x) - Y_{n+1/2}(2x)].$$

$$15. \sum_{k=0}^{[n/2]} \frac{(n-k-1)!}{k!} x^{2k} = \frac{1}{n} x^{n+1} Q_n(2x).$$

$$16. \sum_{k=0}^n \frac{k!}{(2k)!} x^k = 1 + \frac{\sqrt{\pi x}}{2} e^{x/2} \left[\frac{2^n}{\sqrt{\pi} (2n-1)!} \Gamma\left(n+\frac{1}{2}, \frac{x}{2}\right) - \operatorname{erfc}\left(\sqrt{\frac{x}{2}}\right) \right] \quad (x \geq 0).$$

$$17. \sum_{k=0}^{[n/2]} \frac{(2k-1)!}{(2n-2k+1)!} x^{2k} = \frac{1}{2} \pi x^n \left[E_{n+1} \left(\frac{1}{x} \right) + H_{n+1} \left(\frac{1}{x} \right) \right].$$

$$18. \sum_{k=0}^{[n/2]} \frac{[2n-2k-1]!}{(2k+1)!} x^{2k} = (-1)^n \frac{\pi}{2} x^n [E_{-n-1}(x) + H_{-n-1}(x)].$$

$$19. \sum_{k=0}^n \frac{(2n-k)!}{k! (n-k)!} x^k = \frac{1}{\pi} x^n + 1/2 e^{x/2} K_{n+1/2} \left(\frac{x}{2} \right).$$

$$20. \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(v+n-k)}{k! (n-2k)!} x^{2k} = \Gamma(v) x^n C_n^v \left(\frac{1}{2x} \right).$$

{4.1.7.}

* 4.1.8.

$$21. \sum_{k=1}^n (-1)^k \frac{(2n-k)!}{(n-k)! \Gamma\left(\frac{1}{2} - n + k\right)} x^{2k} = \frac{n!}{(2n)! \Gamma\left(\frac{1}{2} - n\right)} \left\{ J_{n+1/2}^2(x) - J_{n-1/2}^2(x) \right\}.$$

$$22. \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k! (n-2k)!} x^{2k} = \frac{2x^n}{n} T_n \left(\frac{1}{2x} \right).$$

$$23. \sum_{k=0}^n (-1)^k \frac{\Gamma\left(n+k+\frac{1}{2}+\delta\right)}{(2k+\delta)! (n-k)!} x^{2k+\delta} = (-1)^n 2^{2n+\delta} \frac{n!}{(2n+\delta)!} \Gamma\left(n+\frac{1}{2}+\delta\right) P_{2n+\delta} \left(\frac{x}{2} \right) \quad (n=0 \text{ или } 1).$$

$$24. \sum_{k=0}^n (-1)^k \frac{\Gamma(k+v)}{(2k+\delta)! (n-k)!} x^{2k+\delta} = (-1)^n \frac{\pi}{\sin \pi v \Gamma(n+1+\delta-v)} C_{2n+\delta}^{v-\delta} \left(\frac{x}{2} \right) \quad (n=0 \text{ или } 1).$$

$$25. \sum_{k=0}^n \frac{x^k}{(2k)! [(n-k)!]^2} = \frac{(x-4)^n}{(2n)!} P_{2n} \left(\sqrt{\frac{x}{x-4}} \right).$$

$$26. \sum_{k=0}^n (-1)^k \frac{(n+k-1)!}{(2k)! (n-k)!} x^k = \frac{1}{n} T_n \left(1 - \frac{x}{2} \right).$$

$$27. \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(2k+1)! (n-k)!} x^k = \frac{1}{n+1} U_{n+1} \left(1 - \frac{x}{2} \right).$$

4.1.8. Суммы вида $\sum_{k=0}^n (-1)^k \left[\frac{k}{n} \right]$.

Всюду в этом пункте [a] означает целую часть a.

$$1. \sum_{k=1}^{[n/2]} \left[\frac{k}{2} \right] = \frac{1}{4} \left[\frac{n}{2} \right]^2 + \frac{1}{8} \{1 - (-1)^{[n/2]}\} \{1 + 2(-1)^{[n/2]}\}.$$

$$2. \sum_{k=1}^{[n/2]} (-1)^k \left[\frac{k}{2} \right] = \frac{1}{4} \left[\frac{n}{2} \right] \{1 + (-1)^{[n/2]}\}.$$

$$3. \sum_{k=1}^{[n/2]} (-1)^k \left[\frac{k}{2} \right] = \frac{(-1)^{[n/2]}}{4} \{1 + (-1)^n\} \{1 - (-1)^{[n/2]}\} + \frac{(-1)^{[n/2]}}{4} \{1 - (-1)^n\} \{n + (-1)^{[n/2]}\}.$$

4. $\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \left[\frac{k}{2} \right] = \frac{1}{2} (-1)^{[(n-2)/4]} \{1 - (-1)^{[n/2]}\} \left[\frac{n+2}{4} \right] +$
 $+ \frac{1}{2} (-1)^{[(n-2)/4]} \{(-1)^{[(n+2)/4]} - 1\}.$
5. $\sum_{k=1}^n \left[\frac{k}{3} \right] = \frac{1}{6} (n-1)(n+4) - \left[\frac{2n-1}{3} \right] - \frac{1}{6} \{(-1)^{[2n/3]} + (-1)^n + [n/3]\}.$
6. $\sum_{k=1}^n (-1)^k \left[\frac{k}{3} \right] = \frac{1}{4} \{(-1)^{[n/3]} + 2(-1)^{[n/3]} \left[\frac{n}{3} \right] - 1\} -$
 $- (-1)^{[(n-1)/3]} \frac{1 - (-1)^n + (-1)^{[n/3]} \left[\frac{n-1}{3} \right]}{2}.$
7. $\sum_{k=0}^{n-1} \left[\varepsilon + \frac{mk}{n} \right] = [n\varepsilon] + \frac{1}{2} (m-1)(n-1) \quad [m, n - \text{взаимно простые числа}].$

4.2. БИНОМИАЛЬНЫЕ КОЭФФИЦИЕНТЫ

Обозначения: a, b — любые комплексные числа, при которых биномиальные коэффициенты существуют (относительно определения и свойств биномиальных коэффициентов см. приложение 1); $l, m, n = 0, 1, 2, \dots$

4.2.1. Суммы вида $\sum (\pm 1)^k \binom{a}{b_k}$.

1. $\sum_{k=0}^n \binom{n}{k} = 2^n, \quad 2. \sum_{k=0}^n (-1)^k \binom{n}{k} = \binom{n-a}{n} = \prod_{k=1}^n \left(1 - \frac{a}{k}\right).$
3. $\sum_{k=0}^n (-1)^k \binom{n}{k} = \delta_{0,n}, \quad 4. \sum_{k=0}^n (-1)^k \binom{m}{k} = (-1)^n \binom{m-1}{n}.$
5. $\sum_{k=m}^n (-1)^k \binom{a}{k} = (-1)^m \binom{a-1}{m-1} + (-1)^n \binom{a-1}{n}.$
6. $\sum_{k=0}^{[n/2]} \binom{n}{k} = 2^{n-1} + \frac{1 + (-1)^n}{4} \binom{n}{n/2}.$
7. $\sum_{k=0}^{[(n-1)/2]} \binom{n}{k} = 2^{n-1} - \frac{1 - (-1)^n}{4} \binom{n}{n/2}.$
8. $\sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^{n-1}, \quad 9. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} = 2^{n/2} \cos \frac{n\pi}{4}.$

10. $\sum_{k=0}^{[n/2]} \binom{2n}{2k} = 2^{2n-1} + \frac{1 + (-1)^n}{2} \binom{2n-1}{n}.$
11. $\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} = 2^{n-1} \quad [n \geq 1].$
12. $\sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} = 2^{n/2} \sin \frac{n\pi}{4}.$
13. $\sum_{k=0}^{[(n-1)/2]} \binom{2n}{2k+1} = 2^{2n-1} + \frac{1 - (-1)^n}{2} \binom{2n-1}{n} \quad [n \geq 1].$
14. $\sum_{k=0}^{[n/3]} \binom{n}{3k} = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right).$
15. $\sum_{k=0}^{[(n-1)/3]} \binom{n}{3k+1} = \frac{1}{3} \left[2^n + 2 \cos \frac{(n-2)\pi}{3} \right].$
16. $\sum_{k=0}^{[n/3]} \binom{n}{3k+2} = \frac{1}{3} \left[2^n + 2 \cos \frac{(n+2)\pi}{3} \right].$
17. $\sum_{k=0}^{[n/4]} \binom{n}{4k} = \frac{1}{4} \left[2^n + 2^{(n+2)/2} \cos \frac{n\pi}{4} \right].$
18. $\sum_{k=0}^{[n/m]} \binom{n}{mk} = \frac{2^n}{m} \sum_{k=1}^m \cos^m \frac{k\pi}{m} \cos \frac{nk\pi}{m}.$
19. $\sum_{k=0}^{[(n-m)/l]} \binom{n}{m+kl} = \frac{1}{l} \sum_{k=1}^l 2 \cos^m \frac{k\pi}{l} \cos \frac{(n-2m)k\pi}{l} \quad [n \geq m \geq 0, l-1 \geq m].$
20. $\sum_{k=0}^{m-n} (-1)^k \binom{m}{n+k} = \binom{m-1}{n-1}, \quad 21. \sum_{k=0}^n (-1)^k \binom{a}{n-k} = \binom{a-1}{n}.$
22. $\sum_{k=0}^{[n/2]} \binom{2n}{n+2k} = 2^{2n-1} + \binom{2n-1}{n}.$
23. $\sum_{k=0}^{[n/2]} \binom{n-k}{k} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.$

24. $\sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} = \frac{(-1)^{[n/2]} + (-1)^{[n/2]+1}}{2}$.
25. $\sum_{k=m}^n \binom{k-a}{k} = (-1)^m \binom{a-2}{m-1} + (-1)^n \binom{a-2}{n}$.
26. $\sum_{k=0}^n \binom{a-k}{b} = \binom{a+1}{b+1} - \binom{a-n}{b+1}$.
27. $\sum_{k=0}^n \binom{a+k}{m} = \binom{a+n+1}{m+1} - \binom{a}{m+1}$.
28. $\sum_{k=0}^n \binom{a+k}{a} = \binom{a+n+1}{n}$.
29. $\sum_{k=0}^n (-1)^k \binom{m+k}{l} = (-1)^n \binom{m}{l-n}$.
30. $\sum_{k=m}^n \binom{k}{a} = \binom{n+1}{a+1} - \binom{m}{a+1}$.
31. $\sum_{k=0}^n \binom{2k+1}{m} = \frac{(-1)^{m+1}}{2^{m+1}} \left[\sum_{k=0}^{m+1} (-1)^k \binom{2n+3}{k} 2^{k+1} \right]$.

4.2.2. Суммы вида $\sum a_k \binom{b_k}{c_k}$.

1. $\sum_{k=1}^n k^n \binom{n}{k} = \left(x \frac{d}{dx} \right)^n [(1+x)^n - 1] \Big|_{x=1}$.
2. $= 2^{n-m} \binom{n}{m} m! + 2 \sum_{k=1}^{m-1} (-1)^k \binom{n}{k} \frac{1}{2^k} \sum_{j=1}^k (-1)^j \binom{k}{j} j^m$.
3. $\sum_{k=0}^n (-1)^k k^m \binom{n}{k} = 0$ [m < n].
4. $\sum_{k=0}^n (-1)^k k^n \binom{n}{k} = (-1)^n n!$.
5. $\sum_{k=1}^n (-1)^k k^{n+1} \binom{n}{k} = \frac{(-1)^n n}{2} (n+1)!$.

6. $\sum_{k=1}^n \binom{n}{k} = 2^n - 1$. 7. $\sum_{k=1}^n k \binom{n}{k} = 2^{n-1} n$.
8. $\sum_{k=1}^n k^2 \binom{n}{k} = 2^{n-2} n (n+1)$. 9. $\sum_{k=1}^n k^3 \binom{n}{k} = 2^{n-3} n^2 (n+3)$.
10. $\sum_{k=1}^n k^4 \binom{n}{k} = 2^{n-4} n (n+1) (n^2 + 5n - 2)$.
11. $\sum_{k=1}^n (-1)^k k \binom{n}{k} = 0$ [n < 1].
12. $\sum_{k=1}^n k \binom{2n}{k} = n 2^{2n-1}$.
13. $\sum_{k=1}^n k \binom{2n+1}{k} = (2n+1) 2^{2n} - \frac{2n+1}{2} \binom{2n}{n}$.
14. $\sum_{k=1}^n k \binom{2n+1}{2k+1} = (2n+1) 2^{2n-1}$ [n > 1].
15. $\sum_{k=0}^n (-1)^k (k+a)^m \binom{n}{k} = 0$ [m = 0, 1, ..., n-1].
16. $\sum_{k=0}^n (-1)^k (k+a)^n \binom{n}{k} = (-1)^n n!$.
17. $\sum_{k=0}^n (-1)^k (k+a)^{n-m} \binom{n}{k} = 0$ [m = 1, 2, ..., n-1].
18. $\sum_{k=0}^n (-1)^k [(k+a)^{n+1} - (k+b)^{n+1}] \binom{n}{k} = (-1)^n (n+1)! (a-b)$.
19. $\sum_{k=0}^n (-1)^k (m+k)^{n+1} \binom{n}{k} = (-1)^n \left(m + \frac{n}{2} \right) (n+1)!$.
20. $\sum_{k=0}^n (-1)^k (m+k)^{n+2} \binom{n}{k} = (-1)^n \frac{(n+2)!}{24} [3a^2 + (12m+1)a + 12m^2]$.
21. $\sum_{k=0}^n (-1)^k (n-k)^{n+2} \binom{n}{k} = \frac{n^2 (n+1)^2 (n+2) (n+3)}{48} n!$.

$$22. \sum_{k=0}^n (-1)^k (k+a)^{n-1/2} \binom{n}{k} =$$

$$= \frac{1}{2\pi} (-1)^{n-1/2} \Gamma\left(n + \frac{1}{2}\right) \int_0^\pi \{\cos[(n+2a)t] - \sin[(n+2a)t]\} t^{n-1/2} \sin^n t dt$$

$$23. \sum_{\substack{k=0 \\ k \neq n/2}}^n (n-2k)^{2m+1} \binom{n}{k} = 0$$

$$[m=0, 1, 2, \dots]$$

$$24. \sum_{\substack{k=0 \\ k \neq n/2}}^n (-1)^k (n-2k)^{2m+1} \binom{n}{k} = 0$$

$$[m=1, 2, 3, \dots]$$

$$25. \sum_{k=0}^n (-1)^k (n-2k)^s \binom{n}{k} = \frac{1}{2} 2^n \Gamma(s+1) \sin \frac{(n-s)\pi}{2} \int_0^\pi t^{s-1} \sin^n t dt$$

$$[n=1, 2, 3, \dots; -1 \leq s < n]$$

$$26. \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k)^n = 2^{n-1} n!$$

$$27. \sum_{k=0}^n (-1)^k (n-2k)^{n+1} \binom{n}{k} = \frac{1}{3} 2^{n-2} n (n+2)!$$

$$28. \sum_{k=0}^n (n-2k)^2 \binom{n}{k} = \frac{n(2n-2)!}{[(n-1)!]^2}$$

$$29. \sum_{k=0}^n (n-2k)^2 \binom{n}{k} = 2^n n \quad 30. \sum_{k=0}^n (-1)^k (n-2k)^2 \binom{n}{k} = 8\delta_{2,n}$$

$$31. \sum_{\substack{k=0 \\ k \neq n/2}}^n (-1)^k (n-2k)^m \binom{n}{k} = 0$$

$$[m+n \text{ — нечетное число, } m=0, \pm 1, \pm 2, \dots]$$

$$32. \sum_{k=1}^n (-1)^k (n-k)^s \binom{2n}{k} = (-1)^{n+1} \frac{1}{n} \Gamma(s+1) 2^{2n-s} \sin \frac{\pi s}{2} \int_0^\pi t^{s-1} \sin^{2n} t dt$$

$$[0 < s < 2n]$$

$$33. \sum_{k=0}^n (-1)^k (n-k)^{2n} \binom{2n}{k} = \frac{1}{2} (2n)!$$

$$34. \sum_{k=0}^n (-1)^k (n-k)^{2n} \binom{2n}{k} = 0$$

$$[n=1, 2, \dots, n-1]$$

$$35. \sum_{k=1}^n k! k^{n-k} \binom{n}{k} = n$$

$$36. \sum_{k=1}^n k^{k-1} (n-k+1)^{n-k} \binom{n}{k} = n (n+1)^{n-1}$$

$$37. \sum_{k=1}^n (\pm 1)^k k^{n-k-1} \binom{n}{k} = \pm (n \pm 1)^{n-1}$$

$$[n \geq 1]$$

$$38. \sum_{k=1}^{n-1} k^{n-k-2} (n-k)^{k-1} \binom{n}{k} = 2n^{n-2}$$

$$[n \geq 2]$$

$$39. \sum_{k=1}^n (-1)^k k^{n-k} (k \mp 1)^{k-1} \binom{n}{k} = \mp (\pm 1)^n$$

$$40. \sum_{k=2}^n (n-k)^{n-k} (k-1)^{k-1} \binom{n}{k} = -(n-1)^n$$

$$41. \sum_{k=0}^n (-1)^k (a+kc)^{k-1} (b+kc)^{n-k} \binom{n}{k} = \frac{1}{a} (b-a)^n$$

$$42. \sum_{k=0}^n \frac{1}{m+k} \binom{n}{k} = (m-1)! \left[\frac{n! (-1)^m}{(m+n)!} - \frac{2^n}{m!} \sum_{k=1}^m (-2)^k \frac{m! n!}{(m-k)! (n+k)!} \right]$$

$$43. \sum_{k=0}^n \frac{(-1)^k k^m}{a+k} \binom{n}{k} = (-1)^m a^{m-1} \binom{a+n}{n}^{-1}$$

$$[m \leq n]$$

$$44. \sum_{k=0}^n \frac{(-1)^k}{m+k} \binom{n}{k} = \frac{1}{m} \binom{m+n}{m}^{-1}$$

$$45. \sum_{k=0}^n \frac{(-1)^k}{k+a} \binom{n}{k} = \frac{n!}{a(a+1) \dots (a+n)}$$

$$46. \sum_{\substack{k=0 \\ k \neq m}}^n \frac{(-1)^k}{m-k} \binom{n}{k} = (-1)^m \binom{n}{m} \left(\sum_{k=1}^{n-m} \frac{1}{k} - \sum_{k=1}^m \frac{1}{k} \right)$$

$$[m=0, 1, \dots, n]$$

$$\begin{aligned}
 47. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} &= \sum_{k=1}^n \frac{1}{k}, & 48. \sum_{k=1}^n \frac{1}{k+1} \binom{n}{k} &= \frac{1}{n+1} (2^{n+1} - 1), \\
 49. \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} &= \frac{n}{n+1}, & 50. \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} &= \frac{(2n)!}{(2n+1)!}, \\
 51. \sum_{k=1}^n \frac{(-1)^{k+1}}{k(k+1)} \binom{n}{k} &= \sum_{k=2}^{n+1} \frac{1}{k}, & 52. \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{m=1}^k \frac{1}{m} &= \frac{1}{n}, \\
 53. \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{m=1}^{2k} \frac{1}{m} &= \frac{1}{2n} + \frac{(2n-2)!}{(2n-1)!}, \\
 54. \sum_{k=0}^n \frac{(-1)^k}{a-k} \binom{n-k}{k} &= \begin{cases} (-1)^n 2/n \\ (-1)^{n-1}/n \end{cases} & \begin{matrix} [n \text{ нечетное}] \\ [n \text{ четное}] \end{matrix} \\
 55. \sum_{k=0}^n \frac{1}{a-k} \binom{a-k}{n-k} &= \frac{1}{a-n} \binom{a}{n}, \\
 56. \sum_{k=1}^n \frac{1}{a+k} \binom{a+k}{k} &= \frac{1}{a} \left[\binom{a+n}{n} - 1 \right], \\
 57. \sum_{k=0}^n (-1)^k \frac{1}{(k+a)^2} \binom{n}{k} &= n! \left(\prod_{k=0}^n \frac{1}{k+a} \right) \left(\sum_{k=0}^n \frac{1}{k+a} \right) & [a \neq 0, -1, -2, \dots, -n], \\
 58. \sum_{k=0}^n (-1)^k \frac{1}{(k+1)^2} \binom{n}{k} &= \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{1}{k}, & 59. \sum_{k=0}^n \frac{n-2k}{(n-2k)^2 - a^2} \binom{n}{k} &= 0.
 \end{aligned}$$

4.2.3. Суммы вида $\sum a_k \binom{b_k}{c_k} x^k$.

$$\begin{aligned}
 1. \sum_{k=0}^n \binom{n}{k} x^k &= (1+x)^n, & 2. \sum_{k=0}^{[n/2]} \binom{n}{2k} x^k &= \frac{(1+\sqrt{x})^n + (1-\sqrt{x})^n}{2}, \\
 3. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} x^k &= \frac{(1+\sqrt{x})^n - (1-\sqrt{x})^n}{2\sqrt{x}}, \\
 4. \sum_{k=0}^{[n/2]} (-3)^k \binom{n}{2k} &= \begin{cases} (-1)^n 2^n \\ (-1)^{n-1} 2^{n-1} \end{cases} & \begin{matrix} [n \text{ четное}] \\ [n \text{ нечетное}] \end{matrix} \\
 5. \sum_{k=0}^{[(n-1)/2]} (-3)^k \binom{n}{2k+1} &= \begin{cases} 0 \\ 2^{n-1} \\ -2^{n-1} \end{cases} & \begin{matrix} [n = 3m], \\ [n = 6m+1, \quad 6m+3], \\ [n = 6m+4, \quad 6m+5]. \end{matrix}
 \end{aligned}$$

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$$\begin{aligned}
 6. \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k &= 2^{-n+1} (1+4x)^{-1/2} [(1+\sqrt{1+4x})^{n+1} - (1-\sqrt{1+4x})^{n+1}], \\
 7. \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{2k} &= x^{n/2} U_n \left(\frac{1}{2x} \right), \\
 8. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^k} \binom{n-k}{k} &= (n+1) 2^{-n}, \\
 9. \sum_{k=0}^n \binom{2n-k}{k} x^k &= \frac{1}{2^n} \frac{(2x+1 \pm \sqrt{1+4x})^{2n+1} + (2x)^{2n+1}}{(2x+1 \pm \sqrt{1+4x})^n (4x+1 \pm \sqrt{1+4x})}, \\
 10. \sum_{k=0}^n 2^k \binom{2n-k}{n} &= 2^{2n}, & 11. \sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{2n-k}{k} &= \frac{2n+1}{2^{2n}}, \\
 12. \sum_{k=0}^n \frac{1}{2^k} \binom{n+k}{k} &= 2^n, & 13. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n+k}{2k} &= (-1)^n (2n+1), \\
 14. \sum_{k=0}^n \frac{1}{2^k} \binom{n+k}{2k} &= \frac{2^{2n+1}+1}{3 \cdot 2^n}, \\
 15. \sum_{k=0}^n (-1)^k \binom{n+k}{2k} x^{2k} &= \sec \left(\arcsin \frac{x}{2} \right) \cos \left[(2n+1) \arcsin \frac{x}{2} \right], \\
 16. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n+k+1}{2k+1} &= (-1)^n (n+1), \\
 17. \sum_{k=0}^n (-1)^k \binom{n+k+1}{2k+1} x^{2k+1} &= \sec \left(\arcsin \frac{x}{2} \right) \sin \left[2(n+1) \arcsin \frac{x}{2} \right], \\
 18. \sum_{k=0}^n k^n \binom{n}{k} x^k &= (1+x)^n \sum_{k=0}^m (-1)^k \binom{n}{k} \frac{x^k}{(1+x)^k} \sum_{l=0}^k (-1)^l \binom{k}{l} l^m, \\
 19. \sum_{k=0}^n \frac{k^m}{2^{2k}} \binom{2k}{k} &= S_m, \\
 S_0 &= \frac{2n+1}{2^{2n}} \binom{2n}{n}, \quad S_1 = \frac{n}{3} S_0, \quad S_2 = \frac{n(3n+2)}{15} S_0, \quad S_3 = \frac{n(15n^2+18n+2)}{15 \cdot 7} S_0, \\
 20. \sum_{k=0}^n \frac{(k+1)^k}{k!} \binom{n}{k} x^k &= x^n \int_0^1 t^{n-1} (1 \pm t)^n dt & [a \neq 0, -1, \dots, -n].
 \end{aligned}$$

$$21. \sum_{k=1}^n \frac{1}{k} \binom{n}{k} x^k = [(1+x)^n - 1] \ln x - n \int_0^x \frac{(1+t)^n - 1}{t} \ln t \, dt.$$

$$22. \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} x^k = \frac{(1+x)^{n+1} - 1}{(n+1)x}.$$

$$23. \sum_{k=0}^{[n/2]} \frac{1}{2k+1} \binom{n}{2k} x^{2k} = \frac{(1+x)^{n+1} - (1-x)^{n+1}}{2(n+1)x}.$$

$$24. \sum_{k=0}^{[(n-1)/2]} \frac{1}{k+1} \binom{n}{2k+1} x^{2k} = \frac{(1+x)^{n+1} + (1-x)^{n+1} - 2}{(n+1)x^2}.$$

$$25. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n-k} \binom{n-k}{k} x^{2k} = \frac{2}{n} x^n T_n \left(\frac{1}{2x} \right).$$

$$26. = \frac{1}{2^n n} [(1 + \sqrt{1-4x^2})^n + (1 - \sqrt{1-4x^2})^n].$$

$$27. \sum_{k=0}^{[n/2]} \frac{6^k}{n-k} \binom{n-k}{k} = \frac{3^n + (-1)^n 2^n}{n}.$$

$$28. \sum_{k=0}^{[n/2]} \frac{(-1)^k}{n-2k} \binom{n-k-1}{k} x^{2k} = \frac{2}{n} x^n T_n \left(\frac{1}{2x} \right).$$

$$29. \sum_{k=0}^n (-1)^k \frac{(k+a)^{n+1} - 1}{k+a-1} \binom{n}{k} = (-1)^n n!$$

$$30. \sum_{k=0}^n \frac{k \cdot k!}{a^{k+1}} \binom{a}{k} = 1 - \frac{(n+1)!}{a^{n+1}} \binom{a}{n+1}.$$

$$31. \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{x^k}{k!} = L_n^\alpha(-x).$$

$$32. \sum_{k=0}^n \binom{n+1}{k} (1^k + 2^k + \dots + n^k) = (n+1)^{n+1} - 1.$$

$$33. \sum_{k=0}^n \binom{2n-k}{n} \frac{x^k}{k!} = \frac{1}{n! \sqrt{x}} x^{n+1/2} {}_2F_2 \left(\frac{x}{2} \right).$$

4.2.4. Суммы вида $\sum a_k \binom{b_k}{c_k}^{-1}$.

$$1. \sum_{k=0}^n \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

$$2. \sum_{k=0}^n (-1)^k \binom{a}{k}^{-1} = \frac{a+1}{a+2} \left[1 + (-1)^n \binom{a+1}{n+1}^{-1} \right].$$

$$3. \sum_{k=0}^n \binom{a+k}{k}^{-1} = \frac{a}{a-1} \left[1 - \binom{a+n}{n+1}^{-1} \right].$$

$$4. \sum_{k=0}^n \binom{a+k}{m}^{-1} = \frac{m}{m-1} \left[\binom{a-1}{m-1}^{-1} - \binom{a+n}{m-1}^{-1} \right].$$

$$5. \sum_{k=1}^n (-1)^{k+1} \binom{2n}{k}^{-1} = \frac{1}{2(n+1)} + \frac{(-1)^{n+1}}{2} \binom{2n}{n}^{-1}.$$

$$6. \sum_{k=0}^{2n-1} (-1)^k \binom{2n}{k}^{-1} = \frac{2n+1}{n+1}, \quad 7. \sum_{k=1}^{2n-1} (-1)^{k+1} k \binom{2n}{k}^{-1} = \frac{n}{n+1}.$$

$$8. \sum_{k=1}^n (n-2k) \binom{n}{k}^{-1} = 1 - (n+1) \binom{n+1}{n+1}^{-1}.$$

$$9. \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1} \binom{2n}{k}^{-1} = (2n+1) \sum_{k=1}^n \frac{1}{(k+n)^2}.$$

$$10. \sum_{k=1}^{2n-1} (-1)^{k+1} \binom{2n}{k}^{-1} \sum_{m=1}^k \frac{1}{m} = \frac{n}{2(n+1)^2} + \frac{1}{2n+2} \sum_{k=1}^{2n} \frac{1}{k}.$$

$$11. \sum_{k=1}^{2n+1} (-1)^{k+1} \binom{2n+1}{k}^{-1} \sum_{m=1}^k \frac{1}{m} = \frac{2n+2}{2n+3} \sum_{k=1}^{2n+2} \frac{1}{k}.$$

$$12. \sum_{k=0}^n \binom{n}{k}^{-1} x^k = (n+1) \left(\frac{x}{1+x} \right)^{n+1} \sum_{k=1}^{n+1} \frac{1}{k} \frac{1+x^k}{1+x} \left(\frac{1+x}{x} \right)^k.$$

$$13. \sum_{k=1}^n \frac{2^{2k-1}}{k} \binom{2n}{k}^{-1} = 2^{2n} \binom{2n}{n}^{-1} - 1.$$

- 4.2.5. Суммы вида $\sum_{k=0}^n (-1)^k \binom{a}{b} \binom{c}{d}$.
1. $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$.
 2. $\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k}^2 = \frac{1}{2} \binom{2n}{n} - \frac{1+(-1)^n}{4} \binom{n}{n/2}$.
 3. $\sum_{k=0}^n (-1)^k \binom{n}{k}^2 = \frac{(-1)^{n/2} [1+(-1)^n]}{2} \binom{n}{n/2}$.
 4. $\sum_{k=0}^n \binom{2n}{k}^2 = \frac{1}{2} \binom{4n}{2n} + \frac{1}{2} \binom{2n}{n}^2$.
 5. $\sum_{k=0}^n (-1)^k \binom{2n}{k}^2 = \frac{(-1)^n}{2} \left[\binom{2n}{n} + \binom{2n}{n}^2 \right]$.
 6. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$.
 7. $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 = \frac{1}{2} \binom{2n}{n} + \frac{(-1)^{n/2} [1+(-1)^n]}{4} \binom{n}{n/2}$.
 8. $\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1}^2 = \frac{1}{2} \binom{2n}{n} - \frac{(-1)^{n/2} [1+(-1)^n]}{4} \binom{n}{n/2}$.
 9. $\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{2k+1}^2 = \frac{1}{4} \binom{4n}{2n} - \frac{(-1)^n}{4} \binom{2n}{n} + \frac{1-(-1)^n}{4} \binom{2n}{n}^2$.
 10. $\sum_{k=0}^n \binom{a}{k} \binom{b}{n-k} = \sum_{k=0}^n (-1)^k \binom{n-a}{k} \binom{b+n-k}{n}$.
 11. $\sum_{k=0}^n \binom{n}{k} \binom{m-n}{k} = \binom{m}{n}$.
 12. $\sum_{k=0}^n \binom{-1/2}{k} \binom{1/2}{k} = \frac{2n+1}{2^{2n}} \binom{2n}{n}^2$.
 13. $\sum_{k=0}^n \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}$.
 14. $\sum_{k=0}^n \binom{-a}{k} \binom{a}{m+k} = \frac{a}{a-m} \binom{-a-1}{n} \binom{a-1}{m+n}$.
 15. $\sum_{k=0}^n \binom{a}{k} \binom{-a}{n-k} = - \sum_{k=n+1}^{\infty} \binom{a}{k} \binom{-a}{n-k}$.

16. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{b}{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{a}{k} \binom{b-a}{n-2k}$.
17. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{n-k} = \frac{(-1)^{n/2} [1+(-1)^n]}{2} \binom{n}{n/2}$.
18. $\sum_{k=0}^n \binom{a}{k} \binom{1-a}{n-k} = \frac{(n-1)(1-a)-m(a-1)}{n(n-1)} \binom{a-1}{m} \binom{-a}{n-m-1}$
 $\{n \geq 2, 1 \leq m \leq n-1\}$
19. $\sum_{k=0}^n \binom{n}{k} \binom{a+b}{a-k} = \binom{a+b+n}{a}$.
20. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n-2n}{a-k} = (-1)^n \binom{a}{n} \binom{2a}{a} \binom{2a}{2n}^{-1}$.
21. $\sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2a-2n}{a-k} = \frac{(-1)^n}{2} \binom{2a}{a} \binom{2a}{2n}^{-1} \left[\binom{n}{n} + \binom{n}{n}^2 \right]$.
22. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{2n-a}{n-k} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-a)$.
23. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2a-n}{a-k} = \frac{(-1)^{n/2} [1+(-1)^n]}{2} \binom{a}{n/2} \binom{2a}{a} \binom{2a}{n}^{-1}$.
24. $\sum_{k=0}^n \binom{n}{k} \binom{n+2a}{k+a} = \binom{2a+2n}{a+n}$.
25. $\sum_{k=0}^n \binom{n}{k} \binom{a}{k-m} = \binom{n+a}{n+m}$.
26. $\sum_{k=0}^n \binom{n}{k} \binom{a}{k-m} = \binom{n+a}{n+m}$.
27. $\sum_{k=0}^n \binom{a}{k} \binom{a}{2n-k} = \frac{1}{2} \binom{2a}{2n} + \frac{1}{2} \binom{a}{n}^2$.
28. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a}{2n-k} = \frac{(-1)^n}{2} \left[\binom{a}{n} + \binom{a}{n}^2 \right]$.
29. $\sum_{k=0}^{2n} (-1)^k \binom{a}{k} \binom{a}{2n-k} = (-1)^n \binom{a}{n}$.
30. $\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{a}{k} \binom{-a}{n-2k} = (-1)^n \binom{a}{n}$.

31. $\sum_{k=0}^n \binom{a}{k} \binom{a}{2n-k+1} = \sum_{k=0}^n \binom{a}{n-k} \binom{a}{n+k+1} = \frac{1}{2} \binom{2a}{2n+1}.$
32. $\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{a}{k} \binom{2n-2a}{n-2k} = \frac{2^n}{n!} \prod_{k=0}^{n-1} (2k+1-a).$
33. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2a}{2k+a} = (-1)^n \binom{2n}{n} \binom{2n+2a}{a} \binom{n+a}{n}^{-1}.$
34. $\sum_{k=0}^{2m-n+1} (-1)^k \binom{n}{k} \binom{n}{2m+1-k} = 0.$
35. $\sum_{k=-n}^n (-1)^k \binom{2n}{k} \binom{2n}{a-k} = (-1)^n \frac{(2n)! (2n)!}{n! n! (m+n)!}.$
36. $\sum_{k=0}^n \binom{a+k}{a} \binom{b+n-k}{b} = \binom{a+b+n+1}{n}.$
37. $\sum_{k=0}^n \binom{a+k}{a} \binom{b-k}{b-n} = \binom{a+b+1}{n}.$
38. $\sum_{k=0}^n (-1)^k \binom{a+k}{a} \binom{a+n-k}{a} = \binom{a+n/2}{n/2} \frac{1+(-1)^n}{2}.$
39. $\sum_{k=l}^{n-m} \binom{k}{l} \binom{n-k}{m} = \binom{n+l}{l+m+1}.$
40. $\sum_{k=m}^{n-m} (-1)^k \binom{k}{m} \binom{n-k}{m} = (-1)^m \frac{1+(-1)^n}{2} \binom{n/2}{m}.$
41. $\sum_{k=0}^{n-1} \binom{k}{m} \binom{k+l}{l} = \binom{n}{m} \binom{n+l}{l} \frac{n-m}{l+m+1}.$
42. $\sum_{k=0}^n \binom{n+2k+1}{n} \binom{n+2m-2k+1}{n} = \frac{1}{2} \binom{2m+2n+3}{2n+1} - \frac{1}{2} \binom{m+n+1}{n+l}.$
43. $\sum_{k=0}^n \binom{a+kc}{k} \binom{b-kc}{n-k} = \sum_{k=0}^n \binom{a+b-k}{n-k} c^k.$

44. $\sum_{k=0}^n \binom{a+kb}{k} \binom{m-a-kb}{n-k} = \begin{cases} \frac{b^{m+1} (b-1)^{n-m-1}}{b-1} & \{0 \leq m \leq n-1\}, \\ \{m=n\}. \end{cases}$
45. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^{n+m} \delta_{n,m}.$ 46. $\sum_{k=m}^n \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m}.$
47. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+kb}{m} = (-1)^n b^a \delta_{n,m}.$ 48. $\sum_{k=m}^n (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^n \delta_{n,m}.$
49. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{b-k}{b-n} = \binom{b-a}{n}.$ 50. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a+n-k-1}{a-1} = 0.$
51. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+m+n-k}{m} = 0.$ 52. $\sum_{k=0}^n \binom{a}{k} \binom{k/2}{m} = \frac{n}{m} 2^{n-2m} \binom{n-m-1}{m-1}.$
53. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a+n-k}{a-1} = (-1)^n \binom{a}{n+1}.$ 54. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k/2}{m} = (-1)^{n+2m-1} \left[\binom{2m-n-1}{m-1} - \binom{2m-n-1}{m} \right].$
55. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+k}{m} = (-1)^n \binom{a}{m-n}.$ 56. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-k}{m} = \binom{a-n}{m-n}.$
57. $\sum_{k=0}^n \binom{4n}{2k} \binom{2n-k}{n} = \frac{2^{2n+1}}{3} \binom{3n}{n}.$ 58. $\sum_{k=0}^n (-1)^k \binom{a}{k} \binom{2n-k}{n} = \binom{2n-a}{n}.$

$$59. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{a-k}{a-n} = \delta_{0,n}. \quad 60. \sum_{k=0}^n (-1)^k \binom{m}{k} \binom{m+n-k-1}{m-1} = 0.$$

$$61. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{m} = (-1)^n 2^{2n-m} \binom{n}{m-n}.$$

$$62. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n+m} = 2^{n-m} \binom{n}{m}.$$

$$63. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-2k}{2n-m} = 2^m \binom{n}{m}.$$

$$64. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{2n-n} = 2^n \binom{n}{n}.$$

$$65. \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \binom{2n-2k+a}{n-1} = \binom{n-a-2}{n-1}.$$

$$66. \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m}{k} \binom{m+n-2k-1}{m-1} = \binom{m}{n}.$$

$$67. \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{4n-2k+1}{2n+1} = (n+1) 2^{2n}.$$

$$68. \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k-3}{n} = \frac{(n-1)n}{2}.$$

$$69. \sum_{k=0}^{\lfloor (m-1)(n-1)/m \rfloor} (-1)^k \binom{n}{k} \binom{mn-mk-m}{n} = \binom{m+n-2}{n-1}.$$

$$70. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m-2k}{n-1} = 0 \quad [m \geq 2n].$$

$$71. \sum_{k=0}^n (-1)^k \binom{2k}{k} \binom{n+k}{2k} = (-1)^n. \quad 72. \sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} = 2^{2n}.$$

$$73. \sum_{k=0}^n (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = [1 + (-1)^n] 2^{n-1} \binom{n}{n/2}.$$

$$74. \sum_{k=0}^n \binom{4k}{2k} \binom{4n-4k}{2n-2k} = 2^{2n-1} \binom{2n}{n} + 2^{2n-1}.$$

$$75. \sum_{k=0}^{n-1} \binom{4k+2}{2k+1} \binom{4n-4k-2}{2n-2k-1} = 2^{4n-1} - 2^{2n-1} \binom{2n}{n}.$$

$$76. \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{2k} \binom{a}{n-2k} = \frac{1}{2} \binom{2a}{n} + \frac{(-1)^{n/2} [1 + (-1)^{n/2}]}{4} \binom{a}{n/2}.$$

$$77. \sum_{k=0}^n \binom{a}{2k} \binom{a}{2n-2k} = \frac{1}{2} \binom{2a}{2n} + \frac{(-1)^n}{2} \binom{a}{n}.$$

$$78. \sum_{k=0}^{n-1} \binom{n}{2k} \binom{m+k}{n-1} = \binom{2m+1}{n-1}.$$

$$79. \sum_{k=m}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{k}{m} = 2^{n-2m-1} \frac{n}{n-2m} \binom{n-m-1}{m}.$$

$$80. \sum_{k=0}^n \binom{2a}{2k} \binom{a-k}{a-n} = \frac{n}{a+n} 2^{2n} \binom{a+n}{2n}.$$

$$81. \sum_{k=0}^n \binom{2a+1}{2k} \binom{a-k}{a-n} = 2^{2n} \binom{a+n}{2n}.$$

$$82. \sum_{k=0}^n \binom{a}{2k} \binom{a+n-k-1}{a-1} = \binom{a+2n-1}{2n}.$$

$$83. \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{2k} \binom{2n-a}{n-2k} = \frac{1}{2} \left[\binom{2n}{n} + (-1)^n 2^{2n} \binom{(a-1)/2}{n} \right].$$

$$84. \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{a+k}{n-1} = \binom{2a+1}{n-1}. \quad 85. \sum_{k=0}^n \binom{1n+1}{2k} \binom{2n-k}{n} = 2^{2n} \binom{3n}{n}.$$

$$86. \sum_{k=0}^n \binom{2a+1}{2k+1} \binom{a-k}{2n} = \frac{2a+1}{2n+1} 2^{2n} \binom{a+n}{2n}.$$

$$87. \sum_{k=0}^n \binom{a}{2k+1} \binom{a+n-k-1}{a-1} = \binom{a+2n}{2n+1}.$$

$$68. \sum_{k=0}^{n-1} \binom{2a}{2k+1} \binom{a-k-1}{a-n} = \frac{n}{a+n} 2^{2n} \binom{a+n}{2n}.$$

$$69. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \binom{a+k}{a-1} = \binom{2a}{n-1}.$$

$$70. \sum_{k=m}^{[(n-1)/2]} \binom{n}{2k+1} \binom{b}{m} = 2^{n-2m-1} \binom{n-m-1}{m}.$$

$$91. \sum_{k=0}^n \binom{2a}{2k+1} \binom{a-k}{a-n} = \frac{2a+1}{2n+1} 2^{2n} \binom{a+n}{2n}.$$

$$92. \sum_{k=0}^{[(n-2m)/2]} \binom{n}{2k+1} \binom{n-1}{2m-2k} = \frac{1}{2} \binom{2n-1}{2m+1} + \frac{(-1)^m}{2} \binom{n-1}{m}.$$

$$93. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \binom{2n-a}{n-2k-1} = \frac{1}{2} \binom{2n}{n} - (-1)^n 2^{2n-1} \binom{a-1}{n/2}.$$

$$94. \sum_{k=0}^{[(n-1)/2]} \binom{a}{2k+1} \binom{a}{n-2k-1} = \frac{1}{2} \binom{2a}{n} - \frac{(-1)^{n/2} [1 + (-1)^n]}{4} \binom{a}{n/2}.$$

4.2.6. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k}$.

$$1. \sum_{k=1}^n k \binom{n}{k} \binom{a}{k} = n \binom{a+n-1}{n-1}.$$

$$2. \sum_{k=0}^n k^m \binom{n}{k}^3 = S_m, \\ S_0 = \binom{2n}{n}, \quad S_1 = \frac{n}{2} \binom{2n}{n}, \quad S_2 = n^2 \binom{2n-2}{n-1}, \quad S_3 = \frac{n^2(n+1)}{2} \binom{2n-2}{n-1}.$$

$$3. \sum_{k=0}^{[(n-1)/2]} (n-2k)^2 \binom{n}{k}^3 = n \binom{2n-2}{n-1}.$$

$$4. \sum_{k=1}^n \binom{n}{k}^2 \sum_{m=1}^k \frac{1}{m} = \binom{2n}{n} \left(\sum_{k=1}^n \frac{1}{k} - \sum_{k=n+1}^{2n} \frac{1}{k} \right).$$

$$5. \sum_{k=1}^n k \binom{a+k}{m} \binom{a-k}{m} = \frac{m+1}{2} \left[\binom{a+1}{m+1} \binom{a}{m+1} - \binom{a+n+1}{m+1} \binom{a-n}{m+1} \right].$$

$$6. \sum_{k=0}^n [mn - (a+k)k] \binom{a}{k} \binom{b}{m-k} = (n+1)(m-n) \binom{a}{n+1} \binom{b}{m-n}.$$

$$7. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} \binom{n+k}{n} = 2 \sum_{k=1}^n \frac{1}{k}.$$

$$8. \sum_{k=n}^n \frac{(-1)^k}{m-k} \binom{m-k}{n} \binom{n}{k} = 0 \quad [2n < m]$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k-1}{n-1} \sum_{m=1}^{n+k-1} \frac{1}{m} = \frac{(-1)^n}{n}.$$

$$10. \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{n+k-1}{n-1} \sum_{m=1}^k \frac{1}{m} = \frac{(-1)^n}{n}.$$

$$11. \sum_{k=0}^n \frac{(-1)^k}{n+k} \binom{n}{k} \binom{a+k}{n} \sum_{m=1}^k \frac{1}{m} = \frac{(-1)^n}{n^2}.$$

$$12. \sum_{k=0}^n \frac{1}{a+k} \binom{2k}{k} \binom{2n-2k}{n-k} = 2^n \frac{(2a+1)(2a+3) \dots (2a+2n-1)}{a(a+1) \dots (a+n)}.$$

$$13. \sum_{k=0}^n \frac{(-1)^k}{a+k} \binom{2k}{k} \binom{n+k}{2k} = \frac{(-1)^n}{a} \binom{a-1}{n} \binom{a+n}{n}^{-1}.$$

$$14. \sum_{k=m}^n \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{n-m+1}{2(n+1)} \binom{2n}{m} \binom{2n-2m+2}{n-m+1}.$$

$$15. \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k-1} = \binom{2n+1}{n-1}.$$

$$16. \sum_{k=0}^n \frac{1}{2k-1} \binom{2k}{k} \binom{2n-2k}{n-k} = -\delta_{n,0}.$$

$$17. \sum_{k=0}^n \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1}.$$

$$18. \sum_{k=0}^n \frac{1}{(k+1)(n-k+1)} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{(2n+2)!}{(n+1)!(n+2)!}.$$

$$19. \sum_{k=0}^n \frac{1}{(2k-1)(2n-2k-1)} \binom{2k}{k} \binom{2n-2k}{n-k} = (-1)^n 2^{2n} \binom{1}{n}.$$

$$20. \sum_{k=0}^n \frac{1}{(2k-1)(2n-2k+1)} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1}{n(2n+1)} 2^{2n+1} \binom{2n}{n}.$$

$$21. \sum_{k=0}^n \frac{(a+kc) \binom{b-ka}{n-k}}{(a+kc) \binom{b-ka}{n-k}} \frac{p+qk}{a(a+b)(b-nc)} = \frac{p(a+b-nc)+naq}{a(a+b)(b-nc)} \binom{a+b}{n}.$$

$$22. \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \binom{ka}{k} \binom{na-ka}{n-k} = \frac{2a}{n} \binom{na}{n} \sum_{k=1}^{n-1} \frac{1}{na-n+k}.$$

$$23. \sum_{k=0}^n \frac{1}{(mk+1)(mn-mk+1)} \binom{mk+1}{k} \binom{mn-mk+1}{n-k} = \frac{2}{2+mn} \binom{2+mn}{n}.$$

$$24. \sum_{k=0}^n \frac{2k+1}{n+k+1} \binom{a-k-1}{a-n-1} \binom{a+k}{a-n} = \binom{a}{n}.$$

$$25. \sum_{k=1}^n \binom{m}{k} k^n \sum_{l=0}^{n-k} (-1)^l \binom{m-k}{l} = m^n$$

[7] > 1

$$26. \sum_{k=0}^{[2n/2]} (-1)^k \sum_{l=0}^{[k/2]} \binom{n-k}{k-l} \binom{k-l}{l} = \frac{1}{2} [1 + (-1)^{[n/2]}].$$

4.2.7. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k} x^k$.

$$1. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} x^k = \sum_{k=0}^n \binom{n}{k} \binom{a+n-k}{n} (x-1)^k.$$

$$2. \sum_{k=0}^n \binom{n}{k} \binom{a+k}{k} (b-c)^{n-k} c^k = \sum_{k=0}^n \binom{n}{k} \binom{a}{k} b^{n-k} c^k.$$

$$3. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} 2^k = \sum_{k=0}^n \binom{n}{k} \binom{a+k}{n}.$$

$$4. \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (x-1)^k.$$

$$5. \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} (x-1)^{n-k}.$$

$$6. \sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n P_n \left(\frac{1+x}{1-x} \right).$$

$$7. \sum_{k=0}^n 2^k \binom{n}{k}^2 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{3n-2k}{2n}.$$

$$8. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1/2}{k} x^{2k} = P_{2n}(\sqrt{1-x^2}).$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+3/2}{k} x^{2k} = (1-x)^{-1/2} P_{2n+1}(\sqrt{1-x^2}).$$

$$10. \sum_{k=0}^n \binom{n}{k} \binom{k}{m} x^k = \binom{n}{m} x^m (1+x)^{n-m}.$$

$$11. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{2k} = (2x)^n P_n \left(\frac{1}{x} \right).$$

$$12. \sum_{k=0}^n \binom{n}{k} \binom{n+k}{n} x^k = P_n(1+2x).$$

$$13. \sum_{k=m}^n \frac{(-1)^k}{2^k} \binom{n}{k} \binom{n+k}{m+n} = \frac{(-1)^{[m+n/2]}}{2^n} \binom{n}{(n-m)/2}$$

[n-m — четное число]

$$14. \sum_{k=0}^n (-2)^k \binom{a}{k} \binom{2n-k}{n} = \frac{2^n}{n!} \prod_{k=1}^n (2k - a - 1).$$

$$15. \sum_{k=0}^n (-1)^k 2^{n-k} \binom{n}{k} \binom{n+2m+k}{n+m} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{n+m}{n/2} \binom{n+2m}{m} \binom{n+m}{n}^{-1}.$$

$$16. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2m-n+k}{m} \frac{1}{2^k} = (-1)^{n/2} [1 + (-1)^n] \frac{1}{2^{n+1}} \binom{m}{n/2} \binom{2m-n}{m} \binom{m}{m-n}^{-1}.$$

$$17. \sum_{k=n-m}^n \frac{(-1)^k}{2^k} \binom{n}{k} \binom{n+k}{2n-m} = \frac{(-1)^{m/2+n}}{2^{m+1}} [1 + (-1)^m] \binom{n}{m/2}.$$

18. $\sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} = \frac{1}{2^{2n}} \binom{2n}{n}.$
19. $\sum_{k=0}^{2n} \frac{(-1)^k}{3^k} \binom{2n}{k} \binom{2n+2k}{n+k} = \frac{1}{3^{2n}} \binom{2n}{n}.$
20. $\sum_{k=0}^{\lfloor (2m-n)/2 \rfloor} \frac{1}{2^{2k}} \binom{m}{k} \binom{m-k}{n-m+k} = 2^{n-2m} \binom{2m}{n}.$
21. $\sum_{k=m}^n (-1)^k 2^{n-k} \binom{n}{k} \binom{2k}{k-m} = \frac{(-1)^n + (-1)^n}{2} \binom{n}{(n-n)/2}.$
22. $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^{2k}} \binom{a}{k} \binom{a-k}{n-2k} = \frac{1}{2^n} \binom{2a}{n}.$
23. $\sum_{k=0}^{2m} (-2)^k \binom{n}{k} \binom{2n-k}{2m-k} = (-1)^m \binom{n}{m}.$
24. $\sum_{k=0}^n (-2)^k \binom{n}{k} \binom{2n-2k}{2m-k} = \binom{n}{m}.$
25. $\sum_{k=0}^n (-1)^k 2^{2k} \binom{a}{k} \binom{2a-2k}{n-k} = (-1)^n \binom{2a}{n}.$
26. $\sum_{k=0}^n (-1)^k 2^{2k} \binom{a}{k} \binom{4a-2k}{n-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{2a}{n/2}.$
27. $\sum_{k=0}^n (-2)^k \binom{a}{k} \binom{b-2k}{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{a}{k} \binom{b-2k}{n-2k}.$
28. $\sum_{k=0}^n (-2)^k \binom{a}{k} \binom{2a-k}{n-k} = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{a}{n/2}.$
29. $\sum_{k=0}^{2n+1} (-2)^k \binom{n+k-1}{n-1} \binom{2m+2n+k}{2m+1-k} = 0.$
30. $\sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} \left(\frac{x+1}{x-1}\right)^k = \frac{2^n}{(x-1)^n} P_n^{(\alpha, \beta)}(x).$

31. $\sum_{k=m}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^{2k}} \binom{n-k}{k} \binom{k}{m} = \frac{(-1)^n}{2^n} \binom{n+1}{2m+1}.$
32. $\sum_{k=0}^n (-1)^k 2^{2k} \binom{k}{m} \binom{n+k}{2k} = (-1)^n \binom{n+m}{2m} 2^{2n} \frac{2n+1}{2m+1}.$
33. $\sum_{k=0}^n (-1)^k 2^{2k} \binom{k}{m} \binom{n+k}{2k+1} = (-1)^{n-1} \binom{n+m}{2m+1} 2^{2n}.$
34. $\sum_{k=0}^n \frac{(-1)^k}{n+k} 2^{2k} \binom{k}{m} \binom{n+k}{2k} = (-1)^n \frac{1}{n+m} 2^{2n} \binom{n+m}{2m}.$
35. $\sum_{k=0}^n \frac{(-1)^k}{2k+1} 2^{2k} \binom{k}{m} \binom{n+k}{2k} = \frac{(-1)^n}{2n+1} 2^{2n} \binom{n+m}{2m}.$
36. $\sum_{k=0}^n \frac{1}{2^{2k}} \binom{2k}{k} \binom{2n-k}{n} = \frac{1}{2^{2n}} \binom{4n+1}{2n}.$
37. $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^{2k}} \binom{n}{2k} \binom{2k}{k} = \frac{1}{2^n} \binom{2n}{n}.$
38. $\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} x^{2k} = 2^{2n} x^n P_n \left[\frac{1}{2} \left(x + \frac{1}{x} \right) \right].$
39. $\sum_{k=0}^n 2^{2k} \binom{a}{2k} \binom{a-2k}{n-k} = \binom{2a}{2n}.$
40. $\sum_{k=0}^n 2^{2k+1} \binom{a+1}{2k+1} \binom{a-2k}{n-k} = \binom{2a+2}{2n+1}.$
41. $\sum_{k=1}^n \frac{(-2)^k k}{2n-k} \binom{n}{k} \binom{2n-k}{n} = 2^{2n} \binom{n-1-a/2}{n}.$
42. $\sum_{k=0}^n \frac{(-1)^k}{2^k} \frac{n-k}{n+k} \binom{2a}{n-k} \binom{n+k}{n} = 2^n \binom{a}{n}.$
43. $\sum_{k=m}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{(n-k) 2^{2k}} \binom{n-k}{k} \binom{k}{m} = \frac{(-1)^m 2^{1-n}}{n} \binom{n}{2m}.$

$$44. \sum_{k=0}^n k! \binom{n}{k} \binom{n-k}{n} x^k = y_n(2x).$$

$$45. \sum_{k=0}^{[n/2]} (-1)^k k! \binom{n}{k} \binom{n-k}{k} x^{2k} = {}_2F_2 \left(\frac{1}{2x} \right).$$

4.2.8. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k}^{-1}$.

$$1. \sum_{k=m}^n \binom{a}{k} \binom{b}{k}^{-1} = \frac{b+1}{b-a+1} \left[\binom{a}{m} \binom{b+1}{m}^{-1} - \binom{a}{n+1} \binom{b+1}{n+1}^{-1} \right].$$

$$2. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+k}{a}^{-1} = \frac{a}{a+n}.$$

$$3. \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{k+a-1}{a}^{-1} = -a \sum_{k=0}^{n-1} \frac{1}{k+a}.$$

$$4. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{l}^{-1} = \frac{l}{n+l} \binom{n+m}{m-l}^{-1}.$$

$$5. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k}^{-1} = \frac{1+(-1)^n (2n+1)}{2(n+1)}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n+1}{2k}^{-1} = 1 + \frac{(-1)^n - 1}{2(n+2)}.$$

$$7. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n+1}{2k+1}^{-1} = \frac{1-(-1)^n}{2(n+2)} + (-1)^n.$$

$$8. \sum_{k=0}^n (-1)^k \binom{2n}{2k} \binom{n}{k}^{-1} = \frac{1+(-1)^n}{2(1-n)}.$$

$$9. \sum_{k=0}^n \binom{n}{k} \binom{n+2a}{k+a}^{-1} = \frac{2a+n+1}{2a+1} \binom{2a}{a}^{-1}.$$

$$10. \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2n+2a+1}{k+a}^{-1} = 0.$$

$$11. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2n+2a}{k+a}^{-1} = \binom{a+n}{n} \binom{2a}{a}^{-1} \binom{2a+2n}{2n}^{-1}.$$

$$12. \sum_{k=0}^n (-1)^k \binom{2n+1}{2k} \binom{n}{k}^{-1} = (-1)^n \left[1 + \frac{1-(-1)^n}{2n} \right].$$

$$13. \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} \binom{n}{k}^{-1} = \frac{n}{n-1} + \frac{(-1)^n - 1}{2n(n-1)}.$$

$$14. \sum_{k=0}^n (-1)^k \binom{2n+1}{2k+1} \binom{n}{k}^{-1} = 1 + \frac{1-(-1)^n}{2n}.$$

$$15. \sum_{k=0}^n (-1)^k k^n \binom{n}{k} \binom{a+k}{k}^{-1} = S_m.$$

$$S_0 = \frac{a}{a+n}, \quad S_1 = \frac{-n}{2(2n-1)}.$$

$$16. \sum_{k=1}^n (-1)^k k^2 m \binom{n}{k} \binom{n+k}{n}^{-1} = 0$$

{1 < 2m < 2n-1}.

$$17. \sum_{k=1}^n k \binom{n}{k} \binom{a}{k}^{-1} = \frac{n(1+a)}{(n-a-1)(n-a-2)}.$$

$$18. \sum_{k=1}^n k^2 \binom{n}{k} \binom{a}{k}^{-1} = -\frac{n(1+a)(a-n+1)}{(n-a-1)(n-a-2)(n-a-3)}.$$

$$19. \sum_{k=1}^n k \binom{n}{k} \binom{2n-1}{k}^{-1} = \frac{2n}{n+1}.$$

$$20. \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} \binom{k+n+1}{n+1}^{-1} = (n+1) \binom{n+1/2}{n}^{-1}.$$

$$21. \sum_{k=0}^n \frac{4k+1}{2n+2k+1} \binom{2n}{n+k} \binom{2n+2k}{n+k}^{-1} = 1.$$

$$22. \sum_{k=0}^n \frac{a+n+1-2k}{a+n+1-k} \binom{a}{k} \binom{a+n-k}{n}^{-1} = 1.$$

$$23. \sum_{k=0}^n \binom{n}{k} \binom{k+m}{k}^{-1} x^k = \left[m + (x+1)^{n+m} - \sum_{k=0}^m \binom{n+m}{k} x^k \right] x^{-m} \binom{n+m}{n}^{-1}.$$

$$24. \sum_{k=0}^n (-1)^k 2^{2k} \binom{n}{k} \binom{2k}{k}^{-1} = \frac{1}{1-2n}.$$

$$25. \sum_{k=0}^n \frac{(-1)^k}{k} \binom{n}{k} (2k)^{-1} = 2 \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k}.$$

$$26. \sum_{k=0}^{2n+1} (-1)^k 2^{2k} \binom{3n+2}{n+k+1} \binom{2n+2k+1}{n+k}^{-1} = \frac{1}{3} \binom{3n+2}{n+1} \binom{2n+1}{n}^{-1}.$$

$$27. \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \binom{n}{k} \binom{m}{k}^{-1} = \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^{n-1} \frac{1}{k}.$$

$$0 \leq m \leq n.$$

$$28. \sum_{k=0}^n \binom{n}{k}^{-1} = \frac{3(n+1)^2}{2n+3} \binom{2n+2}{n+1}^{-1} \sum_{k=1}^{n+1} \frac{1}{k} \binom{2k}{k}.$$

$$29. \sum_{k=1}^n k 2^k \binom{n}{k} \binom{2n}{k}^{-1} = 2n+1 - 2^{2n} \binom{2n}{n}^{-1}.$$

$$30. \sum_{k=1}^n k 2^k \binom{n}{k} \binom{2n+1}{k}^{-1} = 2n+1 - 2^{2n+1} \binom{2n+1}{n}^{-1}.$$

§2.9. Суммы вида $\sum a_k \binom{b_k}{c_k} \binom{d_k}{e_k} \binom{f_k}{g_k}$.

$$1. \sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \frac{(-1)^{n/2} [1 + (-1)^n]}{2} \binom{n}{n/2} \binom{3n/2}{n}.$$

$$2. \sum_{k=0}^n \binom{n}{k}^3 \binom{a+k}{2n} = \binom{a}{n}^3. \quad 3. \sum_{k=0}^n \binom{n}{k}^3 \binom{a+2n-k}{2n} = \binom{a+n}{n}^3.$$

$$4. \sum_{k=0}^n \binom{n}{k}^3 \binom{2a+n-k}{2a} = \binom{a+n}{n}^3.$$

$$5. \sum_{k=0}^n \binom{n}{k}^3 \binom{k}{n-m} = \binom{n}{m} \binom{n+m}{m}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k}^3 \binom{2a-n}{a-k} = (-1)^{n/2} \frac{1 + (-1)^n}{2} \binom{n}{n/2} \binom{a+n/2}{n} \binom{2a}{a} \binom{2a-1}{n}^{-1}.$$

$$7. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{n+k}{n}^3 = \binom{2n}{n}.$$

$$8. \sum_{k=0}^n (-1)^k \binom{2n}{k} \binom{2n-2m}{n-k}^3 = (-1)^m \binom{2m}{m} \binom{2n-m}{n}^3 \binom{2n-m}{m}^{-1}.$$

$$9. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{a}{n-k} = \frac{1 + (-1)^n}{2} \binom{a}{n/2} \binom{-a-1}{n/2}.$$

$$10. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m-a-1}{k+l} \binom{a}{m-l-k} = (-1)^l \binom{m+n}{l-m} \binom{a}{m}.$$

$$11. \sum_{l=0}^n (-1)^l \binom{n}{l} \binom{-a+m-1}{m+k} \binom{a-m-n}{l-m-k} = (-1)^m \binom{l+n}{l-m} \binom{a}{l}.$$

$$12. \sum_{k=-m}^n (-1)^k \binom{m+n}{m+k} \binom{n+l}{n+k} \binom{l+m}{l+k} = \frac{(l+m+n)!}{l! m! n!}.$$

$$13. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2m}{m-n+k} \binom{2l}{l-n+k} = (-1)^n \frac{(l+m+n)! (2l)! (2m)! (2n)!}{l! m! n! (l+m)! (l+n)! (m+n)!}.$$

$$14. \sum_{k=0}^n \binom{m}{k} \binom{n}{k} \binom{m+n+k}{m+n} = \binom{m+n}{n}^3.$$

$$15. \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{a+m+n+k}{m+n} = \binom{a+m+n}{n} \binom{a+m+n}{m}.$$

$$16. \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{m+n+l-k}{m+n} = \binom{m+l}{m} \binom{n+l}{n}.$$

$$17. \sum_{k=0}^n \binom{a}{k} \binom{b}{k} \binom{a+b+n-k}{n-k} = \binom{a+n}{n} \binom{b+n}{n}.$$

$$18. \sum_{k=0}^n \binom{a}{k} \binom{n}{n-k} \binom{k}{m} = \binom{a}{m} \binom{a+n-m}{n-m}.$$

$$19. \sum_{k=0}^n \binom{n}{k} \binom{a}{m-k} \binom{m-k}{n} = \binom{a}{m-n} \binom{a}{n}.$$

$$20. \sum_{k=0}^n \binom{n}{k} \binom{m}{n-k} \binom{a+n-k}{m+n} = \binom{a}{m} \binom{a}{n}.$$

$$21. \sum_{k=0}^{\min(m,n)} \binom{a}{k} \binom{b}{m-n+k} \binom{a+b+n-k}{a+b} = \binom{a+m}{n} \binom{b+n}{m}.$$

$$22. \sum_{k=0}^n \sum_{l=0}^m (-1)^{k+l} \binom{n}{n-k} \binom{m}{m-l} \binom{k+l}{k} = \delta_{n,m}.$$

$$23. \sum_{k=0}^n \binom{n-a+b}{k} \binom{n+a-b}{n-k} \binom{a+k}{m+n} = \binom{a}{m} \binom{b}{n}.$$

$$24. \sum_{k=0}^n \binom{a}{k} \binom{a+m-2}{k+m-1} \binom{a-k-1}{a-n-1} = \binom{a+m-2}{n+m-1} \binom{a+n+m-1}{n}.$$

$$25. \sum_{k=m \wedge n(l, m)}^n (-1)^k \binom{n}{k} \binom{k}{l} \binom{k}{m} = (-1)^n \binom{n}{m} \binom{m}{n-l}.$$

$$26. \sum_{k=n}^n (-1)^k \binom{n}{k} \binom{k}{m} \binom{2n-k}{m} = (-1)^n \binom{n}{m}^2.$$

$$27. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{1+(-1)^n}{2} \binom{n}{n/2}^2.$$

$$28. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{m} = 2^{n-m} \binom{n}{m} \binom{n+m}{m}.$$

$$29. \sum_{k=0}^{n-m} \binom{2n}{k} \binom{n-k}{m} \binom{m+n-k-1}{m-1} = \binom{2n-1}{n} \binom{n}{m}.$$

$$30. \sum_{k=0}^n (-1)^k \binom{a+b+1}{k} \binom{a+n-k}{a} \binom{b+n-k}{b} = \binom{a}{n} \binom{b}{n}.$$

$$31. \sum_{k=0}^n \frac{(-1)^k}{k+a} \binom{2n}{k} \binom{2n-k}{n}^2 = \frac{1}{a} \binom{2n}{n} \binom{2n+a}{n} \binom{a+n}{n}^{-1}.$$

$$32. \sum_{k=0}^n \frac{(-1)^k}{2n-k+1} \binom{2n}{k} \binom{2n-k}{n}^2 = \frac{1}{2n+1}.$$

$$33. \sum_{k=m}^{[n/2]} \frac{1}{2^{2k}} \binom{n}{2k} \binom{2k}{k} \binom{k}{m} = \frac{1}{2^m} \binom{n}{m} \binom{2n-2m}{n}.$$

$$34. \sum_{k=1}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k+m} = \frac{(-1)^n + (-1)^m}{2^{2n-1}} \binom{n+m}{(n+m)/2} \binom{n-m}{(n-m)/2}.$$

$$35. \sum_{k=1}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} \binom{n+k}{n} = \frac{1+(-1)^n}{2^{2n+1}} \binom{n}{n/2}.$$

$$4.2.10. \text{ Суммы вида } \sum_{k=0}^n \binom{a}{k} \binom{b}{k} \binom{c}{k}^{-1}.$$

$$1. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{mn+n}{k}^{-1} = \frac{[(mn)!]^2}{(mn-n)! (mn+n)!}.$$

$$2. \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k} \binom{m}{k}^{-1} = 2 \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^m \frac{1}{k} \quad [m \geq n \geq 1].$$

$$3. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{b}{k}^{-1} = \binom{b-a}{n} \binom{b}{n}^{-1}.$$

$$4. \sum_{k=0}^n \binom{n}{k} \binom{a}{k} \binom{b+k}{k}^{-1} = \binom{a+b+n}{n} \binom{b+n}{n}^{-1}.$$

$$5. \sum_{k=0}^n (-1)^k \binom{a}{k} \binom{b}{k} \binom{a+b-1}{k}^{-1} = (-1)^n \binom{a-1}{a} \binom{b-1}{b} \binom{a+b-1}{a}^{-1}.$$

$$6. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-m}{k} \binom{a}{k}^{-1} = 0 \quad [m < n].$$

$$7. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+a}{a} \binom{k+b}{b}^{-1} = \prod_{k=0}^{a-1} \left(1 - \frac{a+1}{b-k+1}\right).$$

$$8. \sum_{k=0}^{2n} (-1)^k \frac{2k+1}{n+k+1} \binom{2n}{k} \binom{2k}{k} \binom{n+k}{k}^{-1} = 1.$$

$$9. \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \binom{n+1}{k} \binom{m}{k}^{-1} = \sum_{k=1}^{m-1} \frac{1}{k} - 2 \sum_{k=1}^n \frac{1}{k} \quad [1 \leq m \leq n+1].$$

$$10. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{k}^{-1} x^k = (-1)^n \binom{2n}{n}^{-1} x^n P_n \left(1 - \frac{2}{x}\right).$$

$$11. \sum_{k=0}^n \binom{n}{k} \binom{2k}{k}^{-1} x^k = \binom{2n}{n}^{-1} (x-4)^n P_{2n} \left(\sqrt{\frac{x}{x-4}}\right).$$

$$12. \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k}^{-1} x^{2k} = T_n \left(1 - \frac{x^2}{2}\right).$$

13. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{2k}{k}^{-1} x^{2k} = \binom{a-1/2}{n}^{-1} \left(-\frac{x}{4}\right)^n C_n^{1/2+a-n} \left(1-\frac{2}{x}\right).$
14. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a-n+1/2}{k} \binom{a}{k}^{-1} x^{2k} = \binom{a}{2n}^{-1} \left(-\frac{x}{4}\right)^n C_{2n}^{1+a-2n} \left(\sqrt{1-\frac{1}{x}}\right).$
15. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2a-n+1}{k} \binom{2k}{k}^{-1} x^{2k} = (-1)^n \binom{2a+1}{n}^{-1} C_{2n}^{a-n-1/2} (1-2x).$
16. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{-1/2}{k} \binom{a-1/2}{k}^{-1} x^{2k} = \binom{2n}{n}^{-1} (4x)^n P_n \left(\frac{x^2+1}{2x}\right).$
17. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a}{k} \binom{2k}{k}^{-1} x^{2k} = \binom{1-a}{n}^{-1} C_{2n}^{a-n} \left(\frac{x}{2}\right).$
18. $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+1/2}{k} \binom{2k}{k}^{-1} x^{2k} = (-1)^n \binom{2n}{n}^{-1} 2^{1/2} P_n \left(\frac{x}{2}\right).$
19. $\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{a}{k}^{-1} x^{2k} = \binom{a}{n}^{-1} x^n C_n^{1+a-n} \left(\frac{1}{2x}\right) \quad [a > -1/2+n].$
20. $\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{n-1/2}{k}^{-1} x^{2k} = \binom{2n}{n}^{-1} (4x)^n P_n \left(\frac{1}{2x}\right).$
21. $\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \binom{2k}{k} \binom{n}{k}^{-1} x^{2k} = x^n L_n \left(\frac{1}{2x}\right).$
22. $\sum_{k=0}^n \frac{(-1)^k}{2^{2k}} \binom{n}{k} \binom{2k}{k} \binom{m+k}{k}^{-1} = \frac{1}{2^{2n}} \binom{2m+2n}{m+n} \binom{2m}{m}^{-1}.$
23. $\sum_{k=0}^n \frac{(-1)^k}{2^{2k+1}} \binom{n}{k} \binom{a+k}{k} \binom{2k}{k}^{-1} = \frac{2^{2n}}{2n+1} \binom{n-a-1/2}{n} \binom{2n}{n}^{-1}.$
24. $\sum_{k=0}^n \frac{(-1)^k 2^{2k}}{a+k} \binom{n}{k} \binom{a+k}{k} \binom{2k}{k}^{-1} = \frac{(-1)^n}{a} \binom{2a}{2n} \binom{a}{n}^{-1}.$
25. $\sum_{k=1}^n \frac{(-1)^{k+1} 2^{2k}}{k} \binom{n}{k} \binom{2k}{k}^{-1} = 2 \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k}.$

26. $\sum_{k=0}^n \frac{(-1)^k}{2^{2k+1}} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k}^{-1} = \frac{(-1)^n}{2^{2n+1}}.$
27. $\sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{l}{k} \binom{m+n+l}{k}^{-1} = \binom{m+n}{n} \binom{n+l}{n} \binom{n+m+l}{n}^{-1}.$
28. $\sum_{k=0}^n \frac{(-1)^k}{k} \binom{n}{k} \binom{2m+n-1}{k} \binom{m}{k}^{-1} = 2\psi(m) - 2\psi(m+n).$

4.3. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

4.3.1. СУММЫ $\sum_{k=0}^n \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)}$

1. $\sum_{k=0}^n \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} = \text{cosech} \frac{x}{2} \text{sh} \frac{n+1}{2} x \left\{ \frac{\text{sh} \{(nx/2)+a\}}{\text{ch} \{(nx/2)+a\}} \right\}.$
2. $\sum_{k=0}^n (-1)^k \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} =$

$$= \frac{\text{sech}(x/2)}{2} \left[\frac{\text{sh} \{a-(x/2)\}}{\text{ch} \{a-(x/2)\}} + (-1)^n \frac{\text{sh} \{(2nx+x+2a)/2\}}{\text{ch} \{(2nx+x+2a)/2\}} \right].$$
3. $\sum_{k=1}^n k \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} =$

$$= -\text{cosech} \frac{x}{2} \left[(n+1) \frac{\text{sh} \{(nx+a)\}}{\text{ch} \{(nx+a)\}} - n \frac{\text{sh} \{(nx+x+a)\}}{\text{ch} \{(nx+x+a)\}} - \frac{\text{sh} a}{\text{ch} a} \right].$$
4. $\sum_{k=1}^n (-1)^k k \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} =$

$$= \text{sech} \frac{x}{2} \left[(-1)^n (n+1) \frac{\text{sh} \{(nx+a)\}}{\text{ch} \{(nx+a)\}} + (-1)^n n \frac{\text{sh} \{(nx+x+a)\}}{\text{ch} \{(nx+x+a)\}} - \frac{\text{sh} a}{\text{ch} a} \right].$$
5. $\sum_{k=0}^n \binom{n}{k} \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} = 2^n \text{ch}^n \frac{x}{2} \left\{ \frac{\text{sh} \{(nx/2)+a\}}{\text{ch} \{(nx/2)+a\}} \right\}.$
6. $\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\text{sh}(kx+a)}{\text{ch}(kx+a)} =$

$$= 2^{n-1} \text{sh}^n \frac{x}{2} \left[1 + (-1)^n \frac{\text{sh} \{(nx/2)+a\}}{\text{ch} \{(nx/2)+a\}} - \right.$$

$$\left. - 2^{n-1} \text{sh}^n \frac{x}{2} \left[1 + (-1)^n \frac{\text{sh} \{(nx/2)+a\}}{\text{ch} \{(nx/2)+a\}} \right] \right].$$

7. $\sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{sh} (n-k)x =$

$$= \frac{(-1)^{n-1}}{4} 2^{2n+1} x \int_0^{\infty} \frac{1}{t^2 + 4t} \sin^{2n} t \, dt + 2^{2n-1} \operatorname{sh} \frac{x}{2}.$$
8. $\sum_{k=0}^n (-1)^k \binom{2n+1}{k} \operatorname{sh} (2n-2k+1)x = 2^{2n} \operatorname{sh} 2^{n+1} x.$
9. $\sum_{k=0}^{n-1} \frac{(-1)^k}{n-k} \binom{2n}{k} \operatorname{sh} (n-k)x = 2^{2n-1} \int_0^x \frac{[\operatorname{ch}(x/2)]^{2n}}{[\operatorname{sh}(x/2)]} dx - \frac{(-1)^n}{2} \binom{2n}{n/2}.$
10. $\sum_{k=0}^n (-1)^k \binom{2n}{k} \operatorname{ch} (n-k)x = 2^{2n-1} \operatorname{sh} \frac{x}{2} + \frac{(-1)^n}{2} \binom{2n}{n}.$
11. $\sum_{k=0}^n \frac{1}{2n-2k+1} \binom{2n+1}{k} \operatorname{sh} (2n-2k+1)x = 2^{2n} \int_0^x \operatorname{ch}^{2n+1} x \, dx.$
12. $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \operatorname{ch} (n-2k)x = 2^{n-1} \operatorname{ch}^n x + \frac{1}{4} [1 + (-1)^n] \binom{n}{n/2}.$
13. $\sum_{k=0}^{n-1} r^k \frac{\operatorname{sh} kx}{\operatorname{ch} kx} =$

$$= \frac{1}{1-2r \operatorname{ch} x + r^2} \left[r^{n+1} \frac{\operatorname{sh} (n+1)x}{\operatorname{ch} (n+1)x} - r^n \frac{\operatorname{sh} nx}{\operatorname{ch} nx} \pm r \frac{\operatorname{sh} x}{\operatorname{ch} x} + \frac{[0]}{[1]} \right].$$

4.3.2. Суммы, содержащие $\operatorname{sech} x$ и $\operatorname{th} x$.

1. $\sum_{k=0}^{n-1} 2^{2k} \operatorname{sech}^2 2^k x = \operatorname{cosech}^2 x - 2^{2n} \operatorname{cosech}^2 2^n x.$
2. $\sum_{k=1}^n \frac{1}{2^{2k}} \operatorname{sech}^2 \frac{x}{2^k} = \frac{1}{2^{2n}} \operatorname{cosech}^2 \frac{x}{2^n} - \operatorname{cosech}^2 x.$
3. $\sum_{k=1}^n \frac{1}{2^k} \operatorname{th} \frac{x}{2^k} = \operatorname{cth} x - \frac{1}{2^n} \operatorname{cth} \frac{x}{2^n}.$
4. $\sum_{k=0}^{n-1} 2^k \operatorname{th} 2^k x = 2^n \operatorname{cth} 2^n x - \operatorname{cth} x.$
5. $\sum_{k=1}^n \frac{1}{2^{2k}} \operatorname{th}^2 \frac{x}{2^k} = \operatorname{cth}^2 x - \frac{2}{3} \left(1 - \frac{1}{2^{2n}} \right) - \frac{1}{2^{2n}} \operatorname{cth}^2 \frac{x}{2^n}.$

4.4. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

4.4.1. Суммы вида $\sum_{k=0}^n \frac{\sin(kx+a)}{\cos(kx+a)}$.

1. $\sum_{k=0}^n \frac{\sin(kx+a)}{\cos(kx+a)} = (-1)^m \frac{d^{2m}}{dx^{2m}} \sum_{k=0}^n \frac{\sin(kx+a)}{\cos(kx+a)},$
2. $\sum_{k=1}^n k^{2m+1} \frac{\sin(kx+a)}{\cos(kx+a)} = \varepsilon \frac{d^{2m+1}}{dx^{2m+1}} \sum_{k=1}^n \frac{\cos(kx+a)}{\sin(kx+a)}$
 $[\varepsilon = \pm 1, \text{ если } m \text{ — нечетное; } \varepsilon = \mp 1, \text{ если } m \text{ — четное}].$
3. $\sum_{k=0}^n (-1)^k k^{2m} \frac{\sin(kx+a)}{\cos(kx+a)} = (-1)^m \frac{d^{2m}}{dx^{2m}} \sum_{k=0}^n (-1)^k \frac{\sin(kx+a)}{\cos(kx+a)},$
4. $\sum_{k=1}^n (-1)^k k^{2m+1} \frac{\sin(kx+a)}{\cos(kx+a)} = \varepsilon \frac{d^{2m+1}}{dx^{2m+1}} \sum_{k=1}^n (-1)^k \frac{\cos(kx+a)}{\sin(kx+a)}$
 $[\varepsilon = \pm 1, \text{ если } m \text{ — нечетное; } \varepsilon = \mp 1, \text{ если } m \text{ — четное}].$
5. $\sum_{k=0}^n \frac{\sin(kx+a)}{\cos(kx+a)} = \operatorname{cosec} \frac{x}{2} \operatorname{sh} \frac{(n+1)x}{2} \frac{\sin[(nx/2)+a]}{\cos[(nx/2)+a]}.$
6. $\sum_{k=0}^n (-1)^k \frac{\sin(kx+a)}{\cos(kx+a)} = \sec \frac{x}{2} \sin \frac{(n+1)(x+\pi)}{2} \frac{\sin[n(x+\pi)/2+a]}{\cos[n(x+\pi)/2+a]}.$
7. $\sum_{k=1}^n k \frac{\sin(kx+a)}{\cos(kx+a)} =$

$$= \operatorname{cosec}^2 \frac{x}{2} \left[(n+1) \frac{\sin(nx+a)}{\cos(nx+a)} - n \frac{\sin(nx+x+a)}{\cos(nx+x+a)} - \frac{\sin a}{\cos a} \right].$$
8. $\sum_{k=1}^n (-1)^k k \frac{\sin(kx+a)}{\cos(kx+a)} =$

$$= \sec^2 \frac{x}{2} \left[(-1)^n (n+1) \frac{\sin(nx+a)}{\cos(nx+a)} + (-1)^n n \frac{\sin(nx+x+a)}{\cos(nx+x+a)} - \frac{\sin a}{\cos a} \right].$$
9. $\sum_{k=0}^n \binom{n}{k} \frac{\sin(kx+a)}{\cos(kx+a)} = 2^n \cos^{\frac{x}{2}} \frac{\sin[(nx/2)+a]}{\cos[(nx/2)+a]}.$
10. $\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\sin(kx+a)}{\cos(kx+a)} = (-1)^n 2^n \sin^{\frac{x}{2}} \frac{\sin[n(x+\pi)/2+a]}{\cos[n(x+\pi)/2+a]}.$

$$11. \sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{\sin kx}{\cos kx} = 2^{n-1} \left[\cos^n \frac{x}{4} \frac{\sin (nx/4)}{\cos (nx/4)} + (-1)^n \sin^n \frac{x}{4} \frac{\sin [n(2\pi+x)/4]}{\cos [n(2\pi+x)/4]} \right].$$

$$12. \sum_{k=0}^{[n-1]/2} \binom{n}{2k+1} \frac{\sin (2k+1)x}{\cos (2k+1)x} = 2^{n-1} \left[\cos^n \frac{x}{2} \frac{\sin (nx/2)}{\cos (nx/2)} - (-1)^n \sin^n \frac{x}{2} \frac{\sin [n(\pi+x)/2]}{\cos [n(\pi+x)/2]} \right].$$

$$13. \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \frac{\sin kx}{\cos kx} = 2^{n-1} \left[\cos^n \frac{x}{4} \frac{\sin [(n-2)x/4]}{\cos [(n-2)x/4]} - (-1)^n \sin^n \frac{x}{4} \frac{\sin [(2n\pi+nx-2x)/4]}{\cos [(2n\pi+nx-2x)/4]} \right].$$

$$14. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{\sin kx}{\cos kx} = 2^{n-1} \left[\cos^n \frac{\pi+x}{4} \frac{\sin [n(\pi+x)/4]}{\cos [n(\pi+x)/4]} + (-1)^n \sin^n \frac{\pi+x}{4} \frac{\sin [(3n\pi+nx)/4]}{\cos [(3n\pi+nx)/4]} \right].$$

$$15. \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \frac{\sin kx}{\cos kx} = 2^{n-1} \left[(-1)^n \sin^n \frac{\pi+x}{4} \frac{\cos [(3n\pi+nx-2x)/4]}{\sin [(3n\pi+nx-2x)/4]} + \cos^n \frac{\pi+x}{4} \frac{\cos [(n\pi+nx-2x)/4]}{\sin [(n\pi+nx-2x)/4]} \right].$$

$$16. \sum_{k=0}^{n-1} (\pm 1)^k \binom{2n}{k} \cos (2n-2k)x = -\frac{(\pm 1)^n}{2} \binom{2n}{n} + (\pm 1)^n 2^{2n-1} \frac{\cos x}{\sin x}.$$

$$17. \sum_{k=0}^n (\pm 1)^k \binom{2n+1}{k} \cos (2n-2k+1)x = (\pm 1)^n 2^{2n} \frac{\cos x}{\sin x}.$$

$$18. \sum_{k=0}^{[n/2]} \binom{n}{k} \cos (n-2k)x = 2^{n-1} \cos^n x + \frac{1+(-1)^n}{4} \binom{n}{n/2}.$$

$$19. \sum_{k=1}^n (-1)^k \binom{2n}{n-k} k^{2m-1} \sin 2kx = (-1)^m 2^{2n-2m} \frac{d^{2m-1}}{dx^{2m-1}} \sin^{2n} x.$$

$$20. \sum_{k=1}^n (-1)^k \binom{2n}{n-k} k^{2m} \cos 2kx = (-1)^m 2^{2n-2m-1} \frac{d^{2m}}{dx^{2m}} \sin^{2n} x.$$

$$21. \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} (2k+1)^{2m-1} \cos (2k+1)x = (-1)^{n+1} 2^{2n} \frac{d^{2m-1}}{dx^{2m-1}} \sin^{2n+1} x.$$

$$22. \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} (2k+1)^{2m} \sin (2k+1)x = (-1)^n 2^{2n} \frac{d^{2m}}{dx^{2m}} \sin^{2n+1} x.$$

$$23. \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{n-k} \binom{2n}{k} \sin (n-k)x = (-1)^n 2^{2n-1} \int_0^x \frac{\cos (x/2)}{\sin (x/2)} dx - \frac{(\pm 1)^n}{2} x \binom{2n}{n}.$$

$$24. \sum_{k=0}^n \frac{(\pm 1)^k}{2n-2k+1} \binom{2n+1}{k} \frac{\sin (2n-2k+1)x}{\cos (2n-2k+1)x} = (\pm 1)^n 2^{2n} \int_0^x \frac{\cos x}{\sin x} dx.$$

$$25. \sum_{k=0}^n \binom{2k}{k} \frac{2n-2k}{n-k} \cos (n-2k)x = 4^n P_n(\cos x).$$

$$26. \sum_{k=0}^n \frac{\Gamma(k+\alpha) \Gamma(n-k+\alpha)}{k! (n-k)!} \cos (n-2k)x = \Gamma^2(\alpha) C_n^\alpha(\cos x).$$

$$27. \sum_{k=1}^n r^k \frac{\sin (kx+a)}{\cos (kx+a)} = (1-2r \cos x + r^2)^{-1} \left[-r^2 \frac{\sin x}{\cos x} + r \frac{\sin (x+a)}{\cos (x+a)} + r^{n+2} \frac{\sin (nx+a)}{\cos (nx+a)} - r^{n+1} \frac{\sin [(n+1)x+a]}{\cos [(n+1)x+a]} \right].$$

4.4.2. Суммы вида $\sum_{k=0}^n a_k \frac{\sin kx}{\cos kx}^m$

$$1. \sum_{k=1}^n \frac{\sin kx}{\cos kx}^{2m} = \frac{1}{2^{2m-1}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \times \frac{\cos (n+1)x \sin nx}{\sin kx} + \frac{n}{2^{2m}} \binom{2m}{m}.$$

$$2. \sum_{k=1}^n \frac{\sin kx}{\cos kx}^{2m+1} = -\frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{m-k} \frac{\sin [(2k+1)(n+1)x/2]}{\cos [(2k+1)(n+1)x/2]} \frac{\sin [(2k+1)nx/2]}{\sin [(2k+1)x/2]}.$$

3. $\sum_{k=1}^m (-1)^{k-1} \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m} =$
 $= \frac{1}{2^m} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \frac{\cos kx + (-1)^{m-1} \cos (2n+1)kx}{\cos kx} + \frac{1 - (-1)^m}{2^{2m+1}} \binom{2m}{m}.$
4. $\sum_{k=1}^n (-1)^{k-1} \left\{ \frac{\sin kx}{\cos kx} \right\}^{2m+1} = \frac{1}{2^{2m+1}} \sum_{k=0}^m (-1)^k \binom{2m+1}{m-k} \times$
 $\times \frac{\left\{ \frac{\sin [(2k+1)x/2]}{\cos [(2k+1)x/2]} + (-1)^{n-1} \frac{\sin [(2n+1)(2k+1)x/2]}{\cos [(2n+1)(2k+1)x/2]} \right\}}{\cos [(2k+1)x/2]}.$
5. $\sum_{k=1}^n (\pm 1)^k a^k b^n \left\{ \frac{\sin kx}{\cos kx} \right\}^m = \left(a \frac{d}{da} \right)^m \sum_{k=1}^n (\pm 1)^k a^k \left\{ \frac{\sin kx}{\cos kx} \right\}^m.$
6. $\sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^2 = \frac{n}{2} + \frac{1}{2} \sin nx \cos (n+1)x \operatorname{cosec} x.$
7. $\sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^3 = \frac{3}{4} \sin nx \operatorname{cosec} \frac{x}{2} \frac{\sin [(n+1)x/2]}{\cos [(n+1)x/2]} +$
 $+ \frac{1}{4} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2} \frac{\sin [3(n+1)x/2]}{\cos [3(n+1)x/2]}.$
8. $\sum_{k=1}^n \left\{ \frac{\sin kx}{\cos kx} \right\}^4 =$
 $= \frac{1}{8} [3n + 4 \sin nx \cos (n+1)x \operatorname{cosec} x + \sin 2nx \cos 2(n+1)x \operatorname{cosec} 2x].$
9. $\sum_{k=0}^{n-1} (-1)^k \cos^m \frac{\pi k}{n} = \frac{1}{2} [1 - (-1)^{m+1}] \quad [m = 0, 1, 2, \dots, n-1]$
10. $\sum_{k=0}^{n-1} (-1)^k \cos^n \frac{\pi k}{n} = \frac{n}{2^{n-1}}.$
11. $\sum_{k=1}^{[n/2]} \cos^{2m} \frac{\pi k}{n} = \frac{n}{2^{2m+1}} \binom{2m}{m} - \frac{1}{2} \quad [n < 2]$
12. $\sum_{k=1}^n \sin^{2m} \frac{\pi k}{2n} = \frac{1}{2} + \frac{n}{2^{2m}} \binom{2m}{m} \quad [n < 2]$

13. $\sum_{k=1}^n \left\{ \frac{\sin (2\pi k/n)}{\cos (2\pi k/n)} \right\}^m = [1 + (-1)^m] n \binom{m}{[m/2]} \left(\frac{1}{2} \right)^{m-1} \frac{1}{2^{m+1}}$
 $[m, l = 1, 2, 3, \dots; ml < n].$
14. $\sum_{k=0}^{n-1} \left\{ \frac{\sin (2\pi k^2/n)}{\cos (2\pi k^2/n)} \right\} = \frac{1}{2} n \left(1 + \cos \frac{\pi n}{2} + \sin \frac{\pi n}{2} \right).$
- 4.4.3. Суммы вида $\sum a_k \left\{ \frac{\sin b_k x}{\cos b_k x} \right\}^m.$
1. $\sum_{k=1}^n 2^k \sin^2 \frac{x}{2^k} = 2^n \sin^2 \frac{x}{2^n} - \sin^2 x.$
2. $\sum_{k=0}^n \frac{(-1)^k}{3^k} \cos^3 3^k x = \frac{3}{4} \left[\cos x + \frac{(-1)^n}{3^{n+1}} \cos 3^{n+1} x \right].$
3. $\sum_{k=1}^n \frac{(-1)^k}{2^{2k}} \cos^2 2^k x = \frac{(-1)^n}{2^{2n+2}} \cos^2 2^{n+1} x - \frac{1}{4} \cos^2 2x + \frac{1}{20} [1 - (-1)^n].$
- 4.4.4. Суммы вида $\sum a_k \left\{ \frac{\sin kx}{\cos kx} \right\}^l \left\{ \frac{\sin ky}{\cos ky} \right\}^m.$
1. $\sum_{k=0}^{n-1} \sin \frac{lk\pi}{n} \sin \frac{mk\pi}{n} = 0$
 $[\text{либо числа } l-m, l+m \text{ не делятся на } 2n, \text{ либо оба делятся на } 2n].$
2. $= \frac{n}{2} \quad [l-m \text{ делится, } l+m \text{ не делится на } 2n].$
3. $= -\frac{n}{2} \quad [l-m \text{ не делится, } l+m \text{ делится на } 2n].$
4. $\sum_{k=0}^{n-1} \sin \frac{lk\pi}{n} \cos \frac{mk\pi}{n} = 0.$
5. $\sum_{k=0}^{n-1} \cos \frac{lk\pi}{n} \cos \frac{mk\pi}{n} = 0$
 $[m, l - \text{целые; } m+l, m-l \text{ не делятся на } n; m \neq l].$
6. $\sum_{k=0}^{n-1} \cos \frac{lk\pi}{n} \cos \frac{mk\pi}{n} = 0$
 $[l-m, l+m \text{ не делятся на } n].$
7. $= n/2 \quad [\text{либо одно из чисел } l-m, l+m \text{ делится на } n].$
8. $= n \quad [l-m, l+m \text{ делятся на } n].$
9. $\sum_{k=0}^{n-1} \sin \frac{lk\pi}{n} \sin \frac{mk\pi}{n} = \begin{cases} 0, & m \neq l, \\ n, & m = l \neq 0 \end{cases} \quad [m, l < n].$

$$10. \sum_{k=0}^{2n-1} \sin \frac{km\pi}{n} \cos \frac{kl\pi}{n} = 0$$

{m, l < n}

$$11. \sum_{k=0}^{2n-1} \cos \frac{km\pi}{n} \cos \frac{kl\pi}{n} = 0$$

{m ≠ 0}

$$12. = n$$

{m = l не делится на n}

$$13. = 2n$$

{m = l делится на n}

$$14. \sum_{k=0}^{n-1} (-1)^k \cos \frac{2km\pi}{n} \cos \frac{k\pi}{n} = \frac{n}{2} \binom{n}{m}$$

{1 ≤ m ≤ n-1}

$$15. \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \sin^{2k} x \cos^{n-2k} x = \cos nx.$$

$$16. \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \sin^{2k+1} x \cos^{n-2k-1} x = \sin nx.$$

$$17. \sum_{k=0}^n 2^k \sin \frac{x}{2^k} \sin^2 \frac{x}{2^{k+1}} = 2^{n-1} \sin \frac{x}{2^n} - \frac{1}{4} \sin 2x.$$

$$18. \sum_{k=0}^n (-2)^k \sin \frac{x}{2^k} \cos^3 \frac{x}{2^k} = \frac{1}{8} \sin 4x + (-2)^{n-2} \sin \frac{x}{2^{n-1}}.$$

4.4.5. Разные суммы, содержащие $\sin x$, $\cos x$.

$$1. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n}\pi} \sin^{2m+1} \frac{2k+1}{2n}\pi = n \sec nx \sin^m x$$

{m < n}

$$2. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n}\pi} \sin \frac{2k+1}{2n}\pi \cos^m \frac{2k+1}{2n}\pi = n \sec nx \cos^m x$$

{m < n}

$$3. \sum_{k=1}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{k\pi}{n}} \cos^{2m} \frac{k\pi}{n} = \frac{1}{4} \operatorname{cosec} \frac{x}{2} \left(\operatorname{cosec} \frac{x}{2} - 2n \operatorname{cosec} nx \cos^{2m-1} \frac{x}{2} \right)$$

{1 ≤ m ≤ n/2}

$$4. \sum_{k=1}^n \frac{(-1)^k}{\cos x - \cos \frac{2k\pi}{2n+1}} \cos^{2m+1} \frac{2k\pi}{2n+1} =$$

$$= \frac{1}{4} \operatorname{cosec} \frac{x}{2} \left[\operatorname{cosec} \frac{x}{2} - (2n+1) \operatorname{cosec} (2n+1) \frac{x}{2} \cos^{2m} \frac{x}{2} \right]$$

{m < n}

{0, 1, 2, ...}

$$5. \sum_{k=1}^n \frac{(-1)^k}{\cos x - \cos \frac{2k\pi}{2n+1}} \cos \frac{2k\pi}{2n+1} \sin^{2m} \frac{2k\pi}{2n+1} =$$

$$= -\frac{1}{4} (2n+1) \operatorname{cosec} \frac{(2n+1)x}{2} \sin^{2m-1} \frac{x}{2}$$

{m < 2n+2}

$$6. \sum_{k=0}^{n-1} \frac{(-1)^k}{1 - 2x \cos \frac{2k+1}{2n}\pi + x^2} \sin \frac{2k+1}{2n}\pi = \frac{nx^{n-1}}{1+x^{2n}}$$

$$7. \sum_{k=0}^{n-1} \frac{(-1)^k}{\cos x - \cos \frac{2k+1}{2n}\pi} \cos \frac{2k+1}{2n}\pi \sin \frac{2k+1}{2n}\pi = n \cos mx \sec nx$$

{m < n}

$$8. \sum_{k=0}^{[(n-2)/2]} \frac{1}{\cos x - \cos [(2k+1)\pi/n]} = \frac{n}{2} \operatorname{cosec} x \lg \frac{nx}{2} - \frac{1}{8} [1 - (-1)^n] \sec^2 \frac{x}{2}$$

$$9. \sum_{k=0}^{n-1} \frac{2^k}{2 \cos (2^k x) - 1} \sin (2^k x) = \frac{2^n}{2^n \cos (2^n x) + 1} \sin (2^n x) - \frac{\sin x}{2 \cos x + 1}$$

$$10. \sum_{k=2}^n \prod_{m=1}^{k-1} \sin \frac{m\pi}{k} = 3 - \frac{n+2}{2^{n-1}}$$

$$11. \sum_{k=2}^n \prod_{m=1}^{k-1} \cos \frac{m\pi}{k} = -\frac{1}{5 \cdot 2^n} \{2^n + (-1)^{[n/2]} + 3(-1)^{[(n+1)/2]}\}$$

$$12. \sum_{k=1}^n \prod_{m=1}^{2k-1} \sin \frac{m\pi}{2k+1} = \frac{1}{9} \left[11 - \frac{1}{2^{2n}} (6n+11) \right]$$

$$13. \sum_{k=1}^n (-1)^{k+1} \prod_{m=1}^{2k} \sin \frac{m\pi}{2k+1} = \frac{1}{25} \left[13 + \frac{(-1)^{n+1}}{2^{2n}} (10n+13) \right]$$

$$14. \sum_{k=1}^n \prod_{m=1}^{2k} \cos \frac{m\pi}{2k+1} = \frac{(-1)^n}{5 \cdot 4^n} - \frac{1}{5}$$

$$15. \sum_{k=1}^n (-1)^{k+1} \prod_{m=0}^{2k} \cos \frac{m\pi}{2k+1} = \frac{1}{3 \cdot 4^n} - \frac{1}{3}$$

$$16. \sum_{k=1}^n \prod_{m=0}^{2k-1} \sin \frac{2m+1}{2k}\pi = 2 - \frac{1}{2^{n-1}}$$

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17. $\sum_{k=1}^n (-1)^{k-1} \prod_{m=0}^{k-1} \sin \frac{2m+1}{2k} \pi = \frac{2}{3} + \frac{(-1)^{n-1}}{3 \cdot 2^{n-1}}.$
18. $\sum_{k=2}^n \prod_{m=0}^{k-1} \cos \frac{2m+1}{2k} \pi = \frac{1}{3 \cdot 2^n} [(-1)^{(n-1)/2} + 3(-1)^{[n/2]} - 2^n + 1].$
19. $\sum_{k=1}^n \prod_{m=0}^{k-1} (-1)^m \sin \frac{2m+1}{2k} \pi = \frac{2}{3} + \frac{(-1)^{[n/2]}}{3 \cdot 2^{n-1}} [1 - 2(-1)^n].$
20. $\sum_{k=1}^n \prod_{m=0}^{k-1} \sin \frac{2m+1}{2k} \pi = \frac{2^{n-1} - 1}{2^{n-1} (2^2 - 1)}.$
21. $\sum_{k=1}^n \prod_{m=1}^{2k} \cos \frac{m\pi}{2k+1} = (-1)^l \frac{2^{2nl} - (-1)^{nl}}{2^{2nl} [2^{2l} - (-1)^l]}.$
22. $\sum_{k=1}^n \prod_{m=0}^{2k-1} \cos \frac{2m+1}{4k} \pi = (-1)^l \frac{2^{2nl} - (-1)^{nl}}{2^{2nl} [2^{2l} - (-1)^l]}.$

4.4.6. Суммы, содержащие $\sec x$ и $\operatorname{cosec} x$.

1. $\sum_{k=0}^n \sec \frac{k\pi}{n} = 0.$
2. $\sum_{k=0}^n \sec \frac{2k\pi}{2n+1} = (-1)^n \frac{2n+1}{2} + \frac{1}{2}.$
3. $\sum_{k=0}^{n-1} \sec^2 \left(\frac{k\pi}{n} + x \right) = n^2 \operatorname{cosec}^2 \left(\frac{n\pi}{2} + nx \right).$
4. $\sum_{k=1}^{[(n-1)/2]} \sec^2 \frac{k\pi}{n} = \frac{1}{12} (4n^2 - 7) + \frac{(-1)^{n+1}}{12} (2n^2 + 1).$
5. $\sum_{k=1}^{n-1} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{n^2 - 1}{3}.$
6. $\sum_{k=1}^{[(n-1)/2]} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{1}{6} (n^2 - 1) - \frac{1}{4} [1 + (-1)^n].$
7. $\sum_{k=0}^{[(n-2)/2]} \operatorname{cosec}^3 \frac{2k+1}{2n} \pi = \frac{n^3}{2} + \frac{1}{4} [(-1)^n - 1].$

(1.6)

8. $\sum_{k=1}^n \frac{\pi k}{2n+1} = \frac{n(2n+2)}{3}.$
9. $\sum_{k=0}^{n-1} \frac{\{\sec [x + (k\pi/n)]\}^2}{\{\operatorname{cosec} [x + (k\pi/n)]\}} = n^2 \frac{\{\operatorname{cosec} [n(2x + \pi)/2]\}^2}{\{\operatorname{cosec} nx\}}.$
10. $\sum_{k=0}^{n-1} \frac{\{\sec [x + (2k\pi/n)]\}^2}{\{\operatorname{cosec} [x + (2k\pi/n)]\}} = \frac{1}{2} [1 - (-1)^n] n^2 \frac{\{\operatorname{cosec} [n(2x + \pi)/2]\}^2}{\{\operatorname{cosec} nx\}} + \frac{1}{4} [1 + (-1)^n] n^2 \frac{\{\operatorname{cosec} [(2nx + \pi)/4]\}^2}{\{\operatorname{cosec} (nx/2)\}}.$
11. $\sum_{k=1}^{2n} (-1)^k \sin^2 \frac{k\pi}{2n+1} \sec \frac{k\pi}{2n+1} = (-1)^n (2n+1) \quad [n \geq 1].$
12. $\sum_{k=0}^{n-1} \cos \left[(2m+1) \left(x + \frac{k\pi}{n} \right) \right] \operatorname{cosec} \left(x + \frac{k\pi}{n} \right) = n \operatorname{ctg} nx$
 $\{m = 0, 1, \dots, n-1; l \text{ и } n - \text{взаимно простые числа}\}.$
13. $\sum_{k=1}^n \cos \left(x + \frac{k\pi}{n} \right) \operatorname{cosec}^2 \left(x + \frac{k\pi}{n} \right) = n^3 \cos nx \operatorname{cosec}^3 nx.$
14. $\sum_{k=1}^n \frac{\{\sec (kx+a) \sec [(k+1)x+a]\}}{\{\operatorname{cosec} (kx+a) \operatorname{cosec} [(k+1)x+a]\}} =$
 $= \operatorname{cosec} x \left[\pm \frac{\{\operatorname{ctg} [(n+1)x+a]\}}{\{\operatorname{ctg} [(n+1)x+a]\}} \mp \frac{\{\operatorname{ctg} (x+a)\}}{\{\operatorname{ctg} (x+a)\}} \right].$
15. $\sum_{k=0}^{n-1} (-1)^k \frac{\{\sec (kx+a) \sec [(k+2)x+a]\}}{\{\operatorname{cosec} (kx+a) \operatorname{cosec} [(k+2)x+a]\}} =$
 $= \frac{1}{2} \sec x \left[\frac{\{\sec (x+a) \sec a\}}{\{\operatorname{cosec} (x+a) \operatorname{cosec} a\}} - (-1)^n \frac{\{\sec (nx+a) \sec [(n+1)x+a]\}}{\{\operatorname{cosec} (nx+a) \operatorname{cosec} [(n+1)x+a]\}} \right].$
16. $\sum_{k=1}^n \operatorname{cosec} \frac{k}{2n} = \operatorname{ctg} \frac{x}{2n+1} - \operatorname{ctg} x.$
17. $\sum_{k=1}^n \frac{1}{2^{2k}} \sec^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{2^{2n}} \operatorname{cosec}^2 \frac{x}{2^n}.$
18. $\sum_{k=0}^{n-1} \frac{1}{2^k} \sin^2 \frac{x}{2^{k+1}} \operatorname{cosec} \frac{x}{2^k} = \frac{1}{2} \operatorname{cosec} x - \frac{1}{2^{n+1}} \operatorname{cosec} \frac{x}{2^n}.$

$$19. \sum_{k=0}^n \frac{1}{2^{2k}} \sin \frac{x}{2^k} \sec^3 \frac{x}{2^k} = \frac{1}{2^{2n}} \cos \frac{x}{2^n} \operatorname{cosec}^3 \frac{x}{2^n} - 8 \cos 2x \operatorname{cosec}^3 2x.$$

4.4.7. Суммы, содержащие $\operatorname{tg} a_k$ и $\operatorname{ctg} a_k$.

$$1. \sum_{k=0}^{n-1} \left\{ \operatorname{tg} [x + (km\pi/n)] \right\} = \mp n \left\{ \operatorname{ctg} [(2n\pi + \pi)/2] \right\}$$

[m, n — взаимно простые числа].

$$2. \sum_{k=0}^{n-1} (-1)^k \left\{ \operatorname{tg} [(2k+1)\pi/(4n)] \right\} = (\mp 1)^{n-1} n.$$

$$3. \sum_{k=0}^{2n-1} (-1)^k \operatorname{ctg} \frac{(2k+1)\pi}{8n} = 2n.$$

$$4. \sum_{k=0}^{n-1} \left\{ \operatorname{ctg} [x + (km\pi/n)] \right\}^2 = n^2 - n + n^2 \left\{ \operatorname{ctg} [(2n\pi + \pi)/2] \right\}^2.$$

$$5. \sum_{k=1}^{(n-1)/2} \operatorname{tg}^2 \frac{k\pi}{n} = \frac{1}{6} (n-1) [2n-1 - (-1)^n (n+1)].$$

$$6. \sum_{k=1}^{n-1} \operatorname{ctg}^2 \frac{k\pi}{n} = \frac{1}{3} (n-1) (n-2).$$

$$7. \sum_{k=1}^{(n-1)/2} \operatorname{ctg}^2 \frac{k\pi}{n} = \frac{1}{6} (n-1) (n-2).$$

$$8. \sum_{k=1}^n \operatorname{ctg}^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}.$$

$$9. \sum_{k=1}^n \operatorname{tg}^2 \frac{k\pi}{2n+1} = \frac{n}{3} (2n+1) (4n^2 + 6n - 1).$$

$$10. \sum_{k=1}^n \operatorname{ctg}^2 \frac{k\pi}{2n+1} = \frac{n}{45} (2n-1) (4n^2 + 10n - 9).$$

$$11. \sum_{k=0}^n \frac{1}{2^k} \operatorname{tg} \frac{x}{2^k} = \frac{1}{2^n} \operatorname{ctg} \frac{x}{2^n} - 2 \operatorname{ctg} 2x.$$

$$12. \sum_{k=0}^n \frac{1}{2^{2k}} \operatorname{tg}^2 \frac{x}{2^k} = \frac{1}{3 \cdot 2^{2n-1}} (2^{2n+2} - 1) + 4 \operatorname{ctg}^2 2x - 2^{-2n} \operatorname{ctg}^2 \frac{x}{2^n}.$$

$$13. \sum_{k=0}^{n-1} \left\{ \operatorname{tg} (kx+a) \operatorname{tg} [(k+1)x+a] \right\} = \operatorname{ctg} x \left[\pm \left\{ \operatorname{tg} (nx+a) \right\} \mp \operatorname{tg} a \right] - n.$$

$$14. \sum_{k=0}^{n-1} \operatorname{tg} \frac{x}{2^{k+1}} \sec^3 \frac{x}{2^k} = \operatorname{tg} x - \operatorname{tg} \frac{x}{2^n}.$$

$$15. \sum_{k=0}^{n-1} 2^k \sin \frac{x}{2^{k+1}} \operatorname{tg}^2 \frac{x}{2^{k+1}} = \operatorname{tg} x - 2^n \operatorname{tg} \frac{x}{2^n}.$$

$$16. \sum_{k=0}^{n-1} (-1)^k \binom{n}{2k} \operatorname{tg}^{2k} x = \frac{\cos nx}{\cos^n x}.$$

$$17. \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{2k+1} \operatorname{tg}^{2k+1} x = \frac{\sin nx}{\cos^n x}.$$

$$18. \sum_{k=1}^n \prod_{m=1}^{k-1} \operatorname{tg} \frac{m\pi}{2k+1} = (-1)^n (n+1) - 1.$$

$$19. \sum_{k=1}^n \prod_{m=1}^{k-1} \operatorname{tg} \frac{2m+1}{4k} \pi = -\frac{1 - (-1)^n}{2}.$$

$$20. \sum_{k=1}^n \prod_{m=1}^{k-1} \operatorname{ctg} (x_k - x_m) = \sin \frac{n\pi}{2}.$$

$$21. \sum_{k=1}^n \operatorname{ctg} x_k \prod_{m=1}^{k-1} \operatorname{ctg} (x_k - x_m) = \sin \frac{(n+1)\pi}{2} + (-1)^{n+1} \prod_{k=1}^n \operatorname{ctg} x_k.$$

4.4.8. Суммы, содержащие логарифмы.

$$1. \sum_{k=1}^n \ln (ka+b) = n \ln a + \ln \Gamma \left(\frac{b}{a} + n + 1 \right) - \ln \Gamma \left(\frac{b}{a} + 1 \right).$$

$$2. \sum_{k=1}^n (a+k) \ln (a+k) = \int_0^{n+1} \ln \Gamma(a+t) dt + (n+1)a + \frac{1}{2}(n+1)(n - \ln 2\pi)$$

[$a > 0$].

$$3. \sum_{k=0}^n (-1)^k \binom{n}{k} \ln (ka+b) = \int_0^1 (1-t)^n t^{b-1} \frac{dt}{\ln t}$$

[$a, b > 0$].

$$4. \sum_{k=0}^n (-1)^k \binom{n}{k} \ln (b-ka) = \int_0^1 (1-t)^n t^{b-na-1} \frac{dt}{\ln t}$$

[$a > 0, b > nx$].

5. $\sum_{k=1}^n \frac{(-1)^k k^{2m}}{(n-k)! (n+k)!} \ln 2k = (-1)^{m-1} 2^{2m-1} \frac{(2m)!}{(2n)!} \int_0^1 t^{-2m-1} \ln t^{1+t} dt$
6. $\sum_{k=0}^n \frac{(-1)^k (2k+1)^{2m+1}}{(n-k)! (n+k+1)!} \ln (2k+1) =$
 $= (-1)^{m-1} 2^{2m} \frac{(2m+1)!}{(2n+1)!} \int_0^1 t^{-2m-2} \ln t^{1+t} dt \quad [m, n = 1, 2, 3, \dots]$
7. $\sum_{k=2}^n k^m \ln (k!) = \frac{1}{m+1} \left[B_{m+1}(n+1) \ln (n!) - \sum_{k=1}^n B_{m+1}(k) \ln k \right]$
8. $\sum_{k=2}^n \ln (k!) = (n+1) \ln (n+1)! - \sum_{k=2}^{n+1} k \ln k$
9. $\sum_{k=2}^n k \ln (k!) = \frac{n(n+1)}{2} \ln (n!) - \frac{1}{2} \sum_{k=2}^n k(k-1) \ln k$
10. $\sum_{k=2}^n k^2 \ln (k!) = \frac{n}{6} (n+1)(2n+1) \ln (n!) - \frac{1}{6} \sum_{k=2}^n k(k-1)(2k-1) \ln k$
11. $\sum_{k=0}^{n-1} \ln \Gamma(ka+b) =$
 $= \int_0^1 \left[\left(b-1 + \frac{n-1}{2} a \right) e^{at} - \frac{n}{e^t-1} + \frac{1-e^{-nat}}{(1-e^{-a})(e^t-1)} e^{(1-b)t} \right] \frac{dt}{t}$
12. $\sum_{k=0}^n (-1)^{k-1} \ln \Gamma(ka+b) = \int_0^1 (1-t^{-a}) t^{na+b-1} \frac{dt}{(1-t) \ln t} \quad [a < 0; b > -na]$
13. $\sum_{k=0}^{n-1} \left[1 - (k+1) \ln \frac{2k+3}{2k+1} \right] = n - n \ln (2n+1) + \ln \frac{(2n)!}{n! 2^n}$
14. $\sum_{k=1}^{n-1} \cos kx \ln \sin \frac{k\pi}{2n} = -\frac{\pi}{2x} \int_0^{1/2} \sin 2nxt \operatorname{ctg} x t dt$
15. $\sum_{k=1}^{(n-1)/2} \cos \frac{2mk\pi}{n} \ln \sin \frac{k\pi}{n} = \frac{1}{2} \left[\ln 2n + \psi \left(\frac{m}{n} \right) + C \right] + \frac{\pi}{4} \operatorname{ctg} \frac{m\pi}{n}$
 $[m = 1, 2, \dots, n-1; n = 2, 3, 4, \dots]$

$$\sum_{k=1}^n \frac{(-1)^{k-1}}{\cos \frac{m(2k+1)\pi}{n}} \ln \sin \frac{2k+1}{2n} \pi = -\frac{1}{2} \psi \left(\frac{m}{n} \right) + \frac{\pi}{4} \operatorname{cosec} \frac{m\pi}{n}$$

[m = 1, 2, ..., n-1; n = 2, 3, 4, ...]

4.4.9. Суммы, содержащие арктангенсы.

1. $\sum_{k=1}^n \operatorname{arctg} \frac{2}{k^2} = -\frac{\pi}{4} - \operatorname{arctg} n - \operatorname{arctg} (n+1)$
2. $\sum_{k=1}^n \operatorname{arctg} \frac{a}{1+k(k+1)a^2} = \operatorname{arctg} \frac{na}{1+(n+1)a^2}$
3. $\sum_{k=0}^n \operatorname{arctg} \frac{a}{1+a(ka+b)+(ka+b)^2} = \operatorname{arctg} (na+b) - \operatorname{arctg} b$
4. $\sum_{k=1}^n \operatorname{arctg} \frac{2k}{2+k^2+k^4} = \operatorname{arctg} \frac{n+n^2}{2+n+n^2}$
5. $\sum_{k=1}^{2n} \operatorname{arctg} \left(x \sec \frac{2k\pi}{2n+1} \right) = \operatorname{arctg} \{ \operatorname{sh} (2n \operatorname{Arsh} x) \operatorname{sech} [(2n+1) \operatorname{Arsh} x] \}$
6. $\sum_{k=1}^{n-1} (-1)^k \operatorname{arctg} \frac{x - \cos \frac{2k+1}{2n} \pi}{\sin \frac{2k+1}{2n} \pi} = \operatorname{arctg} x^n - \frac{\pi}{4} [1 + (-1)^n]$
7. $\sum_{k=0}^{n-1} (-1)^k \operatorname{arctg} \frac{x \sin \frac{2k+1}{2n} \pi}{1 - x \cos \frac{2k+1}{2n} \pi} = \operatorname{arctg} x^n$

Глава 5. РЯДЫ

5.1. ЧИСЛОВЫЕ РЯДЫ

5.1.1. Введение.

В этом разделе приведены суммы рядов вида $\sum_k a_k$, где a_k — комплексные числа или функции некоторых параметров, причем зависимость от параметров такова, что они не записываются в виде степенных рядов (см. 5.2) или тригонометрических рядов (см. 5.4).

В некоторых случаях для вычисления суммы ряда $\sum_k a_k$ можно ввести вспомогательную сумму $\sum_k a_k x^k$, а затем воспользоваться формулами из (5.2) и теоремой Абеля:

Пусть ряд $\sum_k a_k x^k$ сходится при $|x| < 1$. Если сходится также ряд $\sum_{k=0}^{\infty} a_k$ и $0 < x < 1$, то

$$\lim_{x \rightarrow 1-0} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

При суммировании рядов могут оказаться полезными следующие формулы, в которых функции $f(x)$, $F(x)$, $G(x)$, $M(x)$ и $\varphi(x)$ связаны соотношениями

$$F(x) = \int_0^{\infty} e^{-xt} f(t) dt,$$

$$M(s) = \int_0^{\infty} t^{s-1} G(t) dt,$$

$$F(s) \Gamma(-s) = \int_0^{\infty} e^{-st} \varphi(t) dt.$$

$$1. \sum_{k=1}^{\infty} F(k) = \int_0^{\infty} \int_0^{\infty} \frac{h(x, y) f(y)}{e^x - 1} dx dy,$$

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$$\int h(x, y) e^{-px} dx = e^{-py} \varphi(p).$$

$$2. \sum_{k=1}^{\infty} F(k) = \int_0^{\infty} \frac{f(x)}{e^x - 1} dx.$$

$$3. \sum_{k=1}^{\infty} (-1)^k F(k) = - \int_0^{\infty} \frac{f(x)}{e^x + 1} dx.$$

$$4. \sum_{k=0}^{\infty} G(k+a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(s) \zeta(s, a) ds$$

$|c| > 1$.

$$5. \sum_{k=1}^{\infty} F(k^{1/2}) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} dx \int_0^{\infty} y \frac{e^{-x^2/4y} f(y)}{e^y - 1} dy.$$

$$6. \sum_{k=1}^{\infty} F(\ln k) = \int_0^{\infty} dx \int_0^{\infty} \frac{x^{s-1} f(y)}{\Gamma(y) (e^x - 1)} dy + \int_1^{\infty} f(y) \zeta(y) dy.$$

$$7. \sum_{k=0}^{\infty} (-1)^k \frac{a^k}{k!} F(k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) a^s \Gamma(-s) ds = \varphi(\ln a).$$

$$8. \sum_{k=0}^{\infty} F(k\pi) x^k = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t) \frac{1 - x \cos t}{1 - 2x \cos t + x^2} dt.$$

$$9. \sum_{k=1}^{\infty} F(k\pi) x^k = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(t) \frac{1 - x \sin t}{1 - 2x \cos t + x^2} dt.$$

$$10. F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t) \cos xt dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(t) \sin xt dt.$$

$$5.1.2. \text{Ряды вида } \sum_{k=1}^{\infty} \frac{(k+1)^k}{k!}.$$

$$1. \sum_{k=1}^{\infty} \frac{1}{k!} = \zeta(1) = \frac{1}{\Gamma(1)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

$[\operatorname{Re} s > 1]$.

$$2. \sum_{k=1}^{\infty} \frac{1}{k!} = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(1) = S_n$$

$[n = 2, 3, 4, \dots]$

$$S_{2n} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}| \quad [n = 1, 2, 3, \dots]$$

$$S_2 = \frac{\pi^2}{6} = 1,64493407, \quad S_3 = 1,20205690,$$

$$S_4 = \frac{\pi^4}{90} = 1,08232323, \quad S_5 = 1,03692776.$$

$$S_6 = \frac{\pi^6}{945} = 1,0173406, \quad S_7 = 1,008349277,$$

$$S_8 = \frac{\pi^8}{9450} = 1,00407736, \quad S_9 = 1,00200839,$$

$$S_{10} = \frac{5 \cdot 2^9 \pi^{10}}{33 \cdot 10!} = 1,00099458.$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = (2^{1-s}-1) \zeta(s) = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t+1} dt = T_s \quad [\operatorname{Re} s > 0].$$

$$T_{2n} = \frac{1}{(2n)!} (1-2^{2n-1}) \pi^{2n} |B_{2n}| \quad [n=1, 2, 3, \dots]$$

$$T_1 = -\ln 2 = -0,69314718, \quad T_2 = -\frac{\pi^2}{12} = -0,82246703,$$

$$T_3 = -0,90154268, \quad T_4 = -\frac{7\pi^4}{720} = -0,94703283,$$

$$T_5 = -0,97211977, \quad T_6 = -\frac{31\pi^6}{32 \cdot 945} = -0,98555109,$$

$$T_7 = -0,99259382, \quad T_8 = -\frac{127\pi^8}{128 \cdot 9450} = -0,99623300,$$

$$T_9 = -0,99809430, \quad T_{10} = -\frac{73\pi^{10}}{5 \cdot 33 \cdot 8! \cdot 512} = -0,99903951.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{[k/2]}}{k} = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^{[k\sqrt{2}]}}{k} = -0,5154.$$

5.1.3. Ряды вида $\sum \frac{(\pm 1)^n}{(k+a)^s}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} = \zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-e^{-t}} dt \quad [\operatorname{Re} s > 1].$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k+a)^n} = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(a) \quad [n=2, 3, 4, \dots]$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+a)^2} = -\int_0^1 \frac{t^{a-1}}{1-t} \ln t dt.$$

$$4. \sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^n} = (-1)^{n-1} \frac{\pi}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} \operatorname{ctg} a\pi.$$

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$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s} = 2^{-s} \left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right) \right] = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1+e^{-t}} dt \quad [\operatorname{Re} s > 0].$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^n} = \frac{(-1)^{n-1}}{(n-1)!} \beta^{(n-1)}(a) \quad [n=1, 2, \dots].$$

$$7. \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^n} = (-1)^{n-1} \pi \frac{d^{n-1}}{da^{n-1}} \operatorname{cosec} a\pi \quad [n=1, 2, 3, \dots].$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^2} = -\int_0^1 \frac{t^{a-1}}{1+t} \ln t dt.$$

5.1.4. Ряды вида $\sum \frac{(\pm 1)^k}{(2k+1)^s}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = (1-2^{-s}) \zeta(s) = A_s \quad [\operatorname{Re} s > 1].$$

$$A_{2n} = \frac{1}{2^{2n}} (2^{2n}-1) \pi^{2n} |B_{2n}|.$$

$$A_2 = \frac{\pi^2}{8} = 1,23370055, \quad A_3 = 1,05179974,$$

$$A_4 = \frac{\pi^4}{96} = 1,01467803, \quad A_5 = 1,00452576,$$

$$A_6 = \frac{\pi^6}{960} = 1,00144708, \quad A_7 = 1,00047155,$$

$$A_8 = \frac{17\pi^8}{256 \cdot 630} = 1,00015518, \quad A_9 = 1,00005135,$$

$$A_{10} = \frac{71\pi^{10}}{85 \cdot 8! \cdot 1124} = 1,00001704.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = 2^{-s} \left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right) \right] = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\operatorname{ch} t} dt = \bar{A}_s \quad [\operatorname{Re} s > 0].$$

$$\bar{A}_{2n+1} = \frac{1}{2^{2n+1}} \left(\frac{\pi}{2} \right)^{2n+1} |E_{2n}|.$$

$$\bar{A}_1 = \frac{\pi}{4} = 0,78539816, \quad \bar{A}_2 = G = 0,91596559,$$

$$\bar{A}_3 = \frac{\pi^3}{32} = 1,96894615, \quad \bar{A}_4 = 0,9889155,$$

$$\bar{A}_5 = \frac{\pi^5}{64 \cdot 4!} = 0,99615783, \quad \bar{A}_6 = 0,99868522,$$

$$A_7 = \frac{64\pi^7}{256 \cdot 6!} = 0,9995451, \quad A_8 = 0,99984999,$$

$$A_9 = \frac{1385\pi^8}{1024 \cdot 8!} = 0,99994968, \quad A_{10} = 0,99998316.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^{[k/2]}}{(2k+1)^s} = 8^{-s} \left[2^s \left(s, \frac{1}{8} \right) + 2^s \left(s, \frac{3}{8} \right) - 2^{2s} (2^s - 1) \zeta(s) \right] = C_s$$

(Re s > 0)

$$C_{2n+1} = \frac{(-1)^n \pi^{2n+1}}{(2n+1)! \sqrt{2}} \left[B_{2n+1} \left(\frac{1}{4} \right) - 2^{2n+1} B_{2n+1} \left(\frac{1}{8} \right) \right],$$

$$C_1 = -\frac{\pi \sqrt{2}}{4}, \quad C_3 = \frac{3\pi^3 \sqrt{2}}{2^4}, \quad C_5 = \frac{19\pi^5 \sqrt{2}}{2^{14}}, \quad C_7 = \frac{307\pi^7 \sqrt{2}}{5 \cdot 2^{18}}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^{[k/2]}}{(2k+1)^n} = \hat{C}_n,$$

$$\hat{C}_1 = \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1), \quad \hat{C}_2 = \frac{\pi^2 \sqrt{2}}{16}, \quad \hat{C}_3 = \frac{11\pi^4 \sqrt{2}}{3 \cdot 2^9},$$

$$\hat{C}_5 = \frac{361\pi^6 \sqrt{2}}{15 \cdot 2^{14}}, \quad \hat{C}_7 = \frac{24611\pi^8 \sqrt{2}}{9 \cdot 35 \cdot 2^{20}}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^{[k/3]}}{2k+1} = \frac{5\pi}{12}.$$

5.1.5. Ряды вида $\sum \frac{(-1)^k}{kn+m}$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{ka+b} = \int_0^1 \frac{t^{b-1}}{t^a+1} dt = \frac{1}{a} \beta\left(\frac{b}{a}\right).$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} = \frac{\pi}{2n} \operatorname{cosec} \frac{m\pi}{n} - \frac{2}{n} \sum_{k=0}^{[(n-2)/2]} \cos\left[(2k+1)\frac{m\pi}{n}\right] \ln\left[\sin(2k+1)\frac{\pi}{2n}\right]$$

(n=2, 3, 4, ..., m=1, 2, ..., n-1; см. приложение 1.1)

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+nl+m} = (-1)^l \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} - \sum_{k=0}^{l-1} \frac{(-1)^k}{kn+m} \right].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+m} = (-1)^{m+1} \ln 2 + (-1)^{m+1} \sum_{k=1}^{m-1} \frac{(-1)^k}{k}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2, \quad 6. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+2} = \frac{1}{3} \left(\frac{\pi}{3} - (-1)^0 \ln 2 \right)$$

[n=1 или 2]

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+2 \pm 1} = \frac{1}{4\sqrt{2}} [\pi \mp 2 \ln(1+\sqrt{2})].$$

5.1.6. Ряды вида $\sum \frac{(-1)^k}{(k+a)(k+b)}$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+b)} = - \int_0^1 dy \int_0^1 \frac{x^{b-1-y/cy-1}}{1+x^a} \ln x dx.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+b)} = \pm \frac{1}{b-a} \left\{ \psi(b) - \psi(a) \right\} \quad [a=b \text{ см. в 5.1.3].}$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+b)} = \frac{1}{b-a} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k+b} \right] \quad [\text{см. 5.1.5}].$$

$$4. \sum_{k=-\infty}^{\infty} \frac{1}{(k+a)(k+b)} = \frac{\pi}{b-a} (\operatorname{ctg} \pi a - \operatorname{ctg} \pi b) \quad [a=b \text{ см. в 5.1.3.5}].$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+a+1)} = \pm \frac{1}{a} + 2\beta(a) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(a+m)(kn+nl+m)} = \frac{1}{nl} \sum_{k=0}^{l-1} \frac{1}{kn+m}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kn+m)(kn+m+n)} = \frac{1}{n} \left[2 \sum_{k=0}^{\infty} \frac{(-1)^k}{kn+m} - \frac{1}{m} \right].$$

$$8. \sum_{k=1}^{\infty} \frac{1}{(k+l)(k+m)} = \frac{1}{m-l} \sum_{k=1}^{m-l} \frac{1}{k+l}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+l)(k+m)} = \frac{1}{m-l} [(-1)^m - (-1)^l] \ln 2 +$$

$$+ \frac{(-1)^m}{m-l} \sum_{k=1}^{m-1} \frac{(-1)^k}{k} - (-1)^l \sum_{k=1}^{l-1} \frac{(-1)^k}{k(k+m-l)} \quad [l < m].$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{k(k+a)} = \frac{1}{a} \left\{ \psi(a+1) + \gamma \right\}.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{k(k+a)} = \frac{1}{a^2} - \frac{\pi}{a \sin \pi a} \left\{ \frac{\cos \pi a}{1} \right\}.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k(kn+m)} = \frac{1}{m} \left[\psi \left(\frac{m}{n} + 1 \right) + \gamma \right].$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k(kn+nl+m)} = \frac{n}{nl+m} \left[\sum_{k=1}^l \frac{1}{kn+nl} + \frac{1}{n} \sum_{k=l+1}^{\infty} \frac{1}{k(kn+m)} \right].$$

$$14. \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{1}{n} \sum_{k=1}^n \frac{1}{k}.$$

$$15. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+n)} = \frac{1}{n} [(-1)^n - 1] \ln 2 + \frac{(-1)^n}{n} \sum_{k=1}^n \frac{(-1)^k}{k}.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kn+m)^2} = \frac{1}{n^2} \left\{ \psi' \left(\frac{m}{n} \right) \right\}.$$

5.1.7. Ряды вида $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(kn+m)}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1. \quad 2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} = 1 - 2 \ln 2.$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}. \quad 4. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)} = -\frac{1}{4}.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \frac{11}{18}. \quad 6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+3)} = \frac{5}{18} - \frac{2}{3} \ln 2.$$

$$7. \sum_{k=1}^{\infty} \frac{1}{k(k+4)} = \frac{25}{48}. \quad 8. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+4)} = \frac{7}{48}.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2 \ln 2. \quad 10. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)} = 2 - \frac{\pi}{2} - \ln 2.$$

$$11. \sum_{k=1}^{\infty} \frac{1}{k(2k+3)} = \frac{8}{9} - \frac{2}{3} \ln 2.$$

$$12. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+3)} = -\frac{4}{9} + \frac{\pi}{6} - \frac{1}{3} \ln 2.$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 3 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+1)} = 3 - \frac{\pi}{\sqrt{3}} - 2 \ln 2.$$

$$15. \sum_{k=1}^{\infty} \frac{1}{k(3k+2)} = \frac{3}{4} - \frac{\pi}{4\sqrt{3}} - \frac{3}{4} \ln 3.$$

$$16. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+2)} = \frac{3}{4} - \frac{\pi}{2\sqrt{3}}.$$

$$17. \sum_{k=1}^{\infty} \frac{1}{k(4k+1)} = 4 - \frac{\pi}{2} - 3 \ln 2.$$

$$18. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+1)} = 4 - \frac{\pi}{\sqrt{2}} - \ln 2 - \sqrt{2} \ln(1+\sqrt{2}).$$

$$19. \sum_{k=1}^{\infty} \frac{1}{k(4k+3)} = \frac{4}{9} + \frac{\pi}{6} - \ln 2.$$

$$20. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+3)} = \frac{4}{9} - \frac{\pi}{3\sqrt{2}} - \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k(8k+1)} = 8 - \frac{\pi}{2} (\sqrt{2}+1) - 4 \ln 2 - \sqrt{2} \ln(1+\sqrt{2}).$$

$$22. \sum_{k=1}^{\infty} \frac{1}{k(8k+3)} = \frac{8}{9} - \frac{\pi}{6} (\sqrt{2}-1) - \frac{4}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$23. \sum_{k=1}^{\infty} \frac{1}{k(8k+5)} = \frac{8}{25} - \frac{\pi}{10} (\sqrt{2}-1) - \frac{4}{5} \ln 2 + \frac{\sqrt{2}}{5} \ln(1+\sqrt{2}).$$

$$24. \sum_{k=1}^{\infty} \frac{1}{k(8k+7)} = \frac{8}{49} + \frac{\pi}{28} (\sqrt{2}+1) - \frac{4}{7} \ln 2 - \frac{\sqrt{2}}{7} \ln(1+\sqrt{2}).$$

5.1.9. Ряды вида $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+m)} = \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{1}{k}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)} = 2 \ln 2. \quad 3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)} = \frac{\pi}{2} - \ln 2.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)} = \frac{3}{4} \ln 3 + \frac{\pi}{4\sqrt{3}}. \quad 5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+1)} = \frac{\pi}{2\sqrt{3}}.$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)} = \frac{3}{2} \ln 3 - \frac{\pi}{2\sqrt{3}}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+2)} = \frac{\pi}{\sqrt{3}} - 2 \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)} = \frac{\pi}{6} + \ln 2.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)} = -\frac{1}{3} \ln 2 + \frac{1}{3\sqrt{2}} [\ln(3+\sqrt{2}) + \pi].$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)} = 3 \ln 2 - \frac{\pi}{2}.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)} = -\ln 2 + \frac{\pi}{\sqrt{2}} - \frac{1}{\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+1)} = \frac{\pi}{13} (\sqrt{2}+1) + \frac{4}{7} \ln 2 + \frac{\sqrt{2}}{7} \ln(\sqrt{2}+1).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+3)} = \frac{\pi}{10} (\sqrt{2}-1) + \frac{4}{5} \ln 2 - \frac{\sqrt{2}}{5} \ln(\sqrt{2}+1).$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+5)} = \frac{4}{3} \ln 2 - \frac{\pi}{6} (\sqrt{2}-1) - \frac{\sqrt{2}}{3} \ln(\sqrt{2}+1).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(8k+7)} = 4 \ln 2 - \frac{\pi}{2} (\sqrt{2}+1) + \sqrt{2} \ln(\sqrt{2}+1).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(k+2)(2k+1)} = \frac{1}{3} + \frac{2}{3} \ln 2.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)(2k+1)} = \frac{\pi}{6} - \frac{1}{3} + \frac{1}{3} \ln 2.$$

$$6.1.9. \text{Ряды вида } \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)(kn+m)}.$$

$$1. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(2k+3)} = \frac{1}{2}. \quad 2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} = \frac{\pi}{4} - \frac{1}{2}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+1)} = \frac{\pi}{2\sqrt{3}} - 2 \ln 2 + \frac{3}{2} \ln 3.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(3k+1)} = \pi \left(\frac{\sqrt{3}}{3} - \frac{1}{2} \right) + \ln 2.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+2)} = \frac{\pi}{2\sqrt{3}} + 2 \ln 2 - \frac{3}{2} \ln 3.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(3k+2)} = \pi \left(\frac{1}{2} - \frac{\sqrt{3}}{3} \right) + \ln 2.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)} = \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+1)} = \frac{\pi}{2} (\sqrt{2}-1) - \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)} = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+3)} = \frac{\pi}{4} (1-\sqrt{2}) + \frac{\sqrt{2}}{2} \ln(\sqrt{2}-1).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+5)} = \frac{2}{3} - \frac{\pi}{12} - \frac{1}{6} \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+5)} = -\frac{2}{15} + \frac{\pi}{20} (1+\sqrt{2}) + \frac{1}{5\sqrt{2}} (\sqrt{2}-1).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+7)} = \frac{2}{15} + \frac{\pi}{20} - \frac{1}{10} \ln 2.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+1)} = \frac{\pi}{12}(\sqrt{2}+1) + \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{6} \ln(\sqrt{2}+1).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+3)} = \frac{\pi}{4}(\sqrt{2}-1) + \ln 2 - \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+5)} = \frac{\pi}{4}(\sqrt{2}-1) - \ln 2 + \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(8k+7)} = \frac{\pi}{12}(\sqrt{2}+1) - \frac{1}{3} \ln 2 - \frac{\sqrt{2}}{6} \ln(\sqrt{2}+1).$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)} = \frac{1}{5} + \frac{\pi}{20} + \frac{1}{10} \ln 2.$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)} = -\frac{1}{5} + \frac{\pi}{20}(1+\sqrt{2}) + \frac{1}{5\sqrt{2}} \ln(1+\sqrt{2}).$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)} = \frac{1}{3} - \frac{\pi}{12} + \frac{1}{6} \ln 2.$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)} = -\frac{1}{3} + \frac{\pi}{12}(1+\sqrt{2}) - \frac{1}{3\sqrt{2}} \ln(1+\sqrt{2}).$$

5.1.10. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)} = \frac{\pi}{3\sqrt{3}}. \quad 2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)} = \frac{\pi}{3} \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)} = \frac{1}{3}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+4)} = \frac{1}{3} - \frac{2}{9} \ln 2 - \frac{\pi\sqrt{3}}{3}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+5)} = \frac{1}{8} + \frac{\pi}{12\sqrt{3}}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+5)} = \frac{\pi}{6\sqrt{3}} - \frac{1}{8}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(6k+1)} = \frac{\pi}{3\sqrt{3}} + \frac{2}{3} \ln 2.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(6k+1)} = \frac{\pi}{3} \left(1 - \frac{\sqrt{3}}{3}\right) - \frac{1}{3} \ln 2 + \frac{1}{3\sqrt{3}} \ln(2+\sqrt{3}).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(6k+5)} = \frac{2}{9} \left(\frac{\pi}{\sqrt{3}} - \ln 2\right).$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(6k+5)} = \frac{\pi}{9} \left(\frac{\sqrt{3}}{3} - 1\right) + \frac{1}{18\sqrt{3}} \ln(7+4\sqrt{3}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+4)} = \frac{1}{2} - \frac{\pi}{6\sqrt{3}}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+4)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{2}.$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+5)} = \frac{1}{6}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+5)} = \frac{2}{9} \left(\frac{\pi}{\sqrt{3}} - \ln 2\right) - \frac{1}{6}.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(6k+1)} = \frac{2\pi}{9\sqrt{3}} + \frac{2}{9} \ln 2.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(6k+1)} = \frac{\pi}{9} \left(1 - \frac{\sqrt{3}}{3}\right) + \frac{1}{9} \ln 2 + \frac{1}{9\sqrt{3}} \ln(2+\sqrt{3}).$$

5.1.11. Ряды вида $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k+l)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(4k+m)(4k+4l+m)} = \frac{1}{4l} \sum_{k=0}^{l-1} \frac{1}{4k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} = \frac{\pi}{8}.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)} = \frac{1}{4\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)} = \frac{1}{4}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)} = -\frac{1}{4} + \frac{1}{8\sqrt{2}} [\pi + \ln(3+2\sqrt{2})].$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+7)} = \frac{1}{18} + \frac{\pi}{24}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+7)} = \frac{\pi}{12\sqrt{2}} - \frac{1}{18}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+1)} = \frac{\pi\sqrt{2}}{8} + \frac{1}{4} \ln 2 + \frac{\sqrt{2}}{4} \ln(\sqrt{2}+1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+3)} = \frac{\pi}{8} (2-\sqrt{2}) - \frac{1}{4} \ln 2 + \frac{\sqrt{2}}{4} \ln(1+\sqrt{2}).$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+5)} = \frac{\pi\sqrt{2}}{24} - \frac{1}{12} \ln 2 + \frac{\sqrt{2}}{12} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(8k+7)} = \frac{\pi}{40} (2+\sqrt{2}) - \frac{1}{20} \ln 2 - \frac{\sqrt{2}}{20} \ln(1+\sqrt{2}).$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)} = \frac{1}{2} - \frac{\pi}{8}.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+5)} = \frac{\pi}{4\sqrt{2}} - \frac{1}{2}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)} = \frac{1}{12}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+7)} = \frac{\pi}{8\sqrt{2}} - \frac{1}{4\sqrt{2}} \ln(1+\sqrt{2}) - \frac{1}{12}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+1)} = \frac{\pi}{40} (2+\sqrt{2}) + \frac{1}{20} \ln 2 + \frac{\sqrt{2}}{20} \ln(1+\sqrt{2}).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+3)} = \frac{\pi\sqrt{2}}{24} + \frac{1}{12} \ln 2 - \frac{\sqrt{2}}{12} \ln(1+\sqrt{2}).$$

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$$18. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+5)} = \frac{\pi}{8} (2-\sqrt{2}) + \frac{1}{4} \ln 2 - \frac{\sqrt{2}}{4} \ln(1+\sqrt{2}).$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(8k+7)} = \frac{\pi}{8} \sqrt{2} - \frac{1}{4} \ln 2 - \frac{\sqrt{2}}{4} \ln(1+\sqrt{2}).$$

5.1.12. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(5k+1)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(5k+m)(5k+5l+m)} = \frac{1}{5l} \sum_{k=0}^{l-1} \frac{1}{5k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(5k+1)(5k+6)} = \frac{1}{5}. \quad 3. \sum_{k=0}^{\infty} \frac{1}{(5k+2)(5k+7)} = \frac{1}{10}.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(5k+3)(5k+8)} = \frac{1}{15}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(5k+4)(5k+6)} = \frac{1}{2} - \sqrt{2+\sqrt{5}} \frac{\pi}{10}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(5k+4)(5k+6)} = -\frac{1}{2} + \frac{2\pi}{5\sqrt{10-2\sqrt{5}}}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(5k+4)(5k+9)} = \frac{1}{20}.$$

5.1.13. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+1)(kn+m)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(6k+m)(6k+6l+m)} = \frac{1}{6l} \sum_{k=0}^{l-1} \frac{1}{6k+m}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(6k+1)(6k+5)} = \frac{\pi\sqrt{3}}{24}.$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(6k+1)(6k+5)} = \frac{1}{12\sqrt{3}} \ln(2+\sqrt{3}).$$

$$4. \sum_{k=0}^{\infty} \frac{1}{(6k+1)(6k+7)} = \frac{1}{6}.$$

5.1.16. Ряды вида $\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k(kn_1+m_1)(kn_2+m_2)}$.

1. $\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = \frac{\pi^2}{6} - 1$.
2. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)} = 2 \ln 2 - \frac{\pi^2}{12} - 1$.
3. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = 2 - \frac{\pi^2}{6}$.
4. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)^2} = 2 - \frac{\pi^2}{12} - 2 \ln 2$.
5. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}$.
6. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)} = \frac{5}{4} - 2 \ln 2$.
7. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+3)} = \frac{7}{36}$.
8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+3)} = \frac{1}{12} - \frac{1}{3} \ln 2$.
9. $\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+3)} = \frac{5}{36}$.
10. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)(k+3)} = -\frac{19}{36} - \frac{1}{3} \ln 2$.
11. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} = 3 - 4 \ln 2$.
12. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+1)} = 3 - \pi$.
13. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+3)} = \frac{4}{3} \ln 2 - \frac{7}{9}$.
14. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+3)} = \frac{17}{9} - \frac{\pi}{3} - \frac{4}{3} \ln 2$.
15. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+1)} = 4 - \frac{\pi\sqrt{3}}{4} - \frac{9}{4} \ln 3$.
16. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+1)} = 4 - 2 \ln 2 - \frac{\pi\sqrt{3}}{2}$.
17. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+2)} = 1 + \frac{\pi\sqrt{3}}{4} - \frac{9}{4} \ln 3$.
18. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+2)} = \frac{5}{4} - \frac{\pi\sqrt{3}}{2} + 2 \ln 2$.

[5.1.16]

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19. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(4k+1)} = 5 - \frac{2\pi}{3} - 4 \ln 2$.
20. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(4k+1)} = 5 - \frac{2\sqrt{2}}{3} \pi - \frac{2}{3} \ln 2 - \frac{\sqrt{2}}{3} \ln(1+\sqrt{2})$.
21. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(4k+3)} = \frac{7}{9} + \frac{2\pi}{3} - 4 \ln 2$.
22. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(4k+3)} = \frac{13}{3} + \frac{2}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2})$.
23. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)(2k+3)} = \frac{5}{18} - \frac{1}{3} \ln 2$.
24. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(2k+3)} = \frac{1}{3} + \ln 2 - \frac{\pi}{3}$.
25. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)(3k+1)} = 2 - \frac{\pi\sqrt{3}}{2} + 4 \ln 2 - \frac{9}{2} \ln 3$.
26. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(3k+1)} = 5 + \pi(1-\sqrt{3}) - 4 \ln 2$.
27. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)(3k+2)} = \frac{7}{4} + \frac{\pi\sqrt{3}}{4} - 4 \ln 2 + \frac{9}{4} \ln 3$.
28. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(3k+2)} = \frac{7}{4} - \pi\left(1 - \frac{\sqrt{3}}{2}\right) - 2 \ln 2$.
29. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)(4k+1)} = 6 - \pi - 4 \ln 2$.
30. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(4k+1)} = 6 + \frac{\pi}{2}(1-2\sqrt{2}) - 2\sqrt{2} \ln(1+\sqrt{2})$.
31. $\sum_{k=1}^{\infty} \frac{1}{k(2k+1)(4k+3)} = \frac{10}{9} - \frac{\pi}{3}$.

$$32. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)(4k+3)} = \frac{2}{3} + \frac{\pi}{2}(\sqrt{2}-1) - \frac{1}{3} \ln 2 - \sqrt{2} \ln(1+\sqrt{2}).$$

$$33. \sum_{k=1}^{\infty} \frac{1}{k(4k+1)(4k+3)} = \frac{4}{9} - \frac{\pi}{3} - \ln 2.$$

$$34. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(4k+1)(4k+3)} = \frac{16}{9} - \frac{\pi}{3\sqrt{2}} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

5.1.17. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(kn_1+m_1)(kn_2+m_2)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)} = 2 \ln 2 - 1.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(2k+3)} = 1 - \ln 2.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(3k+1)} = \frac{\pi\sqrt{3}}{6} - 4 \ln 2 + \frac{9}{4} \ln 3.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(3k+1)} = 2 \ln 2 - \pi \left(1 - \frac{\sqrt{3}}{2}\right).$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(3k+1)} = \frac{\pi\sqrt{3}}{2} + 4 \ln 2 - \frac{9}{2} \ln 3.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(3k+2)} = 4 \ln 2 + \pi(1-\sqrt{3}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)} = \frac{\pi}{3}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+1)} = \frac{\pi}{6}(2\sqrt{2}-3) + \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+3)} = \pi - 4 \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+3)} = \frac{\pi}{2}(1-2\sqrt{2}) + \ln 2 + 2\sqrt{2} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+5)} = \frac{\pi}{3} + \frac{8}{3} \ln 2 - \frac{8}{3}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+5)} = \frac{8}{3} + \pi \left(\frac{1}{6} - \sqrt{2}\right) - \ln 2 - 2\sqrt{2} \ln(1+\sqrt{2}).$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+7)} = \frac{4}{5} \ln 2 - \frac{\pi}{15} - \frac{8}{45}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)(4k+7)} = -\frac{8}{45} + \frac{\pi}{5} \left(1 - \frac{\sqrt{2}}{3}\right) - \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{15} \ln(1+\sqrt{2}).$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)} = \frac{\pi}{15} + \frac{4}{5} \ln 2 - \frac{2}{5}.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+1)} = \frac{2}{5} - \frac{\pi}{30}(3-2\sqrt{2}) - \frac{1}{3} \ln 2 + \frac{2\sqrt{2}}{15} \ln(1+\sqrt{2}).$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)} = \frac{8}{3} \ln 2 - \frac{\pi}{3} - \frac{2}{3}.$$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+3)} = \frac{2}{3} + \frac{\pi}{3}(1-\sqrt{2}) - \ln 2 + \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)(3k+2)} = \frac{\pi\sqrt{3}}{4} - \frac{3}{4} \ln 3.$$

$$20. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+1)(3k+2)} = 2 \ln 2 - \frac{\pi}{2\sqrt{3}}.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)(3k+4)} = \frac{3}{2}(\ln 3 - 1).$$

$$22. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(3k+2)(3k+4)} = \frac{3}{2} - 2 \ln 2.$$

$$23. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)} = \frac{\pi}{3} - \ln 2.$$

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)(4k+3)} = -\frac{\pi}{6\sqrt{2}} + \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{2} \ln(1+\sqrt{2}).$$

$$25. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+5)} = \frac{\pi}{6} + \ln 2 - 1.$$

$$26. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)(4k+5)} = \frac{\pi}{3\sqrt{2}} - \frac{1}{3} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$27. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+7)} = \frac{1}{3} \ln 2 - \frac{2}{27}.$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)(4k+5)} = 3 \ln 2 - 2.$$

$$29. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)(4k+5)} = 2 - \ln 2 - \frac{1}{\sqrt{2}} \ln(3+2\sqrt{2}).$$

$$30. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+3)(4k+7)} = \ln 2 - \frac{\pi}{6} - \frac{1}{9}.$$

$$31. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+3)(4k+7)} = \frac{1}{6} + \frac{\pi}{6\sqrt{2}} - \frac{1}{3} \ln 2 - \frac{\pi}{3\sqrt{2}} \ln(1+\sqrt{2}).$$

$$32. \sum_{k=0}^{\infty} \frac{1}{(k+1)(6k+5)(6k+7)} = \frac{3}{2} \ln 3 + 2 \ln 2 - 3.$$

$$33. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(6k+5)(6k+7)} = 3 - \ln 2 - \frac{1}{\sqrt{3}} \ln(2+\sqrt{3}).$$

5.1.18. Ряды вида $\sum \frac{(-1)^k}{(2k+m)(kn_1+m_1)(kn_2+m_2)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{12}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(2k+5)} = \frac{\pi}{8} - \frac{1}{3}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(2k+7)} = \frac{11}{180}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(2k+7)} = \frac{\pi}{24} - \frac{4}{45}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)} = \frac{\pi}{10} + \frac{1}{5} \ln 2 - \frac{1}{10}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)(4k+1)} = \left(\frac{1}{3\sqrt{2}} - \frac{1}{10}\right)\pi - \frac{1}{10} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)} = \frac{\pi}{6} - \frac{1}{3} \ln 2 - \frac{1}{6}.$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+3)} = \frac{1}{6} + \frac{\pi}{12} (1-2\sqrt{2}) - \frac{\sqrt{2}}{3} \ln(\sqrt{2}-1).$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+5)} = \frac{5}{6} - \frac{\pi}{6} - \frac{1}{3} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+3)(4k+5)} = -\frac{5}{6} + \pi \left(\frac{1}{3\sqrt{2}} - \frac{1}{8}\right) + \frac{1}{6} \ln 2 + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+7)} = \frac{7}{30} - \frac{\pi}{10} + \frac{1}{5} \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)} = \frac{1}{2} \ln 2.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(4k+1)(4k+3)} = (\sqrt{2}-1) \frac{\pi}{4}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+5)} = \frac{\pi}{12} + \frac{1}{6} \ln 2 - \frac{1}{6}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+1)(4k+5)} = \frac{1}{6} + \frac{\pi}{12} \left(\frac{\sqrt{2}}{2} - 1\right) + \frac{1}{6\sqrt{2}} \ln(1+\sqrt{2}).$$

16. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+7)} = \frac{\pi}{30} + \frac{1}{10} \ln 2 - \frac{1}{45}.$
17. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)(4k+5)} = \frac{\pi}{6} - \frac{1}{6} \ln 2 - \frac{1}{3}.$
18. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+3)(4k+5)} = \frac{1}{3} - \frac{\pi}{12} (1-2\sqrt{2}) + \frac{1}{3\sqrt{2}} \ln(1+\sqrt{2}).$
19. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+3)(4k+7)} = \frac{\pi}{20} - \frac{1}{10} \ln 2 - \frac{1}{30}.$
20. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+5)(4k+7)} = \frac{4}{15} - \frac{\pi}{15} - \frac{1}{30} \ln 2.$
21. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+3)} = \frac{\pi}{15} - \frac{1}{30} \ln 2 - \frac{1}{15}.$
22. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+3)} = \frac{1}{15} - \frac{\pi}{60} (1+\sqrt{2}) + \frac{4}{15} \ln(1+\sqrt{2}).$
23. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+5)} = \frac{3}{10} - \frac{\pi}{20} - \frac{1}{10} \ln 2.$
24. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+5)} = \frac{\pi}{20} \left(\frac{3\sqrt{2}}{2} - 1 \right) - \frac{3}{10} + \frac{3\sqrt{2}}{20} \ln(1+\sqrt{2}).$
25. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+7)} = \frac{4}{45} + \frac{1}{10} \ln 2 - \frac{\pi}{30}.$
26. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+1)(4k+7)} = \frac{\pi}{20} \left(1 - \frac{2\sqrt{2}}{3} \right) - \frac{4}{45} + \frac{\sqrt{2}}{10} \ln(1+\sqrt{2}).$
27. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)(4k+5)} = \frac{2}{3} - \frac{\pi}{6} - \frac{1}{6} \ln 2.$
28. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)(4k+5)} = \frac{\pi}{12} (2\sqrt{2}-1) - \frac{2}{3} + \frac{\sqrt{2}}{3} \ln(1+\sqrt{2}).$

29. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+3)(4k+7)} = \frac{1}{6} - \frac{\pi}{12} + \frac{1}{6} \ln 2.$
 30. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)(4k+3)(4k+7)} = \frac{\pi}{12} (1-\sqrt{2}) - \frac{1}{6} + \frac{\sqrt{2}}{6} \ln(1+\sqrt{2}).$
- 5.1.19. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+m)(3k+n_1)(3k+n_2)}.$
1. $\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+4)} = \frac{\pi\sqrt{3}}{18} - \frac{1}{6}.$
 2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)(3k+4)} = \frac{1}{6} - \frac{\pi}{9\sqrt{3}} + \frac{2}{9} \ln 2.$
 3. $\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+5)} = \frac{\pi}{12\sqrt{3}} - \frac{1}{24}.$
 4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+2)(3k+5)} = \frac{1}{4} - \frac{7\pi}{36\sqrt{3}} + \frac{13}{36} \ln 2.$
 5. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+2)(3k+7)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{60}.$
 6. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(3k+2)(3k+7)} = \frac{2}{15} \ln 2 - \frac{1}{40}.$
 7. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(3k+4)(3k+7)} = \frac{1}{24}.$
 8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+1)(3k+4)(3k+7)} = \frac{2\pi}{27\sqrt{3}} + \frac{2}{27} \ln 2 - \frac{11}{72}.$
 9. $\sum_{k=0}^{\infty} \frac{1}{(2k+3)(3k+5)(3k+8)} = \frac{1}{12}.$
 10. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)(3k+5)(3k+8)} = \frac{2\pi}{27\sqrt{3}} - \frac{2}{27} \ln 2 - \frac{13}{180}.$

5.1.20. Ряды вида $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k+n)(4k+m_1)(4k+m_2)}$.

1. $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+5)} = \frac{\pi}{16} - \frac{1}{8}.$
2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+5)} = \frac{1}{8} - \frac{\pi}{16\sqrt{2}} + \frac{1}{16\sqrt{2}} \ln(3+2\sqrt{2}).$
3. $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+7)} = \frac{\pi}{48} - \frac{1}{72}.$
4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+7)} = \frac{1}{72} - \frac{\pi}{48\sqrt{2}} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$
5. $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)(4k+7)} = \frac{7}{72} - \frac{\pi}{48}.$
6. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)(4k+7)} = \frac{\pi}{48\sqrt{2}} - \frac{7}{72} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$
7. $\sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)(4k+7)} = \frac{5}{24} - \frac{\pi}{16}.$
8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+5)(4k+7)} = \frac{\pi}{16\sqrt{2}} - \frac{5}{24} + \frac{1}{8\sqrt{2}} \ln(1+\sqrt{2}).$

5.1.21. Ряды вида $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(kn_1+m_1)(kn_2+m_2)(kn_3+m_3)}$.

1. $\sum_{k=0}^{\infty} \frac{1}{(k+a)(k+a+1)(k+a+2)(k+a+3)} = \frac{1}{2a(a+1)(a+2)}$
 $[a \neq 0, -1, -2, \dots].$
2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)(k+a+1)(k+a+2)(k+a+3)} =$
 $= \frac{4}{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} - \frac{7}{6a} + \frac{2}{3(a+1)} - \frac{1}{6(a+2)}.$
3. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+a)^2(k+b)^2} =$
 $= \pm \frac{2}{(b-a)^2} \left\{ \psi(a) - \psi(b) \right\} \mp \frac{1}{(b-a)^2} \left\{ \psi'(a) + \psi'(b) \right\}.$

4. $\sum_{k=1}^{\infty} \frac{1}{k^2(k+n)^2} = \frac{\pi^2}{3n^2} - \frac{1}{n^2} \sum_{k=1}^n \frac{1}{k^2} - \frac{2}{n^3} \sum_{k=1}^n \frac{1}{k}.$
5. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+n)^2} = \frac{2}{n^2} [1 - (-1)^n] \ln 2 - \frac{\pi^2}{12n^2} [1 + (-1)^n] +$
 $+ \frac{2(-1)^{n+1}}{n^3} \sum_{k=1}^n \frac{(-1)^k}{k} + (-1)^{n+1} \sum_{k=1}^n \frac{(-1)^k}{k^2}.$
6. $\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2} = \frac{\pi^2-9}{3}.$
7. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)^2} = 4 \ln 2 - 3.$
8. $\sum_{k=1}^{\infty} \frac{1}{k^2(k+2)^2} = \frac{\pi^2}{12} - \frac{11}{16}.$
9. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+2)^2} = \frac{5}{16} - \frac{\pi^2}{24}.$
10. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{18}.$
11. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)(k+3)} = \frac{8}{9} - \frac{4}{3} \ln 2.$
12. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)(2k+3)} = \frac{17}{9} - \frac{8}{3} \ln 2.$
13. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(2k+1)(2k+3)} = \frac{5}{9} - \frac{\pi}{3} + \frac{2}{3} \ln 2.$
14. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(3k+1)(3k+2)} = \frac{1}{18} - \frac{\pi}{6\sqrt{3}}.$
15. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(3k+1)(3k+2)} = \frac{11}{4} - 4 \ln 2.$
16. $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2(2k+1)(2k+3)} = \frac{1}{2} - \frac{\pi^2}{24}.$
17. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2(2k+1)(2k+3)} = \frac{\pi}{4} - \frac{1}{2} - \frac{\pi^2}{24}.$

18. $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (3k+2) (3k+4)} = \frac{1}{2} - \frac{\pi}{6\sqrt{3}} - \frac{\pi^2}{54}.$
19. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2 (3k+2) (3k+4)} = \frac{\pi}{3\sqrt{3}} - \frac{1}{2} - \frac{\pi^2}{108}.$
20. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (k+2) (2k+1) (2k+3)} = \frac{8}{3} \ln 2 - \frac{5}{3}.$
21. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (k+2) (2k+1) (2k+3)} = \frac{5}{3} - \frac{\pi}{3} - \frac{2}{3} \ln 2.$
22. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (k+2) (2k+1) (2k+5)} = \frac{1}{9}.$
23. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (k+2) (2k+1) (2k+5)} = \frac{5}{9} - \frac{2}{3} \ln 2.$
24. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1)} = \frac{1}{5} + \frac{2\pi}{15} - \frac{2}{5} \ln 2.$
25. $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (2k+3) (4k+1)} =$
 $= \frac{1}{3} \ln 2 + \frac{\pi}{5} \left(\frac{2\sqrt{2}}{3} - 1 \right) + \frac{4\sqrt{2}}{15} \ln(1+\sqrt{2}) - \frac{1}{5}.$
26. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3)} = \frac{2\pi}{3} - \frac{10}{3} \ln 2 + \frac{1}{3}.$
27. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (2k+3) (4k+3)} =$
 $= \frac{\pi}{3} (1 - 2\sqrt{2}) + \frac{4\sqrt{2}}{3} \ln(1+\sqrt{2}) - \frac{1}{3}.$
28. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+5)} = \frac{2\pi}{3} + \frac{10}{3} \ln 2 - \frac{13}{3}.$
29. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+3) (4k+5)} = \frac{\pi}{3} - \frac{10}{3} \ln 2 + \frac{4}{3}.$

30. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+3)} = 2 \ln 2 - \frac{\pi}{3}.$
31. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (4k+1) (4k+3)} =$
 $= \frac{\pi}{2} \left(\frac{4\sqrt{2}}{3} - 1 \right) - \frac{1}{3} \ln 2 - \frac{2\sqrt{2}}{3} \ln(1+\sqrt{2}).$
32. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+5)} = \frac{2}{3} - \frac{2}{3} \ln 2.$
33. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+3) (4k+1) (4k+3)} = \frac{\pi}{5} - \frac{14}{15} \ln 2 + \frac{2}{15}.$
34. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+3) (4k+1) (4k+5)} = \frac{4\pi}{15} + \frac{6}{5} \ln 2 - \frac{8}{5}.$
35. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+3) (4k+3) (4k+5)} = \frac{\pi}{3} + \frac{10}{3} \ln 2 - \frac{10}{3}.$
36. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (3k+1) (3k+2) (3k+4)} = \frac{1}{2} + \frac{\pi}{4\sqrt{3}} - \frac{3}{4} \ln 3.$
37. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (3k+1) (3k+2) (3k+4)} = \frac{1}{2} - \frac{\pi}{2\sqrt{3}} + \frac{1}{3} \ln 2.$
38. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (4k+1) (4k+3) (4k+5)} = \frac{\pi}{12} - \ln 2 + \frac{1}{2}.$
39. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 (2k+3)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$
40. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 (2k+3)^2} = \frac{1}{2} - \frac{\pi}{8}.$
41. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 (2k+5)^2} = \frac{\pi^2}{64} - \frac{1}{9}.$
42. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 (2k+5)^2} = \frac{1}{8} G - \frac{11}{144}.$

43. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 (4k+1) (4k+3)} = \frac{\pi}{2} - \frac{\pi^2}{8}.$
44. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 (4k+1) (4k+3)} = \frac{1}{4\sqrt{2}} \ln(3+2\sqrt{2}) - \frac{1}{4} G.$
45. $\sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+1) (4k+3)} = \frac{1}{30} + \frac{4}{15} \ln 2 - \frac{\pi}{30}.$
46. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) (2k+3) (4k+1) (4k+3)} =$
 $= -\frac{1}{20} + \frac{\pi}{60} (8\sqrt{2}-7) - \frac{\sqrt{2}}{2} \ln(1+\sqrt{2}).$
47. $\sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+1) (4k+5)} = \frac{\pi}{15} + \frac{2}{15} \ln 2 - \frac{7}{30}.$
48. $\sum_{k=0}^{\infty} \frac{1}{(2k+1) (2k+3) (4k+3) (4k+5)} = \frac{\pi}{6} - \frac{1}{2}.$
49. $\sum_{k=0}^{\infty} \frac{1}{(2k+1) (4k+1) (4k+3)^2} = 0,112742.$
50. $\sum_{k=0}^{\infty} \frac{1}{(2k+1) (4k+1) (4k+3) (4k+5)} = \frac{1}{6} \ln 2 - \frac{\pi}{24} + \frac{1}{12}.$
51. $\sum_{k=0}^{\infty} \frac{1}{(2k+3) (4k+1) (4k+3) (4k+5)} = \frac{7\pi}{120} + \frac{1}{20} \ln 2 - \frac{11}{60}.$
52. $\sum_{k=0}^{\infty} \frac{1}{(8k+1) (8k+3) (8k+5) (8k+7)} = \frac{\pi}{95} (2+\sqrt{2})^{-1}.$
53. $\sum_{k=0}^{\infty} \frac{1}{(4k+1)^2 (4k+3)^2} = \frac{\pi(\pi-2)}{32}.$
54. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)^2 (4k+3)^2} = 0,11037723.$
55. $\sum_{k=0}^{\infty} \frac{1}{(4k+3)^2 (4k+7)^2} = \frac{\pi}{16} + \frac{\pi^2}{32} - \frac{1}{2}.$

5.1.22. Ряды вида $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(km_1 + n_1)(km_2 + n_2) \dots (km_d + n_d)}.$

1. $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (2k+1) (2k+3)^2} = 7 - \frac{5\pi^2}{12} - 4 \ln 2.$
2. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1)^2 (2k+3)^2} = \frac{3}{2} - 2 \ln 2.$
3. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+3)} = \frac{22}{15} \ln 2 - \frac{4\pi}{15} - \frac{1}{15}.$
4. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+5)} = \frac{17}{15} - \frac{2\pi}{15} - \frac{14}{15} \ln 2.$
5. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+1) (4k+7)} = \frac{19}{135} - \frac{2}{15} \ln 2.$
6. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3) (4k+5)} = \frac{7}{3} - \frac{10}{3} \ln 2.$
7. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+3) (4k+7)} = \frac{2}{15} - \frac{14}{15} \ln 2 - \frac{11}{45}.$
8. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (2k+3) (4k+5) (4k+7)} = \frac{4\pi}{15} + \frac{22}{15} \ln 2 - \frac{83}{45}.$
9. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+3) (4k+5)} = \frac{4}{3} \ln 2 - \frac{\pi}{6} - \frac{1}{3}.$
10. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1) (2k+1) (4k+1) (4k+3) (4k+5)} =$
 $= \frac{1}{12\sqrt{2}} \left[\frac{35}{6} \pi - 6 \ln(3+2\sqrt{2}) + \frac{1}{3} (1 - \ln 2) \right].$
11. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+3) (4k+7)} = \frac{8}{15} \ln 2 - \frac{\pi}{10} - \frac{1}{135}.$
12. $\sum_{k=0}^{\infty} \frac{1}{(k+1) (2k+1) (4k+1) (4k+5) (4k+7)} = \frac{43}{15} - \frac{4}{15} \ln 2 - \frac{\pi}{30}.$

$$13. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{30} - \frac{16}{15} \ln 2 + \frac{29}{45}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{13}{15} - \frac{\pi}{30} - \frac{16}{15} \ln 2.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+7)} = \frac{\pi}{30} - \frac{4}{15} \ln 2 + \frac{13}{135}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+5)(4k+7)} = \frac{\pi}{10} + \frac{8}{15} \ln 2 - \frac{91}{135}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{6} - \frac{4}{3} \ln 2 - \frac{13}{9}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{13}{54} - \frac{1}{3} \ln 2.$$

$$19. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{1}{15} \ln 2 - \frac{\pi}{20} + \frac{2}{15}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+3)(4k+7)} = \frac{1}{45} + \frac{1}{15} \ln 2 - \frac{\pi}{60}.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{24} + \frac{1}{15} \ln 2 - \frac{1}{5}.$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(4k+1)(4k+5)(4k+7)} = \frac{\pi}{60} + \frac{1}{15} \ln 2 - \frac{4}{45}.$$

$$23. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{7}{180} + \frac{1}{30} \ln 2 - \frac{\pi}{60}.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(2k+3)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{\pi}{60} + \frac{1}{30} \ln 2 - \frac{13}{180}.$$

5.1.23. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{(kn_1+m_1)(kn_2+m_2) \dots (kn_l+m_l)}$, $l=6, 7, 8, 9$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2} = 10 - \pi^2.$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)^2} = 10 - 12 \ln 2 - \frac{3}{2} \pi^2 (3).$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2(k+2)^2} = \frac{\pi^4}{4} - \frac{39}{16}.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)^2(k+2)^2} = \frac{\pi^4}{24} - \frac{7}{16}.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2(2k+1)^2} = 2\pi^4 + \frac{\pi^2}{3} - 23.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2(k+1)^2(2k+1)^2} = 12 \ln 2 - 23 + 16G.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{6}{5} \ln 2 - \frac{\pi}{15} - \frac{3}{5}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{2}{5} - \frac{\pi}{15} - \frac{7}{135}.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+1)(4k+3)(4k+5)} = \frac{67}{135} - \frac{\pi}{15} - \frac{2}{5} \ln 2.$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(2k+3)(4k+3)(4k+5)(4k+7)} = \frac{47}{45} - \frac{\pi}{15} - \frac{6}{5} \ln 2.$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{2}{5} \ln 2 - \frac{\pi}{30} - \frac{22}{135}.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+3)(4k+1)(4k+3)(4k+5)(4k+7)} = \frac{52}{135} - \frac{\pi}{30} - \frac{2}{5} \ln 2.$$

$$13. \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2(2k+3)^2} = \frac{1}{2} - \frac{2\pi^2}{64}.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2(2k+3)^2} = -\frac{1}{2} + \frac{3\pi}{32} - \frac{\pi^2}{128}.$$

$$8. \sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = (-1)^n \sum_{k=0}^{n-2} \binom{m+k-1}{k} \zeta(n-k) + \\ + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} \zeta(m-k) + (-1)^{m-1} \binom{m+n-1}{n-1}.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k^n (k+1)^n} = \\ = \frac{(-1)^n}{(n-1)!} \sum_{k=1}^{[n/2]} \frac{(2n-2k-1)! (2\pi)^{2k}}{(n-2k)! (2k)!} B_{2k} + (-1)^{n-1} \binom{2n-1}{n-1}.$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n (k+1)^n} = (-1)^{n-1} \sum_{k=0}^{n-2} \binom{m+k-1}{k} (1-2^{1-n+k}) \zeta(n-k) + \\ + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} (1-2^{1-m+k}) \zeta(m-k) + \\ + (-1)^n \binom{m+n-1}{n-1} + (-1)^{m-1} \binom{m+n-2}{m-1} 2 \ln 2.$$

$$11. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n (k+1)^n} = 2(-1)^{n-1} \sum_{k=1}^{[(n-1)/2]} \binom{2n-2k-2}{n-1} (1-2^{-2k}) \zeta(2k+1) + \\ + (-1)^n \binom{2n-1}{n-1} + (-1)^{n-1} \binom{2n-2}{n-1} 2 \ln 2.$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^m (2k+3)^n} = \\ = \frac{1}{2^{m+n}} \left[(-1)^n \sum_{k=0}^{n-2} \binom{m+k-1}{k} (2^{n-k}-1) \zeta(n-k) + \right. \\ \left. + \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} (2^{m-k}-1) \zeta(m-k) + \right. \\ \left. + (-1)^{m-1} \sum_{k=0}^{m-1} \binom{m+k-1}{k} 2^{n-k} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n (2k+3)^n} = \\ = \frac{(-1)^n}{2^{2n} (n-1)!} \sum_{k=1}^{[n/2]} \frac{(2n-2k-1)! (2\pi)^{2k} (2^{2k}-1)}{(n-2k)! (2k)!} B_{2k} + \frac{(-1)^{n-2}}{2}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m (2k+3)^n} = \\ = \frac{(-1)^{m-1}}{2^{m+n}} \sum_{k=0}^{n-2} \binom{m+k-1}{k} 2^k \left[\zeta\left(n-k, \frac{1}{4}\right) - \zeta\left(n-k, \frac{3}{4}\right) \right] + \\ + \frac{1}{2^{m+n}} \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} 2^k \left[\zeta\left(m-k, \frac{1}{4}\right) - \zeta\left(m-k, \frac{3}{4}\right) \right] + \\ + \frac{(-1)^m}{2^m} \sum_{k=0}^{n-1} \binom{m+k-1}{k} \left(\frac{1}{2}\right)^k + \frac{(-1)^{m-1}}{2^{m+n}} \binom{m+n-2}{m-1}.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n (2k+3)^n} = \frac{(-1)^n}{2} + \\ + \frac{(-1)^{n-1}}{2^{2n} (n-1)!} \sum_{k=0}^{[(n-1)/2]} \frac{(2n-k-2)!}{(n-k-1)! (2k)!} \pi^{2k} B_{2k}.$$

5.1.25. Ряды $\sum_{k=0}^{\infty} \frac{(-1)^k k^{\pm m}}{(k^2 \pm a^2)^s}$.

$$1. \sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2)^s} = a^{-2s} + \frac{2\sqrt{\pi}}{\Gamma(s)} (2a)^{1/2-s} \int_0^{\infty} \frac{t^{s-1} J_{s-1/2}(at)}{e^{t^2} - 1} dt \\ [Re s > 1/2; |a| < 1].$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(k^2 + a^2)^2 - b^2} = \frac{1}{2b} [\psi(a+b) - \psi(a-b)].$$

$$3. \sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2 \pm b^2)} = \frac{\pi}{b} \left[\frac{\{ \operatorname{ch} 2\pi b \}}{\{ \cos 2\pi a \}} - \cos 2\pi a \right]^{-1} \{ \operatorname{sh} 2\pi b \}.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{k^2 \pm a^2} = \pm \frac{1}{2a^2} \pm \frac{\pi}{2a} \left[\frac{\operatorname{ch} \pi a}{\operatorname{cig} \pi a} \right].$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k^2 - n^2} = \frac{3}{4n^2} \quad [n = 1, 2, 3, \dots].$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2 \pm a^2} = \pm \frac{1}{2a^2} \pm \frac{\pi}{2a} \left[\frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right].$$

7. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - n^2} = \frac{1}{4n^2} [2 + (-1)^n]$ $\{n = 1, 2, 3, \dots\}$
8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - 4n^2} = \frac{3}{16n^2}$
9. $\sum_{k \neq n/2}^{\infty} \frac{1}{k^2 - \frac{n^2}{4}} = \frac{1}{2n^2} [5 + (-1)^n]$ $\{n = 1, 2, 3, \dots\}$
10. $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 \pm a^2} = \frac{1}{2} \left\{ \beta(ia) + \beta(-ia) \right\}$
11. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1} = -0,269611$, 12. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2 + 1/4} = -0,516128$
13. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 + a^2)} = \frac{1}{2a^3} \left\{ \psi(1+ia) + \psi(1-ia) - 2\psi(1) \right\}$
14. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 + 1)} = 0,671866$, 15. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 + 1)} = -0,423567$
16. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 + 1/4)} = 0,993317$, 17. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 + 1/4)} = -0,708078$
18. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/4)} = 4(2 \ln 2 - 1)$, 19. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k^2 - 1/4)} = -1,227411$
20. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/9)} = \frac{27}{2} (\ln 3 - 1)$
21. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/6)} = -108 + 54 \ln 3 + 72 \ln 2$
22. $\sum_{k=1}^{\infty} \frac{1}{k^3 (i^3 \pm a^2)} = \frac{1}{2a^3} \pm \frac{\pi^2}{6a^3} - \frac{\pi}{2a^3} \{ \operatorname{cth} \pi a \}$
23. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 (k^2 \pm a^2)} = \frac{1}{2a^4} \pm \frac{\pi^2}{12a^4} - \frac{\pi}{2a^4} \{ \operatorname{sh} \pi a \}^{-1}$

24. $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^3 + a^3} = \frac{1}{3} \beta(a) + \frac{1}{3} \beta(ae^{2\pi i/3}) + \frac{1}{3} \beta(ae^{4\pi i/3})$
25. $\sum_{k=0}^{\infty} \frac{1}{(k^2 \pm a^2)^2} = \frac{1}{2a^4} + \frac{\pi}{4a^3} \{ \operatorname{cth} \pi a \} + \frac{\pi^2}{4a^3} \{ \operatorname{cosech} \pi a \}^2$
26. $\sum_{k=0}^{\infty} \frac{1}{(k^2 \pm a^2)(k^2 \pm b^2)} = \frac{\pi}{2ab} \cdot \frac{1}{a^2 - b^2} \left\{ a \{ \operatorname{cth} \pi b \} - b \{ \operatorname{cth} \pi a \} \right\} - \frac{1}{a^2 b^2}$
27. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2 \pm a^2)^2} = \frac{1}{2a^4} + \frac{\pi}{4a^3} \{ \operatorname{cosech} \pi a \} \left\{ 1 + \pi a \{ \operatorname{ctg} \pi a \} \right\}$
28. $\sum_{k=1}^{\infty} \frac{k}{(k^2 \pm a^2)^2} = \frac{1}{4aa} [\Psi'(-aa) - \Psi'(aa)]$ $\left[a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$
29. $\sum_{k=1}^{\infty} \frac{k}{(k^2 + 1)^2} = 0,397117$, 30. $\sum_{k=1}^{\infty} \frac{k}{(k^2 + 1/4)^2} = 0,825042$
31. $\sum_{k=1}^{\infty} \frac{k}{(k^2 - 1/4)^2} = 2$
32. $\sum_{k=1}^{\infty} \frac{1}{k(k^2 - 1/4)^2} = 16 \left(\frac{3}{2} - 2 \ln 2 \right)$
33. $\sum_{k=1}^{\infty} \frac{1}{k^3 (k^2 \pm a^2)^2} = \frac{\pi^2}{6a^5} \pm \frac{1}{a^5} + \frac{3\pi}{4a^4} \{ \operatorname{cth} \pi a \} + \frac{\pi^2}{4a^4} \{ \operatorname{cosech} \pi a \}^2$
34. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 (k^2 \pm a^2)^2} = \pm \frac{1}{a^5} - \frac{\pi^2}{12a^5} + \frac{\pi}{4a^4} \{ \operatorname{cosech} \pi a \} \left\{ 3 + \pi a \{ \operatorname{ctg} \pi a \} \right\}$
35. $\sum_{k=1}^{\infty} \frac{k^2}{(k^2 \pm a^2)^2} = \frac{\pi}{4a} \left\{ \pm \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \mp \pi a \{ \operatorname{cosech} \pi a \}^2 \right\}$
36. $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{(k^2 \pm a^2)^2} = \frac{\pi}{4a} \left\{ \pm 1 \mp \pi a \{ \operatorname{cth} \pi a \} \{ \operatorname{cosech} \pi a \} \right\}$
37. $\sum_{k=1}^{\infty} \frac{1}{(k^2 \pm a^2)^3} = \frac{1}{2a^6} \pm \frac{3\pi^2}{16a^6} \{ \operatorname{cosech} \pi a \}^2 \pm$
 $\pm \frac{3\pi}{16a^5} \{ \operatorname{cth} \pi a \} \pm \frac{\pi^2}{a^5} \{ \operatorname{cosech} \pi a \}^2 \{ \operatorname{cth} \pi a \}$

$$38. \sum_{k=1}^{\infty} \frac{k^2}{(k^2 \pm a^2)^3} = \frac{\pi}{16a^3} \left\{ \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} + \frac{\pi}{16a^2} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \left(1 - 2\pi a \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right) \right\}.$$

$$39. \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{(k^2 \pm a^2)^3} = \frac{\pi}{16a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \left(\frac{\pi^2 a^2}{4} - 2\pi^2 a^2 \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 + 1 + \pi a \frac{\operatorname{cth} \pi a}{\operatorname{ctg} \pi a} \right).$$

5.1.26. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^2 m}{[(2k+1)^2 \pm a^2]^n}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pm a^2} = \frac{\pi}{4a} \left\{ \frac{\operatorname{th} (\pi a/2)}{\operatorname{tg} (\pi a/2)} \right\}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - n^2} = \frac{1}{8n^2} [1 - (-1)^n].$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 - a^2} = \frac{1}{4a} \left[\frac{\pi}{\cos (\pi a/2)} - 2\beta \left(\frac{a+1}{2} \right) \right].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1} = 0,425841. \quad 5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1/4} = 0,71864956.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 - 1/4} = 1,246450.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{(2k+1)^2 \pm a^2} = \frac{\pi}{4} \left\{ \frac{\operatorname{sech} (\pi a/2)}{\sec (\pi a/2)} \right\}.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 + 1]} = 0,547701.$$

$$9. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 + 1/4]} = 0,85069754.$$

$$10. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 - 1/4]} = 1,386294.$$

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 \pm a^2]} = \pm \frac{\pi}{4a^2} \left[1 - \left\{ \frac{\operatorname{sech} (\pi a/2)}{\sec (\pi a/2)} \right\} \right].$$

$$12. \sum_{k=0}^{\infty} \frac{1}{(2k+1) [(2k+1)^2 - a^2]} = \frac{1}{4a^2} \left[2\psi \left(\frac{1}{2} \right) - \psi \left(\frac{1-a}{2} \right) - \psi \left(\frac{1+a}{2} \right) \right].$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 + 1]} = 0,4723807.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 - 1/4]} = 2 \ln 2.$$

$$15. \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^2 - 1]} = \frac{3}{4} - \frac{\pi}{4}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 [(2k+1)^2 \pm a^2]} = \pm \frac{\pi^2}{8a^2} + \frac{\pi}{4a^3} \left\{ \frac{\operatorname{th} (\pi a/2)}{\operatorname{tg} (\pi a/2)} \right\}.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + a^2]} = \frac{1}{4a^3} \left[\beta \left(\frac{1-ia}{2} \right) - \beta \left(\frac{1+ia}{2} \right) \right] - \frac{1}{4a^2} \beta' \left(\frac{1}{2} \right).$$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + 1]} = 0,490124.$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 + 1/4]} = 0,789264.$$

$$20. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 [(2k+1)^2 - 1/4]} = 1,32193954.$$

$$21. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 \pm a^2]^2} = \pm \frac{\pi}{8a^2} \left\{ \frac{\operatorname{th} (\pi a/2)}{\operatorname{tg} (\pi a/2)} \right\} + \frac{\pi^2}{16a^2} \left\{ \frac{\operatorname{sech} (\pi a/2)}{\sec (\pi a/2)} \right\}^2.$$

$$22. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)^2 - 1/4]^2} = 1,766036.$$

$$23. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 \pm a^2]^2} = \frac{1}{16a^3} \left[\psi' \left(\frac{1-a}{2} \right) - \psi' \left(\frac{1+a}{2} \right) \right] \quad \left[a = \left\{ \frac{1}{2} \right\} \right]$$

$$24. \sum_{k=0}^{\infty} \frac{2k+1}{[(2k+1)^2 + 1]^2} = 0,293986.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(k^2 - a^2)^2} = \frac{1}{2a^3} + \frac{3\pi}{16a^2} (\operatorname{ctg} \pi a + \operatorname{cth} \pi a) + \frac{\pi^2}{16a^2} (\operatorname{cosec}^2 \pi a + \operatorname{cosech}^2 \pi a).$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2 - a^2)^2} = \frac{1}{2a^3} + \frac{3\pi}{16a^2} (\operatorname{cosec} \pi a + \operatorname{cosech} \pi a) + \frac{\pi^2}{16a^2} (\operatorname{cosec} \pi a \operatorname{ctg} \pi a + \operatorname{cosech} \pi a \operatorname{cth} \pi a).$$

$$10. \sum_{k=0}^{\infty} \frac{2k+2a+1}{[k^2 + (2a+1)k + a^2 + a + b^2]^2 + b^2} = \frac{1}{a^2 + b^2}.$$

$$11. \sum_{k=0}^{\infty} \frac{k^2 + (2a+1)k + a^2 + a - b^2}{[k^2 + (2a+1)k + a^2 + a + b^2]^2 + b^2} = \frac{a}{a^2 + b^2}.$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k [k^2 + (2a+1)k + a^2 + a + b^2]}{[k^2 + (2a+1)k + a^2 + a - b^2]^2 + (2k+2a+1)b^2} = \frac{1}{2(a^2 + b^2)}.$$

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k [k^2 + (2a+1)k + a^2 + a + b^2] (2k+2a+1)}{[k^2 + (2a+1)k + a^2 + a - b^2]^2 + (2k+2a+1)b^2} = \frac{a}{a^2 + b^2}.$$

$$14. \sum_{k=1}^{\infty} \frac{k}{4k^2 + 1} = \frac{1}{4}, \quad 15. \sum_{k=1}^{\infty} \frac{2k^2 - 1}{4k^2 + 1} = \frac{1}{2}.$$

5.1.28. Ряды вида $\sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)^2 + a^2]^n}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 + a^2} = \frac{\pi}{2\sqrt{2}a^2} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a} - \frac{\pi}{4\sqrt{2}a^2} \frac{\operatorname{sh} (\pi a/\sqrt{2}) + \sin (\pi a/\sqrt{2})}{\operatorname{ch} (\pi a/\sqrt{2}) - \cos (\pi a/\sqrt{2})}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - a^2} = \frac{\pi}{8a^2} \left(\operatorname{tg} \frac{\pi a}{2} - \operatorname{th} \frac{\pi a}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1} = 0,489093. \quad 4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + 1/4} = 0,930130.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 - 1/4} = 1,055002$$

$$6. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 + a^2]^2} = \frac{3\pi}{8\sqrt{2}a^3} \frac{\operatorname{sh} \sqrt{2} \pi a + \sin \sqrt{2} \pi a}{\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a} - \frac{3\pi}{16\sqrt{2}a^3} \frac{\operatorname{sh} (\pi a/\sqrt{2}) + \sin (\pi a/\sqrt{2})}{\operatorname{ch} (\pi a/\sqrt{2}) - \cos (\pi a/\sqrt{2})} + \frac{\pi^2}{4a^3} \frac{\operatorname{sh} \sqrt{2} \pi a \sin \sqrt{2} \pi a}{(\operatorname{ch} \sqrt{2} \pi a - \cos \sqrt{2} \pi a)^2} - \frac{\pi^2}{16a^3} \frac{\operatorname{sh} (\pi a/\sqrt{2}) \sin (\pi a/\sqrt{2})}{[\operatorname{ch} (\pi a/\sqrt{2}) - \cos (\pi a/\sqrt{2})]^2}.$$

$$7. \sum_{k=0}^{\infty} \frac{1}{[(2k+1)^2 - a^2]^2} = -\frac{3\pi}{32a^3} \left(\operatorname{tg} \frac{\pi a}{2} - \operatorname{th} \frac{\pi a}{2} \right) + \frac{\pi^2}{64a^3} \left(\operatorname{sec}^2 \frac{\pi a}{2} - \operatorname{sech}^2 \frac{\pi a}{2} \right).$$

5.1.29. Ряды вида $\sum_{k=0}^{\infty} \frac{b_n}{1 + a^{2n}}$.

(Обозначение: $q = \exp [-\pi K'(k)/K(k)]$).

$$1. \sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1 + q^n} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [E(k) - k^2 K(k)].$$

$$2. \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} = \frac{1}{2\pi} K(k) - \frac{1}{4}.$$

$$3. \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^{2n}} = \frac{1}{2\pi^2} K(k) [K(k) - E(k)] \quad (|q| < 1).$$

$$4. \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1 - q^{2n+1}} = \frac{1}{2\pi} K(k) - \frac{1}{4} \quad (|q| < 1).$$

$$5. \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1 - q^n)^2} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [E(k) - k^2 K(k)].$$

$$6. \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2} = -\frac{1}{8} + \frac{1}{2\pi^2} K(k) [2E(k) - K(k)].$$

$$7. \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} = \frac{1}{24} + \frac{1}{6\pi^2} K(k) [(2 - k^2) K(k) - 3E(k)] \quad (|q| < 1).$$

$$8. \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2} = \frac{1}{2\pi^2} K(k) [E(k) - (1 - k^2) K(k)].$$

$$9. \sum_{n=1}^{\infty} \frac{q^{2n+1}}{(1 - q^{2n+1})^2} = \frac{1}{2\pi^2} K(k) [K(k) - E(k)] \quad (|q| < 1).$$

$$10. \sum_{k=1}^{\infty} \frac{q^{2k}}{1 \pm 2q^{2k} \cos 2x + q^{4k}} =$$

$$= \mp \frac{1}{4} \operatorname{cosec} 2x \left[\frac{\{\theta_1(x, q)\}^{-1}}{\{\theta_1(x, q)\}} - \frac{\partial}{\partial x} \frac{\{\theta_1(x, q)\}}{\{\theta_1(x, q)\}} \pm \frac{\{g(x)\}}{\{c_1 g(x)\}} \right] \quad (|q| < 1).$$

$$11. \sum_{k=1}^{\infty} \frac{q^{2k+1}}{1 \pm 2q^{2k+1} \cos 2x + q^{4k+2}} = \mp \frac{1}{4} \operatorname{cosec} 2x \left\{ \frac{\{\theta_2(x, q)\}^{-1}}{\{\theta_2(x, q)\}} - \frac{\partial}{\partial x} \frac{\{\theta_2(x, q)\}}{\{\theta_2(x, q)\}} \right\}$$

$$\quad (|q| < 1).$$

$$12. \sum_{k=0}^{\infty} \frac{2^k}{1+q^{2k}} \left\{ \frac{q^k}{1} \right\} = \frac{1}{q-1} \left\{ \frac{-q}{1} \right\} \quad \left\{ \left\{ \frac{q}{1} \right\} < 1 \right\}.$$

$$13. \sum_{k=0}^{\infty} \frac{1}{1-q^{2k+1}} q^{2k} = \frac{1}{1-q} \left\{ \frac{q}{1} \right\} \quad \left\{ \left\{ \frac{q}{1} \right\} < 1 \right\}.$$

5.1.30. Ряды вида $\sum_{k=1}^{\infty} \frac{a_k}{k^b}$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^b} = 1,291285997. \quad 2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^b} = -0,78343051.$$

$$3. \sum_{k=1}^{\infty} \frac{k!}{k^b} = 1,8796386. \quad 4. \sum_{k=1}^{\infty} (-1)^k \frac{k!}{k^b} = -0,65383160.$$

5.1.31. Ряды вида $\sum_{k=1}^{\infty} \frac{a_k}{k!}$.

(См. также ряды вида $\sum_{k=1}^{\infty} \frac{a_k}{k!} x^k$ (п. 5.2) при $x = \pm 1$.)

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+a)} = \gamma(a, 1).$$

$$2. \sum_{k=0}^{\infty} \frac{k^n}{k!} = e S_n$$

$$S_{0,1}=1, \quad S_2=2, \quad S_3=5, \quad S_4=15, \quad S_5=52, \quad S_6=203,$$

$$S_7=877, \quad S_8=4140, \quad S_{n+1} = \frac{1}{n+1} \sum_{k=0}^n \frac{1}{k!} S_{n-k}.$$

$$3. \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} = \frac{1}{e} A_n,$$

$$A_0=1, \quad A_1=-1, \quad A_2=0, \quad A_3=1, \quad A_4=1, \quad A_5=-2.$$

5.1.32. Ряды вида $\sum a_k \left(\sum_{m=1}^{N(k)} b_m \right)$.

$$1. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{1}{2} \ln^2 2 - \frac{\pi^2}{12}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left(\sum_{m=0}^k \frac{1}{2m+1} \right) = \frac{\pi^2}{16}.$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{1}{2} \ln^2 2.$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{\pi}{8} \ln 2.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{\pi^2}{6} (3).$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left(\sum_{m=1}^{2k-1} \frac{1}{m} \right) = \frac{1}{2} + 2 \ln 2.$$

$$7. \sum_{k=1}^{\infty} \frac{1}{(k+1)^3} \left(\sum_{m=1}^k \frac{1}{m} \right) = \frac{\pi^4}{360}.$$

5.2. СТЕПЕННЫЕ РЯДЫ

5.2.1. Введение.

В этом разделе приведены суммы рядов вида $\sum a_k x^k$. Ряды, в которых a_k содержит тригонометрические функции, логарифмы и т. д., помещены в соответствующих разделах этой главы.

Многие ряды могут быть записаны в виде обобщенных гипергеометрических рядов:

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x)$$

и суммируются тем самым к соответствующей обобщенной гипергеометрической функции. Отметим, что в теории степенных рядов естественно возникает понятие p -функции, частным случаем которой является факториальная функция $(\Gamma(k+1) = k!)$; в связи с этим было сочтено целесообразным поместить в этой главе некоторые степенные ряды, коэффициенты которых содержат гамма-

функцию. При этом могут быть полезными следующие формулы (см. также приложение 1.6):

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = (-1)^k \frac{\Gamma(1-a)}{\Gamma(1-a-k)} = a(a+1) \dots (a+k-1),$$

$$k! = (1)_k, \quad (2k)! = (1)_k \left(\frac{1}{2}\right)_k 2^{2k}, \quad (2k+1)! = (1)_k \left(\frac{3}{2}\right)_k 2^{2k},$$

$$(3k)! = \frac{1}{2\pi} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) (1)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k 3^{3k+1/2},$$

$$(2k)!! = (1)_k 2^k, \quad (2k+1)!! = \left(\frac{3}{2}\right)_k 2^k,$$

$$k+a = \frac{\Gamma(a+1)}{\Gamma(a)} \frac{(a+1)_k}{(a)_k}, \quad (a+1)(a+2) \dots (a+k) = (a+1)_k.$$

Если известна сумма ряда

$$\sum_{k=0}^{\infty} a_k x^k = S(x),$$

то с помощью почленного интегрирования и дифференцирования (в области сходимости) можно найти также следующие суммы:

$$\sum_{k=0}^{\infty} \frac{dx}{nk+m} x^{nk+m} = \int_0^x x^{m-1} S(x^n) dx,$$

$$\sum_{k=0}^{\infty} (nk+m) a_k x^{nk+m} = x \frac{d}{dx} [x^{m-1} S(x^n)]$$

и

$$\sum_{k=0}^{\infty} k^m a_k x^k = \left(x \frac{d}{dx}\right)^m S(x).$$

Другие способы суммирования степенных рядов изложены в [20, 23, 42].

5.2.2. Ряды вида $\sum P(k) x^k$

Условие: $|x| < 1$.

$$1. \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

$$2. \sum_{k=0}^{\infty} (\pm 1)^k k^n x^k = \left(x \frac{d}{dx}\right)^n \frac{1}{1-x}.$$

$$3. = \sum_{k=1}^n \frac{(-1)^k k!}{(1-x)^{k+1}} \left[\sum_{m=0}^{k-1} (-1)^m \binom{k}{m} m^n \right].$$

$$4. \sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}, \quad 5. \sum_{k=1}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}.$$

$$6. \sum_{k=1}^{\infty} k^3 x^k = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

$$7. \sum_{k=1}^{\infty} k^4 x^k = \frac{x(x^3+6x^2-x-2)}{(1-x)^5}.$$

$$8. \sum_{k=1}^{\infty} k(k+1) x^k = \frac{2x}{(1-x)^3}, \quad 9. \sum_{k=0}^{\infty} (k+1)^2 x^k = \frac{1+x}{(1-x)^3}.$$

$$10. \sum_{k=0}^{\infty} (k+1)(k+2) \dots (k+n-1) x^k = \frac{(n-1)!}{(1-x)^n}.$$

5.2.3. Ряды вида $\sum \frac{x^k}{(k+a)^s}$

$$1. \sum_{k=0}^{\infty} \frac{x^k}{(k+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-xe^{-t}} dt = \Phi(x, s, a)$$

[Re a > 0; -1 < x < 1; Re s > 0; если x = 1, то Re s > 1].

$$2. \sum_{k=1}^{\infty} \frac{x^k}{k^2} = x \Phi(x, 2, 1).$$

$$3. \sum_{k=1}^{\infty} \frac{x^k}{k^n} = \text{Li}_n(x) \quad [n > 2, |x| < 1; n = 1, -1 < x < 1].$$

$$4. \sum_{k=0}^{\infty} \frac{x^{k+m}}{k+a} = \int_0^x \frac{x^{a-1}}{1-x} dx \quad [\text{Re } a > 0; 1 < x < 1].$$

$$5. \sum_{k=-\infty}^{\infty} \frac{1}{k+a} e^{ik\pi b} = \pi \operatorname{cosec} a \pi \exp[i(2m-b+1) \pi]$$

[m — целое число, для которого 2m < a < 2m+1].

$$6. \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k+a} e^{ik\pi b} = \pi \operatorname{cosec} a \pi \exp[i(2m-b) \pi]$$

[m — целое число, для которого 2m-1 < a < 2m].

5.2.4. Ряды вида $\sum \frac{(\pm 1)^k}{nk+m} x^k$

$$1. \sum_{k=0}^{\infty} \frac{1}{nk+m} x^k = \frac{1}{n} x^{-m/n} \left[2 \sum_{k=1}^{[(n-1)/2]} \sin \frac{2km}{n} \pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2k\pi}{n}}{1-x^{1/n} \cos \frac{2k\pi}{n}} - \right.$$

$$\left. - \sum_{k=1}^{[(n-1)/2]} \cos \frac{2km\pi}{n} \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2k\pi}{n} + 1 \right) - \right.$$

$$\left. - \ln(1-x^{1/n}) + (-1)^{m+1} \frac{1+(-1)^n}{2} \ln(1+x^{1/n}) \right]$$

[|x| < 1; m < n; m, n = 1, 2, 3, ...].

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{nk+m} x^{nk+m} = \frac{1}{x^{m/n}} \left[\sum_{j=1}^{[(m-1)/n]} (-1)^{j-1} \frac{x^{(m-jn)/n}}{m-jn} + \right. \\ \left. + \frac{2}{n} \sum_{k=0}^{[(n-2)/2]} \sin \frac{2k+1}{n} \pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2k+1}{n} \pi}{1-x^{1/n} \cos \frac{2k+1}{n} \pi} - \right. \\ \left. - \frac{1}{n} \sum_{k=0}^{[(n-2)/2]} \cos \frac{2k+1}{n} \pi \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2k+1}{n} \pi + 1 \right) + \right. \\ \left. + (-1)^{n-1} \frac{1-(-1)^n}{2n} \ln(1+x^{1/n}) \right] \quad [m, n = 1, 2, 3, \dots; -1 < x < 1].$$

$$3. \sum_{k=0}^{\infty} \frac{1}{k+m} x^{k+m} = -\ln(1-x) - \sum_{k=1}^{m-1} \frac{1}{k} x^k \quad [-1 < x < 1; m = 1, 2, 3, \dots].$$

$$4. \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x) \quad [-1 < x < 1].$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+m} x^{2k+m} = \sum_{k=1}^{[(m-1)/2]} (-1)^{k-1} \frac{x^{m-2k}}{m-2k} + \\ + \sin \frac{m\pi}{2} \operatorname{arctg} x - \frac{1}{2} \cos \frac{m\pi}{2} \ln(1-x^2) \quad [|x| < 1].$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+m} x^{3k+m} = \sum_{k=1}^{[(m-1)/3]} (-1)^{k-1} \frac{x^{m-3k}}{m-3k} + \frac{2}{3} \sin \frac{m\pi}{3} \operatorname{arctg} \frac{x\sqrt{3}}{2-x} - \\ - \frac{1}{3} \cos \frac{m\pi}{3} \ln(x^2-x+1) + \frac{(-1)^{m-1}}{3} \ln(1+x) \quad [-1 < x < 1].$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+m} x^{4k+m} = \sum_{k=1}^{[(m-1)/4]} (-1)^{k-1} \frac{x^{m-4k}}{m-4k} + \\ + \frac{1}{2} \sin \frac{m\pi}{4} \operatorname{arctg} \frac{x}{\sqrt{2-x}} + \frac{1}{2} \sin \frac{3m\pi}{4} \operatorname{arctg} \frac{x}{\sqrt{2+x}} - \\ - \frac{1}{4} \cos \frac{m\pi}{4} \ln(x^2-x\sqrt{2}+1) + \frac{1}{4} \cos \frac{3m\pi}{4} \ln(x^2+x\sqrt{2}+1) \quad [|x| < 1].$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = \left\{ \frac{2^{-1} \ln[(1+x)/(1-x)]}{\operatorname{arctg} x} \right\} \quad \left[\left\{ \begin{array}{l} |x| < 1 \\ |x| \leq 1 \end{array} \right\} \right].$$

$$9. \sum_{k=0}^{\infty} \frac{1}{3k+1} x^{3k+1} = \\ = \frac{1}{6} \ln(x^2+x+1) - \frac{1}{3} \ln(1-x) + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{x+2} \quad [-1 < x < 1].$$

$$10. \sum_{k=0}^{\infty} \frac{1}{3k+2} x^{3k+2} = \\ = \frac{1}{6} \ln(1-x^3) - \frac{1}{2} \ln(1-x) - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{x+2} \quad [-1 < x < 1].$$

$$11. \sum_{k=0}^{\infty} \frac{1}{4k+1} x^{4k+1} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \operatorname{arctg} x \quad [|x| < 1].$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+1} x^{4k+1} = \\ = \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} \quad [|x| < 1].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{4k+3} x^{4k+3} = \frac{1}{4} \ln \frac{1+x}{1-x} - \frac{1}{2} \operatorname{arctg} x \quad [|x| < 1].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+3} x^{4k+3} = \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} \quad [-1 < x < 1].$$

$$15. \sum_{k=0}^{\infty} \frac{1}{6k+1} x^{6k+1} = \\ = \frac{1}{6} \ln \frac{1+x}{1-x} + \frac{1}{12} \ln \frac{1+x+x^2}{1-x+x^2} + \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{1-x^2} \quad [|x| < 1].$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{6k+1} x^{6k+1} = \frac{1}{3} \operatorname{arctg} x + \frac{1}{6} \operatorname{arctg} \frac{x}{1-x^2} + \frac{1}{12\sqrt{3}} \ln \frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2} \quad [|x| < 1].$$

5.2.5. Ряды вида $\sum_{k=0}^{\infty} \frac{x^k}{(nk+m)(lk+j)}$.

Условие: $|x| \leq 1$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(nk+m)(lk+j)} x^k = \\ = \frac{1}{nj-lm} \left[x^{j/l} \sum_{k=0}^{l-1} \exp\left(-\frac{2ik\pi}{l}\right) \ln\left(1-x^{1/l} \exp \frac{2ik\pi}{l}\right) - \right. \\ \left. - x^{m/n} \sum_{k=0}^{n-1} \exp\left(-\frac{2ik\pi}{n}\right) \ln\left(1-x^{1/n} \exp \frac{2ik\pi}{n}\right) + l \sum_{k=1}^l \frac{1}{j-kl} x^{-k} \right] \\ [m \leq n; nj-lm \neq 0; s - \text{целое число, для которого } sl < l < (s+1)l].$$

$$2. \sum_{k=1}^{\infty} \frac{x^{k+n}}{k(k+n)} = \frac{1}{n} (1-x^n) \ln(1-x) + \frac{1}{n} \sum_{k=1}^n \frac{x^k}{k}.$$

$$3. \sum_{k=1}^{\infty} \frac{x^{k+n}}{(k+m)(k+n)} = \frac{1}{n-m} \left[\ln(1-x) - x^{n-m} \ln(1-x) + \sum_{k=1}^m \frac{1}{k} x^k - x^{n-m} \sum_{k=1}^m \frac{x^k}{k} \right].$$

$$4. \sum_{k=1}^{\infty} \frac{x^k}{k^2} = - \int_0^x \frac{\ln(1-t)}{t} dt = Li_2(x).$$

$$5. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^{k+1} = x + (1-x) \ln(1-x).$$

$$6. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} x^{k+2} = \frac{x}{2} + \frac{x^2}{4} + \frac{1}{2} (1-x^2) \ln(1-x).$$

$$7. \sum_{k=1}^{\infty} \frac{1}{k(k+3)} x^{k+3} = \frac{1}{18} [6x + 3x^2 + 2x^3 + 6(1-x^3) \ln(1-x)].$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(2k+1)} x^{2k+1} = 2x - 2 \left\{ \operatorname{Arth} x \right\} - x \ln(1 \mp x).$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k(2k+2)} x^{2k+2} = \frac{2x}{3} + \frac{2x^3}{9} - \frac{x^3}{3} \ln(1-x^2) - \frac{2}{3} \operatorname{arctg} x.$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(2k+1)} x^{2k+2} = \pm 2x \left\{ \operatorname{Arth} x \right\} + \ln(1 \mp x^2).$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+1)} x^{3k+3} = \frac{\sqrt{3}}{2} x^2 \operatorname{arctg} \frac{x\sqrt{3}}{x+2} + \frac{1}{4} (x^2+2) \ln(1-x^3) - \frac{3}{4} x^2 \ln(1-x).$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)(2k+1)} x^{k+2} = \frac{1}{3} \left[\pm x + \ln(1 \mp x) + 2x^{3/2} \left\{ \operatorname{Arth} \sqrt{x} \right\} \right].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} x^{2k+1} = \frac{1}{2} \int_0^x \frac{1}{t} \ln \frac{1+t}{1-t} dx = \frac{1}{2} [Li_2(x) - Li_2(-x)].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)^2} = \int_0^x \frac{1}{t} \operatorname{arctg} t dt.$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} x^{2k+3} = \frac{1}{2} (1 \mp x^2) \left\{ \operatorname{Arth} x \right\} \mp \frac{x}{2}.$$

$$16. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)} x^{3k+2} = \frac{1}{6} (x-1) [\ln(1-x^3) - 3 \ln(1-x)] + \frac{1}{\sqrt{3}} (x+1) \operatorname{arctg} \frac{x\sqrt{3}}{x+2}.$$

$$17. \sum_{k=0}^{\infty} \frac{1}{(3k+2)(3k+4)} x^{3k+4} = \frac{x}{2} + \frac{1}{12} (1-x^2) [3 \ln(1-x) - \ln(1-x^3)] - \frac{1}{2\sqrt{3}} (1+x^2) \operatorname{arctg} \frac{x\sqrt{3}}{x+2}.$$

$$18. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} x^{4k+3} = \frac{1}{8} \left[(1-x^2) \ln \frac{1-x}{1+x} + 2(x^2+1) \operatorname{arctg} x \right].$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)} x^{4k+3} = \frac{1}{2\sqrt{2}} (x^2-1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + \frac{1}{4\sqrt{2}} (1+x^2) \ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}.$$

$$20. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)} x^{4k+5} = \frac{x}{4} + \frac{1}{16} (x^4-1) \left[\ln \frac{1+x}{1-x} + 2 \operatorname{arctg} x \right].$$

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+5)} x^{4k+5} = -\frac{x}{4} + \frac{1}{16\sqrt{2}} (1+x^4) \left[\ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2 \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} \right].$$

$$22. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+5)} x^{4k+5} = \frac{x}{2} - \frac{1}{8} (x^2-1) \ln \frac{1+x}{1-x} - \frac{1}{4} (x^2+1) \operatorname{arctg} x.$$

$$23. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+5)(4k+5)} x^{4k+5} = \frac{x}{2} + \frac{1}{4\sqrt{2}} (x^2-1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \frac{1}{8\sqrt{2}} (x^2+1) \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}.$$

$$24. \sum_{k=0}^{\infty} \frac{1}{(4k+3)(4k+7)} x^{4k+7} = \frac{x^7}{12} + \frac{1}{16} (x^4 - 1) \left(\ln \frac{1+x}{1-x} - 2 \operatorname{arctg} x \right).$$

$$25. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(4k+7)} x^{4k+7} = \frac{1}{16\sqrt{2}} (x^4 - 1) \left(2 \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} - \ln \frac{1+x\sqrt{2}+x^3}{1-x\sqrt{2}+x^3} \right).$$

$$26. \sum_{k=0}^{\infty} \frac{1}{k^2+1} x^k = 1 - x \int_0^1 \frac{\ln \ln t}{1-t} dt.$$

5.2.6. Ряды вида $\sum_{k=1}^{\infty} \frac{(\pm 1)^k x^k}{(kn_1+m_1)(kn_2+m_2)\dots(kn_f+m_f)}$, $i \geq 2$.
Условие: $|x| \leq 1$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} x^{k+1} = (x-1) \ln(1-x) + x [Li_2(x) - 1].$$

$$2. \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} x^{k+1} = 2x + (1-x) \ln(1-x) - Li_2(x).$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} x^{k+2} = \frac{3}{4} x^2 - \frac{x}{2} - \frac{1}{2} (1-x)^2 \ln(1-x).$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(2k+1)} x^{2k+1} = 2x - \left(x + \frac{1}{x}\right) \ln(1-x^2) + 2 \ln \frac{1-x}{1+x} \quad (|x| < 1).$$

$$5. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)(2k+1)(2k+3)} x^{2k+3} = \pm x \ln(1 \mp x^2) \mp x \pm (1 \pm x^2) \begin{cases} \operatorname{Arth} x \\ \operatorname{arctg} x \end{cases} \quad (|x| < 1).$$

$$6. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)(4k+3)(4k+5)} x^{4k+5} = \frac{1}{2} (1 \mp x^2) \left\{ \ln \frac{(1+x)/(1-x)}{2^{1/2} \ln[(x^2+x\sqrt{2}+1)/(x^2-x\sqrt{2}+1)]} \right\} \pm \ln(1 \mp x^2) \pm (1-x^2) \begin{cases} \operatorname{arctg} x \\ 2^{-1/2} \operatorname{arctg}[x\sqrt{2}/(1-x^2)] \end{cases} \mp 2.$$

$$7. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)(2k+3)(2k+5)} x^{2k+5} = \frac{1}{24} \left[3(1 \mp x^2)^2 \begin{cases} \operatorname{Arth} x \\ \operatorname{arctg} x \end{cases} - 3x \pm 5x^3 \right].$$

$$8. \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)(4k+5)} x^{4k+5} = \frac{1}{32} \left[(1-x^2)^2 \ln \frac{1+x}{1-x} + 2(1+x^2)^2 \operatorname{arctg} x - 4x \right].$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(4k+3)(4k+5)} x^{4k+5} = \frac{1}{64} \left[\sqrt{2} (x^4+2x^2-1) \ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2\sqrt{2} (x^4-2x^2-1) \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} + 8x \right].$$

$$10. \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2} x^{k+1} = 2(x-1) \ln(1-x) - 3x + (x+1) Li_2(x).$$

$$11. \sum_{k=1}^{\infty} \frac{x^{k+2}}{k^2(k+2)^2} = \frac{1}{4} (x^2+1) Li_2(x) + \frac{1}{4} (x^2-1) \ln(1-x) - \frac{x}{2} - \frac{3x^3}{16}.$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} x^{k+3} = \frac{1}{36} [11x^3 - 15x^2 + 6x + 6(1-x)^3 \ln(1-x)].$$

$$13. \sum_{k=1}^{\infty} \frac{x^{k+3}}{k^2(k+1)^2(k+2)^2} = \frac{1}{4} \left[(1+4x+4x^2) Li_2(x) - 4x - \frac{23}{4} x^2 + 3(1-x^2) \ln(1-x) \right].$$

5.2.7. Ряды вида $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^k}{(nk+m)!} \cdot \sum_{l=0}^{\infty} \frac{(\pm 1)^l x^l}{(kl+a)!}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{(nk+m)!} x^k = \frac{1}{n x^m} \left[\sum_{k=1}^n \frac{e^{\theta_k x^{1/n}}}{\theta_k^n} - n \sum_{k=1}^{\infty} \frac{x^{(m-nk)/n}}{(m-nk)!} \right]$$

[θ_k — один из корней степени n из 1: $\theta_k = \sqrt[n]{1} = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} = e^{2ik\pi/n}$].

$$2. \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x. \quad 3. \sum_{k=0}^{\infty} \frac{\ln^k x}{k!} x^k = \ln^k x.$$

$$4. \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \frac{\sin[(k\pi)/4]}{\cos[(k\pi)/4]} \right\} x^k = e^{x/\sqrt{2}} \frac{\sin(x/\sqrt{2})}{\cos(x/\sqrt{2})}.$$

$$5. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k)!} x^{2k} = \begin{cases} \operatorname{ch} x \\ \cos x \end{cases}. \quad 6. \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)!} x^{2k+1} = \begin{cases} \operatorname{sh} x \\ \sin x \end{cases}.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^{(k+\delta)/2}}{(2k+1)!} x^{2k+1} = \sqrt{2} \begin{cases} \operatorname{sh}(x/\sqrt{2}) \cos(x/\sqrt{2}) \\ \operatorname{ch}(x/\sqrt{2}) \sin(x/\sqrt{2}) \end{cases} \quad \left[\delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

8. $\sum_{k=0}^{\infty} \frac{1}{(3k)!} x^{3k} = \frac{1}{3} \left[e^x + 2e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) \right].$
9. $\sum_{k=0}^{\infty} \frac{1}{(3k+e)!} x^{3k+1} = \frac{1}{3} e^x - \frac{2}{3} e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x - \frac{(-1)^e \pi}{3}\right) \quad [e = 1, 2, 3].$
10. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k)!} x^{4k} = \begin{cases} 2^{-1} (\operatorname{ch} x + \cos x) \\ \cos(x/\sqrt{2}) \operatorname{ch}(x/\sqrt{2}) \end{cases}.$
11. $\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} x^{4k+1} = \frac{1}{2} (\operatorname{sh} x + \sin x).$
12. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)!} x^{4k+1} = \frac{1}{\sqrt{2}} \left(\operatorname{sh} \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}} + \operatorname{ch} \frac{x}{\sqrt{2}} \sin \frac{x}{\sqrt{2}} \right).$
13. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(4k+2)!} x^{4k+2} = \begin{cases} 2^{-1} (\operatorname{ch} x - \cos x) \\ \sin(x/\sqrt{2}) \operatorname{sh}(x/\sqrt{2}) \end{cases}.$
14. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)!} x^{4k+3} = \frac{1}{\sqrt{2}} \left(\sin \frac{x}{\sqrt{2}} \operatorname{ch} \frac{x}{\sqrt{2}} - \cos \frac{x}{\sqrt{2}} \operatorname{sh} \frac{x}{\sqrt{2}} \right).$
15. $\sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+3)!} = \frac{1}{2} (\operatorname{sh} x - \sin x).$
16. $\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu + k + 1)} = E_{\mu}(x; \mu).$
17. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k \Gamma(1/2)}{\Gamma(k/2 + 1)} x^k = e^{\pm x^2} \begin{cases} \operatorname{erfc}(-x) \\ 1 + \operatorname{erfi}(x) \end{cases}.$
18. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{\Gamma(k + 3/2)} x^{2k+1} = e^{\pm x^2} \begin{cases} \operatorname{erf}(x) \\ \operatorname{erfi}(x) \end{cases}.$
19. $\sum_{k=0}^{\infty} \frac{x^{2k+2} \pm 1}{\Gamma(2k+2 \pm 1/2)} = \frac{1}{2} [e^{\pm x^2} \operatorname{erf}(x) \mp e^{-x^2} \operatorname{erfi}(x)].$
20. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{\Gamma(k + \nu + 1)} x^k = \frac{1}{\Gamma(\nu + 1)} x^{-\nu} e^{\pm x} \int_0^x t^{\nu-1} e^{\mp t} dt,$

[5.2.7]

[5.2.8]

$$21. \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k + \nu + 1)} x^{2k+1} = E_{\nu}(2x, 0).$$

5.2.9. Ряды вида $\sum \frac{a_n}{(nk+m)! (k+a)!} x^k.$

1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+a)} x^k = x^{-a} \gamma(a, x).$
2. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k-n)} x^k = \frac{1}{n!} (-x)^n [\psi(n+1) - \ln x] - x^n \Gamma(-n, x)$
[$\arg x < \pi$].
3. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k! k} x^k = -C - \ln x + \operatorname{Ei}(\pm x) = \int_0^x \frac{1}{t} (e^{\pm t} - 1) dt.$
4. $\sum_{k=0}^{\infty} \frac{x^{k+n}}{k! (k+n)} = (-1)^n (n-1)! \left[1 - e^x \sum_{m=0}^{n-1} \frac{(-1)^m x^m}{m!} \right].$
5. $\sum_{k=0}^{\infty} \frac{1}{k! (k+1)} x^{k+1} = e^x - 1.$ 6. $\sum_{k=0}^{\infty} \frac{1}{k! (k+2)} x^{k+2} = (x-1)e^x + 1.$
7. $\sum_{k=1}^{\infty} \frac{k}{(k+n)!} x^{k+n} = (x-n)e^x + (n-x) \sum_{k=0}^{n-1} \frac{x^k}{k!} + \frac{x^n}{(n-1)!}.$
8. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k! (2k+1)} x^{2k+1} = \frac{\sqrt{\pi}}{2} \begin{cases} \operatorname{erfi}(x) \\ \operatorname{erf}(x) \end{cases}.$
9. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k+1)! (2k+1)} x^{2k+1} = \sqrt{\pi} x \begin{cases} \operatorname{erfi}(x) \\ \operatorname{erf}(x) \end{cases} \mp e^{\pm x^2}.$
10. $\sum_{k=0}^{\infty} \frac{x^k}{k! \sqrt{k+1}} = \frac{2}{\sqrt{\pi}} \int_0^x \cos t (\exp(-t^2) - 1) dt.$
11. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)! (k+a)} x^{2k} = 2x^{-a} \left[\Gamma(2a) \cos \pi a - \int_0^{\infty} t^{2a-1} \cos t dt \right].$
12. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{(2k)! k} = 2 \begin{cases} \operatorname{chi}(x) \\ \operatorname{ci}(x) \end{cases} - 2 \ln x - 2C.$

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$$13. \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(2k)!(4k+1)} = \frac{\sqrt{\pi}}{4} [\operatorname{erf}(x) + \operatorname{erfi}(x)].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(k+a)} x^{2k} = 2x^{-2a} \left[-\Gamma(2a-1) \cos \pi a - \int_0^{\infty} t^{2a-1} \sin t \, dt \right].$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!(2k+1)} = \frac{\operatorname{Shi}(x)}{\operatorname{Si}(x)} = \int_0^{\frac{\pi}{2}} \frac{1}{t} \left\{ \frac{\operatorname{sh} t}{\sin t} \right\} dt.$$

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(2k+1)^2} x^{2k+1} = - \int_0^{\frac{\pi}{2}} \ln t \left\{ \frac{\operatorname{sh} t}{\sin t} \right\} dt.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)} = \sqrt{\frac{\pi}{2}} C(x^2).$$

$$18. \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(2k+1)!(4k+3)} = \frac{\sqrt{\pi}}{4} [\operatorname{erf}(x) - \operatorname{erfi}(x)].$$

$$19. \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)} = \sqrt{\frac{\pi}{2}} S(x^2).$$

$$20. \sum_{k=0}^{\infty} \frac{x^{2k+2}}{(k+1)\Gamma(k+3/2)} = 2 \int_0^{\frac{\pi}{2}} e^{x^2 \sin^2 t} \operatorname{erf}(t) \, dt.$$

$$21. \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(2k+1)\Gamma(2k+3/2)} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \operatorname{erfi}(x).$$

$$22. \sum_{k=0}^{\infty} \frac{k}{(2k+1)!} x^{2k+1} = 2 \sqrt{\frac{x}{\pi}} \int_0^{\sqrt{x}} e^{t^2} \operatorname{erf}(t) \, dt.$$

$$23. \sum_{k=0}^{\infty} \frac{(1+1^2)(1+3^2) \dots [1+(2k+1)^2]}{(2k+2)!} x^{2k+2} = \operatorname{ch}(\operatorname{arcsin} x) \sec(\operatorname{arcsin} x) - 1.$$

$$24. \sum_{k=1}^{\infty} \frac{[(1^2-\gamma^2)(3^2-\gamma^2) \dots (2k-1)^2-\gamma^2]}{(2k+1)!} x^{2k} = \frac{1}{\gamma^2} \sin(\gamma \operatorname{arcsin} x) - 1.$$

5.2.9. Ряды вида $\sum_{k=0}^{\infty} \frac{k^n}{(k+m)!} x^k$.

Обозначения: $x = ae^{-a}$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = \frac{a^n}{a^n} e^{ax} (2k+ax)^k \Big|_{x=0} = 0.$$

$$2. \sum_{k=1}^{\infty} \frac{k^n}{k!} x^k = \left(x \frac{d}{dx} \right)^n e^{-x} = S_n.$$

$$3. S_n = \sum_{k=1}^n \frac{(-1)^k}{k!} \left[\sum_{m=1}^k (-1)^m \binom{k}{m} m^n \right] x^k e^x,$$

$$S_1 = x e^x, \quad S_2 = x(x+1) e^x, \quad S_3 = x(x^2+3x+1) e^x.$$

$$4. \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} x^k = S_{n+1}.$$

$$S_{-1} = a, \quad S_0 = \frac{a}{1-a}, \quad S_1 = \frac{a}{(1-a)^2}, \quad S_2 = \frac{a(2a+1)}{(1-a)^3}.$$

$$5. \sum_{k=1}^{\infty} \frac{k^k}{(k+1)!} x^k = \frac{e^x - 1}{a} - 1.$$

$$6. \sum_{k=1}^{\infty} \frac{k^{k+1}}{(k+1)!} x^k = - \frac{(1-a)e^x - 1}{a(1-a)} \quad (0 < |a| < 1).$$

$$7. \sum_{k=1}^{\infty} \frac{k^k}{(k+2)!} x^k = \frac{1-2a+4ae^x-e^{2x}}{4a^2} - \frac{1}{2}.$$

$$8. \sum_{k=1}^{\infty} \frac{k^{k+1}}{(k+2)!} x^k = \frac{1+2ae^x-e^{2x}}{2a^2}.$$

5.2.10. Ряды вида $\sum_{k=0}^{\infty} \frac{a_k}{\Gamma(ku+b)\Gamma(kc+d)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} x^{2k} = x^{-n} \frac{f_n(2x)}{f_n(2x)}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! \Gamma[1+(n-k)/2]} = \frac{1}{\sqrt{\pi}} 2^{(n+1)/2} e^{-x^2/2} D_{-n-1} \left(\frac{x\sqrt{2}}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k)!]^2} x^{2k} = \frac{1}{2} \frac{f_0(2x) + f_0(2x)}{2 \operatorname{ber}(2x)}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(2k+1)!} x^{4k+1} = \frac{1}{2} \frac{f_1(2x) + f_1(2x)}{2 \operatorname{bei}'(2x)}.$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)!]^2} x^{4k+1} = \frac{1}{2} \frac{f_0(2x) - f_0(2x)}{2 \operatorname{bei}(2x)}.$$

6. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{(2k+1)!(2k+2)!} x^{2k+2} = \frac{1}{2} \left\{ \frac{J_0'(2x) + J_2'(2x)}{2 \operatorname{ber}'(2x)} \right\}.$
7. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{k! \Gamma(k+v+1)} = x^{-v} \left\{ \frac{J_v(2x)}{J_v(2x)} \right\}.$
8. $\sum_{k=0}^{\infty} \frac{1}{k! \Gamma(v-k+1)} x^k = \frac{1}{\Gamma(v+1)} (1+x)^v.$
9. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{\Gamma(k+3/2) \Gamma(k+v+3/2)} = x^{-v-1} \left\{ \frac{L_v(2x)}{H_v(2x)} \right\}.$
10. $\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k/2+n/2+1) \Gamma(k/2-n/2+1)} = e^{-ix\pi/2} [J_n(2ix) + iE_n(2ix)].$
11. $\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k/2+1) \Gamma(k/2+n+1)} = x^{-n} [J_n(2ix) - iH_n(2ix)].$
12. $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(k+v+1) \Gamma(k-v+1)} = J_{2v}(2x) \cos v\pi + E_{2v}(2x) \sin v\pi.$
13. $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(k!)^2 (k+2)} = J_2'(2x).$
14. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+\mu) \Gamma(k+\nu)} x^k = 2^{2\mu-2\nu} x^{-\nu} \int_0^{2\sqrt{x}} t^{2\nu-1} J_{\mu-1}(t) dt \quad [\operatorname{Re} \nu > 0].$
15. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+v+1)} \left\{ \frac{\sin \left(\frac{3v}{4} + \frac{3k}{2} \right) \pi}{\cos \left(\frac{3v}{4} + \frac{3k}{2} \right) \pi} \right\} x^{2k} = x^{-v} \left\{ \frac{\operatorname{ber}_v(2x)}{\operatorname{ber}_v(2x)} \right\}.$
16. $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k! \Gamma(k+v+1) (k-n)} = \frac{2(-1)^n x^{2n}}{n! \Gamma(n+v+1)} \left[\frac{1}{2} \psi(n+1) + \frac{1}{2} \psi(n+v+1) - \ln x \right] - 2^{-\mu} x^{2n} \int_0^{\infty} t^{\mu} J_v(t) dt$
 $[\operatorname{Re} \mu < 1/2, \mu + v = -2n-1].$
17. $\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2) \Gamma(k+\mu) \Gamma(k+\nu)} x^k = 2^{\mu-2\nu+1/2} x^{-\nu} \int_0^{2\sqrt{x}} t^{2\nu-1/2} J_{\mu-1/2}(t) dt \quad [\operatorname{Re} \nu > 0].$

$$18. \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\Gamma(k+3/2) \Gamma(k+v+3/2) (k-n+1)!} = \frac{2(-1)^n x^{2n-1}}{\Gamma(n+1/2) \Gamma(n+v+1/2)} \left[\ln x - \frac{1}{2} \psi\left(n+\frac{1}{2}\right) - \frac{1}{2} \psi\left(n+v+\frac{1}{2}\right) \right] - 2^{-1} x^{2n-1} \int_0^{\infty} t^{\mu} H_v(t) dt \quad [\operatorname{Re} \mu < 1/2, \mu + v = -2n].$$

5.2.11. Ряды в виде $\sum_{k=0}^{\infty} \frac{\Gamma(kx+b)}{\Gamma(kx+d)} x^k$.

1. $\sum_{k=0}^{\infty} \frac{(m+k)!}{(n+k)!} x^k = \frac{m!}{(n-1)! (1-x)^{n-1}} \sum_{k=0}^{m-n} (-1)^k \binom{m-n}{k} \frac{x^k}{n+k} \quad [n \leq m].$
2. $= \frac{m!}{(n-1)!} \left(\frac{n-1}{m} \right) \left[\frac{(-1)^{n-m}}{x^n} (1-x)^{n-1} \ln(1-x) + \frac{1}{x^m} \sum_{k=1}^{n-m} (-1)^k \binom{n-m-1}{k-1} \sum_{l=1}^{m+k-1} \frac{x^{l-k}}{l} \right] \quad [n > m].$
3. $\sum_{k=0}^{\infty} \frac{(m+k)!}{l!} x^k = \frac{m!}{(1-x)^{m+1}}.$
4. $\sum_{k=0}^{\infty} \frac{(nk)!}{(nk+m+1)!} x^k =$
 $= \frac{1}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \frac{1}{x^{k+1}} \left[\sum_{l=1}^{\lfloor k/n \rfloor} \frac{x^{l-k} (k-ln+1)/n}{k-ln+1} + \right.$
 $\left. + \frac{2}{n} \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} \sin \frac{2l}{n} (k+1) \pi \operatorname{arctg} \frac{x^{1/n} \sin \frac{2l}{n} \pi}{1-x^{1/n} \cos \frac{2l}{n} \pi} + \right.$
 $\left. + \frac{1}{n} \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \cos \frac{2l}{n} (k+1) \pi \ln \left(x^{2/n} - 2x^{1/n} \cos \frac{2l}{n} \pi + 1 \right) + \right.$
 $\left. + \frac{1}{n} \ln(1-x) + (-1)^{k-1} \frac{1+(-1)^n}{2n} \ln(1+x) \right].$
5. $\sum_{k=0}^{\infty} \frac{k!}{(m+k)!} x^k =$
 $= \frac{1}{(m-1)!} \left[\sum_{k=0}^{m-1} (-1)^k \frac{(1-x)^k}{(m-k-1)! x^{k+1}} + (-1)^m \frac{(1-x)^{m-1}}{x^m} \ln(1-x) \right].$

6. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k)!} x^{2k} = 1 - \frac{\sqrt{\pi}}{2} e^{-x^2/4} \left\{ \operatorname{erf} \left(\frac{x}{2} \right) \right\}$
7. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(2k+1)!} x^{2k+1} = \sqrt{\frac{\pi}{2}} e^{-x^2/4} \left\{ \operatorname{erfi} \left(\frac{x}{2} \right) \right\}$
8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k)!} x^{4k} = 1 - \frac{\sqrt{x}}{2} J_{3/2} \left(\frac{x^2}{2}, 0 \right)$
9. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)!} x^{4k+1} = \sqrt{\frac{x}{2}} J_{1/2} \left(\frac{x^2}{2}, 0 \right)$
10. $\sum_{k=0}^{\infty} \frac{\Gamma(k\nu + \mu)}{\Gamma(k\nu + \mu + n + 1)} x^k = \frac{1}{n!} \int_0^1 \frac{t^{\mu-1}}{1 - xt^\nu} (1-t)^n dt$ $\{|\mu| \leq 1, n \in \mathbb{N}\}$
11. $\sum_{k=1}^{\infty} \frac{\Gamma(k\nu + 1)}{\Gamma(k\nu + k + 1)} x^k = x \int_0^1 t^\nu \exp \{xt^\nu (1-t)\} dt$ $\{\nu > 0\}$
12. $\sum_{k=0}^{\infty} \frac{\Gamma(k\nu + 1)}{k!} x^k = \int_0^{\infty} \exp \{xt^\nu - t\} dt$ $\{0 \leq \nu < 1\}$
13. $\sum_{k=0}^{\infty} \frac{\Gamma(k\nu)}{k!} (-x)^k = \frac{1}{\nu} \left[\int_0^{\infty} e^{-t} \operatorname{Ei} \{-xt^\nu\} dt - (1-\nu) C - \ln x \right]$ $\{\operatorname{Re} x > 0, 0 < \operatorname{Re} \nu < 1\}$
14. $\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k}{2} + \nu\right)}{k!} x^k = 2^{1-\nu} \Gamma(2\nu) e^{x^2/4} D_{-2\nu} \left(-\frac{x}{\sqrt{2}} \right)$
15. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \frac{\Gamma(k+\nu)}{\Gamma(k+\alpha)} x^k = \frac{\Gamma(\nu)}{\Gamma(\alpha)} \int_0^{\infty} e^{-at} t^{\alpha-1} (1 \mp x e^{-t})^{-\nu} dt$ $\{\operatorname{Re} \nu, \operatorname{Re} \alpha > 0\}$
16. $\sum_{k=0}^{\infty} \frac{\Gamma(k+\nu)}{k!} x^k = \Gamma(\nu) (1-x)^{-\nu}$ $\{|x| \leq 1; x \neq 1 \text{ при } \nu > 0\}$
17. $\sum_{k=1}^{\infty} \frac{(k+\nu) \Gamma(k+2s)}{k!} x^k = \frac{\nu \Gamma(2s) \Gamma(1+s)}{(1-x)^{2s+1}}$ $\{|x| \leq 1; x \neq 1 \text{ при } \nu > 1/2\}$

5.2.12. Ряды вида $\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + b) \Gamma(k\alpha + d) \Gamma(k\alpha + f)} x^k$.

1. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k x^{2k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu+1)} = x^{-2\nu} \left\{ \operatorname{ber}_\nu^2(2x) + \operatorname{bei}_\nu^2(2x) \right\}$
2. $\sum_{k=0}^{\infty} \frac{x^{4k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu+2)} = x^{-2\nu-1} [\operatorname{ber}_\nu(2x) \operatorname{bei}_\nu'(2x) - \operatorname{ber}_\nu'(2x) \operatorname{bei}_\nu(2x)]$
3. $\sum_{k=0}^{\infty} \frac{x^{4k}}{k! \Gamma(k+\nu+1) \Gamma(2k+\nu)} = 2x^{1-2\nu} [\operatorname{ber}_\nu(2x) \operatorname{ber}_\nu'(2x) + \operatorname{bei}_\nu(2x) \operatorname{bei}_\nu'(2x)]$
4. $\sum_{k=0}^{\infty} \frac{(-1)^k \cos \{(k+\nu)\pi/2\}}{k! \Gamma(k/2 + \nu/2 + 1) \Gamma(k/2 - \nu/2 + 1)} x^{2k} = J_\nu(2x) I_{-\nu}(2x)$
- 5.2.13. Ряды вида $\sum_{k=0}^{\infty} \frac{a_k \Gamma(k\alpha + b)}{\Gamma(k\alpha + d) \Gamma(k\alpha + f)} x^k$.
1. $\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} x^k = (1-4x)^{-1/2}$ $[-1/4 \leq x < 1/4]$
2. $\sum_{k=0}^{\infty} \frac{(2k+1)!}{(k!)^2} x^k = (1-4x)^{-3/2}$ $[-1/4 \leq x < 1/4]$
3. $\sum_{k=0}^{\infty} \frac{(-1)^{k/2} (2k)!}{(k!)^2} (x)^k = (1+16x^2)^{-1/2} [(1+16x^2)^{1/2} \pm 4x]^{1/2}$
4. $\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2 k} x^k = \frac{1}{2} - x - \frac{1}{2} (1-4x)^{1/2}$ $[-1/4 \leq x < 1/4]$
5. $\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{(k!)^2} \frac{(2k)!}{(2k-1)!} x^{2k} = -\frac{1}{2} \ln |x| - 4 \int_0^{2x} \frac{\{\operatorname{Arsech} t\}}{\{\operatorname{Arcosech} t\}} \frac{dt}{t}$ $\{|x| < 1/2\}$
6. $\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 (2k-1)!} x^k = (1-4x)^{1/2}$ $[-1/4 \leq x < 1/4]$
7. $\sum_{k=0}^{\infty} \frac{(\pm 1)^k}{(k!)^2 (2k+1)!} x^{2k+1} = \frac{x}{2} \{\operatorname{Arsh}(2x)\} + \frac{1}{4} (1-4x^2)^{1/2}$ $\{|x| < 1/2\}$

$$8. \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 (2k-1)!} x^{2k-1} = -\frac{1}{x} (1-4x^2)^{1/2} - 2 \operatorname{arcsin}(2x)$$

[5.2.13]

[|x| < 1/2]

$$9. \sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+1)!} x^{k+1} = \frac{1}{2} [1 - (1-4x)^{1/2}]$$

[|x| < 1/4]

$$10. \sum_{k=0}^{\infty} \frac{(\pm 1)^k (2k)!}{(k!)^2 (2k+1)} x^{2k+1} = \frac{1}{2} \left\{ \operatorname{arcsin}(2x) \right\}$$

[|x| < 1/2]

$$11. \sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+2)!} x^{k+2} = \frac{1}{12} [(1-4x)^{3/2} + 6x - 1]$$

[|x| < 1/4]

$$12. \sum_{k=0}^{\infty} \frac{(2k+1)!}{k! (k+2)!} x^{k+2} = \frac{1}{4} [1 - 2x - (1-4x)^{1/2}]$$

[|x| < 1/4]

$$13. \sum_{k=0}^{\infty} \frac{(2k+1)!}{[(k+1)!]^2} x^{k+1} = -\ln [1 + (1-4x)^{1/2}] + \ln 2$$

[|x| < 1/4]

$$14. \sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+3)!} x^{k+3} = \frac{1}{120} [30x^2 - 10x + 1 - (1-4x)^{3/2}]$$

[|x| < 1/4]

$$15. \sum_{k=0}^{\infty} \frac{(2k+1)!}{k! (k+3)!} x^{k+3} = \frac{1}{24} [6x - 1 - 6x^2 + (1-4x)^{3/2}]$$

[|x| < 1/4]

$$16. \sum_{k=0}^{\infty} \frac{(2k+2)!}{k! (k+2)!} x^k = 4(1-4x)^{-1/2} [1 + (1-4x)^{1/2}]^{-2}$$

[|x| < 1/4]

$$17. \sum_{k=0}^{\infty} \frac{(2k+2)!}{k! (k+3)!} x^k = \frac{8}{3} [1 + (1-4x)^{1/2}]^{-3}$$

[|x| < 1/4]

$$18. \sum_{k=0}^{\infty} \frac{(2k)!}{k! (k+1)! (2k+3)} x^{2k+3} = \frac{1}{8} [4x - 2x(1-4x^2)^{1/2} - \operatorname{arcsin}(2x)]$$

[|x| < 1/2]

$$19. \sum_{k=0}^{\infty} \frac{(\pm 1)^k (2k+1)!}{(k!)^2 (2k+3)} x^{2k+3} = \pm \frac{1}{8} [2x(1 \mp 4x^2)^{-1/2} + \frac{\operatorname{arcsin}(2x)}{\operatorname{Arsh}(2x)}]$$

[|x| < 1/2]

$$20. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+3)!} x^k = 8(1-4x)^{-3/2} [1 + (1-4x)^{1/2}]^{-3}$$

[|x| < 1/4]

5.2.14]

$$21. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+2)!} x^k = (1-4x)^{-3/2} [(1-4x)^{1/2} + 1]^{-2} [8(1-4x)^{1/2} + 4]$$

[|x| < 1/4]

$$22. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+1)!} x^k = 6(1-4x)^{-3/2}$$

[|x| < 1/4]

$$23. \sum_{k=0}^{\infty} \frac{(2k+3)!}{k! (k+4)!} x^{k+4} = \frac{1}{10} [1 - 2x - (1-4x)^{1/2}]^2$$

[|x| < 1/4]

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k (3k+1)!}{k! (2k+1)! (2k+1)} x^{2k+1} = a \cdot \operatorname{clg} y$$

[x = y(1+y); |x| < 2/3^{3/2}]

$$25. \sum_{k=0}^{\infty} \frac{(\pm 1)^k (4k)!}{[(2k)!]^2} x^{2k} = \left[\frac{1 + (1 \mp 16x^2)^{1/2}}{2(1 \mp 16x^2)} \right]^{1/2}$$

[|x| < 1/4]

$$26. \sum_{k=0}^{\infty} \frac{(4k+1)!}{[(2k)!]^2} x^{2k} = \frac{1}{2} [(1-4x)^{-3/2} + (1+4x)^{-3/2}]$$

[|x| < 1/4]

$$27. \sum_{k=0}^{\infty} \frac{(\pm 1)^k \Gamma(k+v+1/2)}{k! \Gamma(k+2v+1)} x^k = \sqrt{x} e^{-x/2} I_v \left(\frac{x}{2} \right)$$

$$28. \sum_{k=0}^{\infty} \frac{k! x^k}{\Gamma(k-v+3/2) \Gamma(k+v+3/2)} = \frac{1}{2} e^{x/2} \int_{-1}^1 e^{xt/2} P_{v-1/2}(t) dt$$

$$29. \sum_{k=0}^{\infty} \frac{\Gamma(kv+\mu)}{k! \Gamma(kv-k+\mu) \Gamma(kv+\mu)} x^k = \frac{y^\mu}{(1-y)y+\nu}$$

[x = (y-1)/y^\nu; |x| < |(y-1)/y^\nu|]

$$30. \sum_{k=0}^{\infty} \frac{\Gamma(kv+\mu)}{k! \Gamma(kv-k+\mu) \Gamma(kv+\mu)} x^k = \frac{1}{\mu-1} y^{1-\mu}$$

[x = (y-1)/y^\nu; |x| < |(y-1)/y^\nu|]

$$31. \sum_{k=0}^{\infty} \frac{\Gamma(k+\mu)}{\Gamma(k+\mu-v+1/2) \Gamma(k+\mu+v+1/2)} x^k = \frac{\Gamma(\mu)}{\Gamma(\mu-v-1/2) \Gamma(\mu+v-1/2)} \left(\frac{x}{2} \right)^{-1-\mu} e^{x/2} I_{\mu-3/2} \left(\frac{x}{2} \right)$$

$$5.2.14. \text{ Ряды вида } \sum_{k=0}^{\infty} a_k \frac{\Gamma(ka+b) \Gamma(kc+d)}{\Gamma(ke+f)} x^k$$

$$L. \sum_{k=0}^{\infty} \frac{(\pm 1)^k (4k)!}{(2k)!} x^{2k} = \frac{4}{4+x^2} \pm 4x(4+x^2)^{-3/2} \left\{ \frac{\operatorname{arcsin}(x/2)}{\operatorname{Arsh}(x/2)} \right\}$$

[|x| < 2]

$$2. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+1)!} x^{2k+1} = \frac{1}{2} (4-x^2)^{-1/2} \left\{ \arcsin(x/2) \right\} \quad (|x| < 2).$$

$$3. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+2)!} x^{2k+2} = 2 \left\{ \operatorname{Arsh}(x/2) \right\}^2 \quad (|x| \leq 2).$$

$$4. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+3)!} x^{2k+3} = \frac{1}{4} (4-x^2)^{1/2} \left\{ \operatorname{Arsh}(x/2) \right\} + 2 \left\{ \operatorname{Arsh}(x/2) \right\}^3 \quad (|x| \leq 2).$$

$$5. \sum_{k=0}^{\infty} (\pm 1)^k \frac{k!(k+1)!}{(2k+3)!} x^{2k+3} = \frac{1}{2} x + 2 \left(4-x^2 \right)^{1/2} \left\{ \operatorname{Arsh}(x/2) \right\} \quad (|x| \leq 2).$$

5.2.15. Ряды вида $\sum_{k=0}^{\infty} a_k \frac{\Gamma(kb+b)\Gamma(kc+c)}{\Gamma(hk+f)\Gamma(hk+g)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k (k!)^2}{(2k)!(2k+1)!} x^{2k} = 1 - \frac{\pi}{2} \left(\frac{x}{2} \right)^{1/2} H_0 \left(\frac{x}{2} \right) \quad (x=0 \text{ или } 1).$$

$$2. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 x^{2k+1} = \pi H_0 \left(\frac{x}{2} \right).$$

$$3. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 \frac{x^{2k+1}}{2k+1} = \frac{\pi^2}{4} - \pi \int_{x/2}^{\infty} H_0(t) \frac{dt}{t}.$$

$$4. \sum_{k=0}^{\infty} (-1)^k \left[\frac{k!}{(2k+1)!} \right]^2 \frac{x^{2k+3}}{2k+3} = 2\pi H_1 \left(\frac{x}{2} \right).$$

$$5. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(k!)^2}{(2k+1)!(2k+2)!} x^{2k+2} = 2\pi \int_0^{x/2} \frac{H_0(t)}{H_0(t)} dt.$$

$$6. \sum_{k=0}^{\infty} (-1)^k \frac{k!(k+1)!}{[(2k+3)!]^2} x^{2k+2} = 2 \ln x - 4 + 2C + \pi \int_{x/2}^{\infty} H_0(t) \frac{dt}{t}.$$

5.2.16. Ряды вида $\sum_{k=0}^{\infty} a_k \frac{\Gamma(kb_1+c_1) \dots \Gamma(kb_l+c_l)}{\Gamma(kd_1+e_1) \dots \Gamma(kd_r+e_r)} x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} x^k = e^{2x} I_0(2x).$$

$$2. \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(k!)^2} x^{2k} = J_0^2(2x).$$

$$3. \sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k)!}{[k!(2k+1)!]^2} x^{2k+1} = \frac{1}{2} \int_0^{2x} \frac{\{ \operatorname{erfi}(t) \}}{\operatorname{Arsh}(t)} \frac{dt}{t} \quad (|x| < 1/2).$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k (2k)! x^{2k}}{(k!)^2 \Gamma(k+v+1) \Gamma(k-v+1)} = J_v(\alpha) J_{-v}(2x).$$

$$5. \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+1) x^{2k}}{k! \Gamma(k+\mu) \Gamma(k+\nu) \Gamma(k+\mu-\nu+1)} = x^{2-1} J_{\mu-\nu}(2x) J_{\nu-1}(2x).$$

$$6. \sum_{k=0}^{\infty} \frac{[(2k)!]^2}{(k!)^4} x^{2k} = \frac{2}{\pi} K(4x) \quad (|x| < 1/4).$$

$$7. \sum_{k=0}^{\infty} \frac{1}{2k-1} \frac{[(2k)!]^2}{(k!)^4} x^{2k} = -\frac{2}{\pi} E(4x) \quad (|x| < 1/4).$$

$$8. \sum_{k=0}^{\infty} \left[\frac{(2k)!}{k!(k+1)!} \right]^2 x^{2k+2} = \frac{1}{2\pi} (1+4x) E \left(\frac{4x}{1+4x} \right) - \frac{1}{4}.$$

$$9. \sum_{k=0}^{\infty} \frac{(2k)!(2k+1)!}{[k!(k+1)!]^2} x^{2k+2} = \frac{1}{4} - \frac{1}{2\pi} E(4x) \quad (0 < x < 1/4).$$

$$10. \sum_{k=0}^{\infty} \left[\frac{(2k+1)!}{k!(k+2)!} \right]^2 x^{2k+2} = -\frac{1}{16\pi} \left[n+4x-2(1+4x) E \left(\frac{4x}{1+4x} \right) \right] \quad (0 < x < 1/4).$$

5.2.17. Ряды вида $\sum_{k=0}^{\infty} a_k \left(\sum_{m=1}^k \frac{(-1)^m}{m!} \right)^p$.

$$1. \sum_{k=1}^{\infty} \left(\sum_{m=1}^k \frac{1}{m} \right) x^k = \frac{1}{x-1} \ln(1-x) \quad (|x| < 1).$$

$$2. \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-1} \frac{1}{m} \right) x^k = \frac{1}{2} \ln^2(1-x) \quad (-1 \leq x < 1).$$

$$3. \sum_{k=2}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-1} \frac{(-1)^{m-1}}{m} \right) x^k = \operatorname{Li}_2 \left(\frac{1-x}{2} \right) - \frac{x^2}{2} + \ln 2 \left[\frac{1}{2} \ln 2 - \ln(1-x) \right] \quad (|x| < 1).$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k} \left(\sum_{m=1}^{k-1} (-1)^m \frac{1}{m!} \right) x^k = \ln(1+x) \ln(-x) \quad (|x| < 1).$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\sum_{m=0}^{k-1} \frac{1}{2m+1} \right) x^{2k} = \frac{\{\operatorname{Arth} x\}^2}{\{\operatorname{arctg} x\}^2} \quad |x| < 1.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{2k+1} = \ln(1+x^2) \operatorname{arctg} x - x \operatorname{arctg}^2 x + \int_0^x \operatorname{arctg}^2 t \, dt \quad |x| < 1.$$

$$7. \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left(\sum_{m=1}^{2k} \frac{1}{m} \right) x^{2k+1} = -\frac{1}{2} \operatorname{arctg} x \ln(1+x^2) \quad |x| < 1.$$

$$8. \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\sum_{m=0}^{2k} \frac{(-1)^m}{2m+1} \right) x^{2k+1} = \frac{1}{2} \operatorname{arctg} x \ln \frac{1+x}{1-x} \quad |x| < 1.$$

$$9. \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{2k} = \frac{1}{2} \ln x \ln^2(1-x) + \ln(1-x) \operatorname{Li}_2(1-x) - \operatorname{Li}_2(1-x) + \operatorname{Li}_2(1).$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\sum_{m=1}^k \frac{1}{m} \right) x^k = e^{x^2} [C + \ln x - \operatorname{Ei}(-x^2)].$$

$$11. \sum_{k=1}^{\infty} \frac{x^{2k}}{k!} \sum_{m=0}^{k-1} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} e^{x^2} \left[3 + \frac{2}{\sqrt{\pi}} \operatorname{erf}(x) \right].$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^k m^2 \right) x^k = \frac{1}{4} x^2 (x^2 + 8x + 14x + 4).$$

$$13. \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} \right) x^{2k} = \left(C + \frac{1}{2} \ln x^2 \right) \left\{ \frac{I_0(2x) + K_0(2x)}{J_0(2x) - (2/2) Y_0(2x)} \right\}.$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k)!]^2} \left(\sum_{m=1}^{2k} \frac{1}{m} \right) x^{4k} = \ker(2x) + \left(C + \frac{1}{2} \ln x \right) \operatorname{ber}(2x) - \frac{\pi}{4} \operatorname{bei}(2x).$$

$$15. \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)!]^2} \left(\sum_{m=1}^{2k+1} \frac{1}{m} \right) x^{4k+2} = \operatorname{kei}(2x) + \left(C + \frac{1}{2} \ln x \right) \operatorname{bei}(2x) + \frac{\pi}{4} \operatorname{ber}(2x).$$

$$16. \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+2)!} \left(\sum_{m=1}^k \frac{1}{m^2} \right) x^{2k+2} = \frac{2}{3} \arcsin^2 \frac{x}{2}.$$

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+1)!} \left(\sum_{m=1}^{k+1} \frac{1}{m} \right) x^{2k} = \frac{1}{2x} [\ln x + 2C] J_1(2x) - \pi Y_1(2x) - \frac{1}{2x^2} J_0(2x).$$

5.2.18. Разные ряды вида $\sum a_k x^k$.

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+2k+1}}{[(\mu+1)^2 - \nu^2] [(\mu+3)^2 - \nu^2] \dots [(\mu+2k+1)^2 - \nu^2]} = s_{\mu, \nu}(x).$$

$$2. \sum_{k=0}^{\infty} \frac{(2\mu+1)(2\mu+3) \dots (2\mu+2k+1)}{[(\mu+1)^2 - \nu^2] [(\mu+2)^2 - \nu^2] \dots [(\mu+k+1)^2 - \nu^2]} x^{\mu+2k+1} = (2\mu+1) e^{\pi i} s_{\mu, \nu}(x).$$

$$3. \sum_{k=1}^{\infty} a^k x^k = \frac{x \ln a}{2 \sqrt{x}} \int_0^{\infty} \frac{1}{(x-x')^{3/2}} e^{-(\ln^2 a)/(4x')} dt \quad |x| < 1, \quad 0 < a < 1.$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k^2} x^k = x \int_0^1 t^{-x^2} dt.$$

$$5. \sum_{k=0}^{\infty} \left[\sum_{m=0}^k \frac{(-1)^m}{2m+1} \binom{p}{k-m} \right] x^{2k+1} = (1+x^2)^p \operatorname{arctg} x.$$

$$6. \sum_{k=1}^{\infty} (-1)^{k-1} \left[\sum_{m=0}^{(k-1)/2} \frac{(-1)^m}{h-m} \binom{p}{m} \right] x^k = (1+x^2)^p \ln(1+x).$$

$$7. \sum_{k=1}^{\infty} \left[\sum_{m=0}^{(k/2)} \frac{(-1)^m}{k-m} \binom{k-m}{m} \right] x^k = -\ln(1-x+x^2).$$

$$8. \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k)!} \left[\sum_{m=1}^n (-1)^m \binom{2n}{n-m} m^{2k} \right] x^{2k} = 2^{2n-1} \sin^{2n} \frac{\pi}{2}.$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[\sum_{m=0}^n \binom{n}{m} (n-2m)^{2k} \right] x^{2k} = 2^n \cos^n x.$$

$$10. \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\sum_{m=0}^n (-1)^m \binom{2n+1}{n-m} (2m+1)^{2k+1} \right] x^{2k+1} = 2^{2n} \sin^{2n+1} x.$$

$$11. \sum_{k=n}^{\infty} \frac{1}{k!} \binom{k}{j} \left[\sum_{m=1}^n (-1)^m \binom{n}{m} m^k \right] x^k = \frac{x^j}{j!} \sum_{k=1}^n (-1)^k \binom{n}{k} k! \psi_k^j(x).$$

$$\begin{aligned}
 12. \sum_{k=1}^{\infty} \left\{ \sum_{n=0}^{\lfloor k/n \rfloor} (-1)^n \binom{k-nm+p-1}{p-1} \binom{p}{m} \right\} x^k &= (1+x+x^2+\dots+x^{m-1})^p - 1. \\
 13. \sum_{k=1}^{\infty} \frac{x^k}{a^k-1} &= \frac{x}{a^2-1} {}_2F_1(a; a^2; x). \\
 14. \sum_{k=0}^{\infty} \frac{x^k}{a^k+1} &= \frac{1}{2} {}_2F_1(a, -1; -a; x). \\
 15. \sum_{k=0}^{\infty} \frac{x^k}{a^{2k+1}-1} &= \frac{1}{a-1} {}_2F_1(a^2, a; a^3; x). \\
 16. \sum_{k=0}^{\infty} \frac{x^k}{a^k+b} &= \\
 &= \frac{1}{b} \int_0^{\infty} \frac{\sin(t \ln b) - x \sin(t \ln ab)}{1-2x \cos(t \ln a) + x^2} \cos \pi t \, dt + \frac{1}{2b(1-x)} \quad |a, b > 0|.
 \end{aligned}$$

5.3. РЯДЫ С ПОКАЗАТЕЛЬНОЙ И ГИПЕРБОЛИЧЕСКИМИ ФУНКЦИЯМИ

5.3.1. Ряды вида $\sum_{k=0}^{\infty} a_k \frac{1}{e^{ak} \pm 1}$

$$\begin{aligned}
 1. \sum_{k=1}^{\infty} \frac{k^{2n-1}}{e^{2\pi k} - (-1)^k} &= \frac{1}{4\pi} (2^{2n} - 1) B_{2n} - \frac{1}{4\pi} \delta_{n,1/2}. \\
 2. \sum_{k=1}^{\infty} \frac{k^{2n+1}}{(-1)^{k+1} e^{2\pi k} - 1} &= \frac{B_{2n+2}}{8\pi + 4}. \\
 3. \sum_{k=1}^{\infty} \frac{k^{4n+4}}{e^{2\pi k} - 1} &= \frac{1}{2} \frac{B_{4n+4}}{4n+2} \quad [n > 0]. \\
 4. \sum_{k=1}^{\infty} \frac{k}{e^{2\pi k} - 1} &= \frac{1}{24} - \frac{1}{8\pi}. \\
 5. \sum_{k=1}^{\infty} \frac{k^5}{(e^{2\pi k} - 1)(4k^4 + \pi^4)} &= \frac{1}{90} + \frac{\pi^3}{64} \cdot \frac{(-1)^6 + \operatorname{ch} \pi \pi}{(-1)^6 - \operatorname{ch} \pi \pi} + \frac{\pi^2}{32} \sum_{k=1}^n \frac{1}{2k^3 - 2kn + n^2}. \\
 6. \sum_{k=1}^{\infty} \frac{1}{k^{4n+3} (e^{2\pi k} - 1)} &= -\frac{1}{2} \zeta(4n+3) - \frac{(2\pi)^{4n+3}}{(4n+4)! 4} \sum_{k=0}^{2n+2} \binom{4n+4}{2k} B_{2k} B_{4n-2k+4}.
 \end{aligned}$$

$$\begin{aligned}
 7. \sum_{k=0}^{\infty} \frac{(2k+1)^{4n+1}}{e^{(2k+1)\pi} + 1} &= (2^{4n+1} - 1) \frac{B_{4n+2}}{8n+4}. \\
 8. \sum_{k=0}^{\infty} \frac{2k+1}{[e^{(2k+1)\pi} + 1] [(2k+1)^4 + 4\pi^4]} &= \\
 &= -\frac{\pi}{32n^3} \frac{(-1)^n - \operatorname{ch} \pi \pi}{(-1)^n + \operatorname{ch} \pi \pi} + \frac{1}{8\pi} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^2 - 2n(2k+1) + 2n^2}. \\
 5.3.2. Ряды вида $\sum_{k=0}^{\infty} a_k q^{(ka+b)^2}$.
 Условие: $|q| < 1$.
 1. $\sum_{k=-\infty}^{\infty} q^{(k+a)^2} = \theta_3(\pi a, e^{\pi i / \ln q}) \ln^{-1/2} \frac{1}{q}.$
 2. $\sum_{k=0}^{\infty} (\pm 1)^k q^{k^2} = \frac{1}{2} + \frac{1}{2} \frac{\theta_3(0, q)}{\theta_4(0, q)}.$
 3. $\sum_{k=0}^{\infty} q^{k(k+1)} = \frac{1}{2q^{1/4}} \theta_2(0, q).$
 4. $\sum_{k=1}^{\infty} (\pm 1)^k k q^{k^2} = \frac{1}{V \pi} (-\ln q)^{-3/2} \int_0^{\infty} t \exp\left(\frac{t^2}{\ln q}\right) \left\{ \operatorname{ctg} \frac{t}{\ln q} \right\} dt \quad |0 < q < 1|.$
 5. $\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} = \frac{1}{2} q^{-1/4} \frac{\partial}{\partial q} \theta_2(a, q) \Big|_{a=0}.$
 6. $\sum_{k=0}^{\infty} q^{(2k+1)^2} = \frac{1}{2} \theta_3(0, q^4).$
 7. $\sum_{k=0}^{\infty} (-1)^k q^{(2k+1)^2} = \frac{2q}{V \pi} \int_0^{\infty} \frac{q^t \cos 4t \sqrt{-\ln q} + 1}{q^2 + 2q^t \cos 4t \sqrt{-\ln q} + 1} e^{-t^2} dt \quad |q < 1|.$
 8. $\sum_{k=0}^{\infty} (2k+1) q^{(2k+1)^2} = -\frac{1}{2V \pi} (-\ln q)^{-3/2} \int_0^{\infty} \exp\left(\frac{t}{\ln q}\right) \operatorname{cosec} 2V t \, dt.$
 9. $\sum_{k=1}^{\infty} \frac{1}{k^n} e^{-k^2} = S_n$$$

$$S_0 = 0,36631860, \quad S_1 = 0,37247207, \quad S_2 = 0,36902569.$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{-k^2} = T_2,$$

$$T_0 = 0,34968710, \quad T_2 = 0,36331424, \quad T_4 = 0,36673624.$$

$$11. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-k^2} = U_2,$$

$$U_0 = 0,36800285, \quad U_2 = 0,36792068, \quad U_4 = 0,36789315, \quad U_6 = 0,36789093.$$

$$12. \sum_{k=0}^{\infty} (-1)^k e^{-k^2} = 0,36775603. \quad 13. \sum_{k=0}^{\infty} (4k^2 - 1) e^{-k^2} = \frac{3}{2}.$$

$$14. \sum_{k=0}^{\infty} e^{-\pi(2k+1)^2} = \frac{(2^{1/4} - 1)}{2^{1/4} \pi^{3/4}} \Gamma\left(\frac{1}{4}\right) = 0,04521392.$$

$$15. \sum_{k=-\infty}^{\infty} (\pm 1)^k a^{2k} q^{k^2} = \frac{\theta_2(-i \ln a, q)}{\theta_1(-i \ln a, q)}.$$

$$16. \sum_{k=-\infty}^{\infty} (\pm 1)^k a^{2k} q^{k^2} = \frac{1}{a} \frac{\theta_2(-i \ln a, q)}{\theta_1(-i \ln a, q)}.$$

5.3.3. Ряды, содержащие $\operatorname{sh} a_k x$ и $\operatorname{ch} a_k x$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k} r^k \frac{\operatorname{sh} kx}{\operatorname{ch} kx} = \operatorname{Arth} \left(\frac{r \operatorname{sh} x}{1 - r \operatorname{ch} x} \right) \left(\frac{1}{0} \right) - \frac{1}{2} \ln(1 - 2r \operatorname{ch} x + r^2) \left(\frac{0}{1} \right).$$

$$|r| < 1, \quad |x| < \frac{1}{2} \ln \frac{1+r}{1-r}.$$

$$2. \sum_{k=0}^{\infty} \frac{1}{k!} r^k \frac{\operatorname{sh} kx}{\operatorname{ch} kx} = e^{r \operatorname{ch} x} \frac{\operatorname{sh}(r \operatorname{sh} x)}{\operatorname{ch}(r \operatorname{sh} x)}.$$

$$3. \sum_{k=0}^{\infty} e^{-a^2 k^2} \operatorname{ch} kx = \frac{1}{2} + \frac{\sqrt{\pi}}{2a} e^{-x^2/(4a^2)} \theta_2 \left(\frac{\pi x}{2a^2}, e^{-\pi^2/a^2} \right).$$

$$4. \sum_{n=1}^{\infty} \frac{\operatorname{ch} nx}{\operatorname{ch} 2nx + a} = \frac{1}{x \sqrt{2(a+1)}} K(k) \operatorname{dn} u - \frac{1}{2(a+1)}.$$

$$\left[e = \frac{\pi K'(k)}{K(k)}, \quad a = \operatorname{ch} \frac{\pi u}{K'(k)}, \quad |\operatorname{Im} \operatorname{Arch} a| < \operatorname{Re} 1 \right].$$

$$5. \sum_{k=0}^{\infty} \frac{r^k}{\operatorname{ch}(kx+a)+b} =$$

$$= \frac{2}{\sqrt{1-b^2}} \int_0^{\infty} \frac{\cos at - r \cos(a-b)t}{1 - 2r \cos xt + r^2} \operatorname{sh}(t \operatorname{arccos} b) \operatorname{cosech} \pi t \, dt$$

$$|r| < 1.$$

5.3.4. Ряды вида $\sum a_n \operatorname{cosech} b_n x$.

$$\text{интегралы, } x = \frac{\pi K'(k)}{K(k)}, \quad k^2 < 1.$$

$$1. \sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{cosech} \left(n - \frac{1}{2} \right) x = \frac{1}{\pi} K(k).$$

$$2. \sum_{n=1}^{\infty} n \operatorname{cosech} nx = \frac{1}{\pi^2} K(k) [K(k) - E(k)].$$

$$3. \sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{cosech} nx = \frac{1}{\pi^2} K(k) E(k) - \frac{1-E^2}{2\pi^2} K^2(k).$$

$$4. \sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{cosech} (2n-1)x = -\frac{1}{8} \ln(1-k^2).$$

$$5. \sum_{n=1}^{\infty} \operatorname{cosech}^2 nx = \frac{1}{6} + \frac{2(2-k^2)}{3\pi^2} K^2(k) - \frac{2}{\pi^2} K(k) E(k).$$

$$6. \sum_{n=1}^{\infty} \operatorname{cosech}^2 \left(n - \frac{1}{2} \right) x = \frac{2}{\pi^2} K(k) [K(k) - E(k)].$$

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\operatorname{cosech} na - \operatorname{cosech} \frac{n\pi}{a} \right) = \frac{1}{2} \left(\frac{\pi}{a} - a \right) \quad [a > 0].$$

5.3.5. Ряды вида $\sum a_k \operatorname{cosech} k\pi$.

$$1. \sum_{k=1}^{\infty} k \operatorname{cosech} k\pi = \frac{1}{32\pi^2} \left(\frac{1}{4} \right) - \frac{1}{4\pi}.$$

$$2. \sum_{k=1}^{\infty} (-1)^k k \operatorname{cosech} k\pi = -\frac{1}{4\pi}.$$

$$3. \sum_{k=1}^{\infty} (-1)^k k^{2n+1} \operatorname{cosech} k\pi = 0 \quad [n = 1, 2, 3, \dots].$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \operatorname{cosech} k\pi =$$

$$= \pi^{2n+1} \sum_{k=0}^{2n+2} \frac{(-1)^k}{(2k)!(4n-2k+4)!} (1-2^{2k-1})(1-2^{2n-2k+1}) B_{2k} B_{2n-2k+4}$$

$$[n = 1, 2, 3, \dots].$$

5. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \operatorname{cosech} k\pi = \frac{\pi^2}{90}.$
6. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \operatorname{cosech} k\pi = \frac{13\pi^2}{450 \cdot 540}.$
7. $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^4 + a^4} \operatorname{cosech} k\pi = \frac{\pi}{2a^2} [\operatorname{ch}(\sqrt{2}\pi a) - \cos(\sqrt{2}\pi a)]^2 - \frac{1}{4\pi a^2}.$
8. $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{k^4 + a^4} \operatorname{cosech} k\pi = -\frac{\pi a^2}{4} \operatorname{cosec} \pi a \operatorname{cosech} \pi a.$
9. $\sum_{k=1}^{\infty} \operatorname{cosech}^2 k\pi = \frac{1}{6} - \frac{1}{24\pi}.$
10. $\sum_{k=0}^{\infty} \operatorname{cosech}^2 \frac{2k+1}{2} \pi = -\frac{1}{2\pi} + \frac{1}{16\pi^3} \Gamma^2\left(\frac{1}{4}\right).$
11. $\sum_{k=1}^{\infty} \frac{1}{k^2 + (k+1)^2} \operatorname{cosech} k\pi \operatorname{cosech} (k+1)\pi =$
 $= \frac{1}{2\pi} \operatorname{cosech} \pi \left(2 + 2\pi \operatorname{cth} \pi - \pi^2 \operatorname{th}^2 \frac{\pi}{2} \right).$
12. $\sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{cosech} k\pi \operatorname{cth} k\pi = \frac{1}{12}.$
13. $\sum_{k=1}^{\infty} \operatorname{cosech}^4 k\pi = -\frac{11}{90} + \frac{1}{3\pi} + \frac{1}{1920\pi^5} \Gamma^2\left(\frac{1}{4}\right).$
14. $\sum_{k=0}^{\infty} \operatorname{cosech}^4 \frac{2k+1}{2} \pi = \frac{1}{2\pi} - \frac{1}{24\pi^3} \Gamma^2\left(\frac{1}{4}\right) + \frac{1}{1920\pi^5} \Gamma^2\left(\frac{1}{4}\right).$

5.3.6. Ряды вида $\sum_{k=0}^{\infty} a_k \operatorname{sech} k\pi$.

Обозначение: $a = \frac{\pi K'(k)}{K(k)}$, $k^2 < 1$.

1. $\sum_{n=-\infty}^{\infty} \operatorname{sech}(\pi x + a) = \frac{2}{\pi} K(k) \operatorname{dn} \left[\frac{2a}{\pi} K'(k) \right]$ $|\operatorname{Im}(2a/\pi)| < \operatorname{Re}(\pi/\pi)$
2. $\sum_{n=-\infty}^{\infty} \operatorname{sech} n\pi = \frac{2}{\pi} K(k).$

3. $\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} n\pi = \frac{2(1-k^2)^{1/2}}{\pi} K(k).$
4. $\sum_{n=-\infty}^{\infty} \operatorname{sech} \left(n - \frac{1}{2} \right) \pi = \frac{2k}{\pi} K(k).$
5. $\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 n\pi = \frac{4}{\pi^2} K(k) E(k).$
6. $\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 \left(n - \frac{1}{2} \right) \pi = \frac{2}{\pi^2} K(k) E(k) - \frac{2(1-k^2)}{\pi^2} K^2(k).$
7. $\sum_{n=0}^{\infty} 2^n \operatorname{sech}^2 2^{n+1} \pi = \pm \operatorname{cosech}^2 \pi + \frac{1}{4} \left(\frac{\pi}{2} \right).$

5.3.7. Ряды вида $\sum_{k=0}^{\infty} a_k \operatorname{sech} b_k \pi$.

1. $\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \operatorname{sech} \frac{2k+1}{2} \pi = 0.$
2. $\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = 0$ $[n=1, 2, 3, \dots]$
3. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{1}{4(4n)!} \left(\frac{\pi}{2} \right)^{4n+1} \sum_{k=0}^{2n} \binom{4n}{2k} E_{2k} E_{4n-2k}.$
4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8}.$
5. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi^2}{768}.$
6. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{23\pi^3}{1720320}.$
7. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} =$
 $= (-1)^n \frac{\pi^{2n+1}}{2} \sum_{k=0}^{2n} \frac{2^{2k+1} - 1}{(2k+1)! (5n-2k)! (k+1)!} B_{2k+2} B_{4n-2k} \cos \frac{2k+1}{3} \pi$
 $[n=1, 2, 3, \dots]$

6. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = \frac{\pi}{24}.$
9. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \operatorname{sech} \frac{2k+1}{2} \pi \sqrt{3} = \frac{\pi^7}{23\,040}.$
10. $\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^3}{(2k+1)^4 - a^4} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8} \sec \frac{\pi a}{2} \operatorname{sech} \frac{\pi a}{2}.$
11. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^4 - a^4]} \operatorname{sech} \frac{2k+1}{2} \pi = \frac{\pi}{8a^4} \left(\sec \frac{\pi a}{2} \operatorname{sech} \frac{\pi a}{2} - 1 \right).$
12. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1) [(2k+1)^4 + a^4]} \operatorname{sech} \frac{2k+1}{2} \pi =$
 $= \frac{\pi}{8a^4} \left[1 - \frac{2}{\cos(\pi a/\sqrt{2}) + \operatorname{ch}(\pi a/\sqrt{2})} \right].$
13. $\sum_{k=1}^{\infty} \operatorname{sech}^2 k\pi = \frac{1}{2\pi} - \frac{1}{2} + \frac{1}{16\pi^4} \Gamma^2\left(\frac{1}{4}\right).$
14. $\sum_{k=0}^{\infty} \operatorname{sech}^2 \frac{2k+1}{2} \pi = \frac{1}{2\pi}.$
15. $\sum_{k=1}^{\infty} \operatorname{sech}^4 k\pi = -\frac{1}{2} + \frac{1}{3\pi} + \frac{1}{24\pi^2} \Gamma^4\left(\frac{1}{4}\right) + \frac{1}{192\pi^6} \Gamma^6\left(\frac{1}{4}\right).$
16. $\sum_{k=0}^{\infty} \operatorname{sech}^4 \frac{2k+1}{2} \pi = \frac{1}{3\pi} - \frac{1}{364\pi^6} \Gamma^6\left(\frac{1}{4}\right).$

5.3.8. Ряды вида $\sum a_k \operatorname{th} b_k.$

1. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \operatorname{th} \frac{2k+1}{2} \pi =$
 $= \frac{1}{2} \pi^{2n+1} \sum_{k=0}^{2n} \frac{(-1)^k (2^{2k} - 1) (2^{4n-2k-3} - 1)}{(2k+2)! (4n-2k+2)!} B_{2k} B_{4n-2k+2}.$
2. $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \operatorname{th} \frac{2k+1}{2} \pi = \frac{\pi^3}{32}.$
3. $\sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^4 - a^4} \operatorname{th} \frac{2k+1}{2} \pi = \frac{\pi}{8a^4} \operatorname{tg} \frac{\pi a}{2} \operatorname{th} \frac{\pi a}{2}.$

4. $\sum_{k=1}^{\infty} \frac{1}{2^k} \operatorname{th} \frac{x}{2^k} = \operatorname{cth} x - \frac{1}{x}.$
 5. $\sum_{k=0}^{\infty} 2^k (1 - \operatorname{th} 2^k x) = \operatorname{cth} x - 1 \quad [x > 0].$
 6. $\sum_{k=1}^{\infty} \frac{1}{2^k} \operatorname{th}^2 \frac{x}{2^k} = \operatorname{cth}^2 x - \frac{1}{x^2} - \frac{2}{3}.$
- 5.3.9. Ряды вида $\sum a_k \operatorname{cth} k\pi.$
1. $\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \operatorname{cth} k\pi = \frac{1}{2} (2\pi)^{2n+1} \sum_{k=1}^n \frac{(-1)^{k+1}}{(2k)! (2n-2k+2)!} B_{2k} B_{2n-2k+2} +$
 $+ \frac{(-1)^n 2^{2n+1} \pi^{2n+1}}{(2n+2)!} B_{2n+2}.$
 2. $\sum_{k=1}^{\infty} \frac{1}{k^3} \operatorname{cth} k\pi = \frac{7}{180} \pi^3.$
 3. $\sum_{k=1}^{\infty} \frac{1}{k^5} \operatorname{cth} k\pi = \frac{19}{56\,700} \pi^5.$
 4. $\sum_{k=1}^{\infty} \frac{k}{k^4 + a^4} \operatorname{cth} k\pi = \frac{\pi \operatorname{ch}(\pi a/\sqrt{2}) + \cos(\pi a/\sqrt{2})}{4a^3 \operatorname{ch}(\pi a/\sqrt{2}) - \cos(\pi a/\sqrt{2})} - \frac{1}{4\pi a^4}.$
 5. $\sum_{k=1}^{\infty} \frac{k}{k^4 - a^4} \operatorname{cth} k\pi = \frac{1}{4\pi a^4} - \frac{\pi}{4a^3} \operatorname{ctg} \pi a \operatorname{cth} \pi a.$

5.4. ТРИГОНОМЕТРИЧЕСКИЕ РЯДЫ

5.4.1. Введение.

В этом разделе помещены ряды, содержащие различные тригонометрические функции, в том числе ряды Фурье. Отметим, что в некоторых случаях тригонометрические ряды сводятся к рядам другого вида, например, к степенным (5.2), к рядам, содержащим показательную и гиперболические функции (5.3).

При суммировании тригонометрических рядов могут оказаться полезными следующие формулы:

$$\sum_{k=1}^{\infty} F\left(\frac{k}{2}\right) \sin kx = \frac{1}{2} \sin x \int_0^{\pi} \frac{f(t) dt}{\sin t - \cos x} \quad (0 \leq x \leq 2\pi),$$

$$\sum_{k=1}^{\infty} F(k) \cos kx = \frac{1}{2} \int_0^{\pi} \frac{\cos x - \cos t}{\sin t - \cos x} f(t) dt \quad (0 \leq x \leq 2\pi),$$

где

$$F(x) = \int_0^{\infty} f(t) e^{-xt} dt.$$

Ряды, содержащие логарифмы, приведены в 5.5.1.

5.4.2. Ряды вида $\sum_{k=1}^{\infty} \frac{(-1)^k (\sin kx)}{k^s}$

1. $\sum_{k=1}^{\infty} \frac{1}{k^s} \frac{\sin kx}{\cos kx} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} (e^t \frac{\sin x}{\cos x} - \frac{0}{1})}{1 - 2e^t \cos x + e^{2t}} dt$
[$x \neq 0$; $\operatorname{Re} s > 0$ или $|x| > 0$, $\operatorname{Re} s > 1$]
2. $= \frac{(2\pi)^s}{4\Gamma(s)} \left\{ \sec(\pi s/2) \right\} \left[\zeta\left(1-s, \frac{x}{2\pi}\right) + \zeta\left(1-s, 1-\frac{x}{2\pi}\right) \right]$
[$\operatorname{Re} s > 1$; $0 < x < 2\pi$]
3. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} \frac{\sin kx}{\cos kx} = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} (e^t \frac{\sin x}{\cos x} + \frac{0}{1})}{1 + 2e^t \cos x + e^{2t}} dt$ [$\operatorname{Re} s > 0$]
4. $= \frac{(2\pi)^s}{4\Gamma(s)} \left\{ \sec(\pi s/2) \right\} \left[\zeta\left(1-s, \frac{x+\pi}{2}\right) + \zeta\left(1-s, \frac{\pi-x}{2}\right) \right]$
[$\operatorname{Re} s > 1$; $-\pi < x < \pi$]
5. $\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin kx = \frac{(-1)^{n-1}}{2(2n+1)!} (2\pi)^{2n+1} B_{2n+1}\left(\frac{x}{2\pi}\right)$
[$0 \leq x < 2\pi$ при $n=1, 2, 3, \dots$; $0 < x < 2\pi$ при $n=0$]
6. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n+1}} \sin kx = \frac{(-1)^{n-1}}{2(2n+1)!} (2\pi)^{2n+1} B_{2n+1}\left(\frac{x+\pi}{2\pi}\right)$
[$-\pi < x < \pi$; $n=0, 1, 2, \dots$]
7. $\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \cos kx = \frac{(-1)^{n-1}}{2(2n)!} (2\pi)^{2n} B_{2n}\left(\frac{x}{2\pi}\right)$
[$0 \leq x < 2\pi$; $n=1, 2, 3, \dots$]
8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} \cos kx = \frac{(-1)^{n-1}}{2(2n)!} (2\pi)^{2n} B_{2n}\left(\frac{x+\pi}{2\pi}\right)$
[$-\pi < x < \pi$]
9. $\sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin(kx+a)}{\cos(kx+a)} = \pm \frac{\pi-s}{2} \left\{ \frac{\cos a}{\sin a} \right\} - \ln \left(2 \cos \frac{x}{2} \right) \left\{ \frac{\sin a}{\cos a} \right\}$
[$0 < x < 2\pi$]
10. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{\sin(kx+a)}{\cos(kx+a)} = \pm \frac{x}{2} \left\{ \frac{\cos a}{\sin a} \right\} + \ln \left(2 \cos \frac{x}{2} \right) \left\{ \frac{\sin a}{\cos a} \right\}$
[$-\pi < x < \pi$]
11. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin kx = -\int_0^x \ln \left(\frac{2 \sin(t/2)}{2 \cos(t/2)} \right) dt$
[$-\pi < x < \pi$]
12. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos kx = \frac{1}{12} \left\{ \frac{3x^2 - 6\pi x + 2\pi^2}{3x^2 - \pi^2} \right\}$
[$0 \leq x \leq 2\pi$]

$$13. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx = \frac{x}{12} \left\{ \frac{x^2 - 3\pi x + 2\pi^2}{x^2 - \pi^2} \right\} \quad \left[\begin{array}{l} 0 \leq x \leq 2\pi \\ -\pi \leq x \leq \pi \end{array} \right]$$

5.4.3. Ряды вида $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+a)^s} \frac{\sin(kx+b)}{\cos(kx+b)}$

1. $\sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \frac{\sin kx}{\cos kx} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at} \left(\frac{0}{1} \pm e^{it} \frac{\sin x}{\cos x} \right)}{1 - 2e^{it} \cos x + e^{2it}} dt$
[$\operatorname{Re} a > 0$; $\operatorname{Re} s > 0$ при $x \neq 0$, $\operatorname{Re} s > 1$ при $x=0$]
2. $\sum_{k=0}^{\infty} \frac{1}{k+a} \frac{\sin kx}{\cos kx} = -\beta(a) \frac{\sin(\pi-x)a}{\cos(\pi-x)a} + \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin[-xa+(a-1/2)t]}{\cos[-xa+(a-1/2)t]} \operatorname{cosec} \frac{t}{2} dt$
3. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+a} \frac{\sin kx}{\cos kx} = -\beta(a) \frac{\sin ax}{\cos ax} + \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos[-xa+(a-1/2)t]}{\sin[-xa+(a-1/2)t]} \sec \frac{t}{2} dt$
4. $\sum_{k=-\infty}^{\infty} \frac{1}{k+a} \frac{\sin kx}{\cos kx} = \pi \operatorname{cosec} \pi a \frac{\sin[(2n+1)\pi-x]a}{\cos[(2n+1)\pi-x]a}$
[$2\pi n < x < 2\pi(n+1)$]
5. $\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{k+a} \frac{\sin kx}{\cos kx} = \pi \operatorname{cosec} \pi a \frac{\sin(2n\pi-x)a}{\cos(2n\pi-x)a}$
[$(2n-1)\pi < x < (2n+1)\pi$]
6. $\sum_{k=-\infty}^{\infty} \frac{1}{kx+a} \frac{\sin(kx+a)}{\cos(kx+a)} = \frac{\pi}{\sin \pi a/x} \frac{\sin[(2n+1)\pi a/x]}{\cos[(2n+1)\pi a/x]}$
[$2\pi n < x < (2n+2)\pi$]
7. $\sum_{k=0}^{\infty} \frac{1}{km+n} \frac{\sin(kx+a)}{\cos(kx+a)} =$

$$= \frac{1}{2\pi n} \sum_{k=0}^{n-1} [(2k-m)\pi + x] \left\{ \frac{\cos[a-(x+2k\pi)/m]}{\sin[a-(x+2k\pi)/m]} \right\} -$$

$$- \frac{1}{m} \sum_{k=0}^{n-1} \frac{\sin[a-(x+2k\pi)/m]}{\cos[a-(x+2k\pi)/m]} \ln \left(2 \left| \sin \frac{x+2k\pi}{2m} \right| \right) +$$

$$+ \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{1}{km-m} \left\{ \frac{\sin(a-kx)}{\cos(a-kx)} \right\}$$

[$m > 0$; $0 < x < 2\pi$; $\delta(n-m) = \begin{cases} 1, & n > m \\ 0, & n \leq m \end{cases}$]

$$8. \sum_{k=0}^{\infty} \frac{1}{km+n} \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} =$$

$$= \frac{1}{2m^2} \sum_{k=0}^{m-1} [(2k-m)\pi+x] \left\{ \frac{\cos[a+(x+2k\pi)/m]}{\sin[a+(x+2k\pi)/m]} \right\} -$$

$$- \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \frac{\sin[a+(x+2k\pi)/m]}{\cos[a+(x+2k\pi)/m]} \right\} \ln \left| 2 \sin \frac{x+2k\pi}{2m} \right| +$$

$$+ \sum_{k=0}^{[n/m]} \frac{1}{km-n} \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} \quad [m > 0; 0 < x < 2\pi; n/m \neq 0, 1, 2, \dots].$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{km+n} \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} =$$

$$= \frac{1}{2m^2} \sum_{k=0}^{m-1} [(2k-m+1)\pi+x] \left\{ \frac{\cos[a-(x+2k\pi+\pi)/m]}{\sin[a-(x+2k\pi+\pi)/m]} \right\} -$$

$$- \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \frac{\sin[a-(x+2k\pi+\pi)/m]}{\cos[a-(x+2k\pi+\pi)/m]} \right\} \ln \left| 2 \sin \frac{x+(2k+1)\pi}{2m} \right| +$$

$$+ \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{(-1)^k}{km-n} \left\{ \frac{\sin(a-kx)}{\cos(a-kx)} \right\}$$

$$[m > 0; -\pi < x < \pi; \delta(n-m) = \begin{cases} 1, & n > m \\ 0, & n \leq m \end{cases}].$$

$$10. \sum_{k=0}^{\infty} \frac{(-1)^k}{km-n} \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} =$$

$$= \frac{1}{2m^2} \sum_{k=0}^{m-1} [(2k-m+1)\pi+x] \left\{ \frac{\cos[a+(x+2k\pi+\pi)/m]}{\sin[a+(x+2k\pi+\pi)/m]} \right\} -$$

$$- \frac{1}{m} \sum_{k=0}^{m-1} \left\{ \frac{\sin[a+(x+2k\pi+\pi)/m]}{\cos[a+(x+2k\pi+\pi)/m]} \right\} \ln \left| 2 \sin \frac{x+(2k+1)\pi}{2m} \right| +$$

$$+ \sum_{k=0}^{[n/m]} \frac{(-1)^k}{km-n} \left\{ \frac{\sin(kx+a)}{\cos(kx+a)} \right\} \quad [m > 0; -\pi < x < \pi; n/m \neq 0, 1, 2, \dots].$$

$$11. \sum_{k=1}^{\infty} \frac{1}{k+n} \left\{ \frac{\sin kx}{\cos kx} \right\} = \frac{1}{2} (\pi-x) \left\{ \frac{\cos nx}{\sin nx} \right\} -$$

$$- \sum_{k=1}^n \frac{1}{k} \sin kx \left\{ \frac{\cos nx}{\sin nx} \right\} \pm \left[\ln \left(2 \sin \frac{x}{2} \right) + \sum_{k=1}^n \frac{1}{k} \cos kx \right] \left\{ \frac{\sin nx}{\cos nx} \right\}$$

$$[n=1, 2, 3, \dots; 0 < x < 2\pi].$$

$$12. \sum_{k=1}^{\infty} \frac{(-1)^k}{k+n} \left\{ \frac{\sin kx}{\cos kx} \right\} = (-1)^{n+1} \left[\sum_{k=1}^n \frac{(-1)^k}{k} \sin kx + \frac{x}{2} \right] \left\{ \frac{\cos nx}{\sin nx} \right\} \pm$$

$$\pm (-1)^n \left[\ln \left(2 \cos \frac{x}{2} \right) + \sum_{k=1}^n \frac{(-1)^k}{k} \cos kx \right] \left\{ \frac{\sin nx}{\cos nx} \right\}$$

$$[n=1, 2, 3, \dots; -\pi < x < \pi].$$

$$13. \sum_{k=0}^{\infty} \frac{1}{2k+1} \left\{ \frac{\sin kx}{\cos kx} \right\} = \frac{\pi}{4} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} \mp \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} \ln \operatorname{ctg}^2 \frac{x}{4}$$

$$[0 < x < 2\pi].$$

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left\{ \frac{\sin kx}{\cos kx} \right\} = -\frac{1}{4} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} \ln \operatorname{ctg}^2 \left(\frac{x+\pi}{4} \right) \mp \frac{\pi}{4} \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\}$$

$$[-\pi < x < \pi].$$

$$5.4.4. \text{Ряды вида } \sum \frac{(\pm 1)^k}{(kn_1+m_1)(kn_2+m_2)} \left\{ \frac{\sin kx}{\cos kx} \right\}.$$

$$1. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left\{ \frac{\sin kx}{\cos kx} \right\} = \pm (\pi-x) \sin \frac{x}{2} \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} \pm$$

$$\pm 2 \sin \frac{x}{2} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} \ln \left(2 \sin \frac{x}{2} \right) + \frac{0}{1}$$

$$[0 \leq x \leq 2\pi].$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \left\{ \frac{\sin kx}{\cos kx} \right\} = -x \cos \frac{x}{2} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} \pm$$

$$\pm 2 \cos \frac{x}{2} \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} \ln \left(2 \cos \frac{x}{2} \right) + \frac{0}{1}$$

$$[-\pi \leq x \leq \pi].$$

$$3. \sum_{k=1}^{\infty} \frac{1}{k(k+2)} \left\{ \frac{\sin kx}{\cos kx} \right\} = \mp \frac{1}{2} [1+(x-\pi) \sin x] \left\{ \frac{\sin x}{\cos x} \right\} -$$

$$- \sin x \ln \left(2 \sin \frac{x}{2} \right) \left\{ \frac{\cos x}{\sin x} \right\} + \frac{1}{4} \frac{0}{1}$$

$$[0 \leq x \leq 2\pi].$$

$$4. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left\{ \frac{\sin(k+1)x}{\cos(k+1)x} \right\} =$$

$$= \mp 2 \sin \frac{x}{2} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} \ln \left(2 \sin \frac{x}{2} \right) + \left\{ \frac{\sin x}{\cos x} \right\} - (x-x) \sin \frac{x}{2} \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\}$$

$$[0 \leq x \leq 2\pi].$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \left\{ \frac{\sin(k+1)x}{\cos(k+1)x} \right\} = \mp x \cos \frac{x}{2} \left\{ \frac{\cos(x/2)}{\sin(x/2)} \right\} +$$

$$+ \left\{ \frac{\sin x}{\cos x} \right\} - 2 \cos \frac{x}{2} \left\{ \frac{\sin(x/2)}{\cos(x/2)} \right\} \ln \left(2 \cos \frac{x}{2} \right)$$

$$[-\pi \leq x \leq \pi].$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+2)} \left\{ \frac{\sin kx}{\cos kx} \right\} = \pm \frac{1}{2} (1 - x \sin x) \left\{ \frac{\sin x}{\cos x} \right\} - \\ - \sin x \ln \left(2 \cos \frac{x}{2} \right) \left\{ \frac{\cos x}{\sin x} \right\} + \frac{1}{4} \{0\} \quad [-\pi \leq x \leq \pi]$$

$$7. \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)} \left\{ \frac{\sin kx}{\cos kx} \right\} = \frac{1}{4} \sin \frac{x}{2} \ln \lg^2 \frac{x}{4} \left\{ \frac{\cos x}{\sin x} \right\} \pm \\ \pm \frac{\pi}{4} \sin \frac{x}{2} \left\{ \frac{\sin x}{\cos x} \right\} + \frac{1}{2} \left\{ \frac{\sin x}{\cos x} \right\} \quad [0 < x < 2\pi]$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} \left\{ \frac{\sin kx}{\cos kx} \right\} = \frac{1}{4} \cos \frac{x}{2} \ln \lg^2 \frac{x+\pi}{4} \left\{ \frac{\cos x}{\sin x} \right\} \pm \\ \pm \frac{\pi}{4} \cos \frac{x}{2} \left\{ \frac{\sin x}{\cos x} \right\} + \frac{1}{2} \left\{ \frac{\sin x}{\cos x} \right\} \quad [-\pi < x < \pi]$$

5.4.5. Ряды вида

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k^2 \pm a^2)^m} \left\{ \frac{\sin kx}{\cos kx} \right\}$$

$$1. \sum_{k=1}^{\infty} \frac{1}{k^2 \pm a^2} \cos kx = \pm \frac{\pi}{2a} \left\{ \frac{\operatorname{ch}[(\pi-x)a]}{\cos[(\pi-x)a]} \operatorname{cosech} \pi a \right\} + \frac{1}{2a^2} \quad [0 \leq x \leq 2\pi]$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 \pm a^2} \cos kx = \pm \frac{\pi}{2a} \left\{ \frac{\operatorname{ch} ax \operatorname{cosech} \pi a}{\cos ax \operatorname{cosec} \pi a} \right\} + \frac{1}{2a^2} \quad [-\pi < x < \pi]$$

$$3. \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{k^2 - n^2} \cos kx = \frac{1}{2n^2} + \frac{\cos nx}{4n^2} - \frac{(\pi-x) \sin nx}{2n} \quad [0 < x < 2\pi]$$

$$4. \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 - 1} \cos kx = \frac{1}{2} - \frac{1}{4} \cos x - \frac{x \sin x}{2} \quad [-\pi \leq x \leq \pi]$$

$$5. \sum_{k=1}^{\infty} \frac{k}{k^2 \pm a^2} \sin kx = \frac{\pi}{2} \left\{ \frac{\operatorname{sh}(\pi-x)a \operatorname{cosech} \pi a}{\sin(\pi-x)a \operatorname{cosec} \pi a} \right\} \quad [0 < x < 2\pi]$$

$$6. \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{k}{k^2 - n^2} \sin kx = \frac{1}{2} (\pi-x) \cos nx - \frac{1}{4n} \sin nx \quad [0 < x < 2\pi; n=1, 2, 3, \dots]$$

$$7. \sum_{k=2}^{\infty} \frac{k}{k^2 - 1} \cos kx = -\frac{1}{2} - \frac{\cos x}{4} - \cos x \ln \left(2 \sin \frac{x}{2} \right) \quad [0 < x < 2\pi]$$

$$8. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 \pm a^2} \sin kx = \frac{\pi}{2} \left\{ \frac{\operatorname{sh} ax \operatorname{cosech} \pi a}{\sin ax \operatorname{cosec} \pi a} \right\} \quad [-\pi < x < \pi]$$

$$9. \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 - 1} \sin kx = \frac{\sin x}{4} + \frac{x \cos x}{2} \quad [-\pi \leq x \leq \pi]$$

$$10. \sum_{k=1}^{\infty} \frac{k}{4k^2 - 1} \cos kx = -\frac{1}{4} - \frac{1}{4} \cos \frac{x}{2} \ln \lg \frac{x}{4} \quad [0 < x < 2\pi]$$

$$11. \sum_{k=1}^{\infty} \frac{1}{(k^2 \pm a^2)^2} \cos kx = \\ = \frac{1}{2a^4} + \frac{\pi}{4a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\} \left\{ \frac{\operatorname{ch}(\pi-x)a}{\cos(\pi-x)a} \right\} + \frac{\operatorname{sh}(\pi-x)a}{\sin(\pi-x)a} \left\{ \frac{\operatorname{ch} ax}{\cos ax} \right\} + \\ + \frac{\pi^2}{4a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\} \left\{ \frac{\operatorname{ch} ax}{\cos ax} \right\}$$

$$12. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k^2 \pm a^2)^2} \cos kx = \\ = \frac{1}{2a^4} + \frac{\pi}{4a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\} \left\{ \frac{\operatorname{ch} ax}{\cos ax} \right\} + \frac{a(x+\pi) \operatorname{sh} ax}{\sin ax} \left\{ \frac{\operatorname{ch} ax}{\cos ax} \right\} + \\ + \frac{\pi^2}{4a^3} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 \left\{ \frac{\operatorname{ch}(x+\pi)a}{\cos(x+\pi)a} \right\}$$

$$13. \sum_{k=1}^{\infty} \frac{k}{(k^2 \pm a^2)^2} \sin kx = \\ = \frac{\pi^2}{4a} \left\{ \frac{\operatorname{sh} ax}{\sin ax} \right\} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 - \frac{\pi x}{4a} \left\{ \frac{\operatorname{ch}(\pi-x)a \operatorname{cosech} \pi a}{\cos(\pi-x)a \operatorname{cosec} \pi a} \right\}$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k k}{(k^2 \pm a^2)^2} \sin kx = \\ = -\frac{\pi^2}{4a} \left\{ \frac{\operatorname{sh}(x+\pi)a}{\sin(x+\pi)a} \right\} \left\{ \frac{\operatorname{cosech} \pi a}{\operatorname{cosec} \pi a} \right\}^2 + \frac{\pi}{4a} (x+\pi) \left\{ \frac{\operatorname{ch} ax \operatorname{cosech} \pi a}{\cos ax \operatorname{cosec} \pi a} \right\}$$

$$15. \sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{(k^2 + a^2)^{1/2} + k}{k^2 + a^2} \right]^{1/2} \sin \frac{k\pi}{3} = \left(\frac{6}{\pi} \right)^{1/2} \int_0^{\infty} \frac{1}{1+2 \operatorname{ch} t^2} \left\{ \frac{\cos at^2}{\sin at^2} \right\} dt$$

5.4.6. Ряды вида $\sum_{k=1}^{\infty} \frac{1}{(k^2 \pm a^2)^m} \left\{ \frac{\sin(2k+1)x}{\cos(2k+1)x} \right\}$

$$1. \sum_{k=0}^{\infty} \frac{1}{(k+a)^2} \left\{ \frac{\sin(2k+1)x}{\cos(2k+1)x} \right\} = \frac{1}{\Gamma(s)} \left\{ \frac{\sin x}{\cos x} \right\} \int_0^{\infty} \frac{t^{s-1} e^{-t} (1 \pm e^{-t})}{1 - 2e^{-t} \cos 2x + e^{-2t}} dt \quad [\operatorname{Re} a, \operatorname{Re} s > 0]$$

$$7. \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(s-1)t} \left(\frac{\sin 3x}{\cos 3x} - \frac{\sin x}{\cos x} \right)}{e^{st} - 2e^t \cos 2x + 1} dt$$

[$\operatorname{Re} s > 0$]

$$8. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \sin(2k+1)x = \frac{(-1)^n \pi^{2n+1}}{4(2n)!} E_{2n} \left(\frac{x}{\pi} \right)$$

[$0 < x < \pi$; $n=0, 1, 2, \dots$]

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n}} \sin(2k+1)x = \frac{(-1)^{n-1} \pi^{2n}}{4(2n-1)!} E_{2n-1} \left(\frac{x}{\pi} + \frac{1}{2} \right)$$

[$-\pi/2 \leq x \leq \pi/2$; $n=1, 2, 3, \dots$]

$$10. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} \cos(2k+1)x = \frac{(-1)^n \pi^{2n}}{4(2n-1)!} E_{2n-1} \left(\frac{x}{\pi} \right)$$

[$0 \leq x \leq \pi$; $n=1, 2, 3, \dots$]

$$11. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \cos(2k+1)x = \frac{(-1)^n \pi^{2n+1}}{4(2n)!} E_{2n} \left(\frac{x}{\pi} + \frac{1}{2} \right)$$

[$-\pi/2 < x < \pi/2$; $n=0, 1, 2, \dots$]

$$12. \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{\pi}{4} \frac{f(1)}{f(0)} + \frac{1}{4} \frac{f(0)}{f(1)} \ln \operatorname{ctg}^2 \frac{x}{4}$$

[$0 < x < \pi$]

$$13. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{\pi}{4} \frac{f(0)}{f(1)} - \frac{1}{4} \frac{f(1)}{f(0)} \ln \operatorname{ctg}^2 \left(\frac{x+\pi}{4} \right)$$

[$-\pi/2 < x < \pi/2$]

$$14. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin(2k+1)x = -\frac{1}{2} \int_0^x \ln \operatorname{tg} \frac{t}{2} dt$$

[$0 \leq x < \pi$]

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x = \frac{\pi}{8} (x-2|x|)$$

[$-\pi \leq x < \pi$]

$$16. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin(2k+1)x = \begin{cases} \pi/4 & \\ \pi(\pi-x)/4 & \end{cases}$$

[$-\pi/2 \leq x \leq \pi/2$]
[$\pi/2 \leq x \leq 3\pi/2$]

$$17. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos(2k+1)x = -\frac{1}{2} \int_0^x \ln \operatorname{tg} \frac{t}{2} dt$$

[$-\pi/2 < x \leq \pi/2$]

$$18. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \sin(2k+1)x = \frac{\pi x}{8} (\pi - |x|)$$

[$-\pi \leq x \leq \pi$]

$$14. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos(2k+1)x = \frac{\pi}{32} (\pi^2 - 4x^2)$$

[$-\pi/2 \leq x \leq \pi/2$]

$$15. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sin(2k+1)x = \sin x - 2 \sin x \ln(2 \sin x)$$

[$0 < x < \pi$]

$$16. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \sin(2k+1)x = \sin x - 2x \cos x$$

[$-\pi/2 < x < \pi/2$]

$$17. \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \cos(2k+1)x = \cos x - \pi \sin x + 2x \sin x$$

[$0 < x < \pi$]

$$18. \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)} \cos(2k+1)x = \cos x - 2 \cos x \ln(2 \cos x)$$

[$-\pi/2 < x < \pi/2$]

$$19. \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(2k+3)} \sin(2k+1)x = \frac{\pi}{8} \sin 2x$$

[$-\pi/2 < x < \pi/2$]

$$20. \sum_{k=0}^{\infty} \frac{1}{(2k-1)(2k+3)} \cos(2k+1)x = -\frac{\pi}{8} \sin 2x$$

[$0 \leq x \leq \pi$]

$$21. \sum_{k=0}^{\infty} \frac{(2k+1)}{(2k-1)(2k+3)} \sin(2k+1)x = \frac{\pi}{4} \cos 2x$$

[$0 < x < \pi$]

$$22. \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(2k+3)} \cos(2k+1)x = \frac{\pi}{4} \cos 2x$$

[$-\pi/2 \leq x \leq \pi/2$]

$$23. \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(4k^2-1)(2k+3)} \cos(2k+1)x = \frac{\pi}{4} \cos^2 x$$

[$-\pi/2 \leq x \leq \pi/2$]

$$24. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 \pm a^2} \sin(2k+1)x = \frac{\pi}{4a} \left\{ \frac{\operatorname{sh} ax \operatorname{sech}(a\pi/2)}{\sin ax \sec(a\pi/2)} \right\}$$

[$-\pi/2 < x < \pi/2$]

$$25. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \pm a^2} \cos(2k+1)x = \frac{\pi}{4a} \left\{ \frac{\operatorname{sh}(\pi-2x)a \operatorname{sech}(a\pi/2)}{\sin(\pi-2x)a \sec(a\pi/2)} \right\}$$

[$0 \leq x \leq \pi$]

$$26. \sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^2 \pm a^2} \sin(2k+1)x = \frac{\pi}{4} \left\{ \frac{\operatorname{ch}(\pi-2x)a \operatorname{sech}(a\pi/2)}{\cos(\pi-2x)a \sec(a\pi/2)} \right\}$$

[$0 < x < \pi$]

$$27. \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{(2k+1)^2 + a^2} \cos(2k+1)x = \frac{\pi}{4} \frac{\operatorname{ch} ax \operatorname{sech}(a\pi/2)}{\cos ax \sec(a\pi/2)}$$

$$[-\pi/2 < x < \pi/2]$$

$$28. \sum_{k=0}^{\infty} \frac{1}{(ka+b)(ka+b+1)\dots(ka+b+n)} \frac{\sin[(2k+1)x]}{\cos[(2k+1)x]} = \frac{1}{n!} \frac{\sin x}{\cos x} \int_0^1 \frac{t^{b-1}(1-t)^n(1+t^n)}{1-2t^a \cos 2x + t^{2a}} dt \quad [a, b > 0; 0 < x < \pi]$$

$$29. \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k)!} \frac{\sin[(2k+1)x]}{\cos[(2k+1)x]} = \frac{1}{\sqrt{2} \sin x} \frac{\sin[(2x+\pi)/4]}{\cos[(2x+\pi)/4]} - \frac{\sin x}{\cos x} \quad [0 < x < \pi]$$

$$30. \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!}{(2k)!} \frac{\sin[(2k+1)x]}{\cos[(2k+1)x]} = \frac{1}{\sqrt{2} \cos x} \frac{\sin(x/2)}{\cos(x/2)} - \frac{\cos x}{\sin x} \quad [-\pi/2 < x < \pi/2]$$

5.4.7. Ряды вида $\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\sin kx}{\cos kx}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\sin kx}{\cos kx} = e^{\cos x} \frac{\sin(\sin x)}{\cos(\sin x)}.$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\sin kx}{\cos kx} = e^{-\cos x} \frac{\sin(\sin x)}{\cos(\sin x)}.$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{\sin kx}{\cos kx} = \frac{\sin[\sin(x/2)] \operatorname{sh}[\cos(x/2)]}{\cos[\sin(x/2)] \operatorname{ch}[\cos(x/2)]}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\sin kx}{\cos kx} = \frac{\sin[\cos(x/2)] \operatorname{sh}[\sin(x/2)]}{\cos[\cos(x/2)] \operatorname{ch}[\sin(x/2)]}.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\sin kx}{\cos kx} = \cos \frac{x}{2} \frac{\sin[\sin(x/2)] \operatorname{ch}[\cos(x/2)]}{\cos[\sin(x/2)] \operatorname{sh}[\cos(x/2)]} + \sin \frac{x}{2} \frac{\cos[\sin(x/2)] \operatorname{sh}[\cos(x/2)]}{\sin[\sin(x/2)] \operatorname{ch}[\cos(x/2)]}.$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\sin kx}{\cos kx} = \cos \frac{x}{2} \frac{\cos[\cos(x/2)] \operatorname{sh}[\sin(x/2)]}{2 \sin[\cos(x/2)] \operatorname{ch}[\sin(x/2)]} + \sin \frac{x}{2} \frac{\sin[\cos(x/2)] \operatorname{ch}[\sin(x/2)]}{2 \cos[\cos(x/2)] \operatorname{sh}[\sin(x/2)]}.$$

5.4.8. Ряды вида $\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\sin(kx+a)}{\cos(kx+a)}$.

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\sin(kx)}{(k+2)} \frac{\sin kx}{\cos kx} = \frac{1}{2} (1 - \cos x) \left[-\frac{\sin x}{\cos x} \ln(2 - 2 \cos x) + (\pi - x) \frac{\cos x}{\sin x} \right] + \frac{3}{4} \left\{ \frac{0}{1} \right\} + \frac{1}{2} \frac{\sin x}{\cos x} \quad [0 < x < 2\pi].$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\sin kx}{(k+2)} \frac{\sin kx}{\cos kx} = \frac{1}{2} (1 + \cos x) \left[+x \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \ln(2 + 2 \cos x) \right] + \frac{3}{4} \left\{ \frac{0}{1} \right\} + \frac{1}{2} \frac{\sin x}{\cos x} \quad [-\pi < x < \pi].$$

$$3. \sum_{k=0}^{\infty} \frac{1}{(kr+b)(kr+b+1)\dots(kr+b+n)} \frac{\sin(kx+a)}{\cos(kx+a)} = \frac{1}{n!} \int_0^1 \frac{t^{b-1}(1-t)^n}{1-2t^r \cos x + t^{2r}} \left[\frac{\sin a}{\cos a} + \frac{\sin(x-a)}{\cos(x-a)} \right] dt \quad [r, b > 0].$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(kr+b)(kr+b+1)\dots(kr+b+n)} \frac{\sin(kx+a)}{\cos(kx+a)} = \frac{1}{n!} \int_0^1 \frac{t^{b-1}(1-t)^n}{1+2t^r \cos x + t^{2r}} \left[\frac{\sin a}{\cos a} + \frac{\sin(x-a)}{\cos(x-a)} \right] dt \quad [r, b > 0].$$

$$5. \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k)!} \cos kx = \left(2 \sin \frac{x}{2} \right)^{-3/2} \cos \frac{\pi-x}{4} - 1 \quad [0 < x < 2\pi].$$

$$6. \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{(2k)!} \cos kx = \left(2 \cos \frac{x}{2} \right)^{-3/2} \cos \frac{x}{4} - 1 \quad [-\pi < x < \pi].$$

$$7. \sum_{k=0}^{\infty} \frac{[(2k)!]^2}{(n!)^4 2^{4k}} \frac{\sin(2k+1/2)x}{\cos(2k+1/2)x} = \frac{1}{\pi} \frac{K[\sin(x/2)]}{K[\cos(x/2)]} \quad [0 < x < \pi].$$

$$8. \sum_{k=0}^{\infty} \frac{(2k)! (2k+1)!}{(k!)^3 (k+1)! 2^{4k}} \frac{\sin(2k+3/2)x}{\cos(2k+3/2)x} = \frac{4}{\pi} \frac{E[\sin(x/2)]}{K[\cos(x/2)]} - \frac{2}{\pi} \frac{K[\sin(x/2)]}{E[\cos(x/2)]} \quad [0 < x < \pi].$$

$$9. \sum_{k=0}^{\infty} \frac{(u)_k (u+v-1)_k}{k! (v)_k} \frac{\sin (u+v+2k-1)x}{\cos (u+v+2k-1)x} =$$

$$= \frac{\Gamma(v) \pi^{1/2}}{2\Gamma(u+v-1)} e^{2i \sin x} \frac{P_{v-1/2}^{u-1/2}(\cos x)}{2Q_{v-1/2}^{u-1/2}(\cos x)} \quad [0 < x < \pi]$$

$$10. \sum_{k=0}^{\infty} \frac{(2k+2m)! (k+m+n)! (k+n+1)!}{k! (k+m)! (2k+2n+2)!} \cos (2k+m+n+1)x =$$

$$= 2^{-2} \pi^{-2m} (\sin x)^{-m} Q_m^n(\cos x) \quad [0 < x < \pi; m, n = 0, 1, 2, \dots]$$

$$11. \sum_{k=1}^{\infty} \frac{\sin kx}{k^4 + 4a^2} = \frac{\pi}{16a^2} \left[1 - \frac{\operatorname{ch} ax \cos a(\pi-x) + \cos ax \operatorname{ch} a(\pi-x)}{\operatorname{ch} \pi a + \cos \pi a} \right]$$

$$[0 < x < \pi]$$

$$12. \sum_{k=1}^{\infty} \frac{(k-1)^k}{\Gamma(v+1+k) \Gamma(v+1-k)} \cos kx =$$

$$= \frac{2^{2v-1}}{\Gamma(2v+1)} \frac{\{\cos(x/2)\}^{2v}}{\{\sin(x/2)\}^{2v}} - \frac{1}{2\Gamma^2(v+1)} \quad \left[\operatorname{Re} v > -\frac{1}{2}; \begin{cases} -\pi < x < \pi \\ 0 < x < 2\pi \end{cases} \right]$$

$$13. \sum_{k=0}^{\infty} \frac{(k-1)^k}{\Gamma(v+k+3/2) \Gamma(v-k+1/2)} \frac{\sin (2k+1)x}{\cos (2k+1)x} =$$

$$= \frac{2^{2v-1}}{\Gamma(2v+1)} \frac{\sin x}{\cos x} e^{2iv} \quad \left[\operatorname{Re} v > -\frac{1}{2}; \begin{cases} 0 < x < \pi \\ -\pi < x < -\pi/2 \end{cases} \right]$$

Б. 4.9. Ряды вида $\sum_{k=0}^{\infty} \frac{r^k}{(k+a)^2} \frac{\sin kx}{\cos kx}$.

Условие: $|r| < 1$.

$$1. \sum_{k=0}^{\infty} \frac{r^k}{(k+a)^2} \frac{\sin kx}{\cos kx} = \frac{1}{2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} [\Phi(re^{-ix}, s, a) - \Phi(re^{ix}, s, a)].$$

$$2. \sum_{k=1}^{\infty} \frac{r^k}{k^2} \frac{\sin kx}{\cos kx} = \frac{r}{\Gamma(s)_0} \int_0^{1-r} \frac{e^t \left(\frac{\sin x}{\cos x} \right) - r \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}}{1 - 2re^t \cos x + r^2 e^{2t}} dt$$

$$[x \neq 0, \operatorname{Re} s > 0] \text{ или } [x] \geq 0, \operatorname{Re} s > 1].$$

$$3. \sum_{k=0}^{\infty} r^k \frac{\sin kx}{\cos kx} = (1 - 2r \cos x + r^2)^{-1} \left[\pm r \frac{\sin x}{\cos x} + \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \right].$$

$$4. \sum_{k=0}^{\infty} r^{2k+1} \frac{\sin (2k+1)x}{\cos (2k+1)x} = r (1 \pm r^2) [(1+r^2)^3 - 4r^2 \cos^2 x]^{-1} \frac{\sin x}{\cos x}.$$

$$5. \sum_{k=0}^{\infty} (-1)^k r^{2k+1} \frac{\sin (2k+1)x}{\cos (2k+1)x} = r (1 \mp r^2) [(1 \mp r^2)^3 - 4r^2 \sin^2 x]^{-1} \frac{\sin x}{\cos x}.$$

$$6. \sum_{k=1}^{\infty} k r^k \frac{\sin kx}{\cos kx} = \frac{r}{(1 - 2r \cos x + r^2)^2} \left[(1-r^2) \frac{\sin x}{\cos x} - 2r (1-r \cos x) \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \right].$$

$$7. \sum_{k=0}^{\infty} \frac{r^k}{km+n} \frac{\sin kx}{\cos kx} =$$

$$= \pm \frac{1}{m r^{n/m}} \sum_{k=0}^{m-1} \frac{\cos [n(x+2k\pi)/m]}{\sin [n(x+2k\pi)/m]} \operatorname{arctg} \frac{r^{1/m} \sin [(x+2k\pi)/m]}{1 - r^{1/m} \cos [(x+2k\pi)/m]} \pm$$

$$\pm \frac{1}{2m r^{n/m}} \sum_{k=0}^{m-1} \frac{\sin [n(x+2k\pi)/m]}{\cos [n(x+2k\pi)/m]} \ln \left(1 - 2r^{1/m} \cos \frac{x+2k\pi}{m} + r^{2/m} \right) \pm$$

$$\mp \delta(n-m) \sum_{k=1}^{[(n-1)/m]} \frac{1}{r^k (km-n)} \frac{\sin kx}{\cos kx}$$

$$[\delta(n-m) = 1 \text{ при } n=m, \delta(n-m)=0 \text{ при } n \neq m; m>0; 0 < x < 2\pi].$$

$$8. \sum_{k=0}^{\infty} \frac{r^k}{km-n} \frac{\sin kx}{\cos kx} =$$

$$= \frac{r^{n/m}}{m} \sum_{k=0}^{m-1} \frac{\cos [n(x+2k\pi)/m]}{\sin [n(x+2k\pi)/m]} \operatorname{arctg} \frac{r^{1/m} \sin [(x+2k\pi)/m]}{1 - r^{1/m} \cos [(x+2k\pi)/m]} -$$

$$- \frac{r^{n/m}}{2m} \sum_{k=0}^{m-1} \frac{\sin [n(x+2k\pi)/m]}{\cos [n(x+2k\pi)/m]} \ln \left(1 - 2r^{1/m} \cos \frac{x+2k\pi}{m} + r^{2/m} \right) +$$

$$+ \sum_{k=0}^{[n/m]} \frac{r^k}{km-n} \frac{\sin kx}{\cos kx} \quad [n/m \neq 0, 1, 2, \dots; m>0; 0 < x < 2\pi].$$

$$9. \sum_{k=0}^{\infty} \frac{(-1)^k r^k}{km \pm n} \frac{\sin kx}{\cos kx} = \sum_{k=0}^{\infty} \frac{r^k}{km \pm n} \frac{\sin k(x+\pi)}{\cos k(x+\pi)} \quad [m>0; -\pi < x < \pi].$$

$$10. \sum_{k=1}^{\infty} \frac{(-1)^k r^k}{k+n} \sin kx = \frac{(-1)^k}{r^n} \left[\sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} r^{n-k}}{n-k} \sin kx + \right.$$

$$\left. + \cos nx \operatorname{arctg} \frac{r \sin x}{1 - r \cos x} \pm \frac{1}{2} \sin nx \ln (1 \pm 2r \cos x + r^2) \right].$$

$$11. \sum_{k=1}^{\infty} \frac{(-1)^k r^k}{k+n} \cos kx = \frac{(-1)^k}{r^n} \left[\sum_{k=1}^{n-1} \frac{(-1)^{n-k-1} r^{n-k}}{n-k+1} \cos (k-1)x + \right.$$

$$\left. + \sin nx \operatorname{arctg} \frac{r \sin x}{1 - r \cos x} \mp \frac{1}{2} \cos nx \ln (1 \pm 2r \cos x + r^2) \right].$$

12. $\sum_{k=1}^{\infty} \frac{r^k}{k} \sin kx = \operatorname{arctg} \left(\frac{r \sin x}{1 - r \cos x} \right).$
13. $\sum_{k=1}^{\infty} \frac{r^k}{k} \cos kx = -\frac{1}{2} \ln(1 - 2r \cos x + r^2).$
14. $\sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \frac{\sin kx}{\cos kx} = \frac{1}{2} \cos x \left\{ \operatorname{arctg} \left[\frac{2r}{1-r^2} \frac{\sin(x/2)}{\cos(x/2)} \right] \right\} +$
 $-\frac{1}{2} \sin x \left\{ \operatorname{Arth} \left[\frac{2r}{1-r^2} \frac{\cos(x/2)}{\sin(x/2)} \right] \right\}.$
15. $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1} \frac{\sin kx}{\cos kx} = \frac{1}{2} \cos x \left\{ \operatorname{arctg} \left[\frac{2r}{1-r^2} \frac{\sin(x/2)}{\cos(x/2)} \right] \right\} +$
 $-\frac{1}{2} \sin x \left\{ \operatorname{arctg} \left[\frac{2r}{1-r^2} \frac{\cos(x/2)}{\sin(x/2)} \right] \right\}.$
6. $\sum_{k=0}^{\infty} \frac{r^{2k+1}}{2k+1} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{2} \operatorname{arctg} \left[\frac{2r}{1-r^2} \frac{\sin x}{\cos x} \right].$
17. $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{2} \operatorname{arctg} \left[\frac{2r}{1-r^2} \frac{\sin x}{\cos x} \right].$
- 5.4.10. Ряды вида $\sum_{k=1}^{\infty} \frac{r^k}{k!} \frac{\sin(kx+a)}{\cos(kx+a)}.$
1. $\sum_{k=1}^{\infty} \frac{r^k}{k!} \frac{\sin(kx+a)}{\cos(kx+a)} = e^{r \cos x} \frac{\sin(r \sin x + a)}{\cos(r \sin x + a)}.$
2. $\sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} \frac{\sin kx}{\cos kx} = \frac{\operatorname{sh}[r \cos(x/2)] \sin[r \sin(x/2)]}{\operatorname{ch}[r \cos(x/2)] \cos[r \sin(x/2)]}.$
3. $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{(2k)!} \frac{\sin kx}{\cos kx} = \frac{\operatorname{sh}[r \sin(x/2)] \sin[r \cos(x/2)]}{\operatorname{ch}[r \sin(x/2)] \cos[r \cos(x/2)]}.$
4. $\sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \frac{\sin kx}{\cos kx} = \frac{\operatorname{ch}[r \cos(x/2)] \sin[r \sin(x/2)]}{\operatorname{sh}[r \cos(x/2)] \cos[r \sin(x/2)]} \cos \frac{x}{2} +$
 $-\frac{\operatorname{sh}[r \cos(x/2)] \cos[r \sin(x/2)]}{\operatorname{ch}[r \cos(x/2)] \sin[r \sin(x/2)]} \sin \frac{x}{2}.$
5. $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{(2k+1)!} \frac{\sin kx}{\cos kx} = \frac{\operatorname{sh}[r \sin(x/2)] \cos[r \cos(x/2)]}{\operatorname{ch}[r \sin(x/2)] \sin[r \cos(x/2)]} \cos \frac{x}{2} +$
 $-\frac{\operatorname{ch}[r \sin(x/2)] \sin[r \cos(x/2)]}{\operatorname{sh}[r \sin(x/2)] \cos[r \cos(x/2)]} \sin \frac{x}{2}.$

6. $\sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{\sin(r \sin x) \operatorname{ch}(r \cos x)}{\cos(r \sin x) \operatorname{sh}(r \cos x)}.$
7. $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{(2k+1)!} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{\cos(r \cos x) \operatorname{sh}(r \sin x)}{\sin(r \cos x) \operatorname{ch}(r \sin x)}.$
8. $\sum_{k=0}^{\infty} \frac{r^{2k}}{(n k)!} \frac{\sin kx}{\cos kx} = \frac{1}{n} \exp \left(r^{1/n} \cos \frac{x}{n} \right) \frac{\sin \left[r^{1/n} \sin \frac{x}{n} \right]}{\cos \left[r^{1/n} \sin \frac{x}{n} \right]} +$
 $-\frac{1}{2n} [1 + (-1)^n] \exp \left(-r^{1/n} \cos \frac{x}{n} \right) \frac{\sin \left[r^{1/n} \sin \frac{x}{n} \right]}{\cos \left[r^{1/n} \sin \frac{x}{n} \right]} +$
 $+\frac{1}{n} \sum_{k=1}^{(n-1)/2} \left[\exp \left(r^{1/n} \cos \frac{2k\pi+x}{n} \right) \frac{\sin \left(r^{1/n} \sin \frac{2k\pi+x}{n} \right)}{\cos \left(r^{1/n} \sin \frac{2k\pi+x}{n} \right)} + \right.$
 $\left. + \exp \left(r^{1/n} \cos \frac{2k\pi-x}{n} \right) \frac{\sin \left(r^{1/n} \sin \frac{2k\pi-x}{n} \right)}{\cos \left(r^{1/n} \sin \frac{2k\pi-x}{n} \right)} \right].$
9. $\sum_{k=0}^{\infty} \frac{r^{2k}}{(k!)^2} \frac{\sin kx}{\cos kx} = \frac{1}{2} \left\{ I_0(2re^{-ix/2}) + I_0(2re^{ix/2}) \right\}.$
10. $\sum_{k=0}^{\infty} \frac{r^{2k+1}}{k!(k+1)!} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{2} \left\{ I_1(2re^{-ix}) + I_1(2re^{ix}) \right\}.$
- 5.4.11. Ряды вида $\sum_{k=1}^{\infty} a_k q^{k-1} \frac{\sin kx}{\cos kx}.$
- Условие: $|q| < 1.$
1. $\sum_{k=1}^{\infty} (-1)^k q^{k-1} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{2} q^{-1/2} \frac{\theta_1(x, q)}{\theta_2(x, q)}.$
2. $\sum_{k=1}^{\infty} (\pm 1)^k q^{k-1} \cos kx = \frac{1}{2} + \frac{1}{2} \frac{\theta_3(x/2, q)}{\theta_4(x/2, q)}.$
- 5.4.12. Ряды вида $\sum_{k=1}^{\infty} a_k e^{-ka} \frac{\sin kx}{\cos kx}.$
1. $\sum_{k=1}^{\infty} (\pm 1)^k e^{-ka} \sin kx = -\frac{1}{2} \frac{\sin x}{\cos x - \operatorname{ch} a}.$
2. $\sum_{k=1}^{\infty} (\pm 1)^k e^{-ka} \cos kx = \frac{1}{2} \frac{\operatorname{sh} a}{\operatorname{ch} a \pm \cos x} - \frac{1}{2}.$
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$$3. \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)a} \sin(2k+1)x}{k} = \frac{\sin x}{\operatorname{ch} 2a + \cos 2x} \frac{\operatorname{sh} a}{\operatorname{sh} a}.$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)a} \cos(2k+1)x}{k} = \frac{\cos x}{\operatorname{ch} 2a + \cos 2x} \frac{\operatorname{sh} a}{\operatorname{ch} a}.$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-ka} \sin kx = -\operatorname{arctg} \frac{\sin x}{\cos x + e^a}.$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-ka} \cos kx = \frac{a}{2} - \frac{1}{2} \ln [2 (\operatorname{ch} a + \cos x)].$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-(2k+1)a} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \frac{1}{2} \operatorname{arctg} \left[\frac{1}{\operatorname{sh} a} \frac{\sin x}{\cos x} \right].$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-(2k+1)a} \frac{\sin(2k+1)x}{\cos(2k+1)x} = \\ = \frac{1}{4} \ln \left(\operatorname{ch} a + \frac{\sin x}{\cos x} \right) - \frac{1}{4} \ln \left(\operatorname{ch} a - \frac{\sin x}{\cos x} \right).$$

5.4.13. Ряды вида $\sum_{k=0}^{\infty} \frac{r^k}{k} \frac{\sin kx}{\cos kx}$.

$$1. \sum_{k=1}^{\infty} \frac{r^k}{k} \frac{\sin kx}{\cos kx} = \\ = \frac{1}{2^{2m}} \sum_{k=1}^m (-1)^{k-1} \binom{2m}{m-k} \ln(1 - 2r \cos kx + r^2) - \frac{1}{2^{2m}} \binom{2m}{m} \ln(1-r) \\ \quad (-1 \leq r \leq 1).$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^{k-1} r^k}{k} \sin^{2m+1} kx = \\ = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{m-k} \operatorname{arctg} \frac{r \sin(2k+1)x}{1 + r \cos(2k+1)x} \\ \quad (-1 \leq r \leq 1).$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^{k-1} r^k}{k} \cos^{2m+1} kx = \\ = \frac{1}{2^{2m+1}} \sum_{k=0}^m \binom{2m+1}{m-k} \ln [1 + 2r \cos(2k+1)x + r^2] \\ \quad (-1 \leq r \leq 1).$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k-1} r^k}{k} \frac{\sin kx}{\cos kx} = \\ = \frac{1}{2^{2m}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \ln(1 - 2r \cos 2kx + r^2) + \frac{1}{2^{2m}} \binom{2m}{m} \ln(1+r) \\ \quad (-1 \leq r \leq 1).$$

$$5. \sum_{k=0}^{\infty} \frac{r^k}{k!} \frac{\sin(kx+a)}{\cos(kx+a)} = \frac{(-1)^{[m/2]} 2^{1-m} m!}{k! (m-k)!} \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k!} \times \\ \times \exp [r \cos(m-2k)x] \cos [a \sin(m-2k)x] + (m-2k)a - \frac{\pi}{4} \frac{1 - (-1)^m}{0} \Big].$$

5.4.14. Ряды вида $\sum_{k=0}^{\infty} \frac{q^k}{k!} \frac{\sin kx}{\cos kx}$.

Обозначения: $q = e^{-\pi K'(k)/K(k)}$, $u = \frac{2K(k)x}{\pi}$, $Q_k = \prod_{n=0}^{\infty} (1 - q^{2n})$.

Условия: $|q| < 1$, $|k| < 1$.

$$1. \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{1 + q^{2k}} \sin 2kx = \frac{1}{4} \frac{\operatorname{ctg} x}{\operatorname{tg} x} + \frac{1}{2\pi} K(k) \frac{\operatorname{sn} u \operatorname{dn} u \operatorname{ns} u}{(1 - k^2) \operatorname{sn} u \operatorname{nc} u \operatorname{nd} u} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

$$2. \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + (-q)^n} \sin 2nx = \frac{1}{4} \frac{\operatorname{tg} x}{\operatorname{ctg} x} + \frac{1}{2\pi} K(k) \frac{\operatorname{sn} u \operatorname{dn} u \operatorname{nc} u}{\operatorname{cn} u \operatorname{nd} u \operatorname{ns} u} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{1 - q^{2k}} \sin 2kx = \frac{1}{4} \frac{\partial_1(x, q)}{\partial_2(x, q)} + \frac{\partial}{\partial x} \frac{\partial_2(x, q)}{\partial_1(x, q)} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-k \cdot q)].$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{1 + q^{2k}} \cos 2kx = \frac{1}{2\pi} K(k) \left\{ \frac{\operatorname{dn} u}{1 - k^2 \operatorname{nd} u} \right\} - \frac{1}{4} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{1 + q^{2k}} \sin 2kx = \frac{1}{4} \frac{\operatorname{ctg} x}{\operatorname{tg} x} + \frac{1}{2\pi} K(k) \left\{ \frac{\operatorname{cn} u}{1 - k^2 \operatorname{sc} u} \right\} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

$$6. \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{1 - q^{2k}} \sin 2kx = \frac{1}{4} \frac{\partial_1(x, q)}{\partial_2(x, q)} + \frac{\partial}{\partial x} \frac{\partial_2(x, q)}{\partial_1(x, q)} + \frac{1}{4} \frac{\operatorname{ctg} x}{\operatorname{tg} x} \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

$$7. \sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}} \cos 2kx = \frac{1}{2\pi^2} K^2(k) (1 - k^2 \operatorname{sn}^2 u) - \frac{1}{2\pi^2} K(k) E(k) \\ \quad [2 \operatorname{Im} \tau < \operatorname{Re}(-\ln q)].$$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1-q^{2n}} \cos 2nx = \frac{1}{2\pi^2} \{K(k) [K(k) - E(k)] - [kK(k) \operatorname{sn} u \operatorname{nd} u]^2\}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
9. $\sum_{n=1}^{\infty} \frac{n q^{2n}}{1-q^{2n}} \cos 2nx = \frac{1}{8} \operatorname{cosec}^2 x + \frac{1}{2\pi^2} K(k) \{K(k) [1 - \operatorname{ns}^2 u] - E(k)\}$
 $[1: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
10. $\sum_{n=1}^{\infty} \frac{q^n}{n(1+q^{2n})} \sin 2nx = \frac{1}{2} \operatorname{am} u - \frac{x}{2}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
11. $\sum_{n=1}^{\infty} \frac{q^n}{n(1+q^n)} \cos 2nx = \frac{1}{2} \ln \left(\frac{1}{2} k^{1/2} q^{-1/4} \operatorname{cosec} x \operatorname{sn} u \right)$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
12. $\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n [1 + (-1)^n q^n]} \cos 2nx =$
 $= \frac{1}{2} \ln \left(\frac{\operatorname{sn} u}{\operatorname{cn} u} \right) - \frac{1}{2} \ln \left(\frac{\sin x}{\sqrt{1-k^2} \cos x} \right) - \frac{1}{2} \ln (2q^{1/4} k^{-1/2})$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
13. $\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n(1-q^{2n})} \cos 2nx = \frac{1}{2} \ln Q_0 - \frac{1}{2} \ln \left\{ \frac{\theta_2(x/n)}{\theta_2(x/n)} \right\}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
14. $\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{n(1-q^{2n})} \cos 2nx = \frac{1}{2} \ln 2Q_0 + \frac{1}{8} \ln q + \frac{1}{2} \ln \left(\frac{\ln \sin x - \ln \theta_1(x/n)}{\ln \cos x - \ln \theta_2(x/n)} \right)$
 $[1: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
15. $\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^{2n+1}} \sin (2n+1)x = \frac{kK(k)}{2\pi \sqrt{q}} \left\{ \frac{\sin u}{\sqrt{1-l^2} \operatorname{sd} u} \right\}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
16. $\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1+q^{2n+1}} \cos (2n+1)x = \frac{kK(k)}{2\pi \sqrt{q}} \left\{ \operatorname{cd} u \right\}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
17. $\sum_{n=1}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} \sin (2n+1)x = \pm \frac{1}{4} \operatorname{cosec} x + \frac{1}{2\pi} K(k) \left\{ \operatorname{ds} u \right\}$
 $[1: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$
18. $\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1+q^{2n+1}} \cos (2n+1)x = \pm \frac{1}{4} \sec x + \frac{1}{2\pi} K(k) \left\{ \frac{\sqrt{1-k^2} \operatorname{nc} u}{\operatorname{dc} u} \right\}$
 $[2: | \operatorname{Im} x | < \operatorname{Re}(-\ln q)].$

5.4.15. Ряды вида $\sum a_k \prod \sin kx, \prod \cos ky.$

1. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx \sin ky = \frac{1}{4} \ln \left\{ \frac{\sin [(x+y)/2] \operatorname{cosec} [(x-y)/2]}{\cos [(x+y)/2] \sec [(x-y)/2]} \right\}$
 $\left[\begin{matrix} x \pm y \neq 0, 2\pi, 4\pi, \dots \\ x \pm y \neq \pi, 3\pi, 5\pi, \dots \end{matrix} \right]$
2. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cos kx \cos ky = -\frac{1}{4} \ln [4 (\cos x + \cos y)^2]$
 $\left[\begin{matrix} x \pm y \neq 0, 2\pi, 4\pi, \dots \\ x \pm y \neq \pi, 3\pi, 5\pi, \dots \end{matrix} \right]$
3. $\sum_{k=1}^{\infty} \frac{1}{k} \sin kx \cos ky = \begin{cases} -1/2, & 0 < x < y, \\ (\pi - 2y)/4, & x = y, \\ (\pi - x)/2, & y < x < \pi \end{cases}$
 $[0 < y < \pi].$
4. $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin kx \sin ky = \begin{cases} x(\pi - y)/2, & -y \leq x \leq y, \\ y(\pi - x)/2, & y \leq x \leq 2\pi - y \end{cases}$
 $[0 < y < \pi].$
5. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \sin kx \sin ky = \frac{\pi y}{2}$
 $[|x \pm y| \leq \pi].$
6. $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kx \cos ky = \frac{1}{12} \begin{cases} 3x^2 + 3(y - \pi)^2 - \pi^2, & 0 \leq x \leq y, \\ 3y^2 + 3(x - \pi)^2 - \pi^2, & y \leq x \leq \pi \end{cases}$
 $[0 \leq y < \pi].$
7. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \cos ky = \frac{1}{12} \begin{cases} 3(x^2 + y^2) - \pi^2, & -(\pi - y) \leq x \leq \pi - y, \\ 3(x - \pi)^2 + 3(y - \pi)^2 - \pi^2, & \pi - y \leq x \leq \pi + y \end{cases}$
 $[0 < y < \pi].$
8. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \sin kx \cos ky = \frac{x}{12} (\pi^2 - x^2 - 3y^2)$
 $[|x \pm y| \leq \pi].$
9. $\sum_{k=1}^{\infty} \frac{a^k}{k} \sin kx \sin ky = \frac{1}{4} \ln \frac{4a \sin^2 [(x+y)/2] + (a-1)^2}{4a \sin^2 [(x-y)/2] + (a-1)^2}$
 $[0 < a < 1].$
10. $\sum_{k=1}^{\infty} \frac{(-1)^k q^k}{k(1-q^{2k})} \sin kx \sin ky = \frac{1}{4} \ln \frac{\theta_1[(x+y)/2, q]}{\theta_1[(x-y)/2, q]}$
 $\left[i = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right].$
11. $\sum_{k=1}^{\infty} \frac{(-1)^k q^{2k}}{k(1-q^{2k})} \sin kx \sin ky =$
 $= \frac{1}{4} \ln \frac{\theta_1[(x+y)/2, q]}{\theta_1[(x-y)/2, q]} - \frac{1}{4} \ln \left\{ \frac{\sin [(x+y)/2] \operatorname{cosec} [(x-y)/2]}{\cos [(x+y)/2] \sec [(x-y)/2]} \right\}$
 $\left[i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; | \operatorname{Im} x | + | \operatorname{Im} y | < -\frac{1}{2} \operatorname{Re} \ln q \right].$

$$12. \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \sin^2 ky = \begin{cases} \pi/4, & 0 < x < 2y, \\ 0, & 2y < x < \pi \end{cases} \quad [0 < x; y < \pi]$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin ky \sin ky \cos kz = \frac{1}{4} \begin{cases} 2(\pi-y)x, & 0 < x < y-z, \\ (x+y-z)\pi - 2xy, & y-z < x < y+z, \\ 2(\pi-x)y, & y+z < x < \pi \end{cases} \quad [0 < z < y < \pi/2]$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \sin kx \sin ky \sin kz = \frac{\pi y z}{2}.$$

[Точка (x, y, z) лежит внутри многогранника $|x|+|y|+|z| < \pi$].

$$15. \sum_{k=1}^{\infty} \frac{1}{k^3} \sin^2 kx \sin^2 ky = \frac{\pi x}{2} \quad [0 \leq x \leq y \leq \pi/2]$$

$$16. \sum_{k=1}^{\infty} \frac{1}{k^3} \sin^4 kx \sin^2 ky = \frac{\pi x}{8} \quad [0 \leq y \leq 2x \leq \pi]$$

$$17. \sum_{k=1}^{\infty} \frac{1}{k^3} \sin^4 kx \sin^2 ky = \frac{\pi x^2}{6}.$$

$$18. \sum_{k=1}^{\infty} \frac{1}{k^n} \sin^n kx = -\frac{\pi^n}{n!} \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{k} B_n \left[(n-2k) \frac{x}{2\pi} \right] - \\ - \frac{1}{4} (-1)^{n/2} [1 + (-1)^n] \left[\left(\frac{n}{2} \right)! \right]^{-2} \pi^n B_n \\ [0 < x < 2\pi \text{ при } n=1; 0 \leq x \leq 2\pi/n \text{ при } n>1]$$

$$19. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin^{n+2m} kx = \frac{\pi^n}{2} \delta_{m,0} \quad [|x| < \pi(2m+n); n=1, 2, 3, \dots]$$

$$20. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \prod_{j=1}^{n+2m} \sin kx_j = \frac{1}{2} \delta_{m,0} \prod_{j=1}^n x_j \\ \left[n \geq 1; \sum_{j=1}^n |x_j| < \pi \text{ при } m=0; \sum_{j=1}^{n+2m} |x_j| \leq \pi \text{ при } m>0 \right]$$

$$21. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \left(\prod_{j=1}^{n+2m} \sin kx_j \right) \left(\prod_{l=1}^p \cos ky_l \right) = \frac{1}{2} \delta_{m,0} \prod_{j=1}^n x_j \\ \left[n \geq 1; \sum_{j=1}^{n+2m} |x_j| + \sum_{l=1}^p |y_l| < \pi \right]$$

5.4.16. Разные ряды, содержащие тригонометрические функции.

$$1. \sum_{k=1}^{\infty} 3^k \sin^2 \frac{x}{3^k} = \frac{3}{4} (x - \sin x).$$

$$2. \sum_{k=1}^{\infty} 2^{2k} \sin^4 \frac{x}{2^k} = x^2 - \sin^2 x.$$

$$3. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \sin \frac{m\pi}{2k+1} \right) x^k = \frac{x(12-x)}{(4-x)^2} \quad [|x| < 4]$$

$$4. \sum_{k=1}^{\infty} \frac{(-1)^k}{3^k} \cos^3 3^k x = \frac{3}{4} \cos x.$$

$$5. \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}} \cos^4 2^k x = \frac{1}{20} - \frac{1}{4} \cos^2 2x.$$

$$6. \sum_{k=1}^{\infty} \frac{\cos kx \sec^k x}{2^k k^2} = \frac{\pi^2}{12} - \frac{x^2}{2} - \frac{1}{2} \ln^2 (2 \cos x) \quad [-\pi/3 < x < \pi/3]$$

$$7. \sum_{k=1}^{\infty} \frac{2^k \cos^k x}{k^2} \cos kx = \left(x - \frac{\pi}{2} \right)^2 \quad [\pi/3 < x < 2\pi/3]$$

$$8. \sum_{k=1}^{\infty} \prod_{m=1}^{2k} \cos^m \frac{m\pi}{2k+1} = \frac{1}{2^{2k-1}} [1 + (-1)^k 2^{2k}].$$

$$9. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \cos \frac{m\pi}{2k+1} \right) x^k = -\frac{x}{4+x} \quad [|x| < 4]$$

$$10. \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \sec^2 \frac{x}{2^k} = \cot^2 x - \frac{1}{x^2}.$$

$$11. \sum_{n=-\infty}^{\infty} (\pm 1)^n \operatorname{cosec} (nx+a) = \pm \frac{2k}{\pi} K(k) \int_{\operatorname{arctan} u}^{\operatorname{arctan} u} \frac{du}{1 \mp u^2} \\ [x = -i\pi K'(k)/K(k), u = iK'(k) - 2\pi K(k)/\pi; | \operatorname{Im} (u/K(k)) | < \operatorname{Re} (K'(k)/K(k))]$$

$$12. \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \operatorname{cosec} (k\pi \sqrt{2}) =$$

$$= \frac{(-1)^n \pi^{2n+1}}{2(\sqrt{2}-1)[1+(\sqrt{2}-1)^{2n}]} \sum_{k=0}^{n-1} \frac{(2^{2k}-2)(2^{2n-2k+1}-2)}{(2k)!(2n-2k+2)!} (\sqrt{2}-1)^{2k} B_{2k} B_{2n-2k+2} \\ [n=1, 2, 3, \dots]$$

$$13. \sum_{k=1}^{\infty} \frac{1}{k^2} \operatorname{cosec} \left(k\pi \frac{1}{2} \right) = -\frac{13\pi^3}{360 \cdot 2}.$$

$$14. \sum_{k=0}^{\infty} \frac{1}{2^k} \lg \frac{x}{2^k} = \frac{1}{x} - \operatorname{ctg} x.$$

$$15. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \lg \frac{2k+1}{2} \cdot \pi \sqrt{3} = \frac{\pi^3}{96} (3-2\sqrt{3}).$$

$$16. \sum_{k=0}^{\infty} \frac{1}{2^k} \lg^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{x^2} - \frac{1}{3}.$$

$$17. \sum_{k=1}^{\infty} \left(\prod_{m=1}^{2k} \lg \frac{m\pi}{2k+1} \right) x^k = -\frac{x(3+x)}{(1+x)^3} \quad [x < 1].$$

$$18. \sum_{l=-\infty}^{\infty} (-1)^l \operatorname{ctg} (l x + a) = \frac{2}{\pi i} K(k) \operatorname{dn} u \quad [\text{см. условие к 5.4.16.11}].$$

$$19. \sum_{k=1}^{\infty} \frac{1}{k^3} \operatorname{ctg} \left(\frac{1+\sqrt{5}}{2} k\pi \right) = -\frac{\pi^3}{45 \cdot 5}.$$

$$20. \sum_{k=1}^{\infty} \prod_{m=1}^{2k} \operatorname{ctg} \frac{m\pi}{2k+1} = \frac{\pi}{4} - 1.$$

$$21. \sum_{k=1}^{\infty} \frac{1}{k} \cos kx \lg \frac{k\pi}{2n+1} = \frac{\pi}{2} \begin{cases} -1/(2n+1), & x \text{ кратно } 2\pi/(2n+1), \\ (-1)^m - 1/(2n+1), & x \text{ не кратно } 2\pi/(2n+1) \end{cases}$$

[m — целая часть $(2n+1)x/(2\pi)$; $n \geq 1$; $0 < x < 2\pi$].

5.5. РЯДЫ С ЛОГАРИФИЧЕСКОЙ И ОБРАТНЫМИ ТРИГОНОМЕТРИЧЕСКИМИ ФУНКЦИЯМИ

5.5.1. Ряды вида $\sum a_k \ln b_k$.

$$1. \sum_{k=1}^{\infty} \frac{1}{k^s} \ln ka = \zeta'_s(s) \ln a - \zeta'_s(s).$$

$$2. \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} \ln k = 2^{1-s} \zeta'_s(s) \ln 2 + (1-2^{1-s}) \zeta'_s(s).$$

$$3. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \ln k = C \ln 2 - \frac{\ln^2 2}{2}, \quad 4. \sum_{k=2}^{\infty} \frac{(-1)^k}{k+1} \ln k = 0,096015.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \ln (2k+1) = -2^{-s} \zeta'_s(s) \ln 2 - (1-2^{-s}) \zeta'_s(s).$$

$$6. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \ln (2k+1) a = \frac{\pi}{4} \ln a - \frac{\pi C}{4} - \frac{1}{4} \int_0^{\infty} \ln t \operatorname{sech} t \, dt.$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \ln [(2k+1)^2 a^2 + 1] = \frac{\pi}{2} \int_1^{\infty} \frac{1}{t} \operatorname{sech} \frac{t\pi}{2a} \, dt.$$

$$8. \sum_{k=1}^{\infty} \frac{1}{k} x^k \ln (k+1) = - \int_0^1 \ln \left(\frac{1-xt}{1-x} \right) \frac{dt}{\ln t} \quad [x < 1].$$

$$9. \sum_{k=1}^{\infty} \frac{1}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^{a-1} - t^{b-1}) \ln (1+xt) \frac{dt}{\ln t}.$$

$$10. \sum_{k=1}^{\infty} (-1)^k \binom{n}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^{a-1} - t^{b-1}) (1-xt)^n \frac{dt}{\ln t}.$$

$$11. \sum_{k=1}^{\infty} \binom{n+k-1}{k} x^k \ln \frac{k+a}{k+b} = - \ln \frac{a}{b} + \int_0^1 (t^a - t^b) (1-xt)^{-n} \frac{dt}{t \ln t}.$$

$$12. \sum_{k=1}^{\infty} \frac{\Gamma(k+a)}{k!} \ln (k+1) = \Gamma(a) \int_0^1 (1-t)^{-a} \frac{dt}{\ln t} \quad [a < -1].$$

$$13. \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k! 2^k (2k+1)} \ln (k+1) a = \frac{\pi}{2} \ln \frac{a}{2} - \int_0^1 \frac{\arccos t}{\ln t} \, dt.$$

$$14. \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}} \ln (2k+1) = \frac{1}{8} (\pi - 2C - 2 \ln 2) \left[\zeta \left(\frac{1}{2}, \frac{1}{4} \right) - \zeta \left(\frac{1}{2}, \frac{3}{4} \right) \right].$$

$$15. \sum_{k=1}^{\infty} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = Ca + \ln \Gamma(a+1).$$

$$16. \sum_{k=-\infty}^{\infty} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = \ln (\pi a \operatorname{cosec} \pi a).$$

$$17. \sum_{k=-\infty}^{\infty} (-1)^{k+1} \left(\frac{a}{k} - \ln \frac{k+a}{k} \right) = \ln \left(\frac{2}{\pi a} \lg \frac{a\pi}{2} \right).$$

$$18. \sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln \frac{k+a+1}{k+a} \right) = C + \ln(a+1),$$

$$19. \sum_{k=0}^{\infty} \left(\frac{1}{k+a} - \ln \frac{k+a+1}{k+a} \right) = \ln a - \psi(a),$$

$$20. \sum_{k=1}^{\infty} \left(\frac{1}{k} - 2 \ln \frac{2k+2}{2k+1} \right) = C - \ln \frac{\pi}{4},$$

$$21. \sum_{k=0}^{\infty} \left[1 - (k+1) \ln \frac{2k+3}{2k+1} \right] = \frac{1}{2} (\ln 2 - 1),$$

$$22. \sum_{k=0}^{\infty} [2 - (2k+2a+1)] \ln \frac{k+a+1}{k+a} = \ln 2\pi + (2a-1) \ln a - 2 \ln \Gamma(a) - 2a.$$

$$23. \sum_{k=0}^{\infty} (\pm 1)^k \left[\frac{1}{2k-a+1} \ln(2k-a+1) \pi \mp \frac{1}{2k+a+1} \ln(2k+a+1) \pi \right] = \\ = \frac{\pi}{2} (\ln \pi - C) \left\{ \frac{\operatorname{tg}(\pi a/2)}{\sec(\pi a/2)} \right\} - \int_0^{\pi/4} (\ln \ln \operatorname{ctg} t) \left\{ \frac{\sec 2t}{1} \right\} [\operatorname{ctg}^a t \mp \operatorname{tg}^a t] dt,$$

$$24. \sum_{k=1}^{\infty} \frac{1}{k} \sin kx \ln ka = \\ = \frac{1}{2} (x - \pi) \left(C - \ln \frac{a}{2\pi} \right) + \frac{\pi}{2} \ln \left| \frac{1}{\pi} \sin \frac{x}{2} \right| + \pi \ln \Gamma \left(\frac{x}{2\pi} \right) \quad [0 < x < 2\pi],$$

$$25. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin kx \ln ka = \frac{\pi}{2} \left(C - \ln \frac{a}{2\pi} \right) + \frac{\pi}{2} \ln \Gamma \left\{ \frac{(\pi+x)/(2\pi)}{(1-x)/(2\pi)} \right\} \\ [-\pi < x < \pi],$$

$$26. \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x \ln(2k+1)a = \\ = \frac{\pi}{4} \left(\ln \frac{a}{4\pi} - C + \ln \operatorname{tg} \frac{x}{2} - 2 \ln \Gamma \left\{ \frac{(\pi+x)/(2\pi)}{x/(2\pi)} \right\} \right) \quad [0 < x < \pi],$$

$$27. \sum_{k=0}^{\infty} \sin(2k+1)x \ln \frac{k}{k+1} = \frac{\pi}{2} \cos x + \left[\psi \left(\frac{x}{\pi} \right) + C + \ln 2\pi \right] \sin x \\ [0 < x < \pi],$$

5.5.2. Ряды вида $\sum a_k \operatorname{arctg} b_k$.

$$1. \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \operatorname{arctg} ka = -\frac{a}{2} + \frac{\pi}{2} \int_1^{\infty} \frac{1}{t} \operatorname{cosech} \frac{t\pi}{a} dt.$$

$$2. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{a}{2k+1} = \frac{1}{2} \operatorname{arctg} \left(\operatorname{sh} \frac{a\pi}{2} \right).$$

$$3. \sum_{k=0}^{\infty} \left[\operatorname{arctg} \left(\frac{a}{k+b} \right) - \frac{a}{k+b} \right] = a\psi(b) - \arg \Gamma(b+ia) \\ [b+ia \neq 0, -1, -2, \dots],$$

$$4. \sum_{k=-\infty}^{\infty} (\pm 1)^k \left[\operatorname{arctg} \left(\frac{a}{k+b} \right) - \frac{a}{k} \right] = \left\{ \operatorname{arctg} (\operatorname{th} a\pi \operatorname{ctg} b\pi) \right\} - \operatorname{arctg} \frac{a}{b},$$

$$5. \sum_{k=1}^{\infty} (\pm 1)^k \operatorname{arctg} \frac{2ab}{k^2+a^2-b^2} = \operatorname{arctg} \frac{a}{b} - \left\{ \operatorname{arctg} (\operatorname{th} a\pi \operatorname{ctg} b\pi) \right\} - \left\{ \operatorname{arctg} (\operatorname{sh} a\pi \operatorname{cosec} b\pi) \right\},$$

$$6. \sum_{k=0}^{\infty} \operatorname{arctg} \frac{a}{k^2+(2b+1)k+a^2+b^2+b} = \operatorname{arctg} \frac{a}{b}.$$

$$7. \sum_{k=1}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+1)a}{k^2+l-a^2} = \operatorname{arctg} a - \pi.$$

$$8. \sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{arctg} \frac{4ka}{4k^2-a^2-1} = \operatorname{arctg} a.$$

$$9. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+2b+1)a}{k^2+(2b+1)k+b^2+b-a^2} = \operatorname{arctg} \frac{a}{b}.$$

$$10. \sum_{k=1}^{\infty} \operatorname{arctg} \frac{ac}{(k+b)^2-c(k+b)+a^2} = \arg [\Gamma(b+ia) \Gamma(b-c+1-ia)].$$

$$11. \sum_{k=1}^{\infty} (\pm 1)^k \operatorname{arctg} \frac{2}{k^2} = \pm \frac{\pi}{4} \left\{ \frac{3}{1} \right\}.$$

$$12. \sum_{k=1}^{\infty} \operatorname{arctg} \frac{1}{k^2+l+1} = \frac{\pi}{4}.$$

$$13. \sum_{k=0}^{\infty} \operatorname{arctg} \frac{2ab}{(2k+1)^2+a^2-b^2} = \operatorname{arctg} (\operatorname{th} a\pi \operatorname{tg} b\pi).$$

$$14. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} \frac{(2k+1)a}{(2k+1)^2 - a^2} = \operatorname{arctg} \left(\operatorname{sh} \frac{a\pi}{4} \sec \frac{\sqrt{3}a\pi}{4} \right).$$

$$15. \sum_{k=0}^{\infty} (-1)^k (2k+1) \operatorname{arctg} \frac{8n^2}{(2k+1)^2} = 4n (-1)^{n-1} \operatorname{arctg} (e^{-n\pi}).$$

$$16. \sum_{k=0}^{\infty} (-1)^k \operatorname{arctg} q^{n+1,2} = \frac{1}{4} \arcsin k \quad \left[|q| < 1; q = \exp \left\{ -\frac{\pi K'(k)}{K(k)} \right\} \right].$$

5.6. КРАТНЫЕ РЯДЫ

$$1. \sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2l)^k} = \ln 2. \quad 2. \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \frac{1}{(l+a)^k} = \frac{1}{1+a}.$$

$$3. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-1)^{2k}} = \frac{1}{4} \ln 2.$$

$$4. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-1)^{2k+1}} = \frac{\pi}{8} - \frac{1}{2} \ln 2.$$

$$5. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(4l-2)^{2k}} = \frac{\pi}{8}.$$

$$6. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{k+l}}{(k+l)^2} kl = \frac{1}{6} \left(\ln 2 - \frac{1}{4} \right).$$

$$7. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} x^{k+l} = \frac{1}{1-2x} \quad (-1 < x < 1/2).$$

$$8. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(k+l+m)!}{k!l!m!} x^{k+l+m} = \frac{1}{1-3x} \quad (-1 < x < 1/3).$$

$$9. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{k^2(k^2 - kl + l^2)} = \frac{\pi^2 \sqrt{3}}{30}.$$

$$10. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{k^6(k^2 + l^2)} = \frac{13\pi^6}{28350}.$$

$$11. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k+1)^2[(2k+1)^2 + (2l+1)^2]} = \frac{\pi^4}{32}.$$

$$12. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{(2k+1)^2[(2k+1)^2 + (2l+1)^2]} = \frac{7\pi^6}{23040}.$$

$$13. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{l^2 - k^2}{(l^2 + k^2)^2} = \frac{\pi}{4}. \quad 14. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l-1)^2 - (2k-1)^2}{[(2l-1)^2 + (2k-1)^2]^2} = \frac{\pi}{16}.$$

$$15. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l-1)^2 - (2k)^2}{[(2l-1)^2 + (2k)^2]^2} = \frac{1}{16} \left[\pi + \pi^2 - \frac{1}{8\pi} \Gamma^2 \left(\frac{1}{4} \right) \right].$$

$$16. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(2l)^2 - (2k-1)^2}{[(2l)^2 + (2k-1)^2]^2} = \frac{1}{16} \left[\pi + \pi^2 + \frac{1}{8\pi} \Gamma^2 \left(\frac{1}{4} \right) \right].$$

$$17. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l} (\beta)_k (\beta')_l}{(\gamma)_{k+l} k!l!} x^k y^l = F_3(\alpha, \beta, \beta', \gamma; x, y) \quad \left[|x|, |y| < 1 \right].$$

$$18. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l} (\beta)_k (\beta')_l}{(\gamma)_k (\gamma')_l k!l!} x^k y^l = F_3(\alpha, \beta, \beta', \gamma, \gamma'; x, y) \quad \left[|x| + |y| < 1 \right].$$

$$19. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_k (\alpha')_l (\beta)_k (\beta')_l}{(\gamma)_{k+l} k!l!} x^k y^l = F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) \quad \left[|x|, |y| < 1 \right].$$

$$20. \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_{k+l} (\beta)_{k+l} (\beta')_l}{(\gamma)_k (\gamma')_l k!l!} x^k y^l = F_3(\alpha, \beta, \gamma, \gamma'; x, y) \quad \left[|x| + |y| < 1 \right].$$

Другие ряды вида (17)–(20) см. в [1], т. 1, стр. 219.

$$21. \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(k)}{\Gamma(2l+k+2)} x^{2l+k} = Si(x).$$

$$22. \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \binom{k+l-1}{k} \binom{k+l-1}{l} \frac{x^k y^l}{k+l-1} = \frac{1}{2} [1 - x - y - \sqrt{1 - 2(x+y) + (x-y)^2}].$$

$$23. \sum_{1 \leq k_1 < k_2 < \dots < k_n} \frac{1}{k_1 k_2 \dots k_n} = \frac{\pi^{2n}}{(2n+1)!}.$$

$$24. \sum_{k_1, k_2, \dots, k_n=0}^{\infty} \frac{(-1)^{k_1+k_2+\dots+k_n}}{k_1! k_2! \dots k_n! \Gamma(k_1+k_2+\dots+k_n+v+1)} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} = (x_1+x_2+\dots+x_n)^{-v-2} J_v(2\sqrt{x_1+x_2+\dots+x_n}).$$

Глава 6. ПРОИЗВЕДЕНИЯ

6.1. КОНЕЧНЫЕ ПРОИЗВЕДЕНИЯ

6.1.1. Степенная и алгебраические функции.

1. $\prod_{k=1}^n (k+a)^s = [(a+1)_n]^s$. 2. $\prod_{k=1}^n k^s = (n!)^s$.
 3. $\prod_{k=1}^n (2k+1)^s = 2^{-ns} \left[\frac{(2n+1)!}{n!} \right]^s$. 4. $\prod_{k=1}^n \left(1 + \frac{a}{k+b} \right) = \frac{(a+b+1)_n}{(b+1)_n}$.
 5. $\prod_{k=1}^n \left(1 - \frac{a^2}{k^2} \right) = \frac{1}{(n!)^2} (1+a)_n (1-a)_n$.
 6. $\prod_{k=1}^n \left[1 - \frac{a^2}{(2k+1)^2} \right] = \left[\left(\frac{3}{2} \right)_n \right]^{-2} \left(\frac{3+a}{2} \right)_n \left(\frac{3-a}{2} \right)_n$.
 7. $\prod_{k=2}^n \frac{k^3-1}{k^3+1} = \frac{2}{3} \left(1 + \frac{1}{n^2+n} \right)$.
 8. $\prod_{k=1}^n (k+a)^k = \Gamma^n(n+a+1) \prod_{k=1}^n \frac{1}{\Gamma(k+a)}$.
 9. $\prod_{k=0}^{n-1} (1+x^{2k}) = \frac{1-x^{2n}}{1-x}$. 10. $\prod_{k=1}^n (a - e^{2\pi k i / n}) = a^n - 1$.
- 6.1.2. Тригонометрические функции.
1. $\prod_{k=0}^{n-1} \frac{\sin [x + (k\pi/n)]}{\cos [x + (k\pi/n)]} = 2^{1-n} \frac{\sin nx}{\sin [n(2x+\pi)/2]}$.
 2. $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = 2^{1-n} n$.

3. $\prod_{k=1}^{[(n-1)/2]} \frac{\sin (k\pi/n)}{\cos (k\pi/n)} = 2^{1-n/2} \frac{n^{1/2}}{\left\{ \frac{1+(-1)^n}{2} n^{1/2} + \frac{1-(-1)^n}{2} \right\}}$.
4. $\prod_{k=0}^{n-1} \frac{\sin [x + 2\pi k/n]}{\cos [x + 2\pi k/n]} = (-2)^{1-n} \frac{\cos [n(2x-\pi)/2] - \cos (n\pi/2)}{-\cos (n\pi/2) + \cos nx}$.
5. $\prod_{k=0}^{[(n-1)/2]} \frac{\sin [(2k+1)\pi/(2n)]}{\cos [(2k+1)\pi/(2n)]} = 2^{1-n/2} \frac{1}{\left\{ \frac{1-(-1)^n}{2} n^{1/2} + \frac{1+(-1)^n}{2} \right\}}$.
6. $\prod_{k=1}^{2n-1} \cos \frac{k\pi}{n} = 2^{1-2n} [(-1)^n - 1]$.
7. $\prod_{k=0}^{n-1} \left[a^2 - 2ab \cos \left(x + \frac{2\pi k}{n} \right) + b^2 \right] = a^{2n} - 2a^n b^n \cos nx + b^{2n}$.
8. $\prod_{k=1}^{n-1} \left(a^2 - 2a \cos \frac{\pi k}{n} + 1 \right) = \frac{a^{2n} - 1}{a^2 - 1}$.
9. $\prod_{k=1}^n \left(a^2 - 2a \cos \frac{2\pi k}{2n+1} + 1 \right) = \frac{a^{2n+1} - 1}{a^2 - 1}$.

6.2. БЕСКОНЕЧНЫЕ ПРОИЗВЕДЕНИЯ

6.2.1. Степенная и алгебраические функции.

1. $\prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right) = \infty$. 2. $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k} \right) = 0$.
3. $\prod_{k=1}^{\infty} \left[1 + (-1)^k \frac{a}{k+b} \right] = 2^{-b} \Gamma^n \left[\frac{b+1}{(a+b+2)/2}, \frac{(b-a+1)/2} \right]$.
4. $\prod_{k=1}^{\infty} \left[1 + (-1)^k \frac{a}{2k+1} \right] = \frac{\sqrt{2}}{a+1} \sin \frac{(a+1)\pi}{2}$.
5. $\prod_{k=1}^{\infty} \left[1 - \frac{a^2}{(k+b)^2} \right] = \Gamma \left[\frac{b+1}{b+a+1}, \frac{b-a+1}{b-a+1} \right]$.
6. $\prod_{k=1}^{\infty} \left(1 \pm \frac{a^k}{k^2} \right) = \frac{1}{\pi a} \left\{ \frac{\operatorname{sh} \pi a}{\sin \pi a} \right\}$. 7. $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2} \right) = \frac{1}{2}$.

$$8. \prod_{k=0}^{\infty} \left[1 \pm \frac{a^2}{(2k+1)^2} \right] = \left\{ \frac{\operatorname{ch}(\pi a/2)}{\cos(\pi a/2)} \right\}. \quad 9. \prod_{k=1}^{\infty} \left[1 - \frac{1}{(2k+1)^2} \right] = \frac{\pi}{4}.$$

$$10. \prod_{k=-\infty}^{\infty} \left[1 \pm \frac{a^2}{(k+b)^2} \right] = 1 \pm \operatorname{cosec}^2 \pi b \left\{ \frac{\operatorname{sh} \pi a}{\sin \pi a} \right\}^2.$$

$$11. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k+b} \right) \left(1 - \frac{a}{k+c} \right) = \Gamma \left[\frac{b+1}{a+b+1}, \frac{c+1}{c-a+1} \right].$$

$$12. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k+b} \right) \left(1 - \frac{a}{k-b} \right) = \frac{b}{a+b} \sin(a+b) \pi \operatorname{cosec} \pi a.$$

$$13. \prod_{k=1}^{\infty} \left(1 \pm \frac{1}{k^2} \right) = \frac{1}{3\pi} \left\{ \frac{3}{1} \right\} \operatorname{ch} \frac{\pi \sqrt{3}}{2}.$$

$$14. \prod_{k=1}^{\infty} \left(1 + \frac{a}{k} \right) \left(1 + \frac{b+1}{k+c} \right) \left(1 - \frac{a+b+1}{k+b} \right) = -\frac{\sin \pi a}{\pi} \Gamma \left[\frac{b, c}{b+c+2} \right].$$

$$15. \prod_{k=0}^{\infty} \left(1 - \frac{a}{2k+1} \right) \left(1 - \frac{a}{2k+2} \right) \left(1 + \frac{a}{k+1} \right) = 2^{-a} \frac{\sin \pi a}{\pi a}.$$

$$16. \prod_{k=1}^{\infty} \frac{k(k+\alpha+\beta)}{(k+\alpha)(k+\beta)} = \frac{\alpha\beta}{\alpha+\beta} B(\alpha, \beta) \quad [\alpha, \beta \neq 0, -1, -2, \dots].$$

$$17. \prod_{k=1}^{\infty} \left(1 \pm \frac{a^4}{k^4} \right) = \frac{1}{2\pi^2 a^2} \left\{ 2 \sin \pi a \operatorname{sh} \pi a \right\}.$$

$$18. \prod_{k=0}^{\infty} \left[1 \pm \frac{a^4}{(2k+1)^4} \right] = \frac{1}{2} \left\{ \frac{\cos(\pi a/2) + \operatorname{ch}(\pi a/2)}{2 \cos(\pi a/2) \operatorname{ch}(\pi a/2)} \right\}.$$

$$19. \prod_{k=1}^{\infty} \left[1 + \frac{1}{(4k-1)^2-1} \right] \left[1 - \frac{1}{(4k+1)^2} \right] = \frac{16}{\pi^2} \Gamma^4 \left(\frac{5}{4} \right).$$

$$20. \prod_{k=1}^{\infty} \left(1 - \frac{x^a}{k^a} \right) = -\frac{1}{x^a} \prod_{k=0}^{a-1} \frac{1}{\Gamma(-a^k x)} \quad [a = e^{2\pi i / m}, a = 2, 3, 4, \dots].$$

$$21. \prod_{k=1}^{\infty} \left[1 - \frac{(-1)^k}{(2k+1)^{2a+1}} \right] = \frac{1}{2} \frac{1}{(2\eta)!} \left(\frac{\pi}{2} \right)^{2a+1} |E_{2a}|.$$

$$22. \prod_{k=1}^{\infty} \left[1 + \left(\frac{a}{k+b} \right)^n \right] \left[1 + \left(\frac{a}{k-b} \right)^n \right] = \\ = \frac{b^n}{a^n + b^n} (2 \sin \pi b)^{-n} \exp \left(2\pi a \cos \frac{\pi}{n} \right) \prod_{k=1}^{[n/2]} \left[1 - 2\alpha_k \cos(b+\beta_k) + \alpha_k^2 \right]^{1/2} \times \\ \times \left[1 - 2\alpha_k \cos(b-\beta_k) + \alpha_k^2 \right]^{1/2} \\ [0 < b < 1; \alpha_k = \exp(-2\pi a \sin[(2k-1)\pi/n]), \beta_k = 2\pi a \cos[(2k-1)\pi/n]].$$

$$23. \prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^s \left(1 - \frac{s}{k+a} \right) = \frac{\Gamma(a+1)}{\Gamma(a-s+1)}.$$

$$24. \prod_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^s \left(1 + \frac{s}{k} \right)^{-1} = \Gamma(s+1).$$

6.2.2. Произведения, содержащие $(1 \pm x^n)^n$.
Обозначение: $x = \exp[-\pi K'(k)/K(k)]$, $0 < k^2 < 1$.

$$1. \prod_{n=1}^{\infty} (1-x^{2n}) = \left[\frac{4}{\pi} \pi^{1/2} k^{1/2} (1-k^2) K(k) \right]^{1/4}.$$

$$2. \prod_{n=1}^{\infty} (1+x^n)^2 = \frac{1}{2} x^{-1/2} k^{1/2} (1-k^2)^{1/2}.$$

$$3. \prod_{n=1}^{\infty} (1-x^n)^2 = 4\pi^{-3/2} x^{-1/4} k^{1/2} (1-k^2) K^2(k).$$

$$4. \prod_{n=1}^{\infty} (1-x^{2n})^2 = \frac{1}{2} x^{-1/4} \frac{d}{da} \vartheta_1(a, x) \Big|_{a=0}.$$

$$5. \prod_{n=1}^{\infty} (1+x^{2n})^2 = \frac{1}{4} x^{-1/2} k (1-k^2)^{-1/4}.$$

$$6. \prod_{n=1}^{\infty} (1-x^{2n})^2 = 2\pi^{-3/2} x^{-1/2} k (1-k^2)^{1/2} K^2(k).$$

$$7. \prod_{n=1}^{\infty} (1+x^{2n+1})^2 = 2x^{1/4} k^{-1/2} (1-k^2)^{-1/2}.$$

$$8. \prod_{n=1}^{\infty} (1-x^{2n+1})^2 = 2x^{1/4} k^{-1/2} (1-k^2)^{1/2}.$$

9. $\prod_{n=1}^{\infty} (1 + ax^{2n} + x^{4n}) = \frac{1}{x} x^{-1/8} (2k)^{1/3} (1 - k^2)^{1/8} \times$
 $\times \operatorname{sech} \left[\frac{1}{2} \operatorname{arccos} \left(-\frac{a}{2} \right) \right] K(k) \vartheta_1 \left[\frac{1}{2} \operatorname{arccos} \left(-\frac{a}{2} \right), x \right] \left[\frac{\partial}{\partial b} \vartheta_1(b, x) \right]_{b=0}^{-1}.$
 $= (16xk^2)^{-1/8} (1 - k^2)^{-1/12} \operatorname{sech} \left(\frac{1}{2} \operatorname{arccos} \frac{a}{2} \right) \vartheta_3 \left(\frac{1}{2} \operatorname{arccos} \frac{a}{2}, x \right) \vartheta_2^{-1}(0, x).$
10. $\prod_{n=0}^{\infty} (1 + ax^{2n+1} + x^{4n+2}) =$
 $= 2^{1/8} x^{1/12} k^{-1/8} (1 - k^2)^{-1/12} \vartheta_3 \left(\frac{1}{2} \operatorname{arccos} \frac{a}{2}, x \right) \vartheta_2^{-1}(0, x).$
 $= 2^{1/3} x^{1/12} k^{-1/8} (1 - k^2)^{-1/6} \vartheta_4 \left[\frac{1}{2} \operatorname{arccos} \left(-\frac{a}{2} \right), x \right] \vartheta_1^{-1}(0, x).$
11. $\prod_{n=0}^{\infty} (1 + x^{n/2}) (1 - x^{n+1}) = \left(\frac{2}{x} \right)^{1/3} x^{-1/16} [kK(k)]^{1/3}.$
12. $\prod_{k=1}^{\infty} (1 - x)^{2k} (1 + x^{2k-1})^2 = \vartheta_6(0, x).$
13. $\prod_{n=0}^{\infty} \frac{1 - abx^n}{1 - bx^n} = {}_1U_0(a, x), \quad 14. \prod_{n=1}^{\infty} \frac{1 - x^n}{1 + x^n} = \vartheta_4(0, x).$
15. $\prod_{n=1}^{\infty} \frac{1 - x^n}{1 - x^{2n-1}} = [(2\pi)^{-1} x^{-1/2} k^{1/2} K(k)]^{1/3}.$
16. $\prod_{n=1}^{\infty} \left(\frac{1 - x^{2n}}{1 + x^{2n}} \right)^{1/2} = \frac{2}{x} (1 - k^2)^{1/4} K(k).$
17. $\prod_{n=1}^{\infty} \left(\frac{1 + x^{2n}}{1 - x^{2n}} \right)^{1/2} = \frac{1}{2} x^{-1/4} k^{1/2} (1 - k^2)^{-1/4}.$
18. $\prod_{n=1}^{\infty} \left(\frac{1 - x^{2n}}{1 - x^{2n-1}} \right)^2 = \frac{1}{x} x^{-1/4} k^{1/2} K(k).$
19. $\prod_{n=1}^{\infty} \left(\frac{1 + x^{2n}}{1 + x^{2n-1}} \right)^4 = \frac{h}{4} x^{-1/8}, \quad 20. \prod_{n=0}^{\infty} \left(\frac{1 - x^{2n+1}}{1 + x^{2n+1}} \right)^4 = (1 - k^2)^{1/2}.$

6.2.3. Показательная функция.

$$1. \prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} \quad [|x| < 1].$$

2. $\prod_{k=1}^{\infty} (1 + a e^{-2^k}) = \frac{1}{2} (1 + \operatorname{cth} a)$ [a > 0].
3. $\prod_{k=1}^{\infty} \left(1 + \frac{2}{k} \right)^{(-1)^{k+1}k} = \frac{\pi^2}{12} \prod_{k=2}^{\infty} \left(1 + \frac{2}{k} \right)^{(-1)^k k} = \frac{\pi}{2e}.$
4. $\prod_{k=1}^{\infty} (1 - e^{-2\pi k/15}) = 3^{1/4} e^{-\pi^2/15}.$
5. $\prod_{k=0}^{\infty} \{1 + e^{-(2k+1)\pi}\} = 2^{1/4} e^{-\pi/24}.$
6. $\prod_{k=1}^{\infty} \left(1 + \frac{a}{k+b} \right)^{a/k} = \frac{\Gamma(b+1)}{\Gamma(b-a+1)} e^{ax} \quad \{\operatorname{Re} b, \operatorname{Re}(b-a) > 0\}.$
7. $\prod_{k=0}^{\infty} \left(1 + \frac{a}{k+b} \right)^{a/(b-bi)} = \frac{\Gamma(b)}{\Gamma(b+bi)} e^{a\varphi(b)},$
8. $\prod_{k=-\infty}^{\infty} \left(1 + \frac{a}{k+b} \right)^{a/k} = \frac{h}{a+b} \sin[\pi(a+b)] \operatorname{sech} \pi b.$
9. $\prod_{k=-\infty}^{\infty} \left(1 \pm \frac{a}{k} \right)^{a/k} = \frac{1}{\pi i} \sin \pi a.$
10. $\prod_{k=-\infty}^{\infty} \left(1 - \frac{a}{2k+1} \right)^{a/(2k+1)} = \cos \frac{\pi a}{2}.$

6.2.4. Тригонометрические функции.

1. $\prod_{k=1}^{\infty} \sec \frac{\pi}{k} = 8,70003663.$
2. $\prod_{k=1}^{\infty} \frac{1}{2\pi} (2k+1) \sin \frac{\pi}{2k+1} = 0,18055054.$
3. $\prod_{k=2}^{\infty} \left(1 - \frac{\pi^2}{2k^2} \right) \sec \frac{\pi}{k} = 0,86883742.$
4. $\prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \frac{\sin x}{x} \quad [|x| < \pi].$
5. $\prod_{k=1}^{\infty} \left(1 - \frac{4}{3} \sin^2 \frac{x}{3^k} \right) = \frac{\sin x}{x}.$

Приложение 1. НЕКОТОРЫЕ ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ И ИХ СВОЙСТВА

1. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

$$\sin x = -\sin(-x) = \pm \sqrt{1 - \cos^2 x} = \pm \sqrt{\frac{1 - \cos 2x}{2}} =$$

$$= 2 \cos^2 \left(\frac{x}{2} - \frac{\pi}{4} \right) - 1 = \frac{2 \operatorname{tg}(x/2)}{1 + \operatorname{tg}^2(x/2)} = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \cos(-x) = \pm \sqrt{1 - \sin^2 x} = \pm \sqrt{\frac{1 + \cos 2x}{2}} =$$

$$= 2 \cos^2 \frac{x}{2} - 1 = \frac{1 - \operatorname{tg}^2(x/2)}{1 + \operatorname{tg}^2(x/2)} = \frac{e^{ix} + e^{-ix}}{2},$$

$$\operatorname{tg} x = -\operatorname{tg}(-x) = \frac{\sin x}{\cos x} = \pm \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} = \frac{\sin 2x}{1 + \cos 2x} =$$

$$= \frac{1 - \cos 2x}{\sin 2x} = \pm \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} = \frac{2 \operatorname{tg}(x/2)}{1 - \operatorname{tg}^2(x/2)} = i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$\operatorname{ctg} x = -\operatorname{ctg}(-x) = \frac{\cos x}{\sin x} = \pm \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}} = \frac{1 + \cos 2x}{\sin 2x} =$$

$$= \frac{\sin 2x}{1 - \cos 2x} = \pm \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}} = \frac{1 - \operatorname{tg}^2(x/2)}{2 \operatorname{tg}(x/2)} = i \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}},$$

$$\sec x = \sec(-x) = \frac{1}{\cos x} = \pm \sqrt{1 + \operatorname{tg}^2 x} = \frac{1 + \operatorname{tg}^2(x/2)}{1 - \operatorname{tg}^2(x/2)},$$

$$\operatorname{cosec} x = -\operatorname{cosec}(-x) = \frac{1}{\sin x} = \pm \sqrt{1 + \operatorname{ctg}^2 x} = \frac{1 + \operatorname{ctg}^2(x/2)}{2 \operatorname{ctg}(x/2)},$$

$$\sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \operatorname{tg}^2 x = 1, \quad \operatorname{cosec}^2 x - \operatorname{ctg}^2 x = 1.$$

2. Формулы дифференцирования.

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \operatorname{tg} x}{dx} = \sec^2 x, \quad \frac{d \sec x}{dx} = \frac{\sin x}{\cos^2 x},$$

$$\frac{d \cos x}{dx} = -\sin x, \quad \frac{d \operatorname{ctg} x}{dx} = -\operatorname{cosec}^2 x, \quad \frac{d \operatorname{cosec} x}{dx} = -\frac{\cos x}{\sin^2 x}.$$

1.1. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

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3. Формулы Эйлера и Муавра.

$$e^{\pm ix} = \cos x \pm i \sin x,$$

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \quad [n = 0, \pm 1, \pm 2, \dots].$$

4. Связь с гиперболическими функциями.

$$\sin(ix) = i \operatorname{sh} x, \quad \cos(ix) = \operatorname{ch} x,$$

$$\operatorname{tg}(ix) = i \operatorname{th} x, \quad \operatorname{ctg}(ix) = -i \operatorname{cth} x,$$

$$\sec(ix) = \operatorname{sech} x, \quad \operatorname{cosec}(ix) = -i \operatorname{cosech} x.$$

5. Формулы сложения.

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\sin(x \pm iy) = \sin x \operatorname{ch} y \pm i \operatorname{sh} y \cos x,$$

$$\cos(x \pm iy) = \cos x \operatorname{ch} y \mp i \sin x \operatorname{sh} y,$$

$$\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}, \quad \operatorname{ctg}(x \pm y) = \frac{1 \mp \operatorname{tg} x \operatorname{tg} y}{\operatorname{tg} x \pm \operatorname{tg} y},$$

$$\operatorname{tg}(x \pm iy) = \frac{\operatorname{tg} x \pm i \operatorname{th} y}{1 \mp i \operatorname{tg} x \operatorname{th} y} = \frac{\sin 2x \pm i \operatorname{sh} 2y}{\cos 2x \mp \operatorname{ch} 2y},$$

$$\operatorname{ctg}(x \pm iy) = \frac{1 \mp i \operatorname{ctg} x \operatorname{ctg} y}{\operatorname{ctg} x \pm i \operatorname{cth} y} = \frac{\sin 2x \mp i \operatorname{sh} 2y}{\operatorname{ch} 2y - \cos 2x}.$$

6. Формулы приведения.

$$\sin(x \pm n\pi) = (-1)^n \sin x, \quad \cos(x \pm n\pi) = (-1)^n \cos x,$$

$$\operatorname{tg}(x \pm n\pi) = \operatorname{tg} x, \quad \operatorname{ctg}(x \pm n\pi) = \operatorname{ctg} x,$$

$$\sin\left(x \pm \frac{2n+1}{2}\pi\right) = \pm (-1)^n \cos x, \quad \cos\left(x \pm \frac{2n+1}{2}\pi\right) = \mp (-1)^n \sin x,$$

$$\operatorname{tg}\left(x \pm \frac{2n+1}{2}\pi\right) = -\operatorname{ctg} x, \quad \operatorname{ctg}\left(x \pm \frac{2n+1}{2}\pi\right) = -\operatorname{tg} x,$$

$$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\sin x \pm \cos x),$$

$$\cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (\cos x \mp \sin x),$$

$$\operatorname{tg}\left(x \pm \frac{\pi}{4}\right) = \frac{\sin x \pm \cos x}{\cos x \mp \sin x} = \frac{\operatorname{tg} x \pm 1}{1 \mp \operatorname{tg} x},$$

$$\operatorname{ctg}\left(x \pm \frac{\pi}{4}\right) = \frac{\cos x \mp \sin x}{\sin x \pm \cos x} = \frac{\operatorname{ctg} x \mp 1}{1 \pm \operatorname{ctg} x}.$$

7. Суммы и разности тригонометрических функций.

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2},$$

$$\cos x \pm \cos y = \pm 2 \left\{ \cos \left[\frac{(x+y)/2}{2} \right] \cos \left[\frac{(x-y)/2}{2} \right] \right\}$$

$$\operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \operatorname{ctg} x \pm \operatorname{ctg} y = -\frac{\sin(x \mp y)}{\sin x \sin y},$$

$$\operatorname{tg} x - \operatorname{ctg} y = -\frac{\cos(x+y)}{\cos x \sin y},$$

$$a \cos x + b \sin x = r \sin(x+\varphi) = r \cos(x-\psi),$$

где

$$r = \sqrt{a^2 + b^2}, \quad \sin \varphi = a/r, \quad \cos \varphi = b/r, \quad \sin \psi = b/r, \quad \cos \psi = a/r, \\ \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y) \sin(x-y), \\ \sin^2 x - \cos^2 y = -\cos(x+y) \cos(x-y), \\ \sec^2 x + \csc^2 x = \sec^2 x \csc^2 x.$$

8. Произведения тригонометрических функций.

$$\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)],$$

$$\cos x \cos y = \frac{1}{2} [\cos(x-y) + \cos(x+y)],$$

$$\sin x \cos y = \frac{1}{2} [\sin(x-y) + \sin(x+y)].$$

9. Степени тригонометрических функций.

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{2n} = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos 2(n-k)x + \frac{1}{2^{2n}} \binom{2n}{n},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^{n-k} \binom{2n+1}{k} \left\{ \begin{matrix} \sin(2n-2k+1)x \\ \cos(2n-2k+1)x \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^2 = \frac{1}{2} \cos 2x + \frac{1}{2},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^3 = \frac{1}{4} \left\{ \begin{matrix} \sin 3x \\ \cos 3x \end{matrix} \right\} + \frac{3}{4} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^4 = \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^5 = \frac{1}{16} \left\{ \begin{matrix} \sin 5x \\ \cos 5x \end{matrix} \right\} + \frac{5}{16} \left\{ \begin{matrix} \sin 3x \\ \cos 3x \end{matrix} \right\} + \frac{5}{8} \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^6 = \frac{1}{32} \cos 6x + \frac{3}{16} \cos 4x + \frac{15}{32} \cos 2x + \frac{5}{16}.$$

10. Тригонометрические функции кратных аргументов.

$$\sin nx = \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{2k+1} \sin^{2k+1} x \cos^{n-2k-1} x,$$

$$= \sin x \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n-1}{2k} 2^{n-2k-1} \cos^{n-2k-1} x,$$

$$\cos nx = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \sin^{2k} x \cos^{n-2k} x,$$

$$= 2^{n-1} \cos^n x + \frac{n}{2} \sum_{k=0}^{(n-2)/2} \frac{(-1)^{k+1}}{k+1} \binom{n-k-2}{k} 2^{n-2k-2} \cos^{n-2k-2} x,$$

$$\sin 2nx = 2n \cos x \left[\sin x + \sum_{k=1}^n (-1)^k \frac{(n^2-1^2)(n^2-2^2) \dots (n^2-k^2)}{(2k-1)!} 4^k \sin^{2k-1} x \right],$$

$$= (-1)^{n-1} \cos x \left[2^{2n-1} \sin^{2n-1} x + \right.$$

$$\left. + \sum_{k=1}^n (-1)^k \frac{(2n-k-1)(2n-k-2) \dots (2n-2k)}{k!} 2^{2n-2k-1} \sin^{2n-2k-1} x \right],$$

$$\cos 2nx = 1 + \sum_{k=1}^n (-1)^k \frac{n^2(n^2-1^2) \dots [n^2-(k-1)^2]}{(2k)!} 4^k \sin^{2k} x,$$

$$= (-1)^n \left[2^{2n-1} \sin^{2n} x - 2^{2n-2} n \sin^{2n-2} x + \right.$$

$$\left. + n \sum_{k=2}^n (-1)^k \frac{(2n-k-1)(2n-k-3) \dots (2n-2k+1)}{k!} 2^{2n-2k} \sin^{2n-2k} x \right],$$

$$\sin(2n+1)x = (2n+1)x \times$$

$$\times \left[\sin x + \sum_{k=1}^n (-1)^k \frac{1(2n+1)^2-1^2}{(2k+1)!} \{ (2n+1)^2-3^2 \} \dots \{ (2n+1)^2-(2k-1)^2 \} \sin^{2k+1} x \right],$$

$$= (-1)^n \left[2^{2n} \sin^{2n+1} x - 2^{2n-2} (2n+1) \sin^{2n-1} x + \right.$$

$$\left. + \frac{2n+1}{2} \sum_{k=2}^n (-1)^k \frac{(2n-k)(2n-k-1) \dots (2n-2k+2)}{k!} 2^{2n-2k+1} \sin^{2n-2k+1} x \right],$$

$$\cos(2n+1)x = \cos x \times$$

$$\times \left[1 + \sum_{k=1}^n (-1)^k \frac{\{ (2n+1)^2-1^2 \} \{ (2n+1)^2-3^2 \} \dots \{ (2n+1)^2-(2k-1)^2 \}}{(2k)!} \sin^{2k} x \right],$$

$$= (-1)^n \cos x \left[2^{2n} \sin^{2n} x - \right.$$

$$\left. - \sum_{k=1}^n \frac{(2n-k)(2n-k-1) \dots (2n-2k+1)}{k!} 2^{2n-2k} \sin^{2n-2k} x \right],$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = 2 \cos^2 x - 1,$$

$$\left\{ \begin{matrix} \sin 3x \\ \cos 3x \end{matrix} \right\} = \pm 3 \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} - \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^3,$$

$$\sin 4x = \cos x (4 \sin x - 8 \sin^3 x),$$

$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1,$$

$$\left\{ \begin{matrix} \sin 5x \\ \cos 5x \end{matrix} \right\} = \pm 5 \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\} - 20 \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^3 + 16 \left\{ \begin{matrix} \sin x \\ \cos x \end{matrix} \right\}^5,$$

$$\sin 6x = \cos(6 \sin x - 32 \sin^3 x + 32 \sin^5 x),$$

$$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1,$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}, \quad \operatorname{tg} 3x = \frac{3 \operatorname{tg} x - \operatorname{tg}^3 x}{1 - 3 \operatorname{tg}^2 x}, \quad \operatorname{tg} 4x = \frac{4 \operatorname{tg} x - 4 \operatorname{tg}^3 x}{1 - 6 \operatorname{tg}^2 x + \operatorname{tg}^4 x}.$$

II. Тригонометрические функции некоторых углов.

$$1. \sin n\pi = 0, \quad \cos n\pi = (-1)^n, \quad \operatorname{tg} n\pi = 0.$$

$$2. \begin{cases} \sin(n\pi/2) \\ \cos(n\pi/2) \end{cases} = (-1)^{\lfloor n/2 \rfloor} \frac{1 + (-1)^n}{2}, \quad \operatorname{tg} \frac{n\pi}{2} = \frac{1 - (-1)^n}{1 + (-1)^n}.$$

$$3. \sin \frac{n\pi}{3} = (-1)^{\lfloor n/3 \rfloor} \frac{\sqrt{3}}{4} [1 + (-1)^{\lfloor n/3 \rfloor}], \quad \begin{cases} \{n - \text{четное}\} \\ \{n - \text{нечетное}\} \end{cases}$$

$$\cos \frac{n\pi}{3} = (-1)^{\lfloor (n+1)/3 \rfloor} \frac{3 + (-1)^{\lfloor (n+1)/3 \rfloor}}{4}, \quad \begin{cases} \{n - \text{четное}\} \\ \{n - \text{нечетное}\} \end{cases}$$

$$\operatorname{tg} \frac{n\pi}{3} = (-1)^{\lfloor (n+2)/3 \rfloor} \frac{\sqrt{3}}{2} [1 - (-1)^{\lfloor (n+1)/3 \rfloor}]$$

$$= (-1)^{\lfloor n/3 \rfloor} \frac{1}{2} [1 + (-1)^{\lfloor (n+1)/3 \rfloor}] \quad \{n - \text{нечетное}\}.$$

$$4. \sin \frac{n\pi}{4} = (-1)^{\lfloor n/4 \rfloor} \frac{1 - (-1)^{\lfloor n/2 \rfloor}}{2}$$

$$= (-1)^{\lfloor (n-1)/4 \rfloor} \frac{1}{\sqrt{2}} \quad \{n - \text{нечетное}\}.$$

$$\cos \frac{n\pi}{4} = (-1)^{\lfloor n/4 \rfloor} \frac{1 + (-1)^{\lfloor n/2 \rfloor}}{2}$$

$$= (-1)^{\lfloor (n+1)/4 \rfloor} \frac{1}{\sqrt{2}} \quad \{n - \text{нечетное}\}.$$

$$\operatorname{tg} \frac{n\pi}{4} = (-1)^{\lfloor n/4 \rfloor} \frac{1 - (-1)^{\lfloor n/2 \rfloor}}{1 + (-1)^{\lfloor n/2 \rfloor}}$$

$$= (-1)^{\lfloor (2n+1)/4 \rfloor} \quad \{n - \text{нечетное}\}.$$

$$5. \sin \frac{n\pi}{5} = (-1)^{\lfloor n/5 \rfloor} \frac{1 - (-1)^{\lfloor 3n/5 \rfloor}}{8} \sqrt{10 - (-1)^{\lfloor n/2 \rfloor} 2\sqrt{5}}$$

$$= (-1)^{\lfloor (n+2)/5 \rfloor} \frac{1 + (-1)^{\lfloor (1n-1)/5 \rfloor}}{8} \sqrt{10 - (-1)^{\lfloor n/2 \rfloor} 2\sqrt{5}} \quad \{n - \text{нечетное}\}.$$

$$\cos \frac{n\pi}{5} = (-1)^{\lfloor (n+2)/5 \rfloor} \frac{\sqrt{5} + (-1)^{\lfloor n/2 \rfloor}}{4} \quad \{n \neq 0, \pm 5, \pm 10, \dots\}.$$

$$6. \sin \frac{n\pi}{6} = (-1)^{\lfloor n/6 \rfloor} \frac{\sqrt{3}}{4} [1 - (-1)^{\lfloor (n+1)/3 \rfloor}] \quad \{n - \text{нечетное}\},$$

$$= (-1)^{\lfloor n/6 \rfloor} \frac{3 - (-1)^{\lfloor (n+1)/3 \rfloor}}{4} \quad \{n - \text{нечетное}\}.$$

$$\cos \frac{n\pi}{6} = (-1)^{\lfloor (n+2)/6 \rfloor} \frac{3 + (-1)^{\lfloor (n+1)/3 \rfloor}}{4}$$

$$= (-1)^{\lfloor (n+1)/6 \rfloor} \frac{\sqrt{3}}{2} [1 - (-1)^{\lfloor (n+1)/3 \rfloor}] \quad \{n - \text{нечетное}\}.$$

$$\operatorname{tg} \frac{n\pi}{6} = (-1)^{\lfloor n/6 \rfloor} \frac{\sqrt{3}}{2} [1 - (-1)^{\lfloor (n+1)/3 \rfloor}]$$

$$= (-1)^{\lfloor n/6 \rfloor} \frac{2}{\sqrt{3} [1 + (-1)^{\lfloor (n+1)/3 \rfloor}]} \quad \{n - \text{нечетное}\}.$$

$$7. \sin \frac{n\pi}{8} = (-1)^{\lfloor (n+1)/8 \rfloor} \frac{1}{2} \sqrt{2 - (-1)^{\lfloor n/4 \rfloor} - (-1)^{\lfloor (3n+1)/4 \rfloor}}$$

$$= (-1)^{\lfloor n/8 \rfloor} \frac{1}{2} \sqrt{2 - (-1)^{\lfloor (n+1)/4 \rfloor} \sqrt{2}} \quad \{n - \text{нечетное}\}.$$

$$\cos \frac{n\pi}{8} = (-1)^{\lfloor (n+5)/8 \rfloor} \frac{1}{2} \sqrt{2 + (-1)^{\lfloor n/4 \rfloor} - (-1)^{\lfloor (3n+1)/4 \rfloor}}$$

$$= (-1)^{\lfloor (n+3)/8 \rfloor} \frac{1}{2} \sqrt{2 + (-1)^{\lfloor (n+1)/4 \rfloor} \sqrt{2}} \quad \{n - \text{нечетное}\}.$$

$$\operatorname{tg} \frac{n\pi}{8} = (-1)^{\lfloor (2n+1)/8 \rfloor} \frac{2 - (-1)^{\lfloor n/4 \rfloor} - (-1)^{\lfloor (3n+1)/4 \rfloor}}{(-1)^{\lfloor n/4 \rfloor} - (-1)^{\lfloor (3n+1)/4 \rfloor}}$$

$$= (-1)^{\lfloor (2n+1)/8 \rfloor} \sqrt{2 - (-1)^{\lfloor (n+1)/4 \rfloor}} \quad \{n - \text{нечетное}\}.$$

$$8. \sin \frac{\pi}{10} = \frac{1}{4} (\sqrt{5} - 1), \quad \sin \frac{3\pi}{10} = \frac{1}{4} (\sqrt{5} + 1),$$

$$\cos \frac{\pi}{10} = \frac{1}{4} \sqrt{10 + 2\sqrt{5}}, \quad \cos \frac{3\pi}{10} = \frac{1}{4} \sqrt{10 - 2\sqrt{5}},$$

$$\operatorname{tg} \frac{\pi}{10} = \frac{1}{\sqrt{5}} \sqrt{5 - 2\sqrt{5}}, \quad \operatorname{tg} \frac{3\pi}{10} = \frac{1}{\sqrt{5}} \sqrt{5 + 2\sqrt{5}},$$

$$\operatorname{ctg} \frac{\pi}{10} = \sqrt{5 + 2\sqrt{5}}, \quad \operatorname{ctg} \frac{3\pi}{10} = \sqrt{5 - 2\sqrt{5}}.$$

$$9. \begin{cases} \sin(\pi/12) \\ \cos(\pi/12) \end{cases} = \frac{1}{4} \{\sqrt{6} \mp \sqrt{2}\}, \quad \begin{cases} \sin(5\pi/12) \\ \cos(5\pi/12) \end{cases} = \frac{\sqrt{3} \pm 1}{2\sqrt{2}},$$

$$\begin{cases} \operatorname{tg}(\pi/12) \\ \operatorname{ctg}(\pi/12) \end{cases} = 2 \mp \sqrt{3}, \quad \begin{cases} \operatorname{tg}(5\pi/12) \\ \operatorname{ctg}(5\pi/12) \end{cases} = 2 \pm \sqrt{3},$$

$$\begin{cases} \sin(7\pi/12) \\ \cos(7\pi/12) \end{cases} = \frac{1}{2\sqrt{2}} (1 \pm \sqrt{3}), \quad \begin{cases} \sin(11\pi/12) \\ \cos(11\pi/12) \end{cases} = \frac{1}{2\sqrt{2}} (-1 \pm \sqrt{3}),$$

$$\begin{cases} \operatorname{tg}(7\pi/12) \\ \operatorname{ctg}(7\pi/12) \end{cases} = -2 \pm \sqrt{3}, \quad \begin{cases} \operatorname{tg}(11\pi/12) \\ \operatorname{ctg}(11\pi/12) \end{cases} = -2 \pm \sqrt{3}.$$

$$10. \sin \frac{\pi}{15} = \frac{1}{8} [-\sqrt{3}(\sqrt{5}-1) + \sqrt{10+2\sqrt{5}}],$$

$$\cos \frac{\pi}{15} = \frac{1}{8} (\sqrt{5}-1 + \sqrt{3}\sqrt{10+2\sqrt{5}}),$$

$$\operatorname{tg} \frac{\pi}{15} = \frac{1}{4} (3 - \sqrt{5}) (2\sqrt{3} - \sqrt{10-2\sqrt{5}}),$$

$$\begin{aligned}\operatorname{ctg} \frac{\pi}{15} &= \frac{1}{4} (1 + \sqrt{5}) (2\sqrt{3} + \sqrt{10 - 2\sqrt{5}}), \\ \sin \frac{2\pi}{15} &= \frac{1}{8} [\sqrt{3}(\sqrt{5} + 1) - \sqrt{10 - 2\sqrt{5}}], \\ \cos \frac{2\pi}{15} &= \frac{1}{8} (\sqrt{5} + 1 + \sqrt{3}\sqrt{10 - 2\sqrt{5}}), \\ \operatorname{tg} \frac{2\pi}{15} &= \frac{1}{4} (3 + \sqrt{5}) (-2\sqrt{3} + \sqrt{10 + 2\sqrt{5}}), \\ \operatorname{ctg} \frac{2\pi}{15} &= \frac{1}{4} (\sqrt{5} - 1) (2\sqrt{3} + \sqrt{10 + 2\sqrt{5}}), \\ 11. \quad \frac{\sin(\pi/16)}{\cos(\pi/16)} &= \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \\ \frac{\operatorname{tg}(\pi/16)}{\operatorname{ctg}(\pi/16)} &= \sqrt{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}} (\sqrt{2} + 1), \\ \frac{\sin(3\pi/16)}{\cos(3\pi/16)} &= \frac{1}{2} \sqrt{2 + \sqrt{2 - \sqrt{2}}}, \\ \frac{\operatorname{tg}(3\pi/16)}{\operatorname{ctg}(3\pi/16)} &= \sqrt{2} \sqrt{2 - \sqrt{2 + \sqrt{2}}} (\sqrt{2} - 1), \\ 12. \quad \frac{\sin(\pi/20)}{\cos(\pi/20)} &= \frac{\sqrt{2}}{8} (\sqrt{5} + 1 + \sqrt{10 - 2\sqrt{5}}), \\ \frac{\operatorname{tg}(\pi/20)}{\operatorname{ctg}(\pi/20)} &= \sqrt{5} + 1 + \sqrt{5 + 2\sqrt{5}}, \\ \frac{\sin(3\pi/20)}{\cos(3\pi/20)} &= \frac{\sqrt{2}}{8} (\sqrt{10 + 2\sqrt{5}} + 1 + \sqrt{5}), \\ \frac{\operatorname{tg}(3\pi/20)}{\operatorname{ctg}(3\pi/20)} &= \sqrt{5} - 1 + \sqrt{5 - 2\sqrt{5}}, \\ 13. \quad \frac{\sin(\pi/24)}{\cos(\pi/24)} &= \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{3}}}, \\ \frac{\operatorname{tg}(\pi/24)}{\operatorname{ctg}(\pi/24)} &= (\sqrt{2} + 1) (\sqrt{3} + \sqrt{2}), \\ \frac{\sin(5\pi/24)}{\cos(5\pi/24)} &= \frac{1}{2} \sqrt{2 + \sqrt{2 - \sqrt{3}}}, \\ \frac{\operatorname{tg}(5\pi/24)}{\operatorname{ctg}(5\pi/24)} &= (\sqrt{2} - 1) (\sqrt{3} + \sqrt{2}), \\ 14. \quad \sin \frac{\pi}{30} &= \frac{1}{8} [-(\sqrt{5} + 1) + \sqrt{3}\sqrt{10 - 2\sqrt{5}}], \\ \cos \frac{\pi}{30} &= \frac{1}{8} [\sqrt{3}(\sqrt{5} + 1) + \sqrt{10 - 2\sqrt{5}}], \\ \operatorname{tg} \frac{\pi}{30} &= \frac{1}{4} (\sqrt{5} - 1) (\sqrt{10 + 2\sqrt{5}} - 2\sqrt{3}), \\ \operatorname{ctg} \frac{\pi}{30} &= \frac{1}{4} (3 + \sqrt{5}) (\sqrt{10 + 2\sqrt{5}} + 2\sqrt{3}), \\ \sin \frac{7\pi}{30} &= \frac{1}{8} [-(\sqrt{5} - 1) + \sqrt{3}\sqrt{10 + 2\sqrt{5}}],\end{aligned}$$

$$\begin{aligned}\cos \frac{7\pi}{30} &= \frac{1}{8} [\sqrt{3}(\sqrt{5} - 1) + \sqrt{10 + 2\sqrt{5}}], \\ \operatorname{tg} \frac{7\pi}{30} &= \frac{1}{4} (\sqrt{5} + 1) (2\sqrt{3} - \sqrt{10 - 2\sqrt{5}}), \\ \operatorname{ctg} \frac{7\pi}{30} &= \frac{1}{4} (3 - \sqrt{5}) (2\sqrt{3} + \sqrt{10 - 2\sqrt{5}}), \\ 15. \quad \frac{\sin(3\pi/40)}{\cos(3\pi/40)} &= \frac{1}{8} [\sqrt{2 + \sqrt{2}\sqrt{10 - 2\sqrt{5}} + \sqrt{2 + \sqrt{2}}(1 + \sqrt{5})], \\ \frac{\operatorname{tg}(3\pi/40)}{\operatorname{ctg}(3\pi/40)} &= \frac{1}{4} (1 + \sqrt{5} + 2\sqrt{2}) [\sqrt{10 - 2\sqrt{5}} + \sqrt{2}(\sqrt{5} - 1)], \\ 16. \quad \frac{\sin(\pi/60)}{\cos(\pi/60)} &= \frac{1}{8} [\sqrt{(2 + \sqrt{3})(3 - \sqrt{5})} + \sqrt{(2 + \sqrt{3})(5 + \sqrt{5})}], \\ \frac{\operatorname{tg}(\pi/60)}{\operatorname{ctg}(\pi/60)} &= \frac{1}{4} (2 + \sqrt{3}) (1 + 2\sqrt{3} + \sqrt{5})(\sqrt{10 - 2\sqrt{5}} + 2).\end{aligned}$$

1.2. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

$$\begin{aligned}\operatorname{sh} x &= -\operatorname{sh}(-x) = \pm \sqrt{\operatorname{ch}^2 x - 1} = \pm \sqrt{\frac{1}{2}(\operatorname{ch} 2x - 1)} = \frac{e^x - e^{-x}}{2}, \\ \operatorname{ch} x &= \operatorname{ch}(-x) = \sqrt{\operatorname{sh}^2 x + 1} = \sqrt{\frac{1}{2}(\operatorname{ch} 2x + 1)} = 2 \operatorname{ch}^2 \frac{x}{2} - 1 = \frac{e^x + e^{-x}}{2}, \\ \operatorname{th} x &= -\operatorname{th}(-x) = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x + 1} = \frac{\operatorname{ch} 2x - 1}{\operatorname{sh} 2x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \\ \operatorname{cth} x &= -\operatorname{cth}(-x) = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x - 1} = \frac{\operatorname{ch} 2x + 1}{\operatorname{sh} 2x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \\ \operatorname{sech} x &= \frac{1}{\operatorname{ch} x}, \quad \operatorname{cosech} x = \frac{1}{\operatorname{sh} x}, \\ \operatorname{ch}^2 x - \operatorname{sh}^2 x &= 1, \quad \operatorname{th}^2 x + \operatorname{sech}^2 x = 1, \quad \operatorname{cth}^2 x - \operatorname{cosech}^2 x = 1, \\ (\operatorname{sh} x + \operatorname{ch} x)^n &= \operatorname{sh} nx + \operatorname{ch} nx \quad (n = 0, \pm 1, \pm 2, \dots).\end{aligned}$$

2. Формулы дифференцирования.

$$\begin{aligned}\frac{d \operatorname{sh} x}{dx} &= \operatorname{ch} x, & \frac{d \operatorname{cosech} x}{dx} &= -\frac{\operatorname{ch} x}{\operatorname{sh}^2 x}, \\ \frac{d \operatorname{ch} x}{dx} &= \operatorname{sh} x, & \frac{d \operatorname{th} x}{dx} &= \operatorname{sech}^2 x, \\ \frac{d \operatorname{sech} x}{dx} &= -\frac{\operatorname{sh} x}{\operatorname{ch}^2 x}, & \frac{d \operatorname{cth} x}{dx} &= -\operatorname{cosech}^2 x.\end{aligned}$$

3. Связь с тригонометрическими функциями.

$$\operatorname{sh}(ix) = i \sin x, \quad \operatorname{ch}(ix) = \cos x, \quad \operatorname{th}(ix) = i \operatorname{tg} x, \quad \operatorname{cth}(ix) = -i \operatorname{ctg} x.$$

4. Формулы сложения.

$$\begin{aligned}\operatorname{sh}(x \pm y) &= \operatorname{sh} x \operatorname{ch} y \pm \operatorname{sh} y \operatorname{ch} x, \\ \operatorname{ch}(x \pm y) &= \operatorname{ch} x \operatorname{ch} y \pm \operatorname{sh} x \operatorname{sh} y,\end{aligned}$$

$$\operatorname{sh}(x \pm iy) = \operatorname{sh} x \cos y \pm i \sin y \operatorname{ch} x,$$

$$\operatorname{ch}(x \pm iy) = \operatorname{ch} x \cos y \pm i \operatorname{sh} x \sin y,$$

$$\operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x \operatorname{th} y} = \frac{\operatorname{sh} 2x \pm \operatorname{sh} 2y}{\operatorname{ch} 2x \pm \operatorname{ch} 2y},$$

$$\operatorname{th}(x \pm iy) = \frac{\operatorname{th} x \pm i \operatorname{tg} y}{1 \pm i \operatorname{th} x \operatorname{tg} y} = \frac{\operatorname{sh} 2x \pm i \sin 2y}{\operatorname{ch} 2x \pm \cos 2y},$$

$$\operatorname{cth}(x \pm iy) = \frac{\operatorname{sh} 2x \mp i \sin 2y}{\operatorname{ch} 2x - \cos 2y}.$$

5. Суммы и разности гиперболических функций.

$$\operatorname{sh} x \pm \operatorname{sh} y = 2 \operatorname{sh} \frac{x \pm y}{2} \operatorname{ch} \frac{x \mp y}{2},$$

$$\operatorname{ch} x \pm \operatorname{ch} y = 2 \left\{ \begin{array}{l} \operatorname{ch} [(x+y)/2] \operatorname{ch} [(x-y)/2] \\ \operatorname{sh} [(x+y)/2] \operatorname{sh} [(x-y)/2] \end{array} \right\},$$

$$\operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x \operatorname{ch} y}, \quad \operatorname{cth} x \pm \operatorname{cth} y = \pm \frac{\operatorname{sh}(x \mp y)}{\operatorname{sh} x \operatorname{sh} y},$$

$$\operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{ch}^2 x - \operatorname{ch}^2 y = \operatorname{sh}(x+y) \operatorname{sh}(x-y),$$

$$\operatorname{sh}^2 x + \operatorname{ch}^2 y = \operatorname{ch}(x+y) \operatorname{cth}(x-y).$$

6. Произведения гиперболических функций.

$$\operatorname{sh} x \operatorname{sh} y = \frac{1}{2} [\operatorname{ch}(x+y) - \operatorname{ch}(x-y)],$$

$$\operatorname{ch} x \operatorname{ch} y = \frac{1}{2} [\operatorname{ch}(x+y) + \operatorname{ch}(x-y)],$$

$$\operatorname{sh} x \operatorname{ch} y = \frac{1}{2} [\operatorname{sh}(x+y) + \operatorname{sh}(x-y)].$$

7. Степени гиперболических функций.

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^{2n} = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \operatorname{ch} 2(n-k)x \pm \frac{(-1)^n}{2^{2n}} \binom{2n}{n},$$

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \left\{ \begin{array}{l} \operatorname{sh}(2n-2k+1)x \\ \operatorname{ch}(2n-2k+1)x \end{array} \right\},$$

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^2 = \frac{1}{2} (\operatorname{ch} 2x \mp 1), \quad \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^3 = \frac{1}{4} (\operatorname{sh} 2x \mp 2 \operatorname{sh} x),$$

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^4 = \frac{1}{8} (\operatorname{ch} 4x \mp 4 \operatorname{ch} 2x \mp 3),$$

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^5 = \frac{1}{16} (\operatorname{sh} 5x \mp 5 \operatorname{sh} 3x \mp 10 \operatorname{sh} x),$$

$$\left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^6 = \frac{1}{32} (\operatorname{ch} 6x \mp 6 \operatorname{ch} 4x \mp 15 \operatorname{ch} 2x \mp 10).$$

8. Гиперболические функции кратных аргументов.

$$\operatorname{sh} nx = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} \operatorname{sh}^{2k+1} x \operatorname{ch}^{n-2k-1} x,$$

$$= \operatorname{sh} x \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} \binom{n-k-1}{k} \operatorname{ch}^{n-2k-1} x,$$

$$\operatorname{ch} nx = \sum_{k=0}^{[n/2]} \binom{n}{2k} \operatorname{sh}^{2k} x \operatorname{ch}^{n-2k} x,$$

$$= 2^{n-1} \cos^n x + 2 \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k+1} \binom{n-k-2}{k-2} 2^{n-2k-3} \operatorname{ch}^{n-2k-3} x,$$

$$\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x, \quad \operatorname{ch} 2x = 2 \operatorname{ch}^2 x - 1,$$

$$\left\{ \begin{array}{l} \operatorname{sh} 3x \\ \operatorname{ch} 3x \end{array} \right\} = 4 \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^3 \pm 3 \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\},$$

$$\operatorname{sh} 4x = \operatorname{ch} x (4 \operatorname{sh} x + 8 \operatorname{sh}^3 x), \quad \operatorname{ch} 4x = 8 \operatorname{ch}^4 x - 8 \operatorname{ch}^2 x + 1,$$

$$\left\{ \begin{array}{l} \operatorname{sh} 5x \\ \operatorname{ch} 5x \end{array} \right\} = 16 \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^5 \pm 20 \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\}^3 \pm 5 \left\{ \begin{array}{l} \operatorname{sh} x \\ \operatorname{ch} x \end{array} \right\},$$

$$\operatorname{sh} 6x = \operatorname{ch} x (32 \operatorname{sh}^5 x + 32 \operatorname{sh}^3 x + 6 \operatorname{sh} x),$$

$$\operatorname{ch} 6x = 32 \operatorname{ch}^6 x - 48 \operatorname{ch}^4 x + 18 \operatorname{ch}^2 x - 1.$$

1.3. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

1. Некоторые соотношения.

Главные значения обратных тригонометрических функций определяются неравенствами

$$-\pi/2 \leq \arcsin x \leq \pi/2; \quad 0 \leq \arccos x \leq \pi \quad |x| \leq 1$$

$$-\pi/2 < \operatorname{arctg} x < \pi/2; \quad 0 < \operatorname{arctg} x < \pi \quad [-\infty < x < \infty].$$

$$\arcsin(-x) = -\arcsin x, \quad \arccos(-x) = \pi - \arccos x,$$

$$\operatorname{arctg}(-x) = -\operatorname{arctg} x, \quad \operatorname{arctg}(-x) = \pi - \operatorname{arctg} x,$$

$$\sin(\arcsin x) = x, \quad \cos(\arccos x) = x,$$

$$\operatorname{tg}(\operatorname{arctg} x) = x, \quad \operatorname{ctg}(\operatorname{arctg} x) = x.$$

$$\arcsin(\sin x) = x - 2n\pi \quad [2n\pi - \pi/2 \leq x \leq 2n\pi + \pi/2],$$

$$= -x + (2n+1)\pi \quad [(2n+1)\pi - \pi/2 \leq x \leq (n+1)\pi + \pi/2].$$

$$\arccos(\cos x) = x - 2n\pi \quad [2n\pi \leq x \leq (2n+1)\pi],$$

$$= -x + 2(n+1)\pi \quad [(2n+1)\pi \leq x \leq 2(n+1)\pi].$$

$$\operatorname{arctg}(\operatorname{tg} x) = x - n\pi \quad [n\pi - \pi/2 < x < n\pi + \pi/2],$$

$$\operatorname{arctg}(\operatorname{ctg} x) = x - n\pi \quad [n\pi < x < (n+1)\pi].$$

$$\lim_{x \rightarrow \pm \infty} \operatorname{arctg} x = \pm \pi/2, \quad \lim_{x \rightarrow \pm \infty} \operatorname{arctg} x = \pi/2 \mp \pi/2.$$

2. Связь с логарифмической и обратными гиперболическими функциями.

$$\operatorname{arcsin} z = -i \ln (iz + \sqrt{1-z^2}) = -i \operatorname{Arsh} (iz),$$

$$\operatorname{arccos} z = -i \ln (z + \sqrt{z^2-1}) = -i \operatorname{Arch} z,$$

$$\operatorname{arctg} z = -\frac{i}{2} \ln \frac{1+iz}{1-iz} = -i \operatorname{Arth} (iz),$$

$$\operatorname{arcctg} z = -\frac{i}{2} \ln \frac{iz-1}{iz+1} = i \operatorname{Areth} (iz).$$

3. Формулы дифференцирования.

$$\frac{d}{dx} \operatorname{arcsin} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \operatorname{arccos} x = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx} \operatorname{arctg} x = \frac{1}{1+x^2}, \quad \frac{d}{dx} \operatorname{arcctg} x = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x \sqrt{x^2-1}}, \quad \left[\left\{ \frac{0}{\pi/2} \right\} < \operatorname{arcsec} x < \left\{ \frac{\pi/2}{\pi} \right\} \right],$$

$$\frac{d}{dx} \operatorname{arccosec} x = \frac{1}{x \sqrt{x^2-1}}, \quad \left[\left\{ \frac{-\pi/2}{0} \right\} < \operatorname{arccosec} x < \left\{ \frac{0}{\pi/2} \right\} \right].$$

4. Связь между обратными тригонометрическими функциями.

$$\operatorname{arcsin} x = \frac{\pi}{2} - \operatorname{arccos} x,$$

$$= \operatorname{arccos} \sqrt{1-x^2} \quad (0 \leq x \leq 1),$$

$$= -\operatorname{arccos} \sqrt{1-x^2} \quad (-1 \leq x \leq 0),$$

$$= \operatorname{arctg} \frac{x}{\sqrt{1-x^2}} \quad (x^2 < 1),$$

$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} \quad (0 < x \leq 1),$$

$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} - \pi \quad (-1 \leq x < 0),$$

$$= \frac{1}{2} \operatorname{arcsin} (2x \sqrt{1-x^2}) \quad (x^2 \leq 1/2),$$

$$= \frac{\pi}{2} - \frac{1}{2} \operatorname{arcsin} (2x \sqrt{1-x^2}) \quad (1/2 \leq x \leq 1),$$

$$= -\frac{\pi}{2} - \frac{1}{2} \operatorname{arcsin} (2x \sqrt{1-x^2}) \quad (-1 \leq x < -1/2),$$

$$= -\frac{\pi}{2} + \frac{1}{2} \operatorname{arcsin} (2x \sqrt{1-x^2}) \quad (-1/2 \leq x < 0),$$

$$\operatorname{arccos} x = \frac{\pi}{2} - \operatorname{arcsin} x,$$

$$= \operatorname{arcsin} \sqrt{1-x^2} \quad (0 \leq x \leq 1),$$

$$= \pi - \operatorname{arcsin} \sqrt{1-x^2} \quad (-1 \leq x \leq 0),$$

$$= \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad (0 < x \leq 1),$$

$$= \operatorname{arcctg} \frac{x}{\sqrt{1-x^2}} \quad (-1 \leq x < 0),$$

$$= \operatorname{arccos} \sqrt{1-x^2} \quad (x^2 \leq 1/2),$$

$$= \pi - \operatorname{arccos} \sqrt{1-x^2} \quad (1/2 \leq x \leq 1),$$

$$= -\pi - \operatorname{arccos} \sqrt{1-x^2} \quad (-1 \leq x < -1/2),$$

$$= -\pi + \operatorname{arccos} \sqrt{1-x^2} \quad (-1/2 \leq x < 0),$$

$$\operatorname{arccos} x = \pi + \operatorname{arctg} \frac{\sqrt{1-x^2}}{x}$$

$$(-1 \leq x < 0),$$

$$= \operatorname{arcctg} \frac{x}{\sqrt{1-x^2}}$$

$$(-1 \leq x \leq 1),$$

$$= \frac{1}{2} \operatorname{arccos} (2x^2 - 1)$$

$$(0 \leq x \leq 1),$$

$$= \pi - \frac{1}{2} \operatorname{arccos} (2x^2 - 1)$$

$$(-1 \leq x < 0),$$

$$\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arcctg} x,$$

$$= \operatorname{arcsin} \frac{x}{\sqrt{1+x^2}},$$

$$(x \geq 0),$$

$$= \operatorname{arccos} \frac{1}{\sqrt{1+x^2}}$$

$$(x \leq 0),$$

$$= -\operatorname{arccos} \frac{1}{\sqrt{1+x^2}}$$

$$(x > 0),$$

$$= \frac{\pi}{2} - \operatorname{arctg} \frac{1}{x}$$

$$(x < 0),$$

$$= -\frac{\pi}{2} - \operatorname{arctg} \frac{1}{x}$$

$$(x > 0),$$

$$= \operatorname{arcctg} \frac{1}{x}$$

$$(x < 0),$$

$$= \operatorname{arctg} \frac{1}{x} - \pi$$

$$(x > 1),$$

$$= \frac{\pi}{4} - \operatorname{arctg} \frac{1-x}{1+x}$$

$$(x < -1),$$

$$= -\frac{3}{4}\pi - \operatorname{arctg} \frac{1-x}{1+x}$$

$$(x < -1),$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2}$$

$$(x > 1),$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} + \frac{\pi}{2}$$

$$(x < -1),$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} - \frac{\pi}{2}$$

$$(x < -1),$$

$$= -\frac{\pi}{2} - \frac{1}{2} \operatorname{arcsin} \frac{2x}{1+x^2}$$

$$(x < -1),$$

$$= \frac{1}{2} \operatorname{arcsin} \frac{2x}{1+x^2}$$

$$(-1 \leq x \leq 1),$$

$$= \frac{\pi}{2} - \frac{1}{2} \operatorname{arcsin} \frac{2x}{1+x^2}$$

$$(x > 1),$$

$$= \frac{1}{2} \operatorname{arccos} \frac{1-x^2}{1+x^2}$$

$$(x \geq 0),$$

$$= -\frac{1}{2} \operatorname{arccos} \frac{1-x^2}{1+x^2}$$

$$(x \leq 0),$$

$$\begin{aligned}
 \operatorname{arctg} x &= \frac{\pi}{2} - \operatorname{arctg} \frac{1}{x}, \\
 &= \arcsin \frac{1}{\sqrt{1+x^2}} & [x > 0], \\
 &= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} & [x < 0], \\
 &= \arccos \frac{1}{\sqrt{1+x^2}}, \\
 &= \operatorname{arctg} \frac{1}{x} & [x > 0], \\
 &= \pi + \operatorname{arctg} \frac{1}{x} & [x < 0].
 \end{aligned}$$

5. Суммы и разности обратных тригонометрических функций.

$$\begin{aligned}
 \arcsin x + \arcsin y &= \arcsin (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & [xy \leq 0 \text{ или } x^2 + y^2 \leq 1], \\
 &= \pm \pi - \arcsin (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & [xy > 0 \text{ или } x^2 + y^2 > 1], \\
 &= \pm \arccos [V(1-x^2)(1-y^2) - xy] & \left[\begin{cases} x, y > 0 \\ x, y < 0 \end{cases}, x^2 + y^2 > 1 \right], \\
 &= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{V(1-x^2)(1-y^2) - xy} & [xy \leq 0 \text{ или } x^2 + y^2 \leq 1], \\
 &= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{V(1-x^2)(1-y^2) - xy} \pm \pi & \left[\begin{cases} x, y > 0 \\ x, y < 0 \end{cases}, x^2 + y^2 > 1 \right], \\
 \arccos x + \arccos y &= 2\pi \begin{cases} 0 \\ 1 \end{cases} \pm \arccos [xy - V(1-x^2)(1-y^2)] & \left[\begin{cases} x + y > 0 \\ x + y < 0 \end{cases} \right], \\
 \arccos x - \arccos y &= \mp \arccos [xy + V(1-x^2)(1-y^2)] & \left[\begin{cases} x > y \\ x < y \end{cases} \right], \\
 \operatorname{arctg} x + \operatorname{arctg} y &= \operatorname{arctg} \frac{x+y}{1-xy} & [xy < 1], \\
 &= \pm \pi + \operatorname{arctg} \frac{x+y}{1-xy} & \left[\begin{cases} x > 1 \\ x < 1 \end{cases}, xy > 1 \right], \\
 \operatorname{arctg} x - \operatorname{arctg} y &= \operatorname{arctg} \frac{x-y}{1+xy} & [xy > -1], \\
 &= \pm \pi + \operatorname{arctg} \frac{x-y}{1+xy} & \left[\begin{cases} x > 0 \\ x < 0 \end{cases}, xy < -1 \right].
 \end{aligned}$$

1.4. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

1. Связь с логарифмической и обратными тригонометрическими функциями.

$$\begin{aligned}
 \operatorname{Arsh} z &= \ln(z + \sqrt{z^2 + 1}) = -i \operatorname{arcsin}(iz), \\
 \operatorname{Arch} z &= \ln(z + \sqrt{z^2 - 1}) = i \arccos z,
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{Arth} z &= \frac{1}{2} \ln \frac{1+z}{1-z} = -i \operatorname{arctg}(iz), \\
 \operatorname{Arcth} z &= \frac{1}{2} \ln \frac{z+1}{z-1} = -i \operatorname{arctg}(-iz).
 \end{aligned}$$

2. Функции отрицательного аргумента.

$$\begin{aligned}
 \operatorname{Arsh}(-z) &= -\operatorname{Arsh} z, & \operatorname{Arch}(-z) &= \operatorname{Arch} z, \\
 \operatorname{Arth}(-z) &= -\operatorname{Arth} z, & \operatorname{Arcth}(-z) &= -\operatorname{Arcth} z.
 \end{aligned}$$

3. Формулы дифференцирования.

$$\begin{aligned}
 \frac{d}{dx} \operatorname{Arsh} \frac{x}{a} &= \frac{1}{\sqrt{x^2 + a^2}}, \\
 \frac{d}{dx} \operatorname{Arch} \frac{x}{a} &= \frac{\pm 1}{\sqrt{x^2 - a^2}} & [\operatorname{Arch}(x/a) \geq 0, x/a > 1], \\
 \frac{d}{dx} \operatorname{Arth} \frac{x}{a} &= \frac{a}{a^2 - x^2} & [x^2 < a^2], \\
 \frac{d}{dx} \operatorname{Arcth} \frac{x}{a} &= \frac{a}{a^2 - x^2} & [x^2 > a^2].
 \end{aligned}$$

4. Связь между обратными гиперболическими функциями.

$$\begin{aligned}
 \operatorname{Arsh} x &= \operatorname{Arch} \sqrt{x^2 + 1}, & \operatorname{Arch} x &= \operatorname{Arsh} \sqrt{x^2 - 1}, \\
 &= \operatorname{Arth} \frac{x}{\sqrt{x^2 + 1}}, & &= \operatorname{Arth} \frac{\sqrt{x^2 - 1}}{x}, \\
 \operatorname{Arth} x &= \operatorname{Arsh} \frac{x}{\sqrt{1 - x^2}}, \\
 &= \operatorname{Arch} \frac{1}{\sqrt{1 - x^2}}, \\
 &= \operatorname{Arcth} \frac{1}{x}.
 \end{aligned}$$

5. Суммы и разности обратных гиперболических функций.

$$\begin{aligned}
 \operatorname{Arsh} x \pm \operatorname{Arsh} y &= \operatorname{Arsh} (x\sqrt{1+y^2} \pm y\sqrt{1+x^2}), \\
 \operatorname{Arch} x \pm \operatorname{Arch} y &= \operatorname{Arch} [xy \pm \sqrt{(x^2-1)(y^2-1)}], \\
 \operatorname{Arth} x \pm \operatorname{Arth} y &= \operatorname{Arth} \frac{x \pm y}{1 \pm xy}, \\
 \operatorname{Arsh} x \pm \operatorname{Arch} y &= \operatorname{Arsh} [xy \pm \sqrt{(x^2+1)(y^2-1)}], \\
 &= \operatorname{Arch} [y\sqrt{x^2+1} \pm x\sqrt{y^2-1}], \\
 \operatorname{Arth} x \pm \operatorname{Arcth} y &= \operatorname{Arth} \frac{xy \pm 1}{y \pm x}, \\
 &= \operatorname{Arcth} \frac{y \pm x}{xy \pm 1}.
 \end{aligned}$$

1.5. БИНОМИАЛЬНЫЕ КОЭФФИЦИЕНТЫ

$$\binom{a}{k} = (-1)^k \frac{(a)_k}{k!} = \frac{a(a-1)\dots(a-k+1)}{k!} \quad [k=1, 2, 3, \dots].$$

$$\binom{a}{0} = 1, \quad \binom{a}{-k} = 0 \quad [k=1, 2, 3, \dots; a \neq -1, -2, -3, \dots].$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad [k=1, 2, \dots, n].$$

$$\binom{n}{k} = 0 \quad [k=-1, -2, -3, \dots \text{ или } k > n].$$

$$\binom{n}{k} = \binom{n}{n-k} \quad [n \geq k \geq 0].$$

$$\binom{a}{k} = (-1)^k \binom{-a+k-1}{k}, \quad \binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

$$\binom{a}{b} = \binom{a}{a-b} \quad [b \neq -k, k+\sigma, \text{ где } k=-1, -2, -3, \dots].$$

$$\binom{a}{b+1} = \frac{a}{b+1} \binom{a-1}{b} = \frac{a-b}{b+1} \binom{a}{b}, \quad \binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1},$$

$$\binom{n+a}{n} \binom{b}{n+a} = \binom{b}{n} \binom{b-n}{a} = \binom{b}{a} \binom{b-a}{n},$$

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^{2n}} \binom{2n}{n} = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$

$$\binom{1/2}{n} = \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} = (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \frac{1}{n},$$

$$\binom{n+1/2}{2n+1} = (-1)^n 2^{-2n-1} \binom{2n}{n}, \quad \binom{2n+1/2}{n} = 2^{-2n} \binom{4n+1}{2n},$$

$$\binom{n}{1/2} = \frac{2^{2n+1}}{\pi} \binom{2n-1}{n}, \quad \binom{n}{n/2} = \frac{2^{2n}}{\pi} \binom{n}{(n-1)/2}.$$

1.6. СИМВОЛ ПОХАММЕРА $(a)_k$

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} = (-1)^k \frac{\Gamma(1-a)}{\Gamma(1-a-k)},$$

$$(a)_0 = 1, \quad (a)_{-k} = \frac{\Gamma(a-k)}{\Gamma(a)} = \frac{(-1)^k}{(1-a)_k} \quad [a \neq 1, 2, \dots, k; k=1, 2, 3, \dots].$$

$$(n)_k = \frac{(n+k-1)!}{(n-1)!}, \quad (1)_k = k!,$$

$$(1/2)_k = 2^{-2k} \frac{(2k)!}{k!}, \quad (3/2)_k = 2^{-2k} \frac{(2k+1)!}{k!},$$

$$(a)_{n+k} = (a)_n (a+n)_k, \quad (a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k},$$

$$(a)_{nk} = \binom{a}{n}_k \binom{a+1}{n}_k \dots \binom{a+n-1}{n}_k n^{nk}, \quad (a)_{2k} = \binom{a}{2}_k \binom{a+1}{2}_k 2^{2k},$$

$$(a+nk)_{nk} = \frac{(a)_{nk+nk}}{(a)_{nk}}, \quad (a+k)_k = \frac{(a)_{2k}}{(a)_k}, \quad (a+m)_k = \frac{(a)_k (a+k)_m}{(a)_m}.$$

$$(a-mk)_{nk} = (-1)^{mk} (a)_{nk-mk} (1-a)_{mk} \quad [n \geq m].$$

$$= (-1)^{mk} \frac{(1-a)_{mk}}{(1-a)_{mk-nk}} \quad [n < m].$$

$$(a-k)_k = (-1)^k (1-a)_k, \quad (a-m)_k = \frac{(1-a)_m (a)_k}{(1-a-k)_m},$$

$$(a+mk)_{n-k} = \frac{(a)_n (a+n)_{m-k-k}}{(a)_{mk}}, \quad (a+k)_{n-k} = \frac{(a)_n}{(a)_k},$$

$$(a-mk)_{n-k} = \frac{(-1)^{mk} (a)_n (1-a)_{mk}}{(1-a-n)_{n-k}}, \quad (a-k)_{n-k} = \frac{(-1)^k (a)_n (1-a)_k}{(1-a-n)_{n-k}}.$$

Приложение II. НЕКОТОРЫЕ СПЕЦИАЛЬНЫЕ ФУНКЦИИ И ИХ СВОЙСТВА

II.1. ГАММА-ФУНКЦИЯ $\Gamma(z)$

Определение: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ [Re z > 0].

При Re z ≤ 0 функция $\Gamma(z)$ определяется как аналитическое продолжение интеграла.

$\Gamma(z)$ — аналитическая функция во всей z -плоскости, за исключением точек $z=0, -1, -2, \dots$, в которых она имеет простые полюсы.

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!, \quad \Gamma(1) = \Gamma(2) = 1.$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}, \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$$

$$\Gamma\left(n + \frac{1}{3}\right) = \frac{1 \cdot 4 \cdot 7 \dots (3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right), \quad \Gamma\left(\frac{1}{3}\right) = 2,678938 \dots$$

$$\Gamma\left(n + \frac{2}{3}\right) = \frac{2 \cdot 5 \cdot 8 \dots (3n-1)}{3^n} \Gamma\left(\frac{2}{3}\right), \quad \Gamma\left(\frac{2}{3}\right) = 1,354118 \dots$$

$$\Gamma\left(n + \frac{1}{4}\right) = \frac{1 \cdot 5 \cdot 9 \dots (4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right), \quad \Gamma\left(\frac{1}{4}\right) = 3,625600 \dots$$

$$\Gamma\left(n + \frac{3}{4}\right) = \frac{3 \cdot 7 \cdot 11 \dots (4n-1)}{4^n} \Gamma\left(\frac{3}{4}\right), \quad \Gamma\left(\frac{3}{4}\right) = 1,225417 \dots$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{\sin \pi z}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

$$\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z}.$$

$$\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)} = \binom{z}{w}, \quad \frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n.$$

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=1}^{n-1} \Gamma\left(z + \frac{k}{n}\right).$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

$$\Gamma(3z) = \frac{5^{3z-1/2}}{2^{2z}} \Gamma(z) \Gamma\left(z + \frac{1}{3}\right) \Gamma\left(z + \frac{2}{3}\right).$$

$$|\Gamma(ix)|^2 = \frac{\pi}{x \operatorname{sh} \pi x}, \quad \left|\Gamma\left(\frac{1}{2} + ix\right)\right|^2 = \frac{\pi}{\operatorname{ch} \pi x},$$

$$|\Gamma(1+ix)|^2 = \frac{\pi x}{\operatorname{sh} \pi x}, \quad \Gamma\left(\frac{1}{4} + ix\right) \Gamma\left(\frac{3}{4} - ix\right) = \frac{\pi \sqrt{2}}{\operatorname{ch} \pi x + i \operatorname{sh} \pi x}.$$

II.2. ФУНКЦИЯ $\psi(z)$

Определение:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

$\psi(z)$ — аналитическая функция во всей z -плоскости, за исключением точек $z=0, -1, -2, \dots$, в которых она имеет простые полюсы.

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{k+z}, \quad \psi(z-n) = \psi(z) - \sum_{k=1}^n \frac{1}{z-k},$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}.$$

$$\psi(z) - \psi(-z) = -\pi \operatorname{ctg} \pi z - \frac{1}{z}, \quad \psi(z) - \psi(1-z) = -\pi \operatorname{ctg} \pi z,$$

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \operatorname{ctg} \pi z, \quad \psi\left(\frac{1}{2} + z\right) - \psi\left(\frac{1}{2} - z\right) = \pi \operatorname{tg} \pi z.$$

$$\psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n, \quad \psi(2z) = \frac{1}{2} \psi(z) + \frac{1}{2} \psi\left(z + \frac{1}{2}\right) + \ln 2.$$

$$\operatorname{Re} \psi(ix) = \operatorname{Re} \psi(1-ix), \quad \operatorname{Im} \psi(ix) = \frac{1}{2x} + \frac{\pi}{2} \operatorname{cth} \pi x.$$

$$\operatorname{Im} \psi(1+ix) = -\frac{1}{2x} + \frac{\pi}{2} \operatorname{cth} \pi x, \quad \operatorname{Im} \psi\left(\frac{1}{2} + ix\right) = \frac{\pi}{2} \operatorname{th} \pi x.$$

$$\psi(n+1) = -C + \sum_{k=1}^n \frac{1}{k}, \quad \psi(1) = -C.$$

$$\psi\left(\frac{1}{2} + n\right) = -C - 2 \ln 2 + 2 \sum_{k=0}^{n-1} \frac{1}{2k+1}, \quad \psi\left(\frac{1}{2}\right) = -C - 2 \ln 2.$$

$$\psi\left(\frac{1}{3}\right) = -C - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3, \quad \psi\left(\frac{2}{3}\right) = -C + \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3,$$

$$\psi\left(\frac{1}{4}\right) = -C - \frac{\pi}{2} - 3 \ln 2, \quad \psi\left(\frac{3}{4}\right) = -C + \frac{\pi}{2} - 3 \ln 2.$$

$$\psi\left(\frac{p}{q}\right) = -C - \ln(2q) - \frac{\pi}{2} \operatorname{ctg} \frac{p\pi}{q} + 2 \sum_{k=1}^{[(q-1)/2]} \cos \frac{2pk\pi}{q} \ln \sin \frac{k\pi}{q} \quad [p=1, 2, \dots, q-1; q=2, 3, 4, \dots].$$

$$\psi(a) = -C + (a-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+a)} \quad [\text{см. 5.1.7, 5.1.8}].$$

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}}.$$

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad [n=1, 2, 3, \dots].$$

$$\psi'(1) = \frac{\pi^2}{6}, \quad \psi^{(m)}(n+1) = (-1)^m m! \left[-\zeta(m+1) + \sum_{k=1}^n \frac{1}{k^{m+1}} \right],$$

$$\psi^{(n)}(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}, \quad \psi^{(n)}(z+m) = \psi^{(n)}(z) + (-1)^n n! \sum_{k=0}^{m-1} \frac{1}{(k+z)^{n+1}}$$

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1}-1) \zeta(n+1) \quad [n=1, 2, 3, \dots].$$

$$\psi\left(\frac{1}{2} \pm n\right) = \frac{\pi^2}{2} + 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}, \quad \psi\left(\frac{1}{2}\right) = \frac{\pi^2}{2}.$$

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1},$$

$$\psi^{(n)}(1-z) + (-1)^{n+1} \psi^{(n)}(z) = (-1)^n \frac{d^n}{dz^n} \operatorname{ctg} \pi z.$$

II.3. ФУНКЦИЯ $\beta(z)$

Определение:

$$\beta(z) = \frac{1}{2} \left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right) \right],$$

$$\beta(z) = \int_0^1 \frac{t^{z-1}}{1+t} dt \quad [\operatorname{Re} z > 0].$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+z} \quad [\text{см. 5.1.5}].$$

$$\beta(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+z)(2k+z+1)} \quad [\text{см. 5.1.9}].$$

$$\beta(z+1) = \frac{1}{z} - \beta(z), \quad \beta(1-z) = \frac{2\pi^2}{\sin \pi z} - \beta(z).$$

$$\beta(nz) = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \beta\left(z + \frac{k}{n}\right).$$

$$\beta(n+1) = (-1)^n \ln 2 + (-1)^n \sum_{k=1}^n \frac{(-1)^k}{k}, \quad \beta(1) = \ln 2.$$

$$\beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin(p\pi/q)} - \sum_{k=0}^{[(q-1)/2]} \cos \frac{(2k+1)p\pi}{q} \ln \left(2 - 2 \cos \frac{(2k+1)\pi}{q}\right) \\ [p=1, 2, 3, \dots, q-1; q=2, 3, 4, \dots].$$

II.4. ДЗЕТА-ФУНКЦИЯ РИМАНА $\zeta(s)$

Определение:

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} \quad [\operatorname{Re} z > 1].$$

При $\operatorname{Re} z \leq 1$, $z \neq 1$, $\zeta(z)$ определяется как аналитическое продолжение этого ряда.

$$\zeta(2n) = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}| \quad [n=1, 2, 3, \dots].$$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \quad [n=1, 2, 3, \dots].$$

$$\zeta(-2n) = 0 \quad [n=1, 2, 3, \dots].$$

$$\zeta'(0) = -\frac{1}{2} \ln 2\pi.$$

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad [n=2, 3, 4, \dots \text{ см. 5.1.2}].$$

$$\lim_{z \rightarrow 1} \left[\zeta(z) - \frac{1}{z-1} \right] = C.$$

II.5. МНОГОЧЛЕНЫ БЕРНУЛЛИ $B_n(x)$

Определение:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad [|t| < 2\pi].$$

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad [\text{см. 11.6}].$$

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{12}x^3 - \frac{1}{6}x, \quad B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^3 + \frac{1}{42}.$$

$$B_n(0) = B_n, \quad B_n(1) = (-1)^n B_n, \quad B_n\left(\frac{1}{2}\right) = -(1-2^{1-n})B_n.$$

II.6. ЧИСЛА БЕРНУЛЛИ B_n

Определение:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

$$B_n = \sum_{k=0}^n \binom{n}{k} B_k, \quad B_0 = 1, \quad B_n = B_n(0).$$

$$B_{2n+1} = \frac{(-1)^{n-1} (2n)!}{2^{2n-1} \pi^{2n}} \zeta(2n), \quad B_{2n+1} = 0 \quad [n=1, 2, 3, \dots].$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = -\frac{1}{30}, \quad B_4 = \frac{1}{42},$$

$$B_5 = -\frac{1}{30}, \quad B_6 = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

II.7. МНОГОЧЛЕНЫ ЭЙЛЕРА $E_n(x)$

Определение:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!} \quad [|t| < \pi].$$

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x,$$

$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \quad E_4(x) = x^4 - 2x^3 + x,$$

$$E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{1}{2}, \quad E_6(x) = x^6 - 3x^5 + 5x^4 - 3x.$$

II.8. ЧИСЛА ЭЙЛЕРА E_n

Определение:

$$\frac{1}{\operatorname{ch} t} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}.$$

$$E_n = 2^n E_n\left(\frac{1}{2}\right), \quad E_{2n+1} = 0,$$

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

$$E_{12} = -50521, \quad E_{14} = 2702765.$$

Приложение III. ТАБЛИЦА ФУНКЦИЙ V_l и $V_l^\pm$

Обозначение:

 $r=p/q$; p и q — взаимно простые натуральные числа; $m=2p+q+1$; $n=p+q+1$.

$$V_1^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_1^+ y^{j,n} F_{n-1} \left(1, \Delta(p, a_1), \Delta(q, b_1); z \right),$$

$$V_2^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_2^+ y^{h,n} F_{n-1} \left(1, \Delta(p, a_2), \Delta(q, b_2); z \right),$$

$$V_3^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_3^+ y^{h,n} F_{n-1} \left(1, \Delta(p, a_3), \Delta(q, b_3); z^{-1} \right),$$

$$V_4^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_4^+ y^{j,n} F_{n-1} \left(1, \Delta(p, a_4), \Delta(q, b_4); z^{-1} \right),$$

$$V_5^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_5^+ y^{j,q+1} F_q \left(1, \Delta(q, a_1); z \right),$$

$$V_6^+ = \sum_{h=0}^{p-1} E_6^+ y^{h,q+1} F_q \left(1, \Delta(q, a_2); z \right),$$

$$V_7^+ = \sum_{h=0}^{p-1} E_7^+ y^{h,q+1} F_q \left(1, \Delta(q, a_3); z^{-1} \right),$$

$$V_8^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_8^+ y^{j,q+1} F_q \left(1, \Delta(q, a_4); z^{-1} \right),$$

$$V_9^+ = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_9^+ y^{j,m} F_{m-1} \left(1, \Delta(p, a_1), \Delta(p, b_1), \Delta(q, c_1); z \right),$$

$$V_{10}^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{10}^+ y^{h,m} F_{m-1} \left(1, \Delta(p, a_2), \Delta(p, b_2), \Delta(q, c_2); z \right),$$

$$V_{11}^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{11}^+ y^{h,m} F_{m-1} \left(1, \Delta(p, a_3), \Delta(p, b_3), \Delta(q, c_3); z \right),$$

$$V_{12}^+ = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{12}^+ y^{h,m} F_{m-1} \left(1, \Delta(p, a_4), \Delta(p, b_4), \Delta(q, c_4); z^{-1} \right),$$

$$V_{13}^- = \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{13}^- y^{h,m} F_{m-1} \left(1, \Delta(p, a_5), \Delta(p, b_5), \Delta(q, c_5); z^{-1} \right),$$

$$V_{14}^- = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{14}^- y^{j,m} F_{m-1} \left(1, \Delta(p, a_6), \Delta(p, b_6), \Delta(q, c_6); z^{-1} \right),$$

$$V_{15}^+ = E_{15}^+ y^{k,k} F_3 \left(a_k, b_k, c_k, d_k; z \right) \quad [k=1, 2, 3, 4],$$

$$V_{16}^+ = E_{16}^+ y^{k,k} F_3 \left(a_k, b_k, c_k, d_k; z^{-1} \right) \quad [k=5, 6, 7, 8],$$

$$V_{17}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{17}^+ \frac{\sin [\gamma (lr + \alpha - 1)]}{\sin [(lr + \alpha) \pi]} y^{lr + \alpha - 2},$$

$$V_{18}^+ = \sum_{l=0}^{\infty} \Gamma \left(\frac{\alpha - l - 1}{r} \right) E_{18}^+ \sin (l\gamma) (-1)^l y^{l-2},$$

$$V_{19}^+ = \sum_{l=0}^{\infty} \Gamma \left(\frac{\alpha + l}{r} \right) E_{19}^+ \sin [(l+1)\gamma] (-1)^l y^{l-2},$$

$$V_{20}^+ = -\frac{\pi}{\sin \gamma} \sum_{l=0}^{\infty} \frac{(1-\rho)^l (-1)^l}{l!} E_{20}^+ \frac{\sin [\gamma (\alpha - 1 + r(\rho - l - 1))]}{\sin [(l + r(\rho - l - 1)) \pi]} y^{r(\rho - l - 1) - 2},$$

$$V_{21}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{21}^+ \frac{\sin [(lr + \alpha - 1) \gamma]}{(1-\sigma)^l \sin [(lr + \alpha) \pi] \sin \gamma} z^{lr + \alpha - 2},$$

$$V_{22}^+ = \sum_{l=0}^{\infty} (-1)^l E_{22}^+ \frac{\sin (l\gamma)}{\sin \gamma} \Gamma \left[\frac{(\alpha - l - 1)/r}{1 - \sigma - (\alpha - l - 1)/r} \right] z^{l-2},$$

$$V_{23}^+ = \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \sin \gamma} E_{23}^+ \frac{\sin [(\alpha - lr/2 - hr/2 - 1) \gamma]}{\sin [(\alpha - (lr + hr)/2) \pi]} \Gamma \left[\frac{(l+h)/2}{1 - \sigma - (l+h)/2} \right] z^{\alpha - r(l+h)/2 - 2},$$

$$V_{24}^+ = \sum_{l=0}^{\infty} (-1)^l E_{24}^+ \frac{\sin [(l+1) \gamma]}{\sin \gamma} \Gamma \left[\frac{(\alpha + l)/2, h-2(\alpha + l)/r}{1 - \sigma - (\alpha + l)/r} \right] z^{l-2},$$

$$V_{25}^+ = \pi \sum_{l=0}^{\infty} (-1)^l \frac{\sin [\lambda l - l\gamma] \sin [\gamma (1 - \alpha - lr)]}{\sin \lambda \sin \gamma \sin [(\alpha + lr) \pi]} y^{lr + \alpha - 2},$$

$$V_{26}^+ = \pi \sum_{l=0}^{\infty} (-1)^l \frac{\sin (l\gamma) \sin [\lambda r^{-1} (\alpha - l - 1 - r)]}{\sin \lambda \sin [(\alpha - l - 1) \pi/r] \sin \gamma} y^{l-2},$$

$$\begin{aligned}
 V_{10} &= \pi \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\lambda l) \sin[\gamma(\alpha - r - l - 1)]}{\sin \lambda \sin \gamma \sin[(\alpha - l - l)r\pi]} y^{r-l-2}, \\
 V_{17} &= \frac{\pi}{r} \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\gamma + l\gamma) \sin[\lambda r^{-1}(r - \alpha - l)]}{\sin \lambda \sin \gamma \sin[(\alpha + l)\pi/r]} y^{r-l-2}, \\
 V_{18} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{18} \frac{(\alpha + lr - 1) \cos[\gamma(\alpha + lr - 2)]}{\sin^2 \gamma \sin[(\alpha + l)\pi]} y^{r-l-2}, \\
 V_{19} &= \sum_{l=0}^{\infty} (-1)^l E_{19} \Gamma\left(\frac{\alpha - l - 1}{r}\right) \frac{l \cos(\gamma l - \gamma)}{\sin^2 \gamma} y^{r-l-2}, \\
 V_{20} &= \sum_{l=0}^{\infty} (-1)^l E_{20} \Gamma\left(\frac{\alpha + l}{r}\right) \frac{(l + 1) \cos(\gamma l + 2\gamma)}{\sin^2 \gamma} y^{r-l-2}, \\
 V_{21} &= \frac{\pi}{2 \sin^2 \gamma} \sum_{l=0}^{\infty} \frac{(-1)^l (1 - \rho) [\alpha - r(1 - \rho + l) - 1] \cos[\gamma(\alpha - r(1 - \rho + l) - 2)]}{l! \sin[(\alpha - r(1 - \rho + l))\pi]} y^{r-l-2}, \\
 V_{22} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{22} \frac{(lr + \alpha - 1) \cos[\gamma(lr + \alpha - 2)]}{\sin^2 \gamma \sin[(lr + \alpha)\pi] (1 - \sigma)_p} z^{r-l-2}, \\
 V_{23} &= \sum_{l=0}^{\infty} (-1)^l E_{23} \frac{l \cos(l\gamma - \gamma)}{\sin^2 \gamma} \Gamma\left[\frac{(\alpha - l - 1)/r}{1 - \sigma - (\alpha - l - 1)/r}\right] z^{r-l-2}, \\
 V_{24} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{24} \times \\
 &\quad \frac{(\alpha - r(h + l)/2 - 1) \cos[\gamma(\alpha - r(h + l)/2 - 2)]}{\sin^2 \gamma \sin[(\alpha - r(h + l)/2)\pi]} \Gamma\left[\frac{(l + h)/2}{1 - \sigma - (l + h)/2}\right], \\
 V_{25} &= \sum_{l=0}^{\infty} (-1)^l E_{25} \frac{(l + 1) \cos[(l + 2)\gamma]}{\sin^2 \gamma} \Gamma\left[\frac{(\alpha + l)/r}{1 - \sigma - (\alpha + l)/r}\right] z^{r-l-2}, \\
 V_{26} &= \frac{\pi}{2} \sum_{l=0}^{\infty} (-1)^l \frac{\sin(\lambda l + \lambda) \cos[\gamma(\alpha + lr - 2)] (1 - \alpha - lr)}{\sin \lambda \sin^2 \gamma \sin[(\alpha + l)r\pi]} y^{r-l-2}, \\
 &\quad \sum_{l=0}^{\infty} (-1)^l \frac{l \cos(\gamma l - \gamma) \sin[\lambda r^{-1}(\alpha - l - 1 - r)]}{\sin \lambda \sin^2 \gamma \sin[(\alpha - l - 1)\pi/r]} y^{r-l-2}, \\
 V_{27} &= \frac{r - l - 1}{\sin \lambda \sin^2 \gamma \sin[(\alpha - l - r)\pi]} y^{r-l-2}.
 \end{aligned}$$

$$\begin{aligned}
 V_{28} &= \frac{\pi}{2r} \sum_{l=0}^{\infty} (-1)^l \frac{(l + 1) \cos(\gamma l + 2\gamma) \sin[\lambda r^{-1}(r - \alpha - l)]}{\sin \lambda \sin^2 \gamma \sin[(\alpha + l)\pi/r]} y^{r-l-2}, \\
 V_{29} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{29} \frac{\sin[\gamma(lr + \lambda r + \alpha - 1)]}{\sin[(lr + \lambda r + \alpha)\pi]} y^{r-l-2}, \\
 V_{30} &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{30} \frac{(lr + \lambda r + \alpha - 1) \cos[\gamma(lr + \lambda r + \alpha - 2)]}{\sin[(lr + \lambda r + \alpha)\pi]} y^{r-l-2}, \\
 V_{31} &= \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{31} y^{n_h} q_{11} F_{q,p} \left(\begin{matrix} 1, \Delta(q, a); \\ \Delta(p, 1+h), \Delta(q, b) \end{matrix} \middle| z \right), \\
 V_{32} &= \sum_{l=0}^{q-1} \frac{(-1)^l}{l!} E_{32} y^{n_l} v_{q+1} F_{q,p} \left(\begin{matrix} 1, \Delta(q, a_1); \\ \Delta(p, b_1), \Delta(q, 1+j) \end{matrix} \middle| z \right), \\
 V_{33} &= \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{33} y^{n_j} u_k v_{k+1} F_{2q,p} \left(\begin{matrix} 1, \Delta(q, a_k), \Delta(q, b_k); \\ \Delta(p, c_k), \Delta(q, d_k), \Delta(q, 1+j) \end{matrix} \middle| z \right), \\
 &\quad [k = 1, 2], \\
 V_{34} &= \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{34} y^{n_h} s_{2q+1} F_{2q,p} \left(\begin{matrix} 1, \Delta(q, a_2), \Delta(q, b_2); \\ \Delta(p, 1+h), \Delta(q, c_2), \Delta(q, d_2) \end{matrix} \middle| z \right), \\
 V_{35} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{35} y^{n_h} v_{2q+1} \times \\
 &\quad \times q_{11} F_{q,p} \left(\begin{matrix} 1, \Delta(q, a_1); \\ \Delta(q, b_1), \Delta(p, c_1), \Delta(p, 1+h) \end{matrix} \middle| z \right), \\
 V_{36} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{36} y^{n_j} v_{2q+1} \times \\
 &\quad \times q_{11} F_{q,p} \left(\begin{matrix} 1, \Delta(q, a_2); \\ \Delta(q, 1+j), \Delta(p, b_2), \Delta(p, c_2) \end{matrix} \middle| z \right), \\
 V_{37} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{37} y^{n_h} v_{12} F_{2p} \left(\begin{matrix} 1; \\ \Delta(p, 1+h), \Delta(p, a_1) \end{matrix} \middle| z \right), \\
 V_{38} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} E_{38} y^{n_j} v_{12} F_{2p} \left(\begin{matrix} 1; \\ \Delta(p, a_2), \Delta(p, b_2) \end{matrix} \middle| z \right), \\
 V_{39} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{39} y^{n_h} v_{12} \times \\
 &\quad \times q_{11} F_{2q,p} \left(\begin{matrix} 1, \Delta(q, a_1), \Delta(q, b_1); \\ \Delta(q, c_1), \Delta(q, d_1), \Delta(p, e_1), \Delta(p, 1+h) \end{matrix} \middle| z \right),
 \end{aligned}$$

$$\begin{aligned}
 V_{20} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{2k} q^j \tilde{b}^{qj} \times \\
 &\quad \times {}_{2q+1}F_{2q+2p} \left(1, \Delta(q, a_2), \Delta(q, b_2); \right. \\
 &\quad \left. \Delta(q, 1+j), \Delta(q, c_2), \Delta(p, d_2), \Delta(p, e_2) \right) z^j, \\
 V_{21} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{2k} q^j \tilde{b}^{qj} \times \\
 &\quad \times {}_{2q+1}F_{2q+2p} \left(1, \Delta(q, a_2), \Delta(q, b_2); \right. \\
 &\quad \left. \Delta(q, 1+j), \Delta(q, c_2), \Delta(p, d_2), \Delta(p, e_2) \right) z^j, \\
 V_{30} &= y^a \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{3k} \frac{\tilde{b}^{cl} \sin a_l}{\Gamma(c_l) \sin b_l}, \\
 V_{31} &= y^a \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_{3k} \Gamma \left[\frac{c_l}{2} \right] \tilde{b}^{cl} \sin d_l, \\
 V_{40} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{4k} q^h \tilde{a}^{q^h} {}_{q+1}F_{2p} \left(1, \Delta(q, a_1); \right. \\
 &\quad \left. \Delta(p, b_1), \Delta(p, 1+h) \right) z^h, \\
 V_{41} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{4k} q^j \tilde{a}^{q^j} {}_{2p+1}F_{2p} \left(1, \Delta(p, a_2), \Delta(p, b_2); \right. \\
 &\quad \left. \Delta(q, 1+j) \right) z^j, \\
 V_{42} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{h=0}^{p-1} \frac{(-1)^h}{h!} E_{4k} q^h \tilde{a}^{q^h} {}_{1}F_{2p+q} \left(1; \right. \\
 &\quad \left. \Delta(p, 1+h), \Delta(p, a_1), \Delta(q, b_1) \right) z^h, \\
 V_{43} &= \sum_{k=0}^{[(n-1)/2]} e \binom{n}{k} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} E_{4k} q^j \tilde{a}^{q^j} {}_{2p+1}F_{2p+q} \left(1; \right. \\
 &\quad \left. \Delta(p, a_2), \Delta(p, b_2), \Delta(q, 1+j) \right) z^j.
 \end{aligned}$$

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УКАЗАТЕЛЬ ОБОЗНАЧЕНИЙ ФУНКЦИЙ И ПОСТОЯННЫХ

$A(x)$ — алгебраическая функция или степенная функция с произвольным показателем

$$\operatorname{am} u \equiv \operatorname{am}(u, k) \text{ — эллиптическая функция Якоби, } u = \int_0^{\operatorname{am} u} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

$\arccos z, \operatorname{arccos} z, \operatorname{arcsin} z, \operatorname{arcsin} z, \operatorname{arctg} z, \operatorname{arctg} z$ — обратные тригонометрические функции

$\operatorname{Arch} z, \operatorname{Arco} \operatorname{sech} z, \operatorname{Arsech} z, \operatorname{Arsh} z, \operatorname{Arth} z, \operatorname{Arth} z$ — обратные гиперболические функции

$\arg z$ — аргумент комплексного числа z ($z = |z| e^{i \arg z}$)

B_n — числа Бернулли

$B_n(z)$ — многочлены Бернулли

$\operatorname{bei}_\nu(z), \operatorname{ber}_\nu(z), \operatorname{lei}(z) = \operatorname{bei}_0(z), \operatorname{ber}(z) = \operatorname{ber}_0(z)$ ($\operatorname{ber}_\nu(z) + i \operatorname{bei}_\nu(z) =$

$= J_\nu(i \sqrt{2} z)$) — функции Кельвина

$C = 0,5772156649 \dots$ — постоянная Эйлера

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt \text{ — косинус-интеграл Френеля}$$

$$C(x, \alpha) = \int_0^x t^{\alpha-1} \cos t dt \quad [\operatorname{Re} \alpha < 1] \text{ — обобщенный косинус-интеграл Френеля}$$

$$C_n^{(\lambda)}(z) = \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-z}{2}\right) \text{ — многочлены Гегенбауэра}$$

$\operatorname{cd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u}$, $\operatorname{sn} u = \operatorname{sn}(\operatorname{am} u)$ — эллиптические функции Якоби

$\operatorname{ch} z$ — гиперболическая функция

$$\operatorname{chl}(x) = C + \ln x + \int_0^x \frac{\operatorname{ch} t - 1}{t} dt \text{ — интегральный гиперболический косинус}$$

$$\operatorname{cl}(x) = - \int_x^\infty \frac{\cos t}{t} dt \text{ — интегральный косинус}$$

$$\operatorname{Cl}_2(z) = - \int_0^z \ln\left(2 \sin \frac{x}{2}\right) dx \text{ — интеграл Клаузена}$$

$\cos z, \operatorname{cosec} z = \frac{1}{\sin z}, \operatorname{ctg} z$ — тригонометрические функции

$\operatorname{cosech} z = \frac{1}{\operatorname{sh} z}, \operatorname{cth} z$ — гиперболические функции

$$D(k) = \int_0^{\pi/2} \frac{\sin^2 \varphi d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = D\left(\frac{\pi}{2}, k\right) - \text{полный эллиптический интеграл}$$

$$D(\varphi, k) = \int_0^{\varphi} \frac{\sin^2 \varphi d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} - \text{эллиптический интеграл}$$

$$D_v(z) = 2\sqrt{2}e^{-z^2/4} \psi\left(-\frac{v}{2}, \frac{1}{2}, \frac{z^2}{2}\right) - \text{функция параболического цилиндра}$$

$$dn u = \frac{dn u}{\sin u}, \quad dn u = \sqrt{1-k^2 \sin^2 u} \quad (am u), \quad ds u = \frac{dn u}{\sin u} - \text{эллиптические функции Якоби}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} d\varphi - \text{полный эллиптический интеграл 2-го рода}$$

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2 \sin^2 \varphi} d\varphi - \text{эллиптический интеграл 2-го рода}$$

E_n — числа Эйлера

$E_n(z)$ — многочлены Эйлера

$$E_v(z) = \frac{1}{\pi} \int_0^{\pi} \sin(v\varphi - z \sin \varphi) d\varphi - \text{функция Вебера}$$

$$E_p(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})} \quad [\rho > 0] - \text{функция Миттаг-Леффлера}$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt - \text{интегральная показательная функция}$$

$e = 2,718281828459 \dots$

$e^x = \exp x$ — показательная функция

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt - \text{интеграл вероятности}$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt - \text{дополнительный интеграл вероятности}$$

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt - \text{интеграл вероятности мнимого аргумента}$$

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} - \text{эллиптический интеграл 1-го рода}$$

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} - \text{гипергеометрическая функция Гаусса}$$

Гаусса

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!} - \text{обобщенная гипергеометрическая функция}$$

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} - \text{вырожденная гипергеометрическая функция}$$

$$F_i(z, \zeta) \quad [i = 1, 2, 3, 4] - \text{функции Аппеля}$$

$$F_1(a, b, c; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_k (b)_l}{(c)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$F_2(a, b, c, d; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_k (b)_l (c)_l}{(c)_k (d)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$F_3(a, b, c, d, e; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_k (b)_l (c)_l}{(c)_k (d)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$F_4(a, b, c, d, e, f; z, \zeta) = \sum_{k, l=0}^{\infty} \frac{(a)_k (b)_l (c)_l}{(c)_k (d)_{k+l}} \frac{z^k \zeta^l}{k! l!} \quad [|z|, |\zeta| < 1]$$

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} = 0,9159655942 \dots - \text{постоянная Каталана}$$

$$H_v(z) = \frac{z}{\Gamma(v+1/2)} \left(\frac{z}{2}\right)^{v+1/2} \frac{1}{\Gamma(v+3/2)} {}_1F_2(1; 3/2, v+3/2; -z^2/4) - \text{функция Струве}$$

$$H_v^{(1)}(z) = J_v(z) + iY_v(z), \quad H_v^{(2)}(z) = J_v(z) - iY_v(z) - \text{функции Бесселя 3-го рода (функции Ганкеля 1-го и 2-го рода)}$$

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} - \text{многочлены Эрмита}$$

$$h_{\mu, \nu}(z) = \frac{z^{1+\nu} e^{-z^2}}{(1+\nu)! \Gamma(\mu+\nu+1)} {}_2F_2(1, \mu+3/2; \mu-\nu+2, \mu+\nu+2; 2z)$$

$$H_{\mu, \nu}(z) = h_{\mu, \nu}(z) - \frac{\Gamma(\mu-\nu+1) \Gamma(\mu+\nu+1)}{2^{1+\nu} \Gamma(\mu+3/2)} \left[I_{\nu}(z) + \frac{\sin(\nu-\mu)\pi}{\pi \cos \mu\pi} K_{\nu}(z) \right]$$

$$h_{\pm 1/2}^{(1)}(z), \quad h_{\pm 1/2}^{(2)}(z), \quad h_{\pm 1/2}(z) = \operatorname{hei}_{\pm 1/2}(z), \quad \operatorname{hei}(z) = \operatorname{hei}_0(z) \quad (\operatorname{hei}_{\pm 1/2}(z) + i \operatorname{hei}_{\mp 1/2}(z) = H_{\pm 1/2}^{(1)}(e^{3\pi i/4} z)) - \text{функции Кельвина}$$

$$I_{\nu}(z) = \frac{1}{\Gamma(\nu+1/2)} \left(\frac{z}{2}\right)^{\nu} {}_1F_1\left(\nu+1/2; \frac{z^2}{4}\right) = e^{-z^2/4} J_{\nu}(i\sqrt{z^2/4}) - \text{модифицированная функция Бесселя 1-го рода (функция Бесселя мнимого аргумента)}$$

$\operatorname{Im} z$ — мнимая часть комплексного числа $z = x + iy$ ($\operatorname{Im} z = y$)

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1/2)} {}_1F_1\left(\nu+1/2; \frac{z^2}{4}\right) - \text{функция Бесселя 1-го рода}$$

$$J_v(z) = \frac{1}{\pi} \int_0^{\pi} \cos(v\varphi - z \sin \varphi) d\varphi - \text{функция Ангера}$$

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = K\left(\frac{\pi}{2}, k\right) - \text{полный эллиптический интеграл 1-го рода}$$

$K_\nu(z) = \frac{\pi [I_{-\nu}(z) - I_\nu(z)]}{2 \sin \pi \nu}$ [$\nu \neq n$], $K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z)$ [$n = 0, \pm 1, \pm 2, \dots$] — функция Макдональда (модифицированная функция Бесселя 3-го рода)
 k, k' [$0 \leq k \leq 1$; $k' = \sqrt{1-k^2}$] — модуль и дополнительный модуль в эллиптических интегралах

$k_\nu(z) = \frac{1}{\Gamma(\frac{\nu}{2} + 1)} \Psi_{\nu/2, 1/2}(2z)$ — функция Бейтмена

$\ker_\nu(z)$, $\ker_\nu(z)$, $\ker(z) = \ker_0(z)$, $\ker(z) = \ker_0(z)$ ($\ker_\nu(z) + i \ker_\nu(z) = e^{-\nu \pi i/2} K_\nu(e^{\pi i/4} z)$) — функции Кельвина

$L(x) = -\int_0^x \ln \cos t \, dt$ — функция Лобачевского

$\mathbb{L}_\nu(z) = e^{-(\nu + 1/2)\pi i/2} H_\nu(\sqrt{z/2})$ — модифицированная функция Струве

$L_n(z) = L_n^+(z)$ — многочлены Лагерра

$L_n^\nu(z) = \frac{z^{-\nu} e^{-z}}{n!} \frac{d^n}{dz^n} (e^{z/2} z^n)$ — обобщенные многочлены Лагерра

$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ [$|z| < 1$]

$= \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} dt}{e^t - z}$ [$\operatorname{Re} s > 0$; $|\arg(1-z)| < \pi$] — полилогарифм порядка s

$\mathbb{B}(z) = \operatorname{Ei}(\ln z)$, $\Pi(x) = \int_0^x \frac{dt}{\ln t}$ — интегральный логарифм, $\ln z = \ln|z| + i \arg z$ — логарифм

галый логарифм [$z = |z| e^{i(\arg z + 2\pi k)}$, $k = 0, \pm 1, \pm 2, \dots$]

$M_{\kappa, \mu}(z) = z^{\mu+1/2} {}_2F_1(1/2 - \kappa + \mu; 2\mu + 1; z)$ — вырожденная гипергеометрическая функция Уиттекера

$\operatorname{sn} u$, $\operatorname{nd} u$, $\operatorname{ps} u$, $\operatorname{cs} u = \frac{1}{\operatorname{sn} u}$, $\operatorname{pd} u = \frac{1}{\operatorname{sn} u}$, $\operatorname{ps} u = \frac{1}{\operatorname{sn} u}$ — эллиптические функции

Якоби

$O_n(z) = \sum_{k=0}^{[n/2]} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$ — многочлены Неймана

$P_n(z) = \frac{z^n - 1}{n!} \frac{d^n}{dz^n} (z^n - 1)^n$ — многочлены Лежандра

$P_\nu(z) = P_\nu^\mu(z) = {}_2F_1(-\nu, 1+\nu; 1; \frac{1-z}{2})$ [$|\arg(1+z)| < \pi$] — функция Лежандра 1-го рода

$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{\mu/2} {}_2F_1(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2})$ [$|\arg(z \pm 1)| < \pi$]

$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+z}{1-z}\right)^{\mu/2} {}_2F_1(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2})$ [$-1 < z < 1$] — присоединенная функция Лежандра 1-го рода

$P_n^{(\alpha, \beta)}(z) = \frac{(-1)^n}{2^n n!} (1-z)^{-\alpha} (1+z)^{-\beta} \frac{d^n}{dz^n} [(1-z)^{\alpha+n} (1+z)^{\beta+n}]$ — многочлены Якоби

$Q_\nu(z) = Q_\nu^0(z)$ — функция Лежандра 2-го рода

$Q_\nu^\mu(z) = \frac{(-1)^{\mu+1}}{2^{\mu+1} \Gamma(\frac{\mu+1}{2})} \Gamma\left[\frac{\mu+\nu+1}{2}\right] e^{-\nu \pi i/2} {}_2F_1((\mu+\nu+1)/2, (\mu+\nu)/2+1; \nu+3/2; z^2)$ [$|\arg(z \pm 1)|, |\arg z| < \pi$]

$Q_\nu^\mu(z) = \frac{1}{2} e^{-i\mu\pi} \left[e^{-i\mu\pi/2} Q_\nu^\mu(x+i0) + e^{i\mu\pi/2} Q_\nu^\mu(x-i0) \right] =$

$= \frac{\pi}{2 \sin \pi \mu} \left[P_\nu^\mu(x) \cos \pi \mu - \Gamma\left[\frac{\nu+\mu+1}{2}\right] P_\nu^{-\mu}(x) \right]$ [$-1 < x < 1$] — присоединенная функция Лежандра 2-го рода

$R(x)$ — рациональная функция

$\operatorname{Re} z$ — действительная часть комплексного числа $z = x + iy$ ($\operatorname{Re} z = x$)

$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{t} dt$ — синус-интеграл Френеля

$S(x, \alpha) = \int_0^x t^{\alpha-1} \sin t \, dt$ [$\operatorname{Re} \alpha < 1$] — обобщенный синус-интеграл Френеля

$S_{\mu, \nu}(z) = S_{\mu, \nu}(z) + 2^{2\mu-1} \Gamma\left[\frac{\nu, \mu+\nu+1}{2}\right] (z/2)^{-\nu} {}_2F_1(1-\nu, -z^2/4; 2\mu-1; x) \times$
 $\times \Gamma\left[\frac{-\nu, (\mu+\nu-1)/2}{(1-\mu-\nu)/2}\right] (z/2)^\nu {}_2F_1(1+\nu, -z^2/4; \text{— функция Ломмеля}$

$S_{\mu, \nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_2F_2(1; (\mu-\nu+3)/2, (\mu+\nu+3)/2; -z^2/4; \text{— функция Ломмеля}$

$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt$ — интегральный синус

$\operatorname{si}(x) = \operatorname{Si}(x) - \frac{\pi}{2} = -\int_x^\infty \frac{\sin t}{t} dt$ — интегральный синус

$\operatorname{se} z = \frac{1}{\cos z}$ — тригонометрическая функция

$\operatorname{sch} z = \frac{1}{\operatorname{ch} z}$ — гиперболическая функция

$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$

$\operatorname{sh} z$ — гиперболическая функция

$\operatorname{shi} x = \int_0^x \frac{\operatorname{sh} t}{t} dt = -i \operatorname{Si}(ix)$ — интегральный гиперболический синус

$\sin z$ — тригонометрическая функция

$\operatorname{sn} u = \operatorname{sn}(\operatorname{am} u)$ — эллиптическая функция Якоби

$T_n(z) = \cos(n \arccos z) = \frac{1}{2} [(z + \sqrt{z^2-1})^n + (z - \sqrt{z^2-1})^n]$ — многочлены Чебышева 1-го рода

$\operatorname{tg} z = \frac{\sin z}{\cos z}$ — тригонометрическая функция

$\operatorname{th} z = \frac{\operatorname{sh} z}{\operatorname{ch} z}$ — гиперболическая функция

$U_n(z) = \frac{\sin[(n+1) \arccos z]}{\sqrt{1-z^2}}$ — многочлены Чебышева 2-го рода

$U_{\nu, \mu}(z) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^{\nu+2k} J_{\nu+2k}(\mu) \text{ — функция Ломмеля двух переменных}$

$W_{\kappa, \mu}(z) = z^{\mu+1/2} e^{-z/2} {}_2F_1(1/2 - \kappa + \mu, 2\mu + 1; z)$ — вырожденная гипергеометрическая функция Уиттекера

$Y_\nu(z) = \frac{\cos \nu \pi J_\nu(z) - J_{-\nu}(z)}{\sin \nu \pi}$ [$\nu \neq n$], $Y_n(z) = \lim_{\nu \rightarrow n} Y_\nu(z)$ [$n = 0, \pm 1, \pm 2, \dots$] — функция Неймана (функция Бесселя 2-го рода)

$\nu_n(z)$ — многочлены Бесселя

$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ — бета-функция

$\beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right]$

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ [$\operatorname{Re} x > 0$] — гамма-функция

$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$ — дополнительная неполная гамма-функция

$\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$ — неполная гамма-функция

$\Gamma \left[\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \right] = \frac{\prod_{k=1}^m \Gamma(a_k)}{\prod_{l=1}^n \Gamma(b_l)}$

$\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \dots, \frac{a+k-1}{k}$

$\Delta(2, a) = \frac{a}{2}, \frac{a+1}{2}$

$\delta_{m,n} = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases}$ — символ Кронекера

$\zeta(s) = \sum_{k=1}^\infty \frac{1}{k^s}$ [$\operatorname{Re} s > 1$] — дзета-функция Римана

$\zeta(s, z) = \sum_{k=0}^\infty \frac{1}{(v+k)^s}$ [$\operatorname{Re} s > 1$; $v \neq 0, -1, -2, \dots$] — обобщенная дзета-функция

Римана

$\theta_0(v, x) = 1 + 2 \sum_{k=0}^\infty (-1)^k e^{-k^2 \pi^2 x} \cos 2k\pi v$

$\theta_1(v, x) = 2 \sum_{k=0}^\infty (-1)^k e^{-(k+1/2)^2 \pi^2 x} \sin(2k+1)\pi v$

$\theta_2(v, x) = 2 \sum_{k=0}^\infty e^{-(k+1/2)^2 \pi^2 x} \cos(2k+1)\pi v$

$\theta_3(v, x) = 1 + 2 \sum_{k=1}^\infty e^{-k^2 \pi^2 x} \cos 2k\pi v$

$\theta_4(v, x) = \theta_0(v, x)$ — эллиптические тета-функции

$\lambda(x, \alpha) = \int_0^\alpha x^{-t} \Gamma(t+1) dt$

$\Psi(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt$

$\Pi(\varphi, n, k) = \int_0^\pi \frac{d\varphi}{(1-n \sin^2 \varphi) \sqrt{1-k^2 \sin^2 \varphi}}$ — эллиптический интеграл 3-го рода

$\Phi(z, s, v) = \sum_{k=0}^\infty \frac{z^k}{(v+k)^s}$ [$|z| < 1$; $v \neq 0, -1, -2, \dots$]

${}_p\Phi_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!}$ — базисный гипергеометрический ряд

$(a)_q, k = (1-a)(1-aq) \dots (1-aq^{k-1})$ [$k = 1, 2, 3, \dots$],

$(c)_q, q = 1$.

$\Psi(a, b; z) = \Gamma \left[\begin{matrix} 1-c \\ 1+a-c \end{matrix} \right] F_1(a; c; z) + \Gamma \left[\begin{matrix} c-1 \\ a \end{matrix} \right] z^{1-c} F_1(1+a-c; 2-c; z)$ — вырожденная гипергеометрическая функция

$\Phi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ — пси-функция

$\Psi^{(n)}(z) = \frac{d^n}{dz^n} \Phi(z)$

УКАЗАТЕЛЬ ОБОЗНАЧЕНИЙ СИМВОЛОВ

$(a)_k = a(a+1) \dots (a+k-1)$ [$k=1, 2, 3, \dots$], $(a)_0 = 1$ — символ Похгаммера

$n! = 1 \cdot 2 \cdot 3 \dots (n-1)n$, $0! = 1$

$(2n)! = 2 \cdot 4 \cdot 6 \dots (2n) = 2^n n!$

$(2n+1)! = 1 \cdot 3 \cdot 5 \dots (2n+1) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(n + \frac{3}{2}\right) = \left(\frac{3}{2}\right)_n 2^n$

$n! = \begin{cases} (2k)! & n = 2k, \\ (2k+1)! & n = 2k+1, \end{cases} \quad 0! = (-1)! = 1$

$\binom{n}{k} = \frac{n(n-1) \dots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \frac{(-1)^k (-n)_k}{k!}$ — биномиальные коэффициенты

цененты

$\binom{n}{0} = 1$

$\operatorname{Re} a, \operatorname{Re} b > c$ означает $\operatorname{Re} a > c$ и $\operatorname{Re} b > c$

$[x] = n$ [$n \leq x < n+1$, $n=0, \pm 1, \pm 2, \dots$] — целая часть числа x

$\bar{z} = x - iy$ [$z = x + iy$]

$|z| = \sqrt{x^2 + y^2}$

$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$

$\prod_{k=m}^n a_k = a_m a_{m+1} \dots a_n$ [$n \geq m$]
 $= 1$ [$n < m$]

$\prod_{k=1}^{\infty} a_k(x) = \lim_{n \rightarrow \infty} \prod_{k=1}^n a_k(x)$

$\sum_{k=m}^n a_k = a_m + a_{m+1} + \dots + a_n$ [$n \geq m$]
 $= 0$ [$n < m$]

$\sum_{k=1}^{\infty} a_k(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k(x)$

УКАЗАТЕЛЬ РАЗЛОЖЕНИЙ НЕКОТОРЫХ ФУНКЦИЙ В РЯДЫ

ЭЛЕМЕНТАРНЫЕ ФУНКЦИИ

$(1+x)^n$ 4.2.3.1, 5.1.29.12—13, 5.2.2.1—10, 5.2.10.8, 5.2.11.17

$(1-x)^{-n}$ 5.2.11.3

$(1-x)^{\pm n/2}$ 5.2.13.1—2, 5.2.13.6

$(1+x+\dots+x^m)^p$ 5.2.18.12

e^x 5.2.7.2

a^x 5.2.7.3

$\operatorname{sh} x$ 5.2.7.6, 5.4.5.8, 5.4.6.24—25

$\operatorname{th}^{2n+1} x$ 4.3.1.8

$\operatorname{li}^{2n} x$ 4.3.1.10

$\operatorname{ch} x$ 5.2.7.5, 5.4.5.1—2, 5.4.6.26—27

$\operatorname{ch}^{2n} x$ 4.3.1.12

$\operatorname{cosech} x$ 4.3.2, 5.1.25, 5.1.27.5, 5.3.5.8, 5.3.6.7, 5.4.5.5, 5.4.5.8

$\operatorname{cosech}^2 x$ 5.3.6.7

$\operatorname{sech} x$ 4.3.2, 5.1.26

$\operatorname{th} x$ 4.3.2, 5.1.26, 5.1.28.1

$\operatorname{cth} x$ 4.3.2, 5.1.25, 5.1.26.16, 5.1.27.3—4, 5.3.8.4—6

$e^r \operatorname{ch} x \operatorname{sh}(r \operatorname{sh} x)$ 5.3.3.2

$e^r \operatorname{ch} x \operatorname{ch}(r \operatorname{sh} x)$ 5.3.3.2

$\operatorname{sh} x \operatorname{sech} y$ 5.4.6.24

$\operatorname{ch} x \operatorname{sech} y$ 5.4.6.27

$\operatorname{acch} x \operatorname{sec} x$ 5.3.7.10—11

$\frac{1}{x^2} \operatorname{th} x \operatorname{tg} x$ 5.3.8.3

$\frac{1}{x^2} \operatorname{cth} x \operatorname{ctg} x$ 5.3.9.5

$\sin x$ 5.2.7.6, 5.4.3.4—6, 5.4.5.8, 5.4.6.19—20, 5.4.6.24—25

$\sin^{2n} x$ 4.4.1.16, 5.2.18.8

$\sin^{2n+1} x$ 4.4.1.17, 5.2.19.10

$\cos x$ 5.2.7.5, 5.4.3.4—5, 5.4.5.1—2, 5.4.6.21—22, 5.4.16.4, 5.4.6.26—27

$\cos^2 x$ 5.4.6.23, 5.4.16.5

$\cos^n x$ 4.4.1.16—18, 5.2.18.9

$e^x \sin x$ 5.2.7.4

$e^x \cos x$ 5.2.7.4

$\operatorname{sh} x \sin x$ 5.2.7.13

$\operatorname{ch} x \sin x$ 5.2.7.7

$\operatorname{sh} x \cos x$ 5.2.7.7

$\operatorname{ch} x \cos x$ 5.2.7.10

$\operatorname{sh} x - \sin x$ 5.2.7.15

$\operatorname{sh} x + \sin x$ 5.2.7.11

$\operatorname{ch} x + \cos x$ 5.2.7.10

$\operatorname{ch} x - \cos x$ 5.2.7.13

$\frac{\sin x}{\cos x + \operatorname{ch} u}$ 5.4.12.1

$\frac{1}{\cos x + \operatorname{ch} u}$ 5.4.12.2

$e^{\pm i} \cos x \sin(\sin x)$, $e^{\pm i} \cos x \cos(\sin x)$ 5.4.7.1—2

$\sin x \sec y$ 5.4.6.24

$\cos x \sec y$ 5.4.6.27

$(\cos x)^{-1/2} \sin \frac{x}{2}$ 5.4.6.30

$(\cos x)^{-1/2} \cos \frac{x}{2}$ 5.4.6.30, 5.4.9.6

$\operatorname{cosec} x$ 5.1.25, 5.1.27, 5.2.3.5—6, 5.3.5.8, 5.4.5.5, 5.4.5.8, 5.4.16.10, 5.4.16.16

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